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Sebastian Wiederrecht

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Kurzfassung

Matchingminoren sind eine Spezialisierung des herkömmlichen Minorenbegriffs für Graphen, die die Existenz und elementare strukturelle Eigenschaften von perfekten Matchings erhalten. Ein Teilgraph H eines Graphen G heißt konform, wenn G und $G - H$ perfekte Matchings besitzen und ein Konten von Grad genau zwei heißt bikontrahierbar. Eine Bikontraktion eines bikonrahierbaren Kontens ist die Operation bei der beide, mit dem Knoten inzidenten, Kanten gleichzeitig kontrahiert werden. Ein Matchinminor schließlich ist ein Graph H', der durch eine Reihe von Bikontraktionen aus einem konformen Subgraphen des Graphen G hervorgeht. Ähnlich wie reguläre Minoren gewisse Eigenschaften von Graphen beschreiben, so können Matching Minoren Eigenschaften von Graphen mit perfekten Matchings beschreiben.

In der vorliegenden Arbeit werden fundamentale Teile des Graphminorenprojekts von Robertson und Seymour für das Studium von Matching Minoren adaptiert und Verbindungen zur Strukturtheorie gerichteter Graphen aufgezeigt. Im Einzelnen entwickeln wir matchingtheoretische Analogismen zu etablierten Resultaten des Graphminorenprojekts wie folgt: Wir charakterisieren die Existenz eines Kreuzes über einem konformen Kreis mittels topologischer Eigenschaften und schlussfolgern, dass das Zwei-Pfade-Problem für alternierende Pfade in bipartiten Graphen mit perfekten Matchings in Polynomialzeit lösbar ist. Weiter entwickeln wir eine Theorie zu perfekter Matchingweite, einem Weiteparameter für Graphen mit perfekten Matchings, der von Norine eingeführt wurde. Hier zeigen wir, wie sich Matchigminoren und perfekte Matchingweite zu einander verhalten, wir zeigen, dass das Disjunkte Alternierende Pfade Problem auf bipartiten Graphen mit beschränkter Weite in Polynomialzeit lösbar ist. Weiter zeigen wir, dass jeder bipartite Graph mit hoher perfekter Matchingweite ein großes Gitter als Matchingminor enthalten muss, was wir dann verwenden, um zu zeigen, dass das Erkennen von bipartiten und planaren Matchingminoren in Polynimialzeit durchführbar ist. Wir

erweitern die Erdős-Pósa Eigenschaft für Minoren auf Matchingminoren in bipartiten Graphen und charakterisieren so alle Klassen beschränkter perfekter Matchingweite. Schließlich vereinen wir alle diese Ergebnisse zu einem Analogon des Flat Wall Theorem und geben eine qualitative Beschreibung aller bipartiter Graphen an, die, für ein festes $t \in \mathbb{N}$, den $K_{t,t}$ als Matchingminor ausschließen.

Parallel zu diesen Ergebnissen entwickeln wir eine Theorie für unendliche Antiketten von Butterflyminoren in gerichteten Graphen, die mit der oben genannten Theorie für Matchingminoren korrespondiert.

Abstract

Matching minors are a specialisation of usual graph minors which preserves the existence, and the elementary structural properties of perfect matchings. A subgraph H of a graph G is conformal if G and $G - H$ both have a perfect matching, and a vertex of degree exactly two is called bicontractible. A bicontraction of a bicontractible vertex is the operation of contracting both of its incindent edges, at the same time. Finally, a matching minor is a graph H' that can be obtained by a series of bicontractions from a conformal subgraph of G. Similar to how ordinary minors capture certain structural properties of graphs, matching minors are able to capture structural properties of graphs with perfect matchings.

In this thesis fundamental parts of the Graph Minors Project by Robertson and Seymour are adapted for the study of matching minors and connections to structural digraph theory are identified. To be precise, we develop matching theoretic analogues to established results from the Graph Minors Project as follows: We characterise the existence of a cross over conformal cycles via topological properties and deduce that the Two-Paths-Problem for alternating paths in bipartite graphs with perfect matchings is solvable in polynomial time. Furthermore, we develop a theory for perfect matching width, a width parameter for graphs with perfect matchings introduced by Norine. Here we show how matching minors and perfect matching width interact, we show that the Disjoint Alternating Paths Problem can be solved in polynomial time on bipartite graphs of bounded width. Moreover, we show that every bipartite graph of large perfect matching width contains a large grid as a matching minor, which we then use to show that recognising planar and bipartite graphs with perfect matchings as matching minors is solvable in polynomial time. We extend the Erdős-Pósa property for minors to matching minors in bipartite graphs and thereby characterise all classes of bipartite graphs with perfect matchings of bounded perfect matching width. Finally, we combine all of the results above to obtain an analogue to the Flat Wall Theorem and thus give a

qualitative description of all bipartite graphs which, for some fixed $t \in \mathbb{N}$, exclude $K_{t,t}$ as a matching minor.

In parallel to those results we develop a theory for infinite anti chains of butterfly minors in digraphs which corresponds to the theory of matching minors as described above.

Preface

A major role in the widespread interest in structural graph theory, as popularised by Robertson and Seymour in their Graph Minors Project, is probably played by the large area of algorithmic applications that arose from the notion of treewidth and its dual versions. Treewidth itself can already be used to find many, often times even straightforward, parametrised algorithms for problems which are otherwise computationally hard. It is hard to say what would have happened, if the Graph Minors Project would have produced only the structural results known today, but without most of their algorithmic application, it might be fair to think that the influence of treewidth would not be nearly as far reaching as it is today.

In fact, we can find an example for such a phenomenon when we consider the directed analogue of treewidth. Directed Treewidth, as its undirected cousin, aims to decompose a given directed graph into a tree-like shape using small cut sets. However, the tree-like shape itself, as well as the cut sets, are considerably less well behaved than their undirected brethren, a fact that makes most of the proofs involved much more complicated while simultaneously limiting the algorithmic power of directed treewidth. As a result, considerably less attention is received even by results as strong as the Directed Grid Theorem and thus, structural digraph theory is often accompanied by an uncertainty about which definitions are the correct ones to work with in the field

An even more niche topic is the field of structural matching theory, which, as we will explain in the first part of this thesis, can be seen as a common generalisation of both directed and signed graphs. Indeed, while matchings and even perfect matchings find many applications in graph theory as a whole, few of the deeper structural theorems beyond the algorithmically important ones like the Gallai-Edmonds Structure Theorem have found a home outside structural matching theory itself. Even milestone breakthroughs like the resolution of the bipartite Pfaffian

Recognition Problem appear to be relatively unknown outside of the community, despite of its many algorithmic applications. Surely, the often times long and complicated proofs, involving lots of case distinctions, are one reason for this lack of interest, which can probably not be helped without developing significantly better tools.

Another reason might be accessibility. Besides the stellar monograph by Lovász and Plummer [LP09], there is little to no literature to be found on the topic except for the journal articles themselves. Moreover, the book by Lovász and Plummer has last been updated in 2009. Since then however, we have seen a surge in new results, especially regarding the theory of matching minors which is still in its infancy, even when compared to the theory of butterfly minors in directed graphs. To address this problem of accessibility, this thesis contains an extensive chapter meant as an introduction to structural matching theory as a whole. Of course the main focus still lies on the topics relevant to this thesis, but as our main topic itself is the theory of matching minors, a good portion[1] of what might be considered to be the state of the art, at the time of writing, has found its way into the first chapter. We hope that such an extended survey might help to make the topic as a whole more accessible.

[1] The significantly more complicated topic of non-bipartite matching covered graphs is only discussed sparingly, even in the introduction. This topic has many interesting problems and techniques to offer and there sadly was no room to cram more of it into this document. The interested reader is referred to the PhD thesis of Norine [Nor05] and the PhD thesis of Kothari [Kot16].

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It hard to grasp the full extent of support one gets over the course of a five year project such as this one. Many people have contributed to this research, some of them even without having any knowledge of doing so or of mathematics in general and even I might not be able to name them all.

First I want to express gratitude to my supervisor Stephan Kreutzer who took a leap of faith in suggesting the topic of matching theory, a relatively unexplored area of research which he himself had little experience with. He provided me with all the freedom I needed, sometimes shielding me from tasks that could have been a distraction. Whenever I was in need of something, got stuck, wanted to go somewhere, or sometimes just wanted to take a day off, he would let me, usually smiling and mocking my way of giving silly names to the things I discovered in a friendly way. This trust in me and my vision provided me with the freedom I needed to find my own path.

Almost as important as Stephan's supervision was the cooperation with Archontia Giannopoulou. During my first year in Berlin she would accompany me on my way of discovering these new concepts, she would tirelessly listen to what I had thought of and more often than not it was one of her remarks that lead to the next breakthrough. She would also always listen when I started doubting myself or felt lost within a project that, at times, could feel far bigger than I could handle, and at other times just so niche and insignificant that I started to struggle with the idea of working in matching theory all together. During my stay in Athens in 2019 we shared several weeks of great adventure, laying the foundation for the Matching Two Paths Theorem and the Matching Flat Wall, at one time even while hiding within an old office of an occupied university building just to have access to a blackboard.

Among my fellow PhD students, not all of them under Stephan's supervision, there are three who made a big difference in my life in the past five years, Meike Hatzel, Maximilian Gorsky, as well as Raphael Steiner. The three of them were pillars I could always lean on, when I needed someone to discuss with, at times they would even hold back their own work to be there and support me. I am sure that, over time the word ‘matching’, coming from my mouth, became an indicator for them that once again I had to tell a story and I needed them to listen and help me process my thoughts, and listen they did! Thank you guys, you are the best and I hope our friendship will last for many years to come!

I would also like to thank all of the other colleagues who have been part of our group at TU Berlin over the past five years, namely Saeed Akhoondian Amiri, Jakub Gajarský, Karl Heuer, O-Joung Kwon, Irene Muzi, Jana Pilz, Roman Rabinovich, Jean-Florent Raymond, and Sebastian Siebertz, all of whom I had fruitful discussions and fun times with.

Of course, this thesis would not have been possible without Archontia Giannopoulou, Maximilian Gorsky, Meike Hatzel, Karl Heuer, Hania Lask, Carina Pukade, Raphael Steiner, and Eva Westebbe who carefully read through these pages and providing helpful remarks along the way while pointing out all the errors which occur while writing such a long text. Finally I need to thank my family and the many, many dear friends I made during my time in Berlin. You really helped to shape this city into a place I feel at home in and without you and your support, I am sure, none of this would have been possible. Thank you.

"Wenn ein anderer z.B. ein Buch, ein Gedicht, einen Roman gelesen hat, das einen starken Eindruck auf ihn machte und ihm die Seele füllte, wenn er nun mit diesem Eindruck in eine Gesellschaft tritt, er sei nun froh oder schwermütig gestimmt, er kann sich mitteilen, und man versteht ihn. Aber wenn ich einen mathematischen Lehrsatz ergründet habe, dessen Erhabenheit und Größe mir auch die Seele füllte, wenn ich nun mit diesem Eindruck in eine Gesellschaft trete, wem darf ich mich mitteilen, wer versteht mich? Nicht einmal ahnen darf ich lassen, was mich zur Bewunderung hinriß, nicht einen von allen Gedanken darf ich mitteilen, die mir die Seele füllen. - Und so muß man denn freilich zuweilen leer und gedankenlos erscheinen, ob man es gleich wohl nicht ist."

Heinrich von Kleist [VK60]

Contents

Chapter 1.

Introduction

Graph minors sit at the heart of many central problems of graph theory including the famous Four Colour Conjecture itself. The notion of graph minors has inspired many different research projects, among those the *Graph Minors Project* by Robertson and Seymour can be seen as one of the most influential works on graph theory to this date. In their series of over twenty papers Robertson and Seymour have adapted and developed numerous ideas, connecting different fields of graph theory and topology, which lead to several powerful discoveries like the resolution of Wagner's Conjecture (see Theorem 2.2.12) and a first polynomial time algorithm for the t-Disjoint Paths Problem. At the heart of this theory sits the idea that any graph whose structure is similar to the structure of a tree is structurally 'simple', while those graphs for which such a description does not exist must contain a large planar minor. The idea of *treewidth*, i.e. a measure for how 'tree-like' a graph is, and its dual notions, especially the Grid Theorem (see Theorem 2.2.25), gave rise to a wide area of both theoretical and algorithmic results.

Inspired by the overall success of graph minors as a concept, the idea of reducing subgraphs by contracting certain areas took hold quickly and was introduced to many different areas of graph theory. In this thesis, we are concerned with two of these fields, namely (bipartite) graphs with perfect matchings and directed graphs. We aim to establish a structure theory similar to the Graph Minors Project for both, *matching minors* in bipartite graphs with perfect matchings, and *butterfly minors* in directed graphs. Both of these fields are not new, and especially the latter has received a fair amount of attention in recent years. Indeed, a directed version of treewidth, as well as a directed version of the grid theorem, based on butterfly minors, already exist (see Section 2.3). However, the field of matching minors is largely underdeveloped, while the area of butterfly minors has some unique challenges due to the increased complexity of directed graph, as opposed to undirected graphs. This thesis seeks to provide a unified framework for both matching minors in bipartite graphs, and butterfly minors in directed graphs, which allows us to generalise large parts of the Graph Minors Project to both settings, while also providing a new approach to some of the unique challenges of the study of butterfly minors.

The Structure of this Thesis

This thesis is split into three different parts.

i) An introductory Part I, which exists to provide context to the results presented in this thesis and also contains a broad survey on previous results regarding matching theory as a whole, and the theory of matching minors, especially in bipartite graphs,

ii) a Part II, which contains the main body of work, together with proofs and explanations of the main results, and

iii) a concluding Part III, which briefly discusses some additional directions of research and wraps up the thesis by discussing the advancements made by our main results regarding some central questions brought up at the end of Part I.

For some of our results there also exist versions which were developed for directed graphs, these results are either directly related to our results, or they act as analogues from other parts of structural graph theory to further strengthen the intuition and motivation behind this thesis. Since most of these analogues come from the already complex Graph Minors Project by Robertson and Seymour which consists of over twenty individual papers, we believe that it is helpful for the reader to have a small introduction to the original concepts from graph minor theory, and then see how these concepts were adapted for the setting of directed graphs. Hence the first chapter of Part I gives an overview of the Graph Minors Project, and, in its second half, provides a brief survey on the advancements of structural digraph theory, with a focus on directed treewidth and related results.

The second chapter is a broad introduction to structural matching theory with a strong emphasis on decomposition theorems and matching minors.

This chapter also contains some preliminary results which are aimed at explaining how alternating paths, and therefore in a broader sense matching minors, are related to the different notions for describing the structure of graphs with perfect matchings. These two early chapters exist to give some context to the reader for the kind of results they can expect in the second part, and what some of the difficulties encountered along the way might be.

Part II is then made up of a series of chapters, arranged in a way such that each chapter builds upon the findings of the previous ones as much as possible. In Chapter 4, we obtain an analogue of the famous Two Paths Theorem for bipartite graphs with perfect matchings, which allows us to solve the 2-Matching Linkage Problem[1] in polynomial time. Chapter 5 then introduces perfect matching width as a variant of treewidth appropriate for the setting of (bipartite) graphs with perfect matchings. Many different aspects of this parameter are discussed, such as its relation to directed treewidth, duality, and algorithmic applications. As a step further, in Chapter 6, we show how the idea of orienting separations to identify areas of high connectivity, better known as tangles, can be applied to structural matching theory and we show that these ideas harmonise well with the perfect matching width from the previous chapter. Following in the footsteps of Robertson and Seymour, we then combine many of the findings from previous chapters in Chapter 7, to give a rough description of all classes of bipartite graphs with perfect matchings that exclude some planar and bipartite matching covered graph as a matching minor. While the results from previous chapters, except for the Two Paths Theorem, can be obtained for the setting of directed graphs without making use of the matching setting, in Chapter 7 we obtain, for the first time, results on directed graphs that go beyond what was previously known. The second part of the thesis is then completed in Chapter 8, where we combine all previous results to obtain an approximate characterisation of bipartite graphs with perfect matchings that exclude a complete bipartite graph as a matching minor, which can be seen as a close relative of the Flat Wall Theorem from the Graph Minors Project.

[1] This is a variant of the 2-Linkage Problem, where both paths are required to be alternating for some perfect matching of the graph.

Chapter 9 finally, as the only chapter of the third part, wraps up the current state of (bipartite) matching minor theory and hints at some possible directions of future research in this field.

Publications and Projects

The different parts and results in this thesis have, for the majority, been part of projects that involved several co-authors and most of what can be found here is, to this date at least, unpublished. In some instances, only the vague idea of a project exists, but not even a manuscript hast been written. To further complicate things, the different projects are intertwined and for the sake of readability many results were reordered to optimise their presentation in the context of this work. Hence, to allow for a quick way of reference and to find out which set of co-authors was involved in the development of a single result, we use this section to present the different projects together with the co-authors involved. We also introduce a small set of indicators which will pop up through the thesis at places, where usually citations would be found. In addition to address credit where credit is due, this hopefully also helps to underline which results are original and which were adopted from previously existing work by other authors.

The following is a list of all publications, preprints, or general projects that were selected to be highlighted in this thesis. Please note that the order is, as far as possible, chronological, and there is no intention to value one result over another. All of these projects have been challenging and interesting on their own and it was a great pleasure working with everyone involved. Wherever possible, we also include, as a citation, the version of the corresponding paper freely available on arXiv.org for better accessibility.

Instead of a numbering system, we use a capitalised letter with a star, from A^* to F^*, and additionally an X^*, to mark all those results which were not part of a specific project, but rather developed exclusively for this thesis by the author[2]. For each of the projects we give a brief summary,

[2] It is possible that, eventually, some of the 'miscellaneous' results marked by X^* will find their way into a publication other than this thesis, but currently no such plans exist.

and highlight the milestones achieved during the project, and hint at the chapters and sections where most of their associated results can be found.

- A* **Project: Matching Connectivity** with *Archontia Giannopoulou*, *Maximilian Gorsky*, and *Stephan Kreutzer*.
  Our project on matching connectivity was an initial and preliminary dive into structural matching theory, mainly aimed at achieving a deeper understanding of the topic, while also getting familiar with the many different concepts. Most results here are small observations and translations between different concepts and settings. Results from this project can be found in Section 3.1.
- B* **Cyclewidth and the Grid Theorem for Perfect Matching Width of Bipartite Graphs** with *Meike Hatzel* and *Roman Rabinovich* [HRW19a, HRW19b].
  This project marked the first major leap forward, as it contains the inception of cyclewidth, a width parameter for directed graphs that acts as a link between directed treewidth and perfect matching width. In this paper, it is shown that the bipartite version of Norine's conjecture regarding grid minors for perfect matching width follows from the Directed Grid Theorem and thus the relatively short paper is the foundation of most of what came after. A majority of the results from this paper can be found in Chapter 5, especially in Section 5.3.
- C* **Braces of Perfect Matching Width** 2 with *Archontia Giannopoulou* and *Meike Hatzel* [GHW19].
  While the discovery of cyclewidth already provided us with some nice intuitions behind the inner workings of perfect matching width, the parameter still seemed relatively distinct from directed treewidth. As a result, the decision was made to investigate a class of braces of fixed perfect matching width to get a better understanding of the parameter. While most of the results from this preprint are centred around the characterisation of braces of perfect matching width two and an efficient algorithm to find a corresponding decomposition, both of which are presented in Section 5.2, the more important insights gained from this project were those on the general behaviour of perfect matching width with regards to tight cut contractions and conformal subgraphs, which can be found in Section 5.1.

D* **Excluding a Planar Matching Minor in Bipartite Graphs** with *Archontia Giannopoulou* and *Stephan Kreutzer*.
In B* it was already shown that cyclewidth and perfect matching width can be seen as equivalent parameters. The step from cyclewidth to directed treewidth however, was made using the Directed Grid Theorem. While this approach yielded the desired result, the bound it produced was less than desirable and it also did not give any valuable insight into how it could be possible to use perfect matching width to achieve any results beyond the grid theorem itself. So probably the most important finding of D* is a relation between the matching porosity of an edge cut in a bipartite graph and the size of a certain kind of separator. This result allowed for much tighter bounds on the relation between perfect matching width and directed treewidth. It also allowed us to design an algorithm to solve a matching version of the disjoint paths problem, test for the existence of matching minors, and replicate a deep theorem from the Graph Minors Project on the Erdős-Pósa property of minors for the setting of bipartite graphs with perfect matchings. Moreover, in this project we gained some insight into what a topological theory of butterfly minors in digraphs might actually look like. Some parts of this project can be found in Section 5.3 and Section 5.4, while the entirety of Chapter 7 is also drawn from this project.

E* **Two Disjoint Alternating Paths in Bipartite Graphs with Perfect Matchings** with *Archontia Giannopoulou*.
During a visit in Athens, Archontia Giannopoulou and the author decided to start working on the next logical step for the matching minor series, after obtaining a nice description of bipartite graphs that exclude a planar matching minor: a version of the Flat Wall Theorem. We had high hopes for this project, since many of the challenges previously encountered by the first attempt at a directed version of the Flat Wall Theorem, appeared to be easily solvable in the setting of bipartite graphs with perfect matchings. A major hurdle the directed version had to face was the lack of a directed version of the Two Paths Theorem. The discovery of such a theorem, as presented in Chapter 4, was therefore a major milestone towards this goal.

- F* **Project: Bidirected Graphs** with *Maximilian Gorsky* and *Raphael Steiner*.
  In parallel to the discovery of a matching theoretic Two Paths Theorem, Maximilian Gorsky, Raphael Steiner and the Author discovered a way to generalise the bridge between directed and bipartite graphs to the setting of bidirected versus general graphs with perfect matchings. As many of the previous advances in bipartite matching theory were at least partially fuelled by the intuition the directed setting could provide, this appeared to be a major leap forward. Especially once we discovered how to generalise the notion of non-evenness to bidirected graphs while preserving its equivalence to being Pfaffian. This result is presented in Section 3.3. While the project still is in its infancy, some preliminary results have already been achieved and some of the major challenges have been identified. We discuss the topic in more depth in Section 3.4 and Section 9.2.
- G* **A Weak Structure Theorem for Bipartite Graphs with Perfect Matchings** with *Archontia Giannopoulou*.
  Finally, in what was probably the most challenging bit of research throughout all results gathered here, we were able to combine all previous results and the insight gained from the directed setting, to give an approximate description of all bipartite graphs with perfect matchings that exclude a complete bipartite graph of fixed size as a matching minor. This final result, which can be seen as a generalisation of the famous result on Pfaffian bipartite graphs by McCuaig et al., can be found in Chapter 8.

The idea how to encode a Pfaffian orientation in a bidirected graph to generalise the notion of being non-even has to be credited to Raphael Steiner and thus the authors contribution to Theorem 3.3.15 is only minor. This however is the only exception, for all other results found in this work and marked by one of the indicators above, including X*, the author is responsible for the majority of the conceptual work and also played a major role in working out the details of the proofs. Especially the original ideas for proofs involving matching theory and the interaction between digraphs and bipartite graphs with perfect matchings, as well as all results marked with X*, are due to the authors work and the contribution of co-authors was usually in form of discussions to refine certain details and clarify definitions or formulations.

Part I.

Background

Chapter 2.

General Background

This thesis aims to generalise several concepts developed for general undirected graphs to the setting of graphs with perfect matchings. We introduce the fundamental concepts of matching theory in Chapter 3 and also give a brief overview on a partial history of the field. Before that however, we need to introduce the blueprints most of the ideas presented in this thesis are build upon. That is, we introduce some general terms and basic notation for graph theory in Section 2.1. Next we give a quick summary of the Graph Minors Project and Wagner's conjecture in Section 2.2. Several key concepts were first introduced in, or at least popularised through, the Graph Minors Project and thus this is the place where we give all necessary definitions. The last piece we need to introduce beforehand is the generalisation of structural graph (minor) theory to the world of directed graphs, or *digraphs* for short, in Section 2.3. Besides introducing basic concepts and discussing how they differ from their undirected analogues, we also briefly touch upon the problem that many concepts which appear indistinguishable for undirected graphs suddenly are split into many, sometimes vastly different, concepts. This discussion is especially important as it gives a first glimpse at the problems one could expect when generalising further, say from digraphs to so called bidirected graphs which we briefly discuss at the end of Chapter 3. Please note that we introduce several different concepts in this chapter which sometimes turn out to be equivalent, or at least closely related. The reason why these concepts are introduced here, sometimes in great detail, is that we aim to give generalisations of these ideas to the setting of (bipartite) graphs with perfect matchings in later chapters.

2.1. General Graph Theory

Before we start with basic definitions in graph theory let us quickly fix some additional, more general notation.

Given two integers $i, j \in \mathbb{Z}$ where $i \leq j$, we denote by $[i, j]$ the *integer interval*, which is the set $\{i, i+1, \ldots, j-1, j\} = \{n \mid i \leq n \leq j, n \in \mathbb{Z}\}$. Given any set X we denote by 2^X the *power set* of X, which is the set of all subsets of X. Let X be finite and $k \in [0, |X|]$, by $\binom{X}{k}$ we denote the set $\{S \subseteq X \mid |S| = k\}$, and by $\binom{X}{\leq k}$ the set $\{S \subseteq X \mid |S| \leq k\}$. Let $S \subseteq X$, we denote by $\overline{S} \coloneqq X \setminus S$ the *complement* of S with respect to X. In case ambiguity arises we write the host set X in the top right index spot, that is $\overline{S}^X$.

Most of the definitions we present here are included to fix notation and to make this thesis as self contained as possible. For any definition not included here, the reader is referred to the book on graph theory by Diestel [Die12].

Definition 2.1.1 (Graph)**.** In general a (undirected) *graph* is a tuple $G = (V, E)$, where V is some ground set and E is a multiset over $\binom{V}{\leq 2} \setminus \{\emptyset\}$. The members of V are called the *vertices*, while the members of E are called the *edges* of G. An edge $e \in E$ with $|e| = 1$ is called a *loop*. If $e, e' \in E$ are different instances of the same member of $\binom{V}{\leq 2}$ we say that e and e' are *parallel edges*. The maximal number of different instances of $e \in \binom{V}{\leq 2}$ that occur in E is called the *multiplicity* of the edge e.

A graph where V is finite is called *finite*, and a graph where $E \subseteq \binom{V}{2}$ is not a multiset is called *simple*. If E contains an element of multiplicity two or more, G is called a *proper multigraph*.

For a graph G given without explicitly stating the tuple, we denote by $V(G)$ the vertex set of G, and by $E(G)$ the edge set of G.

Most graphs in this thesis are finite and simple. Indeed, whenever we encounter a graph for which we allow parallel edges we either state this directly, or use the term *multigraph* to emphasise that we are not necessarily dealing with a simple graph.

If $e \in E(G)$ is an edge of a graph G, we call the vertices $v \in e$ the *endpoints* of e and e and v are said to be *incident*. Two vertices u, $v \in V(G)$ are called *adjacent* if $\{u, v\} \in E(G)$. For better readability we drop the set-notation for edges and usually write uv instead of $\{u, v\}$. Two distinct edges $e, f \in E(G)$ that share an endpoint are said to be *adjacent*. If u and v are two adjacent vertices in G, we say that v is a *neighbour* of u and by $\mathrm{N}_G(u)$ we denote the set of all neighbours of u in G. In case the graph G is understood from the context we sometimes drop the subscript G and just write $\mathrm{N}(v)$. The *degree* of a vertex v, denoted by $\deg_G(v)$, is the number of edges in G which are incident with v. Note that in a simple graph this means that $\deg_G(v) = |\mathrm{N}_G(v)|$. By $\Delta(G)$ we denote the largest degree among all vertices of G, called the *maximum degree*, while the *minimum degree*, denoted by $\delta(G)$, is defined as the smallest degree over all vertices of G.

Let G and H be a graphs. We say that H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$, and $E(H) \subseteq E(G)$. Let $X \subseteq V(G)$, we denote by $G[X]$ the *subgraph of G induced by X*, which is the graph with vertex set X and edge set $\{e \in E(G) \mid e \subseteq X\}$.

Deleting a vertex v from G, denoted by $G - v$, means removing v from G and deleting all incident edges. Formally this means $G - v \coloneqq G[V(G) \setminus \{v\}]$. If $S \subseteq V(G)$ is a set of vertices, we write $G - S$ for the graph obtained from G by deleting all vertices of S. If $e \in E(G)$ is an edge of G, we denote by $G - e$ the graph $(V(G), E(G) \setminus \{e\})$, and if $F \subseteq E(G)$ is a (multi-)set of edges, $G - F$ is the graph obtained from deleting all edges in F from G.

These operations can also be reversed. Let G be a graph, $v \notin V(G)$ a vertex, and $e \in \binom{V(G)}{\leq 2}$ be non-empty. By $G + v$ we denote the graph obtained from G by adding v to the vertex set, while $G + e$ is the graph obtained from G by adding the edge e. Similarly, if S is some set of vertices and $F \subseteq \binom{V(G)}{\leq 2} \setminus \{\emptyset\}$, the graph $G + S$ is the graph obtained from G by adding all vertices of S, and $G + F$ is the graph obtained from G by adding all edges from F. So if H is another graph, we denote by $G + H$, or sometime $G \cup H$, the graph $(V(G) \cup V(H), E(G) \cup E(H))$. We also allow the *intersection* of graphs, which is defined as $G \cap H \coloneqq (V(G) \cap V(H), E(G) \cap E(H))$.

Definition 2.1.2 (Paths and Cycles)**.** Let G be a graph. A *walk* is a sequence

$$T = (v_0, e_0, v_1, e_1, v_2, \ldots, v_\ell, e_\ell, v_{\ell+1})$$

Where $v_i \in V(G)$ for all $i \in [0, \ell+1]$, $e_i \in E(G)$ for all $i \in [0, \ell]$, and $e_i = v_i v_{i+1}$ for all $i \in [0, \ell]$. The vertices v_0 and $v_{\ell+1}$ are the *endpoints* of T, the vertices $v_1, \dots, v_\ell$ are said to be the *internal vertices*, and T is said to be of *length* ℓ. A trail is called *closed* if $v_0 = v_{\ell+1}$. Although a walk formally is not a graph, one can associate naturally the subgraph of G with T which has vertex set $\{v_0, \dots, v_{\ell+1}\}$ and edge set $\{e_0, \dots, e_\ell\}$. In a slight abuse of notation we do not differentiate between the sequence of vertices and edges, and the subgraph.

A *trail* is a walk W where $i \neq j$ implies $e_i \neq e_j$ for all $i, j \in [0, \ell]$, a *closed trail* is a closed walk that is a trail. The walk T is called a *path*, if $i \neq j$ implies $v_i \neq v_j$ for all $i, j \in [0, \ell+1]$. Since every path mus also be a trail, we sometimes abbreviate the notation for walks and give paths as only a sequence of vertices. Moreover, we drop the comma and write

$$T = v_0 v_1 v_2 \dots v_\ell v_{\ell+1}.$$

A closed walk T is called a *cycle*, if $v_0 = v_{\ell+1}$, and $(v_0, e_0, v_1, e_1, v_2, \dots, v_\ell)$ and $(v_1, e_1, v_2, \dots, v_\ell, e_\ell, v_{\ell+1})$ are paths.

Let $X, Y \subseteq V(G)$ be two sets of vertices. An *X-Y-path* is a path P with one endpoint in X, the other endpoint in Y, and which is otherwise disjoint from $X \cup Y$.

If P_1 and P_2 are paths, and there exists a non-empty path Q such that $Q = P_1 \cap P_2$ and Q contains an endpoint of P_1 that is also an endpoint of P_2, we denote by $P_1 P_2$ the path $P_1 + P_2$. In this context we treat edges $e \in E(G)$ as paths of length one. This means, in particular, that if P is a path with endpoint v and e is an edge incident with v such that the other endpoint of e does not belong to P, then Pe is also a path. Suppose P_1 and P_2 are disjoint paths such that v_i is an endpoint of P_i for each $i \in [1, 2]$ while $v_1 v_2 \in E(G)$, then $P_1 P_2 = P_1 v_1 v_2 P_2$ is also a path. At last suppose we have fixed P to be an ordering of vertices as in the definition and $v = v_i \in V(P)$ is some vertex. We denote by Pv the subpath of P starting with v_0 and ending with v, while vP is the subpath of P starting with v and ending in $v_{\ell+1}$.

Definition 2.1.3 (Connected Graph)**.** A graph G is *connected*, if for every pair of distinct vertices $u, v \in V(G)$ there is a path P in G with endpoints u and v.

A *component* of a graph G is a maximal connected subgraph of G.

Let $k \in \mathbb{N}$ be a positive integer. The graph G is said the be *k-connected* if $|V(G)| \geq k+1$, and for every set $S \subseteq V(G)$ with $|S| \leq k-1$, $G-S$ is connected.

We close this first short introduction with two version of Menger's Theorem.

Theorem 2.1.4 (Menger's Theorem, Local Version [Men27])**.** Let G be a graph and $X, Y \subseteq V(G)$ be two sets of vertices, then the maximum number of pairwise disjoint X-Y-paths in G equals the minimum size of a set $S \subseteq V(G)$ such that every X-Y-path in G contains a vertex of S.

Theorem 2.1.5 (Menger's Theorem, Global Version)**.** Let G be a graph and $k \in \mathbb{N}$ be a positive integer. Then G is k-connected if and only if $|V(G)| \geq k+1$ and every pair of vertices is joined by a family of k pairwise internally disjoint paths.

2.2. The Graph Minors Series

The graph minors project by Robertson and Seymour is a series of twenty-three papers[1] in which the two authors develop, mostly qualitative, techniques to describe graphs that exclude a fixed minor.

Definition 2.2.1 (Edge Contraction)**.** Let G be a graph and $e = uv \in E(G)$. Let

$$G' \coloneqq G - u - v + v_e + \{xv_e \mid xu \in E(G) \text{ or } xv \in E(G)\},$$

where $v_e \notin V(G)$. We say that G' is obtained from G by *contraction* of the edge e.

Definition 2.2.2 (Minor)**.** A graph H is said to be a *minor* of a graph G, if H can be obtained from G by a sequence of edge deletions, vertex deletions, and contractions.

Classes of graphs that exclude minors are deeply connected to the historic roots of graph theory itself. The arguably most classical question in graph theory is the so called *Four Colour Theorem*, which states that every planar graph has a proper 4-colouring. Here a graph is called *planar* if it can be drawn on the plane such that no edge crosses another edge or a vertex.

[1] Conveniently named Graph Minors I to Graph Minors XXIII.

There exist two characterisations of planar graphs in terms of forbidden minors. Technically Kuratowski proved a stronger version of the following theorem using so called *topological minors*, while Wagner proved the theorem presented here. However, the contribution of both authors to the field is so significant that both should be credited.

Theorem 2.2.3 (Kuratowski-Wagner Theorem [Wag37, Kur30])**.** A graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor.

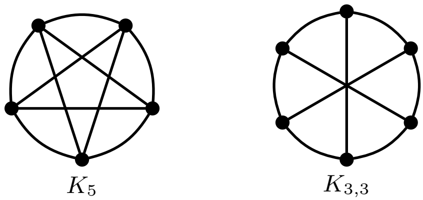


Figure 2.1.: The graphs K_5 and $K_{3,3}$.

So excluding certain minors is directly linked to being embeddable into some fixed surface. Another property that seems to interact, at least, with planarity is the existence of disjoint paths connecting prescribed pairs of vertices.

Definition 2.2.4 (Disjoint Paths and Linkedness)**.** Let G be a graph, $k \in \mathbb{N}$ a positive integer, and $s_1, \dots, s_k, t_1, \dots, t_k \in V(G)$. The question, whether there exist paths $P_1, \dots, P_k$ in G such that these paths are pairwise internally disjoint, and P_i has endpoints s_i and t_i for every $i \in [1, k]$ is called the *k-disjoint paths problem.*

We say that $(s_1, \dots, s_k)$ and $(t_1, \dots, t_k)$ are *linked* if the answer to the k-disjoint path problem with input $(s_1, \dots, s_k, t_1, \dots, t_k)$ is 'yes'.

A graph G is said to be *k-linked* if the answer to every instance of the k-disjoint paths problem in G is 'yes'.

Definition 2.2.5 (Clique Sums and Reductions)**.** Let G_1, and G_2 be graphs, $k \in \mathbb{N}$ be a positive integer, and K be a clique of size k which exists in both G_i. Suppose $G_1 \cap G_2 = K$, then we say that the graph $G_1 + G_2 - F$, where $F \subseteq E(K)$, is obtained from G_1 and G_2 via a *k-clique sum.*

Let $H \subseteq G$ be a subgraph of G and $S \subseteq V(G)$ a set of k vertices such that $G - S$ is not connected, but there exists a component C of $G - S$ such that $H \subseteq G[S \cup V(C)]$. We call the graph G' obtained from $G[S \cup V(C)]$ by adding the edges in $\{uv \mid u, v \in S\}$ an *k-reduction of G towards H*.

If $k \leq 3$ we say that G' was obtained via *elementary H-reduction*.

Suppose there is a sequence $G_0 = G, G_1, G_2, \ldots, G_\ell = G'$ such that G_i is obtained from G_{i-1} via elementary H-reduction, where $H \subseteq G_j$ for all $i \in [1, \ell]$ and $j \in [0, \ell]$. We say that G' is obtained from G by *H-reduction*. Sometimes we call G' itself a *H-reduction*.

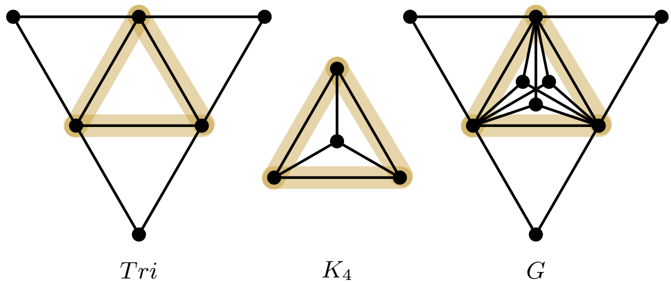


Figure 2.2.: The planar graphs Tri (to the left) and K_4 (in the middle), and the non-planar graph G (to the right) obtained from Tri and three copies of K_4 via the clique sum operation applied to the marked 3-clique.

We say that a cycle C is *C-flat* in a graph G if G can be constructed via k-clique sums, where $k \leq 3$, from graphs $G_1, \ldots, G_\ell$, where the graph G_ℓ that contains C is planar, and C bounds a face of G_ℓ. Please note that this means that G_ℓ is a C-reduction of G. In the case of flat cycles, one immediately finds a flavour of linkedness. A cycle C is said to have a *C-cross* in G, if there exist distinct vertices s_1, s_2, t_1, t_2 that occur on C in the order listed, and paths P_1 and P_2 such that: Both P_i are internally vertex disjoint from C, P_1 and P_2 are vertex disjoint and each P_i has s_i and t_i as its endpoints. A classic theorem, to which we will refer as the *Two Paths Theorem*, links the notions of C-flatness and C-crosses. The theorem has been obtained in many different forms with various techniques by a plethora of authors over time [Jun70, Sey80, Shi80, Tho80, RS90a].

Theorem 2.2.6 (Two Paths Theorem)**.** A cycle C in a graph G has **no** C-cross in G if and only if it is C-flat in G.

Note that the Two Paths Theorem is indeed a characterisation for when (s_1, s_2) and (t_1, t_2) are linked. To find a corresponding pair of paths, one can simply add in the edges s_1s_2, s_2t_1, t_1t_2, and t_2s_1 which then form a 4-cycle C'. We then can ask whether there exists a C'-reduction of G which is planar and in which C' bounds a face. If that is the case, the answer is 'no', otherwise the answer is 'yes'.

Theorem 2.2.7 ([Jun70, Sey80, Shi80, Tho80, RS90a])**.** There exists a polynomial time algorithm for the two disjoint paths problem.

In the Two Paths Theorem we have a first example of the usefulness of clique sums. The following can be seen as a refinement of the Kuratowski-Wagner Theorem. For this we need another special graph. The graph $\mathcal{M}_8$ obtained from the cycle

$$(v_1, e_1, v_2, e_2, \dots, v_7, e_7, v_8, e_8, v_1),$$

by adding the edges v_1v_5, v_2v_6, v_3v_7, and v_4v_8, is called *Wagner's graph*, see Figure 2.3.

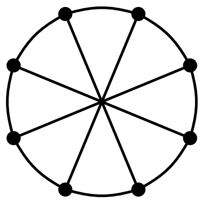


Figure 2.3.: Wagner's graph $\mathcal{M}_8$.

Theorem 2.2.8 (Wagner's Theorem, [Wag37])**.** A graph G does not contain K_5 as a minor if and only if it can be obtained from planar graphs and subgraphs of $\mathcal{M}_8$ by clique sums of order at most three.

There exist many examples of forbidden minors classifications of graph classes and most of them make use of clique sums. Another property these forbidden-minor-characterisations have in common is, that they always forbid a finite list of minors. This phenomenon has already been observed by Wagner, who formulated a fundamental conjecture regarding classes that exclude some graph as a minor.

Definition 2.2.9 (Quasiorder and Partial Order)**.** Let X be a set and let $\leq$ be a binary relation over X. We call $(X, \leq)$ a *quasiorder*, if it meets the following requirements:

i) $x \leq x$ for all $x \in X$ (reflexivity), and
ii) if $x \leq y$, and $y \leq z$ then $x \leq z$ for all $x, y, z \in X$ (transitivity).

If $(X, \leq)$ also satisfies

iii) if $x \leq y$, and $y \leq x$ then $x = y$ for all $x, y \in X$ (antisymmetry),

it is called a *partial order*, and the set X is called a *partially ordered set* or *poset* for short.

Definition 2.2.10 (Chains and Anti-Chains)**.** Let $(X, \leq)$ be a quasiorder and $A \subseteq X$. We call A a *chain* if for every choice of $a, b \in A$ we have $a \leq b$ or $b \leq a$. If on the other hand $a \nleq b$, and $b \nleq a$ for all $a, b \in A$, A is called an *anti-chain*.

Definition 2.2.11 (Well-Quasiorder)**.** Let $(X, \leq)$ be a quasiorder. We say that $(X, \leq)$ is a *well-quasiorder* if all anti-chains $A \subseteq X$ are finite.

With this we can state Wanger's Conjecture, which since has become the Robertson-Seymour Theorem.

Theorem 2.2.12 (Robertson-Seymour Theorem, [RS04])**.** The graph minor relation on the class of all graphs is a well-quasiorder.

In the following we give a short, and therefore incomplete overview on the different techniques and important results obtained in the Graph Minors Project which made the proof of Wagner's Conjecture possible. For a more complete overview the interested reader might want to consult the survey by Lovász [Lov06] or the more in-depth summary by Kawarabayashi and Mohar [KM07].

2.2.1. Treewidth and Duality

The concept of clique sums seems to play an important role in graph minor theory and thus, as a first step, one could ask the question: What can we say about graphs that are build from graphs with few vertices via clique sums. The part where we ask for 'few vertices' is crucial as Wagner's Theorem already illustrates that one might get large and non-trivial classes if one does not restrict the size of the graphs involved in building the graphs. Say we were to construct a graph via clique sums of order at most $k-1$, for some $k \geq 2$, from graphs of size at most k. In case $k=2$ one can easily check that the resulting graph would be a tree. Indeed, the rough idea of this still holds true for values of $k \geq 3$ as these graphs can still be seen as some kind of generalised tree made from slightly thicker edges. By formalising this idea one obtains the powerful parameter of treewidth.

Definition 2.2.13 (Treewidth)**.** Let G be a graph. A *tree decomposition* of G is a tuple (T, β), where T is a tree and $\beta\colon V(T) \to 2^{V(G)}$ is a function, called the *bags* of (T, β), such that the following properties hold:

i) $\bigcup_{t \in V(T)} \beta(t) = V(G)$,
ii) for every $e \in E(G)$ there exists $t_e \in V(T)$ such that $e \subseteq \beta(t_e)$, and
iii) for every $v \in V(G)$, the set $\{t \in V(T) \mid v \in \beta(t)\}$ induces a subtree of T.

The *width* of (T, β) is defined as $\mathrm{width}((T, \beta)) \coloneqq \max_{t \in V(T)} |\beta(t)| - 1$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width over all tree decompositions of G.

Branch Decompositions

Graphs of small treewidth can be seen as graphs with 'well behaved' structure. That is, these graphs are, in spirit, close to trees and one can describe them in an easy to understand way, namely using an optimal[2] tree decomposition. There are several related concepts of 'width', one of which we introduce here. It will play, at least conceptually, an important role in later chapters.

A tree T is said to be *cubic* if every vertex of T is either of degree one, and thus a leaf, or of degree three. By $\mathsf{L}(T)$ we denote the set of leaves of any tree T.

Definition 2.2.14 (General Branch Decomposition)**.** Let X be a set, and $f \coloneqq 2^X \to \mathbb{N}$ be a function such that $f(S) = f(\overline{S})$ for all $S \subseteq X$.

[2] Optimal here means a tree decomposition of minimum width.

A *f-branch decomposition* for X is a tuple (T, δ) where T is a cubic tree and $\delta\colon \mathsf{L}(T) \to X$ is a bijection.

Let $t_1t_2 \in E(T)$ be any edge. We denote by T_{t_i} the unique component of $T - t_1t_2$ that contains t_i for both $i \in [1,2]$. By $\delta(T_{t_i})$ we denote the set $\{\delta(t) \mid t \in \mathsf{L}(T) \cap V(T_{t_i})\}$. Please note that, since δ is a bijection, $(\delta(T_{t_1}), \delta(T_{t_2}))$ is a partition of X. For every edge $e = t_1t_2 \in E(T)$ we associate the bipartition $(\delta(T_{t_1}), \delta(T_{t_2}))$ with e and write $f(e) \coloneqq f(\delta(T_{t_1})) = f(\delta(T_{t_2}))$.

The *f-width* of (T, δ) is then defined as $\mathrm{width}_f(T, \delta) \coloneqq \max_{e \in E(T)} f(e)$, and the *$f$-width* of X, denoted by $\mathrm{width}_f(X)$, is defined as the minimum f-width over all f-branch decompositions of X.

The framework of f-branch decompositions can be used to define many different parameters for all kinds of combinatorial structures.

Definition 2.2.15 (Branchwidth)**.** Let G be a graph and let us define $b\colon 2^{E(G)} \to \mathbb{N}$ as

$$b(F) = |(\bigcup_{e \in F} e) \cap (\bigcup_{e \in \overline{F}} e)|.$$

A *branch decomposition* for G is a b-branch decomposition for $E(G)$, and the *branchwidth* of G, denoted by $\mathrm{bw}(G)$, is the b-width of $E(G)$.

Theorem 2.2.16 ([RS91])**.** Let G be a graph. Then $\mathrm{bw}(G) - 1 \leq \mathrm{tw}(G) \leq \frac{3}{2}\mathrm{bw}(G) - 1$.

Roughly speaking, Theorem 2.2.16 shows that the difference between branchwidth and treewidth is only a qualitative one. In what follows, we will call two parameters *equivalent*, if one of the parameters can be bounded by two functions in the other parameter. Naturally one would prefer these functions to be linear, or even just contain a small constant, but in general we will be fine with any computable function. Many of the results from the graph minors project and its generalisation to the setting of digraphs include statements of this approximate nature. Indeed, because of this one could view the graph minor theory of Robertson and Seymour as an *approximation of structure theory for graphs.*

Brambles and Tangles

In a way classes of bounded branchwidth, and therefore also of bounded treewidth, may be seen as 'structurally simple' graph. We discuss a bit more what that means in Section 2.2.3. So the next natural question to ask would be: how can we describe a graph that is **not** 'structurally simple'? To do this we would like to have some concept off structure we can find in a graph whose treewidth is larger than a certain constant. In this subsections and the two afterwards we explore several concepts on how such a duality could look like and what other parameters one could use to essentially capture the same 'structural simplicity' as treewidth and branchwidth do. Please note that we do not insist on a tight duality. In general whenever we say that two parameters are *dual* if they are equivalent and one of them induces a minimisation problem, while the other one induces a maximisation problem.

Over all, treewidth can be seen as a generalised connectivity measure for graphs. Not only does it interact with separations, but it provides a way to decompose a graph into small pieces. Indeed, even classical decompositions like the block decomposition of a graph can be seen as special versions of tree decompositions. From this point of view one could expect to find a highly connected, in some sense, part in a graph with high treewidth.

Let G be a graph and H_1, H_2 be two connected subgraphs of G. We say that H_1 and H_2 *touch* if $V(H_1) \cap V(H_2) \neq \emptyset$, or there is an edge uv with $u \in V(H_1)$ and $v \in V(H_2)$. A set $S \subseteq V(G)$ is a *hitting set* or *cover* for a family $\mathcal{H}$ of subgraphs of G, if $V(H) \cap S \neq \emptyset$ for all $H \in \mathcal{H}$.

Definition 2.2.17 (Bramble)**.** Let G be a graph. A *bramble* $\mathcal{B} = \{B_1, B_2, \dots, B_\ell\}$ of G is a family of connected and pairwise touching subgraphs B_i of G. The *order* of $\mathcal{B}$ is the size of a minimum hitting set for B.

The *bramble number*, denoted by $\mathrm{br}(G)$, is the maximum order over all brambles in G.

Theorem 2.2.18 ([ST93])**.** Let G be a graph, and $k \in \mathbb{N}$ a positive integer. Then G contains a bramble of order k if and only if $\mathrm{tw}(G) \geq k - 1$. Hence $\mathrm{br}(G) - 1 = \mathrm{tw}(G)$.

So a bramble is a more abstract way to described a highly connected subgraph of a graph G of high treewidth. There is an even more abstract way to capture this kind of high connectivity areas in graphs, and it directly interacts with the notion of connectivity itself.

Definition 2.2.19 (Separation)**.** Let G be a graph. A tuple (X, Y) is a *separation* of G if $X \cup Y = V(G)$, and there is no path from $X \setminus Y$ to $Y \setminus X$ that avoids the vertices in $X \cap Y$.

The set $X \cap Y$ is called the *separator* of (X, Y), and $|X \cap Y|$ is the *order* of the separation (X, Y).

Note that, if (X, Y) is a separation, then so is (Y, X). Let $k \in \mathbb{N}$ be a positive integer. Let us denote by $\mathcal{S}_k(G)$ the set of all separations of order at most $k - 1$ in G. An *orientation* of $\mathcal{S}_k(G)$ is a set $\mathcal{O} \subseteq \mathcal{S}_k(G)$ such that for every $(X, Y) \in \mathcal{S}_k(G)$ exactly one of (X, Y) and (Y, X) belongs to $\mathcal{O}$. If $\mathcal{O}$ is an orientation of $\mathcal{S}_k(G)$, and $(X, Y) \in \mathcal{O}$, we say that X is the *small side*, while Y is the *large side* of (X, Y).

Definition 2.2.20 (Tangle)**.** Let G be a graph and $k \in \mathbb{N}$ be a positive integer. A *tangle of order* k of G is an orientation $\mathcal{T}$ of $\mathcal{S}_k(G)$ such that for every triple $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3) \in \mathcal{O}$ we have

$$X_1 \cup X_2 \cup X_3 \neq V(G).$$

The *tangle number* of G, denoted by $\mathrm{t}(G)$, is the largest integer k such that G has a tangle of order k.

Theorem 2.2.21 ([RS91])**.** Let G be a graph, and $k \in \mathbb{N}$ a positive integer. Then G contains a tangle of order k if and only if $\mathrm{tw}(G) \geq k$. Hence $\mathrm{t}(G) = \mathrm{tw}(G)$.

We discuss the topic of tangles for digraphs and graphs with perfect matchings in Chapter 6.

Cops & Robber Games in Undirected Graphs

Graph searching is an area of graph theory, where the graph itself is seen as a network with some properties in its vertices that has to be searched for a specific property while obeying a fixed set of rules. Many variants of graph searching can be expressed as a combinatorial game for one or two players on the given graph. Indeed, one of the best known category of such games are the so called *fugitive-search games*, where one or more fugitives are positioned somewhere in graph graph and are hunted by a group of searchers, usually imagined as law enforcement. There is a wide variety of these games, see [DKT97] for an overview, but here we are specifically interested in those variants, that correspond to treewidth and related parameters.

Given a graph G let us denote by 2^G the family of all subgraphs of G.

Definition 2.2.22 (Cops & Robber Game)**.** Let G be a graph. A *play* of the cops & robber game on G is a sequence

$$(C_0, R_0), (C_1, R_1), \dots, (C_{\ell-1}, R_{\ell-1}), (C_\ell, R_\ell)$$

such that the following requirements are met:

i) $C_i \subseteq V(G)$, and R_i is a component of $G - C_i$ for all $i \in [0, \ell]$, and
ii) $(V(R_{i-1}) \cap V(R_i)) \setminus (C_{i-1} \cap C_i) \neq \emptyset$ for all $i \in [1, \ell]$.

We call a tuple of the form (C_i, R_i) a *position*, where C_i is the *cop-position* while R_i is the *robber position*.

A *cop-strategy* is a function $c\colon 2^{V(G)} \times 2^G \to 2^{V(G)}$ that assigns to every position a new cop-position. Similarly, a *robber-strategy* is a function $r\colon 2^{V(G)} \times 2^{V(G)} \times 2^G \to 2^G$ that assigns to every tuple of the form (C_i, C_{i+1}, R_i), where (C_i, R_i) is a position and C_{i+1} is a new cop position, a subgraph R_{i+1} of G such that $(C_i, R_i), (C_{i+1}, R_{i+1})$ is a play.

A play $(C_0, R_0), (C_1, R_1), \dots, (C_{\ell-1}, R_{\ell-1}), (C_\ell, R_\ell)$ is *consistent* with a cop-strategy c if $c(C_i, R_i) = C_{i+1}$ for all $i \in [0, \ell-1]$. The play is *consistent* with a robber strategy r if $r(C_i, C_{i+1}, R_i) = R_{i+1}$ for every $i \in [0, \ell - 1]$.

A cop-strategy c is *winning* if for every robber strategy r, for every maximal play that is consistent with c and r there exists an integer $i \in \mathbb{N}$ such that $R_i \subseteq C_i \cap C_{i+1}$. Let $k \in \mathbb{N}$. A robber-strategy r is *winning against k cops* if for every cop strategy c, and every play $(C_0, R_0), (C_1, R_1), \dots, (C_{\ell-1}, R_{\ell-1}), (C_\ell, R_\ell)$ that is consistent with c and r we have

i) $|C_i| \leq k$ for all $i \in [0, \ell]$, and
ii) c is not winning.

The *cop number* of a graph G, denoted by $\mathrm{cops}(G)$, is the smallest integer k such that there is no robber-strategy which is winning against k cops.

The idea behind these kinds of games is that the robber tries to escape the grasp of the police for as long as possible, always using paths that are not currently blocked by cops. Let (C_{i-1}, R_{i-1}) be some position. The game is played as follows: The cops announce their next position, say C_i and remove all cops from the graph that do not occupy vertices from C_i. Then the robber announces her new component R_i which is reachable from her current position R_{i-1} via a path in $G - (C_{i-1} \cap C_i)$. Afterwards cops are placed on all vertices from $C_i \setminus C_{i-1}$.

One might want to impose additional rules on the game to make it more interesting. The most commonly used additional rules are:

i) *Invisibility*: That is, the robber is invisible and the cops must decide on their next position without knowing the current position of the robber. If we want the robber to be invisible, we add *iv* to the subscript of the cop number, so we write $\text{cops}_{iv}(G)$.

ii) *Inertness*: Here the robber is inert which means, she always occupies a vertex and does not move until the cops threaten to occupy her vertex. If we want the robber to be inert, we add *in* to the subscript of the cop number, so we write $\text{cops}_{in}(G)$.

iii) *Robber-Monotone*: A cop-strategy c is robber-monotone if for every play $(C_0, R_0), (C_1, R_1), \ldots, (C_{\ell-1}, R_{\ell-1}), (C_\ell, R_\ell)$ that is consistent with c we have $R_{i+1} \subseteq R_i$ for all $i \in [1, \ell - 1]$. If we want the cops to play robber-monotone, we add *rm* to the subscript of the cop number, so we write $\text{cops}_{rm}(G)$.

iv) *Cop-Monotone*: A cop-strategy c is cop-monotone if for every play $(C_0, R_0), (C_1, R_1), \ldots, (C_{\ell-1}, R_{\ell-1}), (C_\ell, R_\ell)$ that is consistent with c we have $(C_{i+1} \setminus C_i) \cap \bigcup_{j=0}^{i} C_j = \emptyset$ for all $i \in [1, \ell - 1]$. If we want the cops to play cop-monotone, we add *cm* to the subscript of the cop number, so we write $\text{cops}_{cm}(G)$.

A cop-strategy that is neither cop- nor robber-monotone is called *non-monotone*.

In the undirected case, many of these variants turn out to be indistinguishable from one-another.

Theorem 2.2.23 ([ST93, DKT97, Bod98])**.** Let G be a graph. Then $\text{tw}(G) + 1 = \text{cops}(G) = \text{cops}_{rm}(G) = \text{cops}_{cm}(G) = \text{cops}_{iv,in}(G) = \text{cops}_{iv,in,cm}(G)$.

We will revisit cops & robber games in the setting of digraphs, where the above equalities break down into several incomparable parameters.

Grids and Walls

We close this section on treewidth and duality with the arguably most important and powerful of all duality theorems that are known for treewidth. So far, all witnesses or *obstructions* to small treewidth we have seen were relatively abstract. Winning strategies for the robber or tangles of high order guarantee that the treewidth cannot be small, but they do not immediately yield consequence for the structure of the graph G in the sense of subgraphs or minors. Brambles an the other hand give at least some intuition on the connectivity one should be able to find in a small graph of high treewidth. But brambles also remain relatively abstract and do not give us a canonical type of subgraph or minor one could be guaranteed to see in a graph of high treewidth.

Definition 2.2.24 (Grid)**.** Let $k, t \in \mathbb{N}$ be positive integers. The $k \times t$*-grid* is the graph with vertex set

$$\{v_{i,j} \mid i \in [1,k], j \in [1,t]\},$$

and edge set

$$\{v_{i_1,j_1}v_{i_2,j_2} \mid i_1 = i_2 \text{ and } |j_1 - j_2| = 1, \text{ or } j_1 = j_2 \text{ and } |i_1 - i_2| = 1\}.$$

The vertices $v_{1,1}$, $v_{1,t}$, $v_{k,1}$, and $v_{k,t}$ are called the *corners* of the $k \times t$-grid.

Theorem 2.2.25 (Grid Theorem, [RS86b])**.** There exists a function $\mathsf{g}_{\mathrm{undir}} \colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every graph G we have $\mathrm{tw}(G) \leq \mathsf{g}_{\mathrm{undir}}(k)$, or G contains the $k \times k$-grid as a minor.

The function $\mathsf{g}_{\mathrm{undir}}$ started out as something highly exponential, but was eventually brought down to a function polynomial in k [CC16]. This polynomial has since been further improved to $\mathcal{O}(k^9 \operatorname{poly}\log k)$ [CT19].

While the Grid Theorem guarantees us a very structured minor, we would like to go one step further. Let G be a graph and $uv = e \in E(G)$ be some edge. Let $G' \coloneqq G - e + uxv$ where $x \notin V(G)$. We say that G' is obtained from G by *subdivision of* e. Any graph H that can be obtained from G by a sequence of edge-subdivisions is called a *subdivision of* G.

Lemma 2.2.26 (folklore)**.** Let H be a graph with $\Delta(H) \leq 3$. Then any graph G contains a subdivision of H as a subgraph if and only if it contains H as a minor.

In light of the above theorem one only needs to replace the grid in the Grid Theorem with a subcubic graph with similar properties to obtain a theorem that guarantees the existence of a highly structured subgraph in every graph of high treewidth.

Definition 2.2.27 (Wall)**.** Let $k \in \mathbb{N}$ be some positive integer. An *elementary k-wall* is obtained from the $k \times 2k$-grid by deleting all edges with endpoints $v_{2i-1,2j-1}$ and $v_{2i-1,2j}$ for all $i \in [1, \lfloor \frac{k}{2} \rfloor]$ and $j \in [1, k]$, and all edges with endpoints $v_{2i,2j}$ and $v_{2i,2j+1}$ for all $i \in [1, \lfloor \frac{k}{2} \rfloor]$ and $j \in [1, k]$, and then deleting the two resulting vertices of degree one.
A subdivision of an elementary k-wall is called a *k-wall*.

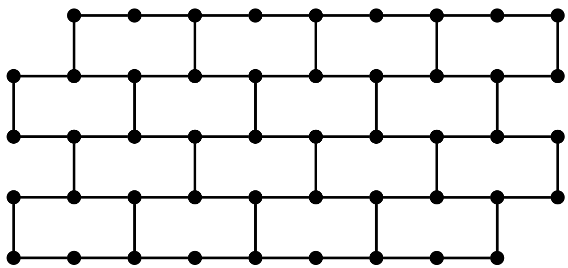


Figure 2.4.: An elementary 5-wall.

Since the transition between a $k \times k$-grid and a $\lfloor \frac{k}{2} \rfloor$-wall is only a factor of two, we may, essentially, use the same function as in the grid theorem to state its wall-variant.

Corollary 2.2.28 (Wall Theorem)**.** There exists a function $\mathfrak{g}_{\mathrm{undir}} : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every graph G we have $\mathrm{tw}(G) \leq \mathfrak{g}_{\mathrm{undir}}(2k)$, or G contains a k-wall as a subgraph.

2.2.2. Graphs Excluding a Minor

With treewidth and its many dual concepts, especially the Grid Theorem, we now have a way to distinguish between 'structurally simple' graphs and those that are more complicated, i.e. those which contain large k-walls.

Excluding a Planar Graph

Let us fix some planar graph H and try to describe the structure of all graphs which exclude H as a minor. For this, a simple observation is almost enough.

Lemma 2.2.29 ([RST94]). Let H be a graph with $|V(G)|+2|E(H)| \leq n$ for some positive integer $n \in \mathbb{N}$. Then H is a minor of the $2n \times 2n$-grid.

This means, that by excluding H as a minor, we also exclude all grid of large enough size as minors as well. Indeed this means that any class of graphs that excludes a planar minor must be of bounded treewidth.

Theorem 2.2.30 ([RS86b]). A proper minor closed class $\mathcal{G}$ of graphs is of bounded treewidth[3] if and only if $\mathcal{G}$ excludes a planar graph as a minor.

So if bounded treewidth means 'structurally simple' as we have assumed throughout these subsections, we may now say that a class of graph is 'structurally simple' if and only if it excludes a planar minor. Since every planar graph H is a minor of some grid, grids of much larger size must actually contain several vertex disjoint subgraphs which have H as a minor. This observation lead to another property closely tied to treewidth and being planar.

Definition 2.2.31 (Erdős-Pósa-Property for Minors). Let H be a graph. We say that H has the *Erdős-Pósa-property for minors* if there exists a function $f_H \colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, every graph G either contains k pairwise vertex disjoint subgraphs all of which have a minor isomorphic to H, or there exists a set $S \subseteq V(G)$ with $|S| \leq f_H(k)$ such that $G - S$ does not have a minor isomorphic to H.

Theorem 2.2.32 ([RS86b]). Let H be a graph. Then H has the Erdős-Pósa-property for minors of and only if it is planar.

Erdős-Pósa-type results have received a lot of attention over the years. The name itself stems from the original result by Erős and Pósa [EP65] which stated the existence of a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for every k, every graph G either contains k pairwise disjoint cycles, or has a set

[3] This means there is some constant $c_{\mathcal{G}}$ such that every graph $G \in \mathcal{G}$ satisfies $\mathrm{tw}(G) \leq c_{\mathcal{G}}$.

$S \subseteq V(G)$ of size $|S| \leq f(k)$ such that $G - S$ has no cycle. Indeed, this original result can be seen as the special case of Theorem 2.2.32 where $H = K_3$. Moreover, this duality of either finding many disjoint objects or a small hitting set for all of them can be seen as a hypergraph-version of the duality between matching and vertex cover as introduced in Chapter 3.

The Weak Structure Theorem

Whenever we exclude a planar graph H as a minor we end up with a class of graphs of bounded treewidth. What can we say if we exclude some complete graph K_t, $t \in \mathbb{N}$ and $t \geq 5$? The case $t = 5$ is solved exactly by Wagner's Theorem, that is any K_5-free graph can be built up from planar graphs and $\mathcal{M}_8$ by small order clique sums. In particular this means that, in case a K_5-minor free graph G has large treewidth, one of the planar graphs it is created from must also have large treewidth. This means, in such a G we can find a subgraph G' which is planar, witnesses high treewidth, and the rest of G is 'attached' to G' by means of small order clique sums. Recall our definition of C-flatness. One could say, that the graph G' is 'flat' in G. In this subsection we present the next iterative step towards an approximate structure theory of graphs excluding some minor: an approximate characterisation of graphs excluding K_t as a minor.

Definition 2.2.33 (Flatness)**.** Let G be a graph and H be a planar graph with outer face D. We say that H is *flat* in G, if there exists a separation (X, Y) such that $X \cap Y \subseteq V(D)$, $H \subseteq G[Y]$, and there is an H-reduction G' of $G[Y]$ such that

i) $D \subseteq G'$,
ii) G' is planar, and
iii) D bounds a face of G'.

Notice that, if G has a K_t-minor, there exist pairwise disjoint sets $X_1, \dots, X_t \subseteq V(G)$ such that $G[X_t]$ is connected and for every pair X_i, X_j, $i \neq j$, there is an edge $u_iu_j \in E(G)$ with $u_i \in X_i$ and $u_j \in X_j$. We say that the $X_1, \dots, X_t$ form a *model* of K_t in G. Let W be some k-wall for $k \geq t$ with horizontal paths $P_1, P_2, \dots, P_k$, and vertical paths $Q_1, Q_2, \dots, Q_k$. We say that a model of a K_t-minor is *grasped* by W if for every X_h there are distinct indices $i_1, \dots, i_t \in [1, k]$, and $j_1, \dots, j_t \in [1, k]$ such that $V(P_{i_\ell} \cap Q_{j_\ell}) \subseteq X_h$ for all $\ell \in [1, t]$.

The following theorem was first obtained in [RS95], but there it was stated in a slightly weaker form and the bounds were not given explicitly. So we credit Kawarabayashi et al. [KTW18] for simplifying the proof and providing explicit bounds.

Theorem 2.2.34 (Flat Wall Theorem, [RS95, KTW18]). Let $r, t \in \mathbb{N}$ be positive integers, $R = 49152t^{24}(40t^2 + r)$, and G be a graph. Then the following is true: If W is an R-wall in G, then either

i) G has a K_t-minor grasped by W, or
ii) there exists a set $A \subseteq V(G)$ with $|A| \leq 12288t^{24}$ and an r-wall $W' \subseteq W - A$ such that W' is flat in $G - A$.

Indeed, one can take the Flat Wall Theorem and obtain an almost-characterisation of all graphs G that exclude K_t as a minor.

Theorem 2.2.35 (Weak Structure Theorem[4], [KTW18]). Let $r, t \in \mathbb{N}$ be positive integers, $R = 49152t^{24}(40t^2 + r)$, and G be a graph. If G has no K_t-minor, then for every R-wall W in G there exists a set $A \subseteq V(G)$ with $|A| \leq 12288t^{24}$ and an r-wall $W' \subseteq W - A$ such that W' is flat in $G - A$. Conversely, if $t \geq 2$, $r \geq 80t^{12}$, and for every R-wall W in G there is a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r-wall $W' \subseteq W - A$ which is flat in $G - A$, then G has no $K_{t'}$-minor, where $t' = 2R^2$.

Beyond the Flat Wall

The Flat Wall Theorem was only an intermediate, although important, point towards the proof of Wagner's Conjecture. The structural contributions that play a role for the content of this thesis have already been presented in the previous subsections, hence we only give a brief summary of the remaining steps in the story of the graph minors project.

A consequence of Theorem 2.2.30 is, that any anti-chain for the graph minor relation that contains a planar graph must be finite [RS86b]. The next step would be to lift the Kuratowski-Wanger theorem to surfaces of arbitrary genus. In [RS90b] it was shown that for any surface Σ the class of graphs $\mathcal{G}$ which has a crossing-free embedding on Σ can be described by a

[4] Traditionally the Flat Wall Theorem and the Weak Structure Theorem are used synonymous, but since there are two slightly different statements it might make sense to differentiate between the two.

finite set of forbidden minors. For the final part of the graph minors project several ingredients were necessary. The first one being tree decompositions of small adhesion. That is a tree decomposition where the intersection of neighbouring bags is of bounded size, but not necessarily the bags themselves. These small sized adhesion sets, which may be seen as small separators in the graph, correspond to the areas where G is constructed as a clique sum of several smaller graphs. Next consider the set A from the Flat Wall Theorem. These sets are called *apex sets* and make up the second ingredient. So after constructing a graph G by means of clique sums from smaller graphs, one is allowed to add a small number of *apex vertices* which might be connected to the whole graph. Now consider the actual building blocks, so those graphs which are glued together by clique sums. Clearly we may assume that those are not of small treewidth as otherwise we could decompose them further. They might also not (yet) fit onto some surface of bounded genus. That is because they might contain subgraphs of high genus which overall do not contribute too much to the structure of the graph. These subgraphs are called *vortices* or *fringes*. When removing these fringes we are left with graphs of bounded genus in every bag of our generalised tree decomposition. As it turns out, that is all one needs to approximate the class of graphs which excludes a fixed graph H as a minor [RS99] in a meaningful way.

2.2.3. Algorithmic Properties, Disjoint Paths, and Minor Testing

The discovery of treewidth has created a huge area of research in the algorithmic community. Especially in the area of fixed parameter tractability [DF12] treewidth found a broad variety of applications. The importance of treewidth was drastically raised by the fact that finding a good tree decomposition itself is tractable on graphs of bounded treewidth [Ree92]. The value of tree decompositions for these fields is that they enable the generalisation of dynamic programming algorithms for trees to graphs of bounded treewidth which are much richer classes of graphs. Indeed model checking for monadic second order formulas on graphs turned out to be fixed-parameter tractable on graphs of bounded treewidth [Cou90]. These kinds of applications however appear to be out of reach for the generalisations of treewidth this thesis deals with. However, besides this broad spectrum of applications for treewidth, the graph minors project has some deep algorithmic implications itself. In fact, one of the earliest applications of treewidth was an algorithm that solves the disjoint paths problem on a graph of treewidth at most k in polynomial time [RS86a]. From Theorem 2.2.30 it immediately follows that deciding whether a graph G contains a graph H as a minor is polynomial time solvable, for fixed H, if H is planar. The algorithm basically goes like this: If the treewidth of G is large, so large in fact that G contains a grid-minor which itself contains H as a minor, then the answer is 'yes'. Otherwise the treewidth of G is bounded by a constant only depending on H and now one can use the disjoint paths algorithm for graphs of bounded treewidth to either construct a minor model of H in G, or refute the existence of one.

In [RS95] this approach was generalised to its full potential as follows:

i) If G has bounded treewidth, say bounded in some function depending on two integers t and r, then one can use dynamic programming on a bounded width tree decomposition of G to solve the disjoint paths problem for a fixed number of paths.
ii) If G has a K_t, for t large enough, as a minor, then the model of K_t can be used to route our paths and we can decide the disjoint paths problem quickly.
iii) If none of the first two properties hold that means that we must find a flat wall W of size r in G by the Flat Wall Theorem. If r is chosen to be large enough it can be shown that removing a vertex from the middle of W does not change the outcome of the disjoint paths problem. Hence we can shrink the graph and reiterate the whole process.

This process eventually terminates and thereby solves the disjoint paths problem for a fixed number of paths in polynomial time. Indeed, using this approach Robertson and Seymour proved the following theorem.

Theorem 2.2.36 ([RS95])**.** *For every graph H there exists a polynomial time algorithm that decides whether a given graph G has a minor isomorphic to H.*

2.3. Structural Digraph Theory

While for undirected graphs definitions for concepts like minors and connectivity appear relatively straight forward, directed graphs, or *digraphs*, are a their own beast. As for the undirected case this is a brief and incomplete introduction, for a more in-depth explanation and missing definitions please consult [BJG18].

Definition 2.3.1 (Digraph)**.** A *digraph* is a tuple $D = (V, E)$, where V is a set, called the *vertices*, and E is a multiset over V^2, the members of E are called the *edges*[5]. In case the tuple is not explicitly stated we denote the vertex-set of D by $V(D)$ and the edge-set by $E(D)$. Given $e = (u, v) \in E$ we call u the *tail* of e, while v is its *head*. For any $e \in E$ we denote its tail by $\mathsf{tail}(e)$ and its head by $\mathsf{head}(e)$.

An edge $e \in E$ with $\mathsf{head}(e) = \mathsf{tail}(e)$ is called a *loop* and two distinct edges $e, e' \in E$ with $\mathsf{head}(e) = \mathsf{head}(e')$ and $\mathsf{tail}(e) = \mathsf{tail}(e')$ are said to be *parallel*. A digraph without parallel edges or loops is called *simple*.

Two edges (u, v) and (v, u) form a *digon*. A simple digraph without digons is called an *oriented graph*.

Given a digraph D, we denote by $\mathrm{un}(D)$ the *underlying undirected graph* of D, which is the graph with vertex set $V(D)$ and edge (multi-)set $\{uv \mid (u, v) \in E(D)\}$.

The digraph D is said to be *finite* if $V(D)$ is finite.

In most cases our digraphs are simple, so if not stated explicitly our digraphs do not have parallel edges or loops. Moreover, we also usually assume $\mathrm{un}(D)$ to be a simple graph. That means even if D contains digons we identify parallel edges and treat $\mathrm{un}(D)$ as a simple graph, except when stated otherwise. All digraphs we consider in this thesis are finite.

Definition 2.3.2 (In- and Out-Neighbourhood)**.** Let D be a digraph and $v \in V(D)$. The *out-neighbourhood* of v is the set $N_D^{\mathsf{out}}(v) := \{u \in V(D) \mid (v, u) \in E(D)\}$, while the *in-neighbourhood* of v is defined as the set $N_D^{\mathsf{in}}(v) := \{u \in V(D) \mid (u, v) \in E(D)\}$.

An edge e is said to be *outgoing* or *emanating* from v if $\mathsf{tail}(e) = v$, and it is said to be *incoming* to v if $\mathsf{head}(e) = v$. The *in-degree* of v, denoted by $\deg_D^{\mathsf{in}}(v)$, is the number of edges that are incoming to v in D, while the *out-degree* of v, denoted by $\deg_D^{\mathsf{out}}(V)$, is the number of edges emanating from v in D. The *total degree* of a vertex v, denoted by $\deg_D(v)$, is the sum of its out- and its in-degree.

[5]The more traditional approach is to call directed edges *arcs* and to denote the arc set by $A(D)$.

Every digraph D can be made into an undirected graph by simply 'forgetting' the orientation of the edges. There is also a way to interpret any undirected graph as a digraph.

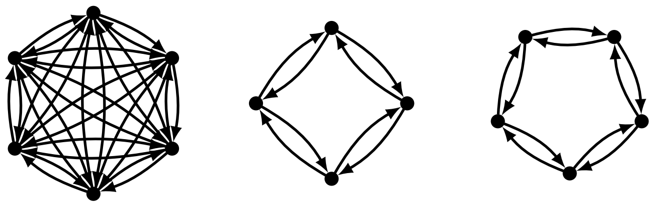
Definition 2.3.3 (Biorientation)**.** Let G be a graph. The digraph

$$\overleftrightarrow{G} := (V(G), \{(u,v),(v,u) \mid uv \in E(G)\})$$

is called the *biorientation* of G.

A digraph D for which a graph G exists with $D = \overleftrightarrow{G}$ is called a *bioriented graph* or *symmetric digraph*.

Two families of bioriented graphs are of particular interest for us. Those are the bioriented cycles, or *bicycles*, $\overleftrightarrow{C}_k$ where $k \in \mathbb{N}$, $k \geq 2$. If k is odd, $\overleftrightarrow{C}_k$ is called an *odd bicycle*, otherwise it is an *even bicycle*. For any $t \in \mathbb{N}$ the digraph $\overleftrightarrow{K}_t$ is called a *complete digraph* or *clique* of order t. See Figure 2.5 for an illustration.



$\overleftrightarrow{K}_6$ $\overleftrightarrow{C}_4$ $\overleftrightarrow{C}_5$

Figure 2.5.: The digraphs $\overleftrightarrow{K}_6$, $\overleftrightarrow{C}_4$, and $\overleftrightarrow{C}_5$.

Definition 2.3.4 (Paths and Cycles)**.** Let D be a digraph. A sequence

$$T = (v_0, e_0, v_1, e_1, \ldots, v_\ell, e_\ell, v_{\ell+1})$$

is a *directed walk* if $\mathrm{un}(T)$ is a walk, and $e_i = (v_i, v_{i+1})$ for all $i \in [0, \ell]$. We say that T *starts* on v_0 and *ends* of $v_{\ell+1}$. It is *closed* if $\mathrm{un}(T)$ is closed. Moreover, T is a *directed trail* if it is a directed walk and $\mathrm{un}(T)$ is a trail, and T is a *directed* path if it is a directed walk and $\mathrm{un}(T)$ is a path.

At last, T is a *directed cycle* if it is a directed walk and $\mathrm{un}(T)$ is a cycle.

As in the undirected case we identify the sequence and the corresponding subgraph. Moreover, in case the edges are uniquely determined by the vertices we omit them. We also adapt the notation of combining paths and starting or ending paths at intermediate vertices from the undirected setting.

In (undirected) graphs we only know one notion of connectivity. For digraphs however, there are two different notions, namely weak and strong connectivity.

Definition 2.3.5 (Weak Connectivity)**.** Let D be a digraph. We say that D is *weakly connected* if $\mathrm{un}(D)$ is connected. A digraph that is not weakly connected is called *disconnected*.

Definition 2.3.6 (Complementary Pair of Directed Paths)**.** Let D be a digraph, and $u, v \in V(D)$ be two distinct vertices. A pair (P_1, P_2) of directed paths is an *complementary pair of directed paths* connecting u and v if P_1 starts on u and ends on v, while P_2 starts on v and ends on u.

Definition 2.3.7 (Strong Connectivity)**.** Let D be a digraph. We say that D is *strongly connected* if every pair of vertices is connected by a complementary pair of paths within D.

A maximal strongly connected subgraph of D is called a *strong component* of D.

Let $k \in \mathbb{N}$ be a positive integer. We say that D is *strongly k-connected* if $|V(D)| \geq k + 1$, and $D - S$ is strongly connected for all $S \subseteq V(D)$ with $|S| \leq k - 1$.

Both versions of Menger's Theorem as stated in the section on undirected graphs are true for digraphs. Let D be a digraph and $X, Y \subseteq V(D)$. A *directed X-Y-path* is a directed path that starts on a vertex from X and ends on a vertex from Y.

Theorem 2.3.8 (Direct Menger's Theorem, Local Version [Men27])**.** Let D be a digraph and $X, Y \subseteq V(D)$ be two sets of vertices, then the maximum number of pairwise disjoint directed X-Y-paths in D equals the minimum size of a set $S \subseteq V(G)$ such that every directed X-Y-path in D contains a vertex of S.

Theorem 2.3.9 (Directed Menger's Theorem, Global Version)**.** Let D be a graph and $k \in \mathbb{N}$ be a positive integer. Then D is strongly k-connected if and only if $|V(D)| \geq k+1$ and for every pair of vertices $u, v \in V(D)$ there exists a family of k pairwise internally disjoint directed $\{u\}$-$\{v\}$-paths.

Note that just by destroying strong connectivity, or more generally, all X-Y-paths in a digraph D, one does not necessarily render D disconnected. Let D be a digraph and $\mathcal{D}$ be the set of all strong components of D. For any choice of $C_1, C_2 \in \mathcal{D}$ we write $C_1 \leq_{\text{top}} C_2$ if either $C_1 = C_2$, or there exists a directed $V(C_1)$-$V(C_2)$-path in D. Then $(\mathcal{D}, \leq_{\text{top}})$ is a partial order, usually called the *topological ordering* of $\mathcal{D}$.

A digraph T without any directed cycle is called a *directed acyclic graph* or *DAG* for short. Note that the digraph obtained from D by contracting every $C \in \mathcal{D}$ into a single vertex is in fact a DAG.

The last remaining fundamental concept we need to introduce here is a notion of minors for digraphs. There exist several different definitions of minors for digraphs which are pairwise incomparable and their usefulness usually depends on the setting. The version of directed minors we are interested in here is the one called *butterfly minor*.

Definition 2.3.10 (Butterfly Minor)**.** Let D be a digraph and $(u, v) \in E(D)$. The edge (u, v) is *butterfly contractible* if $N_D^{\mathsf{out}}(u) = \{v\}$, or $N_D^{\mathsf{in}}(v) = \{u\}$.

Suppose (u, v) is butterfly contractible and let

$$\begin{aligned} D' \coloneqq D - u - v + x + \{(w, x) \mid (w, u) \in E(D) \text{ or } (w, v) \in E(D)\} \\ + \{(x, w) \mid (u, w) \in E(D) \text{ or } (v, w) \in E(D)\}, \end{aligned}$$

where $x \notin V(D)$. We say that D' is obtained from D by *butterfly contraction* of (u, v).

A digraph H is a *butterfly minor* of D if it can be obtained from D by a sequence of edge-deletions, vertex-deletions, and butterfly contractions.

The idea behind butterfly contractions is, that any directed cycle in D that does not use (u, v) still is a directed cycle of D', while any directed cycle of D which uses (u, v) can still be obtained from a directed cycle of D' which contains the vertex x. Moreover, every directed cycle of D' belongs to one of the above categories. In particular this means that two distinct vertices of D, belonging to different strong components of D, can never become strongly connected by means of butterfly contractions.

2.3.1. Generalising Treewidth

When considering digraphs we are suddenly confronted with two different notions of connectivity, namely weak and strong connectivity. It is not immediately clear if a directed version of treewidth should focus on one these two connectivity parameters or try to involve both at the same time. Indeed, it is not clear whether it is even possible to consider weak connectivity and still obtain a parameter that is fundamentally 'directed' in the sense that it differs from the treewidth of the underlying undirected graph in a meaningful way. Several attempts have been made to obtain a generalisation of treewidth in the digraphic world and most of them can, at least qualitatively, be described using directed versions of the cops & robber game.

Directed Cops & Robber Games

Generally speaking, directed versions of the cops & robber game can be defined in the same way as they are defined for undirected graphs. The only difference we make here is the introduction of a *movement mode* for the robber, i.e. we add different rules how the new robber position R_i may be chosen.

Definition 2.3.11 (Reachability)**.** Let D be a digraph and $u \in V(D)$ be any vertex. We say that u *reaches* another vertex $v \in V(D)$ if either $u = v$ or there exists a directed path P starting on u and ending in v in D. By $\text{Reaches}_D(u)$ we denote the set of all vertices of D which are reached by u.
If $X \subseteq V(D)$, we write $\text{Reaches}_D(X)$ for $\bigcup_{x \in X} \text{Reaches}_D(x)$.

Definition 2.3.12 (Directed Cops & Robber Game)**.** Let D be a digraph. A *play* of the cops & robber game on D is a sequence

$$(C_0, R_0), (C_1, R_1), \ldots, (C_{\ell-1}, R_{\ell-1}), (C_\ell, R_\ell)$$

such that $C_i \subseteq V(D)$, and R_i is a strong component of $D - C_i$ for all $i \in [0, \ell]$, and one of the following conditions is met:

i) In $D-(C_{i-1} \cap C_i)$ there exists a strong component R with $V(R_{i-1}) \cup V(R_i) \subseteq V(R)$ (*strong game*), or
ii) $V(R_i) \subseteq \text{Reaches}_{D-(C_{i-1} \cap C_i)}(V(R_{i-1}))$ (*weak game*).

Every other definition regarding strategies, winning, monotonicity, inertness, and invisibility can be directly lifted from the definitions for undirected graphs.

To define the *cop-number* of a digraph D we need to specify which of the two movement modes we allow for our robber. In case we consider the strong game we add s to the subscript, so we write $\mathrm{cops}_s(D)$ for the smallest integer k such that there is no robber-strategy which is winning against k cops in the strong game. Similarly, when considering the weak game we add w to the subscript.

Suppose D is a DAG. Notice that in the strong game the robber can never leave its original strong component, which is a single vertex, and thus a single cop suffices to catch her, even if she is invisible. In the weak game the robber can escape from her current position, say the vertex v, to some out-neighbour of v. However, since D has no directed cycle, eventually she is forced to enter a *sink vertex*, that is a vertex with out-degree zero, from which she cannot escape further. Therefore, even in the weak game one cop suffices to eventually catch her. A valid strategy for any possible definition of the game, even against an invisible robber while having to play a monotone strategy, would be to consider a linearisation of the topological order on the vertex set of D. Now let the single cop simply move from smallest to largest vertex in this order and note that the robber can never enter a vertex which is smaller than her current position in this linear order.

This means that no variant of the cops & robber game as defined above is able to capture the complexity of DAGs. Hence weak connectivity plays only a minor role, if any, in these kinds of games.

To illustrate the complexity that arises from these different versions of the cops & robber game we first introduce the arguably most successful width parameter that can be seen as a generalisation of treewidth to digraphs. In Section 2.3.4 we add some honourable mentions of other width parameters and briefly summarise the current state of interconnectedness between these parameters and the different cops & robber games. It is worth mentioning that this landscape is vastly different from a unifying result such as Theorem 2.2.23.

2.3.2. Directed Treewidth and Duality

The, at least structurally, most successful directed version of treewidth [JRST01] was defined with computability in mind. That is, the authors were looking for a way to generalise the idea of tree decompositions alongside with obstructions like brambles which enable a constant factor approximation of treewidth in polynomial time if the treewidth is fixed. In particular, the authors were looking for a parameter satisfying the following bullet points:

i) The new parameter should serve as a corner point for the study of butterfly minors,
ii) it should be useful for proving theorems in structural digraph theory,
iii) it should have algorithmic applications, i.e. some otherwise computationally hard problems should be tractable on digraphs where the new parameter is bounded, and
iv) it should potentially be useful in practical applications.

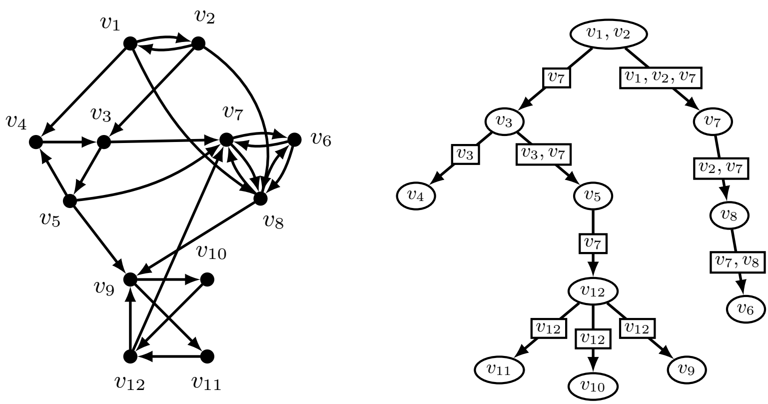


Figure 2.6.: An example digraph D (left) and a directed tree decomposition of width two for D (right).

These are some of the properties that make treewidth such a successful graph parameter [RS86b, AP89, KBVH01].

Let D be a digraph and $X \subseteq V(D)$. A directed walk W is a *directed X-walk* if it starts and ends in X, and contains a vertex of $V(D - X)$.

Definition 2.3.13 (Strong and Weak Guarding)**.** Let D be a digraph and $X, Y \subseteq V(D)$. We say that Y *strongly guards* X if every directed

X-walk in D contains a vertex of Y. The set Y *weakly guards* X if every directed X-$V(D - X)$-path contains a vertex of Y.

An *arborescence* is a digraph $\vec{T}$ obtained from a tree T by selecting a *root* $r \in V(T)$ and orienting all edges of T away from r. If e is a directed edge and v is an endpoint of e we write $v \sim e$.

Definition 2.3.14 (Directed Treewidth)**.** Let D be a digraph. A *directed tree decomposition* for D is a tuple (T, β, γ) where T is an arborescence, $\beta\colon V(T) \to 2^{V(D)}$ is a function that partitions $V(D)$, into sets called the *bags*[6], and $\gamma\colon E(T) \to 2^{V(D)}$ is a function, giving us sets called the *guards*, satisfying the following requirement:

For every $(d, t) \in E(T)$, $\gamma(d, t)$ strongly guards $\beta(T_t) := \bigcup_{t' \in V(T_t)} \beta(t')$.

Here T_t denotes the subarboresence of T with root t. For every $t \in V(T)$ let $\Gamma(t) := \beta(t) \cup \bigcup_{t \sim e} \gamma(e)$. The *width* of (T, β, γ) is defined as

$$\operatorname{width}(T, \beta, \gamma) := \max_{t \in V(T)} |\Gamma(t)| - 1.$$

The *directed treewidth* of D, denoted by $\operatorname{dtw}(D)$, is the minimum width over all directed tree decompositions for D.

We sometimes use a slightly relaxed version of directed tree decompositions. A tuple (T, β, γ) is a *relaxed directed tree decomposition* for a digraph D if it satisfies the definition of a directed tree decomposition for D with the exception that $\beta(t) = \emptyset$ is allowed for $t \in V(T)$.

Theorem 2.3.15 ([JRST01])**.** Let D be a digraph and $k \in \mathbb{N}$. If $\operatorname{dtw}(D) \leq k$ then $\operatorname{cops}_s(D) \leq k + 1$, and $\operatorname{cops}_s(D) \leq k$ then $\operatorname{dtw}(D) \leq 3k - 2$.

Indeed, from the proof of Theorem 2.3.15 one can obtain the following corollary.

Corollary 2.3.16 ([JRST01])**.** Let D be a digraph. Then $\operatorname{cops}_s(D) \leq \operatorname{cops}_{s,rm}(D) \leq 3 \operatorname{cops}_s(D) - 1$.

So, while we might need some additional cops for a robber-monotone strategy, the so called *monotonicity costs* for robber-monotonicity for the

[6]This means $\{\beta(t) \mid t \in V(T)\}$ is a partition of $V(D)$ into non-empty sets.

strong directed cops & robber game are bounded by a factor of 3. As mentioned earlier, these kinds of approximate results are unavoidable and are usually regarded as 'good enough', especially in the setting of digraphs. What might be surprising is that the step from robber- to cop-monotonicity comes with potentially unbounded monotonicity costs.

Theorem 2.3.17 ([KKRS14])**.** For every $k \in \mathbb{N}$, $k \geq 3$, there exists a digraph D_k such that $\mathrm{cops}_s(D_k) \leq 4$ and $\mathrm{cops}_{s,cm}(D_k) \geq k$.

Another observation that poses an inconvenience for directed treewidth is the fact, that it is **not** monotone under the butterfly minor relation. That is, there exist digraphs whose directed treewidth increases after the butterfly contraction of certain edges [Adl07]. Still one can show that the directed treewidth cannot be increased arbitrarily by means of butterfly contractions. A proof of this can be found in Section 5.3.

On the positive side, there exists an efficient algorithm to obtain a directed tree decomposition of bounded width for digraphs of bounded directed treewidth.

A directed tree decomposition (T, β, γ) for a digraph D is *nice* if for every $(t', t) \in E(T)$,

i) $\beta(T_t)$ induces a strong component of $D - \gamma(t', t)$, and
ii) $\gamma(t', t) \cap \beta(T_t) = \emptyset$.

Theorem 2.3.18 ([CLMS19])**.** Let D be a digraph, $k \in \mathbb{N}$, and $\mathrm{dtw}(D) \leq k$. There exists an algorithm with running time $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$ that computes a nice directed tree-decomposition of width at most $3k - 2$ for D.

Please note that the notion of nice directed tree decompositions, together with Theorem 2.3.18 allows to quickly confirm how Corollary 2.3.16 can be proven.

An essential part in the study of tree decompositions and their interaction with graph minor theory in general is played by the existence of obstructions, i.e. substructures one is guaranteed to find in every graph of high treewidth.

Definition 2.3.19 (Directed Bramble)**.** Let D be a digraph. A *directed bramble* in D is a family $\mathcal{B}$ of strongly connected subgraphs of D such that for every pair $B_1, B_2 \in \mathcal{B}$, either $V(B_1) \cap V(B_2) \neq \emptyset$, or there exist edges $(u_1, v_2), (u_2, v_1) \in E(D)$ with $u_i, v_i \in V(B_i)$ for both $i \in [1, 2]$.

A *cover* or *hitting set* for $\mathcal{B}$ is a set $S \subseteq V(D)$ such that $S \cap V(D) \neq \emptyset$ for all $B \in \mathcal{B}$.

The *order* of $\mathcal{B}$ is the minimum size of a hitting set for $\mathcal{B}$. The *directed bramble number* of a digraph D, denoted by $\mathrm{dbr}(D)$, is the maximum order of a bramble in D.

Theorem 2.3.20 ([KO14])**.** Let D be a digraph. Then $\mathrm{dbr}(D) \leq \mathrm{dtw}(D) \leq 6\,\mathrm{dbr}(D) + 1$.

Finally, there exists a directed analogue of the grid theorem for directed treewidth.

Definition 2.3.21 (Cylindrical Grid)**.** Let $k \in \mathbb{N}$ be a positive integer. The *cylindrical grid of order k* is the digraph obtained from the cycles $C_1, \dots C_k$, with

$$C_i = (v_0^i, e_0^i, v_1^i, e_1^i, \dots, e_{2k-3}^i, v_{2k-2}^i, e_{ek-2}^i, v_{2k-1}^i, e_{2k-1}^i, v_0^i)$$

for each $i \in [1, k]$, by adding the directed paths

$$P_i = v_{2i}^1 v_{2i}^2 \dots v_{2i}^{k-1} v_{2i}^k, \text{ and}$$

$$Q_i = v_{2i+1}^1 v_{2i+1}^2 \dots v_{2i+1}^{k-1} v_{2i+1}^k$$

for every $i \in [0, k-1]$.

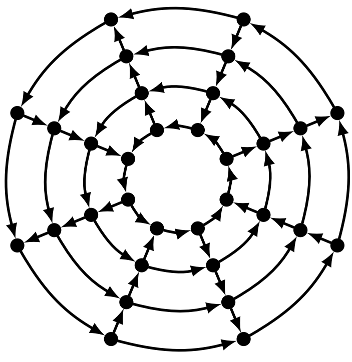


Figure 2.7.: The cylindrical grid of order 4.

Theorem 2.3.22 (Directed Grid Theorem, [KK15])**.** There exists a function $\mathsf{g}_{\mathrm{dir}} : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every digraph D we have $\mathrm{dtw}(D) \leq \mathsf{g}_{\mathrm{dir}}(k)$, or D contains the cylindrical grid of order k as a butterfly minor.

The definition of subdivisions can be generalised seamlessly from the undirected setting to digraphs. As before, one can observe that for digraphs D of maximum total-degree at most three there is no difference between containing a subdivision of D and having D as a butterfly minor.

Lemma 2.3.23 ([AKKW16])**.** Let D and H be digraphs where H has maximum total-degree at most three. Then D contains a subdivision of H if and only if D has a butterfly minor isomorphic to H.

Definition 2.3.24 (Cylindrical Wall)**.** Let $k \in \mathbb{N}$ be a positive integer. An *elementary cylindrical k-wall* W is the digraph obtained from the cylindrical grid G of order $2k$ by deleting the edges $(v_{2i}^{2j}, v_{2i+1}^{2j})$ and $(v_{2i+1}^{2j+1}, v_{2i+2}^{2j+1})$ for every $i \in [0, 2k-1]$ and every $j \in [0, k-1]$.
A *cylindrical k-wall* is a subdivision of W.

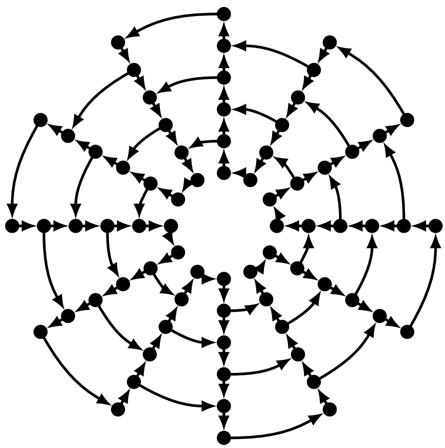


Figure 2.8.: The elementary cylindrical 3-wall.

As in the undirected case, we can translate the Directed Grid Theorem into a Directed Wall Theorem by simply doubling the necessary quantities.

Since this is just a linear factor we may assume $\mathsf{g}_{\mathrm{dir}}$ to already incorporate this factor.

Corollary 2.3.25 (Directed Wall Theorem)**.** There exists a function $\mathsf{g}_{\mathrm{dir}} \colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every digraph D we have $\mathrm{dtw}(D) \leq \mathsf{g}_{\mathrm{dir}}(2k)$, or D contains a cylindrical k-wall.

2.3.3. Disjoint Paths and Butterfly Minor Testing

Although directed treewidth is not exactly monotone under the butterfly minor relation, the existence of Theorem 2.3.22 means that it still is probably the right parameter for the study of butterfly minors. But what about its algorithmic value? A big issue, which we will address in slightly more detail in Section 2.3.5, with most directed width measures that somehow correspond to directed cop & robber games as introduced above is, that they cannot deal with DAGs. Indeed, most of these measures are some small constant, 0 or 1 in many cases, on DAGs no matter how complicated they are. The whole depth of this problem becomes apparent as soon as one realises that for any graph G there exists a DAG T such that $G = \mathrm{un}(T)$. To see this consider any undirected graph G and take any linear ordering π on $V(G)$. On any edge $uv \in E(G)$ we can now impose a direction based on π. Simply replace uv by (u, v) if and only if $\pi(u) \leq \pi(v)$, and (v, u) otherwise. By doing so, no directed edge can ever have its head appear before its tail with respect to π, and thus the resulting digraph D does not have a single directed cycle.

There are two major points attached to these observations:

i) If a problem is tractable on digraphs of bounded directed treewidth, it is necessarily tractable on DAGs.
ii) Many problems are hard on DAGs.

Hence it makes sense to focus on problems that are, at least in some sense, tractable on DAGs, but which still are hard on general digraphs.

Disjoint Paths

An important consequence of the graph minors project is Theorem 2.2.36 that was made possible by the findings from [RS95] which allowed for a polynomial time algorithm for the disjoint paths problem.

Definition 2.3.26 (Disjoint Paths and Linkedness)**.** Let D be a digraph, $k \in \mathbb{N}$ a positive integer, and $s_1, \dots, s_k, t_1, \dots, t_k \in V(D)$. The question whether there exist directed paths $P_1, \dots, P_k$ in G such that these paths are pairwise internally disjoint, and P_i starts on s_i and ends t_i for every $i \in [1, k]$ is called the *directed k-disjoint paths problem.*

We say that $(s_1, \dots, s_k)$ and $(t_1, \dots, t_k)$ are *linked* if the answer to the directed k-disjoint path problem with input $(s_1, \dots, s_k, t_1, \dots, t_k)$ is 'yes'.

A digraph D is said to be *k-linked* if the answer to every instance of the directed k-disjoint paths problem in D is 'yes'.

While the case $k = 2$ in undirected graphs can be solved by a polynomial time algorithm based on Theorem 2.2.6, for digraphs the case is vastly different.

Theorem 2.3.27 ([FHW80])**.** The Directed 2-Disjoint Paths Problem is NP-complete.

In particular this means that a characterisation such as Theorem 2.2.6 is probably[7] impossible. For DAGs however, there exists an analogue of the Two Paths Theorem [Tho85]. And in general, if we fix the number of terminal pairs t to be constant, there exists a polynomial time algorithm for the directed t-disjoint paths problem on DAGs [FHW80]. Hence one could expect the directed disjoint paths problem to be tractable, for a fixed number of paths, on digraphs of bounded directed treewidth. Indeed, in the original paper which introduced directed treewidth, an algorithm was given to solve the directed disjoint paths problem.

Theorem 2.3.28 ([JRST01])**.** Let D be a digraph, and $t, k \in \mathbb{N}$ be two positive integers such that $\mathrm{dtw}(D) \leq k$. There exists an algorithm that decides the directed t-disjoint paths on D in time $|V(D)|^{\mathcal{O}(t+k)}$.

Minor Testing and the Erdős-Pósa-Property for Butterfly Minors

With the disjoint paths problem being tractable on digraphs of bounded directed treewidth, one can now also decide for a fixed digraph H, whether any digraph D with $\mathrm{dtw}(D) \leq k$ has a butterfly minor isomorphic to D.

[7] Under the assumption $\mathsf{P} \neq \mathsf{NP}$.

Theorem 2.3.29 ([AKKW16])**.** Let D and H be digraphs and $k \in \mathbb{N}$ an integer such that $\mathrm{dtw}(D) \leq k$. There exists an algorithm that decides in time $|V(D)|^{\mathcal{O}(|V(H)|^2+k)}$ whether D has a butterfly minor isomorphic to H.

Similar to the undirected case, one can also ask for a generalisation of the Erdős-Pósa-property of directed cycles [RRST96] to butterfly minors.

Definition 2.3.30 (The Erdős-Pósa Property for Butterfly Minors)**.** Let H be a digraph. We say that H has the *Erdős-Pósa property for butterfly minors* if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every digraph D and every $k \in \mathbb{N}$ either D contains k pairwise vertex disjoint subgraphs, all of which have a butterfly minor isomorphic to H, or there exists a set $S \subseteq V(D)$ with $|S| \leq f(k)$ such that $D - S$ does not have H as a butterfly minor.

It was originally expected by the community that it would turn out that all planar digraphs have the Erdős-Pósa property for butterfly minors. However,there exist planar digraphs which are not butterfly minors of any cylindrical grid. This observation is crucial as the approach to proving the Erdős-Pósa property for butterfly minors for a given digraph H is two-fold, as it is in the undirected case: If the directed treewidth is bounded by a function depending on H, then one can use a directed tree decomposition of small width to either find the desired subgraphs, or the set S which eliminates all occurrences of H as a butterfly minor. If however the directed treewidth is to large, the only thing one can guarantee is a large cylindrical grid butterfly minor. But in case H is a butterfly minor of the cylindrical grid of any order, then we do not have any tool that provides us with either a small hitting set, or many disjoint butterfly minor models of H.

Indeed, as it turns out, being a butterfly minor of the cylindrical grid is exactly what determines the Erdős-Pósa property for butterfly minors.

Theorem 2.3.31 ([AKKW16])**.** Let H be a digraph. Then H has the Erdős-Pósa property for butterfly minors if and only if there exists $k \in \mathbb{N}$ such that H is a butterfly minor of the cylindrical grid of order k.

We discuss the topological reasons behind this outcome in more detail in Section 7.2.

2.3.4. Other Directed Width Parameters

As announced above we briefly introduce other attempts at generalising the concept of treewidth to the world of digraphs by utilising cops & robber games. Both concepts we introduce here also come with their own decompositions, but since we only mention them to illustrate the surprising diversity in parameters and the problems arising from this diversity we remain within the realm of cops & robbers.

Before we begin, note that any winning strategy for the weak variant of the directed cops & robber game also gives a winning strategy for the strong variant with the same additional rules. Indeed this means that we obtain the following hierarchy between the different cop numbers on a digraph D:

$$\begin{aligned}\mathrm{cops}_s(D) \leq \mathrm{cops}_w(D) \leq & \, \mathrm{cops}_{w,rm}(D) \\ & \leq \mathrm{cops}_{w,cm}(D) \leq \mathrm{cops}_{w,in,iv,cm}(D)\end{aligned}$$

In particular, by Theorem 2.3.15, this means that if any of these numbers is bounded then so is the directed treewidth of D. Hence, by Theorem 2.3.28, the directed t-disjoint paths problem for fixed t can be solved in polynomial time on any digraph for which at least one of these numbers is bounded.

Definition 2.3.32 (DAG-width, [BDH$^+$12])**.** Let D be a digraph and $k \in \mathbb{N}$. Then D is said to have *DAG-width* at most k if and only if $\mathrm{cops}_{w,cm}(D) \leq k$.

An interesting open problem is the question whether there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{cops}_{w,cm}(D) \leq f(\mathrm{cops}_w(D))$. We have seen that such a function does not exist for the strong variant, but for the weak game it might still be true. What is known however is, that DAG-width and directed treewidth are only related in one direction.

Theorem 2.3.33 ([BDH$^+$12])**.** For every $k \in \mathbb{N}$ there exists a digraph D_k with $\mathrm{dtw}(D_k) = 1$ and $k \leq \mathrm{cops}_w(D_k) \leq \mathrm{cops}_{w,cm}(D_k)$.

As a second example let us consider the weak variant of the directed cops & robber game where the robber is inert and invisible.

Definition 2.3.34 (Kelly-width, [HK08])**.** Let D be a digraph and $k \in \mathbb{N}$. Then D is said to have *Kelly-width* at most k if and only if $\mathrm{cops}_{w,in,iv}(D) \leq k$.

The same family of examples which shows that there exist digraphs of constant directed treewidth and unbounded DAG-width also works for Kelly-width. Whether Kelly-width and DAG-width can be seen as equivalent is another open question in this area.

2.3.5. On the Algorithmic Power of Directed Width Measures

Besides directed treewidth, DAG-width and Kelly-width there are several other directed width measures such as *directed pathwidth* [Bar06], *cycle rank* [Coh68][8], and *DAG-depth* [GHK+14]. But none of these concepts seem to perform anywhere near as good as treewidth for undirected graphs. Indeed, with the exception of some specialised problems, for most of these parameters the directed t-disjoint paths problem appears to be the only one which stays consistently tractable on classes where any of the above parameters is bounded.

An interesting turn in this story is provided by parameters which do not aim to translate treewidth like properties to digraphs, but take a more density based approach. For *clique width* [CER93], which can be defined for directed graphs in the same way it is defined for undirected graphs, there is even an analogue of the powerful algorithmic meta theorem for treewidth [Cou97]. When considering the more restrictive parameter called *directed modular width*, many otherwise hard problems appear to became somewhat tame even on digraphs [SW20].

In their work on algorithmic meta arguments regarding directed width measures, Ganian et al. [GHK+16] introduce a concept to capture the algorithmic abilities of a directed width parameter and reach the conclusion that no directed width measure that is closed under subgraphs and reversing subdivisions can be a powerful tool in terms of algorithmic application. This means that there exists some kind of dichotomy between those graph parameters which are ‘algorithmically useful’ and those which are ‘structurally meaningful’ in the sense of, for example, the theory of butterfly minors.

[8] Oddly enough, the discovery of cycle rank predates even the undirected width measures.

Chapter 3.

An Introduction to Matching Theory

Before we begin with our introduction to matching theory let us formally introduce the object of interest itself.

Definition 3.0.1 (Matching)**.** Let G be a graph and $F \subseteq E(G)$ be a set of edges. The set F is called a *matching* if the edges in F are pairwise disjoint, i.e. no two edges in F share a common endpoint. By $V(F)$ we denote the vertex set obtained by the union over all edges in F and a vertex $v \in V(G)$ is *matched* by F if $v \in V(F)$.
A matching M is *perfect* if it matches all vertices of G. The set of all perfect matching of G is denoted by $\mathcal{M}(G)$.

Since this thesis is mainly oriented towards the structural part of matching theory, and even within the structural part our topics are relatively specialised, we can only give a rough overview over an ever expanding field with many different facets. In fact, matching theory as a topic has grown so large that one can find whole monographs on the topic. For many of the topics and results this chapter touches upon the interested reader is therefore referred to the book 'Matching Theory' by Lovász and Plummer [LP09] for a deeper discussion and related material. Especially algorithmic aspects of matching theory will be neglected by us, so anyone particularly interested in practical application will find this book to be an enlightening read. Still, in our introduction and every once in a while throughout the thesis we will dabble in some algorithmic application - however, the applicability of our findings to the 'real world' might still be debatable.

Matching problems come in numerous forms and there are many known examples like the assignment of jobs to, preferably, the most capable employee, the distribution of a limited set of antibiotic drugs to a number of patients such that no patient is allergic to the drug which is assigned to them, or even just the problem of finding a suitable place to sit at the table for all of your guests for the occasional board game night. Indeed, every injective mapping between any two sets is a special case of the *maximum cardinality matching* which asks for the largest, in terms of edges it contains, matching F one can find in a given graph G.

Definition 3.0.2 (Maximum Matching)**.** Let G be a graph. A matching $F \subseteq E(G)$ is a *maximum matching* of G if for all matchings $M \subseteq E(G)$ we have $|M| \leq |F|$. The size of a maximum matching in G is the *matching number* of G, denoted by $\nu(G)$.

Once the idea of some kind of optimality, in our case the maximum number of edges one can fit into a matching, has taken hold, the realm of mathematical programming, in particular *linear programming*, has gotten close. For an introduction to this topic we recommend the book on combinatorial optimisation by Schrijver [Sch03]. For any optimisation problem that can be formulated as the maximisation of some linear function under the restriction of a set of linear inequalities one can find a *dual* minimisation problem.

Definition 3.0.3 (Vertex Cover)**.** Let G be a graph and $F \subseteq V(G)$ a set of vertices. An edge $e \in E(G)$ is said to be *covered* by F, if F contains at least one endpoint of e. The set F is called a *vertex cover* if every edge of G is covered by F.

A vertex cover F is a *minimum vertex cover* of G if for all vertex covers $N \subseteq V(G)$ we have $|F| \leq |N|$. The size of a minimum vertex cover in G is the *vertex cover number* of G, denoted by $\tau(G)$.

Suppose a graph G contains a matching M of size k, then any vertex cover of G must contain at least k vertices since no vertex can cover two edges from M. Thus we have

$$\nu(G) \leq \tau(G).$$

However, if M is a maximum matching of G, then no edge of G can be completely disjoint from $V(M)$. Therefore $V(M)$ is a vertex cover of G and we obtain a second inequality, namely

$$\tau(G) \leq 2\nu(G).$$

So the size of a maximum matching in a graph G and the size of a minimum vertex cover are closely tied to one another. Indeed, in the language of

Figure 3.1.: A graph G with a maximum matching (on the left) and a minimum vertex cover (on the right).

linear programming these two problems are said to be *dual.* Taking a step back from our, probably a bit restricted, notion of matchings and vertex covers, we can reformulate the same notion in the much more general setting of hypergraphs. A hypergraph is a straightforward generalisation of our notion of graphs by simply allowing any subset of the vertices as edges, sometimes called *hyperedges*, instead of just allowing edges to be sets of cardinality two. For an introduction to hypergraph theory the reader is referred to the book by Berge [Ber84].

Definition 3.0.4 (Hypergraph)**.** A *hypergraph* H is a tuple $H = (V, E)$ where V is a finite set and $E \subseteq 2^V$. We write $V(H) = V$ and $E(H) = E$ as we do for graphs. The set V is the set of *vertices* and E is the set of *edges.* The *rank* of H, denoted by $\mathrm{r}(H)$, is the largest cardinality among all edges in E.

A set $F \subseteq E(H)$ is called a *matching* if its members are pairwise disjoint. We denote the size of a maximum matching in H by $\nu(H)$.

A set $F \subseteq V(H)$ is called a *vertex cover* if F is a hitting set for $E(H)$. We denote the size of a minimum vertex cover in H by $\tau(H)$.

Observe that the notion of matchings and vertex covers in hypergraphs are straightforward generalisations of matchings and vertex covers in graphs. Moreover, please note that the factor 2 from the duality of maximum matching and minimum vertex cover in graphs G is nothing more that the rank of G. So, by using the same arguments, one can obtain a more general duality for maximum matching and minimum vertex cover in hypergraphs H:

$$\nu(H) \le \tau(H) \le \mathrm{r}(H)\nu(H).$$

Hypergraphs are capable of modelling a wide range of combinatorial objects and problems, and generally the problems of finding maximum matchings and minimum vertex covers, oftentimes at least approximations of those, have become known as *packing and covering problems*. The intuition here is that a maximum matching can be seen as the objective to pack as many objects, represented by the hyperedges of the corresponding hypergraph H, as possible into some kind of combinatorial structure. The covering part is the same as the covering that occurs in vertex covers: we want to find a hitting set for all occurrences of the objects we previously wanted to pack. Famous examples of these kinds of packing and covering problems include Menger's Theorem Theorem 2.1.4 and a wide range of Erdős-Pósa-type results [EP65, RST94, AKKW16]. We revisit the latter in Chapter 7. The theme of duality itself, i.e. if we cannot minimise a certain quantity we necessarily find an object of certain size that acts as some kind of witness to the quantity not being small, will come up at several points and with several faces in the following chapters.

Organisation The goal of this chapter is to motivate the development of a matching theoretic structure theory for graphs while also introducing the important notions on which this thesis is built. We start, in Section 3.1, with a brief motivation for the structural study of graphs with perfect matchings and an overview on the development so far, always with the focus on the kind of structure theory we are after. Section 3.2 then lays the foundation of arguably the most prominent application of theoretic results for this thesis, namely a close connection between the structural properties of digraphs and bipartite graphs with perfect matchings. We then specialise further and dedicate Section 3.3 to the introduction of Pólya's Permanent Problem, the question of the number of perfect matchings in a graph and a first structure theorem beyond those from Section 3.1. At last, Section 3.4 is a discussion on the differences between bipartite graphs and non-bipartite graphs and tries to shed some light on the much more complex structure we find in non-bipartite graphs with perfect matchings.

3.1. General Matching Theoretic Background

The first goal of this section is to establish a structure theory for general graphs that shows that we already understand a lot about how maximum matchings are arranged within graphs. This theory consists mainly of two parts, an understanding of how maximum matchings in bipartite graphs work and a decomposition of a given graph G into several parts, one of them being a subgraph of G with a perfect matching. This decomposition, and its existence, has become known as the Gallai-Edmonds Structure Theorem. Before we move on to this theorem, however, we first need to explore the world of bipartite graphs and their matching related properties a bit.

Definition 3.1.1 (Bipartite Graph)**.** A graph B is called *bipartite* if there exists a partition V_1, V_2, called the *colour classes* or *bipartition*, of its vertex set, such that every edge of B has exactly one endpoint in each V_i, $i \in [1, 2]$.

Wherever possible we use the letter 'B' to denote a bipartite graph to avoid ambiguity[1] Please note that the bipartition V_1, V_2 of a bipartite graph B is unique if B is connected. Whenever a bipartite graph occurs and we do not state otherwise, we assume the two colour classes to be given implicitly. For each $i \in [1, 2]$ we denote the vertices of V_i in B by $V_i(B)$. In some cases, and by slightly abusing the notation, we use V_1 and V_2 as if those where more abstract sets from which all vertices of bipartite graphs come. The vertices in V_1 are said to be *black*, while those in V_2 are *white*. This choice of colours[2] is for purely practical reasons as we stick to this convention in all figures where bipartite graphs might occur.

Bipartite graphs are of particular importance in matching theory as they naturally occur as the class of graphs where the duality of maximum matching and minimum vertex cover is tight. The minmax theorem of König is probably the single most important result in all of matching theory and, in some sense, it outshines even the Gallai-Edmonds Structure Theorem and Tutte's Theorem which we present later in this section.

Theorem 3.1.2 (König's Minmax Theorem)**.** A graph G is bipartite if and only if

$$\nu(G') = \tau(G')$$

for all subgraphs $G' \subseteq G$.

[1] This is not always possible, especially when dealing with many different bipartite graphs at the same time different names are necessary.

[2] Or non-colours to be exact.

A proof of Theorem 3.1.2 appears in [LP09] where it can be found on page 4 as Theorem 1.1.1. The fact that this is the first theorem to appear in the book on matching theory by Lovász and Plummer can and should be seen as a strong indicator for its fundamentality. Please note that generally only the 'if' part of this theorem is stated as 'Königs Theorem', the 'only if' part, however, immediately arises from the observation that a graph is bipartite if and only if it is free of odd cycles and the fact that any odd cycle violates the equality of vertex cover and matching number.

The next two results are equally strong, in fact equivalent, to Königs Minmax Theorem, but a bit more technical to state.

Let B be a bipartite graph, $i, j \in [1, 2]$ distinct, and $X \subseteq V_i$. A matching $F \subseteq E(B)$ is said to *match* V_i *into* V_j if every vertex of V_i is matched by F. The *deficiency* of X is the quantity

$$\mathrm{def}_B(X) = \mathrm{def}(X) = |X| - |\,\mathrm{N}_B(X)|,$$

and the *surplus* of X, denoted by $\mathrm{sur}_B(X) = \mathrm{sur}(X)$, is defined to be the value $-\,\mathrm{def}_B(X)$.

Theorem 3.1.3 (Hall's Theorem, [Hal09]). Let B be a bipartite graph, and $i, j \in [1, 2]$ distinct. Then B has a matching of V_i into V_j if and only if $\mathrm{sur}(X) \geq 0$ for all $X \subseteq V_i$.

The connection between positive deficiency and the existence of matchings fully matching one of the two colour classes was discovered even earlier by Frobenius and finds its expression in the famous Marriage Theorem.

Theorem 3.1.4 (Frobenius' Marriage Theorem, [Fro17]). A bipartite graph B has a perfect matching if and only if $|V_1| = |V_2|$ and for each $X \subseteq V_1$, $\mathrm{sur}(X) \geq 0$.

At the point in time Frobenius published his findings, many results, especially on matching theory and bipartite graphs, were formulated in terms of matrices and their determinants. The deep connection between this subworld of linear algebra and matching theory is bound to show up again in more detail in Section 3.3. Theorems 3.1.2 to 3.1.4 might appear to be relatively niche as they only talk about the class of bipartite graphs which is widely regarded as well understood and tame. However, all three theorems can be seen to be pairwise equivalent and, in particular, they are equivalent to Menger's Theorem Theorem 2.1.4 and the famous Max-Flow

Min-Cut Theorem by Ford and Fulkerson [FJF15]. The proof of these equivalences is left to the reader.

Gallai [Gal63, Gal64] and Edmonds [Edm65] independently proved that there is a *canonical*, that is in some sense unique, decomposition of the vertex set of any graph G that fully grasps the structure of all maximum matchings in G.

A graph G is called *factor critical* if $G - v$ has a perfect matching for all $v \in V(G)$. A matching F of G is called *near perfect* if it matches all but one vertex of G.

Let us denote by $\mathrm{comp}(G)$ the set of all connected components of the graph G. If $X \subseteq V(G)$ is a set of vertices, we denote by $\mathrm{comp}_G(X)$ the set $\mathrm{comp}(G[X])$.

For the statement of the Gallai-Edmonds Structure Theorem we need to define three sets. Let G be any graph, then we denote by

i) $D(G)$ the set of all vertices of G which are not matched by at least one maximum matching of G,
ii) $A(G)$ the set of vertices from $V(G) \setminus D(G)$ which have neighbours in $D(G)$, and
iii) $C(G)$ the set $V(G) \setminus (D(G) \cup A(G))$.

Theorem 3.1.5 (Gallai-Edmonds Structure Theorem, [Gal63, Gal64, Edm65])**.** Let G be a graph with the sets $D(G)$, $A(G)$, and $C(G)$ as defined above, then:

i) the components of $G[D(G)]$ are factor critical,
ii) $G[C(G)]$ has a perfect matching,
iii) if B is the bipartite graph with bipartition $A(G)$ and $\mathrm{comp}_G(D(G))$ such that a vertex $v \in A(G)$ is adjacent to a component $K \in \mathrm{comp}_G(D(G))$ if and only if v has a neighbour in K, then B has a matching of $A(G)$ into $\mathrm{comp}_G(D(G))$,
iv) if M is any maximum matching of G, it contains a perfect matching of $G[C(G)]$, a near perfect matching for every $K \in \mathrm{comp}_G(D(G))$, and it matches all vertices in $A(G)$ with vertices from distinct members of $\mathrm{comp}_G(D(G))$, and
v) $\nu(G) = \frac{1}{2}\left(|V(G)| - |\mathrm{comp}_G(D(G))| + |A(G)|\right)$.

Let us discuss some of the immediate consequences of the Gallai-Edmonds Structure Theorem and what we mean when we say it is canonical. This

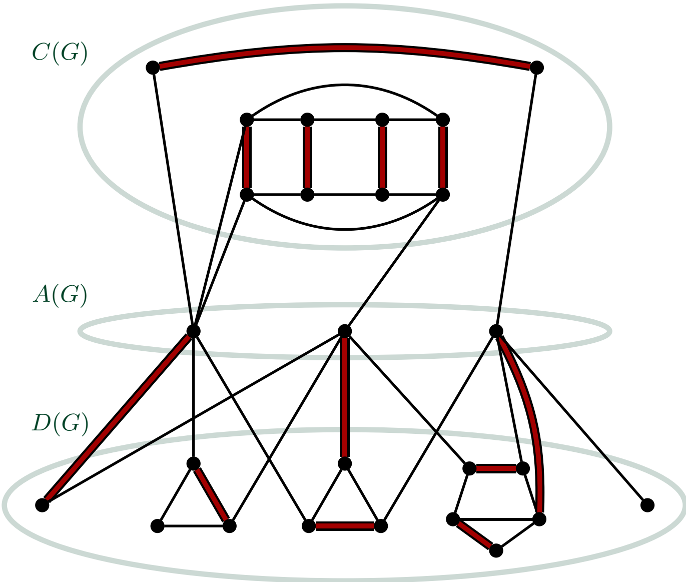


Figure 3.2.: The Gallai-Edmonds decomposition of a graph G with a maximum matching M.

discussion is taken from [LP09], where the interested reader can find some additional points. We say that a bipartite graph B has *positive surplus* (for V_i) if there exists $i \in [1,2]$ such that $\min_{X \subseteq V_i} \operatorname{sur}_B(X) \geq 1$.

(1) If G has no perfect matching, then every edge incident with a vertex of $D(G)$ lies in some maximum matching of G. Moreover, no edge with one endpoint in $A(G)$ and the other one in $A(G) \cup C(G)$ can belong to any maximum matching of G.

(2) If G is factor critical, then $D(G) = V(G)$ and thus the other two sets are empty.

(3) If G has a perfect matching on the other hand, $C(G) = V(G)$ and the other two sets are empty.

(4) In case B is a bipartite graph with $i, j \in [1,2]$ distinct such that B has positive surplus for V_i, we obtain $A(G) = V_i$ and $D(G) = V_j$.

This means that we are faced with three trivial cases for the Gallai-Edmonds Structure Theorem, namely graphs with perfect matchings, factor critical graphs and bipartite graphs of positive surplus. However, if we consider the following construction and the corresponding theorem, one obtains a good idea of why these three cases might be of interest.

Let B be a bipartite graph with $i, j \in [1, 2]$ distinct such that B has positive surplus for V_i. For every $v \in V_j$ let G_v be any factor critical graph and let H be a graph with a perfect matching. We may now construct a graph G as follows:

i) $V(G) \coloneqq V(H) \cup V_i \cup \bigcup_{v \in V_j} G_v$,
ii) keep all edges of H and all G_v, $v \in V_j$,
iii) for every edge $uv \in E(B)$ with $u \in V_i$ select an arbitrary vertex $v' \in V(G_v)$ and introduce the edge uv', and finally
iv) introduce edges within V_i and between $V(H)$ and V_i arbitrarily.

Note that, by constructing G this way, the three sets from the Gallai-Edmonds Structure Theorem are $D(G) = \bigcup_{v \in V_j} V(G_v)$, $A(G) = V_i$, and $C(G) = V(H)$. Furthermore (see [LP09] page 99, Theorem 3.2.3) every graph G can be constructed this way and the graphs H, B, and G_v for $v \in V_j$ are uniquely determined by G.

In some sense this construction is a flavour of things to come and also hints at the greater perspective, as also the notion of treewidth yields a canonical decomposition of graphs that was, eventually, generalised to a canonical way to construct any graph G that excludes some minor as described in Section 2.2.

Hence in order to describe all graphs in terms of the structure of their maximum matchings it suffices, in some sense, to only focus on graphs with perfect matchings, factor critical graphs, and bipartite graphs with positive surplus. For a discussion on the latter two we refer to [LP09] as this thesis is mainly concerned with the structure of graphs with perfect matchings.

3.1.1. Fundamental Decompositions

As demonstrated above, to reach a good understanding of the structure of graphs from the perspective of their maximum matchings, to which we will refer to as *matching structure* from here on, it is desirable to further investigate the structure of graphs with perfect matchings. A key role in the study of graphs with perfect matchings plays the so called Tutte Formula which can be seen as a generalisation of the Marriage Theorem to general graphs.

Given a graph G, a component K of G is said to be *odd* if $|V(K)|$ is odd. By $\mathrm{c}_{\mathrm{odd}}(G)$ we denote the number of odd components in G.

Theorem 3.1.6 (Tutte's Theorem, [Tut47])**.** A graph G has a perfect matching if and only if $\mathrm{c}_{\mathrm{odd}}(G - S) \leq |S|$ for all $S \subseteq V(G)$.

For the study of graphs in, for example, the context of graph minors, it often suffices to restrict the discussion to connected graphs. The notion of connectivity, however, does not necessarily suffice to capture the structure of a graph with a perfect matching, or more general, to capture the structure of the perfect matchings within a graph G.

An edge e of a graph G is called *admissible* if there exists a maximum matching $M \in \mathcal{M}(G)$ such that $e \in M$. We denote the set of all non-admissible edges of G by $E_{\mathrm{no}}(G)$.

Definition 3.1.7 (Elementary and Matching Covered)**.** A graph G is called *elementary* if $G - E_{\mathrm{no}}(G)$ is connected, G is *matching covered* if it is connected and $E_{\mathrm{no}}(G) = \emptyset$.

Let G be any graph with a perfect matching and consider $G - E_{\mathrm{no}}(G)$, then every component of $G - E_{\mathrm{no}}(G)$ must be matching covered (elementary)[3]. We call the components of $G - E_{\mathrm{no}}(G)$ the *elementary components* of G and denote the set $\mathrm{comp}(G - E_{\mathrm{no}}(G))$ by $\mathcal{E}(G)$. See Figure 3.3 for an illustration of a connected graph with a perfect matching and distinct elementary components.

In Section 3.1.1 we describe the structure that arises from the elementary components within a graph. We do this in a very general setting that allows us to handle bipartite and non-bipartite graphs within the same framework. This general approach comes at the price of high technicality, which we will not go into to much detail about, however, the interested reader may consult the papers we cite, especially those by Kita who

[3]The term 'matching covered' is only slightly stronger than the term 'elementary', in fact in most cases these two properties are indistinguishable, especially in bipartite graphs. Sometimes the term 'factor-components' is used, but we try to not digress to much into general factor theory, thus this term is avoided here.

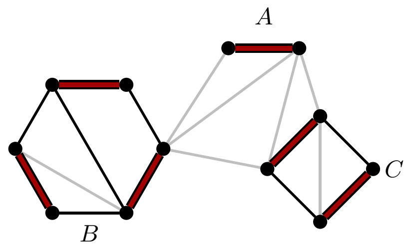


Figure 3.3.: A graph G with a perfect matching M, three different elementary components (black and bold edges), A, B and C, and grey edges, that do not belong to the cover graph. The edge in A is not contained in an alternating cycle with any of the other edges of M. Within the non-trivial elementary components B and C, any edge of M can be replaced. Replaced means that there is another perfect matching of the graph not containing e, but not touching the matchings within the other elementary components.

contributed a lot to the unification of concepts, especially those discussed in Section 3.1.1. Section 3.1.1 is the first time we encounter a way to generate all matching covered graphs, which is then taken one step further by explaining how non-bipartite elementary graphs can be generated from bipartite matching covered graphs and non-bipartite matching covered graphs with additional structure. All of these notions are then unified in terms of the *tight cut decomposition* which can be understood as the matching theoretic version of the decomposition of graphs into their 2-connected components, sometimes called *blocks*.

The Canonical Partition

Let G be any non-bipartite graph and $F \subseteq E(G)$ be a matching, the *defect* of F is the number of vertices of G which are not matched by F, i.e. the number $|V(G) \setminus V(F)|$. We may now define the *deficiency* of any graph to be $\operatorname{def}(G) := |V(G)| - 2\nu(G)$. Hence $\operatorname{def}(G)$ is the number of vertices that are left unmatched by a maximum matching. Note that, by our previous discussion, this definition would in fact produce the sum of the deficiencies obtained from the two different colour classes of a bipartite graph B. The notion of deficiency gives rise to the Berge Formula.

Theorem 3.1.8 (Berge Formula, [Ber58])**.** Let G be a graph. Then

$$\operatorname{def}(G) = \max\{\mathrm{c_{odd}}(G - S) - |S| \mid S \subseteq V(G)\}.$$

Definition 3.1.9 (Barrier)**.** Let G be a graph. A *barrier* is a set $S \subseteq V(G)$ such that $\mathrm{c_{odd}}(G - S) = |S| + \operatorname{def}(G)$.

The importance of barriers can be illustrated best by the following consequence of Tutte's Theorem:

Corollary 3.1.10 (folklore)**.** Let G be a graph with a perfect matching. An edge $e \in E(G)$ is admissible if and only if there does not exist a barrier S of G such that $e \subseteq S$.

Indeed, since we are concerned with graphs with perfect matchings in this thesis, for our purposes it suffices to say, a set $S \subseteq V(G)$ is a barrier if $\mathrm{c_{odd}}(G - S) = |S|$. Please note that this observation also shows that, in case G has a perfect matching, one could also use Tutte's Theorem to define barriers. Moreover, with this, barriers are also well defined for bipartite graphs with perfect matchings. Revisiting the Gallai-Edmonds Structure Theorem, we may restate it as follows:

Theorem 3.1.11 (Gallai-Edmonds Structure Theorem, restated)**.** Let G be a graph, then

i) $A(G)$ is a barrier such that $\mathrm{comp}_G(D(G))$ are exactly the odd components of $G - A(G)$,
ii) each odd component of $G - A(G)$ is factor critical, and
iii) every edge with one endpoint in $A(G)$ and one endpoint in $D(G)$ is admissible, while no edge with one endpoint in $A(G)$ and the other endpoint in $A(G) \cup C(G)$ is admissible.

The original idea of a canonical partition by Lovász [Lov72] was to consider an elementary graph G and its set $\mathcal{B}$ of maximal barriers. However, our goal is to use the barriers to give a more general structure, even for graphs with perfect matchings that are not elementary. Thus we use the work of Kita, who introduced a slight generalisation of the original canonical partition. Indeed, Kita's definitions are also applicable for graphs without perfect matchings, but in this context we are not interested in those any longer, hence we simply omit this part of her definition.

Definition 3.1.12 (Canonical Partition)**.** Let G be a graph and let $u, v \in V(G)$. We say $u \sim_G v$ if u and v are identical, or there exists an elementary component $K \in \mathcal{E}(G)$ with $u, v \in V(K)$, and $\operatorname{def}(G - u - v) > 0$.

Theorem 3.1.13 ([Kit12])**.** For every graph G, $\sim_G$ is an equivalence relation.

Let us denote the family of equivalence classes determined by $\sim_G$ by $\mathcal{B}(G)$. This family is called the *Kotzig-Lovász-decomposition* of G as tribute to Kotzig and Lovász, who showed that for any elementary graph G, $\sim_G$ is exactly the family of its maximal barriers[4]. Indeed, every $S \in \mathcal{B}(G)$ must be a maximal barrier of some elementary component of G. Moreover, if K is an elementary component of G and G has a perfect matching, then $\mathcal{B}_G(K) \coloneqq \{S \mid S \in \mathcal{B}(G), \text{ and } S \subseteq V(K)\}$ is exactly the family of maximal barriers of K and it partitions $V(K)$.

An exercise from [LP09] states that a matching covered graph B is bipartite if and only if $\mathcal{B}(B)$ consists of exactly two classes. Indeed, one can even show $\mathcal{B}(B) = \{V_1, V_2\}$. For non-bipartite graphs this picture is way different. Let us introduce the main concept that later distinguishes bipartite and non-bipartite graphs.

Definition 3.1.14 (Bicritical)**.** Let G be a graph and $n \in \mathbb{N}$ a positive integer. Then G is called *n-bicritical* (or *bicritical in case* $n = 1$) if $|V(G)| \geq 2n$, and $G - S$ has a perfect matching for every set $S \subseteq G$ of size $2i$ for some $i \in [0, n]$.

Note that this means that any n-bicritical graph G is also bicritical. Moreover, every bicritical graph G is necessarily connected. To see this suppose G has two distinct components K^1 and K^2. Since G has a perfect matching, each K^i must have an even number of vertices and thus, if we were to delete one vertex from each K^i, each of them would have an odd number of vertices, making the existence of a perfect matching impossible. Besides bipartite graphs, bicritical graphs pose the other class of graphs whose Kotzig-Lovász-decomposition is trivial.

[4]Hence the ‘B’ in $\mathcal{B}(G)$

Theorem 3.1.15 ([LP09]). Let G be a graph, then the following statements are equivalent.

i) G is bicritical.
ii) G is matching covered and every element of $\mathcal{B}(G)$ contains a single vertex.
iii) If $S \subseteq V(G)$ and $|S| \geq 2$ then $c_{\text{odd}}(G - S) \leq |S| - 2$.

In Section 3.1.1 we show how Lovász and Plummer used $\mathcal{B}(G)$ to construct any elementary graph by using only bipartite matching covered graphs and bicritical graphs. This construction can be seen as the next step in the refinement of the construction of general graphs we obtained from the Gallai-Edmonds Structure Theorem.

The Generalised Dulmange-Mendelsohn Decomposition

Before we go on to the construction of matching covered graphs, we need to establish a relation between the elementary components of a graph with a perfect matching to keep progressing from the most general structure one can find to the next step of refinement.

Let G be any graph and $X \subseteq V(G)$, by G/X we denote the graph obtained from G by contracting X into a single vertex and deleting parallel edges.

Let G be a graph with a perfect matching. A set X is called a *blanket* if it is the disjoint union of vertex sets of some elementary components of G. For any two elementary components $K^1, K^2 \in \mathcal{E}(G)$ we write $K^1 \trianglelefteq K^2$ if there exists a blanket $X \subseteq V(G)$ with $V(K^1) \cup V(K^2) \subseteq X$ such that $G[X]/V(K^1)$ is factor critical.

Theorem 3.1.16 ([Kit12]). Let G be a graph with a perfect matching. Then $\trianglelefteq$ is a partial order over $\mathcal{E}(G)$.

Let $K \in \mathcal{E}(G)$ be some elementary component of the graph G which has a perfect matching. We denote by $\mathcal{U}_G(K) \subseteq \mathcal{E}(G)$ the set of all strict upper bounds of K with respect to $\trianglelefteq$, while $V(\mathcal{U}_G(K))$ denotes the union of the vertex sets of the elementary components in $\mathcal{U}_G(K)$. An important property of the relation $\trianglelefteq$ is that whenever we consider an elementary component K, each component of $G[V(\mathcal{U}_G(K))]$ attaches to a unique barrier of K. One could say that the elementary components in $\mathcal{U}_G(K)$ are arranged in a complicated, tower-like structure above K. Because of this observation, Kita proposed the name *basilica decomposition* for the structure solidified with the following theorem. The basilica structure, including the naming pattern, is based on the earlier work of Kotzig and Lovász on the so called 'Cathedral Structure'.

Theorem 3.1.17 (Basilica Theorem, [Kit12])**.** Let G be a graph with a perfect matching, and let $K \in \mathcal{E}(G)$ be an elementary component of G. Then for every component H of $G[V(\mathcal{U}_G(K))]$, there exists $S \in \mathcal{B}_G(K)$ such that $\mathrm{N}_G(H) \cap V(K) \subseteq S$.

In light of the basilica theorem we may now reverse the direction and associate with every maximal barrier S of an elementary component K of G the set $\mathcal{U}_G(S)$ of elementary components that are contained in a component H of $G[V(\mathcal{U}_G(K))]$ with $\mathrm{N}_G(H) \cap V(K) \subseteq S$. The set $\mathcal{U}_G(S)$ is called the *tower over* S. Let $U_G^\top(S) := V(\mathcal{U}_G(K)) \setminus V(\mathcal{U}_G(S))$, this is known as the *court of* S. Indeed, $U_G^\top(S)$ contains the vertex sets of exactly those components of $G[V(\mathcal{U}_G(K))]$ which do not attach to S. Hence $U_G^\top(S)$ is the collection of all towers over K that are not based on S.

Definition 3.1.18 (Kita-Component)**.** A *Kita-component* of a graph G with a perfect matching is a subgraph of the form $G[S \cup \mathcal{U}_G(S)]$, where $S \in \mathcal{B}(G)$. We say that S is the *base* of $G[S \cup \mathcal{U}_G(S)]$. If C is a Kita-component of G, we denote by $\mathrm{base}_G(C) \in \mathcal{B}(G)$ the base of C, and, given some $S \in \mathcal{B}(G)$, we write $\mathrm{comp}(S)$ for the Kita-component with base S. We write $\mathcal{K}(G)$ for the set of all Kita-components of G.

For the next step we need the abstract notion of a partially ordered set (or poset) with a transitive forbidden relation.

Definition 3.1.19 (Poset with Transitive Forbidden Relation (TFR poset))**.** Let X be a set an $\preceq$ be a partial order over X. Let $\smile$ be a binary relation over X such that,

i) for each $x, y, z \in X$ if $x \preceq y$ and $y \smile z$, then $x \smile z$ (transitivity)
ii) for each $x \in X$, we have $x \not\smile x$ (non-reflexivity), and
iii) for all $x, y \in X$, if $x \smile y$, then $y \smile x$ (symmetry).

We call this poset endowed with the binary relation $\smile$ a *poset with transitive forbidden relation* of *TFR poset* for short. If for any $x, y \in X$ we have $x \smile y$, we say (x, y) is *forbidden*, or x and y form a *forbidden pair*.

Now we just have to pick up the pieces we prepared so far to form a TFR poset for any graph G with a perfect matching. To do this we need to define two binary relations, one of them being the partial order and the other one the transitive forbidden relation.

Definition 3.1.20 (The Kita-Relation)**.** Let us first define an auxiliary relation $\preceq^\circ$, the relation $\preceq$ itself is then derived from $\preceq^\circ$.

Let G be a graph with a perfect matching and let $K^1, K^2 \in \mathcal{K}(G)$. We set $K^1 \preceq^\circ K^2$ if $K^1 = K^2$ or if $\mathrm{N}_G(U_G^\top(\mathrm{base}_G(K^1))) \cap \mathrm{base}_G(K^2) \neq \emptyset$.

We then write $K^1 \preceq K^2$ for any two Kita-components $K^1, K^2 \in \mathcal{K}(G)$ if there exist $C_1, \dots, C_k \in \mathcal{K}(G)$ with $k \geq 1$ such that $\mathrm{base}_G(C_1) = \mathrm{base}_G(K^1)$, $\mathrm{base}_G(C_k) = \mathrm{base}_G(K^2)$, and $C_i \preceq^\circ C_{i+1}$ for all $i \in [1, k-1]$.

So we have $K^1 \preceq^\circ K^2$ if the court of $\mathrm{base}_G(K^1)$ contains the tower over $\mathrm{base}_G(K^2)$. Moreover, if K^1, K^2, K^3 all are Kita-components of G with $K^1 \preceq^\circ K^2 \preceq^\circ K^3$, then we can reach K^3 from K^1 by entering a tower from the court of $\mathrm{base}_G(K^1)$, climbing into the tower over $\mathrm{base}_G(K^2)$, taking the stairs of this tower down to its base and then entering a tower of the court of $\mathrm{base}_G(K^2)$ which eventually leads to the tower over $\mathrm{base}_G(K^3)$. Hence, if $K^1 \preceq K^2$, we can reach the base of K^2 from the base of K^1 by a sequence of 'climbing a tower of the court of some base' and 'entering the tower over some other base and taking the stairs down until the base is reached'. Indeed, this intuition is backed up by some auxiliary results regarding certain paths between towers, bases and towers from the court of those bases.

Definition 3.1.21 (Forbidden Courts)**.** The strategy to define the transitive forbidden relation $\smile$ is similar to before, we first define an auxiliary relation $\smile^\circ$.

Let G be a graph with a perfect matching and let $K^1, K^2 \in \mathcal{K}(G)$. We set $K^1 \smile^\circ K^2$ if $\mathrm{base}_G(K^2) \subseteq V(K^1) \setminus \mathrm{base}_G(K^1)$.

We then write $K^1 \smile K^2$ for any two Kita-components $K^1, K^2 \in \mathcal{K}(G)$ if there exists $K' \in \mathcal{K}(G)$ with $K^1 \preceq K'$ and $K' \smile^\circ K^2$.

To stick with the image of an escheresque basilica with many towers, the forbidden relation means that, although we could also reach the base of towers that sprout off from towers of our court, we should not do so. Instead, whenever one climbs down some stairs one should always leave their current tower and go on to a completely new one. One could also imagine these towers to be connected by small bridges and a court can only be entered safely if no guard was alerted by us climbing up the corresponding tower[5].

Theorem 3.1.22 ([Kit18])**.** For any graph G with a perfect matching, the triple $(\mathcal{K}(G), \preceq, \smile)$ is a TFR poset.

The TFR poset $(\mathcal{K}(G), \preceq, \smile)$ is in fact uniquely determined by G and it describes exactly the interaction between the maximal barriers of different elementary components of G. In the original paper, the canonical structure described by $(\mathcal{K}(G), \preceq, \smile)$ is called the *generalised Dulmage-Mendelsohn decomposition* of G. This name comes from a far simpler description of the interaction between the colour classes, which again are the maximal barriers, of the elementary components of bipartite graphs with perfect matchings.

The original Dulmage-Mendelsohn decomposition can be defined in the style of Kita as follows.

Definition 3.1.23 (Dulmage-Mendelsohn Decomposition)**.** Let B be a bipartite graph with a perfect matching, and $i \in [1, 2]$.

For any two elementary components $K^1, K^2 \in \mathcal{E}(B)$ we set $K^1 \leq_i^\circ K^2$ if $K^1 = K^2$, or there exists an edge with one endpoint in $V_i \cap E(K^2)$ and the other one in $E(K^1) \setminus V_i$.

We then write $K^1 \leq_i K^2$ for any two elementary components of B if there exist $H_1, \ldots, H_k \in \mathcal{E}(B)$, $k \geq 1$, such that $H_1 = K^1$, $H_k = K^2$, and $H_j \leq_i^\circ H_{j+1}$ for all $j \in [1, k-1]$.

In particular, this means $K^1 \leq_1 K^2$ if and only if $K^2 \leq_2 K^1$. This relation, in a way, resembles the topological ordering of strong components of digraphs, as in every step we leave a component going from V_i to V_j, inside the component we pass over to V_i again and may now move to the next component.

Theorem 3.1.24 (Dulmage-Mendelsohn Decomposition, [DM58, DM59, DM63])**.** Let B be a bipartite graph with a perfect matching. Then for any $i \in [1, 2]$, the binary relation $\leq_i$ is a partial order over $\mathcal{E}(B)$.

[5] There seems to be a quite dramatic reason behind this. Perhaps you are imprisoned in this basilica and need to confuse the guards to escape, or something even more sinister is going on!

When studying bipartite graphs one can observe that for any two elementary components $K^1, K^2 \in \mathcal{E}(B)$ of a bipartite graph B with a perfect matching we have $\mathcal{B}_B(K^1) = \{V_1 \cap V(K^1), V_2 \cap V(K^1)\}$, and $K^1 \ntrianglelefteq K^2$ and $K^2 \ntrianglelefteq K^1$. Hence bipartite graphs are those whose basilica structure is degenerate. This observation ties in nicely with our findings from previous subsections.

For any bipartite graph B with a perfect matching and any $i \in [1,2]$ we define $\mathcal{K}_i(B) \coloneqq \{K \in \mathcal{K}(B) \mid \mathrm{base}_B(K) \subseteq V_i\}$ to be exactly those courts whose base belongs to V_i. Let us define the mapping $\mathrm{dm}_i \colon \mathcal{E}(B) \to \mathcal{K}_i(B)$ as $\mathrm{dm}_i(K) \coloneqq \mathrm{comp}(V(K) \cap V_i)$. So we map every elementary K component of B to the Kita-component with base $V(K) \cap V_i$. Observe that now $K^1 \leq_i K^2$ holds if and only if $\mathrm{dm}_i(K^1) \preceq \mathrm{dm}_i(K^2)$. Consequently, the generalised Dulmage-Mendelsohn decomposition indeed generalises the Dulmage-Mendelsohn decomposition for bipartite graphs.

Constructing Non-Bipartite Elementary Graphs

Recall the construction procedure of general graphs implied by the Gallai-Edmonds Structure Theorem. Here we were able to reduce every graph to building blocks from just three different classes of graphs, namely factor-critical graphs, bipartite graphs with positive surplus, and graphs with perfect matchings. The generalised Dulmage-Mendelsohn decomposition together with Corollary 3.1.10 provides us with a good understanding of how graphs with perfect matchings can be decomposed into their elementary components and how those are created from matching covered graphs.

In this intermediate step we want to recapture the spirit of the construction procedure obtained from the Gallai-Edmonds Structure Theorem.

Let G be an elementary non-bipartite graph and $S \in \mathcal{B}(G)$ be a maximal barrier.

Please note that, since G is elementary and S is a maximal barrier, $G - S$ does not have a component with an even number of vertices. This can be seen easily by observing that every perfect matching of G must match one vertex of each odd component of $G - S$ to a vertex of S. Since $|S| = \mathrm{c}_{\mathrm{odd}}(G - S)$ by definition of a barrier, no vertex of S can ever be matched to a vertex of a potential even sized component under any perfect matching of G. Hence if $G-S$ has a component K' with an even number of vertices, then no edge linking a vertex of K' to S can be admissible and thus $E-E_{\mathrm{no}}(G)$ cannot be connected.

Corollary 3.1.10 tells us that any edge with both endpoints in S is non-admissible, so let us ignore those edges for now and call the set of such edges E_S for later use. Let us construct a bipartite graph $\mathsf{B}_{G,S}$ based on S as follows:

i) For every $K \in \mathrm{comp}(G-S)$ introduce the vertex v_K and let $V_1 \coloneqq S$, $V_2 \coloneqq \{v_K \mid K \in \mathrm{comp}(G-S)\}$. We set $V(\mathsf{B}_{G,S}) \coloneqq V_1 \cup V_2$.

ii) For every $s \in S$ and $K \in \mathrm{comp}(G-S)$ introduce the edge sv_K to $\mathsf{B}_{G,S}$ if and only if $\mathrm{N}_G(S) \cap V(K) \neq \emptyset$.

Since every edge in $\mathsf{B}_{G,S}$ has one endpoint in V_1 and the other in V_2, it is a bipartite graph. Moreover, $\mathsf{B}_{G,S}$ is matching covered if and only if $\mathsf{B}_{G,S} + E_S$ is elementary.

So when considering a maximal barrier S of G we are able to create a uniquely determined bipartite graph representing the connections of S to the rest of the graph. What we lose here, however, is the information on which vertices of K exactly have neighbours in S. To model this relation we construct a second auxiliary graph $\mathsf{K}_{G,S,K}$ by simply deleting all vertices of G that do not belong to K or S, then deleting the edges in E_S, and finally by identifying S into a single vertex v_S. It follows immediately from Tutte's Theorem that $\mathsf{K}_{G,S,K}$ has a perfect matching.

In some sense, the graphs $\mathsf{B}_{G,S}$ and $\mathsf{K}_{G,S,K}$, $K \in \mathrm{comp}(G-S)$ encode how the perfect matchings of G interact with S and the components of $G-S$. Indeed, these reduced graphs inherit a lot of structure from G itself.

Theorem 3.1.25 ([LP09])**.** Let G be an elementary and non-bipartite graph, $S \in \mathcal{B}(G)$ with $|S| \geq 2$, and let $K \in \mathrm{comp}(G-S)$. Then

i) the bipartite graph $\mathsf{B}_{G,S}$ is matching covered,

ii) the graph $\mathsf{K}_{G,S,K}$ is elementary, and

iii) $\mathcal{B}(\mathsf{K}_{G,S,K}) = \{\{v_S\}\} \cup \{T \cap V(K) \neq \emptyset \mid T \in \mathcal{B}(G)\}$.

In some sense, the bipartite graph $\mathsf{B}_{G,S}$ acts as some kind of 'frame'[6] for the assembly of the $\mathsf{K}_{G,S,K}$ and thus the recovery of the original graph G. This process can even be iterated for every $K \in \mathrm{comp}(G-S)$ for which

[6] Some might say as a 'brace'.

$\mathsf{K}_{G,S,K}$ is not bicritical and hence has a maximal barrier with at least two vertices, and thus we obtain a decomposition procedure for G into a collection of bipartite and matching covered graphs and a collection of bicritical graphs. This procedure is called the *bicritical decomposition* of G.

By reversing the process described in Theorem 3.1.25, one obtains a construction method similar to the one that uses the Gallai-Edmonds Structure Theorem to generate all graphs.

Definition 3.1.26 (Frame Construction)**.** Let G_0 be a matching covered bipartite graph with more than two vertices, and suppose we are given for each vertex $w \in V_2$ an elementary graph G_w with the property that there exists $\{v_w\} \in \mathcal{B}(G_w)$ with $\deg_{G_w}(v_w) = \deg_{G_0}(w)$.

For each $w \in V_2$ let $A(w) = \mathrm{N}_{G_0}(w)$ and $B(w) = \mathrm{N}_{G_w}(v_w)$.

Now for each $w \in V_2$ do the following:

i) add G_w to G_0 and delete w and v_w, and
ii) add edges between $A(w)$ and $B(w)$ such that every vertex of $A(w) \cup B(w)$ is incident to at least one new edge.
Finally add edges with both endpoints in V_1 arbitrarily.

We close this part with the theorem which solidifies the Frame Construction as a way to obtain all elementary graphs.

Theorem 3.1.27 (Frame Construction Theorem, [LP09])**.** A graph is elementary if and only if it can be built from matching covered bipartite graphs and bicritical graphs by iterating the Frame Construction.

Alternating Paths and Ear-Decompositions

With the generalised Dulmage-Mendelsohn decomposition we have a canonical way to describe the interactions of elementary components within the same graph with a perfect matching. We have also seen how to construct elementary graphs from bipartite matching covered graphs and bicritical (and thus non-bipartite and matching covered) graphs. The next natural step is to further investigate the structure within such an elementary component. Hence from here on out we will mostly be concerned with matching covered or at least elementary graphs. An important tool in matching theory, especially of bipartite graphs with perfect matchings, are alternating paths.

Definition 3.1.28 (Alternating Paths and Cycles)**.** Let G be a graph and F a matching in G. A path P is said to be F-*alternating*, if there exists a subset S of the endpoints of P such that F contains a perfect matching of $P - S$. The path P is *alternating* if there is a maximum matching M of G such that P is M-alternating.

A cycle C is said to be F-*alternating* if F contains a perfect matching of C. The cycle C is said to be *alternating* if G has a maximum matching M for which C is M-alternating.

We are particularly interested in a certain kind of alternating paths. Before we dive deeper into the classical topic of ear-decompositions, let us introduce a notion of connectivity that is suitable for graphs with perfect matchings and, at least for bipartite graphs, equivalent to some classical measures of connectivity in graphs with perfect matchings. From here onward we are interested in a theory and in operations that preserve the property of our graphs to have perfect matchings. To do this, we have to restrict notions like subgraphs and vertex sets to respect the existence of perfect matchings in a meaningful way.

Definition 3.1.29 (Conformal Sets and Subgraphs)**.** Let G be a graph with a perfect matching M. A set $X \subseteq V(G)$ is said to be *conformal* if $G - X$ has a perfect matching, it is M-*conformal* if M contains a perfect matching of $G - X$. A subgraph $H \subseteq G$ is *conformal* if $V(H)$ is conformal and H itself has a perfect matching. We say that H is M-*conformal* if $V(H)$ is M-conformal and M contains a perfect matching of H. If M is a matching and H some graph such that $M \cap E(H)$ is a perfect matching of H, then we call M a *perfect matching*[7] of H.

Please note that a cycle C is alternating in a graph G with a perfect matching if and only if it is conformal. Hence with regards to cycles these two terms are interchangeable, with alternating being the notion most commonly used in the literature. However, since most of our results will be formulated in terms of conformal subgraphs, it is oftentimes convenient to

[7]This is purely for convenience as we work with M-conformal subgraphs on many occasions and this slight alteration in terminology allows us to spare some annoying notation.

not use two different notions and talk about conformal cycles. Moreover, if C is M-conformal, then C must be of even length and $E(C) \setminus M$ is also a perfect matching of C.

Definition 3.1.30 (Symmetric Difference)**.** Let X and Y be two sets. We denote the *symmetric difference* of X and Y by

$$X \Delta Y := (X \setminus Y) \cup (Y \setminus X).$$

So if C is an M-conformal cycle where M is a perfect matching of G, then $M \Delta E(C)$ is again a perfect matching of G. We call this operation of obtaining a new perfect matching from exchanging the edges along an M-conformal cycle the *switching* of M along C.

Definition 3.1.31 (Internally Conformal Paths)**.** Let G be a graph with a perfect matching M. An alternating path P with endpoints u and v is called *internally M-conformal* if it has at least one edge, and $P - u - v$ is an M-conformal subgraph of G. In case $E(P) \subseteq M$ we say that P is *M-trivial.* A path P is said to be *internally conformal* if it is internally M'-conformal for some perfect matching M' of G. Finally, an M-conformal path is *M-covered* if for every edge $e \in E(P) \setminus M$ there exists a perfect matching $M_e \in \mathcal{M}(G)$ with $e \in M_e$.

Let us make some preliminary observations on the inner workings of an elementary graph. Let G be a graph and $F \subseteq E(G)$ be a set of edges. By $G[F]$ we denote the subgraph of G that is *induced by the edge set F*, i.e. the graph with vertex set $V(F)$ and edge set F.

Observation 3.1.32 (folklore)**.** Let G be a graph with two perfect matchings M_1 and M_2. Then every component of $G[M_1 \cup M_2]$ is either isomorphic to K_2, or an even cycle.

Proof. Since every vertex of G is incident with exactly one edge from M_i for each $i \in [1, 2]$, no vertex in $G[M_1 \cup M_2]$ has degree 0 or more than two. Hence every component of $G[M_1 \cup M_2]$ must either be a path or a cycle. If it is a cycle, then this cycle clearly must alternate between edges of M_1 and edges of M_2. So suppose there is a path P which is a component of $G[M_1 \cup M_2]$. Let x be an endpoint of P and let e be the edge of P incident with y. In case $e \in M_1 \cap M_2$, P must be isomorphic to K_2 and thus we are done. So suppose there is $i \in [1, 2]$ such that e belongs exclusively to

M_i. This, however, means that x must be matched by an edge e' of M_j, $j \in [1,2] \setminus \{i\}$. Then e' must be an edge of $G[M_1 \cup M_2]$, contradicting x being an endpoint of a path-component of $G[M_1 \cup M_2]$. $\square$

Observation 3.1.32 has a deep implication on the structure of perfect matchings in a graph G, namely that every perfect matching of G can be transformed into any other perfect matching of G by simply switching the first matching along some cycles. The best possible way to go from one perfect matching to another is an important topic in the theory of reconfigurations of combinatorial structures (see [BBH$^+$19] for more information on this area). This observation has an immediate consequence on the connectivity and the structure of elementary graphs.

Observation 3.1.33 (folklore)**.** Let G be a elementary graph with a perfect matching M and an edge $e \in M$. Then for every perfect matching N of G with $e \notin N$ there exists an N-conformal cycle C with $e \in E(C)$.

Proof. By Observation 3.1.32 the graph $G[M \cup N]$ consists purely of components that are isomorphic to K_2 or an even cycle. As $e \in M \setminus N$, e must be an edge of $G[M \cup N]$ but cannot be contained in a component which is isomorphic to K_2, as both endpoints of e must be matched by N. Hence the component of $G[M \cup N]$ that contains e is the desired cycle. $\square$

With this we are ready to prove a first observation on how alternating paths are responsible for a graph G being elementary.

Lemma 3.1.34 (A*)**.** Let G be a graph with a perfect matching M. Then G is elementary if and only if for every vertex $v \in V(G)$ and every edge $xy \in M$ there exists $z \in \{x, y\}$ such that there is an M-covered path with endpoints v and z in G.

Proof. First, assume that such a path exists for every choice of v and $e = xy$. Since these paths are covered, they also exist in $G - E_{\mathrm{no}}(G)$ and thus for every vertex w we have a path from v to w in $G - E_{\mathrm{no}}(G)$. Hence $G - E_{\mathrm{no}}(G)$ is connected and so G is elementary.

Now for the reverse direction, let us assume that G is elementary and let $v \in V(G)$ and $xy \in M$ be fixed. If $v \in e = xy$, we have an M-trivial path with endpoints v and $z \in e$ and are done, Thus we may assume $v \notin e$.

Towards a contradiction, let us choose v and e to minimise the distance from v to an endpoint of e in $G - E_{\text{no}}(G)$ such that there is no M-covered path with endpoints v and $z \in e$ in G. Without loss of generality, we may assume x to be an endpoint of e in minimal distance to v in $G - E_{\text{no}}(G)$. Let P' be a shortest x-v-path in $G - E_{\text{no}}(G)$. Clearly, P' exists since G is elementary. Now let w be the neighbour of x on P' and $e' \in M$ the edge matching w. If $e' = vw$, we can find an M-conformal cycle containing xy, xw, and vw by Observation 3.1.32 and thus the subpath of this cycle starting in v and ending on e while avoiding w is our desired path. Moreover, by choice of x we also have $e' \neq e$.

Since the distance between v and w is strictly smaller than the distance between v and x, there exists an M-covered path P with endpoints v and $z' \in e'$ in G. By Observation 3.1.32 there exists an M-conformal cycle C containing e' and xw, and thus C must also contain e. Observe that, if Q is a component of $C \cap P$, it is an M-alternating path starting and ending with an edge of M. Moreover, for every $u \in V(C) \setminus e$ there is an M-covered path with endpoints u and $z'' \in e$ in C. Let $t' \in V(C \cap P)$ be the vertex with the smallest distance to v along P and let $t't \in M$ be the edge of M matching t'. Then $t't$ is an edge of C and the M-covered path P'' with endpoints t and $z''' \in e$ on C cannot contain t'. Now $Pt'tP''$ is a covered M-covered path with endpoints v and $z''' \in e$ in G. $\square$

While M-covered paths are a nice way to describe the overall connectivity of elementary graphs, they do not fully capture the kind of connectivity based on cycles which is hinted at by Observation 3.1.33. To better replicate this 'cyclical' part, we ask for internally M-conformal paths between any two edges of M such that both paths together cover all four endpoints of the two edges. It turns out that this gives rise to a kind of connectivity that allows us to fully focus on a single perfect matching, that is, we do not require our paths to be M-covered any more.

Definition 3.1.35 (M-Complementary Pairs of Paths)**.** Let G be a graph with a perfect matching M, and let $e_1 = u_1u_2, e_2 = v_1v_2 \in M$ be two distinct edges. A pair (P_1, P_2) of internally M-conformal paths[8] is said to be an e_1-e_2-*M-complementary pair of paths* if there exist distinct

[8] The two paths P_1 and P_2 do not need to be disjoint.

$i, j \in [1, 2]$ such that P_1 has u_1 and v_i as its endpoints, while P_2 has u_2 and v_j as its endpoints.

Let P_1 and P_2 be two internally M-conformal paths, then (P_1, P_2) is an *M-complementary pair* of paths if e_1 and e_2 are the two edges of M that match the endpoints of P_1 and (P_1, P_2) is an e_1-e_2-M-complementary pair of paths.

Definition 3.1.36 (Matching Connected)**.** Let G be a graph with a perfect matching M. Two edges $e_1, e_2 \in M$ are said to be *matching connected* if either $e_1 = e_2$ or there exists an e_1-e_2-M-complementary pair of paths (P_1, P_2) in G such that for every choice of distinct $e_1', e_2' \in E(P_1 \cup P_2) \cap M$ there exists an e_1'-e_2'-M-complementary pair of paths in G. If $e_1, e_2 \in M$ are matching connected, we write $e_1 \sim_M e_2$. A graph G is said to be *matching connected* if there exists a perfect matching M of G such that the edges of M are pairwise matching connected.

This last part might seem fairly technical, but it is necessary, as there exist non-elementary graphs with perfect matchings and pairs of matching edges that are connected via a complementary pair of paths, see for Figure 3.4 for a small example. However, as we discuss in Section 3.4, these examples must be non-bipartite and thus, if we were only interested in bipartite graphs, it would suffice to define the matching connectivity of two matching edges purely by the existence of a complementary pair of paths.

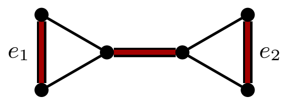


Figure 3.4.: A non-bipartite graph G with a perfect matching M and two matching edges e_1 and e_2 joined by a complementary pair of paths such that e_1 and e_2 belong to different elementary components of G.

In greater context, we claim that the binary relation of matching connectivity partitions the vertex set of any given graph G with a perfect matching into the vertex sets of its elementary components. Hence our goal is to prove the following theorem.

Theorem 3.1.37. Let G be a graph with a perfect matching M. Then $\sim_M$ is an equivalence relation on M.

To prove Theorem 3.1.37 we first need a way to handle the possible types of intersection the two paths in an M-complementary pair can have. These intersections are best described using a sense of direction. So if we are given two edges $e_1, e_2 \in M$ together with an e_1-e_2-M-complementary pair of paths (P_1, P_2), we implicitly treat the two paths P_1 and P_2 to be oriented away from their respective endpoint on e_1 and towards their respective endpoint in e_2.

With respect to their orientation, there are two possible ways P_1 and P_2 can intersect. Either they are both directed in the same way and thus *align* for a short section, or they are oriented in opposing directions and *collide*. Formally, we call every maximal subpath I of $P_1 \cap P_2$ an *overlap* of P_1 and P_2. An overlap I of P_1 and P_2 in which the two paths align is called an *alignment* and similarly an overlap I of P_1 and P_2 in which they collide is called a *collision*. See Figure 3.5 for an illustration.

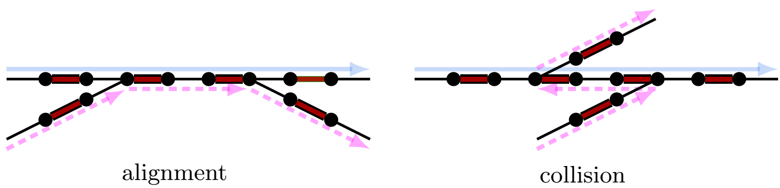


Figure 3.5.: A complementary pair of paths together with their implicit orientations and the two possible ways of intersections.

Every overlap I of P_1 and P_2 splits both P_1 and P_2 into two parts. Given an overlap I of P_1 and P_2, we denote by $s_I \in V(I)$ the endpoint of I that we meet first when we traverse P_1 starting from u_1, and by t_I the other endpoint of I.

Observe that the paths $P_1 s_I$ and $t_I P_1$ are internally M-conformal paths. For P_2 our notation will differ depending on the type of overlap we have in I. If I is an alignment, P_2 is split into the paths $P_2 s_I$, I, and $t_I P_2$, similar to P_1. However, if I is a collision P_2 will be split into $P_2 t_I$, I, and $s_I P_2$.

If I is an alignment of P_1 and P_2 such that $P_1 s_I$ and $t_I P_2$ collide, I is a *trap*. See Figure 3.6 for an illustration.

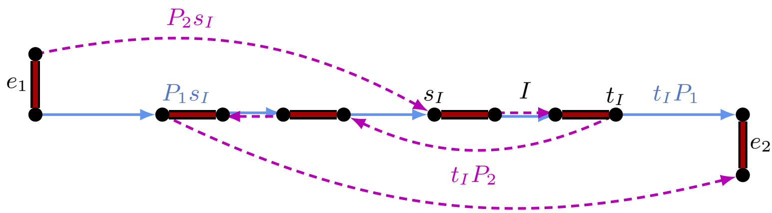


Figure 3.6.: A trap.

Lemma 3.1.38 (A*)**.** Let G be a graph with a perfect matching M, $e_1, e_2 \in M$ a pair of matching edges and (P_1, P_2) an e_1-e_2-M-complementary pair of paths. If I' is an alignment of P_1 and P_2 that is not a trap, then there exists an alignment I of P_1 and P_2 such that $V(P_1 s_I) \cap V(t_I P_2) = \emptyset$.

Proof. Let I' be an alignment of P_1 and P_2 that is not a trap. If $V(P_1 s_{I'}) \cap V(t_{I'} P_2) = \emptyset$ the assertion holds with $I = I'$, so we may assume the contrary. Since I' is not a trap, every overlap of $P_1 s_{I'}$ and $t_{I'} P_2$ is an alignment. Let $I_1, \dots, I_k$ be these alignments ordered by their appearance along P_2. So $t_{I_k} P_2$ and $P_1 s_{I'}$ do not overlap since otherwise they would do so in an alignment I_{k+1}, which does not exist. Therefore, in particular $V(P_1 s_{I_k}) \cap V(t_{I_k} P_2) = \emptyset$ and with $I = I_k$ we are done. □

Lemma 3.1.39 (A*)**.** Let G be a graph with a perfect matching M, $e_1, e_2 \in M$ a pair of matching edges and (P_1, P_2) an e_1-e_2-M-complementary pair of paths such that P_1 has endpoints u_1 and u_2 while P_2 has endpoints v_1 and v_2. If P_1 and P_2 have an alignment that is not a trap, there exists an internally M-conformal path $P \subseteq P_1 \cup P_2$ with endpoints u_1 and v_2.

Proof. By Lemma 3.1.38 the existence of an alignment that is not a trap implies the existence of an alignment I such that $V(P_1 s_I) \cap V(t_I P_2) = \emptyset$. The path I is M-conformal while the paths $P_1 s_I$ and $t_I P_2$ are internally

M-conformal, hence $P_1 s_I + I + t_I P_2$ is the desired internally M-conformal path. $\square$

So as long as we have a single alignment that is not a trap we can choose which endpoint of e_2 we want to connect to u_1 via an internally M-conformal path. The next lemma will handle the case where every alignment is a trap. In particular, this includes the case where there is no alignment at all.

Lemma 3.1.40 (A*). Let G be a graph with a perfect matching M, $e_1, e_2 \in M$ a pair of matching edges and (P_1, P_2) an e_1-e_2-M-complementary pair of paths such that P_1 has endpoints u_1 and u_2 while P_2 has endpoints v_1 and v_2. If all alignments of P_1 and P_2 are traps, then there exists an M-covered path with endpoints u_1 and $z \in e_2$.

Proof. First of all, note that the case where P_1 and P_2 are disjoint and thus do not overlap at all is trivial since in this case $P_1 e_1 P_2 e_2$ forms an M-alternating cycle and so both P_1 and P_2 are covered. Hence we may assume that P_1 and P_2 overlap.

Let $e_1, e_2 \in M$ together with (P_1, P_2) be chosen such that $|V(P_1)| + |V(P_2)|$ is minimum and there does not exist an M-covered path with endpoints u_1 and $z \in e_2$. Now let $e_3 \in E(P_1) \cap E(P_2) \cap M$ be the first edge of P_1, with respect to the orientation of P_1, that belongs to P_2. Let u_3 be the first vertex of e_3 along P_1 and v_3 be its other endpoint.

Suppose e_3 belongs to an alignment A of P_1 and P_2. Then $P_1 s_A$ is internally disjoint from P_2 and therefore completely disjoint from $t_A P_2$ and thus A is not a trap, contradicting our assumption. So e_3 belongs to a collision. Let $P_1' := P_1 u_3$ and $P_1'' := v_3 P_1 = P_1 - P_1'$. Then P_1' and P_1'' are internally M-conformal paths, and, as we have seen before, P_1' is internally disjoint from P_2. Similarly, e_3 splits P_2 into the two internally M-conformal paths $P_2'' := u_3 P_2$ and $P_2' := P_2 v_3$. By construction, P_1' and P_2' are disjoint, and so $C := P_1' e_1 P_2' e_3$ is an M-conformal cycle. Hence both P_1' and P_2' are covered. Hence neither P_1'' nor P_2'' can be covered. Moreover, notice that P_1'' and P_2'' cannot be disjoint since otherwise, by the same arguments as for P_1' and P_2', they would be covered. Thus P_1''

and P_2'' must overlap again. Since both are strict subpaths of P_1 and P_2 respectively, we obtain the following inequality.

$$|V(P_1'')| + |V(P_2'')| < |V(P_1)| + |V(P_2)|$$

Additionally, with P_1' being disjoint from P_2'', any alignment of P_1'' and P_2'' that is not a trap is also an alignment of P_1 and P_2 that is not a trap. Therefore (P_1'', P_2'') is an e_3-e_2-M-complementary pair of paths such that every alignment is a trap. Thus, from the choice of e_1 and e_2, we obtain the existence of an M-covered path P starting on v_3 and ending on a vertex of e_2, say w. Let x be the unique vertex of $V(C) \cap V(P)$ such that xP is internally vertex disjoint from P_1' and P_2'. The edge e_4 of M matching x is contained in C and has a vertex y as its other endpoint. Now either $e_1 = e_4$ or there exists an M-covered path P' with endpoints y and $z' \in e_1$ within C. In the first case, P itself is an M-covered path joining an endpoint of e_1 to an endpoint of e_2, and in the second case, $P' + xy + P$ is such a path. Thus e_1 and e_2 must be contained in the same elementary component and so, by Lemma 3.1.34, there must exist an M-covered path with endpoints u_1 and $z'' \in e_2$. □

As a last step towards Theorem 3.1.37 we show that for any two edges $e_1, e_2 \in M$, $e_1 \sim_M e_2$ is equivalent to e_1 and e_2 belonging to the same elementary component. Since $\{V(K) \mid K \in \mathcal{E}(G)\}$ is a partition of the vertex set of G, it also induces a partition of the edges in M and thus Theorem 3.1.37 follows from this last lemma.

Lemma 3.1.41 (A*)**.** Let G be a graph with a perfect matching M and let $e_1, e_2 \in M$, then $e_1 \sim_M e_2$ if and only if e_1 and e_2 belong to the same elementary component of G.

Proof. First assume G to be elementary and let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be any two distinct edges from M. For $e_1 = e_2$ the claim follows immediately. Hence we may assume $e_1 \neq e_2$. By Lemma 3.1.34 there is an M-covered path P_1 with endpoints u_1 and $z_1 \in e_2$ in G. Without loss of generality let $z_1 = u_2$. By Lemma 3.1.34 there also exists an M-covered path P_2 with endpoints v_1 and $z_2 \in e_2$. If $z_2 = v_2$ we have found our complementary pair, so suppose $z_2 = u_2$. Now Lemma 3.1.34 guarantees the existence of an M-covered path P_3 with endpoints v_2 and $z_3 \in e_1$ in G. By our previous assumptions v_1 cannot be an endpoint of P_3 and thus u_1

must be one. In this case, however, (P_2, P_3) is an e_1-e_2-M-complementary pair. Moreover, since e_1 and e_2 were chosen arbitrarily, we are able to find such a pair of paths between any distinct two edges of M and thus our claim follows.

For the converse direction, suppose that there are distinct elementary components $K^1, K^2 \in \mathcal{E}(G)$ with $e_i \in E(K^i) \cap M$ for both $i \in [1,2]$ such that $e_1 \sim_M e_2$. Let (P_1, P_2) be an e_1-e_2-M-complementary pair of paths that witnesses $e_1 \sim_M e_2$. With this we can find a pair of edges $e_1' = u_1'v_1', e_2' = u_2'v_2' \in M \cap E(P_1 \cup P_2)$ such that $e' = v_1'u_2' \in E(G)$, but e_1' and e_2' belong to different elementary components of G. Then $e' = v_1'u_2'$ is not admissible in G. In what follows, we show that the edge e' cannot exist.

With $e_1 \sim_M e_2$ and $e_1', e_2' \in E(P_1 \cup P_2) \cap M$ there must exist an e_1'-e_2'-M-complementary pair (P_1', P_2') of paths by definition of $\sim_M$. Suppose there is $i \in [1,2]$ such that P_i' has endpoints u_1' and v_2'. Then $P_1'v_2'u_2'v_1'u_1'$ is an M-conformal cycle containing the edge e' which therefore must be admissible. As this is a contradiction, we may assume P_1' to have the endpoints u_1' and u_2', while P_2' has endpoints v_1' and v_2'. Indeed, we may also assume for P_1' and P_2' to intersect since otherwise, they would form an M-conformal cycle that contains edges with endpoints in different elementary components. But since every M-conformal cycle must be contained in a single elementary component of G, this is impossible.

First suppose P_1' and P_2' have an alignment that is not a trap, then, by Lemma 3.1.39 there exists an internally M-conformal path $P \subseteq P_1' \cup P_2'$ with endpoints v_1' and u_2'. Since this path P would again form an M-conformal cycle with e_1', e_2', and e', we may assume that every alignment of P_1' and P_2' is a trap. Under this assumption, however, Lemma 3.1.40 implies the existence of an M-covered path P' linking an endpoint of e_1' to an endpoint of e_2'. Any M-covered path must also exist in $G - E_{\mathrm{no}}(G)$, and thus e_1' and e_2' cannot belong to different elementary components of G. Hence a non-admissible edge $e' \in E(P_1 \cup P_2)$ with endpoints in distinct elementary components cannot exist in G, and therefore e_1 and e_2 must belong to the same elementary component. $\square$

In our proofs towards Theorem 3.1.37, we used certain properties of M-alternating paths in graphs with perfect matchings. Indeed, whenever two

M-alternating paths meet such that their union contains an M-conformal cycle C, we may switch M along this cycle C to obtain a new perfect matching N together with two N-alternating paths that intersect slightly less then before. This relatively simple observation yields a general and very powerful tool for bipartite graphs in particular.

Lemma 3.1.42 (Bipartite Untangling Lemma, [McC01])**.** Let G be a bipartite graph with a perfect matching M and $a_1, a_2 \in V_1$ and $b_1 b_1 \in V_2$ four distinct vertices in G. Let further P_1 and P_2 be two internally M-conformal paths such that P_i has endpoints a_i and b_i. Then there exists a perfect matching $M' \in \mathcal{M}(G)$ together with two internally M'-conformal paths P_1' and P_2' such that:

i) P_i' has endpoints a_i and b_i for both $i \in \{1, 2\}$,
ii) $P_1' + P_2'$ is a subgraph of $P_1 + P_2$,
iii) $M \setminus E(P_1 + P_2) = M' \setminus E(P_1 + P_2)$, and
iv) either $P_1' \cap P_2'$ is an M'-conformal path or P_1' and P_2' are disjoint.

For non-bipartite graphs it is not always possible to completely untangle two internally M-conformal paths, as their intersections may be much more complicated. Indeed certain kinds of alignments and collisions are simply not possible in bipartite graphs, as those could create odd cycles. For a precise definition of the possible types of intersections in general graphs the interested reader should consult the PhD thesis of Norine [Nor05], where a precise description of these kinds of intersections can be found as Theorem 2.3.4 on page 25. While $\sim_M$ partitions the vertex set of any graph with a perfect matching into its elementary components and therefore can be seen as a canonical way to decompose a graph, it does not tell us how to construct matching covered graphs in the first place. To close this gap we start with a generation process for matching covered bipartite graphs which begins, in essence, from an even cycle and then iteratively proceeds by adding alternating paths to the already created portion of the graph.

Definition 3.1.43 (Single Ear)**.** Let G be a graph with a perfect matching. A *single ear* is a path P of odd length such that all internal vertices, if there are any, of P have degree two in G. Let M be a perfect matching of G. The path P is a *single M-ear* if it is internally M-conformal and a single ear.

Theorem 3.1.44 (Theorem 4.1.6 in [LP09])**.** Given any bipartite matching covered graph B, there exists a sequence $B_1 \subset B_2 \subset \cdots \subset B_t$ of matching covered conformal subgraphs of B, such that $B_1 = K_2$, $B_t = B$, and B_{i+1} is obtained from B_i by adding a single ear of B_{i+1} for all $i \in [1, t-1]$.

A sequence as in the above theorem is called a *bipartite ear-decomposition*. The bipartite part here is important, as such an ear-decomposition cannot exist for non-bipartite graphs. Indeed, even K_4 does not have such an ear decomposition. The reason for this is as follows: In each possible ear decomposition, B_2 is an even cycle, as it is obtained from adding a path of odd length to the endpoints of K_2. Hence B_2 is bipartite. Suppose there exists some $i \geq 3$ such that B_i is non-bipartite, but B_{i-1} is. Then the single ear, let it be P, we added to B_{i-1} must have created a cycle of odd length. The only way possible to do this is to attach both endpoints of P to vertices from the same colour class, say V_j of B_{i-1}. Say, for simplicity, P was just an edge (the more general case follows along the same reasoning), then P would be an edge with both endpoints in V_j, which is a maximal barrier of B_{i-1}. Hence by Corollary 3.1.10 the single edge of P is not admissible, thereby rendering B_i, although it is elementary, not matching covered. This means that there is some inherent hurdle which must be overcome to go from bipartite to non-bipartite in a way that preserves being matching covered.

If we were, however, to add two edges e_1 and e_2 at once to a bipartite matching covered graph B such that e_i has both endpoints in V_i for each $i \in [1, 2]$, the result would be a non-bipartite graph that is matching covered. This observation leads to the following definition:

Definition 3.1.45 (Double Ear)**.** Let G be a graph with a perfect matching. A *double ear* is a pair of disjoint paths P_1 and P_2 such that P_i is a single ear of G for both $i \in [1, 2]$.

An *ear* of G is either a single ear, or a double ear of G.

Definition 3.1.46 (Ear-Decomposition)**.** An *ear-decomposition* of a matching covered graph G is a sequence

$$G_1 \subset G_2 \subset \cdots \subset G_t$$

of matching covered conformal subgraphs of G such that $G_1 = K_2$, $G_t = G$, and G_i is the union of G_{i-1} and an ear (single or double) of G_{i+1}.

With the introduction of double ears, the barrier between bipartite and non-bipartite could finally be breached.

Theorem 3.1.47 (Two-Ear Theorem, [LP09])**.** Every matching covered graph has an ear-decomposition.

A subtheme of this chapter is the search for canonical decompositions. While the Two-Ear Theorem provides a nice way to generate all matching covered graphs, a result by Carvalho and Lucchesi [CL96] implies that every matching covered graph G has at least $\Delta(G)!$ many ear-decompositions. Let G be some non-bipartite matching covered graph and $G_1 \subset G_2 \subset \cdots \subset G_t$ be an ear-decomposition of G that uses a minimum amount of double ears. Clearly there must exist some $i \in [1, t-1]$ such that G_{i+1} is obtained from G_i by adding a double ear, since otherwise G would be bipartite. Let $i \in [1, t-1]$ be chosen to be minimal with respect to this property. Then G_j is a bipartite graph for all $j \in [1, i]$. Moreover, if G_{i+1} is bipartite, then G_{i+1} could also be obtained from G_i by adding two single ears consecutively, therefore contradicting our choice for the ear-decomposition. Hence G_j is non-bipartite for all $j \in [i+1, t]$. To describe the next step we need to slightly generalise our idea of subgraphs and introduce a way to subdivide edges without rendering a graph G void of perfect matchings.

Definition 3.1.48 (Bisubdivision)**.** Let G be a matching covered graph and $e \in E(G)$. We call the operation that replaces e by a path of length three, i.e. subdivides e with two vertices, *bisubdividing* the edge e. A matching covered graph H is a *bisubdivision* of G if there exists a sequence

$$H_1,\ H_2, \dots,\ H_t$$

Such that $H_1 = G$, $H_t = H$, and H_i is obtained from H_{i-1} by bisubdividing an edge for all $i \in [2, t]$.

Let us further assume that $G_1 \subset G_2 \subset \cdots \subset G_t$ is an ear-decomposition of G that minimises the number of double ear additions and also adds a double ear as soon as possible, that is it minimises the value of i from above. By definition we have $G_1 = K_2$ and G_2 is a cycle of even length. Suppose G_3 is obtained from G_2 by adding a double ear. The discussion above shows that G_3 is non-bipartite and, moreover, one can check that G_3 must now be a bisubdivision of K_4. In case G_4 is the first graph in the sequence obtained by adding a double ear, then G_4 can be seen to be a bisubdivision of the *triangular prism* $\overline{C_6}$.

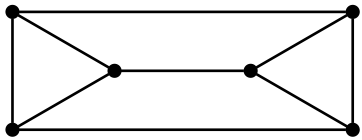


Figure 3.7.: The triangular prism $\overline{C_6}$.

Let us call an ear decomposition $G_1 \subset G_2 \subset \cdots \subset G_t$ *canonical* if either G is bipartite, G_3 is a bisubdivision of K_4, or G_4 is a bisubdivision of $\overline{C_6}$. Please note that the term canonical here is a bit misleading since still every matching covered graph might have many canonical ear-decompositions. However, canonical ear decompositions have proven to be a powerful tool, especially in the study of non-bipartite matching covered graphs. Indeed, a fundamental theorem by Lovász [Lov83] proves that every matching covered graph has such a decomposition.

Theorem 3.1.49 (Canonical Ear-Decomposition Theorem, [Lov83]). Every matching covered graph G has a canonical ear-decomposition.

The Tight Cut Decomposition

While ear-decompositions already yield quite some descriptive power, the story does not end here. The Canonical Ear-Decomposition Theorem provides us with a way to construct every matching covered graph by iteratively adding paths, but, apart from the differentiation between bipartite and non-bipartite, it does not provide us with further restrictions on the structure of these graphs. In particular, ear-decompositions have two drawbacks, the first one being that the property of being bicritical does not seem to be tied directly to ear-decompositions. The second one is that we might end up with bisubdivisions of matching covered graphs. While this is not a problem immediately, it is easy to see that identifying two vertices with a common neighbour of degree two within a matching covered graph preserves the property of being matching covered. Hence whenever we have a bisubdivided matching covered graph G one can obtain a smaller such graph by essentially reversing the bisubdivisions without losing too much information on the structural properties of G. This second part is further explored in detail in Section 3.1.4. However, there is a greater unified approach to decomposing matching covered graphs which, in a way, can be seen as a generalisation of the Frame Construction Theorem.

Definition 3.1.50 (Edge Cut)**.** Let G be a graph and $X \subseteq V(G)$. The set $\partial_G(X) = \partial(X) = \{xy \in E(G) \mid x \in X,\ y \in \overline{X}\}$ is called the *edge cut*, or just *cut*, around the set X. The sets X and $\overline{X}$ are known as the *shores* of the cut $\partial_G(X)$. A set $F \subseteq E(G)$ is called a *cut* in G if there exists a set $X \subseteq V(G)$ such that $F = \partial_G(X)$

Definition 3.1.51 (Generalised Tight Cut)**.** Let G be a graph with a perfect matching, $k \in \mathbb{N}$ a positive integer, and $X \subseteq V(G)$. The edge cut $\partial_G(X)$ is *k-tight* if $|\partial_G(X) \cap M| = k$ for all $M \in \mathcal{M}(G)$. If $\partial_G(X)$ is a k-tight cut, we say that X *induces* a k-tight cut. A k-tight cut is *trivial* if $|X| = k$ or $|\overline{X}| = k$. A cut $\partial_G(X)$ is called a *tight cut* if it is 1-tight, and a tight cut is *trivial* if it is a trivial 1-tight cut.

In this section we are exclusively interested in tight cuts, but eventually we will see that the concept of tight cuts and the decomposition of matching covered graphs they provide, can be seen as a highly specialised version of a much more general concept. The whole generality of k-tight cuts is explored in Chapter 6. As a first observation, note that any k-tight cut $\partial_G(X)$ must satisfy that $|X|$, $|\overline{X}|$ and k all share the same parity. This can be seen by considering a perfect matching M of our graph G. When deleting $V(\partial_G(X) \cap M)$ from G, $M \setminus \partial_G(X)$ must still be a perfect matching of $G - V(\partial_G(X) \cap M)$. Indeed, every edge in $M \setminus \partial_G(X)$ has both endpoints either in $X \setminus V(\partial_G(X) \cap M)$ or in $\overline{X} \setminus V(\partial_G(X) \cap M)$. Hence both of these sets are of even cardinality. Moreover, since we have deleted exactly k vertices from every shore, the parity of all three quantities must be the same.

Tight cuts were first studied for their helpful properties regarding the *perfect matching polytope*[9] of a graph G. These polytopes can be decomposed along tight cuts into smaller polytopes with a richer structure. Using tight

[9] The polytope obtained from the convex combinations of the 0-1-vectors of $\mathbb{R}^{|E(G)|}$ that encode the perfect matchings of G.

cuts enabled Lovász to find a formula for the dimension of the perfect matching polytope of any given graph G [Lov87]. This decomposition into smaller polytopes can also be expressed purely in a graph theoretic sense.

Definition 3.1.52 (Tight Cut Contraction)**.** Let G be a matching covered graph, and $X \subseteq V(G)$ such that X induces a tight cut. We call the graphs $G_X \coloneqq G/X$ and $G_{\overline{X}}/\overline{X} \coloneqq G/\overline{X}$ the *tight cut contractions* of G at X. In case we want to specify the name of the *contraction vertex*, i.e. the vertex X (or $\overline{X}$ respectively) is contracted into, we write $G/(X \to v)$. If $F \subseteq E(G)$ is a tight cut with shores X and Y, then G_X and G_Y are referred to as the two F*-contractions* of G.

Lemma 3.1.53 (folklore)**.** Let G be a matching covered graph and $X \subseteq V(G)$ a set that induces a tight cut in G. Then $G_X \coloneqq G/(X \to v)$ is matching covered.

Proof. To see this, let $e \in E(G_X)$ be any edge. Then one of two cases can occur: either e is incident with v or it is not. In the later case e is also an edge of G and thus, since G is matching covered, there exists a perfect matching $M_e \in \mathcal{M}(G)$ with $e \in M_e$. Let $e' \in M_e \cap \partial_G(X)$ be the unique edge of M_e in the tight cut induced by X and let u be the endpoint of e' that belongs to $\overline{X}$. Then $u \in V(G_X)$ and $uv \in E(G_X)$. Moreover, $E(G_X) \cap M_e$ is a matching of G_X that matches all vertices except u and v. Hence $(E(G_X) \cap M_e) \cup \{uv\}$ is a perfect matching of G_X. Similarly, suppose e is incident with v and let u be its other endpoint. Then in G there exists $e' \in \partial_G(X)$ incident with u and a perfect matching $M_{e'}$ of G with $e' \in M_{e'}$. Note that every vertex of $X \setminus \{u\}$ is matched by an edge of $M_{e'}$ with both endpoints in X. Hence $(M_{e'} \cap E(G_X)) \cup \{uv\}$ is a perfect matching of G that contains e. Also note that G_X must also be connected and thus G_X is matching covered. □

Definition 3.1.54 (Residual Matching)**.** Let G be a graph with a perfect matching M and $X \subseteq V(G)$ be a set of vertices that induces a tight cut. Let $G_X \coloneqq G/(X \to v_X)$, and let u be the unique vertex of $\overline{X}$ which is incident with the edge of M in $\partial_G(X)$. We call the perfect matching $M|_{G_X} \coloneqq (M \cap E(G[\overline{X}])) \cup \{uv_X\}$ the *residual* of M in G_X.

A second and similarly fundamental property of tight cuts is their submodularity. Two cuts $\partial_G(X)$ and $\partial_G(Y)$, where $X, Y \subseteq V(G)$ and G is matching covered, are said to be *laminar* if $X \subseteq Y$, $X \subseteq \overline{Y}$, $\overline{X} \subseteq Y$, or $\overline{X} \subseteq \overline{Y}$. Otherwise $\partial_G(X)$ and $\partial_G(Y)$ are said to *cross*.

An equivalent way to define the notions of crossing and being laminar is via intersections of sets rather than set containment. The *corners* or *quadrants* of the two cuts $\partial_G(X)$ and $\partial_G(Y)$ are the sets

$$X \cap Y,\ X \cap \overline{Y},\ \overline{X} \cap Y, \text{ and } \overline{X} \cap \overline{Y}.$$

The two cuts can be seen to cross if and only if all four quadrants are non-empty. The corner pairs $X \cap Y$ and $\overline{X} \cap \overline{Y}$ as well as $X \cap \overline{Y}$ and $\overline{X} \cap Y$ are said to be *opposing*.

Lemma 3.1.55 ([Lov87])**.** Let G be a matching covered graph and $X, Y \subseteq V(G)$ sets that induce tight cuts in G. Then, if $|X \cap Y|$ is odd, $\partial_G(X \cap Y)$ and $\partial_G(\overline{X} \cap \overline{Y})$ are tight cuts, otherwise $\partial_G(X \cap \overline{Y})$ and $\partial_G(\overline{X} \cap Y)$ are tight. Moreover, between the two opposing quadrants of $\partial_G(X)$ and $\partial_G(Y)$ which do not induce a tight cut there is no edge in G.

In bipartite graphs k-tight cuts have a specific structure to them which allows for a generalisation of Lemma 3.1.55.

Definition 3.1.56 (Minority and Majority)**.** Let B be a bipartite graph, and $X \subseteq V(G)$. If $|X \cap V_1| = |X \cap V_2|$ we say that X is *balanced*, otherwise it is *unbalanced*. Suppose X is unbalanced, then there are $i, j \in [1, 2]$, and $k \in \mathbb{N}$ such that $|X \cap V_i| = |X \cap V_j| + k$. In this case we call $X \cap V_i$ the *majority* of X, denoted by $\mathrm{Maj}(X)$, and $X \cap V_j$ is the *minority*, denoted by $\mathrm{Min}(X)$. We say that k is the *imbalance* of X, and in general we set

$$\mathrm{imbalance}(X) := \begin{cases} 0, & \text{if } X \text{ is balanced, or} \\ k, & \text{if the imbalance of } X \text{ is } k. \end{cases}$$

Lemma 3.1.57 (X*)**.** Let B be a bipartite matching covered graph with a perfect matching, $k \in \mathbb{N}$ a positive integer, and $X \subseteq V(B)$ a set of vertices that induce a k-tight cut. Then there exist $k_1, k_2 \in \mathbb{N}$ such that for every perfect matching $M \in \mathcal{M}(B)$ there are exactly k_i vertices of $X \cap V_i$ which are matched by edges of $\partial_B(X) \cap M$ for both $i \in [1, 2]$.

Proof. Let M be some perfect matching of B and for both $i \in [1, 2]$, let k_i be the number of vertices in $X \cap V_i$ which are matched by edges of $\partial_B(X) \cap V_i$. Then every other edge of M either has both or no endpoint in X. Hence there is a number $n \in \mathbb{N}$ of edges of M with both endpoints in X such that $|X| = k_1 + k_2 + 2n$. Now suppose, towards a contradiction, there exist $k_1', k_2' \in \mathbb{N}$ together with a perfect matching $M' \in \mathcal{M}(B)$ such that for each $i \in [1, 2]$, k_i' is the number of vertices of $X \cap V_i$ that are matched by edges of $\partial_B(X)$, and $k_1' \neq k_1$, which also implies $k_2' \neq k_2$. By the same arguments as before, there exists a number n' such that $|X| = k_1' + k_2' + 2n'$. Indeed, we have $|X \cap V_i| = k_i + n = k_i' + n'$ for both $i \in [1, 2]$. Without loss of generality, let us assume $k_1' > k_1$. Then $n' < n$ since $k_1 + n = k_1' + n'$. But since $\partial_B(X)$ is k-tight, we have $k_1 + k_2 = k = k_1' + k_2'$. Hence

$$|X| = k + 2n > k + 2n' = |X|,$$

which is impossible and thus our claim must hold. □

The following can be seen as a generalisation of an observation first made by Lovász (see the proof of Lemma 1.4 in [Lov87]).

Lemma 3.1.58 (X*)**.** Let B be a bipartite matching covered graph, $k \in \mathbb{N}$ a positive integer, and $X \subseteq V(G)$ a set of imbalance k. Then $\partial_B(X)$ is k-tight if and only if $\mathrm{N}_B(\mathrm{Min}(X)) \subseteq \mathrm{Maj}(X)$.

Proof. Let us first assume X induces a k-tight cut and suppose there is some edge $e \in \partial_B(X)$ such that e has an endpoint in $\mathrm{Min}(X)$. As G is matching covered there exists $M_e \in \mathcal{M}(B)$ such that $e \in M_e$. Now there are $|\mathrm{Min}(X)| - 1$ many vertices of the minority left which can be matched by M_e to vertices of the majority of X. Hence at least $k + 1$ vertices of $\mathrm{Maj}(X)$ cannot be matched by M_e with vertices inside X. This however means that $|\partial_B(X) \cap M_e| \geq k + 2$, contradicting the assumption that X induces a k-tight cut.

For the reverse direction, let us assume $\mathrm{N}_B(\mathrm{Min}(X)) \subseteq \mathrm{Maj}(X)$. Then for every $M \in \mathcal{M}(B)$, every vertex of $\mathrm{Min}(X)$ must be matched with a vertex of $\mathrm{Maj}(X)$, therefore leaving exactly $\mathrm{imbalance}(X) = k$ vertices of $\mathrm{Maj}(X)$ which must be matched via edges of $\partial_B(X)$. Therefore $|\partial_B(X) \cap M| = k$ for all $M \in \mathcal{M}(B)$. □

To see that the above lemma does indeed yield a characterisation of tight cuts in bipartite graphs, simply observe that any set X that induces a tight cut must be of odd cardinality, hence it necessarily must be imbalanced. We call a k-tight cut in a bipartite graph whose shores have an imbalance of order k a *proper k-tight cut*. If $\partial_G(X)$ is a proper k-tight cut for some k we say that $\partial_G(X)$ is a *generalised tight cut*.

Lemma 3.1.59 (X*). Let B be a bipartite graph with a perfect matching, $k_1, k_2 \in \mathbb{N}$ two positive integers, and $X_1, X_2 \subseteq V(G)$ two sets such that, for each $i \in [1, 2]$, X_i induces a proper k_i-tight cut, and for some $j \in [1, 2]$ we have $\mathrm{Maj}(X_1) \cup \mathrm{Maj}(X_2) \subseteq V_j$. Then there exist $k_1', k_2' \in \mathbb{N}$ such that $X_1 \cap X_2$ induces a proper k_1'-tight cut, $\overline{X_1} \cap \overline{X_2}$ induces a proper k_2'-tight cut, and $k_1 + k_2 = k_1' + k_2'$.

Proof. First consider $X_1 \cap X_2$. We claim that any edge in $\partial_B(X_1 \cap X_2)$ must be incident with a vertex of $\mathrm{Maj}(X_1) \cap \mathrm{Maj}(X_2)$. Suppose this is not the case, and let $e \in \partial_B(X_1 \cap X_2)$ be an edge that witnesses this fact. Then e is incident with a vertex of $\mathrm{Min}(X_1) \cap \mathrm{Min}(X_2)$. Let $i \in [1, 2]$ be chosen such that the other endpoint of e lies in $\overline{X_i}$. Such an i must exist as otherwise both endpoints of e would lie in $X_1 \cap X_2$. However, with this the existence of e contradicts Lemma 3.1.58. Hence our claim follows. Now let $M \in \mathcal{M}(B)$ be any perfect matching of B and let $k_1' := |\partial_B(X_1 \cap X_2) \cap M|$. Then, since all edges of $\partial_B(X_1 \cap X_2) \cap M$ must be incident with vertices of $\mathrm{Maj}(X_1) \cap \mathrm{Maj}(X_2)$, it follows that $X_1 \cap X_2$ has an imbalance of order k_1'. This together with the first claim and Lemma 3.1.58 implies that $X_1 \cap X_2$ induces a proper k_1'-tight cut.

The case for $\overline{X_1} \cap \overline{X_2}$ can be made with similar arguments. Here we first observe that every edge in $\partial_B(\overline{X_1} \cap \overline{X_2})$ must be incident to a vertex from $\mathrm{Maj}(X_1) \cup \mathrm{Maj}(X_2)$, and thus all of these edges attach to vertices from one colour class in $\overline{X_1} \cap \overline{X_2}$. Hence if we consider the perfect matching M from before and let k_2' be the number of edges M has in $\partial_B(\overline{X_1} \cap \overline{X_2})$, we observe that $\overline{X_1} \cap \overline{X_2}$ induces a proper k_2'-tight cut.

At last, note that any edge from $M \cap (\partial_B(X_1 \cap X_2) \cup \partial_B(\overline{X_1} \cap \overline{X_2}))$ must also appear in one of $\partial_B(X_1)$ or $\partial_B(X_2)$. Moreover, an edge appears in both $\partial_B(X_1)$ and $\partial_B(X_2)$ if and only if it also appears in $\partial_B(X_1 \cap X_2)$ and $\partial_B(\overline{X_1} \cap \overline{X_2})$ and thus $k_1 + k_2 = k_1' + k_2'$. ☐

The possibility of generalising the notion of tight cuts to the notion of proper k-tight cuts while preserving the submodularity is a crucial tool for matching theory of bipartite graphs, which we revisit in Chapter 6. In

Section 3.4 we show that such a generalisation cannot exist in non-bipartite graphs.

Let us come back to matching covered graphs with tight cuts, more precisely, let us assume G to be a matching covered graph with a non-trivial tight cut induced by the set $X \subseteq V(G)$. Consider the graph $G_X \coloneqq G/(X \to v)$ and let $Y \subseteq G_X$ be a non-trivial tight cut in G_X such that $v \in Y$. Observe that $(Y \setminus \{v\}) \cup X$ must then also induce a non-trivial tight cut in G, and, moreover, this tight cut is laminar with the one induced by X. With this observation, it becomes apparent that we can turn the construction of tight cut contractions into a procedure.

Definition 3.1.60 (Tight Cut Decomposition)**.** Let G be a matching covered graph. We iteratively construct a tree T as follows: First, let T consist of only one vertex, say t_0 and let us associate $G_0 = G$ with t_0. Then select a set $X \subseteq V(G)$ that induces a non-trivial tight cut in G and let G_X, $G_{\overline{X}}$ be its two tight cut contractions. We introduce $t_{1,1}$ and $t_{1,2}$ together with the edges $t_0t_{1,i}$, $i \in [1, 2]$ to our tree T and associate[10] with each of the two tight cut contractions exactly one of the two new vertices.

Suppose now that we have constructed a binary tree T with root t_0 such that each pair of successors of an inner vertex t of T are associated with the two tight cut contractions obtained from the graph associated with t by using a single non-trivial tight cut. If there exists a leaf ℓ whose associated matching covered graph still has a non-trivial tight cut, we construct the two tight cut contractions from it, add two new successors to ℓ and associate each of the two newly obtained tight cut contractions with exactly one of the two new successors of ℓ.

At some point, this procedure stops, and we obtain a tree T as above such that each leaf of T is associated with a matching covered graph without any non-trivial tight cuts, Let $\mathcal{G}$ be the family of all graphs associated with the leaves of T. Here we explicitly allow for $\mathcal{G}$ to be a multiset. Then $\mathcal{G}$ is a *tight cut decomposition* of G. We call the tree T a *tight cut decomposition tree* of G. Moreover, there is a family $\mathcal{T}$ of pairwise laminar non-trivial tight cuts of G such that each member of $\mathcal{T}$ can be associated with an inner vertex of T.

[10] Formally this is a bijection between the vertices of T and a family of graphs.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two multisets of graphs. We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *isomorphic* if there exists a bijection $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ such that for every $F \in \mathcal{F}_1$, F and $f(F)$ are isomorphic.

Let $\mathcal{T}$ be the family of pairwise laminar non-trivial tight cuts of G as above. Then $\mathcal{T}$ must be a maximal such family since otherwise we could find some leaf of the tight cut decomposition tree T whose associated graph still has a non-trivial tight cut. A fundamental result by Lovász shows that the tight cut decomposition $\mathcal{G}$ itself is independent of the choice of a maximal family of pairwise laminar and non-trivial tight cuts in G. Hence the tight cut decomposition is indeed canonical.

Theorem 3.1.61 ([Lov87])**.** Let G be a matching covered graph, then any two tight cut decompositions of G are isomorphic.

Let G be a matching covered graph, $S \in \mathcal{B}(G)$ a maximal barrier of G as well as K be an odd component of $G - S$. Then every perfect matching of G contains exactly one edge that has an endpoint in S and the other one in K. Hence $V(K)$ induces a non-trivial tight cut. Note that the two tight cut contractions are graphs that occur in the frame construction. This means that the tight cut decomposition is in fact a refinement, since it is canonical and also decomposes bipartite graphs, of the decomposition associated with the frame construction.

A natural next step is to further investigate the structure of those graphs whose tight cut decomposition is trivial, i.e. matching covered graphs where every tight cut is trivial.

Definition 3.1.62 (Brace and Brick)**.** Let G be a matching covered graph without a non-trivial tight cut. If G is bipartite it is called a *brace*, otherwise it is called a *brick*.

3.1.2. Braces and Bipartite Graphs

By Theorem 3.1.61 every matching covered graph can be decomposed into a unique[11] list of bricks and braces. Indeed, from Lemma 3.1.58 it follows that any tight cut contraction of a bipartite graph must itself be bipartite. To see this observe that the contraction vertex v can be coloured by the colour of the majority of the set from whose contraction v was

[11] Up to isomorphisms.

obtained. So while every non-bipartite matching covered graph that is not bicritical must necessarily have at least one brace, no bipartite matching covered graph has a brick. For a better understanding, especially of bipartite graphs with perfect matchings, it is therefore desirable to further investigate the structure of braces. For this, the concept of matching extendibility[12] is crucial.

Definition 3.1.63. Let G be a graph with a perfect matching and $F \subseteq E(G)$ a matching. We say that F is *extendible* if there exists $M \in \mathcal{M}(G)$ such that $F \subseteq M$.

For any positive integer $k \in \mathbb{N}$, G is said to be *k-extendible* if it is connected, has at least $2k+2$ vertices, and every matching of size k in G is extendible.

In particular, this means that any connected graph with a perfect matching may be regarded as 0-extendible.

Before we start with a brief discussion of the properties of k-extendible graphs in general, note that any 1-extendible graph is matching covered[13]. The notion of 2-extendibility is especially relevant for this subsection as illustrated by the following theorems.

Theorem 3.1.64 ([Plu80])**.** Any 2-extendible graph is either a brace or a brick.

For bipartite graphs the reverse of Theorem 3.1.64 is true as well.

Theorem 3.1.65 ([LP09])**.** A bipartite graph B is a brace if and only if it is either isomorphic to C_4, or it is 2-extendible.

The case of C_4 is a bit special. Clearly, every matching of size two in C_4 can be extended to, and is in fact, a perfect matching. However, it has fewer vertices and lower connectivity than any other brace and therefore it is usually excluded. In some cases, C_4 is not even regarded as a brace at all. In this work, we count it as a brace, but not as 2-extendible. This is similar to K_2, which is commonly regarded as a block[14], but not as

[12] In the literature this is sometimes referred to as 'extendability'.

[13] Sometimes 1-extendibility and matching covered are used as synonyms in the literature.

[14] A block is a connected graph without a cut vertex.

2-connected, while every block besides K_2 is necessarily 2-connected. The comparison to k-connectivity is probably the best way to understand the tight cut decomposition and the structure of braces. To further illustrate this, we first present some general statements on extendibility and then concentrate more on properties unique to bipartite graphs.

Theorem 3.1.66 ([Plu80])**.** Let $k \in \mathbb{N}$ be a positive integer. Then every k-extendible graph is also $(k-1)$-extendible.

Theorem 3.1.67 ([Plu80])**.** Let $k \in \mathbb{N}$ be a positive integer. Then every k-extendible graph is $(k+1)$-connected.

To fully stress the comparison to k-connectivity we started earlier, let us generalise the concept of matching connectivity.

Definition 3.1.68 (Matching k-Connectivity)**.** Let $k \in \mathbb{N}$ be a positive integer. A graph G is said to be *matching k-connected* if it has a perfect matching, $|V(G)| \geq 2k+2$, and for every conformal set $S \subseteq V(G)$, which is M-conformal for some $M \in \mathcal{M}(G)$, with $|S| \leq 2k-2$, $G-S$ is matching connected.

The notion of matching connectivity itself is indistinguishable from the notion of a graph being elementary, so one could call matching k-connectivity 'k-elementarity' instead. However, there is a direct link to how connectivity in (directed) graphs works which matching k-connectivity resembles in a nice way. Hence we think calling the concept 'connectivity' is justified.

The following theorem is a collection of several different characterisations of k-extendibility in bipartite graphs.

Theorem 3.1.69 ([Plu86, AHLS03])**.** Let B be a bipartite graph and $k \in \mathbb{N}$ a positive integer. The following statements are equivalent.

i) B is k-extendible.
ii) $|V_1| = |V_2|$, and for all non-empty $S \subseteq V_1$, $|\mathrm{N}_B(S)| \geq |S| + k$.
iii) For all sets $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$ with $|S_1| = |S_2| \leq k$ the graph $B - S_1 - S_2$ has a perfect matching.
iv) B is matching k-connected.
v) There is a perfect matching $M \in \mathcal{M}(B)$ such that for every $v_1 \in V_1$, every $v_2 \in V_2$ there are k pairwise internally disjoint internally M-conformal paths with endpoints v_1 and v_2.

vi) For every perfect matching $M \in \mathcal{M}(B)$, every $v_1 \in V_1$, every $v_2 \in V_2$ there are k pairwise internally disjoint internally M-conformal paths with endpoints v_1 and v_2.

While ii) can be seen as a generalisation of Hall's Theorem, iii) is, in a way, an even stronger version of matching k-connectivity. The statements iv) and v) can be seen as matching theoretic versions of Menger's Theorem. We present both versions here to emphasise that having this property for one perfect matching already implies the property for all of them. This is a phenomenon we will encounter again in later chapters, and thankfully so, as this allows to fix a single perfect matching for a proof instead of having to consider many different matchings individually.

So, in some sense, one could say that braces are the matching 2-connected building blocks of bipartite matching covered graphs, similarly to how blocks make up any connected graph. Indeed, the decomposition of any connected graph into its blocks is canonical in the sense that the list of blocks is unique, and the corresponding decomposition has a tree structure just like the tight cut decomposition has for matching covered graphs.

So far we know how to construct any graph from just three families of graphs, one of them being those with a perfect matching. We then discovered how to obtain graphs with perfect matchings from their elementary components and how to obtain those from the frame construction or via an ear-decomposition. Using the tight cut decomposition, we can decompose bipartite elementary graphs further into their braces. A natural question would be whether there is a way to generate all braces. To do this, let us revisit a classical result by Tutte. Let v be some vertex of a graph G, and let $\{Y_1, Y_2\}$ be a partition of $\mathrm{N}_G(v)$ into two non-empty sets. Let

$$G' := (G - v) + x_1x_2 + \{x_iy_i \mid y_i \in Y_i \text{ where } i \in [1,2]\},$$

such that $x_1, x_2 \notin V(G)$. We say that G' is obtained from G by *simple expansion*. If a graph H is obtained from a graph G by simple expansion and $\delta(H) \geq 3$, then H is obtained from G by 3-*expansion*. For $k \geq 3$ the *wheel* $\mathcal{W}_k$ is the graph obtained from a cycle of length k by introducing a single new vertex, and joining this vertex to all vertices of the cycle.

Theorem 3.1.70 ([Tut61]). A simple graph is 3-connected if and only if it can be obtained from a wheel by a sequence of edge additions and 3-expansions.

This theorem is especially beautiful, as it shows that any 3-connected graph can be traced back to a single infinite family of graphs from which all others originate. We close this subsection with a surprisingly similar result for braces.

Definition 3.1.71 (Bipartite Expansion)**.** Let B be a bipartite graph with $\delta(B) \geq 2$, let $x \in V(B)$ be any vertex, and $\{Y_1, Y_2\}$ be a bipartition of $\mathrm{N}_B(x)$ into two non-empty sets. Let

$$B' \coloneqq (B - x) + x_1vx_2 + \{x_iy_i \mid y_i \in Y_i \text{ where } i \in [1,2]\}\,,$$

where $x_1, v, x_2 \notin V(B)$. We say that B' is obtained from B by *bipartite expansion of x (to x_1vx_2).*

Definition 3.1.72 (Bipartite Augmentation)**.** Let B be a bipartite graph with $\delta(B) \geq 2$.

i) Let $v_1 \in V_1$, $v_2 \in V_2$ be two non-adjacent vertices from different colour classes. We say that the graph $B' \coloneqq B + v_1v_2$ is obtained from B by *type I bipartite augmentation.*
ii) Let $i \in [1,2]$, $x, u \in V_i$ be two distinct vertices and B' be obtained from B by bipartite expansion of x to x_1vx_2. If $\delta(B'+uv) \geq 3$ we say that $B' + uv$ is obtained from B by *type II bipartite augmentation.*
iii) Let $i, j \in [1,2]$ be distinct, $x \in V_i$, $y \in V_j$ be non-adjacent, and B' be obtained from B by bipartite expansion of x to x_1ux_2 and y to y_1vy_2. If $\delta(B' + uv) \geq 3$ we say that $B' + uv$ is obtained from B by *type III bipartite augmentation.*
iv) Let $i, j \in [1,2]$ be distinct, $x \in V_i$, $y \in V_j$ be adjacent, and B' be obtained from B by bipartite expansion of x to x_1ux_2 and y to y_1vy_2. If $\delta(B' + uv) \geq 3$ we say that $B' + uv$ is obtained from B by *type IV bipartite augmentation.*

If B' is obtained from B by type i bipartite augmentation for $i \in \{I, II, III, IV\}$, it is obtained from B by *bipartite augmentation.*

Clearly, bipartite augmentations are more complex than edge addition and 3-expansion. In some sense, this increase in complexity is to be expected and will be encountered wherever we aim to generalise some concept from (undirected) structural graph theory to matching theory in the following chapters. Similar to the more complicated generation rules, our graph families from which we start the generation process are more complex as well.

Definition 3.1.73 (McCuaig Braces)**.** An *odd möbius ladder* of *order* $k \geq 1$ is the graph $\mathcal{M}_{4k+2}$ obtained from the cycle

$$(v_0^1, v_0^2, v_1^1, v_1^2, v_2^1, \dots, v_{2k-1}^2, v_{2k}^1, v_{2k}^2, v_0^1)$$

by adding the edges $v_i^1 v_{k+i \pmod{2k+1}}^2$, for all $i \in [0, 2k]$.

An *even prism* of order $k \geq 1$ is the graph $\mathcal{P}_{4(k+1)}$ obtained from the cycles

$$(u_0^1, u_0^2, u_1^1, \dots, u_{2k}^2, u_{2k+1}^1, u_{2k+1}^2, u_0^1), \text{ and}$$
$$(v_0^2, v_0^1, v_1^2, v_1^1, v_2^2, \dots, v_{2k}^1, v_{2k+1}^2, v_{2k+1}^1, v_0^2)$$

by adding the edges $u_i^1 v_i^2$ and $u_i^2 v_i^1$ for all $i \in [0, 2k+1]$.

A *biwheel* of order $k \geq 1$ is the graph $\mathcal{B}_{2k+6}$ obtained from the cycle

$$(v_0^1, v_0^2, v_1^1, \dots, v_{2k+4}^2, v_{2k+3}^1, v_{2k+3}^2, v_0^1)$$

and the vertices u^1, u^2 by adding the edges $u^1 v_i^2$, and $u^2 v_i^1$ for all $i \in [0, 2k+3]$.

Every graph that is isomorphic to an odd möbius ladder, an even prism, or a biwheel is called a *McCuaig brace*. See Figure 3.8 for an illustration.

Theorem 3.1.74 (Brace Generation Theorem, [McC01])**.** A graph G is a brace if and only if it can be obtained from a McCuaig brace by a sequence of bipartite augmentations.

3.1.3. Bricks and Non-Bipartite Graphs

Theorem 3.1.74 and Theorem 3.1.65 give us precise descriptions of bipartite graphs without non-trivial tight cuts. Moreover, with Theorem 3.1.69 we have a pretty powerful description of the inner workings of braces, especially in terms of internally M-conformal paths. However, in the presence of bicriticality, a property no bipartite graph can have[15], extendibility becomes a property that is slightly too strong. To characterise bricks, i.e. non-bipartite graphs without non-trivial tight cuts, one needs some additional tools, as the reverse of Theorem 3.1.64 is not true in general. As an example consider the triangular prism $\overline{C_6}$. This graph does not contain a non-trivial tight cut, but it is also not 2-extendible. See Figure 3.9 for a matching of size two in $\overline{C_6}$ which is not extendible.

[15]Technically K_2 is an exception, but usually we assume that our graphs have more than two vertices.

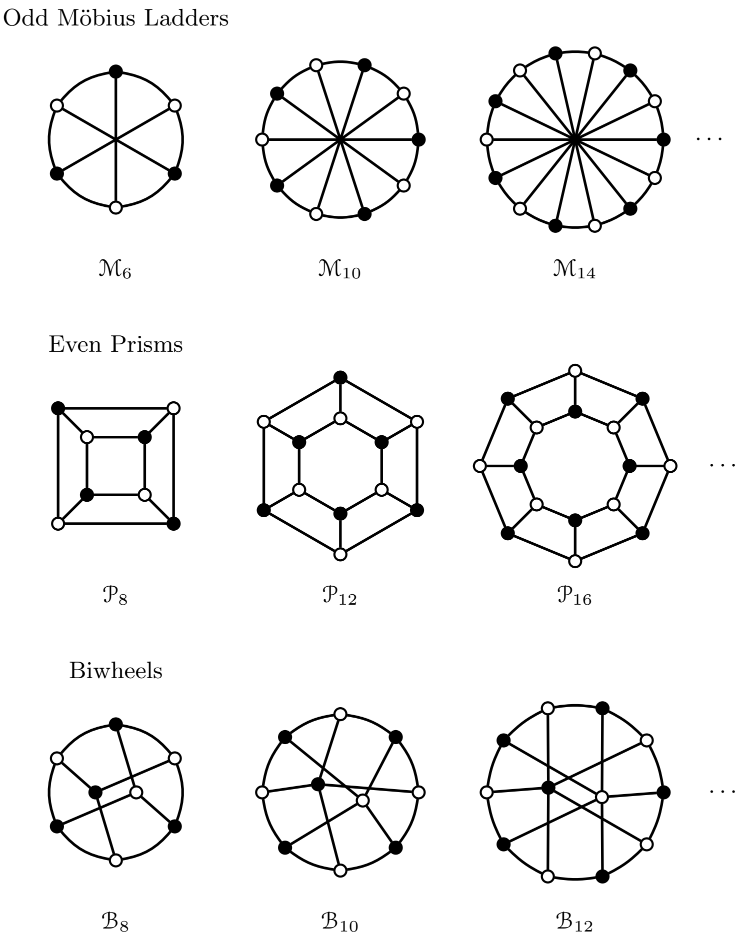


Figure 3.8.: The three infinite families of McCuaig braces.

While bipartite matching covered graphs are one of the two types of building blocks for general elementary graphs in the frame construction, bicritical graphs form the other type. Still, bicritical graphs may contain non-trivial tight cuts, but as it turns out, they are almost free from these cuts. In their studies of the dimension of the perfect matching polytope Edmonds, Pulleyblank, and Lovász proved the following fundamental statement, which they state as the most challenging part of their work.

Theorem 3.1.75 (Tight Cut Lemma, [EPL82])**.** A graph G is a brick if and only if it is 3-connected and bicritical.

This means we only have to add the additional requirement of G being 3-connected in order to get rid of all non-trivial tight cuts. Let G be some matching covered graph and $S = \{s_1, s_2\} \subseteq V(G)$ a set of size two which is not a barrier such that $G - S$ is not connected. Please note that we will always find such a set in a bicritical graph that is not a brick. Let K be a component of $G - S$. Since G is matching covered and S is not a barrier, $G + s_1s_2$ would also be matching covered and thus $G - S$ has a perfect matching. This implies that K has even size, hence $V(K) \cup \{s_1\}$ is odd and, moreover, any perfect matching of G can have at most two edges in $\partial_G(V(K) \cup \{s_1\})$. However, with $V(K) \cup \{s_1\}$ being odd, no perfect matching can have an even number of edges in the cut, hence $\partial_G(V(K) \cup \{s_1\})$ is a non-trivial tight cut. Such a cut is called a 2-*separation cut*. Together with the cuts that arise from maximal barriers that contain at least two vertices, which are called *barrier cuts*, 2-separation cuts form the family of so called *ELP-cuts*. In a bicritical graph G every barrier is trivial and thus if G has an ELP-cut it must be a 2-separation cut. The following theorem illustrates the key role of this observation for the Tight Cut Lemma.

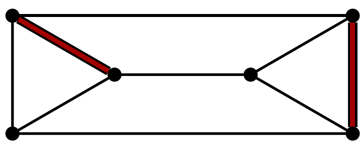


Figure 3.9.: The triangular prism $\overline{C_6}$ with a non-extendible matching of size two.

Theorem 3.1.76 (ELP Theorem, [CLM18])**.** If a matching covered graph G has a non-trivial tight cut, then it has a non-trivial ELP-cut.

Theorem 3.1.67 provides an alternative proof for the fact that every brace is 3-connected.

Let us further investigate the notions of extendibility and bicriticality in non-bipartite graphs. There exists a generalisation of Tutte's Theorem on the existence of perfect matchings in a graph to the extendibility of matchings.

Theorem 3.1.77 ([Yu93])**.** Let $k \in \mathbb{N}$ be a positive integer. A graph G is k-extendible if and only if for all $S \subseteq V(G)$

i) $c_{\text{odd}}(G - S) \leq |S|$ and
ii) $c_{\text{odd}}(G - S) = |S| - 2h$ for $h \in [0, k-1]$ implies $\nu(G[S]) \leq h$.

It is straight forward to see that any k-bicritical graph is also k-extendible. By Theorems 3.1.64 and 3.1.75 we also know that every 2-extendible non-bipartite graph must be 1-bicritical. This observation can actually be generalised for arbitrary k.

Theorem 3.1.78 ([LY98, Fav00])**.** Let $k \in \mathbb{N}$ be a positive integer. Every non-bipartite $2k$-extendible graph is k-bicritical.

One could ask the question whether the notion of matching k-connectivity yields tighter results here, as it seems to lie somewhere between extendability and bicriticality. However, one can easily find examples for graphs with an arbitrarily large gap between the two parameters extendibility and matching connectivity, which implies, by Theorem 3.1.78, that the gap between matching connectivity and bicriticality can also be arbitrarily high. For $k \in \mathbb{N}$, $k \geq 2$, let G_k be the graph consisting of a clique of size $k+2$ together with a stable set of size k such that every possible edge between the clique and the stable set exists. If we delete any two vertices of G_k that are contained in the clique, say x and y, then $G_k - x - y - E_{\text{no}}(G)$ is isomorphic to the complete bipartite graph $K_{k,k}$ which, although too small to be k-extendible, has the property that any matching of size at most k can be extended to a perfect matching. Hence G_k is indeed matching covered. This implies that $G_k - V(F)$ is matching connected for all matchings F that contain at most one edge of the clique. On the other side, if K is the clique of size $k+2$ in G_k, then $c_{\text{odd}}(G_k - V(K)) = |V(K)| - 2$, but K contains a matching of size 2. Thus no two disjoint edges of K can be contained in the same perfect matching of G_k and so G_k is not 2-extendible. This also means that any conformal set S of size at most $2(k-1)$, which is M-conformal for some perfect matching $M \in \mathcal{M}(G)$, contains at least $\frac{1}{2}(|S|-2)$ vertices from the stable set and thus there exists a matching F of size $\frac{1}{2}|S|$ with $V(F) = S$ and F contains at most one edge with both endpoints in K. Hence G_k is matching k-connected.

Still, this observation does not mean that there is no way to capture, at least approximately, the structure of highly bicritical graphs using some notion of connectivity based on alternating paths. For this, we need a classical result by Berge which, in some sense, illustrates the deep connection between matching theory and flow theory.

Let G be a graph, $F \subseteq E(G)$ be a matching and P be a path in G. The path P is *F-augmenting* if it is F alternating and none of its endpoints is matched by F.

Theorem 3.1.79 (Augmenting Path Lemma, [Ber57])**.** Let F be a matching in a graph G, then F is a maximum matching if and only if there exists no F-augmenting path in G.

Lemma 3.1.80 (A*)**.** A graph G is bicritical if and only if it has a perfect matching and for every $M \in \mathcal{M}(G)$ and all pairs of vertices $x, y \in V(G)$ there is an internally M-conformal path with endpoints x and y.

Proof. Let us assume that G is bicritical, $M \in \mathcal{M}(G)$, and $x, y \in V(G)$ such that $x \neq y$. If $xy \in M$, then $M \setminus \{xy\}$ is a perfect matching of $G - x - y$ and xy is an internally M-conformal path connecting x and y. Hence we may further assume $xy \notin M$. So let $x', y' \in V(G)$ such that $xx', yy' \in M$. Since G is bicritical, $G - x' - y'$ has a perfect matching which, in particular, has size $|M|-1$. Moreover, $M' := M \setminus \{xx', yy'\}$ is a matching of G of size $|M|-2$. So M' is not a maximum matching of G and therefore Theorem 3.1.79 guarantees the existence of an M'-augmenting path P in $G - x' - y'$. Additionally, the endpoints of P must be x and y since these are the only two vertices of $G - x' - y'$ not covered by M'. Since P is M'-alternating by definition, it is internally M-conformal in G and thus we have found our path.

For the reverse direction, let us assume that there is an internally M-conformal path between any pair of vertices and for every $M \in \mathcal{M}(G)$. Now consider the vertex pair $x, y \in V(G)$ and any perfect matching $M \in \mathcal{M}(G)$. Again either $xy \in M$ and we are done, or there are $x', y' \in V(G) \setminus \{x, y\}$ such that $xx', yy' \in M$. In the later case let P be an internally M-conformal path with endpoints x' and y'. Then P is an $M \setminus \{xx', yy'\}$-augmenting path in $G - x - y$. Since $M \setminus \{xx', yy'\}$ has size $|M| - 2$ this means by Theorem 3.1.79 that there is a matching of size $|M| - 1$ in $G - x - y$ implying that $G - x - y$ has a perfect matching. Since x and y were arbitrary, this means that G is bicritical. □

The property of having an internally M-conformal path joining any two vertices of G is, similar to bicriticality, inherently non-bipartite, as this implies, among other things, that between any two vertices there is a path of odd length. In bipartite graphs, no two vertices from the same colour class can be linked by such a path.

So while the concept of matching k-connectivity does not seem strong enough to capture the structure of internally conformal paths in non-bipartite graphs, maybe a stronger version fits better into this setting.

Definition 3.1.81 (Strong Matching Connectivity)**.** We call a graph G *strongly matching connected*, if it has a perfect matching and for every pair of vertices $x, y \in V(G)$ and every perfect matching $M \in \mathcal{M}(G)$ there exists an internally M-conformal path with endpoints x and y in G. Moreover, for every positive $k \in \mathbb{N}$, G is said to be *strongly matching k-connected* if $G - S$ is strongly matching connected for every conformal set $S \subseteq V(G)$ with $|S| \leq 2k - 2$ that is M-conformal for some $M \in \mathcal{M}(G)$.

Since in bipartite graphs matching connectivity and extendibility are closely related, it would be nice if we could show at least some link between extendibility and strong matching connectivity in the non-bipartite world. To this end we introduce the following to lemmas.

Lemma 3.1.82 (A*)**.** Let G be an elementary graph and $k \geq 2$. Then $G - E_{\mathrm{no}}(G)$ is k-extendible if and only if $G - V(S) - E_{\mathrm{no}}(G - V(S)) = G - E_{\mathrm{no}}(G) - V(S)$ for all matchings $S \subseteq E(G)$ of size at most $k - 1$.

Proof. Assume that $G - E_{\mathrm{no}}(G)$ is k-extendible and let $S \subseteq E(G)$ be a matching of size $1 \leq k' \leq k - 1$. By Theorem 3.1.66 G is k'-extendible and so there is a perfect matching containing S, hence $V(S)$ is conformal.

Moreover, let e be any edge of $G - E_{\mathrm{no}}(G) - V(S)$, then $S \cup \{e\}$ is a matching of size $k' + 1 \leq k$ and thus there is a perfect matching M_e of G containing $S \cup e$. Hence $M_e \setminus S$ is a perfect matching of $G - V(S)$ and so $\mathcal{E}(G - V(S)) = G - E_{\mathrm{no}}(G) - V(S)$.

For the reverse direction, let $F \subseteq E(G - E_{\mathrm{no}}(G))$ be a matching of size k and $e \in F$ be any edge. Then $S := F \setminus \{e\}$ is a matching of size $k - 1$ and by our assumption $e \in E(G - V(S)) \setminus E_{\mathrm{no}}(G - V(S))$. Hence there must be a perfect matching M_e of $G - V(S)$ containing e and so $M_e \cup S$ is a perfect matching of G. Consequently, for every matching of size k in $G - E_{\mathrm{no}}(G)$ there is a perfect matching containing it, and thus $G - E_{\mathrm{no}}(G)$ is k-extendible. $\square$

Lemma 3.1.83 (A*)**.** A graph G is k-extendible for $k \geq 2$ if and only if $G - x - y$ is $(k-1)$-extendible for every $xy \in E(G)$.

Proof. With G being n-extendible, we know that any edge xy of G is contained in a perfect matching, thus Lemma 3.1.82 yields $G - x - y - E_{\mathrm{no}}(G - x - y) = G - E_{\mathrm{no}}(G) - x - y$. Now, if $G - E_{\mathrm{no}}(G) - x - y$ is not $(k-1)$-extendible, there is a matching F of size $k - 1$ in $G - x - y$ that cannot be extended to a perfect matching of $G - x - y$. But $F \cup \{xy\}$ forms a matching of size k in G which must be contained in a perfect matching M and clearly M contains a perfect matching of $G - x - y$, which is a contradiction.

For the reverse, suppose $G - x - y$ is $(k-1)$-extendible for every edge $xy \in E(G)$. Let F be a matching of size k containing xy, then $F \setminus \{xy\}$ is a matching of size $k - 1$ of $G - x - y$ and thus contained in a perfect matching M of $G - x - y$. Then $M \cup \{xy\}$ is a perfect matching of G and so G is k-extendible. $\square$

The gap between (weak) matching connectivity and extendibility can be arbitrarily high as we have discussed. This is no longer true once we consider strong matching connectivity.

Lemma 3.1.84 (A*)**.** Let G be a strongly matching k-connected graph for some positive integer $k \in \mathbb{N}$. Then G is k-extendible.

Proof. We prove the assertion by induction over k. For $k = 1$ the claim follows immediately from Lemma 3.1.80. So let $k \geq 2$. Invoking

Lemma 3.1.80 yields that G still is bicritical. Hence every edge $xy \in E(G)$ is contained in a perfect matching, and so $G - x - y$ is matching $(k-1)$-connected. Now, by induction $G - x - y$ is $(k-1)$-extendible and since the choice of xy was arbitrary, we are done by Lemma 3.1.83. □

We have seen that extendibility does not seem to capture exactly the essentials of the structure of non-bipartite matching covered graphs. Still, matching connectivity is closely tied to extendibility. By definition however, it seems natural to expect a similar relation between matching connectivity and higher order bicriticality.

Lemma 3.1.85 (A*)**.** Let G be a k-bicritical graph with $k \geq 1$, then G is strongly matching k-connected.

Proof. Let $S \subseteq V(G)$ be an M-conformal set for some $M \in \mathcal{M}(G)$ of size at most $2k-2$, then $G - S$ is still bicritical and thus, by Lemma 3.1.80 it is matching connected. Since the choices of M and S were arbitrary, the claim follows immediately from the definition of strong matching k-connectivity. □

In light of Theorem 3.1.78, we are now able to reverse Lemma 3.1.85 to obtain at least a qualitative relation between strong matching connectivity and bicriticality. It is not known whether this bound can be improved.

Theorem 3.1.86 (A*)**.** Let $k \in \mathbb{N}$ be a positive integer and G a graph. If G is k-bicritical, then G is strongly matching k-connected, and, if G is strongly matching $2k$-connected, G is k-bicritical.

We have seen how the structure of internally M-conformal paths contributes to the structure of bricks. Not much is known beyond the results we presented in this section, and there appears to exist no concept that nicely captures the connectivity properties of bricks. We close this section with a non-bipartite analogue of Theorem 3.1.74 which, as expected, is even more complicated than the bipartite version.

For bricks there exists one exceptional graph: The *Petersen graph* as defined in Figure 3.10.

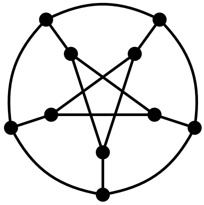


Figure 3.10.: The Petersen graph.

While any brace can be generated from the McCuaig braces by using bipartite augmentations, the Petersen graph seems to be the sole outstanding brick which cannot be fit into such a framework. For non-Petersen bricks, i.e. bricks that are not isomorphic to the Petersen graph, one can formulate the following generation rules.

Definition 3.1.87 (Expansion)**.** Let G be a graph with $\delta(G) \geq 2$, let $x \in V(G)$ be any vertex, and $\{Y_1, Y_2\}$ be a bipartition of $\mathrm{N}_G(x)$ into two non-empty sets. Let

$$G' := (G - x) + x_1 v x_2 + \{x_i y_i \mid y - I \in Y_i \text{ where } i \in [1, 2]\},$$

where $x_1, x_2, v \notin V(G)$. We say that G' is obtained from G by *expansion of x (to $x_1 v x_2$)*.

Please note that the only difference between an expansion and a bipartite expansion is that we allow G to be non-bipartite here.

Definition 3.1.88 (Augmentation)**.** Let G be a graph with $\delta(G) \geq 2$.

i) Let $x, y \in V(G)$ be non-adjacent vertices. We say that the graph $G' := G + xy$ is obtained from G by *type I augmentation.*
ii) Let $x, u \in V(G)$ be two distinct and non-adjacent vertices and G' be obtained from G by expansion of x to $x_1 v x_2$. If $\delta(G' + uv) \geq 3$, we say that $G' + uv$ is obtained from G by *type II augmentation.*
iii) Let $x, u \in V(G)$ be two adjacent vertices and G' be obtained from G by expansion of x to $x_1 v x_2$. If $\delta(G' + uv) \geq 3$, we say that $G' + uv$ is obtained from G by *type III augmentation.*
iv) Let $x, y \in V(G)$ be two distinct and non-adjacent vertices, and G' be obtained from G by expansion of x to $x_1 u x_2$, and y to $y_1 v y_2$. If $\delta(G' + uv) \geq 3$, we say that $G' + uv$ is obtained from G by *type IV augmentation.*

v) Let $x, y \in V(G)$ be two adjacent vertices, and G' be obtained from G by expansion of x to $x_1 u x_2$, and y to $y_1 v y_2$. If $\delta(G' + uv) \geq 3$, we say that $G' + uv$ is obtained from G by *type V augmentation*.

If G' is obtained from G by type i augmentation for some $i \in \{I, II, III, IV, V\}$, it is obtained from G by *augmentation*.

These rules for augmentation come from an analysis of the different cases in the original proofs, as those are worded not for generating bricks but to reduce them to bricks from one of the following five infinite families.

Definition 3.1.89 (Norine-Thomas Bricks)**.** An *odd wheel* of order $k \geq 1$ is the graph $\mathcal{W}_{2k+2}$ obtained from a cycle C_{2k+1} of length $2k+1$ by adding a single vertex and joining it to all vertices of C_{2k+1}.

An *odd prism* of order $k \geq 1$ is the graph $\mathcal{P}_{4k+2}$ obtained from the cycles

$$(u_0, u_1, \ldots, u_{2k-1}, u_{2k}, u_0), \text{ and}$$

$$(v_0, v_1, \ldots, v_{2k-1}, v_{2k}, v_0)$$

by adding the edges $u_i v_i$ for all $i \in [0, 2k]$.

An *even möbius ladder* of order $k \geq 1$ is the graph $\mathcal{M}_{4k}$ obtained from the cycle

$$(v_0, v_1, v_2, \ldots, v_{4k-2}, v_{4k-1}, v_0)$$

by adding the edges $v_i v_{2k+i}$ for all $i \in [0, 2k-1]$.

A *diamond* of order $k \geq 1$ is the graph $\mathcal{D}_{2k+4}$ obtained from the path

$$v_1 v_2 v_3 \ldots v_{2k+1} v_{2k+2},$$

and the vertices u_1, u_2 by adding the edges $u_1 v_{2k+2}$, $u_2 v_1$, and $u_j u_i$ for all $j \in [1, 2]$ and all $i \in [1, 2k+2]$ for which $i \equiv j \bmod 2$.

A *staircase* of order $k \geq 1$ is the graph $\mathcal{S}_{2k+4}$ obtained from the cycle

$$(v_0, v_1, v_2, \ldots, v_{2k-2}, v_{2k-1}, v_0)$$

and the vertices u_1, u_2 by adding the edges $u_1 u_2$, $u_1 v_0$, $u_1 v_1$, $u_2 v_{k+1}$, $u_2 v_{k+2}$, and $v_i v_{2k+3-i}$ for all $i \in [2, k]$. If k is even, $\mathcal{S}_{2k+4}$ is called an *odd staircase*, otherwise it is an *even staircase*.

Every graph that is isomorphic to an odd wheel, an odd prism, an even möbis ladder, a diamond, or a staircase is called a *Norine-Thomas brick*. See Figure 3.11 for an illustration.

Theorem 3.1.90 (Brick Generation Theorem, [NT07])**.** A graph G is a brick if and only if it is isomorphic to the Petersen graph, or it can be obtained from a Norine-Thomas brick by a sequence of augmentations.

3.1.4. Matching Minors

So far we have seen several concepts of identifying sets of vertices into a single vertex or contracting sets of vertices, that all play a role in matching theory. What we have not seen is a precise concept of a minor that fits into the setting of graphs with perfect matchings and, hopefully, unifies at least a majority of the different notions of contraction. The closest we have come so far to such a definition is in terms of conformal bisubdivisions. However, bisubdivisions more closely resemble the notion of *topological minors* or *subdivisions* of graphs. While this is already a powerful concept it might prove to be not general enough in a broader setting.

What do we need to define a concept of matching minors, and what do we expect from such a definition? We want a concept that generalises our already established notion of conformal subgraphs, it should generalise the idea of conformal bisubdivisions, and it should at least interact with the notion of tight cut contractions. Moreover, a definition of minor should immediately yield a minor-version of the Brace and the Brick Generation Theorem in the same way that Theorem 3.1.70 implies that every 3-connected graph contains a wheel as a minor. To ensure this, our operation should reverse both the process of bisubdividing an edge and also the operation of expanding a vertex.

Definition 3.1.91 (Bicontraction)**.** Let G be a graph with a perfect matching and $v \in V(G)$ a vertex of degree two with $\mathrm{N}_G(v) = \{v_1, v_2\}$. Let

$$G' := G - \{v_1, v, v_2\} + u + \{uw \mid w \in \mathrm{N}_{G-v}(v_1) \cup \mathrm{N}_{G-v}(v_2)\},$$

where $u \notin V(G)$. We say that G' is obtained from G by *bicontracting* v.

Let G be a graph with a perfect matching and $v \in V(G)$ be a vertex of degree two. Then $\mathrm{N}_G(v) \cup \{v\}$ is an odd set where every perfect matching of G must match v to exactly one of its two neighbours. So if G has at least six vertices, $\partial_G(\mathrm{N}_G(v) \cup \{v\})$ is a non-trivial tight cut in G. Note that the contraction of its complement always yields C_4. Therefore, as intended, the notion of bicontracting can be seen as a special case of tight cut contractions. Moreover, bicontraction indeed serves as a reverse operation to bisubdividing an edge, as this produces two vertices of degree two, and expanding a vertex, since this produces at least one vertex of degree two.

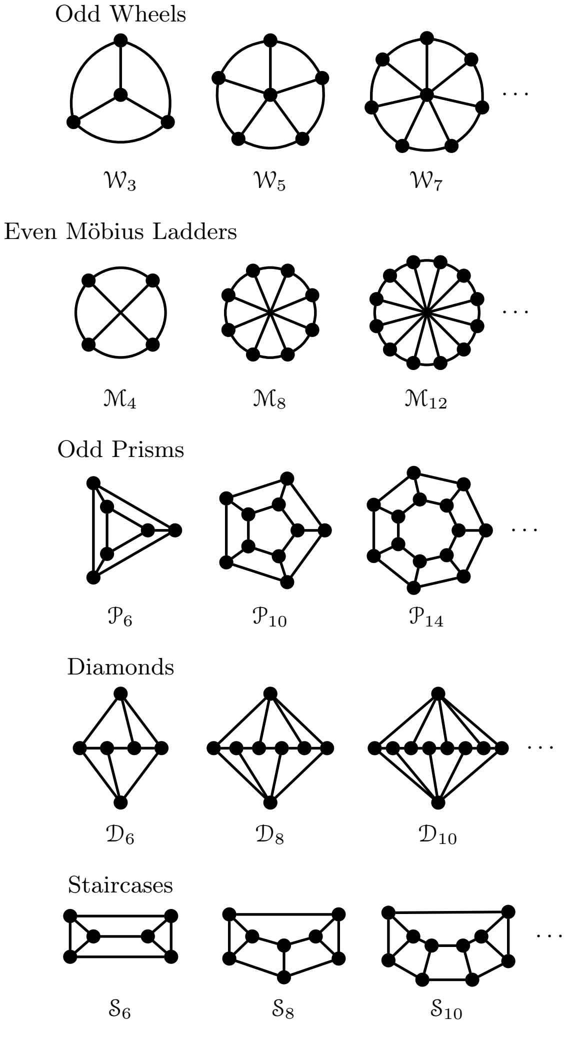


Figure 3.11.: The five infinite families of Norine-Thomas bricks.

Definition 3.1.92 (Matching Minor)**.** Let G and H be graphs with perfect matchings. We say that H is a *matching minor* of G if H can be obtained from a conformal subgraph of G by a sequence of bicontractions.

Let M be a perfect matching of G, if H can be obtained from an M-conformal subgraph of G by a sequence of bicontractions we say that H is an *M-minor* of G.

The graph G is said to be *H-free* if it does not contain H as a matching minor.

Although the idea of matching minors already occurs in the work of McCuaig [McC01], the term itself stems from the work of Norine and Thomas on bricks [NT07]. From the Brace Generation Theorem of McCuaig and the Brick Generation Theorem of Norine and Thomas, it follows now that every brace has a matching minor that is isomorphic to a McCuaig brace, and every brick is either isomorphic to the Petersen graph or contains a matching minor isomorphic to a Norine-Thomas brick. Indeed, the findings of Lovász in his research on ear-decompositions of non-bipartite matching covered graphs yield the following theorem.

Theorem 3.1.93 ([LP09])**.** Every brick has a conformal bisubdivision of (and therefore a matching minor isomorphic to) $K_4 = \mathcal{W}_4$ or $\overline{C_6} = \mathcal{P}_6$.

A similar statement, but of course with different graphs, can be proven regarding braces.

Theorem 3.1.94. Every brace has a conformal bisubdivision of (and therefore a matching minor isomorphic to) the *cube* $\mathcal{P}_8$ or $K_{3,3} = \mathcal{M}_6$.

An important property of any viable minor operation is transitivity. This was first mentioned in a paper of Lucchesi et al. [LdCM15] which aimed at a more unified framework for the Brace Generation Theorem.

Lemma 3.1.95 (Transitivity of Matching Minors, [LdCM15])**.** Let G be a matching covered graph and let H be a matching minor of G. Then any matching minor of H is also a matching minor of G.

Let G and H be matching covered graphs. It is easy to see that H has to be a matching minor of G if G contains a conformal bisubdivision of H. However, it is not necessarily true that G contains a conformal bisubdivision of H if it has H as a matching minor. Similar to topological and (general) minors in undirected graphs, the reverse is a question of vertex degrees in H.

Lemma 3.1.96 ([LDCKM18]). Let G and H be matching covered graphs such that $\Delta(H) = 3$. Then G contains a conformal bisubdivision of H if and only if it contains H as a matching minor.

For bipartite graphs one can actually show that matching minors are enough to capture the whole concept of tight cut contractions. This is a property that cannot be generalised to non-bipartite graphs.

Lemma 3.1.97 ([LdCM15]). Let B be a bipartite matching covered graph, $X \subseteq V(B)$ a set that induces a tight cut, and $B' := B/(X \rightarrow v)$. Then B' is a matching minor of B.

Corollary 3.1.98 ([LdCM15]). Let B be a bipartite matching covered graph. Then every brace of B is a matching minor of B.

Moreover, and this is probably the most important property of matching minors in bipartite graphs, if a bipartite graph B has some brace H as a matching minor, then this property is reflected by at least one brace of B. This means that brace matching minors cannot vanish during the tight cut decomposition procedure.

Lemma 3.1.99 ([LdCM15]). Let B be a bipartite matching covered graph. A brace H is a matching minor of B if and only if it is a matching minor of some brace of B.

While a generalisation of the above lemma to non-bipartite graphs is generally not possible, there is a small special case for which tight cut contractions still preserve the existence of matching minors that are bricks.

Theorem 3.1.100 (Cubic Brick Theorem, [KM16]). Let G be a matching covered graph, H be a cubic brick, and $X \subseteq V(G)$ a set that induces a tight cut in G. Then G contains a conformal bisubdivision of H if and only if one of $G/(X \rightarrow v)$ or $G/(\overline{X} \rightarrow \overline{v})$ contains a conformal bisubdivision of H.

To speak about matching minors in a more formal way, we introduce the concept of models for matching minors. Models, or embeddings, for matching minors have already been used and discussed in [RST99] and [NT07] and the definitions we give here are similar to those of Norine and Thomas. Some parts of these definitions, however, have been changed to better suit our needs in the chapters to come and therefore we provide the necessary proofs.

Let T' be a tree and let T be obtained from T' by subdividing every edge an odd number of times. Then $V(T') \subseteq V(T)$. The vertices of T that belong to T' are called *old*, and the vertices in $V(T) \setminus V(T')$ are called *new*. We say that T is a *barycentric tree*.

Definition 3.1.101 (Matching Minor Model)**.** Let G and H be graphs with perfect matchings. An *embedding* or *matching minor model* of H in G is a mapping

$$\mu\colon V(H) \cup E(H) \to \{F \mid F \subseteq G\}\,,$$

such that the following requirements are met for all $v, v' \in V(H)$ and $e, e' \in E(H)$:

i) $\mu(v)$ is a barycentric subtree in G,
ii) if $v \neq v'$, then $\mu(v)$ and $\mu(v')$ are vertex disjoint,
iii) $\mu(e)$ is an odd path with no internal vertex in any $\mu(v)$, and if $e' \neq e$, then $\mu(e)$ and $\mu(e')$ are internally vertex disjoint,
iv) if $e = u_1u_2$, then the ends of $\mu(e)$ can be labelled by x_1, x_2 such that x_i is an old vertex of $\mu(u_i)$,
v) if v has degree one, then $\mu(v)$ is exactly one vertex, and
vi) $G - \mu(H)$ has a perfect matching, where $\mu(H') \coloneqq \bigcup_{x \in V(H') \cup E(H')} \mu(x)$ for every subgraph H' of H.

If μ is a matching minor model of H in G we write $\mu\colon H \to G$.

While we slightly changed the definition here, the next lemma follows immediately from a result of [NT07] and thus we omit the proof.

Lemma 3.1.102 ([NT07])**.** Let G and H be graphs with perfect matchings. There exists a matching minor model $\mu\colon H \to G$ if and only if H is isomorphic to a matching minor of G.

Lemma 3.1.103 (D*)**.** Let H and G be graphs and $\mu\colon H \to G$ be an embedding of H into G. Then H has a perfect matching if and only if $\mu(H)$ has a perfect matching.

Proof. Suppose H has a perfect matching. We prove our claim by induction on the number c of bicontractions that have to be applied to $\mu(H)$ in order to obtain a graph isomorphic to H. For $c = 0$ this implies $\mu(H) = H$ and H has a perfect matching.

So let $c \geq 1$. Starting with $\mu(H)$ let $b_1, \ldots, b_c$ be the bicontractions that need to be applied and H_i be the graph obtained from $\mu(H)$ by only applying the bicontractions $b_1, \ldots, b_i$. Furthermore, let those bicontractions be ordered in such a way that $H_c = H$ and $H_0 = \mu(H)$, where H_0 is the uncontracted graph and moreover H_i is obtained from H_{i-1} by applying exactly one bicontraction.

Hence H_1 is a matching minor of G that also contains H as a matching minor and H can be obtained from H_1 by applying $b_2, \ldots, b_c$, which are $c-1$ bicontractions, let μ_1 be a corresponding matching model of H in H_1, then $\mu_1(H) = H_1$. By our induction hypothesis, H_1 has a perfect matching. The transition from $\mu(H)$ to H_1 is done by applying b_1 to $\mu(H)$. Let v_0 be the vertex in $\mu(H)$ that is to be bicontracted by b_1, and let v_1, v_2 be its two unique neighbours, and let v be the new vertex in H_1 after the bicontraction. Since H_1 has a perfect matching, there is some vertex $x \in V(H_1) \cap V(\mu(H))$ such that xv is a perfect matching edge in H_1. Therefore, there must be v_i with $i \in \{1, 2\}$, say $i = 1$, such that xv_1 is an edge of $\mu(H)$ by the definition of matching models. Let M be some perfect matching of H_1 containing xv, then $M \setminus \{xv\}$ is a perfect matching of $\mu(H) - x - v_0 - v_1 - v_2$. Let $M' := (M \cup \{xv_1, v_0v_2\}) \setminus \{xv\}$, then M' is a perfect matching of $\mu(H)$.

The reverse direction follows along similar lines and is therefore omitted. □

If $\mu\colon H \to G$ is a matching minor model of a matching covered graph H in G, then both $G - \mu(H)$ and $\mu(H)$ have a perfect matching. Let M be a perfect matching of G such that $M \cap E(G - \mu(H))$ is a perfect matching of $G - \mu(H)$ and $M' := M \cap E(\mu(H))$ is a perfect matching of $\mu(H)$. Then there is a perfect matching of H that 'mimics' the structure of M' in $\mu(H)$. In the following, we explain what we mean with the word 'mimics'.

Lemma 3.1.104 (D*)**.** Let G and H be graphs with perfect matchings, $\mu\colon H \to G$, and M a perfect matching of $\mu(H)$. Then for every $u \in V(H)$, there is a unique vertex $v \in N_H(u)$ such that $\mu(uv)$ is an M-conformal path and for all other edges $e \in E(H)$ incident with u their respective model $\mu(e)$ is internally M-conformal.

Proof. Let $e \in E(H)$ be any edge, then $\mu(e)$ is a path of odd length where every inner vertex has degree two in $\mu(H)$. Thus for every perfect matching M' of $\mu(H)$, $\mu(e)$ either is an internally M'-conformal path, i. e. M' contains a perfect matching of $\mu(e)$ without its endpoints, or M' contains a perfect matching of $\mu(e)$.

For any vertex $u \in V(H)$ let us call $t \in V(\mu(u))$ *exposed* if the edge of M covering t is not an edge of the barycentric tree $\mu(u)$. Please note that for every exposed vertex t of $\mu(u)$ there must be an edge $e \in E(H)$ such that t is an endpoint of $\mu(e)$ and the edge of M covering t is an edge of $\mu(e)$. Moreover, in this case $\mu(e)$ cannot be internally M-conformal and thus must be M-conformal by the observation above. Hence the other endpoint of $\mu(e)$, which is a vertex of $\mu(v)$ for some $v \in V(H)$, must also be exposed. These observations immediately imply that any exposed vertex in $\mu(u)$ must be an old vertex.

Next, observe that every path P in $\mu(u)$ that connects two old vertices and otherwise consists only of new vertices is of even length. Similar to our observation for $\mu(e)$, every inner vertex of P must be covered by an edge of $E(P) \cap M$. Hence there exists exactly one vertex of P that is not covered by an edge of $E(P) \cap M$.

So in order to prove our claim, we have to show that for every $u \in V(H)$ there is exactly one exposed vertex in $\mu(u)$. To do this, we generalise the observation on the even paths within $\mu(u)$ we made above. Let $T = \mu(u)$ be a barycentric tree and let O be the set of old vertices of T. Moreover, let T' be the tree with $V(T') = O$ from which T was constructed by subdividing every edge an odd number of times. Then any two old vertices that are adjacent in T' are linked by a path of even length in T. Hence in a proper 2-colouring of T, all vertices of O receive the same colour. Now let $e \in E(T')$ be any edge and P_e the corresponding path in T, moreover let $P \coloneqq P_e - O$. Then P is a path of even length as well and thus in a proper 2-colouring of T its endpoints receive the same colour. With P being of even length, it has an odd number of vertices, say $2k + 1$, and thus in a proper 2-colouring of T, P has $k + 1$ vertices whose colour is different from the colour of the old vertices and k vertices with the same colour as the old vertices, combining this with our observation above that in each P_e exactly one vertex is not covered by an edge of $E(P_e) \cap M$. This yields that in total $|O| - |E(T')| = 1$ vertices of T must be exposed by every perfect matching of $\mu(u)$. □

In the situation of Lemma 3.1.104 let M be a perfect matching of $\mu(H)$, then for every $u \in V(H)$ there is a unique vertex $v \in V(H)$ such that $\mu(uv)$ is M-conformal. Let

$$M|_H \coloneqq \{uv \in E(H) \mid \mu(uv) \text{ is } M\text{-conformal}\},$$

then $M|_H$ is a perfect matching of H. In a slight extension of our definition of residual matching we call $M|_H$ the *M-residual* matching of H. Moreover, we call a matching minor model $\mu\colon H \to G$ an *M-model* of H in G if $\mu(H)$ is M-conformal. With this, we obtain the following corollary.

Corollary 3.1.105 (D*)**.** Let G and H be graphs with perfect matchings and M a perfect matching of G. Then H is isomorphic to an M-minor of G if and only if there exists an M-model $\mu_M : H \to G$ in G.

3.2. Perfect Matchings, Digraphs, and Bidirected Graphs

It often suffices to fix a single perfect matching M of a graph G and describe its structural properties through the lens of M. Not only can this approach lead to a simplification of the matter at hand or at least the notation, but it can also allow for deeper insight, especially for the application of different branches of structural graph theory.

Definition 3.2.1 (Bidirected Graph)**.** A *bidirected graph* (G, σ) is a graph G with vertex set $V(G)$, edge (multi-)set $E(G)$, a corresponding set of *half-edges*

$$\mathsf{E}(G) \coloneqq \{(u, e), (v, e) \mid e = uv \in E(G)\},$$

and a *signing* $\sigma\colon \mathsf{E}(G) \to \{+, -\}$ of the half-edges.

Two distinct edges e and e', both with endpoints u and v, are said to be *parallel* if $\sigma(u, e) = \sigma(u, e')$[16] and $\sigma(v, e) = \sigma(v, e')$. We call (G, σ) *simple*

[16] Formally we would need to write $\sigma((u, e))$, but we drop the extra set of parenthesis for better readability.

if it has no loops and no parallel edges. In particular, this means that the edge multiplicity of G for a simple bidirected graph (G, σ) is at most 4. An edge $e = uv \in E(G)$ is called *introverted* if $\sigma(u, e) = \sigma(v, e) = +$, *extroverted* if $\sigma(u, e) = \sigma(v, e) = -$, and *normal* otherwise. A natural operation for bidirected graphs is the change of the signs of all half-edges incident with a single vertex.

Definition 3.2.2 (Switching)**.** Let (G, σ) be a bidirected graph and $v \in V(G)$ a vertex. Let furthermore

$$\sigma'(u, e) \coloneqq \begin{cases} \sigma(u, e), & u \neq v, \\ -, & \sigma(u, e) = +,\ u = v, \text{ and} \\ +, & \sigma(u, e) = -,\ u = v. \end{cases}$$

We say that (G, σ') is obtained from (G, σ) by *switching* at the vertex v. Two bidirected graphs (G, σ) and (G, σ') are said to be *switching equivalent* if (G, σ') can be obtained from (G, σ) by a sequence of switchings.

Definition 3.2.3 (Digraphic Bidirected Graph)**.** A bidirected graph (G, σ) is called a *digraph* if all of its edges are normal, it is called *digraphic* if it is switching equivalent to a digraph.

The idea here is simply that one can obtain an actual digraph from any bidirected graph (G, σ) by orienting each edge $e = uv$ away from the half edge (u, e) with positive sign and towards the half-edge (v, e) with negative sign. Clearly the orientation from $+$ to $-$ is arbitrary and may also be chosen to be exactly the other way around, but for sake of consistency we fix the first of these two options. Indeed, in most places throughout this thesis we drop the rather unwieldy notation of bidirected graphs and use standard digraph notation instead whenever we are working with a digraph.

Let G be a graph with a perfect matching M. A map $\zeta \colon V(G) \to \{+, -\}$ is called an *M-signing* of G if for every edge $uv \in M$ we have $\zeta(u) \neq \zeta(v)$. An M-singing is *proper* if it is a proper 2-colouring[17] of G, i.e. if no two adjacent vertices are assigned the same colour by ζ. Note that a graph G has a proper M-signing if and only if it is bipartite.

[17] Here we interpret $+$ and $-$ as colours.

Given a bipartite graph B with a perfect matching M, we denote by ζ_B the M-singing obtained by letting $\zeta_B(u) \coloneqq +$ if and only if $u \in V_1$. Please note that ζ_B is indeed independent of the choice of M.

Definition 3.2.4 (General Bidirection)**.** Let G be a simple graph with a perfect matching M, and let ζ be an M-signing of G. Consider the bidirected graph $\mathcal{D}_{\pm}(G, M, \zeta) \coloneqq (G', \sigma)$ with

$$E(G') \coloneqq \{\{e, f\} \in \binom{M}{2} | \text{there is } g \in E(G) \text{ s.t. } e \cap g \neq \emptyset \neq g \cap f\},$$

$$V(G') \coloneqq M,$$

$$\mathsf{E}(G') \coloneqq \left\{(f, e) \mid f \in M,\ e \in E(G'), \text{ and } e \cap f \neq \emptyset\right\}, \text{ and}$$

$$\sigma(f, e) \coloneqq \zeta(v), \text{ where } v \text{ is the endpoint } e \text{ and } f \text{ have in common,}$$

the M-*bidirection* under ζ.

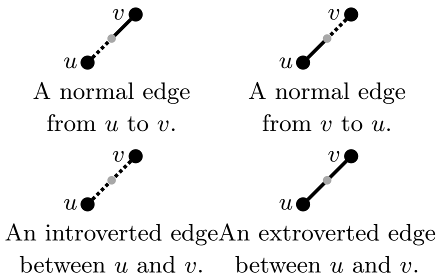


Figure 3.12.: The four different types of edges in a bidirected graph.

For better readability let us introduce the following convention for edges in figures: Each half edge will either be displayed solid, which means its sign is $+$, or it will be dotted, indicating that its label is $-$. To make it easier to distinguish the two half edges of an edge, we place a small gray dot between them. See Figure 3.12 for a depiction of the different types of edges.

This convention deviates a bit from the literature as usually a negatively signed half edge is marked with an arrow head that points away from the endpoint of the half edge, while a positively signed half edge is marked by an arrow head that points towards its endpoint. An illustration of a general M-birection can be found in Figure 3.13.

Since most of the time we work on bipartite graphs, it is convenient to have a short hand for the M-bidirection under ζ_B for any bipartite graph B.

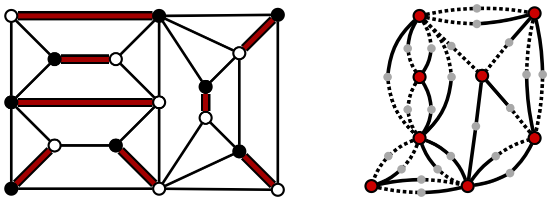


Figure 3.13.: Left: A graph G with a perfect matching M and an M-signing ζ, where $-$ is associated with the solid vertices, while $+$ is the sign of the empty vertices. Right: The arising general M-bidirection $\mathcal{D}_{\pm}(G, M, \zeta)$.

Definition 3.2.5 (M-Direction)**.** Let B be a bipartite graph with a perfect matching M. The *M-direction* of B is the digraph $\mathcal{D}(B, M) := \mathcal{D}_{\pm}(B, M, \zeta_B)$.

In Figure 3.14 we give an example of the M-direction.

Observe that any two M-signings of a graph G can be translated into one another by iteratively flipping the colourings of the two endpoints of one edge of M at a time. This process can be seen to be equivalent to switching signs at vertices of the M-bidirection of G. So even if there are many different possible M-signings for a single perfect matching M of a single graph G, all M-bidirections obtained from them are in fact switching equivalent. Moreover, given any bidirected graph (G, σ), the process can be reversed.

Definition 3.2.6 (Split)**.** Let (G, σ) be a bidirected graph G. We define $\mathcal{S}(G, \sigma)$ to be the graph G' for which a perfect matching M and an M-signing ζ exist such that $\mathcal{D}_{\pm}(G', M, \zeta) = (G, \sigma)$.

If D is a digraph, then $\mathcal{S}(D)$ is the bipartite graph B that has a perfect matching M for which $\mathcal{D}(B, M) = D$.

Consequently, every graph G with at least one perfect matching naturally corresponds to a family of bidirected graphs (or digraphs in case G is bipartite). In what follows, we briefly explain some initial consequences of this correspondence (mostly) for bipartite graphs which we build upon in Part II. The general case of non-bipartite graphs and thus of general bidirected graphs is much more complicated and only few successful ventures into the topic have been made so far.

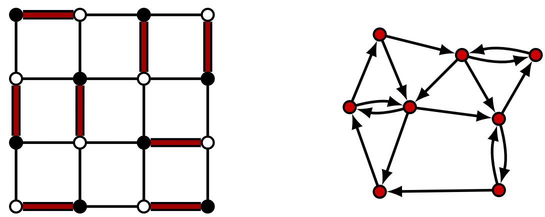


Figure 3.14.: Left: A bipartite graph B with a perfect matching M. Right: The arising M-direction $\mathcal{D}(B, M)$.

Extendiblity, Strong Connectivity, and Directed Separations

To successfully operate on M-bidirections (G, σ) and the corresponding graph H with the perfect matching M, we need to be able to translate fundamental parts of the structure of H induced by M and the structure of (G, σ) into one another. For an initial intuition, let us consider paths, cycles, and subgraphs.

Subgraphs might be the easiest. Simply observe that for any (G', σ), where $G' \subseteq G$, $\mathcal{S}(G')$ must be an M-conformal subgraph of H.

Definition 3.2.7 (Directed Paths and Cycles)**.** Let (G, σ) be a bidirected graph. A path $v_1 e_1 v_2 e_2, \dots, e_{\ell-1} v_\ell$ is called *directed* if for all $i \in [2, \ell - 1]$ we have $\sigma(v_i, e_{i-1}) \neq \sigma(v_i, e_i)$.

A cycle $(v_0, e_0, v_1, e_1, \dots, e_{\ell-1}, v_\ell, e_\ell, v_0)$ is *directed* if every subpath of the cycle is a directed path.

First of all, these definitions correspond to the definitions of directed paths and cycles in digraphs since here we always enter a vertex via an incoming edge, so via the sign $+$, and leave a vertex via an emanating edge, so via the sign $-$. Second, if we interpret every vertex of the bidirected graph (G, σ) as an edge of the perfect matching M, then this definition requires us to always traverse an edge of M before we are allowed to use an edge that does not belong to M. From this observation, it follows that every internally M-conformal path in a graph H with a perfect matching M and some M-signing ζ naturally corresponds to a directed path in $\mathcal{D}_{\pm}(H, M, \zeta)$ and vice versa. Similarly, one can observe that a cycle C in $\mathcal{D}_{\pm}(H, M, \zeta)$ is directed if and only if $\mathcal{S}(C)$ is an M-conformal cycle in H. For an example see Figure 3.15. The M-signing in the figure is not a proper 2-colouring of the cycle, note that by swapping the colours of the endpoints at two of the M-edges one can obtain such a proper 2-colouring and thus transform the generalised M-direction into a digraph.

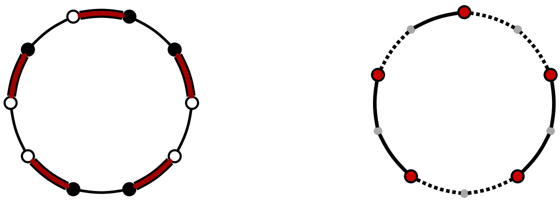


Figure 3.15.: Left: An M-conformal cycle C with an M-signing ζ, where we associate with $-$ the solid vertices, while $+$ is the sign of the empty vertices. Right: The general M-direction $\mathcal{D}_{\pm}(C, M, \zeta)$.

The matter of connectivity on general bidirected graphs is a complicated one. Indeed, there exist several papers (see for example [AFN96, Kit17]) that propose a definition of a generalisation of strong connectivity to the setting of bidirected graphs. Because of this, from here on we will consider only bipartite graphs, as for those some consensus exists.

Recall that a directed graph D is strongly connected if and only if any pair of vertices is joined by a pair of directed paths, i.e. if every vertex v can reach every vertex u via a directed path starting in v and ending in u. Moreover, these paths exist inside strong components, and thus if we fix any pair of vertices u and v and a pair P_1, P_2 of directed paths such that P_1 starts at u and ends in v while P_2 starts at v and ends in u, one can see that P_1 and P_2 correspond to a complementary pair of internally M-conformal paths in $\mathcal{S}(D)$ where M is the perfect matching of B such that $\mathcal{D}(B, M) = D$. Moreover, this complementary pair of paths meets the requirements of our definition of matching connectivity. Indeed, a classical result in structural matching theory links strong connectivity in digraphs and extendibility in bipartite graphs with perfect matchings. The following statement is folklore (a proof can be found in [ZL10], but the result was already known by [RST99]).

Theorem 3.2.8. Let B be a bipartite graph with a perfect matching M and $k \in \mathbb{N}$ be a positive integer. Then B is k-extendible if and only if $\mathcal{D}(B, M)$ is strongly k-connected.

When examining the construction of $\mathcal{D}(B, M)$ more closely, one may also note that the Dulmage-Mendelsohn ordering $\leq_2$ actually corresponds to the so called *topological ordering* of strong components of a digraph. Here we write $K_1 \leq_{\text{top}} K_2$ for strong components K_1 and K_2 of $\mathcal{D}(B, M)$ if there is a directed path starting in K_1 and ending in K_2.

Definition 3.2.9 (Directed Separation)**.** Let D be a digraph. A tuple (X, Y) is a *directed separation* of D if there is no directed path starting in $X \setminus Y$, ending in $X \setminus Y$, and avoiding $X \cap Y$ while $X \cup Y = V(D)$. The set $X \cap Y$ is called the *separator* of (X, Y), and the number $|X \cap Y|$ is called the *order* of the directed separation, it is denoted by $\operatorname{ord}(X, Y)$.
A directed separation (X, Y) is called *trivial* if $X \setminus Y = \emptyset$ or $Y \setminus X = \emptyset$.
Two directed separations (A, B) and (C, D) are said to *cross* if the following sets all are non-empty:

$$A \cap C,\ B \cap D,\ (A \cap D) \setminus (B \cap C),\ \text{and } (B \cap C) \setminus (A \cap D).$$

If (A, B) and (C, D) do not cross, they are called *laminar*.

Definition 3.2.10 (The Split of a Directed Separation)**.** Let B be a bipartite graph with a perfect matching M, $D \coloneqq \mathcal{D}(B, M)$, and (X, Y) be a directed separation in D. We denote by $\mathcal{S}(X, Y)$ the *split of* (X, Y), which is defined as the vertex set

$$\mathcal{S}(X, Y) \coloneqq V(\mathcal{S}(D[X \setminus Y])) \cup (V(\mathcal{S}(D[X \cap Y])) \cap V_1).$$

Lemma 3.2.11 (X*)**.** Let B be a bipartite graph with a perfect matching M, $D \coloneqq \mathcal{D}(B, M)$, and (X, Y) be a directed separation of order $k \geq 1$ in D. Then $\mathcal{S}(X, Y)$ induces a proper k-tight cut in B and $\operatorname{Maj}(\mathcal{S}(X, Y)) \subseteq V_1$.

Proof. First, observe $\mathrm{Maj}(\mathcal{S}(X,Y)) \subseteq V_1$. This follows immediately from the fact that $V(\mathcal{S}(D[X \setminus Y]))$ is balanced, and $|X \cap Y| = k \geq 1$. Next, suppose there is an edge $uv \in E(B)$ with $u \in \mathcal{S}(X,Y) \cap V_2$ and $v \in \overline{\mathcal{S}(X,Y)}$. Then u is not an endpoint of any edge corresponding to a vertex in $X \cap Y$ and neither is v since $v \in V_1$. Let $e_u, e_v \in M$ such that e_x matches $x \in \{u,v\}$. Then (e_v, e_u) is a directed edge in D whose head lies in $X \setminus Y$ and whose tail belongs to $Y \setminus X$, contradicting (X,Y) to be a directed separation. Hence by Lemma 3.1.58 $\mathcal{S}(X,Y)$ induces a proper k-tight cut. □

The process of splitting a directed separation can also be reversed.

Definition 3.2.12 (The M-Direction of a k-Tight Cut)**.** Let B be a bipartite graph with a perfect matching M and $X \subseteq V(G)$ induce a proper k-tight cut such that $\mathrm{Maj}(X) \subseteq V_1$. We denote by $\mathcal{D}(X,M)$ the tuple (Y_1, Y_2), where

$$Y_1 \coloneqq V(\mathcal{D}(B[X \cup V(M \cap \partial_B(X))], M))$$
$$Y_2 \coloneqq V(\mathcal{D}(B[\overline{X} \cup V(M \cap \partial_B(X))], M)).$$

Lemma 3.2.13 (X*)**.** Let B be a bipartite graph with a perfect matching M and $X \subseteq V(G)$ induce a proper k-tight cut such that $\mathrm{Maj}(X) \subseteq V_1$. Then $\mathcal{D}(X,M) = (Y_1, Y_2)$ is a directed separation of order k.

Proof. Since X and $\overline{X}$ partition the vertex set of B, any edge of M that does not belong to $\partial_B(X)$ is either completely contained in X or its complement. Therefore we get $|Y_1 \cap Y_2| = |\partial_B(X) \cap M| = k$. So all that remains is to show that (Y_1, Y_2) is indeed a directed separation. To this end, suppose there is a directed edge $(e_1, e_2) \in E(D)$ with $e_1 \in Y_2 \setminus Y_1$ and $e_2 \in Y_1 \setminus Y_1$. Then (e_1, e_2) corresponds to an edge uv in B with $u \in X \cap V_2$ and $v \in \overline{X} \cap V_1$. This, however, contradicts Lemma 3.1.58 since X induces a proper k-tight cut with $\mathrm{Maj}(X) \subseteq V_1$. Hence (Y_1, Y_2) must be a directed separation of order k. □

Matching Minors and Butterfly Minors

In the previous paragraph, we have seen that any proper k-tight cut in a bipartite graph with a perfect matching M naturally corresponds to a directed separation of order k in $\mathcal{D}(B,M)$ and vice versa. This implies in particular that B has a non-trivial tight cut if and only if $\mathcal{D}(B, M)$ has a directed separation (X, Y) of order 1 where $X \setminus Y \neq \emptyset$, and $Y \setminus X \neq \emptyset$. Having observed this, it seems straight forward to transport the idea of tight cut contraction into the world of digraphs.

Definition 3.2.14 (Directed Tight Cut Contraction)**.** Let D be a digraph, and (X, Y) be a non-trivial directed separation of order 1 in D. Let $\{v\} = X \cap Y$, we set

$$\begin{aligned} D/(X \to v_X) &\coloneqq D - X + v_X + \{(y, v_X) \mid (y, v) \in E(D) \text{ and } y \in Y\} \\ &\quad + \{(v_X, y) \mid (x, y) \in E(D),\ x \in X, \text{ and } y \in Y\}, \text{ and} \\ D/(Y \to v_Y) &\coloneqq D - Y + v_Y + \{(v_Y, x) \mid (v, x) \in E(D) \text{ and } x \in X\} \\ &\quad + \{(x, v_Y) \mid (x, y) \in E(D),\ x \in X, \text{ and } y \in Y\}. \end{aligned}$$

The two digraphs $D_X \coloneqq D/(X \to v_X)$ and $D_Y \coloneqq D/(Y \to v_Y)$ are called the (X, Y)-*contractions* of D.

Let B be a bipartite graph with a perfect matching M and $X \subseteq V(B)$ be a set that induces a non-trivial tight cut in B. Let $B_X \coloneqq B/(X \to v_X)$ and $B_{\overline{X}} \coloneqq B/(\overline{X} \to v_{\overline{X}})$ be the two tight cut contractions obtained from $\partial_B(X)$. Notice that $\mathcal{D}(B_X, M|_{B_X})$ and $\mathcal{D}(B_{\overline{X}}, M|_{B_{\overline{X}}})$ are isomorphic to the two $\mathcal{D}(X, M)$-contractions of $\mathcal{D}(B, M)$. Moreover, the reverse is also true, so the splits of the two (X, Y)-contractions of some digraph D, where (X, Y) is a non-trivial directed separation of order 1, are isomorphic to the two tight cut contractions obtained from $\mathcal{S}(X, Y)$ in $\mathcal{S}(D)$. Also notice that, by Theorem 3.2.8, if D is strongly connected, then its (X, Y)-contractions are strongly connected as well. At last, two tight cuts $\partial_B(Z_1)$, $\partial_B(Z_2)$ are laminar if and only if $\mathcal{D}(Z_1, M)$ and $\mathcal{D}(Z_2, M)$ are laminar. This allows us to transport the whole concept of tight cut decompositions to the world of digraphs.

Definition 3.2.15 (Directed Tight Cut Decomposition)**.** Let D be a strongly connected digraph. We iteratively construct a tree T as follows. First let T consist of only one vertex, say t_0 and let us associate $D_0 = D$ with t_0. Then select a non-trivial directed separation (X, Y) of order 1 in D and let D_X, D_Y be its two (X, Y)-contractions. We introduce $t_{1,1}$ and $t_{1,2}$ together with the edges $t_0 t_{1,i}$, $i \in [1, 2]$ to our tree T and associate[18] with each of the two (X, Y)-contractions exactly one of the two new vertices.

Suppose now that we have constructed a binary tree T with root t_0 such that each pair of successors of an inner vertex t of T are associated with the two S-contractions obtained from the digraph associated with t by using a single non-trivial directed separation S of order 1. If there exists a leaf ℓ whose associated matching covered graph still has a non-trivial tight cut, we construct the two S-contractions from it, add two new successors to ℓ, and associate each of the two newly obtained S-contractions with exactly one of the two new successors of ℓ.

At some point, this procedure stops and we obtain a tree T as above such that each leaf of T is associated with a strongly connected digraph without any non-trivial directed separations of order 1. Let $\mathcal{D}$ be the family of all digraphs associated with the leaves of T. Here we explicitly allow for $\mathcal{D}$ to be a multiset. Then $\mathcal{D}$ is a *directed tight cut decomposition* of D. We call the tree T a *tight cut decomposition tree* of D. Moreover, there is a family $\mathcal{T}$ of pairwise laminar non-trivial directed separations of order 1 of D such that each member of $\mathcal{T}$ can be associated with an inner vertex of T.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two multisets of digraphs. We say that $\mathcal{D}_1$ and $\mathcal{D}_2$ are *isomorphic* if there exists a bijection $f\colon \mathcal{D}_1 \to \mathcal{D}_2$ such that for all $D \in \mathcal{D}_1$, D and $f(D)$ are isomorphic.

The following theorem follows immediately from Theorem 3.1.61 by applying the observations above.

Theorem 3.2.16 (X*)**.** Let D be a strongly connected digraph. Then any two directed tight cut decompositions of D are isomorphic.

This means that any strongly connected digraph D can be decomposed into a unique list of strongly 2-connected[19] digraphs and, possibly, some copies of $\overleftrightarrow{K}_2$, called the *dibraces* of D. A natural question to ask is whether a directed analogue of the Brace Generation Theorem exists. Indeed, in [McC01] McCuaig claims to have found this analogue, but the manuscript mentioned there has not been published or made available anywhere since.

[18] Formally this is a bijection between the vertices of T and a family of digraphs.
[19] Since braces are 2-extendible.

What is known about this connection of digraphs and bipartite matching covered graphs is that the notion of butterfly minors in digraphs is compatible with bipartite matching minors.

Lemma 3.2.17 ([McC00]). Let B and H be bipartite matching covered graphs. Then H is a matching minor of G if and only if there exist perfect matchings $M \in \mathcal{M}(G)$ and $M' \in \mathcal{M}(H)$ such that $\mathcal{D}(H, M')$ is a butterfly minor of $\mathcal{D}(G, M)$.

From this, one can immediately derive directed analogues of Lemma 3.1.97 and Corollary 3.1.98.

Corollary 3.2.18. Let D be a strongly connected digraph, (X, Y) be non-trivial directed separation of order 1, and $D' \coloneqq D/(X \rightarrow v)$. Then D' is a butterfly minor of D.

Corollary 3.2.19. Let D be a strongly connected digraph. Then every dibrace of D is a butterfly minor of D.

So in some sense, butterfly minors in digraphs and matching minors in bipartite graphs with perfect matchings can be seen as the same thing. There appears, however, to be a major difference between the concepts. The butterfly minor relation, although it is a quasi-order of the class of all digraphs, is not a well-quasi order [Liu20]. Even when restricted to only strongly connected digraphs, or even strongly 2-connected digraphs, there exist infinite anti-chains, see Figure 3.17 for an example. For matching minors on bipartite matching covered graphs however, no such antichain is known. Moreover, there exist results (see Section 3.3 for more information) which are able to characterise non-trivial classes of bipartite matching covered graphs by excluding a finite number of graphs as matching minors. The major difference between matching minors and butterfly minors is that butterfly minors can be modelled more precisely as M-minors. Note that a digraph D is a butterfly minor of $\mathcal{D}(B, M)$ if and only if $\mathcal{S}(D)$ is an M-minor of B. Hence when working with the butterfly-minor relation, the perfect matching is fixed and one is only allowed to take M-conformal subgraphs. For matching minors on the other hand, the existence of a perfect matching is enough. Indeed, there is an interesting result by McCuaig regarding this phenomenon.

Lemma 3.2.20 ([McC04])**.** Let B be a bipartite graph with a perfect matching M and $K_{3,3}$ as a matching minor. Then B contains an odd möbius ladder, not necessarily of order 1, as an M-minor.

Lemma 3.2.20 is best possible. The existence of a $K_{3,3}$-matching minor does not guarantee that $K_{3,3}$ is an M-minor of B for every $M \in \mathcal{M}(B)$. In the original version of Lemma 3.2.20, McCuaig even formulated a possible reason for this. He proved that the bisubdivisions of the *rungs*, i.e. the edges of $\mathcal{M}_{4k+2}$ that do not belong to the Hamilton cycle used in its construction, are M-conformal paths in the M-conformal bisubdivision he finds. Let D and J be two digraphs. If J is not a butterfly minor of D, but $\mathcal{S}(J)$ is a matching minor of $\mathcal{S}(D)$, does D still contain some butterfly minor that witnesses this fact? Or in other words: How can we generalise Lemma 3.2.20 to arbitrary bipartite and matching covered graphs? The digraph J is a *proper butterfly minor* of the digraph D if J is a butterfly minor of D and $J \ncong D$. We say that D is *J-minimal* if $\mathcal{S}(D)$ contains $\mathcal{S}(J)$ as a matching minor, but for every proper butterfly minor D' of D, $\mathcal{S}(D')$ is $\mathcal{S}(J)$-free.

Definition 3.2.21 (Canonical Anti-Chain)**.** Let D be a digraph. The family

$$\mathfrak{A}(D) \coloneqq \big\{D' \mid D' \text{ is a } D\text{-minimal digraph}\big\}$$

is called the *canonical anti-chain based on D*.

Lemma 3.2.22 (X*)**.** Let D be a digraph. Then $\mathfrak{A}(D)$ is an anti-chain for the butterfly minor relation.

Proof. Suppose $\mathfrak{A}(D)$ is not an anti-chain for the butterfly minor relation. Then there must exist D_1 and D_2 in $\mathfrak{A}(D)$ such that D_1 is a butterfly minor of D_2. Indeed, D_1 must be a proper butterfly minor of D_2, as otherwise the two digraphs would be isomorphic. Since by definition $\mathcal{S}(D_1)$ contains $\mathcal{S}(D)$ as a matching minor, D_2 cannot be D-minimal, which contradicts $D_2 \in \mathfrak{A}(D)$. Hence $\mathfrak{A}(D)$ must be an anti-chain for the butterfly minor relation. $\square$

Whether $\mathfrak{A}(D)$ is finite or not appears to be unclear. There exist examples for which it is finite, the *Heawood-digraph* F_7 in Figure 3.16 for example. For other digraphs, like $\overleftrightarrow{C}_3$ on the other hand, $\mathfrak{A}(\overleftrightarrow{C}_3)$ is infinite, see Figure 3.17 for an illustration.

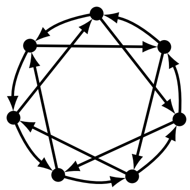


Figure 3.16.: The digraph F_7.

Definition 3.2.23 (Matching Equivalent)**.** Two digraphs D_1 and D_2 are said to be *matching equivalent* if $\mathcal{S}(D_1)$ and $\mathcal{S}(D_2)$ are isomorphic.

Given a digraph D, we denote by $\mathfrak{M}(D)$ the family of all digraphs that are matching equivalent to D.

Lemma 3.2.24 (X*)**.** Let D be a digraph. Then $\mathfrak{M}(D) \subseteq \mathfrak{A}(D)$.

Proof. First note that there cannot be a pair of distinct digraphs $D_1, D_2 \in \mathfrak{M}(D)$ such that D_1 is a proper butterfly minor of D_2. If this were the case, then $\mathcal{S}(D_1)$ would be a proper matching minor of $\mathcal{S}(D_2)$, but by definition $\mathcal{S}(D_1)$ and $\mathcal{S}(D_2)$ must be isomorphic. Hence $\mathfrak{M}(D)$ forms an anti-chain for the butterfly minor relation. Moreover, any proper butterfly minor D' of some digraph in $\mathfrak{M}(D)$ must satisfy that $\mathcal{S}(D')$ is $\mathcal{S}(D)$-free and thus the claim follows. $\square$

At last let us prove that excluding $\mathcal{S}(D)$ as a matching minor is the same as excluding every digraph in $\mathfrak{A}(D)$ as a butterfly minor.

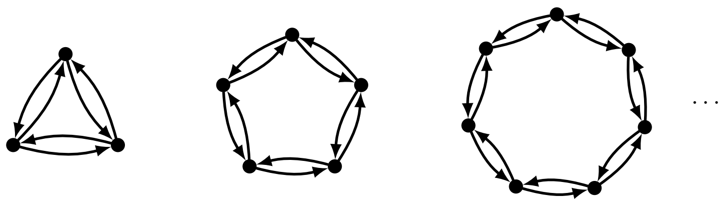


Figure 3.17.: The anti-chain $\mathfrak{A}(\overleftrightarrow{C}_3)$.

Lemma 3.2.25 (X*). Let H and D be digraphs. Then D contains a butterfly minor from $\mathfrak{A}(H)$ if and only if $\mathcal{S}(D)$ contains $\mathcal{S}(H)$ as a matching minor.

Proof. Suppose D contains a butterfly minor from $\mathfrak{A}(H)$, say J, Then by Lemma 3.2.17 $\mathcal{S}(D)$ must contain $\mathcal{S}(J)$ as a matching minor, and by definition of $\mathfrak{A}(H)$, $\mathcal{S}(J)$ must contain $\mathcal{S}(H)$ as a matching minor. For the reverse, assume $\mathcal{S}(D)$ contains $\mathcal{S}(H)$ as a matching minor. Then let J be an H-minimal butterfly minor of D. Clearly, J must exist and $J \in \mathfrak{A}(H)$. □

3.3. Pólya's Permanent Problem and Pfaffian Graphs

Counting the number of perfect matchings in a graph is known to be #P-complete[20], as it is polynomial time equivalent to computing the *permanent* of a square matrix [Val79]. This holds true even in the case of bipartite graphs. Pólya [Pól13] asked whether, given a square matrix A, it is possible to change the signs of some entries of A to obtain a new matrix B such that the determinant of B equals the permanent of A. The question, what properties the matrix A should have such that Pólya's approach can be used became known as *Pólya's Permanent Problem* and was later shown to have many different, but equivalent, formulations. We are particularly interested in three of these formulations. For an overview on even more equivalent formulations and proofs of all of these equivalences consult the outstanding work by McCuaig [McC04] or the survey by Thomas [Tho06].

Definition 3.3.1 (Pfaffian Orientation). Let G be a graph. A digraph $\vec{G}$ is called an *orientation* of G if the underlying undirected multigraph of $\vec{G}$ is simple and isomorphic to G.

An orientation $\vec{G}$ is *Pfaffian* if G has at least one perfect matching, and every conformal cycle C of G has an odd number of directed edges in $\vec{G}$ going in one direction around C and an odd number of directed edges going in the other direction. We say that the cycle C is *oddly oriented.*

[20] A complexity class containing the counting problems associated with decision problems in NP.

A graph that has a Pfaffian orientation is called *Pfaffian*. See Figure 3.18 for an illustration of a Pfaffian orientation.

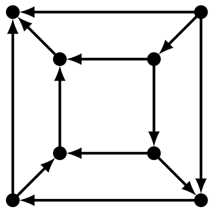


Figure 3.18.: A Pfaffian orientation of the cube.

A nice property of Pfaffian orientations is that is suffices to fix a single perfect matching M and only consider the M-conformal cycles to ensure some orientation $\vec{G}$ is indeed Pfaffian.

Theorem 3.3.2 ([LP09])**.** Let G be graph with a perfect matching M. Then an orientation $\vec{G}$ of G is Pfaffian if and only if every M-conformal cycle in G is oddly oriented.

Definition 3.3.3 (Non-Even Weighting of a Digraph)**.** Let D be a digraph. A weighting $w\colon E(D) \rightarrow \mathbb{F}_2$ is called *non-even* if for every directed cycle C in D we have

$$\sum_{e \in E(C)} w(e) \equiv 1 \ (\text{mod} 2).$$

A digraph is said to be *non-even* if it has a non-even weighting.

The property of being non-even was first discussed in a slightly different manner. Instead of asking for such a weighting, one could ask: Does a given digraph D have the property that any possible subdivision of D contains a directed cycle of even length? Digraphs with this property were called *even*, and one can check that a digraph is not even (or non-even) if and only if it has a non-even weighting. The last important contribution is the theorem of Little [Lit75], which characterises bipartite Pfaffian graphs by excluding the single graph $K_{3,3}$ as a matching minor. Please note that Pólya's Permanent Problem is indeed equivalent to recognising any of the graph classes below.

Theorem 3.3.4 ([Lit75, ST87, McC04])**.** Let B be a bipartite graph with a perfect matching M. The following statements are equivalent.

i) B is Pfaffian.
ii) B does not contain $K_{3,3}$ as a matching minor.
iii) $\mathcal{D}(B, M)$ is non-even.
iv) $\mathcal{D}(B, M)$ does not contain an odd bicycle as a butterfly minor.

Here the relation between forbidding a single matching minor, namely $K_{3,3}$ in the bipartite setting, and forbidding the infinite antichain $\mathfrak{A}(\mathcal{D}(K_{3,3}, M))$ in the directed setting comes up as one of the challenges for the proof of Theorem 3.3.4. Since $K_{3,3}$ is a brace, one can apply Lemma 3.1.99 and obtain the following theorem in the bipartite case from the one above. The general proof, however, is much more involved.

Theorem 3.3.5 ([VY89])**.** Let G be a matching covered graph, $X \subseteq V(G)$ be a set that induces a non-trivial tight cut, and G_1, G_2 be the two graphs obtained by the tight cut contractions of X and $\overline{X}$ in G respectively. Then G is Pfaffian if and only if G_1 and G_2 are Pfaffian.

For the case of bipartite graphs this means one may reduce the problem of describing general bipartite Pfaffian graphs to finding a good characterisation for Pfaffian braces. The term 'good' plays an important role here. With 'good', we mean a characterisation that also provides an algorithm which can recognise a Pfaffian brace in polynomial time. Sadly, none of the characterisations from Theorem 3.3.4 yields such an algorithm, at least not immediately. Eventually, the discovery of a concept akin to clique sums but more suited for the context of graphs with perfect matchings brought with it the first polynomial time algorithm to actually solve the Pfaffian recognition problem for braces.

Definition 3.3.6 (4-Cycle Sum and Trisum)**.** For every $i \in \{1, 2, 3\}$ let G_i be a bipartite graph with a perfect matching and C_i be a conformal cycle of length four in G_i. A 4-*cycle-sum* of G_1 and G_2 at C_1 and C_2 is a graph G' obtained by identifying C_1 and C_2 into the cycle C' and possibly forgetting some of its edges.

If a bipartite graph G'' is a 4-cycle-sum of G' and some bipartite and matching covered graph G_3 at C' and C_3, then G'' is called a *trisum* of G_1, G_2 and G_3.

The significance of the trisum-operation is that, besides one small exception, it provides a way to combine braces into larger braces.

Let T_{10} be the 4-cycle-sum of three $K_{3,3}$ at a 4-cycle C such that no edge of C is in $E(T_{10})$.

Lemma 3.3.7 ([McC04]). Let $k \geq 3$, and let $B, B_1, \dots, B_k$ be bipartite graphs such that B is not isomorphic to T_{10}. Suppose B is a 4-cycle-sum of $B_1, \dots, B_n$ at the 4-cycle C, then G is a brace if and only if $B_1, \dots, B_n$ are braces.

The *Heawood graph* is the bipartite graph associated with the incidence matrix of the Fano plane, see Figure 3.19 for an illustration. Including one exception in form of the Heawood graph, the structure theorem for Pfaffian braces bears a striking resemblance to Wagner's characterisation of K_5 minor free graphs [Wag37].

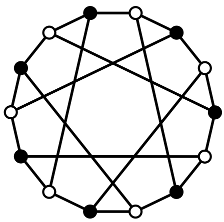


Figure 3.19.: The Heawood graph H_{14}.

Theorem 3.3.8 ([McC04, RST99]). A brace is Pfaffian if and only if it either is isomorphic to the Heawood graph or it can be obtained from planar braces by repeated application of the trisum operation.

As an immediate corollary, one obtains the following bound on the number of edges in a Pfaffian brace.

Corollary 3.3.9 ([Tho06]). If B is a Pfaffian brace, then $|E(B)| \leq 2|V(B)| - 4$.

Corollary 3.3.10 ([McC04, RST99]). There exists an algorithm that decides, given a brace B as input, whether B is Pfaffian in time $\mathcal{O}(|V(B)|^3)$.

Non-Bipartite Pfaffian Graphs and Non-Even Bidirected Graphs

With Theorem 3.3.8 the Pfaffian recognition problem was solved for the case of bipartite graphs and thus a solution to Pólya's Permanent Problem has been found. However, this does not conclude the story. Pfaffian orientations are defined for all graphs with perfect matchings, not just for bipartite ones, and even on non-bipartite graphs a Pfaffian orientation may be used to compute the number of perfect matchings [Tho06].

In general the property of being Pfaffian seems at least somewhat connected to planarity, as the following classic result by Kasteleyn illustrates.

Theorem 3.3.11 ([Kas67])**.** Every planar graph with a perfect matching is Pfaffian.

Beyond this, some characterisations exist, but only for restricted classes of non-bipartite graphs (see for example [FL01, DCLM12]). This work is mainly focused on bipartite graphs and the interaction of graphs with perfect matchings and bidirected graphs and thus we do not go into too much detail here. Instead, we propose a possible generalisation of the notion of non-even to the more general setting of bidirected graphs.

Definition 3.3.12 (Consistent Weighting)**.** Let (G, σ) be a bidirected graph. A weighting $w\colon \mathsf{E}(G) \rightarrow [0, 1]$ of the half-edges of (G, σ) is called *consistent* if for every edge $e = uv \in E(G)$, we have $w(u, e) = w(v, e)$ if e is a normal edge, and $w(u, e) \neq w(v, e)$ if e is introverted or extroverted.

Definition 3.3.13 (Non-Even Bidirected Graph)**.** A bidirected graph (G, σ) is called *even* if, for every consistent weighting w of the half-edges of (G, σ), there exists a directed cycle $C = (v_0, e_0, v_1, e_1, \dots, e_{\ell-1}, v_\ell, e_\ell, v_0)$ in (G, σ) such that

$$\sum_{i=0}^{\ell} w(v_i, e_i) \equiv 0 \ (\mathrm{mod} 2).$$

Equivalently, a bidirected graph (G, σ) is *non-even* if there exists a consistent weighting w of the half-edges of (G, σ) such that for every directed cycle $C = (v_0, e_0, v_1, e_1, \dots, e_{\ell-1}, v_\ell, e_\ell, v_0)$ in (G, σ) we have

$$\sum_{i=0}^{\ell} w(v_i, e_i) \equiv 1 \ (\mathrm{mod} 2).$$

Such a weighting w is called *non-even*.

It should be noted that the parity of $\sum_{i=0}^{\ell} w(v_i, e_i)$ is independent of the choice of traversal around the directed cycle $C = (v_0, e_0, v_1, e_1, \dots, e_{\ell-1}, v_\ell, e_\ell, v_0)$, and therefore, the above is a valid definition. From now on, we will call $w_2(C) \coloneqq \sum_{i=0}^{\ell} w(v_i, e_i) \bmod 2 \in [0, 1]$ the *parity-weight* of the directed cycle C.

Lemma 3.3.14 (F^*)**.** Let (G, σ) be a bidirected graph with a consistent weighting w of its half-edges. For any directed cycle $C = (v_0, e_0, v_1, e_1, \dots, e_{\ell-1}, v_\ell, e_\ell, v_0)$ we have

$$\sum_{i=0}^{\ell} w(v_i, e_i) \equiv \sum_{i=0}^{\ell} w(v_i, e_{i \bmod (\ell+1)})(\mathrm{mod} 2).$$

Proof. Let $I_1 \subseteq [0, \ell]$ be the indices of the normal edges $e_i \in E(C)$ and let $I_2 \coloneqq [0, \ell] \setminus I_1$. Clearly, we have $\sum_{i \in I_1} w(v_i, e_i) = \sum_{i \in I_1} w(v_i, e_{i \bmod (\ell+1)})$, due to the consistency of w. Therefore it remains to be shown that $\sum_{i \in I_2} w(v_i, e_i) \equiv \sum_{i \in I_2} w(v_i, e_{i \bmod (\ell+1)})(\mathrm{mod} 2)$. We observe that in C, due to it being directed, the number of half-edges with the sign $+$ must equal the number of half-edges with the sign $-$. This leads us to the conclusion that there exists a bijection between the extroverted edges e_i, with $i \in I_2$, such that $\sigma(v_i, e_i) = \alpha$ and the introverted edges e_j, with $j \in I_2$ and $i \neq j$, such that $\sigma(v_i, e_i) = -\alpha$. Therefore $|I_2|$ is even. The consistency of w thus lets us conclude that $\sum_{i \in I_2} w(v_i, e_i) \equiv 1(\mathrm{mod} 2)$ if and only if $\sum_{i \in I_2} w(v_i, e_{i \bmod (\ell+1)}) \equiv 1(\mathrm{mod} 2)$. $\square$

Theorem 3.3.15 (F^*)**.** Let H be a graph with a perfect matching M and an M-signing ζ and let $(G, \sigma) \coloneqq \mathcal{D}_{\pm}(H, M, \zeta)$ be its M-bidirection. Then H is Pfaffian if and only if (G, σ) is non-even.

Proof. By Theorem 3.3.2, it suffices to show equivalence of the following two statements:

(A) There is an orientation $\vec{H}$ of H such that every M-conformal cycle in H is oddly oriented.

(B) There is a non-even weighting w of the half-edges of (G, σ).

Let us assume (A) holds and suppose we are given an orientation $\vec{H}$ of G such that every alternating cycle is oddly oriented. Since reorienting all edges incident to one vertex in an orientation of H maintains the property that every M-conformal cycle is oddly oriented, we may assume without loss of generality that for every matching edge $v_1v_2 \in M$, we have $(v_1, v_2) \in E(\vec{H})$ if and only if $\zeta(v_1) = +$ and $\zeta(v_2) = -$. We now define a weighting w of $\mathsf{E}(G)$ as follows: Given an edge $e \in E(G)$ incident to a vertex $v \in V(G)$, let $\vec{e} \in E(\vec{H})$ be the corresponding oriented edge in $\vec{H}$ and $m \in M$ the matching-edge corresponding to v. We now set $w(v, e) := 1$ if and only if either $\vec{e}$ has its tail on the $+$-vertex of m or its head on the $-$-vertex of m. In every other case we set $w(v, e) := 0$. From the relation of ζ and σ it directly follows that w is a consistent weighting of $\mathsf{E}(G)$. Now let $C = (v_0, e_0, v_1, e_1, \ldots, e_{\ell-1}, v_\ell, e_\ell, v_0)$ be a directed cycle in (G, σ). Then by assumption, the M-conformal cycle in $\vec{H}$ corresponding to C is oddly oriented. For each $i \in [0, \ell]$ let $m_i = a_ib_i$ be the matching-edge in M corresponding to v_i such that $\zeta(a_i) = +$, and $\zeta(b_i) = -$. For every edge $e_i = v_iv_{i+1 \bmod (\ell+1)}$ with $i \in [0, \ell]$ on C let $\vec{e}_i \in E(\vec{H})$ be the corresponding non-matching edge in $\vec{H}$. Let us now count the contribution of the subpath m_ia_{i+1} of the M-conformal cycle in H to the number of forward-edges in the orientation $\vec{H}$ when traversing the matching cycle in the cyclical order such that we first visit m_0, m_1, …, m_ℓ, m_0. A case distinction shows that this contribution is 1 if $w(v_i, e_i) = 1$ and 0 or 2 otherwise. This shows that the parity of $\sum_{i=0}^{\ell} w(v_i, e_i)$ equals the parity of the number of forward-edges on the M-conformal matching cycle corresponding to C in $\vec{H}$. Hence this number has to be odd, and therefore w is an odd weighting of the half-edges of (G, σ) certifying that (G, σ) is non-even, and so (B) has been proven.

For the reverse direction, let us assume we are given an odd weighting w of the half-edges of (G, σ). We then define an orientation $\vec{H}$ of H as follows: If $m = ab \in M$ such that $\zeta(a) = +$ and $\zeta(b) = -$, then we let $(a, b) \in E(\vec{H})$. For every non-matching edge $xy \in E(H) \setminus M$ with endpoints in the matching-edges $m_1 = a_1b_1$ and $m_2 = a_2b_2$ such that $\zeta(a_1) = \zeta(a_2) = +, \zeta(b_1) = \zeta(b_2) = -$, we orient xy from x to y if and only if either $w(u, uv) = 1$ and $x = a_1$, or $w(u, uv) = 0$ and $x = a_2$. Here, u and v denote the vertices in G obtained by contracting m_1 and m_2. For every M-conformal cycle in H using matching-edges $m_0, m_1, \ldots, m_\ell$, if $(v_0, e_0, v_1, e_1, v_2, \ldots, e_{\ell-1}, v_\ell, e_\ell, v_0)$ is the corresponding directed cycle in (G, σ) such that v_i is the obtained from the contraction of m_i, the number of forward-edges on this M-conformal cycle when traversing the matching-edges in circular order $m_0, m_1, m_2, \ldots, m_\ell, m_0$ is congruent to the value of $\sum_{i=0}^{\ell} w(v_i, e_i)$ modulo 2, and this has to be odd since w is non-even. Hence every M-conformal cycle in H has an odd number of forward- and backward edges according to the orientation $\overrightarrow{H}$, and (B) is satisfied. □

Hence a valid strategy to solve the problem of recognising Pfaffian graphs in polynomial time, if this is possible, could be to focus on the slightly less complex bidirected graphs. Note that bidirected graphs are a common generalisation of both directed graphs and so called *signed graphs*, which are undirected graphs with a signing of their edges. With bidirected graphs being such a general concept, they are by no means easy to understand. The advantage, at least in some points, over general graphs with perfect matchings, is that we focus on a single perfect matching instead of possibly all of them.

3.4. Differences between Bipartite and Non-Bipartite Graphs

There seems to exist a fundamental difference between bipartite graphs with perfect matchings and non-bipartite graphs with perfect matchings. This difference has become apparent throughout the previous sections and usually manifests in the fact that one needs much more notation and more delicate definitions to describe properties of matchings in non-bipartite graphs. But how can we quantify these differences and is there a reason why proving anything for the non-bipartite case appears to be much more difficult?

i) The existence of a perfect matching in a bipartite graph can be characterised by Hall's Theorem, while general graphs need the more complicated Theorem 3.1.6 by Tutte.
ii) Inspecting Tutte's Theorem closely, one realises that the concept of (maximal) barriers can, in some sense, be seen as a generalisation of the two colour classes in bipartite graphs. While maximal barriers still partition the graph as seen in Theorem 3.1.13, there is no longer a bound on their total number and their interaction can be much more complicated.

iii) The increased complexity of the interaction between maximal barriers can be seen in the difference between the Dulmage-Mendelsohn decomposition, which induces a partial order on the elementary components of a bipartite graph with a perfect matching, and the generalised Dulmage-Mendelsohn decomposition, which makes it much harder to obtain a relation between elementary components themselves.
iv) Being matching covered can be related to the concept of matching connectivity. This connectivity parameter, however, has an almost unnatural (or at least very technical) part in its definition, which is completely unnecessary for bipartite graphs as one can see in Theorem 3.2.8 and the definition of strong connectivity.
v) The problem of matching connectivity from above ties into the hurdles of ear-decompositions. Here there is a strict line that has to be crossed to go from bipartite to non-bipartite, and once it is crossed it cannot be reversed.
vi) For bipartite graphs, especially braces, there exist many equivalent concepts as seen in Theorem 3.1.69 which all boil down to the idea of k-extendibility. Indeed, 2-extendibility is exactly the same as the absence of non-trivial tight cuts, and this concept can even be generalised, ultimately giving rise to a matching version of Menger's Theorem. In non-bipartite graphs however, one needs bicriticality **and** connectivity to get rid of non-trivial tight cuts, k-extendibility and higher order bicriticality do not interact tightly, and it is not clear at all whether there exists a good generalisation of tight cuts.

Overall, best exemplified by ear-decompositions and the difference between braces and bricks, there appears to be a strict dichotomy between the nicely behaved bipartite graphs and the much more complicated non-bipartite graphs. Another good example of this dichotomy is the case of Pfaffian graphs. The problem of recognising a bipartite Pfaffian graph is equivalent to many different problems, even reaching into linear algebra [McC04], while for the general problem not many equivalent versions are known.

Indeed, there is some evidence that even matching minors, which are one of the most powerful concepts available for matching theory so far, might not yield a good way to deal with general Pfaffian graphs. Furthermore, there even exists an infinite anti-chain for the matching minor relation which consists entirely of non-bipartite non-Pfaffian graphs [NT08]. For bipartite graphs it is not known whether there exists any anti-chain of matching covered graphs for the matching minor relation at all.

Especially the problem of infinite matching minor anti-chains might be explainable by looking at the different behaviour of internally M-conformal paths in non-bipartite graphs. There exist two useful tools regarding these paths in the bipartite setting, namely Lemma 3.1.42, and Corollary 3.1.98. While for the first, a generalisation to non-bipartite graphs is known, but it is much more complicated and has several possible outcomes, the second one does not have a non-bipartite counterpart at all.

From what we have seen in Section 3.2, many of the differences between bipartite and non-bipartite graphs carry over to differences between digraphs and non-digraphic bidirected graphs. Using the setting of bidirected graphs, we provide some intuition to why dealing with internally M-conformal paths, and therefore dealing with matching minors, in non-bipartite graphs might be so much more difficult. To do so, we transform our bidirected graphs even further.

Definition 3.4.1 (2-Edge Coloured Graph)**.** A tuple (G, χ) where G is a graph and $\chi\colon E(G) \rightarrow \{+, -\}$ is a function that assigns a *colour*[21] to every edge of G, is called a 2-*edge coloured graph.* A path P in a 2-edge coloured graph (G, χ) is *alternating* if $\{e \in E(P) \mid \chi(e) = +\}$ and $\{e \in E(P) \mid \chi(e) = -\}$ form two matchings of P.

Notice that for every introverted or extroverted edge in a bidirected graph (G, σ), one can replace the signing of its two half edges by a single sign, or colour, of the whole edge. If $e \in E(G)$ is a normal edge however, one can just subdivide it, replacing it with two new edges, where one represents the $+$-half edge of e and the other one represents the $-$-half edge of e. By doing so, all directed paths and cycles are preserved, but now we obtained a bidirected graph without normal edges. Hence the above procedure can be applied to all edges.

Definition 3.4.2. Let (G, σ) be a bidirected graph. We denote by (G_S, σ_S) the bidirected graph obtained from (G, σ) by replacing every

[21] Here we use signs, but this is only for our convenience, in general any two distinguishable objects work just fine.

edge $e = uv \in E(G)$ with $\sigma(u,e) = +, \sigma(v,e) = -$ by a directed path (u, ux, x, xv, v), where we set $\sigma_S(u,u) \coloneqq +$, $\sigma_S(x, ux) \coloneqq +$, $\sigma_S(x, xv) \coloneqq -$, and $\sigma_S(v, xv) \coloneqq -$, while setting $\sigma_S(z,f) \coloneqq \sigma(z,f)$ for all other half-edges (z,f) of G.

By treating σ_S as a singing, or colouring, of the edges instead of the half edges as described above, we obtain a 2-edge colour graph (G_S, σ_S) for every bidirected graph (G, σ) there is. Moreover, as discussed before, there is a correspondence between the alternating paths and cycles of (G_S, σ_S), and the directed paths and cycles of (G, σ).

Definition 3.4.3 (Edge Coloured k-Disjoint Alternating s-t-Paths Problem)**.** Let $k \in \mathbb{N}$ be some positive integer. The *edge coloured k-disjoint alternating paths problem* is the decision problem, given a 2-edge coloured graph (G, χ) and vertices $s, t \in V(G)$, whether there exist k pairwise internally disjoint alternating paths in (G, χ) with endpoints s and t.

Theorem 3.4.4 ([ADF$^+$08])**.** The edge coloured 2-disjoint alternating s-t-paths problem is NP-complete.

Using our construction from above, one can reach the following conclusion.

Definition 3.4.5 (Bidirected k-Disjoint s-t-Paths Problem)**.** Let $k \in \mathbb{N}$ be some positive integer. The *bidirected k-disjoint s-t-paths problem* is the decision problem, given a bidirected graph (G, σ) and vertices $s, t \in V(G)$, whether there exist k pairwise internally disjoint directed paths in (G, σ) with endpoints s and t.

Corollary 3.4.6 (F*)**.** The bidirected 2-disjoint s-t-paths problem is NP-complete.

One can see Theorem 3.4.4 as the continuation of a trend. In undirected graphs, finding the maximum number of pairwise internally disjoint paths between two vertices is polynomial time solvable, a fact that follows from Menger's Theorem. Additionally, even deciding the k-disjoint paths problem is solvable in polynomial time if k is fixed. On digraphs, Menger's Theorem does still hold, but the directed 2-disjoint paths problem is already NP-complete as seen in Theorem 2.3.27. With Corollary 3.4.6 and Theorem 2.3.27 it follows that both problems are NP-complete for general bidirected graphs. So with the next step of generalisation from directed to bidirected graphs, one even loses Menger's Theorem. Indeed, a tight result like Menger's Theorem seems to be unlikely for directed paths in bidirected graphs. What still could exist is some kind of approximative version of Menger's Theorem, either in the form of Erdős-Pósa-type results, still using disjoint paths but with a possibly worse function bounding the size of a hitting set, or in form of a relaxed version of disjointness like a bound on the number of paths any vertex is allowed to appear in.

The (possible) lack of a Menger-type theorem for bidirected graphs poses a huge problem for extensions of digraph structure theory to bidirected graphs, as especially the proof of the Directed Grid Theorem relies heavily on Menger's Theorem.

3.5. A List of Open Questions

We have touched upon many different aspects of structural matching theory in the previous sections. The area is relatively new, and many questions are still wide open. In this last section of the chapter, we collect some of these open questions, for some of which this thesis makes an effort towards a possible solution.

We start with possibly the biggest open question in matching theory:

Question 3.5.1 (The Pfaffian Recognition Problem)**.** Is there a polynomial time algorithm that decides for any given graph G whether it is Pfaffian?

For bipartite graphs, the Pfaffian recognition problem is known to be in P, and by Theorem 3.3.4 the problem of deciding whether a bipartite graph with a perfect matching has $K_{3,3}$ as a matching minor is therefore also in P. This, however, is the only non-trivial[22] graph for which the complexity of the matching minor question is known.

Question 3.5.2 ((Bipartite) Matching Minor Recognition Problem)**.** Let H be a (bipartite) graph with a perfect matching. Is there a polynomial time algorithm that decides, given a fixed (bipartite) graph G with a perfect matching, whether G has H as a matching minor?

[22] Short cycles and paths can be detected easily.

The reason why this question might make sense only in the setting of bipartite graphs is Corollary 3.4.6. So one could conjecture that the matching minor recognition problem for non-bipartite graphs is NP-hard. In undirected graphs, testing for a (rooted) minor and solving the k-disjoint paths problem go hand in hand, so it might make sense to consider a matching version. Here we restrict our attention to the bipartite case, but the general problem is also still wide open.

Definition 3.5.3 (The Bipartite k-Disjoint Alternating Paths Problem)**.** Let B be a bipartite graph with a perfect matching, $k \in \mathbb{N}$ a positive integer, and $s_1, \dots, s_t \in V_1$, $t_1, \dots, t_k \in V_2$. The question whether there exists a perfect matching M of B and internally M-conformal paths $P_1, \dots, P_k$ in B which are pairwise internally disjoint and for all $i \in [1, k]$, P_i has endpoints s_i and t_i is called the *bipartite k-disjoint alternating paths problem* (k-DAPP).

Question 3.5.4. Let $k \in \mathbb{N}$ be a positive integer. Is there a polynomial time algorithm that solves k-DAPP on bipartite graphs?

Please note that in case we were to ask for a fixed perfect matching M of B, whether there are internally M-conformal paths $P_1, \dots, P_k$ as required in k-DAPP, the problem is equivalent to the directed k-disjoint paths problem and therefore NP-complete by Theorem 2.3.27. By also being able to change the perfect matching however, the complexity may be different. Besides testing for a specific matching minor in a bipartite graph, one could also ask for another generalisation of Theorem 3.3.4, namely the structure of bipartite graphs that exclude $K_{t,t}$ for any t as a matching minor.

Question 3.5.5. What is the structure of bipartite matching covered graphs that exclude $K_{t,t}$ as a matching minor for some fixed $t \in \mathbb{N}$?

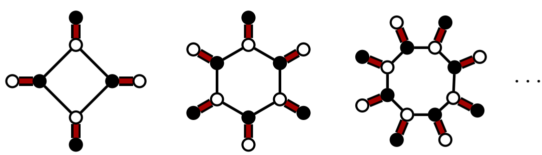


Figure 3.20.: An infinite anti-chain for the matching minor relation of bipartite graphs with prefect matchings.

It is not hard to construct examples of infinite anti-chains of matching minors in bipartite graphs. These anti-chains, however, also have the property that every single member has a unique perfect matching as illustrated in Figure 3.20.

Moreover, all known classes of bipartite graphs with perfect matchings which can be characterised by forbidden matching minors can also be characterised by a finite family of those. The forbidden matching minors usually are braces and thus we reach our final open question.

Question 3.5.6 (Brace Anti-Chains)**.** Is there an infinite anti-chain for the matching minor relation consisting only of braces?

Part II.

A Matching Theoretic Minors Project

Chapter 4.

The Two Paths Theorem

The Two Paths Theorem is an important corner stone of the Graph Minors Project as it represents the link between topological graph theory and the exclusion of a minor, and solving the disjoint paths problem. As a first step to finding, at least partial, answers to the questions asked in Section 3.5 it is therefore a good idea to ask for a matching theoretic analogue of the Two Paths Theorem and whether such an analogue can be used to obtain a polynomial time algorithm. For such a theorem, we first need to decide whether we are interested in 'crosses' over any cycle, of whether conformal cycles suffice. There is a short argument for the later.

Lemma 4.0.1 ([McC04])**.** Let B be a bipartite and planar matching covered graph, then every facial cycle of B is conformal.

If we are interested in a statement like the one of Theorem 2.2.6, then we would expect the cycle which does not have a cross to bound a face in some kind of reduction of the original graph. If this is the case, then Lemma 4.0.1 implies that our cycle is conformal. Indeed, if our cycle is conformal in the end, our reductions should not have changed this and thus it should have been conformal even before applying any sort of reduction. Hence it makes sense to only consider 'crosses' over conformal cycles.

Next we need a notion of reduction that is appropriate for the setting of bipartite graphs with perfect matchings. We have already seen the use of the 4-cycle sum in the characterisation of bipartite Pfaffian graphs in Theorem 3.3.4. This sum operation appears to replicate small order clique sums as it preserves the existence of a matching minor which is a brace without introducing new and more complicated ones. At least to a certain extend. In light of auxiliary results like Lemma 3.3.7 it also seems natural to only consider ‘crosses’ over conformal cycles in braces.

The last piece we need is the definition of a ’cross’ itself.

Let B be a bipartite graph with a perfect matching M and let C be a conformal cycle in B. Let P be an M-alternating path in B. If P is internally M-conformal we say that P is of *type 1*, if M is a perfect matching of P, we say P is of *type 2*, and otherwise exactly one of the end-edges of P must belong to M, in this case P is of *type 3*.

Definition 4.0.2 (Matching Cross)**.** Let B be a bipartite graph with a perfect matching M and let C be a conformal cycle in B. The cycle C is said to have a *matching cross* if there exists a perfect matching M and vertices s_1, s_2, t_1, t_2, called the *pegs* of the cross, that appear on C in the order listed such that there exist paths P_1 and P_2 satisfying the following properties:

- for each $i \in [1, 2]$, P_i has endpoints s_i and t_i and is otherwise disjoint from C,
- P_1 and P_2 are M-alternating, and
- P_1 and P_2 are vertex disjoint.

A matching cross over a conformal cycle C is said to be *strong* if it also meets the following requirements:

- $|V_1 \cap \{s_1, s_2, t_1, t_2\}| = |V_2 \cap \{s_1, s_2, t_1, t_2\}|$, and
- P_1 and P_2 are of the same type

In case C has a matching cross with paths P_1 and P_2 such that $C + P_1 + P_2$ is a conformal subgraph of B we say that C has a *conformal cross*.

Please note that any path of type 1 or 2 must be of odd length and every path of type 3 is of even length. Hence the two paths of a conformal cross are either both of type 1 or 2, or both of type 3. Moreover, if exactly one of the two paths is of type 2, then this path together with one of the two subpaths of C connecting its endpoints forms an alternating cycle. By switching the perfect matching along this cycle we arrive at a perfect matching for which both paths are of the same type. Therefore, if we are faced with a conformal cross we may assume this cross to be strong.

Definition 4.0.3. Let B be a brace and C a conformal cycle in B. A brace H is called a *first order C-reduction* of B, if there exist braces H,

B_1, B_2, and a 4-cycle K such that B is a trisum of H, B_1, and B_2 at K, there exists $i \in [1, 2]$ such that $V(C) \cap V(K) \subseteq V_i$, and $C \subseteq H$. A brace H_ℓ is called a *C-reduction* of B if there is a sequence of braces $H_1, \ldots, H_\ell$ such that $B = H_1$ and H_{i+1} is a first order C-reduction of H_i for all $i \in [1, \ell - 1]$.

We can now state the main result of this chapter, a Two Paths Theorem for braces.

Theorem 4.0.4 (E*)**.** Let B be a brace and C a conformal cycle in B, then C has **no** matching cross in B if and only if B is Pfaffian and there exists a planar C-reduction of B in which C bounds a face.

Some of the intermediate results that lead to Theorem 4.0.4 can be used to solve a slightly altered version of 2-DAPP.

Definition 4.0.5 (The (Bipartite) k-Matching Linkage Problem)**.** Let B be a bipartite graph with a perfect matching, $k \in \mathbb{N}$ a positive integer, and let $s_1, \ldots, s_k \in V_1$ as well as $t_1, \ldots, t_k \in V_2$ be $2k$ pairwise distinct vertices in B. A *matching linkage* in B for the *terminals* $s_1, \ldots, s_k, t_1, \ldots, t_k$ is a perfect matching M and a collection $P_1, \ldots, P_k$ of pairwise disjoint and internally M-conformal paths such that P_i has endpoints s_i and t_i for each $i \in [1, k]$. The *(bipartite) k-Matching Linkage Problem* (k-MLP) is the question whether, given tuples $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_k)$ of vertices as above, there exists a matching linkage for the terminals $s_1, \ldots, s_k, t_1, \ldots, t_k$ in B.

Please note that one can always turn an instance of k-DAPP into polynomially many instances of k-MLP by replacing vertices which appear several times as a terminal with a selection of their distance-2-neighbours[1]. Hence k-DAPP and k-MLP are polynomial time equivalent, the difference is that k-MLP can be easier to work with since the possibility of several terminals being the same vertex does not have to be taken into account.

Theorem 4.0.6 (E*)**.** Let B be a bipartite graph with a perfect matching, and let $s_1, s_2 \in V_1$ as well as $t_1, t_2 \in V_2$ be four distinct vertices. There exists an algorithm that decides 2-MLP for the terminals s_1, s_2, t_1, t_2 in time $\mathcal{O}(|V(B)|^5)$.

[1] If G is a graph and $v \in V(G)$, then a *distance 2-neighbour* of v is a vertex from $\mathrm{N}_G(\mathrm{N}_G(v)) \setminus (\mathrm{N}_G(v) \cup \{v\})$.

Recall that, in case we slightly alter the problem and instead ask for a solution of the 2-MLP for a fixed perfect matching, we obtain a problem equivalent to the directed 2-disjoint paths problem. This means that the additional freedom of being able to choose our perfect matching instead to being bound to a specific one makes the difference[2] between containment in P and being NP-hard.

Organisation of the Chapter and Proof of Theorem 4.0.4 Our proof of Theorem 4.0.4 can be broken down into two essential pieces. The first is Proposition 4.0.7 which characterises the existence of matching crosses over conformal cycles in Pfaffian braces. Section 4.3 is dedicated to its proof, but the planar case, which is handled in Section 4.1, plays a major role.

Proposition 4.0.7 (E*)**.** *Let B be a Pfaffian brace and C a conformal cycle in B. Then there is* **no** *matching cross over C in B if and only if there exists a planar C-reduction of B in which C bounds a face.*

The second part is Proposition 4.0.8, which guarantees conformal crosses over 4-cycles in non-Pfaffian braces, this proposition is proved in Section 4.4.

Proposition 4.0.8 (E*)**.** *Let B be a non-Pfaffian brace and C a 4-cycle in B, then there exists a conformal bisubdivision of $K_{3,3}$ with C as a subgraph.*

In Section 4.2, we establish some preliminary results which are needed for both the Pfaffian and the non-Pfaffian case, especially regarding paths and matching crosses over 4-cycles. An important role, in order to bridge between the existence of matching crosses and Proposition 4.0.8 is held by the following lemma.

Lemma 4.0.9 (E*)**.** *Let B be a brace and C a 4-cycle in B, then there is a conformal cross over C in B if and only if C is contained in a conformal bisubdivision of $K_{3,3}$.*

As a last piece, we need to be able to make use of strong matching crosses over certain conformal cycles to find (not necessarily strong)

[2] That is, under the assumption $\mathsf{P} \neq \mathsf{NP}$.

matching crosses more easily. This is especially useful when paired with Proposition 4.0.8 but also finds applications in other places within this chapter.

Lemma 4.0.10 (E*)**.** Let B be a brace and C a conformal cycle. If C' is a conformal cycle such that there exists an edge e with both endpoints on C with $C' \subseteq C + e$, and there is a perfect matching M of B and M-alternating paths L and R that form a matching cross over C', then there exists a matching cross over C in B that does not use e.

Proof. Since $C' \subseteq C + e$, $C' - e$ forms a subpath of C, hence the order of the endpoints of L and R on C is the same as the order of these vertices on C'. So in case L and R are internally disjoint from C, they immediately form a matching cross over C as well. Suppose exactly one of these paths, say L, intersects C. Let v_R and w_R be the endpoints of R and v_1, w_1 be the endpoints of L. Then let x_1 be the last vertex of L on C we encounter when traversing along L starting with v_1 such that x_1 is separated from w_1 on C by v_2 and w_2. Next let x_2 be the first vertex of L that lies on C we encounter after x_1. Then x_2 must belong to the same component of $C - v_2 - w_2$ as w_1 and thus x_1 and x_2 are separated on C by v_2 and w_2. Since L and R are disjoint and M-alternating, so are x_1Lx_2 and R and thus we have found a matching cross over C. So now assume that also R intersects C. In this case, let y_1 be the last vertex we encounter when traversing along R starting in v_2 such that y_1 and w_2 belong to different components of $C - x_1 - x_2$. Then let y_2 be the first vertex of C we encounter on R after y_1. By choice of y_1, y_1 and y_2 must belong to different components of $C - x_1 - x_2$ and y_1Ry_2 is internally disjoint from C. Hence x_1Lx_2 and y_1Ry_2 form a matching cross over C. Moreover, since e is not contained in either L or R, we have found a matching cross over C which does not contain e. ☐

The four results above combined yield a short proof of Theorem 4.0.4.

Proof of Theorem 4.0.4. Let B be a brace and C a conformal cycle in B. Suppose B is Pfaffian, then Proposition 4.0.7 immediately yields both directions of our claim. Hence we may assume B to be non-Pfaffian. If C is a 4-cycle, then Proposition 4.0.8 guarantees the existence of a conformal cross over C in B. So we may assume C to have length at least six. Let P be a subpath of C of length three, so P consists of exactly four vertices, two of each colour class. Let $a \in V_1$ and $b \in V_2$ be the endpoints of P. If the edge ab does not exist in B we introduce it, please note that introducing an edge does not change the status of B being a brace, nor can $B + ab$ be Pfaffian if B is not. Hence $C' := P + ab$ is a 4-cycle in $B + ab$ such that $C' \subseteq C + ab$. By Proposition 4.0.8 there is a conformal bisubdivision L of $K_{3,3}$ in $B + ab$ that has C' as a subgraph. By Lemma 4.0.9 there is a conformal cross over C'. Please note that, with L being a bisubdivision of $K_{3,3}$, we may choose a perfect matching M of B such that L is M-conformal and $ab \notin M$. An application of Lemma 4.0.10 now yields a matching cross over C in $B + ab$ that does not contain ab. With $ab \notin M$ this means that there is a matching cross over C in B. □

With Theorem 4.0.4, we have a tool that can help us to obtain an algorithmic solution of 2-MLP. In fact, it is Proposition 4.0.8, which provides the important insight. Theorem 2.2.6 can be used to solve the 2-Linkage Problem by introducing a small local construction. In Section 4.5, we describe how a similar construction can be used for the 2-MLP.

4.1. Matching Crosses in Planar Braces

In this section we establish the base case of Theorem 4.0.4 in form of an exact characterisation of the existence of matching crosses over conformal cycles in planar braces.

Proposition 4.1.1 (E*). Let B be a brace and C a conformal cycle in B, then there exists a strong matching cross over C in B if and only if C does not bound a face.

Since every matching cross over C in B also is an ordinary[3] C-cross, the existence of such a cross immediately certifies that it is impossible to draw B such that C bounds a face. We call a cycle C in a planar graph B *separating* if B cannot be drawn in a way such that C bounds a face. So we only need to show the reverse direction.

Since every brace is 3-connected by Theorems 3.1.65 and 3.1.67, we can rely on the uniqueness of plane embeddings for 3-connected graphs [Whi92].

[3] Ordinary here means a standard undirected cross in the sense of Theorem 2.2.6.

Let B be a brace and C a conformal and separating cycle in B. Then by Whitney's Theorem [Whi92] the interior and the exterior of C are the same in every drawing of B (up to the choice of the outer face). In what follows, we always assume B to come with a fixed drawing to avoid ambiguity. We denote the subgraph of B induced by the interior of C together with C itself by B_C^{int} and the subgraph of B induced by the exterior of C together with C is denoted by B_C^{out}. In both graphs C bounds a face and, since C is conformal, both graphs have a perfect matching.

Lemma 4.1.2 (E*)**.** Let B be a planar brace and C a conformal and separating cycle in B, then B_C^{int} and B_C^{out} are matching covered.

Proof. It suffices to show the claim for one of the two graphs. Moreover, by Theorem 3.1.69 it suffices to show for a single perfect matching M, that any pair $a \in V_1$, $b \in V_2$ of vertices is linked by an internally M-conformal path. So let us fix a perfect matching M for which C is M-conformal. Since B is a brace, given $a \in V_1(B_C^{\mathrm{int}})$ and $b \in V_2(B_C^{\mathrm{int}})$, Theorem 3.1.69 guarantees the existence of two internally disjoint and internally M-conformal paths P_1, P_2 from a to b. If one of these paths is disjoint from C, there is nothing to show. Hence we may assume that both meet C and, by a similar argument, both of them need to contain an edge of $B_C^{\mathrm{out}} - E(C)$. Let b_1 be the first vertex of P_1 on C when traversing P_1 from a towards b and let a_1 be the last vertex of P_1 on C. Then a_1 and b_1 separate C into two paths, one of them being M-conformal, let P' be this path. Moreover, P_1b_1 and a_1P_1 are internally M-conformal, and all three paths are contained in B_C^{int}. Hence $P_1b_1P'a_1P_1$ is an internally M-conformal a-b-path in B_C^{int}, and we are done. □

Lemma 4.1.3 (E*)**.** Let B be a planar brace, C be a conformal and separating cycle in B such that $B' \in \{B_C^{\mathrm{int}}, B_C^{\mathrm{out}}\}$ is not a brace, and $\partial_{B'}(X)$ be a non-trivial tight cut in B'. Then $|X \cap V(C)| \geq 3$ and $|\overline{X} \cap V(C)| \geq 3$.

Proof. Suppose $|X \cap V(C)| \leq 1$. By symmetry, it suffices to treat this case. With Lemma 3.1.58 we know that the minority of X has no edge to a vertex of $\overline{X}$, without loss of generality let us assume the majority of $\partial_B(X)$ to be in V_1. If $V(C) \cap X = \emptyset$, then clearly $\partial_B(X)$ must be a non-trivial tight cut in B, which is impossible. Hence there must exist a unique vertex $a \in V(C) \cap X$. Moreover, since this vertex has neighbours in C which do not belong to X, $a \in V_1$. But also in this case $\partial_B(X)$ is non-trivially tight in B and the claim follows. $\square$

Corollary 4.1.4 (E*)**.** Let B be a planar brace, C a conformal and separating 4-cycle in B, then B_C^{int} and B_C^{out} are braces.

Please note that lemmata 4.1.2 and 4.1.3 and Corollary 4.1.4 can be extended to non-planar braces as well. The general version of Corollary 4.1.4 is due to McCuaig.

Lemma 4.1.5 ([McC04])**.** Let B_1 and B_2 be bipartite graphs with a common 4-cycle C and otherwise disjoint. If $B_1 + B_2$ is a brace, then so are B_1 and B_2.

The next lemma is a slightly restated version of Lemma 46 from [McC04] which can be derived with the methods presented there.

Lemma 4.1.6 ([McC04])**.** Let B be a planar brace, C a facial cycle of B, M a perfect matching of B for which C is M-conformal, and $a \in V_1(C)$, $b \in V_2(C)$ two vertices with $ab \notin E(C)$. Then there exists an internally M-conformal a-b-path P in B which is internally disjoint from C.

As a special case, we first assume that we are interested in a (strong) matching cross over some separating cycle C in a planar brace B where both B_C^{int} and B_C^{out} are braces. We also need some deeper insight in how conformal bisubdivisions of $K_{3,3}$ and the cube can appear in braces with respect to cycles of length four.

Lemma 4.1.7 ([McC04])**.** Let B be a brace and C a 4-cycle such that $B - V(C)$ is connected, let $uv \in E(C)$ and $x, y \in V(B) \setminus V(C)$ such that $ux, vy \in E(B)$. Then B contains a conformal bisubdivision of the cube with $C + ux + vy$ as a subgraph, or B contains a conformal bisubdivision of $K_{3,3}$ with C as a subgraph.

The following is a slight weakening of the lemma above.

Corollary 4.1.8 ([McC04])**.** Let B be a brace and C a 4-cycle such that $B - V(C)$ is connected, then B contains a conformal bisubdivision of the cube or $K_{3,3}$ with C as a subgraph.

Please note that a version of Corollary 4.1.8 can be found in [RST99], where the containment of a 4-cycle in a conformal bisubdivision of the cube is referred to as being 'fat' while being a subgraph of some conformal bisubdivision of $K_{3,3}$ is called having a 'C-cross'. Sadly these two are not mutually exclusive as one can see in the example in Figure 4.1. However, there is a deeper connection between the existence of matching crosses, especially conformal ones, and the existence of conformal bisubdivisions of $K_{3,3}$ when it comes to 4-cycles. We revisit this topic in Section 4.2.

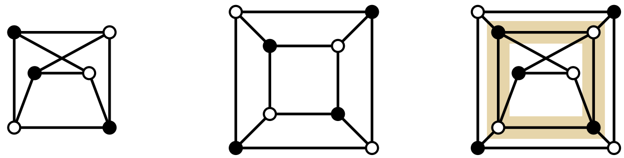


Figure 4.1.: From left to right: $K_{3,3}$, the cube, and a brace with a 4-cycle (the marked one) which is contained in both, a conformal bisubdivision of $K_{3,3}$ and a conformal bisubdivision of the cube. Please note that one could get rid of the fact that the marked cycle is separating by adding additional edges between vertices from different colour classes.

Lemma 4.1.9 (E*)**.** Let B be a planar brace and C a conformal and separating cycle in B such that B_C^{int} and B_C^{out} both are braces. Then there exists a strong matching cross over C in B.

Proof. First, assume C to have at least length 6. In this case, we can select a perfect matching M of B such that C is M-conformal, which in turn implies that M contains perfect matchings M_{int} and M_{out} of B_C^{int} and B_C^{out} respectively. Now select vertices s_1, s_2, t_1, and t_2 such that they appear on C in the order listed where $s_1, s_2 \in V_1$, and $t_1, t_2 \in V_2$. According to Lemma 4.1.6, we may choose an internally M-conformal s_1-t_1-path P_1 in B_C^{int} and an internally M-conformal s_2-t_2-path P_2 in B_C^{out} such that each of the P_i is internally disjoint from C. Then P_1 and P_2 form a C-cross in B, and we are done.

What remains is the case where $C = (s_1, t_1, s_2, t_2)$ is a 4-cycle. By calling upon Corollary 4.1.8 we can find a conformal bisubdivision of the cube in each of the two braces such that each of these cubes contains C as a subgraph. Let H_1 be a conformal bisubdivision of the cube in B_C^{int} and let H_2 be a conformal bisubdivision of the cube in B_C^{out}. One can easily see, that $H := H_1 + H_2$ is a conformal subgraph of B. We can now use H to find the required matching cross over C, as illustrated in Figure 4.2. $\square$

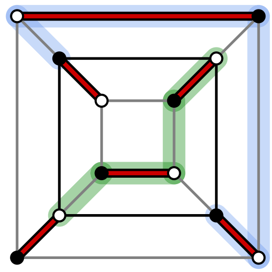


Figure 4.2.: The conformal subgraph H in the proof of Lemma 4.1.9 together with a strong matching cross over the separating 4-cycle C.

Lemma 4.1.10 (E*)**.** Let B be a bipartite matching covered graph, M a perfect matching of B, $e = ab \in M$ with $a \in V_1$, $b \in V_2$, $X \subseteq V_1 \setminus \{a\}$ and $Y \subseteq V_2 \setminus \{b\}$ such that every internally M-conformal X-Y-path in B contains e. Then there exists a tight cut $\partial_B(Z)$ in B with $X \subseteq Z$, $Y \subseteq \overline{Z}$, and $e \in \partial_B(Z)$.

Proof. By assumption, there is no internally M-conformal X-Y-path in $B - a - b$ and thus no vertex of X can share an elementary component with a vertex of Y. Since, by definition, each elementary component would be matching covered and thus, Theorem 3.1.69 would guarantee the existence of such a path. Let $Up(X)$ be the set of all vertices w of $B - a - b$ such that there exist elementary components K_X and K_w with $K_X \leq_2 K_w$ where K_X contains a vertex of X and $w \in V(K_w)$. Then $Y \cap Up(X) = \emptyset$. Moreover, there is no $V_1(Up(X))$-$V_2(\overline{Up(X)})$-path in $B - a - b$ at all. Hence in B $Up(X) \cup \{b\}$ is a set of odd cardinality, where no vertex of V_1 has a neighbour outside of it and the difference between the number of V_1-vertices and the number of V_1-vertices is exactly one. So by Lemma 3.1.58 $\partial_B(Up(X) \cup \{b\})$ is a tight cut with $Y \subseteq \overline{Up(X) \cup \{b\}}$. $\square$

Together with the upcoming lemma, Corollary 4.1.4 and Lemma 4.1.9 imply Proposition 4.1.1.

Lemma 4.1.11 (E*)**.** Let B be a planar brace, C a conformal and separating cycle in B such that $B' \in \left\{B_C^{\mathrm{int}}, B_C^{\mathrm{out}}\right\}$ is not a brace. Then there exists a strong matching cross over C in B.

Proof. By Corollary 4.1.4 we may assume $|V(C)| \geq 6$. Let $\partial_{B'}(X)$ be a non-trivial tight cut in B' maximising $|\overline{X}|$. Without loss of generality let us assume the majority of X to be in V_1. With Lemma 4.1.3 and the fact that for every selection of three distinct edges of C, at least two of them belong to a common perfect matching, one can see that $\partial_{B'}(X)$ separates C into two non-trivial paths Q_1 and Q_2 such that $V(Q_1) \subseteq \overline{X}$ and $V(Q_2) \subseteq X$. Let $a_1, a_2 \in X \cap V_1$ be the endpoints of Q_2 and let $b_1, b_2 \in \overline{X} \cap V_2$ be the endpoints of Q_1 such that $a_i b_i \in E(C)$ for both $i \in [1,2]$. Next let M be a perfect matching such that C is M-conformal and $a_2 b_2 \in M$, moreover let $a_1' b_1, a_1 b_1' \in M$. Since the majority of X is in V_1, there cannot exist an internally M-conformal path starting at some vertex of $V_1 \cap \overline{X}$ and ending in a vertex of $V_2 \cap X$ that avoids $a_2 b_2$. However, with B being a brace and Theorem 3.1.69 there must be two internally disjoint and internally M-conformal a_1'-x-paths for every $x \in X \cap V_2$ and one of them must avoid $a_2 b_2$. Let us choose $b_X \in V_2 \cap X$ such that there exists an internally M-conformal path P from a_1' to b_X with the following properties:

- $a_{\overline{X}}$ is the last vertex of $V(C) \cap \overline{X}$ along P starting in a_1', and
- $V(P) \cap V(Q_2) = \{b_X\}$.

Then $a_{\overline{X}} \in V_1$ and $P_1 := a_{\overline{X}} P$ is an internally M-conformal path which is internally disjoint from C. Moreover, $V(P_1) \cap V(B') = V(P_1) \cap V(C) = \{a_{\overline{X}}, b_X\}$. Let Q_3 be the component of $Q_1 - a_{\overline{X}}$ containing b_1, let Y be the component of $C - a_{\overline{X}} - b_X$ containing Q_3 and at last let $\overline{Y}$ denote the other component of $C - a_{\overline{X}} - b_X$. Let $e \in M$ be the edge covering b_X. What follows is a case distinction on the existence of some internally M-conformal path from $\overline{Y} \cap V_1$ to $Y \cap V_2$.

Case 1: There exists an internally M-conformal path P' from $\overline{Y} \cap V_1$ to $Y \cap V_2$ that avoids e.

If this is the case, let b_Y be the first vertex of Y encountered while traversing along P' starting in $\overline{Y} \cap V_1$. Then let $a_{\overline{Y}}$ be the last vertex of

P' in $\overline{Y} \cap V_1$ encountered before b_Y. Now $P_2 \coloneqq a_{\overline{Y}}P'b_Y$ is an internally M-conformal path with no inner vertex on C that is disjoint from P_1. Moreover, the vertices $a_{\overline{Y}}$, $a_{\overline{X}}$, b_Y, and b_X appear on C in the order listed and thus P_1 and P_2 form a strong matching cross over C in B.

Case 2: All internally M-conformal $(\overline{Y} \cap V_1)$-$(Y \cap V_2)$-paths contain e.

Then the deletion of both endpoints of e in B' leaves at least two elementary components, some containing vertices of $\overline{Y} \cap V_1$ and some of the others containing vertices of $Y \cap V_2$ but never both. Thus, by Lemma 4.1.10, there exists a tight cut $\partial_{B'}(Z)$ with $\overline{Y} \cap V_1 \subseteq Z$, $Y \cap V_2 \subseteq \overline{Z}$, and $e \in \partial_B(Z)$. Indeed, the majority of Z must be in V_2. Since X is odd, one of the two sets $X \cap Z$ and $X \setminus Z$ must be odd and therefore, by Lemma 3.1.55, one of these sets defines a tight cut in B'. Clearly $a_2, b_X \in X \cap Z$ and thus $|X \cap Z| > 1$. Since $\partial_B(Z)$ cannot contain more than two edges of C and by choice of $\partial_B(X)$ there cannot be a non-trivial tight cut $\partial_B(X')$ with $X' \subset X$ in B', hence $X \setminus Z = \{a_1\}$ and $a_1b_X \in M$.

With an argument similar to the one in **Case 1** one can see that, in case there exists an internally M-conformal $(Y \cap V_1)$-$(\overline{Y} \cap V_2)$-path avoiding the edge $e' \in M$ covering $a_{\overline{X}}$, we are done again. So we may also assume that e' meets all internally M-conformal $(Y \cap V_1)$-$(\overline{Y} \cap V_2)$-paths. Then, with arguments as before, one derives the existence of a non-trivial tight cut $\partial_{B'}(Z')$ such that $a_1 \in Z'$ and $\overline{Y} \cap B \subseteq \overline{Z'}$. In the end we arrive at the conclusion that $X \cap Z' = \{a_2\}$ and $e' \in \partial_{B'}(Z')$.

Moreover, this means that there is neither an internally M-conformal $(Y \cap V_1)$-$(\overline{Y} \cap V_2)$-path, nor an internally M-conformal $(\overline{Y} \cap V_1)$-$(Y \cap V_2)$-path in B' after deleting the four vertices in $S \coloneqq e \cup e'$. Hence there cannot be any Y-$\overline{Y}$-path in $B' - S$. This means there must be a face C' of B' containing both $a_{\overline{X}}$ and b_X that is distinct from C. Since B itself is a brace, there must be an internally M-conformal path P_1' from $Y \cap V_1$ to $\overline{Y} \cap V_2$ that avoids e' and this path may be chosen to be internally disjoint from B'. In particular, we may choose the endpoints of P_1' to be disjoint from one of the two paths of C', say R, connecting $a_{\overline{X}}$ and b_X. This is due to the fact that every internal vertex of a path in $C \cap C'$ must be of degree two in B', $B' \neq C$, and $a_{\overline{X}}b_X \notin E(C)$. Since B' is matching covered by Lemma 4.1.2, Lemma 4.0.1 guarantees the existence of a perfect matching M' of B' for which C' is M'-conformal. Let us choose M' such that R is an internally M'-conformal path and let P_2' be an internally M'-conformal subpath of R with both endpoints on C and otherwise disjoint from C. All of these choices are possible since $a_{\overline{X}}$ and b_X belong to different colour classes of B'. At last let us set $M'' := M' \cup (M \setminus E(B'))$. Since B' is an M-conformal subgraph of B, M'' is a perfect matching of B, and by construction, both P_1' and P_2' are internally M''-conformal paths. Now P_1' and P_2' form a strong matching cross over C in B. □

Proof of Proposition 4.1.1. Let C be a conformal cycle in a planar brace. If C does not bound a face, it is separating and thus lemmata 4.1.9 and 4.1.11 guarantee the existence of a strong matching cross over C in B. For the reverse suppose there exists a strong matching cross over C in B, then exactly one path of the cross must lie in the interior of C for every plane embedding of B. Hence C does not bound a face in any plane embedding of B. □

4.2. Paths and Matching Crosses through 4-Cycle Sums

Cycles of length four play a key role in many aspects of bipartite matching theory as we have seen in Theorem 3.3.8. In particular, the 4-cycle sum operation and the fact, that no perfect matching M for which an M-conformal matching cross over C_4 exists can contain a perfect matching of C_4 itself are things to be considered. While tight cut contractions only preserve special types of matching crosses[4], at some point they are not applicable any more to further decompose a given graph. In the spirit of the Two Paths Theorem, we want to decompose our graph further while maintaining a fixed subgraph, in most cases the conformal cycle for which we seek a matching cross. This section, and the following one, exist to describe exactly the interaction between conformal cycles, matching crosses over these cycles, 4-cycle sums, and matching crosses over 4-cycles in both Pfaffian braces which are non-planar and in non-Pfaffian braces.

A brace B is a *maximal* 4-cycle sum at the 4-cycle C of the braces $B_1, \dots, B_\ell$, $\ell \geq 3$, if there do not exist braces $H_1, \dots, H_n$, $n > \ell$ such that B is a 4-cycle sum of $H_1, \dots, H_m$ at C.

[4] It is possible to lose a matching cross if for example the tight cut separates the endpoints of both paths of the cross, in both tight cut contractions, what remains of the paths is now a set of two paths with a common endpoint.

Lemma 4.2.1 (E*)**.** Let B be a Pfaffian brace, $n \geq 3$, $B_1, \dots, B_\ell$ braces such that B is a maximal 4-cycle sum of $B_1, \dots, B_\ell$ at the 4-cycle $C = (a_1, b_1, a_2, b_2)$ and let M be a perfect matching of B_1. Then for every choice of $x \in \{a_1, a_2\}$ and $y \in \{b_1, b_2\}$ there exists a perfect matching M' of B such that $M \setminus E(C) \subseteq M'$ and there are paths P_1 and P_2 in $B - V(B_1 - V(C))$ where

- P_1 has endpoints x and y, while P_2 connects $\{a_1, a_2\} \setminus \{x\}$ to $\{b_1, b_2\} \setminus \{y\}$,
- P_1 and P_2 are disjoint, and
- $P_1 + P_2 + \{e \in M' \mid e \cap V(C) \neq \emptyset\}$ is an M'-conformal subgraph of B.

Proof. By Corollary 4.1.8 for every $i \in [2, \ell]$ C is contained in a conformal bisubdivision of the cube, or of $K_{3,3}$. Since B is Pfaffian, Theorem 3.3.4 implies that the later can never be true and thus for every $j \in [2, \ell]$, C is contained in a conformal bisubdivision H of the cube in B_j. As H is conformal in B_j, every perfect matching of H can be combined with a perfect matching of $B_j - V(H)$ to a perfect matching of B_j. In general, let M_H be a perfect matching of H such that for every $e \in E(C) \cap M_H$ both endpoints of e are covered by edges of $M \setminus E(C)$ and a vertex of C is covered by a non-$E(C)$-edge in M_H if and only if it is covered by an edge of $M \cap E(C)$. Moreover, let M_j be a perfect matching of $B_j - V(H)$, and for each $i \in 2, \ell \setminus \{j\}$ let M_i be a perfect matching of $B_i - V(C)$. The matching M_i clearly exists since the B_i are braces, and therefore they are 2-extendible. Then $(M \setminus E(C)) \cup (M_H \setminus E(C)) \cup \bigcup_{i=1, i \neq j}^{n} M_i$ is a perfect matching of B. Hence it suffices to show that, for any M we are given, we can choose the matching M_j such that we are able to find the desired paths within H. What follows is a discussion of these paths depending on the number of edges in $M \cap E(C)$. We present these matchings together with the paths in figures 4.3 and 4.4. As H is a bisubdivision of the cube, each perfect matching of H mirrors a perfect matching M'_H of the cube in the sense that a bisubdivided edge of the cube is M_H-conformal if and only if the corresponding edge of the cube belongs to M'_H. $\square$

Lemma 4.2.2 (E*)**.** Let B be a brace and C a 4-cycle in B as well as P_1, P_2 two paths that form a conformal cross over C. Then for every $e \in C$, $C + P_1 + P_2$ has a perfect matching M_e such that $\{e\} = M_e \cap E(C)$.

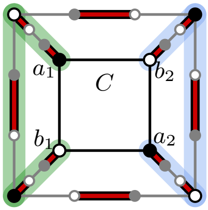


Figure 4.3.: A perfect matching M_H of H where all four vertices of C are matched to vertices of $H - V(C)$. Two paths, as requested by the assertion of Lemma 4.2.1, are marked.

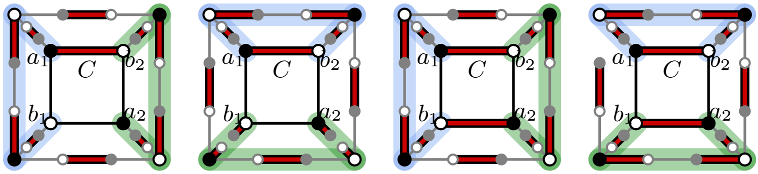


Figure 4.4.: A bisubdivision of the cube together with a perfect matching. Two paths, as requested by the assertion of Lemma 4.2.1, are marked. All other cases, in particular the ones regarding to the exact identity of e, can be derived from this by symmetry.

Proof. By definition, since P_1 and P_2 form a conformal cross over C, there is a perfect matching M of $H := C + P_1 + P_2$. Since C is a 4-cycle, P_1 and P_2 each must connect two vertices of the same colour on C and thus M must contain exactly one edge, say e', of C since the cross is conformal, hence $M_{e'} := M$. Let $e \in E(C) \setminus \{e'\}$ be another edge of C. If e and e' are disjoint, then $P_1 + P_2 + e + e'$ is an M-conformal cycle which contains all edges of $M_{e'}$ and thus, $M_e := E(P_1+P_2+e+e') \setminus M_{e'}$ is a perfect matching containing e. Otherwise, e and e' share exactly one endpoint and e is incident with an endpoint of P_i for some $i \in [1,2]$. Then P_i+e+e' is an $M_{e'}$-conformal cycle and thus $M_e := (M_{e'} \setminus E(P_i+e+e')) \cup (E(P_i+e+e') \setminus M_{e'})$ is a perfect matching as required. □

A last and essential tool before we dive into the more specific cases of Pfaffian and non-Pfaffian braces is the observation on conformal crosses over 4-cycles in form of Lemma 4.0.9.

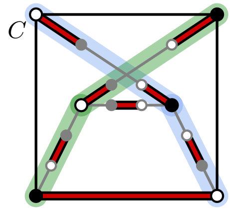


Figure 4.5.: A bisubdivision of $K_{3,3}$ together with a perfect matching M and two disjoint M-alternating paths that form a conformal cross over the 4-cycle C.

Proof of Lemma 4.0.9. First, assume that there exists a conformal bisubdivision H of $K_{3,3}$ that has C as a subgraph. Then we may choose a perfect matching M of B such that it contains a perfect matching of H as seen in Figure 4.5. The conformal cross over C is also presented in the same figure.

For the reverse let M be a perfect matching of B and P_1, P_2 two M-alternating paths that form an M-conformal cross over C. Since both, P_1 and P_2, must be of even length, they contain at least one inner vertex v_1 and v_2 respectively. Moreover, no endpoint of P_1 belongs to the same colour class of an endpoint of P_2, and thus v_1 and v_2 can be chosen such that they belong to different colour classes, and, for each i, the colour class of v_i is different from the colour class of the endpoints of P_i. Let $e \in M \cap E(C)$ be the unique edge of C that belongs to M. With B being a brace and Theorem 3.1.69 there must exist an internally M-conformal v_1-v_2 path Q that avoids e. For each $i \in [1,2]$ let x_i be the endpoint of P_i that is not incident with e. We claim that Q contains a subpath R which is internally disjoint from P_1 and P_2, and for each $i \in [1,2]$ an endpoint of R is an inner vertex of P_i. Let u_1 be the last vertex of Q when traversing along Q starting in v_1, which belongs to P_1. Then, as P_1 is M-alternating and the only vertex of P_1 not covered by M within the path belongs to e, u_1 must be incident to an edge of $M \cap E(P_1)$, and thus $P_1 u_1$ is of even length. Hence u_1 and v_1 belong to the same colour class of B, which is different from the colour of x_1. Clearly, $u_1 Q$ still contains an inner vertex of P_2, let u_2 be the first vertex of P_2 we encounter when traversing along $u_1 Q$ starting in u_1. By the same arguments as above, the edge of

M that covers u_2 must belong to P_2 and thus u_2 and v_2 have the same colour, which is different from v_2. Consequently, $u_2 \neq v_2$ and $R \coloneqq u_1 Q u_2$ is an internally M-conformal path as required. Since R is internally M-conformal and $C + P_1 + P_2$ is M-conformal, $H \coloneqq C + P_1 + P_2 + R$ is also M-conformal. Moreover, for each $i \in [1,2]$, the vertex u_i divides P_i into two subpaths, and as u_i has a different colour than any of the two endpoints of P_i, both of these paths must be of odd length. Hence H is a conformal bisubdivision of $K_{3,3}$ in B. □

4.3. Matching Crosses in Pfaffian Braces

To proceed towards the proof of Theorem 4.0.4, we need to describe how the structure of Pfaffian braces, especially the non-planar Pfaffian braces we obtain via the trisum operation from Theorem 3.3.8 by using planar braces as the base building blocks, behave regarding the existence of matching crosses. The purpose of this section is to establish the Pfaffian part of Theorem 4.0.4 in the form of Proposition 4.0.7. In what follows we are concerned with Pfaffian braces that are not planar. By Theorem 3.3.8 there is a single exception to the Pfaffian braces constructed from planar braces by the trisum operation, namely the Heawood graph. While the Heawood graph does not contain a single 4-cycle, in order to prove Proposition 4.0.7, we have to discuss its cycles.

Lemma 4.3.1 (E*)**.** Let C be a conformal cycle of the Heawood graph H_{14}, then there exists a conformal cross over C in H_{14}.

Proof. It is known that the Heawood graph has, up to automorphisms, exactly one perfect matching, as, for example, the complement of every perfect matching of H_{14} is a Hamilton cycle. See [AAF$^+$04] for a discussion on the matter. Moreover, no conformal cycle in H_{14} is of a length that is a multiple of four, for more details on that please consult [McC00]. Indeed, for every fixed length ℓ of a conformal cycle in H_{14} it suffices to find a cross for one of them, as, again, the graph is highly symmetric. Hence in order to prove the assertion, it suffices to fix a perfect matching and check conformal cycles of length 14, 10, and 6. This is done in Figure 4.6. □

Let us first observe that the 4-cycle itself, on which a trisum operation has been performed, must have a strong matching cross.

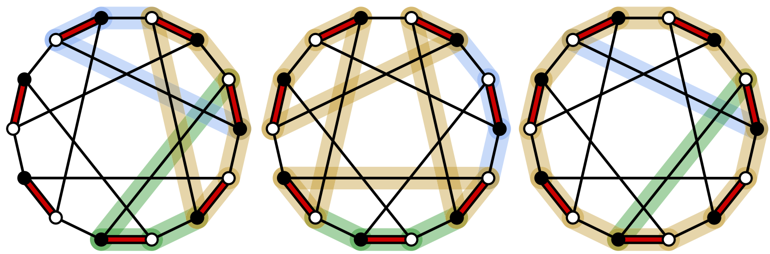


Figure 4.6.: The Heawood graph H_{14} together with a perfect matching and the three, up to symmetry, different conformal cycles in H_{14}. For each of these cycles, we provide a conformal cross.

Lemma 4.3.2 (E*). Let B be a Pfaffian brace that is not the Heawood graph, $\ell \geq 3$, $B_1, \ldots, B_\ell$ braces such that B is a maximal 4-cycle sum of $B_1, \ldots, B_\ell$ at the 4-cycle $C = (a_1, b_1, a_2, b_2)$. Then for any choice of two distinct values $i, j \in [1, \ell]$ and any choice of $x \in \{a_1, a_2\}$ and $y \in \{b_1, b_2\}$, there is a strong matching cross over C with paths P_1 and P_2 in $B_i + B_j$ such that the P_i are M-alternating for some perfect matching M of B for which x and y are the two vertices of C which are covered by edges of $M \cap (E(P_1) \cup E(P_2) \cup E(C))$.

Proof. The lemma is almost identical to Lemma 4.1.9. In both B_i and B_j there exists a conformal bisubdivision of the cube which has C as a subgraph by Corollary 4.1.8 since B cannot contain a conformal bisubdivision of $K_{3,3}$. Hence $B_i + B_j$ contains a conformal bisubdivision of the graph H_{12}, which is obtained by identifying two cubes on one 4-cycle. See Figure 4.2 for an illustration of a bisubdivision of H_{12}. The figure also shows perfect matchings of the respective conformal bisubdivision one can find, together with two paths that make up a strong matching cross as desired. The exact matching cross depending on the choices of x and y can be obtained from the paths illustrated in Figure 4.2 by symmetry. By adjusting the perfect matching such that exactly the requested edges belong to M, our proof is complete. □

Lemma 4.3.3 (E*). Let B be a Pfaffian brace, $\ell \geq 3$, $B_1, \ldots, B_\ell$ braces such that B is a maximal 4-cycle sum of $B_1, \ldots, B_\ell$ at the 4-cycle C and let C' be a conformal cycle in B_1 that also exists in B such that $C' \cap C$ is a non-trivial path. Then there is a strong matching cross over C' in B.

Proof. Since $C \cap C'$ is a non-trivial path, let us call it K, the two cycles share at least two vertices, and K contains at least one vertex of every colour class.

Let us first assume, K has length one. Let M be a perfect matching of B_1 such that C' is M-conformal and M contains as many edges of C as possible. Let $a \in V_1$ and $b \in V_2$ be the two vertices of $C - V(K)$. We divide this case into two subcases, one, where C is M-conformal as well and the second, where for every possible choice of M we have $ab \notin M$.

So first let us assume $ab \in M$. Then, since B_1 is a brace and by Theorem 3.1.69, there exists an internally M-conformal path L' with a as one endpoint and a vertex b' of $V_2(C')$ as its other endpoints such that L' is internally disjoint from C' and avoids K. Similarly, there is an internally M-conformal path R' with endpoints b and $a' \in V_1(C') \setminus V(K)$. By Lemma 3.1.42 there exists a perfect matching M' which coincides with M everywhere outside of $L' + R'$ together with internally M'-conformal paths L and R linking a and b', and b and a' respectively such that either L and R are disjoint, or $L \cap R$ is an M'-conformal path. In the later case, ab together with the a-b-subpath of $L + R$ forms an M'-conformal cycle O. In a slight abuse of notation let us adjust M'[5] in these cases to be the perfect matching $M' \Delta E(O)$. Let u and v be the endpoints of $L \cap R$ such that u appears on L before v when traversing along L starting from a. We also might have to adjust our definition of L and R slightly, depending on the way, L and R currently connect to C'. In case we adjust the perfect matching, we also adjust L to be the M'-alternating path LuR, while R is adjusted to be the path RvL.

In either case, we can now use Lemma 4.3.2 to find a perfect matching M'' of B such that $E(M') \setminus E(C) \subseteq M''$ together with two M''-alternating paths P_1 and P_2 such that these paths are disjoint and are completely contained in $B - V(B_1 - V(C))$. Indeed, P_1 and P_2 can be chosen such that $P_1 L$ and $P_2 R$ form a strong matching cross over C' in B as illustrated in Figure 4.7.

[5] We still keep the name 'M'' for better readability

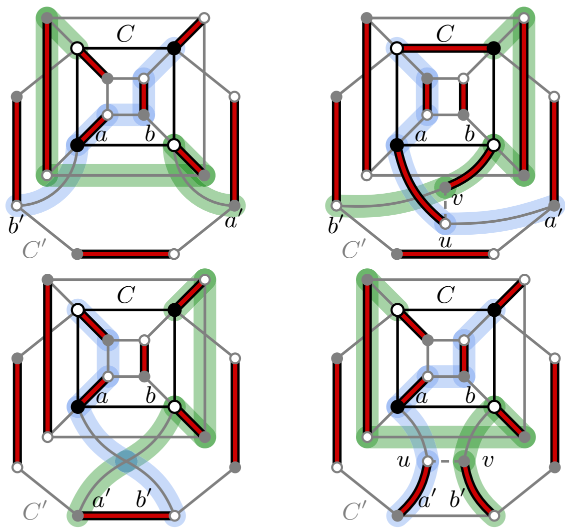


Figure 4.7.: The four different ways to obtain a strong matching cross over C' in the first subcase of the first case in the proof of Lemma 4.3.3.

So now let us assume $ab \notin M$. Then there exist vertices $x, y \in V(B) \setminus (V(C) \cup V(C'))$ such that $ax, by \in M$. Let P be an internally M-conformal path connecting x to some vertex a' of $V_1(C')$ such that P is internally disjoint from C' and avoids K. Similarly we choose Q to be an internally M-conformal path connecting y to some vertex $b' \in V_2(C')$ while avoiding K and being internally disjoint from C'. Suppose one of the two paths contains the initial matching edge of the other. Since these cases are symmetric, it suffices to consider one of them, so let us assume P contains by. Then Py is an internally M-conformal path that is disjoint from C' and does not meet K at all. Hence $Pyba$ is an M-conformal cycle and thus $M \Delta E(Pyba)$ is a perfect matching of B_1 for which $C' + K$ is conformal. Since we ruled this possibility out by the first case this cannot happen, and thus P does not contain by, and neither does Q contain the edge ax. By calling upon Lemma 3.1.42 again, we find a perfect matching M' and paths L' and R' such that $L' \cap R'$ is either empty or an M'-conformal

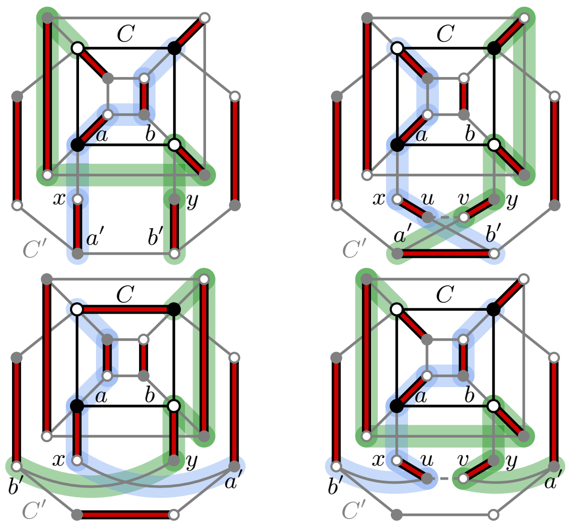


Figure 4.8.: The four different ways to obtain a strong matching cross over C' in the second subcase of the first case in the proof of Lemma 4.3.3.

path, M' equals M outside of axP and byQ, and R and L connect $\{a,b\}$ to $\{a',b'\}$ while being internally disjoint from C' and avoiding K. If case $L' \cap R'$ is empty let $L := L'$ and $R := R'$. Otherwise, let u and v be the endpoints of $L' \cap R'$ such that u is the first vertex of R' one encounters when traversing L' starting in a. Then ab together with the unique a-b-subpath of $L' + R'$ forms an M'-conformal cycle O. By adjusting M' to be the perfect matching $M' \Delta E(O)$ and setting $L := L'uR'$, $R := R'vL'$ we have found two disjoint M'-alternating paths that can be extended to form a strong matching cross over C' in B by using Lemma 4.3.2 as before. Please note that this step might alter the perfect matching M' again with regards to the edges of C. See Figure 4.8 for an illustration of the cases that might arise.

So now let us assume K to be of length two. In this case, there is a unique vertex $u \in V(C)$ that does not belong to C'. For this case, let us choose M to be a perfect matching of B_1 for which C is M-conformal. Since

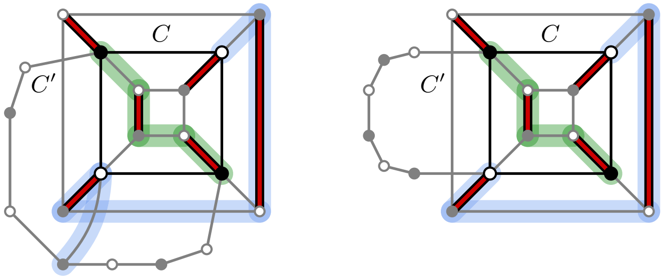


Figure 4.9.: Strong matching crosses over C' in the second and third case in the proof of Lemma 4.3.3.

B is a brace, by Theorem 3.1.69, there exists an internally M-conformal path P connecting v to a vertex of C' while avoiding any vertex in $V(K)$. We then use Lemma 4.3.2 to find paths L' and R together with a perfect matching M' of B such that $L := L'P$ and R form a strong matching cross over C' as illustrated in Figure 4.9.

At last, consider the case where K contains all of C. Here Lemma 4.3.2 yields the strong matching cross over C' in B immediately. □

The above lemma illustrates why we cannot allow to always reduce a brace along a 4-cycle sum. In some cases, even the small separator given by the 4-cycle is enough to provide a matching cross. The following lemmas aim to make this observation more general and exact.

Lemma 4.3.4 (E*)**.** Let B be a Pfaffian brace, $\ell \geq 3$, $B_1, \dots, B_\ell$ braces such that B is a maximal 4-cycle sum of $B_1, \dots, B_\ell$ at the 4-cycle C and let C' be a conformal cycle in B_1 that also exists in B such that $|V(C') \cap V(C)| \geq 2$ and $V(C) \cap V(C')$ contains vertices of both colour classes, then there is a matching cross over C' in B.

Proof. We divide this proof into three cases.

1: $|V(C) \cap V(C')| = 2$ and the vertices in $V(C) \cap V(C')$ are adjacent on C.

2: $|V(C) \cap V(C')| = 3$.

3: $|V(C) \cap V(C')| = 4$.

Case 1: $|V(C) \cap V(C')| = 2$ and the vertices in $V(C) \cap V(C')$ are adjacent on C.

In this case, let x and y be the two adjacent vertices of C belong to C'. In case $xy \in E(C')$, we are done immediately by Lemma 4.3.3. If $xy \notin E(C')$, then x and y divide C' into two paths of odd length, say P_1 and P_2, both with endpoints x and y. If M is a perfect matching of B such that C' is M-conformal, then exactly one of the two paths, say P_1 is also M-conformal and thus $P_1 + xy$ is an M-conformal cycle as well. By Lemma 4.3.3 $P_1 + xy$ has a strong matching cross and thus, by Lemma 4.0.10, C' must have a matching cross in B.

Case 2: $|V(C) \cap V(C')| = 3$.

In case $C \cap C'$ is a subpath of C', we are done immediately by Lemma 4.3.3. Hence we may assume that this is not the case. Next, suppose C' contains exactly one edge of C and there is $xy \in E(C)$ such that $x, y \in V(C')$, but $xy \notin E(C')$. Let z be the remaining vertex of C on C', then x and y separate C' into two paths, where one of them, say P, does not contain z. We may choose a perfect matching M of B such that P is internally M-conformal. Then $K := C' - P + xy$ is also an M-conformal cycle, and by our assumption, $K \cap C$ is a subpath of C. Hence we may apply Lemma 4.3.3 together with Lemma 4.0.10 to obtain a matching cross over C'. At last assume that C' does not contain an edge of C. If we call the vertices of C on C' x, y, and z again such that x and z belong to the same colour class, we again find the path P avoiding z but connecting x and y as before. But we also find a path $Q \subseteq C'$ that connects z and y and avoids x. By choosing a perfect matching M of B such that C' is M-conformal and Q is internally M-conformal, we have found a perfect matching for which $K' := C' - Q + yz$ is an M-conformal cycle. For this cycle, we find a matching cross as discussed above, and by applying Lemma 4.0.10 again, we obtain a matching cross for C' as well.

Case 3: $|V(C) \cap V(C')| = 4$.

Let $C = (a_1, b_1, a_2, b_2)$. If the vertices of C appear on C' in the same order as they do on C, we can use Lemma 4.3.2 to find a strong matching cross over C whose paths are internally disjoint from B_1. Hence we have found a strong matching cross over C' in B.

Hence the vertices of C do not appear on C' in the order listed. The only way this is possible is, if they appear on C' in the order a_1, a_2, b_1,

b_2, or a_1, a_2, b_2, b_1. In both cases we can use Lemma 4.2.1 to obtain a perfect matching M of B and internally M-conformal paths P_1 and P_2 which are internally disjoint from B_1 such that P_1 connects a_1 and b_1 while P_2 connects a_2 and b_2 in case the order of appearance is a_1, a_2, b_1, b_2. Otherwise, P_1 and P_2 may be chosen such that P_1 connects a_1 and b_2 while P_2 connects a_2 and b_1. Either way, the two paths form a strong matching cross over C' in B. □

Let B be a Pfaffian brace, C a 4-cycle in B such that $B - C$ is not connected, and C' be any conformal cycle in B. Suppose there is a matching cross over C' in B such that at least one of the paths of this cross uses C and extends into another component of $B - C$. Let B' be a brace with $C' \subseteq B'$ such that B is made from a set of Pfaffian braces including B' via a 4-cycle sum at C. Then some information on the matching cross over C' in B should also exist in B'. Our goal is to use this information to show that even B' cannot be planar while C' bounds a face.

Let B be a Pfaffian brace, C a 4-cycle in B, M a perfect matching and C' a conformal cycle in B. A tuple (P_1, P_2, Q, M) is a *diffuse C'-M-precross through* C if P_1, P_2 and Q are pairwise disjoint, internally disjoint from C', all are M-alternating paths[6], and for each $i \in [1, 2]$, P_i is a $V(C')$-$V(C)$-path such that the endpoints of the P_i on C' belong to different components of $C' - V(Q)$. Additionally, we require among the endpoints of the P_i on C at least one of each colour class to be covered by an edge of $E(C) \cap M$. A diffuse C'-M-precross through C is *daring* if the endpoints of the P_i on C belong to the same colour class.

Lemma 4.3.5 (E*)**.** Let B be a Pfaffian brace, $\ell \geq 3$, $B_1, \ldots, B_\ell$ braces such that B is a maximal 4-cycle sum of $B_1, \ldots, B_\ell$ at the 4-cycle C and let C' be a conformal cycle in B_1 that also exists in B. If there is a diffuse C'-M_1-precross H through C in B_1, for some perfect matching M_1 of B_1, which is not daring, then there are matching crosses over C' in B_1 and B.

Proof. Let $H = (P_1, P_2, Q, M_1)$. Since H is not daring, we may apply Lemma 4.2.1 to find a perfect matching M of B with $M_1 \setminus E(C) \subseteq M$ together with a path R in B_j for some $j \in [2, \ell]$ such that $P_1 R P_2^{-1}$ is

[6] In particular, P_1 and P_2 are allowed to be single vertices of C'.

M-alternating and disjoint from Q. Note that such a matching exists in particular because either C is M-conformal, or the edge connecting the endpoints of the P_i on C must belong to M. Hence in this case we have found our matching cross over C' in B. Since R is a subpath of an M-alternating path, it itself is M-alternating. Moreover, with H not being daring, the endpoints of R belong to different colour classes, and thus R is either M-conformal or internally M-conformal. Either way, the endpoints, let us call them x and y, are adjacent on C and there exists a perfect matching M' of B_1 such that $E(B_1 - V(C)) \cap M \subseteq M'$ and $xy \in M'$ if and only if R is M-conformal. Hence $P_1 xy P_2^{-1}$ is an M'-alternating path which forms, together with Q, a matching cross over C' in B_1. □

Hence diffuse precrosses can always be extended to actual crosses if they are not daring. Next we inspect daring precrosses more closely. Let B be a Pfaffian brace, C a 4-cycle in B, M a perfect matching and C' an M-conformal cycle in B. A diffuse C'-M-precross through C (P_1, P_2, Q, M) is *successful* if it is daring, and either P_1 is internally M-conformal while P_2 is M-conformal or P_1 and P_2 are both of even length, and the endpoint of P_1 on C' is covered by an edge of $E(P_1) \cap M$ if and only if the endpoint of P_2 on C' is **not** covered by an edge of $E(P_2) \cap M$.

Lemma 4.3.6 (E*)**.** Let B be a Pfaffian brace, $\ell \geq 3$, $B_1, \dots, B_\ell$ braces such that B is a maximal 4-cycle sum of $B_1, \dots, B_\ell$ at the 4-cycle C and let C' be a conformal cycle in B_1 that also exists in B such that there is $i \in [1,2]$ with $V(C) \cap V(C') \subseteq V_i$. Then there is a matching cross over C' in B if and only if one of the following is true

i) there is a matching cross over C' in B_1,
ii) there is a perfect matching M_1 of B_1 such that there exists a diffuse C'-M_1-precross through C, which is successful.

Proof. Let us first prove that in case one of our conditions holds true we find a matching cross over C' in B. First let L and R be the two paths of a matching cross over C' in B_1 and let M be a perfect matching such that L and R are M-alternating. If L and R also exist in B, we are done. So let us assume exactly one of them, say L, contains an edge of C which does not exist in B, note that this is the only possibility why L is not an alternating path in B. Let us traverse along L starting in one of its endpoints, and let x be the first vertex of C we encounter, while y is the last vertex of C on L. In case $xy \in E(C)$, we either have $xy \in M$, or $xy \notin M$, but xLy is M-conformal, hence $xLyx$ is an M-conformal cycle, and we can adjust M to include the edge xy. Then, after possibly adjusting the matching as above, (Lx, yL, R, M) is a diffuse C'-M-precross through C which is not daring. Hence we are done by Lemma 4.3.5. So we may assume x and y not to be adjacent on C and thus they belong to the same colour class. In this case, xLy is an M-alternating path of even length and thus can neither be internally M-conformal, nor M-conformal. However, since L contains an edge of C, there must be a vertex $z \in V(C) \setminus \{x, y\}$ such that one of the edges xz, yz belongs to L. Without loss of generality let us assume $xz \in E(L)$. In case $xz \in M$, (Lx, yL, R, M) is a successful diffuse C'-M-precross through C and we are done by Lemma 4.3.5. So assume $xz \notin M$. Then zLy must be an M-conformal path, and thus $zLyz$ is an M-conformal cycle. Hence we may adjust M such that $yz \in M$ and again (Lx, yL, R, M) is a successful diffuse C'-M-precross through C and we are done by Lemma 4.3.5. Hence we may now assume L and R to contain an edge of C each. Since C only has four vertices, this means there is a unique edge $x_L y_L \in E(L) \cap E(C)$ and a unique edge $x_R y_R \in E(R) \cap E(C)$. Now Lemma 4.2.1 provides us with the two paths in B_2 that are necessary to be combined with Lx_L, $y_L L$ and Rx_R, $y_R R$ respectively in order to obtain a strong matching cross over C' in B.

So now we have to show that the existence of a matching cross in B implies the existence of one of the two structures above. Let R and L be two M-alternating paths for some perfect matching M that form a matching cross over C'. If neither L nor R contains a vertex of C, L and R must be completely contained in B_1 and thus form a strong matching cross over C' in B_1 as well. So let us assume that exactly one of the two paths contains a vertex of C and further assume, without loss of generality, that L is that path. In case L contains exactly one vertex of C, it cannot contain any vertex of $B_i - V(C)$ for any $i \in [2, \ell]$ since C separates the B_j from each other. Hence in this case, L and R again also exist in B_1. So assume that L contains exactly two vertices of C, say v and w. Then either $vw \in E(C)$, or v and w belong to the same colour class. In the first case, either $vw \in E(L)$, or $vLwv$ is an M-conformal cycle, and we may adjust M such that $LvwL$ is M-alternating. In any case, after possibly adjusting M, $LvwL$ and R form a matching cross over C' in B_1 for some perfect matching M_1 where $vw \in M_1$ if and only if vLw is M-conformal. In the second case, vLw must be of even length. In case exactly one of Lv and wL is of even length, there must be a vertex $u \in V(C) \setminus \{v, w\}$ such that u is not covered by an edge of $M \cap E(B_1)$, or $vLw = vuw$. Then we may choose a perfect matching M_1 of B_1 such that $M_1 \setminus E(C) \subseteq M$ and $LvuwL$ is an M_1 alternating path of the same type as L in B_1. Hence there is a matching cross over C' in B_1. Thus we may assume Lv and wL to either both be of odd or both be of even length. In total, this means that L is of even length. Hence, by choosing M_1 as before, (Lv, wL, R, M_1) is a successful diffuse C'-M_1-precross over C' in B_1. In case L contains more than two vertices of C, let v be the first vertex of C on L and w be the last one. Let Q be a shortest v-w-path on C, then Q and vLw are of the same parity, and we can choose a perfect matching M_1 such that both R and $LvQwL$ are M_1 alternating in B_1, thus forming a matching cross over C' in B_1.

With this, we may now assume that both L and R contain vertices of C. If R contains exactly one vertex of C and the edge of M covering this vertex belongs to B_1, this case can be handled the same way as the cases where R does not contain any vertex of C. If, on the other hand, R contains exactly one vertex of C and the edge of M covering this vertex does not belong to B_1 this means this vertex of C is an endpoint of R and thus belongs to C'. Let us assume that both L and R contain exactly one vertex of C each and the edges of M covering these vertices do not belong to B_1. This means that these endpoints of L and R on C must belong to the same colour class by our assumption and thus no edge of M that covers a vertex of C can belong to B_1. But then we may choose a perfect matching M_1 of B_1 such that $M \cap E(B_1) \subseteq M_1$ and C is M_1-conformal, and then L and R sill are M_1-alternating paths in B_1. Hence we have found a matching cross over C' in B_1. Now assume that L contains more than one vertex of C, while R still contains exactly one vertex, say u, of C for which the edge of M covering it does not belong to B_1. Let x be the first vertex of L on C and y be the last vertex. Then xLy is of even length if and only if x and y belong to the same colour class. Suppose this is the case, then for one $z \in \{x, y\}$ the edge of M covering z cannot belong to B_1 since u cannot belong to the same colour class as x and y. Hence there exists a perfect matching M_1 of B_1 with $M \cap E(B_1) \subseteq M_1$ and $zu \in M_1$ and therefore (xL, yL, R, M_1) is a successful C'-M_1-precross through C in B_1. Otherwise, x and y belong to different colour classes and thus xLy is either M-conformal or internally M-conformal. In the first case, $xLyx$ is an M-conformal cycle, and we can adjust M such that $xy \in M$, in the second case, $LxyL$ already is an M-alternating path. Hence there is a perfect matching M_1 of B_1 such that R and $LxyL$ form a matching cross over C' in B_1. Thus we may assume both L and R to contain exactly two vertices of C each. Let v_X, w_X be the two vertices of $VC \cap V(X)$ for each $X \in \{L, R\}$. In case v_L and w_L are adjacent on C, then so are v_R and w_R, and we can choose a perfect matching M_1 such that $v_X w_X \in M_1$ if and only if $v_X X w_X$ is an M-conformal path and $M_1 \setminus E(C) \subseteq M$. Then $Lv_L w_L L$ and $Rv_R w_R R$ form a matching cross over C' in B_1. In case v_L and w_L belong to the same colour class, then so must v_R and w_R. In this case, $v_L L w_L$ and $v_R R w_R$ form a matching cross over C in B and since these paths are alternating, each of them must contain an edge of M. Moreover, we can change M to a perfect matching M' such that M' coincides with M on $E(B_i) \setminus E(C)$ for all $i \in [2, \ell]$, and $C + v_L L w_L + v_R R w_R$ is M'-conformal. Indeed this means that $v_L L w_L$ and $v_R R w_R$ form a conformal cross over C in B and thus, by Lemma 4.0.9 and Theorem 3.3.4, B cannot be Pfaffian which contradicts our assumption. $\square$

A graph that plays a huge role in non-planar Pfaffian graphs that are **not** the Heawood graph is the *Rotunda*. The *Rotunda* is the graph obtained by performing the 4-cycle sum operation on three cubes at a single common 4-cycle C and then forgetting all edges of C. An important observation is the non-planarity of the Rotunda.

Observation 4.3.7. The Rotunda is **not** planar.

From Theorem 3.3.8 and Corollary 4.1.8 one can derive the following.

Corollary 4.3.8. Let $B, B_1, \ldots, B_\ell$, $\ell \geq 3$, be braces such that B is Pfaffian and a 4-cycle sum of $B_1, \ldots, B_\ell$ at a 4-cycle C. Then B contains a conformal bisubdivision of the Rotunda.

Lemma 4.3.9 (E*)**.** Let $B, B_1, \ldots, B_\ell$, $\ell \geq 3$, be braces such that B is Pfaffian and a maximal 4-cycle sum of $B_1, \ldots, B_\ell$ at the 4-cycle C. Moreover, let C' be a conformal cycle in B_1 that also exists in B. If

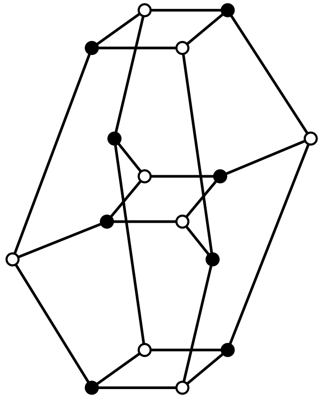


Figure 4.10.: The smallest bipartite and non-planar Pfaffian graph that is not isomorphic to the Heawood graph: The Rotunda.

$V(C) \cap V(C')$ contains a vertex of each colour class, every C'-reduction of B to some brace H is **not** planar.

Proof. Let B be a minimal counterexample to the assertion, that is, the claim holds for every C'-reduction of any C'-reduction of B. Let K be a 4-cycle and $H_1, \dots, H_m$, $m \geq 3$, be braces such that B is a maximal 4-cycle sum of $H_1, \dots, H_m$ at K with $C' \subseteq H_h$ for some $h \in [1, m]$. Moreover, let K be chosen such that $V(K) \cap V(C')$ contains vertices from at most one colour class of B, then $C \neq K$ and C contains at least one vertex of C' which does not belong to K. Hence $|V(K) \cap V(C)| \leq 3$

Let us first observe that for any choice of $Z \in \{C, K\}$, $B + E(Z)$ still is a Pfaffian brace as this does not change B being a maximal 4-cycle sum of the braces associated with Z at the 4-cycle Z, and by Theorem 3.3.8 all of these braces are Pfaffian. Indeed, we claim that $B' \coloneqq B + E(C)$ is a maximal 4-cycle sum of $H_1', \dots, H_m'$, where $H_i' \coloneqq H_i + \{xy \in E(C) \mid x, y \in V(H_i)\}$, at the 4-cycle K. Suppose K does not separate C, i.e. $V(C) \setminus V(K)$ belongs to a unique component of $B' - V(K)$, there is a unique $i \in [1, m]$ such that $V(C) \subseteq V(H_i)$ and in this case our claim holds true. Indeed that means if $|V(C) \cap V(K)| \leq 1$ or $|V(C) \cap V(K)| = 3$ the claim follows immediately as in those cases K does not separate vertices of C. So let us assume

$|V(K) \cap V(C)| = 2$ and K separates C. Let $\{a_1, a_2\} = V(C) \cap V(K)$, note that the a_i belong to the same colour class, say V_1, of B. Let $\{b_1, b_2\} := V(C) \setminus V(K)$ and let $\{c_1, c_2\} := V(K) \setminus V(C)$. Without loss of generality let us assume $b_1 \in V(H'_1)$ and $b_2 \in V(H'_2)$. As $m \geq 3$, there is also some H'_3. For each $i \in [1, 2]$ let L_i be the 4-cycle in H'_i with vertex set $\{a_1, a_2, c_i, b_i\}$. By Lemma 4.1.7 and Theorem 3.3.8, since B' is Pfaffian, in H'_1 there is a conformal bisubdivision R_1 of the cube that contains $L_1 + a_2c_2$ as a subgraph. Similarly, in H'_2 there is a conformal bisubdivision R_2 of the cube which contains $L_2 + a_1c_1$ as a subgraph. Moreover, H'_3 has a conformal bisubdivision R_3 of the cube with K as a subgraph. For each $i \in [1, 3]$ let R'_i be obtained from R_i by removing all inner vertices of the paths that correspond to a bisubdivided edge of K. Then let $R := R'_1 + R'_2 + R'_3$. By construction R is a conformal subgraph of B' and $C \subseteq R$. Careful inspection reveals, that there is a conformal cross over C in R, see Figure 4.11, and thus, by Lemma 4.0.9 there must be a conformal bisubdivision of $K_{3,3}$ in B'. As B' is Pfaffian, this is a contradiction, and thus K can never separate C.

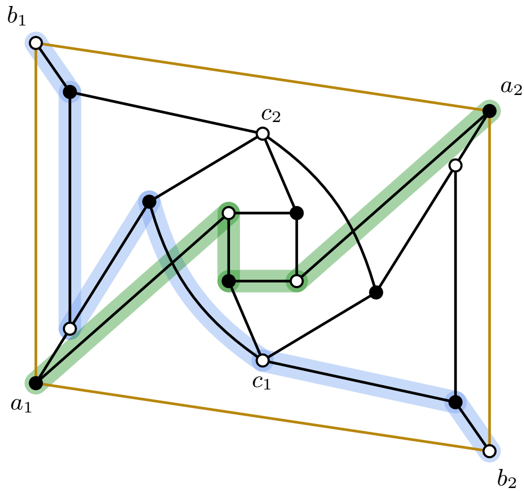


Figure 4.11.: The graph R from the proof of Lemma 4.3.9 with the 4-cycle C as a subgraph and a conformal cross over C.

Consequently the graph $B'' := B' + E(K)$ is Pfaffian and thus, with the same arguments as above, B'' is a 4-cycle sum of $B'_1, \dots, B'_\ell$ at C, where $B'_i := B_i + \{xy \in E(K) \mid x, y \in V(B_i)\}$ for all $i \in [1, \ell]$. Indeed, from the discussion above one can derive that there are $i \in [1, \ell]$ and $j \in [1, m]$ such that $\bigcup_{k \in [1,m] \setminus \{j\}} V(H'_k) \subseteq V(B_i)$. If $i \neq 1$, then, as $C' \subseteq H_h$, we must have $j = h$ and H'_h still contains all B_k for $k \in [1, \ell] \setminus \{i\}$, as well as a C-reduction of B_i. So we may assume $i = 1$. By assumption, we have that $V(C) \cap V(C')$ contains a vertex of each colour class of B and thus, in this case, $\bigcup_{k \in [2,\ell]} V(B'_k) \subseteq V(H'_h)$ implying that H'_h is a 4-cycle sum of at least 3 braces at the cycle C. Consequently, by Corollary 4.3.8, in both cases H'_h contains a conformal bisubdivision of the Rotunda and thus is not planar. This contradicts B being a minimal counterexample and thus completes our proof. □

We can now combine the above lemma with our previous observation on precrosses to rule out any planar C'-reductions if there exists a diffuse C'-M-precross though a 4-cycle C which shares vertices of at most one colour class with C'.

Lemma 4.3.10 (E*)**.** Let B be a Pfaffian brace, M a perfect matching of B, C a 4-cycle, and C' a conformal cycle for which $V(C) \cap V(C')$ contains vertices from at most one colour class of B such that there exists a diffuse C'-M-precross through C in B, then there does **not** exist a C-reduction of B to a brace H such that H is planar and C bounds a face of H.

Proof. Let Q be the path of our C'-M-precross through C and let P_1, P_2 be the two paths connecting C' to C such that their endpoints belong to different components of $C - V(Q)$.

Suppose there is a 4-cycle K in B such that B is a maximal 4-cycle sum of the braces $H_1, \dots, H_m$, $m \geq 3$ at K, H_1 is a non-trivial C'-reduction of H, and $V(C) \subseteq H_1$. Then there is a diffuse C'-M'-precross through C in H_1 for some perfect matching M', or there is at most one path $W \in \{P_1, P_2, Q\}$ such that $V(W) \cap \bigcup_{i=2}^m V(H_i) \setminus V(K) \neq \emptyset$.

Suppose there are two paths $W_1, W_2 \in \{P_1, P_2, Q\}$ such that $V(W_k) \cap \bigcup_{i=2}^m V(H_i) \setminus V(K) \neq \emptyset$ for both $k \in [1, 2]$. Then let W'_k be the subpath of W_k in $\sum_{i=2}^m H_i$. If one of the W'_k is of even length, then so is the other one. Indeed, if both are of even length, then each of them must have an edge incident to one of its endpoints in $\sum_{i=2}^{m} H_i$ that belongs to M. Moreover, as H_1 is a C'-reduction and therefore $V(K) \cap V(C')$ contains vertices from at most one colour class of B, the edges of M that are incident to the other endpoints of the W_k' must belong to H_1. Hence there exists $h \in [2, m]$ such that W_1' and W_2' belong to H_h and, in H_h these paths form a conformal cross over K. Consequently, by Lemma 4.0.9, H_h has a conformal bisubdivision of $K_{3,3}$ which, by Theorem 3.3.8, contradicts B being Pfaffian. Hence the W_k' are of odd length and thus are either internally M-conformal or M-conformal. In either case, for each $k \in [1, 2]$ the endpoints u_k, v_k of W_k' are adjacent on K and there exists a perfect matching M_1' of H_1 with $M_1' \setminus E(C) \subseteq M$ and $u_k v_k \in M_1'$ if and only if W_k' is M-conformal. Thus there is a diffuse C'-M'-precross through C in H_1.

If there is a path $W \in \{P_1, P_2, Q\}$ such that $V(W) \cap \bigcup_{i=2}^{m} V(H_i) \setminus V(K) \neq \emptyset$, then either the endpoints of W', which is the subpath of W starting on the first vertex of K and ending on the last vertex of K when traversing along W, are adjacent on K, or at most one vertex of $V(K) \setminus V(W)$ belongs to another path from $\{P_1, P_2, Q\} \setminus \{W\}$. As we have seen above, no path besides W may leave H_1 through K, hence all edges of the other paths in $\{P_1, P_2, Q\} \setminus \{W\}$ belong to H_1. This is particularly true for the edges of M on these paths. Indeed, this means that all four vertices of K must be matched inside H_1 by M. However, W' is a path of even length and therefore must contain an edge of M that covers one of its endpoints. By definition and our assumption that W' contains exactly two vertices of K, which are of the same colour, no edge of W' belongs to H_1, which is impossible. A set S of vertices with $|S \cap V_1| = |S \cap V_2| = 2$ is called *splitting* if there exist braces $L_1, \ldots, Lq$, $q \geq 3$, such that B is a 4-cycle sum of $L_1, \ldots, L_q$ at a 4-cycle with vertex set S. Let us call a set $S \subseteq V(B)$ with $|S \cap V_1| = |S \cap V_2| = 2$ *well behaved*, if B is a maximal 4-cycle sum of the braces $H_1, \ldots, H_m$, $m \geq 3$ at the 4-cycle K' with vertex set S, H_1 is a non-trivial C'-reduction of H, and $V(C) \subseteq H_1$, or S is not splitting. Let H' be a C'-reduction of B such that no splitting set S in H' is well behaved and let $K_1, \ldots, K_\ell$ be the 4-cycles used to reduce B to H'. Let the K_i be numbered in the order in which the K_i were used to construct a non-trivial C'-reduction of B to some brace J_i in order to eventually reach H'. We claim that either H' is non-planar, or C' does not bound a face of H'. Since B is Pfaffian and contains a 4-cycle it cannot be isomorphic to the Heawood graph. Suppose H' still has a splitting set, then, by Corollary 4.3.8, H' is non-planar. Hence we may assume H' to be planar for the sake of this claim. Next we iteratively construct paths R^i_Q, $R^i_{P_1}$, and $R^i_{P_2}$ such that for each $W \in \{P_1, P_2, Q\}$, R^i_W is a path in J_i and all three paths are disjoint. For each $W \in \{P_1, P_2, Q\}$ let $R^1_W := W$. The construction is pretty straight forward. Suppose in J_i, $i \in [1, \ell-1]$, the subpath of R^i_W starting with its first vertex, u^i_W, on K_{i+1} and ending on its last vertex, v^i_W, has edges that do not belong to J_{i+1}. Then either u^i_W and v^i_W are adjacent and we can set $R^{i+1}_W := R^i_W u^i_W v^i_W R^i_W$, or they are not adjacent, in which case we have seen that there is a path U of length two on K_i such that $R^{i+1}_W := R^i_W u^i_W U v^i_W R^i_W$ is a path and disjoint from the other two paths. If R^i_W has no such subpath we simply set $R^{i+1}_W := R^i_W$. Then the paths R^ℓ_W, for $W \in \{P_1, P_2, Q\}$, are pairwise disjoint paths in H' such that R^ℓ_Q has both endpoints on C' and $R^\ell_{P_1}$ and $R^\ell_{P_2}$ connect C' to C. Moreover, the endpoints of $R^\ell_{P_1}$ and $R^\ell_{P_2}$ on C' belong to different components of $C' - V(R^\ell_Q)$. As all three paths are internally disjoint from C, we can now connect $R^\ell_{P_1}$ and $R^\ell_{P_2}$ on C in order to create an ordinary cross[7]. However, this means that C' cannot bound a face of H' by Theorem 2.2.6. To finalise the proof we have to show that, in case H' is non-planar, we still cannot find a planar C'-reduction of H' such that C' bounds a face. For this note that, by Lemma 4.3.9, no splitting set S can contain vertices of C' from more than one colour class, or otherwise, the claim would follow immediately. Observe that every splitting set S in H' in this case must separate C from C'. Clearly P_1 and P_2 are separated by S. If also Q is separated by S, we have found two disjoint alternating paths that connect C' to S and that belong to a matching cross over C'. Suppose the two disjoint subpaths of Q that link C' to S both have their endpoints in S in the same colour class. Then, if we were to complete this matching cross we would, in particular, obtain a conformal cross over a 4-cycle with vertex set S. As S is splitting and B Pfaffian, by Lemma 4.0.9 this is impossible. Hence, if C_S is the 4-cycle with vertex set S and H'' is the C'-reduction of H' at S, we either find a matching cross over C', again implying that H'' is not planar, as otherwise, we would be done or find a diffuse C'-M'-precross through C_S.

[7]Ordinary in this context means that our paths are not necessarily alternating for any perfect matching.

Hence, in either case, we simply re-enter a previously discussed case and thus our proof is complete. □

Observation 4.3.11 (E*)**.** Let B be a brace and C a conformal cycle in B. If B is planar there does not exist a C-reduction of B.

Proof. We prove a stronger result, namely, that a planar brace cannot be a maximal 4-cycle sum of three or more braces. Since B is planar, it is Pfaffian and thus does not contain conformal bisubdivisions of $K_{3,3}$. Let us assume B is a maximal 4-cycle sum of the braces $B_1, \dots, B_\ell$, $\ell \geq 3$, at the 4-cycle C. Then the claim follows immediately from Corollary 4.3.8. □

With this, everything is in place to prove Proposition 4.0.7.

Proof of Proposition 4.0.7. Let B be a minimal counterexample to the forward direction of the assertion. So let us assume that B is Pfaffian, there is a conformal cycle C in B which has **no** cross, but every C-reduction of B to a brace H is either non-planar or C bounds **no** face of H. Indeed, we may assume that B is not isomorphic to the Heawood graph since every conformal cycle here has a matching cross Lemma 4.3.1. However, in every C-reduction of B, our assertion holds. We claim that this means there is **no** C-reduction of B.

Suppose there was one and let H be a C-reduction of B. By the minimality of B this means that either H has a C-reduction to some brace H' such that C bounds a face of H' or there is a matching cross over C in H. In the first case, there exists a C-reduction of B to H' and thus we have a contradiction to B being a counterexample. So we may consider the second case and assume that there is a matching cross over C in H. Let B be a maximal 4-cycle sum of the braces $H, B_1, \dots, B_\ell$, $\ell \geq 2$ at the 4-cycle C'. Then, since there is a matching cross over C in H, lemmata 4.3.4 and 4.3.6 imply that there also must exist a matching cross over C in B which again is a contradiction.

Suppose B is planar. By Proposition 4.1.1, this means that C must either bound a face of B or have a strong matching cross in B. Since neither is correct by our assumption, B cannot be planar. So B is neither planar nor does there exist a C-reduction. According to Theorem 3.3.8 B must either be isomorphic to the Heawood graph or be a maximal

4-cycle sum at some 4-cycle K of Pfaffian braces $H_1, \dots, H_m$, $m \geq 3$. The first case is impossible by assumption. If $V(K) \cap V(C)$ contains vertices from at most one colour class of B, there would be a C-reduction in B which we already ruled out, hence we must have $|V(K) \cap V(C)| \geq 2$ and $V(K) \cap V(C)$ contains a vertex of each of the two colour classes. Then Lemma 4.3.4 implies the existence of a matching cross over C in B. So, in either case, we reach a contradiction which means that there is no minimal counterexample and thus our proof is complete.

For the reverse let B be a minimal counterexample to the assertion such that B is Pfaffian, there is a conformal cycle C in B which has a matching cross, but there is a C-reduction of B to H such that H is planar and C bounds a face. First, suppose H is isomorphic to B. Then, since C bounds a face of B, Proposition 4.1.1 implies that there cannot be a matching cross over C in B. Consequently, B is non-planar. Since there is a C-reduction of B, B is not the Heawood graph. Let K be a 4-cycle such that B is a 4-cycle sum of the braces $B_1, \dots, B_\ell$ at K where H is a C-reduction of B_1. With H being a C-reduction of B, this must exist. Thus, as B is a minimal counterexample and H is planar such that C bounds a face, there is no matching cross over C in B_1.

Since B_1 is a C-reduction of B, $V(K) \cap V(C)$ cannot contain vertices from both colour classes of B. Moreover, with B being Pfaffian, by Theorem 3.3.8, none of the B_i can have a conformal bisubdivision of $K_{3,3}$. Hence, by Lemma 4.3.6, there exists a perfect matching M_1 of B_1 such that there is a diffuse C-M_1-precross through K which is daring. However, in this case, Lemma 4.3.10 tells us that no C-reduction of B_1 can be planar such that C bounds a face. As we assumed H to be a C-reduction of B_1, this is a contradiction. □

4.4. Matching Crosses in Non-Pfaffian Braces

With Proposition 4.0.7, we already have one half of Theorem 4.0.4. To prove the non-Pfaffian part of our main result, we essentially need to strengthen Corollary 4.1.8 in the form of Proposition 4.0.8.

In light of Lemma 4.0.9, this means that every 4-cycle in a non-Pfaffian brace has a conformal cross. As a first step, we need to establish that we can always find a perfect matching M in a non-Pfaffian brace such that a prescribed 4-cycle C is M-conformal and there exists an M-conformal bisubdivision of $K_{3,3}$ in B. To do this, we make use of a helpful lemma of McCuaig once more.

Recall the definition of the odd möbius ladders. For a möbius ladder $\mathcal{M}_{4k+2}$ with $k \geq 2$ we call an edge e a *rung* if it lies on two 4-cycles. The *rungs* of a $\mathcal{M}_{4k+2}$ bisubdivision are the paths that correspond to the bisubdivided rungs of the Möbius ladder. The *base cycle* of $\mathcal{M}_{4k+2}$ is the Hamiltoncycle C that consists entirely of non-rung edges. The *base cycle* of a $\mathcal{M}_{4k+2}$ bisubdivision is the cycle that consists entirely of the paths corresponding to the non-rung edges of $\mathcal{M}_{4k+2}$.

We make use of the original, and slightly stronger, statement of Lemma 3.2.20.

Lemma 4.4.1 ([McC04])**.** Let B be a bipartite graph with a conformal bisubdivision of $K_{3,3}$ and M a perfect matching of B, then B contains an M-conformal bisubdivision L of $\mathcal{M}_{4k+2}$ for some $k \geq 1$. Furthermore, the rungs of L are M-conformal in case $k \geq 2$.

Lemma 4.4.2 (E*)**.** Let B be a non-Pfaffian brace and C a 4-cycle in B, then there exists a perfect matching M of B such that C is M-conformal and B has an M-conformal bisubdivision of $K_{3,3}$.

Proof. Let M' be any perfect matching of B for which C is M'-conformal. By Lemma 4.4.1 there exists an M'-conformal bisubdivision L of $\mathcal{M}_{4k+2}$ for some $k \geq 1$. In case $k = 1$ we are done, so assume $k \geq 2$. Let us choose M' such that k is as small as possible. We call a path P in L a *bisubdivided edge* if P corresponds to an edge of $\mathcal{M}_{4k+2}$. Note that since C is M'-conformal, it contains exactly two edges of M'. Moreover, since L is M'-conformal, if L contains a vertex x of C, then it also contains the vertex y of C with $xy \in M$. Indeed, if a bisubdivided edge P of L contains a vertex x of C, then either x is an endpoint of P, or P contains the vertex y of C with $xy \in M$. Let $\{e_1, e_2\} = E(C) \cap M$ and let P_1, P_2 be the subdivided edges of L such that $e_j \in E(P_j)$ if $e_j \in E(L)$. If $e_j \notin E(L)$ for some j, let P_j be chosen arbitrarily. In case $P_1 = P_2$ let us choose P_2 to be any non-P_1-rung of L instead. We show that there exists a perfect matching N of L such that $M \cap (E(P_1) \cup E(P_2)) \subseteq N$ and L contains an N-conformal bisubdivision L' of $\mathcal{M}_{4(k-1)+2}$ such that P_1 and P_2 are subdivided edges of L'. Since $M := (M' \setminus E(L)) \cup N$ is a perfect matching of B for which C is M-conformal, this is a contradiction to the choice of M', and thus we must have had $k = 1$ in the first place.

Let x_j, y_j be the endpoints of P_j.

In case P_1 and P_2 are both rungs, we may assume that x_1, x_2, y_1, y_2 appear on C' in the order listed, and $x_1 \in V_1$. Then C' is divided into four internally disjoint paths $Q_1, \dots, Q_4$ such that Q_1 connects x_1 to x_2, Q_2 connects x_2 to y_1, and so forth. Moreover, every rung of L that has an endpoint on Q_1 also has an endpoint of Q_3, similarly for Q_2 and Q_4. Let us call the number of rungs that are different from P_1 and P_2 and have an endpoint on Q_j, the *length* of Q_j. With $i \geq 2$ at least one of Q_1 and Q_2 has length at least two. Without loss of generality let us assume this to be true for Q_1 and thus also for Q_3. Let R_1 and R_2 be two rungs whose endpoints on Q_1 are internal vertices of Q_1 and consecutive, i.e. no other rung has an endpoint on the subpath of Q_1 connecting R_1 to R_2. Let Q_1' be this subpath and let Q_3' be the corresponding subpath of Q_3. Then, since R_1 and R_2 are M'-conformal by Lemma 4.4.1, $K \coloneqq R_1 Q_1' R_2 Q_2'$ is an M'-conformal cycle. Let $N \coloneqq ((M \cap E(L)) \setminus (E(K) \setminus M)) \cup ((E(K)) \setminus (M \cap E(L)))$, then R_1 and R_2 are internally N-conformal. Let L' be the N-conformal subgraph of L obtained by deleting all inner vertices of R_1 and R_2. Then L' is a bisubdivision of $\mathcal{M}_{4(k-1)+2}$ as required.

Now suppose, without loss of generality, that P_1 is not a rung of L, but P_2 is. Then one of x_1 and y_1 is an endpoint of a bisubdivided edge Q of L such that $Q \neq P_1$ and Q is not a rung, but no rung of L which shares an endpoint of Q is P_2. Let R_1 and R_2 be those rungs and let Q' be the subdivided edge of L that connects the other two endpoints of R_1 and R_2. Not $K \coloneqq Q R_1 Q' R_2$ is an M'-conformal cycle that does not contain a vertex of C. We set $N \coloneqq ((M \cap E(L)) \setminus (E(K) \setminus M)) \cup ((E(K)) \setminus (M \cap E(L)))$ and define L' as the subgraph of L we obtain by deleting the inner vertices of R_1 and R_2. Then L' is again a bisubdivision of $\mathcal{M}_{4(k-1)+2}$ as required.

If both P_1 and P_2 are subpaths of C' we can again find some vertex $z \in \{x_1, x_2, y_2, y_2\}$ such that z is an endpoint of a bisubdivided edge Q that is a subpath of C' and different from P_1 and P_2. Let R_1 and R_2 be the two rungs that share endpoints with Q and let Q' be the bisubdivided edge of L that connects the other two endpoints of R_1 and R_2. Note that

z can be chosen such that Q' is also different from P_1 and P_2. We define K, N, and L' as above, and thus the proof is complete. $\square$

So for every 4-cycle C in a non-Pfaffian brace there is a perfect matching M such that C is M-conformal and there exists an M-conformal bisubdivision of $K_{3,3}$ in B. The next step is to show that we may assume $V(C)$ to be a subset of the vertices of this bisubdivision.

Lemma 4.4.3 (E*)**.** Let B be a non-Pfaffian brace and C a 4-cycle in B, then there exists a perfect matching M of B such that C is M-conformal and there is an M-conformal bisubdivision L of $K_{3,3}$ with $V(C) \subseteq V(L)$.

Proof. By Lemma 4.4.2 there exist a perfect matching M' and an M'-conformal bisubdivision L' of $K_{3,3}$ in B such that C is M' conformal. In case $V(C) \subseteq V(L')$ we are done. Next suppose L' contains exactly one of the two edges in $M' \cap E(C)$, let xy be this edge. Then C contains an internally M'-conformal path P with $V(P) = V(C)$ with endpoints x and y. Let $M := M' \Delta E(C)$, then P is M-conformal and by replacing xy in L' with P we obtain an M-conformal bisubdivision L of $K_{3,3}$ as desired. So from now on, we may assume C and L' to be vertex disjoint. Let $ab \in E(C) \setminus M'$. By using Theorem 3.1.69 and Lemma 3.1.42, we can find two internally M'-conformal paths P_a and P_b such that each P_x has $x \in \{a, b\}$ as an endpoint, has its other endpoint on L' and is otherwise disjoint from L' and C. Moreover, P_a and P_b are either disjoint, or $P_a \cap P_b$ is an M'-conformal path. What follows is a case distinction on how P_a and P_b connect C to L'. For each $x \in \{a, b\}$ let s_x be the endpoint of P_x on L', and let U be the M'-conformal path of length four on C with endpoints a and b.

Case 1: P_a and P_b are disjoint and there exists a bisubdivided edge Q of L' containing both s_a and s_b.

Since L' is an M'-conformal bisubdivision of $K_{3,3}$ and s_a and s_b belong to different colour classes, we can choose M' such that the subpath connecting s_a to s_b on Q is internally M'-conformal. Then we can simply replace Q by $P_a U P_b$ in order to obtain L as desired.

Case 2: $P_a \cap P_b$ is an M'-conformal path, and there exists a bisubdivided edge Q of L' containing both s_a and s_b.

Since L' is an M'-conformal bisubdivision of $K_{3,3}$ and s_a and s_b belong to different colour classes, we can choose M' such that the subpath R connecting s_a to s_b on Q is internally M'-conformal. Let W be the internally M'-conformal subpath of $P_a + P_b$ with endpoints s_a and s_b, then we can replace Q by W in order to obtain a new M'-conformal bisubdivision L'' of $K_{3,3}$ which meets exactly the requirements of the previous case. So by reapplying the arguments from above we can find a conformal bisubdivision L of $K_{3,3}$ as desired.

Case 3: P_a and P_b are disjoint, and there exist bisubdivided edges Q_a and Q_b of L' that share exactly one endpoint such that each Q_x contains s_x for $x \in \{a, b\}$.

Let z be the common endpoint of Q_a and Q_b and let us assume, without loss of generality, that z belongs to the same colour class as s_a. We may choose M' such that Q_a is M'-conformal. Let R_1 be the subpath of Q_a that connects the non-z-endpoint of Q_a to s_a, then R_1 is also M'-conformal. Let u be the non-z-endpoint of Q_b and v be the non-z-endpoint of the third bisubdivided edge Q' of L' that has z as an endpoint. Let $R_2 \coloneqq s_aQ_azQ'v$, as well as R_3, be the path $P_aUP_bs_bQ_bu$. Now R_2 and R_3 are internally M'-conformal and by replacing Q_a, Q_b and Q' with R_1, R_2 and R_3 we have found our desired M'-conformal bisubdivision of $K_{3,3}$.

Case 4: $P_a \cap P_b$ is an M'-conformal path, and there exist bisubdivided edges Q_a and Q_b of L' that share exactly one endpoint such that each Q_x contains s_x for $x \in \{a, b\}$.

Let W be the internally M'-conformal subpath of $P_a + P_b$ with endpoints s_a and s_b. Let z be the common endpoint of Q_a and Q_b and let us assume, without loss of generality, that z belongs to the same colour class as s_a. We may choose M' such that Q_a is M'-conformal. Let R_1 be the subpath of Q_a that connects the non-z-endpoint of Q_a to s_a, then R_1 is also M'-conformal. Let u be the non-z-endpoint of Q_b and v be the non-z-endpoint of the third bisubdivided edge Q' of L' that has z as an endpoint. Let $R_2 \coloneqq s_aQ_azQ'v$ as well as R_3 be the path WQ_bu. Now R_2 and R_3 are internally M'-conformal and by replacing Q_a, Q_b and Q' with R_1, R_2 and R_3 we have found an M'-conformal bisubdivision L'' of $K_{3,3}$ together with two disjoint internally M'-conformal paths, each linking a vertex of $\{a, b\}$ to a common bisubdivided edge of L''. Hence by recurring to the first case, we can finish the argument.

Case 5: P_a and P_b are disjoint and there exist bisubdivided edges Q_a and Q_b of L' that vertex disjoint such that each Q_x contains s_x for $x \in \{a, b\}$. Let Q be the unique bisubdivided edge of L' that shares an endpoint, say v_a, with Q_a and an endpoint, let us call it v_b, with Q_b such that for each $x \in \{a, b\}$, v_x and x belong to the same colour class of B. For each $x \in \{a, b\}$ let F_x be the bisubdivided edge of L' with endpoint v_x that is neither Q nor Q_x. Moreover, let R_1^x be the subpath of Q_x connecting s_x to the non-v_x-endpoint of Q_x. Since L' is an M'-conformal bisubdivision of $K_{3,3}$, we may choose M' such that Q_a and Q_b both are M'-conformal. Then R_1^a and R_1^b are M'-conformal as well. For each $x \in \{a, b\}$ let $R_2^x \coloneqq s_x Q_x v_x F_x$ and let R be the path $P_a U P_b$. Then let L be the graph obtained from L' by replacing Q_a, F_a, Q, Q_b, and F_b with the R_i^x, $i \in [1, 2]$, $x \in \{a, b\}$, and R. It is straight forward to check that L is an M'-conformal bisubdivision of $K_{3,3}$ as required by the assertion.

Case 6: $P_a \cap P_b$ is an M'-conformal path and there exist bisubdivided edges Q_a and Q_b of L' that vertex disjoint such that each Q_x contains s_x for $x \in \{a, b\}$.

As before with the even numbered cases let W be the internally M'-conformal subpath of $P_a + P_b$ with endpoints s_a and s_b. We then repeat the construction from Case 5 in order to obtain an M'-conformal bisubdivision L'' of $K_{3,3}$ together with two disjoint internally M'-conformal paths that meet the requirements of the first case. By reapplying the arguments of the first case, we finally obtain L as desired, and thus our proof is complete. □

Having established Lemma 4.4.3, the next step on the agenda is to analyse how the edges of $E(C) \cap M$ can occur in the M-conformal bisubdivision L of $K_{3,3}$. The goal is to identify the cases where we immediately find a conformal cross over C, and which cases cannot occur in the first place.

Let B be a non-Pfaffian brace, C a 4-cycle in B and M a perfect matching of B such that there exists an M-conformal bisubdivision L of $K_{3,3}$ in B for which $V(C) \subseteq V(L)$. Let $\{ab, a'b'\} = E(C) \cap M$ such that $a, a' \in V_1$ and let P, Q be two odd length M-alternating paths where each $X \in \{P, Q\}$ has endpoints a_X, b_X such that $a_X \in V_1$. We say that $e \in \{ab, a'b'\}$ *occurs* on P, if $e \in E(P)$ and it *occurs in reverse* on P if $P - e$ consists of two paths of even length. Please note that in case both of ab and $a'b'$ occur on P such that exactly one of them occurs in reverse, then no perfect matching M' for which P is M'-conformal or internally M'-conformal can contain both edges. To see this simply observe that an edge occurring in reverse on an M-alternating path P belongs to M if and only if P is M-conformal.

Observation 4.4.4 (E*)**.** Let B be a non-Pfaffian brace, C a 4-cycle in B and M a perfect matching of B such that there exists an M-conformal bisubdivision L of $K_{3,3}$ in B for which $V(C) \subseteq V(L)$. Let $\{ab, a'b'\} = E(C) \cap M$ such that $a, a' \in V_1$ and let P be a bisubdivided edge of L. If both ab and $a'b'$ occur on P, then either both or none of them occurs in reverse.

If P and Q have a common endpoint z and are otherwise disjoint, we say that ab and $a'b'$ are *split* over P and Q if exactly one of ab and $a'b'$ occurs on P and the other one occurs on Q. They are said to be *split nicely* if the shortest $\{a, b\}$-$\{a', b'\}$-subpath R of PzQ has even length, and $z \in V(R)$ does not share the colour of the endpoints of R. In case R is even, and z belongs to the same colour class as the two endpoints of R we say ab and $a'b'$ are *split completely* over P and Q. Please note that, if ab and $a'b'$ are split completely over P and Q, then each of the two edges occurs in reverse on its respective path. Moreover, by the discussion above, P and Q would need both be M-conformal in order to guarantee $ab, a'b' \in M$. Hence they can never be split completely over two bisubdivided edges of L that share an endpoint.

Observation 4.4.5 (E*)**.** Let B be a non-Pfaffian brace, C a 4-cycle in B, and M a perfect matching of B such that there exists an M-conformal bisubdivision L of $K_{3,3}$ in B for which $V(C) \subseteq V(L)$. Let $\{ab, a'b'\} = E(C) \cap M$ such that $a, a' \in V_1$ and let P, Q be two bisubdivided edges of L sharing a single endpoint. If ab and $a'b'$ are split over P and Q, then they are not completely split.

Lemma 4.4.6 (E*)**.** Let B be a non-Pfaffian brace, C a 4-cycle in B and M a perfect matching of B such that there exists an M-conformal bisubdivision L of $K_{3,3}$ in B for which $V(C) \subseteq V(L)$. Let $\{ab, a'b'\} = E(C) \cap M$ such that $a, a' \in V_1$ and let P, Q be two bisubdivided edges of L such that ab occurs on P and $a'b'$ occurs on Q. If P and Q are disjoint, or ab and $a'b'$ are split nicely over P and Q, there exist paths R_1 and R_2 in L that form a conformal cross over C.

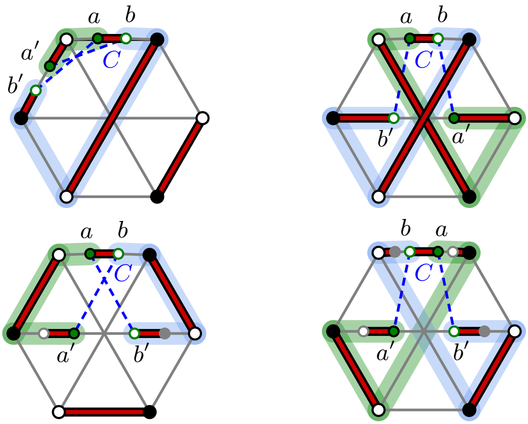


Figure 4.12.: The conformal crosses over the 4-cycle C in a bisubdivision of $K_{3,3}$ from Lemma 4.4.6.

Proof. The proof is essentially another case distinction over the following cases:

i) ab and $a'b'$ are nicely split over P and Q,

ii) P and Q are disjoint and neither ab nor $a'b'$ occurs in reverse on its respective path,

iii) P and Q are disjoint and, without loss of generality, $a'b'$ occurs in reverse on Q, and

iv) P and Q are disjoint and both, ab and $a'b'$, occur in reverse on their respective path.

The perfect matchings of L together with the paths R_1 and R_2 are illustrated in Figure 4.12. Please note that the copies of $K_{3,3}$ depicted in the figure are in fact bisubdivisions. Where necessary, additional subdivision vertices are drawn, but in general, the edges depicted in a light grey may be subdivided an arbitrary, but even, number of times. If in order to depict the respective perfect matching, a bisubdivided edge is marked and bold, this means that the respective path is M-conformal, while an unmarked bisubdivided edge represents an internally M-conformal path. □

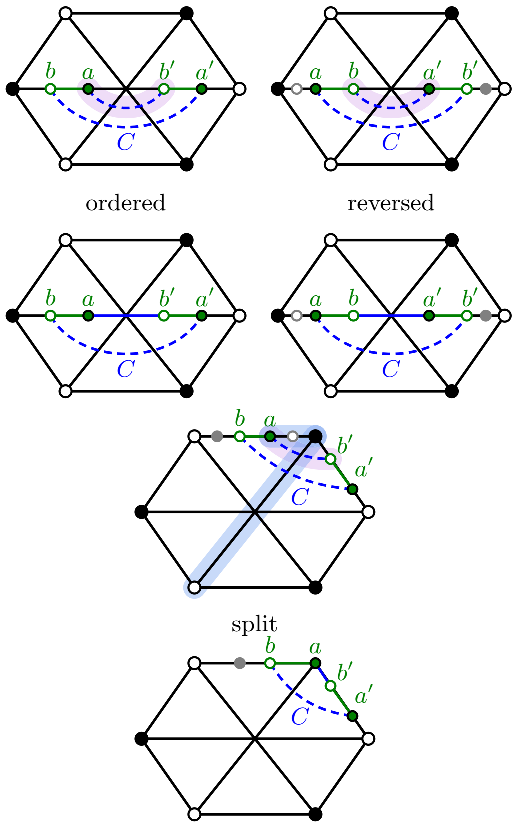


Figure 4.13.: The three possible configurations how the edges ab and $a'b'$ may occur in a bisubdivision of $K_{3,3}$ without immediately yielding a conformal cross over C. The second line of figures shows how these cases can be reduced.

If we can find a conformal bisubdivision of $K_{3,3}$ that fits one of the cases in Lemma 4.4.6, we are done immediately. Hence what remains is a discussion of the three cases which are still left. In Figure 4.13, these cases are illustrated. In two of the three cases, the edges ab and $a'b'$ occur on a single subdivided edge of L, while in the last case, the *split case*, ab and $a'b'$ are split over two subdivided edges in such a way that exactly one of ab and $a'b'$ occurs in reverse on its respective path. In each of the three cases, we can use an edge $e \in E(C) \setminus \{ab, a'b'\}$ in order to further reduce L and make sure that we always find a conformal bisubdivision of $K_{3,3}$ which contains at least three edges of C. Let B be a brace, C a 4-cycle, M a perfect matching of B such that C is M-conformal and L an M-conformal bisubdivision of $K_{3,3}$ that contains the vertices of C. We say that L *splits* C if the way the vertices of C are distributed over the bisubdivided edges of L as they are in the split case in Figure 4.13. As an intermediate step, we want to show that we can always find a bisubdivision of $K_{3,3}$ that splits C. Suppose ab and $a'b'$ occur on a single subdivided edge P of L as in the *ordered* or the *reversed* case from Figure 4.13. Let $u \in V_i$ be an endpoint of P and let Y be the *bisubdivided claw* with centre u in L consisting of the three bisubdivided edges P, Q_1, and Q_2 of L that have u as an endpoint. Let T be the shortest u-$V(C)$-subpath of P. If there exists an internally M-conformal path R that is internally disjoint from L such that R has an endpoint in $V_{3-i}(T)$ and its other endpoint lies in $V_i(L - Y)$ we say that L has a *V_i-jump over C*.

Lemma 4.4.7 (E*)**.** Let B be a brace, M a perfect matching of B, $H \subseteq B$ an M-conformal and matching covered subgraph, and $X \subseteq V(H)$ such that $\partial_H(X)$ is a non-trivial tight cut in H. Then there exists an internally M-conformal path P in B such that P is internally disjoint from H and has its endpoints in the minorities of X and $V(H) \setminus X$.

Proof. The claim follows immediately from Theorem 3.1.69. With B being a brace it is 2-extendible. Let e be the unique edge of M in $\partial_H(X)$. Then there must be an internally M-conformal $\mathrm{Min}(X)$-$\mathrm{Min}(Y)$-path P in B that avoids e. If we choose P to be as short as possible, it cannot contain any vertex of $\mathrm{Min}(X) \cup \mathrm{Min}(Y)$ as an inner vertex. Moreover, since H is M-conformal, no vertex of H can be an inner vertex of P. □

Lemma 4.4.8 (E*)**.** Let B be a non-Pfaffian brace and C a 4-cycle in B such that there is no conformal cross over C in B. Then there exists a perfect matching M of B such that C is M-conformal and there is an

M-conformal bisubdivision L of $K_{3,3}$ such that L splits C, or L has a V_1-jump over C.

Proof. By lemmata 4.4.3 and 4.4.6 and the discussion above we know that there are a perfect matching M of B such that C is M-conformal and an M-conformal bisubdivision L of $K_{3,3}$ that contains the vertices of C such that the way the vertices of C occur in L corresponds to one of the three cases depicted in Figure 4.13. If L splits C we are done already, so let us assume that there is a bisubdivided edge P of L such that the edges $ab, a'b \in E(C) \cap M$ occur on P as in the *ordered* or the *reversed* case from Figure 4.13. Let $u \in V_1$ be an endpoint of P. Consider the three bisubdivided edges P, Q_1, and Q_2 of L that have u as an endpoint. Let us choose L such that the tuple $(|E(P)|, |E(Q_1) \cup E(Q_2)|)$ is lexicographically minimised. For each $Z \in \{P, Q_1, Q_2\}$ let $v_Z \in V_2$ be the endpoint of Z different from u and let $Y := (V(P) \cup V(Q_1) \cup V(Q_2)) \setminus \{v_P, v_{Q_1}, v_{Q_2}\}$. Now every component of $L[Y] - u$ is a path of odd length and thus $|V_1 \cap Y| - |V_2 \cap Y| = 1$, moreover, no vertex of $Y \cap V_2$ has a neighbour in $L - Y$ within L and $|Y \cap V(C)| \geq 3$. Hence $\partial_L(Y)$ defines a non-trivial tight cut in L. For an illustration, see Figure 4.14.

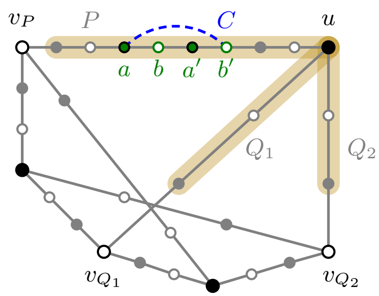


Figure 4.14.: The non-trivial tight cut around the bisubdivided claw centred at u in the proof of Lemma 4.4.8

By Lemma 4.4.7 there exists an internally M-conformal path F in B such that

- F has an endpoint in $V_2 \cap Y$,
- the other endpoint of F lies in $V_1(L) \setminus Y$, and

- F is internally disjoint from L.

Please note that we may change the perfect matching M within the M-conformal subgraph L of B at will, without changing the fact that F is an internally M-conformal path with the properties listed above.

Let y be the endpoint of F in Y and let x be its other endpoint as well as P_x be the subdivided edge of L that contains x in case x is a vertex of degree two in L. What follows is a discussion on the possible positions of x and y in L. For an illustrative overview on the different cases that might appear consult Figure 4.15. Let T_{V_1} be the shortest subpath of P with one endpoint in $V(C)$ and u as its other endpoint. Similarly, let T_{V_2} be the shortest subpath of P with one endpoint in $V(C)$ and v_P as its other endpoint. At last, let w_1 and w_2 be the two degree three vertices in $V_1(L)$ that are different from u, let w_1 be the endpoint of P_x that lies in V_1. Given any two vertices v_1, v_2 of degree three in L that belong to different colour classes let us denote by $E_{v_1v_2} = E_{v_2v_1}$ the subdivided edge of L with endpoints v_1 and v_2.

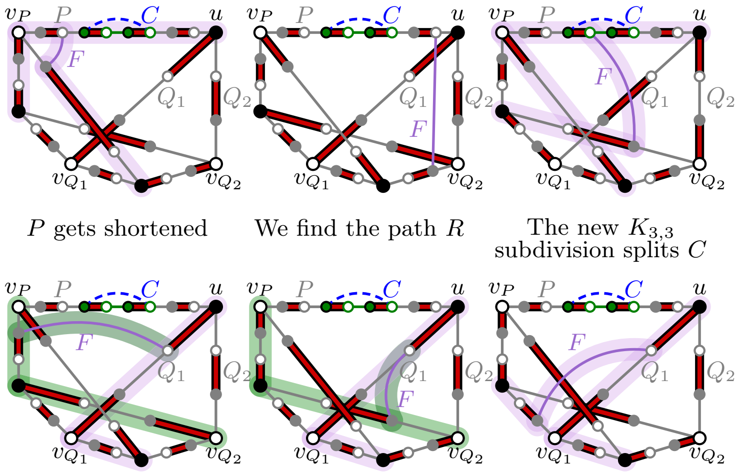
Case 1: $y \in V(T_{V_2} - C)$

Suppose P_x contains the vertex v_P, let W be the third bisubdivided edge with endpoint v_P. Then choose M such that both P and P_x are internally M-conformal. Now we may replace the three subdivided edges of L with v_P as an endpoint by the following three M-alternating paths in order to obtain an M-conformal bisubdivision L' of $K_{3,3}$, where the subdivided edge P' that contains $V(C)$ is strictly shorter than P, thereby violating the minimal choice of L. We set $P' := yPu$ and the other paths are yPv_PW and $yFxP_xw_1$

So we may assume P_x does not contain v_P which means that there is $i \in [1, 2]$ such that P_x has v_{Q_i} as an endpoint. We now aim for a bisubdivision L' of $K_{3,3}$ in which both x and y are vertices of degree three. As before, the bisubdivided edge of L' that contains $V(C)$ will be shorter than P and thus provide a contradiction. We now replace the paths P, P_x, $E_{v_Pw_1}$, $E_{v_Pw_2}$, and $E_{w_1v_{Q_{3-i}}}$ by the paths yPu, F, $yPv_PE_{v_Pw_2}$, $xP_xv_{Q_i}$, and $xP_xw_1E_{w_1v_{Q_{3-i}}}$ to obtain the graph L'. Since L is a bisubdivision of $K_{3,3}$, we may choose M such that L' is M-conformal and thus we are done with this case.

Case 2: $y \in V(T_AV - 1 - C)$

In this case F is a V_1-jump over C and thus we are done immediately.



P gets shortened | We find the path R | The new $K_{3,3}$ subdivision splits C

Q_1 gets shortened | Q_1 gets shortened | Q_1 gets shortened

Figure 4.15.: Examples of the cases occuring in the proof of Lemma 4.4.8

Case 3: $y \in V(C)$

In essence, we can repeat the construction from the first case to obtain an M-conformal bisubdivision L' of $K_{3,3}$. Since $y \in V(C)$ we end up with some L' in which the edges ab and $a'b'$ occur on two different subdivided edges that share the endpoint y. Thus L' splits C and we can close this case.

Case 4: $y \notin V(P_x)$

We may assume $y \in V(Q_1)$ as $y \in V(Q_2)$ can be handled analogously. Instead of P as in the first case we reduce the length of Q_1 while maintaining the lengths of P and Q_2 in order to obtain a contradiction. The main idea of the construction, however, remains the same as in the first case and thus we omit the exact construction here. In Figure 4.15, the possible ways to obtain the new $K_{3,3}$-bisubdivision L' are illustrated.

Combining all of these cases, this means that $\partial_L(Y)$ cannot be a non-trivial tight cut. Since otherwise we are either done since we find a path F that allows us to change L into L' which splits C, or F is a V_1-jump over C. However, by construction $|V(C) \cap Y| \geq 3$ and $|V(L) \setminus Y| \geq 3$ and thus this is impossible. It follows that L itself must already split C and so we are done. $\square$

Lemma 4.4.9 (E*)**.** Let B be a non-Pfaffian brace and C a 4-cycle in B such that there is no conformal cross over C in B. If there exists a perfect matching M of B such that C is M-conformal and there is an M-conformal bisubdivision L of $K_{3,3}$ that has a V_1-jump over C, then there exists a perfect matching M' of B such that C is M'-conformal and there is an M'-conformal bisubdivision L' of $K_{3,3}$ that either splits C, or has both, a V_1-jump and a V_2-jump over C.

Proof. The proof is a slight alteration of the proof of the previous lemma. Since L has an V_1-jump over C, there exists a bisubdivided edge P of L such that $V(C) \subseteq V(P)$. Let $u \in V_1$ and $v \in V_2$ be the endpoints of P. Let a_1, a_2, b_1, and b_2 be the four degree three vertices of L aside from u and v such that $a_1, a_2 \in V_1$. Now let $Y := V(P-u) \cup V(E_{va_1} - a_1) \cup V(E_{va_2} - a_2)$. By the same arguments as in the previous lemma, $\partial_L(Y)$ must be a non-trivial tight cut. Similar to before we choose L such that the tuple $(|E(P)|, |E(E_{va_1}) \cup E(E_{va_2})|)$ is lexicographically minimised. By using the same case distinction as in the proof of Lemma 4.4.8 we either reach a contradiction, find a conformal $K_{3,3}$-bisubdivision L' that splits C, or the path F yielded by Lemma 4.4.7 is a V_2-jump over C in L. The major difference between this lemma and Lemma 4.4.8 is, that we have to maintain the existence of a V_1-jump over C. In the technique from the proof of the previous lemma, there are two possible ways, the existence of an V_1-jump over C in the newly constructed $K_{3,3}$-bisubdivision L' is threatened[8]. Let R be an V_1-jump over C for L.

The easier to handle case is the one in which the newly found path F in **Case 4** of the case distinction intersects R. However, since R and F are internally M-conformal, let z be the first vertex of R on F, then Rz still is internally M-conformal, and thus in this case, L' still has a V_1-jump over C.

The more complicated case is a subcase of **Case 1**. Let T_{V_1} be the shortest subpath of P with one endpoint in $V(C)$ and u as its other endpoint. Similarly, let T_{V_2} be the shortest subpath of P with one endpoint in $V(C)$ and v as its other endpoint. If F has its endpoint in Y on the

[8]Note that in case L' splits C we are done.

subpath of T_{V_1} that connects $V(C)$ to R, then no subpath of R can be a V_1-jump over C for L'. However, in this case, we have found a conformal $K_{3,3}$-bisubdivision where the path in which both M-edges of C occur is shorter than in L. Among those bisubdivisions choose L' to be one that lexicographically minimises $(|E(P')|, |E(E_{u'b'_1}) \cup E(E_{u'b'_2})|)$, where the vertices marked with a $'$ are those of L' that naturally correspond to the vertices of L. similarly we define P'. By reapplying the case distinction of Lemma 4.4.8 to L' we either find a $K_{3,3}$-bisubdivision L'' that splits C, of we find a new V_1-jump over C for L' which would contradict our choice of L in the first place since $|E(P')| < |E(P)|$. Hence if we cannot find a conformal $K_{3,3}$-bisubdivision that splits C, we always find one that has both, a V_1-jump and a V_2-jump over C. □

Lemma 4.4.10 (E*)**.** Let B be a non-Pfaffian brace and C a 4-cycle in B such that there is no conformal cross over C in B. Then there exists a perfect matching M of B such that C is M-conformal and there is an M-conformal bisubdivision L of $K_{3,3}$ that splits C.

Proof. By Lemma 4.4.8 we either find a conformal bisubdivision L' of $K_{3,3}$ that splits C, in which case we are done, or we find one with a V_1-jump over C. Then Lemma 4.4.9 might again yield the existence of a conformal bisubdivision L of $K_{3,3}$ that splits C if it does not we find M and L such that L has a V_1-jump R_{V_1} and a V_2-jump R_{V_2} over C. Let P be the bisubdivided edge of L that contains the vertices of C. We may assume L to be a conformal $K_{3,3}$-bisubdivision that minimises the length of P among all conformal bisubdivisions of $K_{3,3}$ for which ab and $a'b'$ occur on a single bisubdivided edge P. By Lemma 3.1.42 we may assume that R_{V_1} and R_{V_2} are either disjoint, or $R_{V_1} \cap R_{V_2}$ is an M-conformal path. For each $X \in \{V_1, V_2\}$ let v_X be the endpoint of R_X that does not belong to the bisubdivided edge P. We have to consider the cases how v_{V_1} and v_{V_2} occur on the bisubdivided edges of L and for each of these cases we need to look at R_{V_1} and R_{V_2} being disjoint or meeting in an M-conformal path. Let $u \in V_1$ and $v \in V_2$ be the endpoints of P and let $a_1, a_2 \in V_1$, $b_1, b_2 \in V_2$ be the remaining four vertices of degree three in L. Then R_{V_1} cannot have an endpoint on E_{ub_1} or E_{ub_2}, while R_{V_2} cannot have an endpoint on E_{va_1} or E_{va_2}. Our goal is to show that R_{V_1} and R_{V_2} can be used to produce a contradiction to the choice of L with respect to the minimality of P.

Let us first consider the cases where at least one of R_{V_1} and R_{V_2} has an endpoint on one of the E_{ub_i} or E_{va_i}. By symmetry, we just need to consider the case where R_{V_1} meets E_{va_1} and R_{V_2} meets E_{ub_1}, and the case where R_{V_1} meets E_{va_1} while R_{V_2} meets an arbitrary other bisubdivided edge of L, say $E_{a_1b_1}$. Please note that in all of these cases, it does not play a role whether ab and $a'b'$ occur in reverse on P or not. Hence we only treat the case where ab and $a'b'$ are not reversed. In Figure 4.16 we give exemplary constructions of a new conformal $K_{3,3}$-bisubdivision L' which still has a bisubdivided edge P' containing ab and $a'b'$, but with $|E(P')| < |E(P)|$ this contradicts the choice of L.

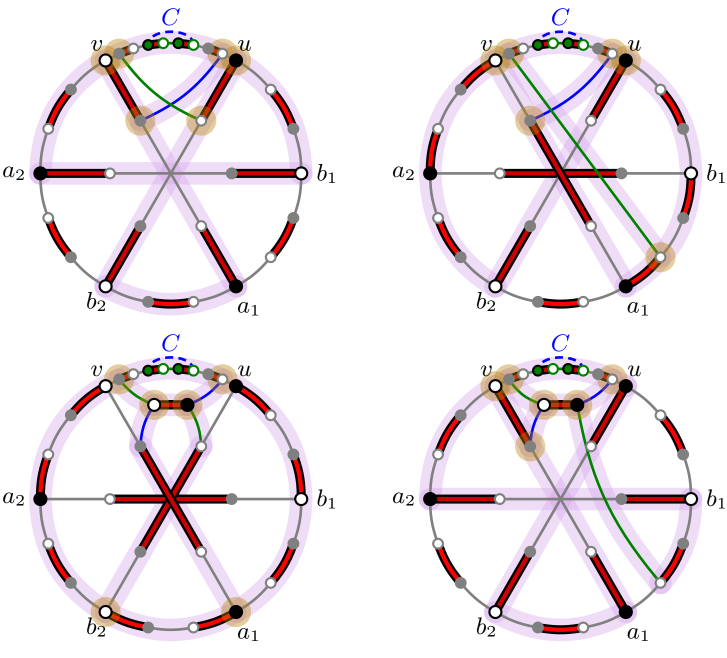


Figure 4.16.: The construction of the new conformal $K_{3,3}$-bisubdivision in the first case of the proof of Lemma 4.4.10.

For the next case we assume v_{V_1} and v_{V_2} to be vertices of a common bisubdivided edge Q of L. According to the previous discussion, Q cannot share an endpoint with P and by symmetry, it suffices to only consider one possible choice for Q, so let $Q := E_{a_2b_1}$. The path Q is split into three, possibly trivial, subpaths by the vertices v_{V_1} and v_{V_2}. Since Q is of odd length, either zero or exactly two of these subpaths are of even length, and these are exactly the two cases we need to distinguish. Figure 4.17 shows how to construct the new conformal $K_{3,3}$-bisubdivision L' which yields the desired contradiction.

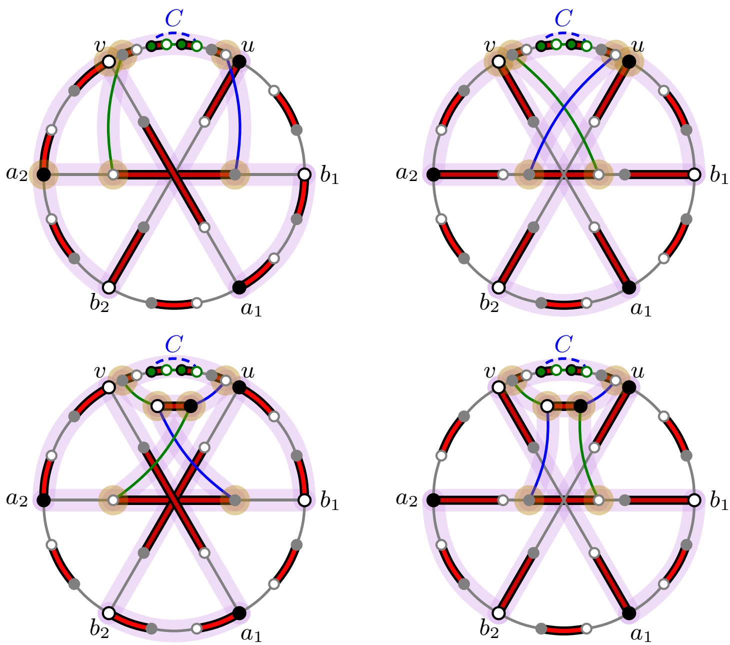


Figure 4.17.: The construction of the new conformal $K_{3,3}$-bisubdivision in the second case of the proof of Lemma 4.4.10.

For the last case, we may assume v_{V_1} and v_{V_2} to belong to different bisubdivided edges Q_1 and Q_2 such that neither Q_i shares an endpoint with P. Here we need to distinguish between Q_1 and Q_2 sharing an endpoint and being disjoint. Figure 4.18 illustrates the construction of L'.

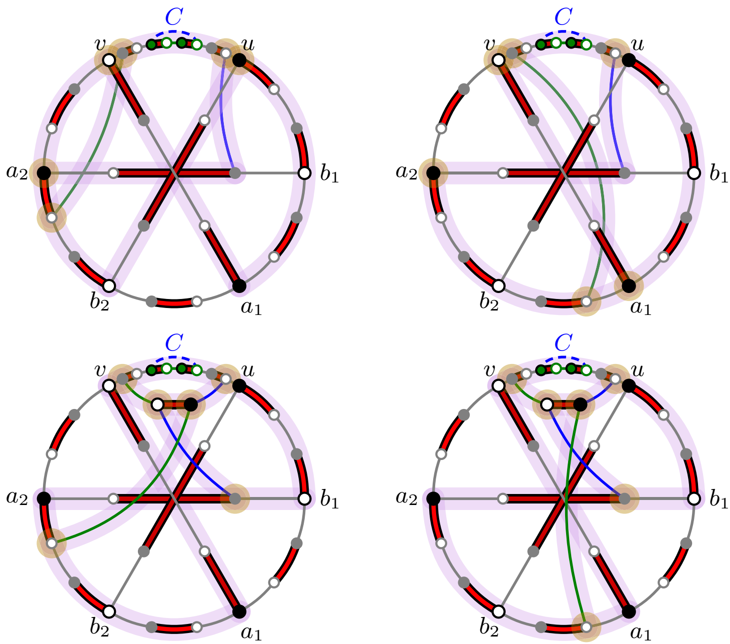


Figure 4.18.: The construction of the new conformal $K_{3,3}$-bisubdivision in the third case of the proof of Lemma 4.4.10.

So whenever we find both a V_1-jump and a V_2-jump in L, we are able to find a conformal bisubdivision L' of $K_{3,3}$ with a bisubdivided edge P' that contains all of C but is shorter than P in the previous bisubdivision. Thus by choosing L with minimal P, we still find a non-trivial tight cut as in the proofs of lemmata 4.4.8 and 4.4.9, but neither of these tight cuts may yield a V_1-jump. Hence we must be able to construct a conformal bisubdivision of $K_{3,3}$ that splits C. $\square$

With this we are ready to close this section with the proof of Proposition 4.0.8.

Proof of Proposition 4.0.8. Suppose B is a counterexample, so there exists a 4-cycle C in B such that C is not a subgraph of a conformal $K_{3,3}$-

bisubdivision. By Lemma 4.0.9 this means that there is no conformal cross over C in B and thus, by Lemma 4.4.10 there exists a conformal bisubdivision L of $K_{3,3}$ that splits C. As we have seen in Figure 4.13 we may choose L such that one of the degree three vertices in L belongs to C, let us call that vertex u. Let P_1, P_2, and P_3 be the bisubdivided edges of L that have u as an endpoint and let v_i be the other endpoint of P_i for all $i \in [1,3]$. Let $Y := \bigcup_{i=1}^{3} V(P_i - v_i)$ and let us choose L among all conformal bisubdivisions of $K_{3,3}$ in B that split C to be one where $|Y|$ is minimal. As L splits C we still have $|Y \cap V(C)| \geq 3$ and $|V(L) \setminus Y| \geq 3$ and thus Y is, as we have seen before, a non-trivial tight cut whose majority is exactly the colour class u belongs to. Without loss of generality let us assume the minority of Y to be in V_1. Observe that $C \cap L$ forms an M-conformal path that must contain internal vertices of two different bisubdivided edges of L. By Lemma 4.4.7 there must exist an internally M-conformal path F that has one endpoint in $Y \cap V_1$ and the other one in $V_2(L - Y)$ such that F is internally disjoint from L. Recall the constructions illustrated in Figure 4.15 and suppose the endpoint of F in Y is an interior vertex of $C \cap Y$. If this is the case, we find a conformal bisubdivision L' of $K_{3,3}$ in which ab and $a'b'$ belong to two different bisubdivided edges which do not share an endpoint. By Lemma 4.4.6 this means we find a conformal bisubdivision of $K_{3,3}$ which contains C as a subgraph, contradicting B being a counterexample. Hence F cannot contain an inner vertex of $C \cap L$. But in this case, we can find a conformal $K_{3,3}$-bisubdivision L' that splits C such that Y', which is defined for L' in the same way as Y is defined for L, contains fewer vertices than Y which contradicts our choice of L. So either way we reach a contradiction and thus there cannot be a counterexample to our claim. $\square$

4.5. An Algorithm for 2-MLP

To obtain an algorithmic solution for 2-MLP, we use Proposition 4.0.8 together with Corollary 3.3.10. On a high level, we run into the following problems: First, we do not know for which perfect matching M of B we might be able to find a solution for 2-MLP and since there is a potentially exponential number of perfect matchings in B it clearly does not suffice to simply test all of them. Indeed, such an approach is doomed from the beginning since trying to solve 2-MLP for a fixed perfect matching is equivalent to the 2-DLP. So we take a slightly different approach. Let a_1, a_2, b_1, b_2 be the four vertices of an instance of 2-MLP. Let $|V(B)| = n$, then B contains $\frac{n}{2}$ vertices of each colour. For each $x \in \{a_1, a_2, b_1, b_2\}$ we may choose from among the $\frac{n}{2}$ vertices of the opposite colour in order to find a neighbour that might be matched to x by some perfect matching of B. In total this means there are at most

$$2\binom{\frac{n}{2}}{2} = 2\frac{\frac{n}{2}(\frac{n}{2}-1)}{2} \in \mathcal{O}(n^2)$$

many choices of edges that might cover our four terminal vertices in a perfect matching of B. Let $F \subseteq E(B)$ be a set of at most four edges such that each vertex from among a_1, a_2, b_1, and b_2 is covered by an edge of F. Next we need to decide whether F is contained in a perfect matching of B, which can be done by the Hopcroft-Karp algorithm in time $\mathcal{O}(n^{\frac{5}{2}})$ [HK73]. In case such a perfect matching exists, we then alter the graph B locally which takes up constant time. The main concern of this section is to introduce this local construction and to show that the existence of a conformal cross over a well-chosen 4-cycle certifies the existence of the desired linkage in a way that makes use of the matching edges in F. The key to deciding whether a conformal cross over our 4-cycle exists is Proposition 4.0.8 in combination with Corollary 3.3.10. In total, this approach decides 2-MLP in time $\mathcal{O}(n^5)$.

Let B be a bipartite graph with a perfect matching, $F \subseteq E(B)$ and $X \subseteq V(B)$. The set F is said to be an X-*cover*, if every edge in F contains a vertex of X and every vertex in X is covered by an edge of F. If F is an extendible set of edges in B and M is a perfect matching B with $F \subseteq M$, M is said to *extend* F. From the discussion above it is clear that there are $\mathcal{O}(|V(B)|^2)$ many X-covers in B for any set $X \subseteq V(B)$ with $|X \cap V_1| = |X \cap V_2| = 2$. Given distinct vertices $a_1, a_2 \in V_1$, $b_1, b_2 \in V_2$, and an extendible $\{a_1, a_2, b_1, b_2\}$-cover $F \subseteq E(B)$ we say that B is an F-*instance* of 2-MLP for (a_1, a_2) and (b_1, b_2) if there exists a perfect matching M of B that extends F such that there are two disjoint internally M-conformal paths P_1 and P_2 such that P_i has endpoints a_i and b_i for each $i \in [1, 2]$.

Definition 4.5.1. Let B be a bipartite graph with a perfect matching, $a_1, a_2 \in V_1$ and $b_1, b_2 \in V_2$ four distinct vertices of B, and F an extendible $\{a_1, a_2, b_1, b_2\}$-cover of size four. Let $ua_2, vb_1 \in F$. We define the following transformation of B with respect to F, (a_1, a_2), and (b_1, b_2).

$$\mathsf{B}_3(B, F, \{a_1, a_2, b_1, b_2\}) \coloneqq B - u - v + a_2b_1$$
$$\mathsf{F}_3(B, F, \{a_1, a_2, b_1, b_2\}) \coloneqq (F \setminus \{ua_2, vb_1\}) \cup \{a_2b_1\}$$

Lemma 4.5.2 (E*)**.** Let B be a bipartite graph with a perfect matching, $a_1, a_2 \in V_1$ and $b_1, b_2 \in V_2$ four distinct vertices of B, and F an extendible $\{a_1, a_2, b_1, b_2\}$-cover of size four. Then B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2) if and only if $\mathsf{B}_3(B, F, \{a_1, a_2, b_1, b_2\})$ is a $\mathsf{F}_3(B, F, \{a_1, a_2, b_1, b_2\})$-instance of 2-MLP for (a_1, a_2) and (b_1, b_2).

Proof. If B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2) there is a perfect matching M that extends F such that there exist disjoint and internally M-conformal paths P_1 and P_2 where P_i has endpoints a_i and b_i for each $i \in [1, 2]$. Let $ua_2, vb_1 \in F$, then $\{u, v\} \cap (V(P_1) \cup V(P_2)) = \emptyset$. Let us add the edges uv and a_2b_1 to B, then $C \coloneqq ua_2b_1vu$ is an M-conformal 4-cycle in B. We set $M' \coloneqq (M' \setminus E(C)) \cup \{uv, a_2b_1\}$, then P_1 and P_2 are internally M'-conformal paths that still exist in $\mathsf{B}_3(B, F, \{a_1, a_2, b_1, b_2\})$ and $M' \setminus \{uv\}$ is a perfect matching of $\mathsf{B}_3(B, F, \{a_1, a_2, b_1, b_2\})$ that extends $\mathsf{F}_3(B, F, \{a_1, a_2, b_1, b_2\})$. Hence $\mathsf{B}_3(B, F, \{a_1, a_2, b_1, b_2\})$ is a $\mathsf{F}_3(B, F, \{a_1, a_2, b_1, b_2\})$-instance of 2-MLP for (a_1, a_2) and (b_1, b_2).

Now assume that $\mathsf{B}_3(B, F, \{a_1, a_2, b_1, b_2\})$ is a $\mathsf{F}_3(B, F, \{a_1, a_2, b_1, b_2\})$-instance of 2-MLP for (a_1, a_2) and (b_1, b_2). As before let M be a perfect matching extending $\mathsf{F}_3(B, F, \{a_1, a_2, b_1, b_2\})$ and let P_1, P_2 be the corresponding internally M-conformal paths. Let $ua_2, vb_1 \in F$, then $M \cup \{uv\}$ is a perfect matching of $B + a_2b_1 + uv$ and $C \coloneqq ua_2b_1vu$ is an $M \cup \{uv\}$-conformal 4-cycle in $B + a_2b_1 + uv$. We set $M' \coloneqq (M' \setminus E(C)) \cup \{ua_2, vb_1\}$, then P_1 and P_2 are internally M'-conformal paths in $B + a_2b_1 + uv$ that still exist in B and M' is also a perfect matching of B. Thus B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2). $\square$

Definition 4.5.3. Let B be a bipartite graph with a perfect matching, $a_1, a_2 \in V_1$ and $b_1, b_2 \in V_2$ four distinct vertices of B, $S \coloneqq \{a_1, a_2, b_1, b_2\}$, and F an extendible S-cover of size at least three such that $a_2b_1 \in F$ if and only if $|F| = 3$. Let $ua_1, vb_2 \in F$. We define the following transformation of B with respect to F, (a_1, a_2), and (b_1, b_2). If $|F| = 3$ use the following construction:

$$\mathsf{B}_2(B, F, S) := B - u - v + a_1 b_2$$
$$\mathsf{F}_2(B, F, S) := (F \setminus \{ua_1, vb_2\}) \cup \{a_1 b_2\}$$

Otherwise, we can first obtain an instance where our extendible cover has size three as required:

$$\mathsf{B}_2(B, F, S) := \mathsf{B}_2(\mathsf{B}_3(B, F, S), \mathsf{F}_3(B, F, S), S)$$
$$\mathsf{F}_2(B, F, S) := \mathsf{F}_2(\mathsf{B}_3(B, F, S), \mathsf{F}_3(B, F, S), S)$$

Lemma 4.5.4 (E*)**.** Let B be a bipartite graph with a perfect matching, $a_1, a_2 \in V_1$ and $b_1, b_2 \in V_2$ four distinct vertices of B, $S := \{a_1, a_2, b_1, b_2\}$, and F an extendible S-cover of size at least three. Then B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2) if and only if $\mathsf{B}_2(B, F, S)$ is a $\mathsf{F}_2(B, F, S)$-instance of 2-MLP for (a_1, a_2) and (b_1, b_2).

Proof. We only have to consider the case $|F| = 3$, since the case $|F| = 4$ follows, by Lemma 4.5.2, with the same arguments.

If B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2) there is a perfect matching M that extends F such that there exist disjoint and internally M-conformal paths P_1 and P_2 where P_i has endpoints a_i and b_i for each $i \in [1, 2]$. Let $ua_1, vb_2 \in F$, then $\{u, v\} \cap (V(P_1) \cup V(P_2)) = \emptyset$. Let us add the edges uv and $a_1 b_2$ to B, then $C := ua_1 b_2 vu$ is an M-conformal 4-cycle in B. We set $M' := (M' \setminus E(C)) \cup \{uv, a_1 b_2\}$, then P_1 and P_2 are internally M'-conformal paths that still exist in $\mathsf{B}_2(B, F, S)$ and $M' \setminus \{uv\}$ is a perfect matching of $\mathsf{B}_2(B, F, S)$ that extends $\mathsf{F}_2(B, F, S)$. Hence $\mathsf{B}_2(B, F, S)$ is a $\mathsf{F}_2(B, F, S)$-instance of 2-MLP for (a_1, a_2) and (b_1, b_2).

Now assume that $\mathsf{B}_2(B, F, S)$ is a $\mathsf{F}_2(B, F, S)$-instance of 2-MLP for (a_1, a_2) and (b_1, b_2). As before let M be a perfect matching extending $\mathsf{F}_2(B, F, S)$ and let P_1, P_2 be the corresponding internally M-conformal paths. Let $ua_1, vb_2 \in F$, then $M \cup \{uv\}$ is a perfect matching of $B + a_1 b_2 + uv$ and $C := ua_1 b_2 vu$ is an $M \cup \{uv\}$-conformal 4-cycle in $B + a_1 b_2 + uv$. We set $M' := (M' \setminus E(C)) \cup \{ua_1, vb_2\}$, then P_1 and P_2 are internally M'-conformal paths in $B + a_1 b_2 + uv$ that still exist in B and M' is also a perfect matching of B. Thus B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2). $\square$

Definition 4.5.5. Let B be a bipartite graph with a perfect matching, $a_1, a_2 \in V_1$ and $b_1, b_2 \in V_2$ four distinct vertices of B, $S := \{a_1, a_2, b_1, b_2\}$, and F an extendible S-cover such that $a_2b_1 \in F$ if and only if $|F| \leq 3$. We define the following transformation of B with respect to F, (a_1, a_2), and (b_1, b_2). If $|F| = 2$, and therefore $F = \{a_1b_2, a_2b_1\}$, use the following construction: Let x, y be two distinct vertices that do not belong to B.

$$\mathsf{B}(B, F, S) := B + x + y + \{xy, xb_1, xb_2, ya_1, ya_2\}$$

Otherwise, we can first obtain an instance where our extendible cover has size two as required above:

$$\mathsf{B}(B, F, S) := \mathsf{B}(\mathsf{B}_2(B, F, S), \mathsf{F}_2(B, F, S), S)$$

In either case let $\mathrm{Cycle}(B, F, S)$ be the 4-cycle a_1yxb_2.

Lemma 4.5.6 (E*)**.** Let B be a bipartite graph with a perfect matching, $a_1, a_2 \in V_1$ and $b_1, b_2 \in V_2$ four distinct vertices of B, $S := \{a_1, a_2, b_1, b_2\}$, and F an extendible S-cover such that $a_2b_1 \in F$ if and only if $|F| \leq 3$. Then B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2) if and only if there exists a conformal cross over $\mathrm{Cycle}(B, F, S)$ in $\mathsf{B}(B, F, S)$.

Proof. In case $|F| \geq 3$ we may replace B and F by $\mathsf{B}_2(B, F, S)$ and $\mathsf{F}_2(B, F, S)$ without influencing the fact whether B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2) by Lemma 4.5.4. Hence, without loss of generality, we may assume $|F| = 2$.

If B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2) there is a perfect matching M that extends F such that there exist disjoint and internally M-conformal paths P_1 and P_2 where P_i has endpoints a_i and b_i for each $i \in [1, 2]$. Then $M' := M \cup \{xy\}$ is a perfect matching of $\mathsf{B}(B, F, S)$ and P_1, P_2 are internally M'-conformal paths in $\mathsf{B}(B, F, S)$ that, in particular, avoid the vertices x and y. Hence $H := P_1 + P_2 + \mathsf{B}(B, F, S)[S \cup \{x, y\}]$ is an M'-conformal subgraph of $\mathsf{B}(B, F, S)$. It is straightforward to see that H is indeed a bisubdivision of $K_{3,3}$ that contains $\mathrm{Cycle}(B, F, S)$ as a subgraph, see Figure 4.19 for an illustration. By Lemma 4.0.9 this means that there is a conformal cross over $\mathrm{Cycle}(B, F, S)$ in $\mathsf{B}(B, F, S)$ and thus we are done with the forward direction.

For the reverse direction let P_1 and P_2 be the two alternating paths that form the conformal cross over $\mathrm{Cycle}(B, F, S)$ in $\mathsf{B}(B, F, S)$ such that P_1 has a_1 as an endpoint while P_2 as b_2 as an endpoint. Then, in particular, P_1 and P_2 are of even length. Since x and y both are of degree exactly

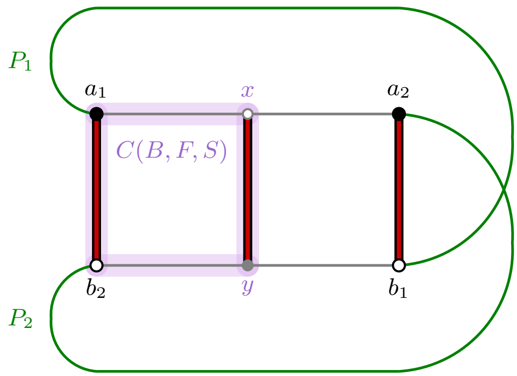


Figure 4.19.: A conformal bisubdivision of $K_{3,3}$ containing the 4-cycle Cycle(B, F, S).

three in $\mathsf{B}(B, F, S)$, $P_1 + P_2$ must contain all neighbours of x and y and thus $S \subseteq V(P_1 + P_2)$. Since P_1 and P_2 form a conformal cross over Cycle(B, F, S), $H := \text{Cycle}(B, F, S) + P_1 + P_2 + a_2b_1$ is a conformal subgraph of $\mathsf{B}(B, F, S)$. Indeed, H is a bisubdivision of $K_{3,3}$ and thus there exists a perfect matching M of $\mathsf{B}(B, F, S)$ such that $a_1b_2, xy, a_2b_1 \in M$ and H is M-conformal. Let $P_1' := a_1P_1b_1$ and $P_2' := b_2P_2a_2$, then the P_i' are disjoint and internally M'-conformal paths. Moreover, $M' \setminus \{xy\}$ is a perfect matching of B that extends F, and thus B is a F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2). $\square$

Our goal is to reduce 2-MLP to the detection of Pfaffian braces. For this we need to make sure that, in case we are dealing with a 'Yes'-instance, the bisubdivision of $K_{3,3}$ cannot vanish somehow.

Lemma 4.5.7 (E*)**.** Let B be a bipartite matching covered graph, $\partial_B(X)$ a non-trivial tight cut in B, and M a perfect matching in B. If P is an internally M-conformal path with both endpoints in X but $E(P) \cap \partial_B(X) \neq \emptyset$, then $E(P) \cap \partial_B(X) \cap M \neq \emptyset$ and $|E(P) \cap \partial_B(X)| = 2$.

Proof. Let $a \in V_1$ and $b \in V_2$ be the two endpoints of P and let us traverse P from a towards b. Let e_1 be the first edge of $E(P) \cap \partial_B(X)$ we encounter this way and let e_2 be the second edge. Moreover let x_i be the endpoint of e_i in $\overline{X}$ and suppose $\{e_1, e_2\} \cap M = \emptyset$. By choice of e_1 and e_2 the path x_1Px_2 lies completely in $\overline{X}$ and is M-conformal. Thus x_1Px_2 must be of odd length and therefore x_1 and x_2 must be of different colour. Hence both $X \cap V_1$ and $X \cap V_2$ must have a neighbour in $\overline{X}$, this, however, contradicts Lemma 3.1.58, and thus one of the two edges must be an edge of M.

Suppose P has more than two edges in $\partial_B(X)$. If the majority of X is in V_1, then the second endpoint, say y_2, of e_1 must be a vertex of V_1 as well and $e_2 \in M$. In this case let $a' := y_2$. If on the other hand the majority of X is in V_2, then $y_2 \in V_2$ and thus $e_1 \in M$ implying $e_2 \notin M$. Hence y_2 must be covered by an edge $e' \in M \cap E(P)$ with second endpoint a'. In either case, $a'P$ is an internally M-conformal path with both endpoints in X and an edge in $\partial_B(X)$. By the arguments above, this means that $E(a'P) \cap \partial_B(X) \cap M \neq \emptyset$, but this means $|M \cap \partial_B(X)| \geq 2$ contradicting $\partial_B(X)$ being a tight cut. Hence $|E(P) \cap \partial_B(X)| = 2$. $\square$

Lemma 4.5.8 (E*)**.** Let B be a bipartite graph with a perfect matching and $H \subseteq B$ a conformal subgraph B such that $V_1(H) = \{a_1, a_2, y\}$, $V_2(H) = \{b_1, b_2, x\}$, $E(H) = \{a_1b_2, a_2b_1, xy, a_1x, a_2x, b_1y, b_2y\}$, and $\deg_B(x) = \deg_B(y) = 3$. Let $C := a_1xyb_2a_1$, then there is a conformal cross over C in B if and only if B has a brace J such that $H \subseteq J$ and J is **not** Pfaffian.

Proof. Let $S := \{a_1, a_2, b_2, b_2\}$. Suppose there is a conformal cross over C in B. This case starts out similar to the reverse direction of the previous lemma. Let P_1 and P_2 be the two alternating paths that form the conformal cross over C in B such that P_1 has a_1 as an endpoint while P_2 as b_2 as an endpoint. Then, in particular, P_1 and P_2 are of even length. Since x and y both are of degree exactly three in B by our assumption, $P_1 + P_2$ must contain all neighbours of x and y and thus $S \subseteq V(P_1 + P_2)$. Since P_1 and P_2 form a conformal cross over C, $H' := C + P_1 + P_2 + a_2b_1$ is a conformal subgraph of B. Indeed, H' is a bisubdivision of $K_{3,3}$

Let us choose B to be a minimal counterexample. In case B is brace, it must be non-Pfaffian since it contains a conformal bisubdivision of $K_{3,3}$ and thus we are done. Hence we may assume that there is a non-trivial tight cut $\partial_B(X)$ in B. If X, or $\overline{X}$, is disjoint from H', one of the two tight cut contractions of $\partial_B(X)$, let us call it B', still contains H' as a conformal subgraph and by choice of B, the assertion holds true for B' and we find a brace J of B' as desired. By Theorem 3.1.61, this means that J is a brace of B and thus B cannot be a counterexample. Hence both X and $\overline{X}$ must contain vertices of H'. Observe that $\partial_{H'}(X)$ is also a tight cut of H'. Now H' has exactly one $K_{3,3}$ and, possibly, a bunch of C_4 as its list of braces. Moreover, the brace J' of H' that is isomorphic to $K_{3,3}$ must contain all six degree three vertices of H', or remainders of them. Indeed, this means that either X or $\overline{X}$ contains at least five vertices of H. By Lemma 4.5.7 this means that at most one of the two paths P_1 and P_2 may have an edge in $\partial_B(X)$. If none of the two paths has an edge in $\partial_B(X)$, then one of the two tight cut contractions of $\partial_B(X)$ contains all of H' as a conformal subgraph, contradicting B being a minimal counterexample as before. Hence we may assume $E(P_1) \cap \partial_B(X) \neq \emptyset$. First, assume $|E(P_1) \cap \partial_B(X)| \geq 2$. We claim that both endpoints of P_1 belong to one of the two shores, say X and $|E(P_1) \cap \partial_B(X)| = 2$. To see this let $Q_1, \ldots, Q_\ell$, $\ell \geq 2$ be the components of $P_1 - \partial_B(X)$ with vertex sets in $\overline{X}$. By Lemma 3.1.58 each Q_j must have both endpoints in the same colour class and thus is of even length. Thus for each Q_j there exists an edge in $M \cap \partial_B(X) \cap E(P_1)$ covering an endpoint of Q_j. Consequently, with $\ell \geq 2$ this contradicts $\partial_B(X)$ being tight. Hence $|E(P_1) \cap \partial_B(X)| \geq 2$. However, if P_1 would have an endpoint in both X and $\overline{X}$, then $|\partial_B(X)|$ would be odd. Also note that in case both endpoints of P_1 are in X, then all of H must be in X since otherwise, we could choose a perfect matching of H' with at least two edges in $\partial_B(X)$. Hence after contracting the shore that does not contain an endpoint of P_1, we obtain a matching covered graph that contains a conformal $K_{3,3}$-bisubdivision with H as a subgraph. In case $|E(P_1) \cap \partial_B(X)| = 1$ exactly one endpoint of P_1 must be contained in, say, $\overline{X}$, while the rest of H belongs to X. Again, after contracting the shore that does not contain an endpoint of P_1 we obtain a matching covered graph that contains a conformal $K_{3,3}$-bisubdivision with H as a subgraph. Hence in neither case B can be a minimal counterexample, and thus no such B can exist.

The reverse follows among similar lines. If there is a non-Pfaffian brace J of B such that $H \subseteq J$, then, by Proposition 4.0.8 there must be a conformal bisubdivision L of $K_{3,3}$ in J that contains C as a subgraph. Indeed, as we have seen before, we can choose L such that $H \subseteq L$ and thus there must be a conformal bisubdivision L' of $K_{3,3}$ in B such that

$H \subseteq L'$. According to Lemma 4.0.9, this means that there is a conformal cross over C in B. $\square$

Lemma 4.5.9 ([RST99])**.** There exists an algorithm that, given a bipartite and matching covered graph B as input, computes a list of all braces of B in time $\mathcal{O}(|V(B)||E(B)|)$.

Proof of Theorem 4.0.6. Let B be a bipartite graph with a perfect matching and $S := \{a_1, a_2, b_1, b_2\}$ be the set of terminals we received as input for the 2-MLP. By the discussion at the start of this section we only have to check for each of the at most $|V(B)|^2$ S-covers F whether they are extendible and whether B is an F-instance of 2-MLP for (a_1, a_2) and (b_1, b_2). To check whether F is extendable we have to check whether $B - V(F)$ has a perfect matching which can be done by the Hopcroft-Karp algorithm in time $\mathcal{O}(n^{\frac{5}{2}})$ [HK73]. So we may assume F to be extendible. In case $F = \{a_1b_1, a_2b_2\}$ we can stop immediately and return the answer 'Yes'.

If $|F \cap \{a_1b_1, a_2b_2\}| = 1$ we can reduce the problem of finding our 2-linkage to the reachability problem in digraphs as follows. Without loss of generality let us assume $a_1a_2 \in F$ and let M be a perfect matching of B that extends F. Moreover, let $e_a \in M$ be the edge covering a_2 while e_b is the edge of M covering b_2. Let $D := \mathcal{D}(B, M)$ be the M-direction of B, let v be the vertex corresponding to a_1b_2, let s be be the vertex corresponding to e_a, and let t be the vertex corresponding to e_b. Then there exists a perfect matching M' of B that extends F such that there is an internally M'-conformal path with endpoints a_2 and b_2 in $B - a_1 - b_1$ if and only if there is an internally M-conformal path P with endpoints a_2 and b_2 in $B - a_1 - b_1$ by Theorem 3.1.69. Finally, such a path exists if and only if there is a directed s-t-path in $D - v$.

Hence we may assume $F \cap \{a_1b_1, a_2b_2\} = \emptyset$. So, by Lemma 4.5.6 we can translate the problem into the decision problem, whether there is a conformal cross over the 4-cycle $\mathrm{Cycle}(B, F, S)$ in $\mathsf{B}(B, F, S)$. Let H be the subgraph of $\mathsf{B}(B, F, S)$ induced by $S \cup \{x, y\}$, then Lemma 4.5.8 allows us to return 'Yes' if and only if $\mathsf{B}(B, F, S)$ has a non-Pfaffian brace J with $H \subseteq J$. Lemma 4.5.9 finds all braces of $\mathsf{B}(B, F, S)$ in time $\mathcal{O}(|V(B)|^3)$ and if there is a brace J with $H \subseteq J$ we can use Corollary 3.3.10 to decide in time $\mathcal{O}(|V(B)|^3)$ whether J is Pfaffian. $\square$

Chapter 5.

Perfect Matching Width

At the heart of the Graph Minors Project sits the idea of decomposing a graph in a tree-like fashion. While towards the end of this series of papers the building blocks become more and more abstract, a first step towards such a decomposition was the idea of treewidth. We briefly discussed in Chapter 2 that graphs of small treewidth can be regarded as 'structurally simple' and treewidth itself serves two main purposes:

It captures the structure of graphs of small treewidth with high accuracy and acts as a foundation to solve related algorithmic problems like the disjoint paths problem and minor testing.

On graphs of high treewidth it gives a rough approximation on the structure of the graph, if it excludes some minor, in the form of the Grid Theorem and, later on, the Flat Wall Theorem.

In this chapter we explore a potential analogue of treewidth for graphs with perfect matchings. This analogue is designed to capture the structure of a graph with regards to its perfect matchings and thus it can be expected to be incomparable with treewidth itself. We dive deeper into the topic of comparing treewidth and our matching version of treewidth in Section 5.5.

5.1. Introducing Perfect Matching Width

Norine and Thomas found an infinite anti-chain of minimally non-Pfaffian bricks [NT08] which forced their study of Pfaffian orientations for general graphs to a momentary halt. It took several years from the first discovery of the unique excluded matching minor for Pfaffian bipartite graphs [Lit75] to the solution of the corresponding recognition problem [RST99] which, as we have seen in Chapter 4, appears to have deep connections to the study of matching minors as a whole. If non-bipartite Pfaffian graphs cannot be described by a finite number of forbidden matching minors, then either the notion of matching minors itself is not suitable, or at least not strong enough, or some other way has to be found to solve the problem. In [NT08] the authors discuss some additional operations one could add to taking conformal subgraphs and bicontracting vertices of degree two to create a stronger and possible more suitable version of minors (see also [Tho06] for more information).

As a slightly different approach, Norine proposed a parameter inspired by treewidth which, in some sense, may be seen as a possible generalisation of the tight cut decomposition.

Definition 5.1.1 (Matching Porosity)**.** Let G be a graph with a perfect matching and $X \subseteq V(G)$ a set of vertices. The *matching porosity* of the cut around X in G is the value

$$\mathrm{mp}(\partial_G(X)) \coloneqq \max_{M \in \mathcal{M}(G)} |M \cap \partial_G(X)|.$$

Note that the function mp is symmetric, so $\mathrm{mp}(\partial_G(X)) = \mathrm{mp}(\partial_G(\overline{X}))$. This means, matching porosity fits into the mould of general branch decompositions.

Definition 5.1.2 (Perfect Matching Width)**.** Let G be a graph with a perfect matching. A *perfect matching decomposition* of G is a mp-branch decomposition (T, δ) over $V(G)$ where T is a cubic tree and $\delta\colon \mathsf{L}(T) \to V(G)$ is a bijection.

The *width* of a perfect matching decomposition (T, δ) is defined as its mp-width, and the *perfect matching width* of G, denoted by $\mathrm{pmw}(G)$, is the minimum width over all perfect matching decompositions for G.

For an example of a perfect matching decomposition consider Figure 5.1. The bipartite graph in our example is of perfect matching width 2 and the decomposition tree contains exactly three edges inducing cuts of matching porosity 1.

In [Nor05] Norine proposed an algorithm that asks for a graph G with a perfect matching and a matching decomposition (T, δ) of width k and tests in time $|V(G)|^{\mathcal{O}(k)}$ whether G is Pfaffian. In fact, Norine's algorithm is not exclusively designed to work on perfect matching decompositions as introduced above, but it can be used for any tree like decomposition

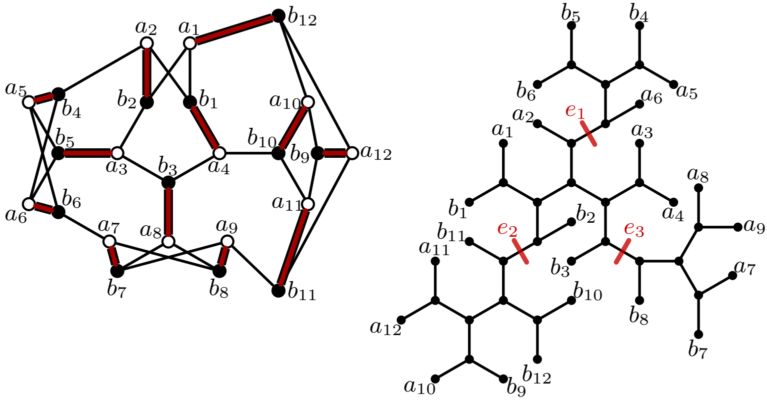


Figure 5.1.: A bipartite matching covered graph B of perfect matching width 2 together with an optimal perfect matching decomposition. The marked edges form a perfect matching of B and the edges e_1, e_2, and e_3 of the decomposition tree induce non-trivial tight cuts.

of subgraphs of G, separated by cuts of small matching porosity, as long as it is guaranteed that each of these subgraphs can be made Pfaffian by deleting a small set of vertices. This general approach strongly resembles the more abstract decompositions presented in the later papers of the Graph Minors Project. Norine conjectured, that the analogy to treewidth does not stop there, in fact he proposed the following conjecture:

Conjecture 5.1.3 (Norine's Matching Grid Conjecture, [Nor05])**.** There exists a function $g\colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, and every graph G with a perfect matching either $\operatorname{pmw}(G) \leq g(k)$, or G contains the $2k \times 2k$-grid as a matching minor.

To solve this conjecture, there are some immediate questions that need answers:

i) How does perfect matching width interact with tight cut contractions and matching minors?
ii) Does matching porosity and thus perfect matching width capture, in some sense, the matching related connectivity of a graph?

iii) Is there a connection between matching porosity and some sense of separation in matching covered graphs?

The reason why the later two questions are relevant is that in both the undirected and the directed version of treewidth, the key role in proving the (Directed) Grid Theorem was played by obstructions such as tangles and brambles and not so much by treewidth itself. Especially the existence of large families of pairwise disjoint paths and therefore Menger's Theorem were of great importance to these proofs. As we have discussed in Section 3.4, for non-bipartite graphs this is a problem as the matching version of disjoint s-t-paths is NP-complete.

Please note that Norine's algorithm requires a perfect matching decomposition of small width to be given as input, so another open problem for this approach would be:

iv) Is it possible to compute a perfect matching decomposition of (reasonably) small width in polynomial time on graphs of small perfect matching width?

Bicontracting a vertex of degree two preserves the parity of distances between vertices in the graph. That means, if Conjecture 5.1.3 is true, any non-bipartite graph of large perfect matching width must contain a conformal and bipartite subgraph which still is of relatively large perfect matching width. This is an immediate consequence of Lemma 3.1.102. Hence we propose the following strategy towards the solution of Conjecture 5.1.3.

i) Solve Conjecture 5.1.3 for bipartite graphs, and then
ii) show that there is a function $g\colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, and every graph G with a perfect matching either $\operatorname{pmw}(G) \leq g(k)$, or G contains a conformal and bipartite subgraph H with $\operatorname{pmw}(H) \geq k$.

Conjecture 5.1.3 then follows from the combination of steps (i) and (ii). In this chapter we are mainly concerned with step (i) and thus we stick to the case of bipartite graphs.

Please note that the notion of subgraph might not be strong enough. Indeed, in non-bipartite matching covered graphs there exists another possible generalisation of tight cuts, called *separating cuts* (see for example [CL00]). It might be possible that one needs to first refine the non-bipartite graph before the non-bipartite subgraph H as above can be found. Still, if this is necessary and cannot be avoided, this should yield a family of graphs with large perfect matching width, where no matching model of the $2k \times 2k$-grid can be found. By Lemma 3.1.102 this would mean that Conjecture 5.1.3 must be further refined.

5.1.1. Basic Properties of Perfect Matching Width

As a first step let us investigate the basic properties of perfect matching width as a parameter and establish some useful tools we might need later.

Let G be some graph with a perfect matching, and $X \subseteq V(G)$. If we select a single vertex $v \in X$, then any perfect matching M of G either matches v with a vertex inside of X, or with a vertex in $\overline{X}$. If v is matched with a vertex in $\overline{X}$, then $|\partial_G(X \setminus \{v\}) \cap M| = |\partial_G(X) \cap M| - 1$. Otherwise we have $|\partial_G(X \setminus \{v\}) \cap M| = |\partial_G(X) \cap M| + 1$. This means, by moving around a single vertex, the matching porosity of the cuts involved cannot change by more than one.

Observation 5.1.4 (B*)**.** Let G be a graph with a perfect matching, $X \subseteq V(G)$, and $v \in X$. Then $\mathrm{mp}(\partial_G(X)) - 1 \leq \mathrm{mp}(\partial_G(X \setminus \{v\})) \leq \mathrm{mp}(\partial_G(X)) + 1$.

The trees of perfect matching decompositions, as for many branch decompositions are cubic, or at least subcubic[1]. Just considering the possible structures of the trees themselves can be a useful tool when dealing with this kind of decompositions.

Definition 5.1.5 (Spine and Odd Edges)**.** Let T be a cubic tree. The *spine* of T is defined as $\mathrm{spine}(T) \coloneqq T - \mathsf{L}(T)$.

The edges in $E(T) \setminus E(\mathrm{spine}(T))$ are called *trivial*. An edge $e \in E(\mathrm{spine}(T))$ is *even*, if the two trees of $T - e$ contain an even number of leaves of T each and it is *odd* otherwise.

Note that, if T is the cubic tree of a perfect matching decomposition of some graph G with a perfect matching, then T has an even number of leaves as G has an even number of vertices. This implies that in T a non-trivial edge e is odd if and only if the two trees of $T - e$ contain an odd number of leaves of T each.

[1] Which means that vertices of degree two are allowed.

The following is a collection of several useful observations on cubic trees with an even number of leaves.

Observation 5.1.6 (C*)**.** Let T be a cubic tree with $|\mathsf{L}(T)| = \ell$ even. Then the following statements are true.

i) $|V(T)| = 2\ell - 2$,
ii) $\text{spine}(T)$ has an even number of vertices,
iii) $\text{spine}(T)$ has an even number of vertices of degree 2, and
iv) $e \in E(\text{spine}(T))$ is an odd edge of T if and only if the two trees of $\text{spine}(T) - e$ contain an even number of vertices each.

If T is a cubic tree, then $\text{spine}(T)$ is a subcubic tree. There is a close correspondence between the occurrence of odd edges in T and vertices of degree 2 in $\text{spine}(T)$.

Lemma 5.1.7 (C*)**.** Let T be a cubic tree with an even number of leaves.

i) If $\deg_v(\text{spine}(T)) = 1$, then v is not incident with an odd edge of T.
ii) If $\deg_v(\text{spine}(T)) = 2$, then v is incident with exactly one odd edge of T.
iii) If $\deg_v(\text{spine}(T)) = 3$, then v is either incident with exactly two odd edges of T or with none.

Proof. If v is of degree 1 in the spine of T, it is adjacent with exactly two leaves of T and thus, by definition, the unique edge incident with v in $\text{spine}(T)$ cannot be odd.

Let v be a vertex of degree 2 in $\text{spine}(T)$ and e_1, e_2 the two edges incident with v in the spine. In T itself v is incident with a third edge e_3 whose other endpoint is a leaf of T. Let k_i be the number of leaves of T contained in the component of $T - e_i$ that does not contain v. Then $|\mathsf{L}(T)| = k_1 + k_2 + k_3$ and $k_3 = 1$. Since the total number of leaves is even and k_3 is odd, exactly one of k_1 and k_2 is odd as well. Thus, exactly one of the two edges e_1 and e_2 is an odd edge of T.

At last we consider a degree 3 vertex v in $\text{spine}(T)$. Let e_1, e_2, e_3 be the three edges of the spine incident with v and let k_i be the number of leaves of T contained in the component of $T - e_i$. In this case $|\mathsf{L}(T)| = k_1 + k_2 + k_3$ and thus neither all three, nor just one of them can be odd. $\square$

Corollary 5.1.8 (C*)**.** Let T be a cubic tree with an even number of leaves. Then spine(T) is cubic if and only if T has no odd edges.

Let T' be the subgraph of T induced by its odd edges. Then no vertex of T' can have degree three as there does not exist a vertex in T which is incident with three odd edges. Therefore every component of T' must be a path and the endpoints of these paths are exactly the degree two vertices of spine(T).

Corollary 5.1.9 (C*)**.** Let T be a cubic tree with an even number of leaves and $E_O \subset E(T)$ the set of odd edges of T. Then $T[E_O]$ is a collection of pairwise disjoint paths. Moreover, the set of endpoints of these paths is exactly the set of degree 2 vertices in spine(T).

What exactly is the interaction between the odd edges of T and perfect matching decompositions based on T? As a first step, we investigate the influence of the existence of odd edges in the cubic tree of a perfect matching decomposition (T, δ) on the parity of the width of (T, δ).

Lemma 5.1.10 (C*)**.** Let G be a matching covered graph and $X \subseteq V(G)$. Then $\mathrm{mp}(\partial_G(X))$ is odd if and only if $|X|$ is odd.

Proof. Let $M \in \mathcal{M}(G)$ be a perfect matching of G that maximises $\partial_G(X)$ and let $k := |M \cap \partial_G(X)|$. Then $G[X] - V(\partial_G(X) \cap M)$ has a perfect matching and therefore an even number of vertices, say n. So in total $|X| = n + k$. Hence $|X| \equiv k \pmod{2}$ and the claim follows. $\square$

Perfect matching width measures, in some sense, the complexity of the interaction of all perfect matchings of a given graph G. We have seen before that sometimes it might be useful to focus on a single perfect matching M, for example to call upon the structure of the M-bidirection of G. A natural question to ask would be, whether we can specialise any given perfect matching decomposition with respect to a single perfect matching M without changing the width of the decomposition too much. Let us consider the cut $\partial_G(X)$ of matching porosity k in G. Then there are at most k vertices in X that are incident with edges in $M \cap \partial_G(X)$. Hence we have to move at most k vertices from one shore to the other in order to obtain a new cut where both shores are M-conformal. This, together with Observation 5.1.4, immediately yields the following observation.

Observation 5.1.11 (B*)**.** Let G be a graph with a perfect matching, $X \subseteq V(G)$ and $M \in \mathcal{M}(G)$. Then there is an M-conformal set $X' \subseteq V(G)$ such that

i) $X \subseteq X'$,
ii) $|X'| \leq |X| + \operatorname{mp}(\partial_G(X))$, and
iii) $\operatorname{mp}(\partial_G(X')) \leq 2\operatorname{mp}(\partial_G(X))$.

One way to utilise this observation and to customise the notion of perfect matching width with regards to a single perfect matching M could be to require that any inner edge of the decomposition induces a bipartition of the vertex set of the decomposed graph G into two M-conformal sets.

Definition 5.1.12 (M-Perfect Matching Width)**.** Let G be a graph with a perfect matching M. An M-decomposition of G is a perfect matching decomposition (T, δ) such that for every $t_1t_2 \in E(\operatorname{spine}(T))$ the sets $\delta(T_{t_1})$ and $\delta(T_{t_2})$ are M-conformal. The *M-perfect matching width*, denoted by $M\text{-}\operatorname{pmw}(G)$, is defined as the minimum width over all M-decompositions of G.

Theorem 5.1.13 (B*)**.** Let G be a graph with a perfect matching M. Then, $\operatorname{pmw}(G) \leq M\text{-}\operatorname{pmw}(G) \leq 2\operatorname{pmw}(G)$.

Proof. Clearly $\operatorname{pmw}(G) \leq M\text{-}\operatorname{pmw}(G)$ as every M-decomposition is also a perfect matching decomposition.

Hence we only need to prove $M\text{-}\operatorname{pmw}(G) \leq 2\operatorname{pmw}(G)$. Let (T, δ) be a perfect matching decomposition of G of minimum width. Now let $X \subseteq V(G)$ such that for all $e \in M$ we have $|e \cap X| = 1$. For every $x \in X$ denote by x_M the vertex from $\overline{X}$ with $xx_M \in M$ and let $X' \subseteq X$ be the set of vertices $x \in X$ such that the path from $\delta^{-1}(x)$ to $\delta^{-1}(x_M)$ in T contains an inner edge (i.e. an edge not incident with a leaf).

Now we construct a new decomposition (T', δ'). We remove $\delta^{-1}(x_M)$ and add two new leaves to the vertex $\delta^{-1}(x)$ in T'. The deletion of $\delta^{-1}(x_M)$ left a vertex of degree two, in order to maintain a cubic tree we contract one of the two edges incident with said degree two vertex. Now $\delta^{-1}(x)$ has two new neighbours a and b which we map to the vertices x_M and x respectively via δ'. Thus the vertex $\delta^{-1}(x)$ has become an inner vertex of T'.

The only additional inner edges in T' are those where the corresponding cut separates a pair of leaves mapped to a matching edge of M containing a vertex in X' from the rest of the graph. So these induce cuts of matching porosity at most two and M-conformal shores.

Consider an inner edge e of T and the two shores X and $\overline{X}$ it induces. The edges of M that have vertices in both shores are at most $\mathrm{pmw}(G)$ many. Therefore by Observation 5.1.4 the porosity of the induced cut has at most doubled. □

Note that the proof of Theorem 5.1.13 gives a procedure that can be performed in polynomial time to obtain from any perfect matching decomposition (T, δ) of a graph G, an M-decomposition of width at most $2\,\mathrm{width}(T, \delta)$.

Tight Cut Contractions and Matching Minors

A fundamental question is, whether we can bound the perfect matching width of some tight cut contraction or matching minor H of a graph G with a perfect matching in terms of $\mathrm{pmw}(G)$. Using the findings from above we can now investigate this topic. We start with an inequality in the other direction, showing that the perfect matching width of a graph G cannot be larger than the perfect matching width of its bricks and braces.

Proposition 5.1.14 (C*)**.** Let G be a graph with a perfect matching. Then

$$\mathrm{pmw}(G) \leq \max_{\substack{H \text{ brick or} \\ \text{brace of } G}} \mathrm{pmw}(H).$$

Proof. Let $\mathcal{L}$ be a maximal family of pairwise laminar tight cuts of G and let $H_1, \ldots, H_t$ be the bricks and braces of G. For all $i \in [1, t]$ let (T_i, δ_i) be an optimal perfect matching decomposition of H_i and let $k_i := \mathrm{pmw}(H_i)$. We say that H_i and H_j are adjacent in $\mathcal{L}$ if there is a tight cut $\partial_G(Z) \in \mathcal{L}$ such that $v_Z \in V(H_i)$ and $v_{\overline{Z}} \in V(H_j)$ where v_Z and $v_{\overline{Z}}$ are the two contraction vertices of $\partial_G(Z)$. As the cuts in $\mathcal{L}$ are pairwise laminar, the adjacency structure of the bricks and braces obtained by the above definition is a tree F. So every vertex of F corresponds to a brick or brace of G together with an optimal perfect matching decomposition and if two vertices in F are adjacent, then the corresponding bricks or braces are separated by exactly one cut of $\mathcal{L}$.

We are going to iteratively construct a perfect matching decomposition of G by merging the (T_i, δ_i). To this end, we slightly relax our definition of the tree F and simply assume that any vertex of F corresponds to a matching covered graph H together with an optimal perfect matching decomposition of H and two vertices x and y of F are adjacent if there is a tight cut $\partial_G(Z) \in \mathcal{L}$ such that $v_{\overline{Z}}$ is a vertex of the graph associated with x, while v_Z is a vertex of the graph associated with y.

Let x be a leaf of F and y the neighbour of x. Then, let H be the matching covered graph corresponding to x and J the matching covered graph corresponding to y. By definition of F, there is a tight cut $\partial_G(Z)$ such that $v_{\overline{Z}} \in V(H)$ and $v_Z \in V(J)$. Let (T_H, δ_H) be the decomposition of H associated with x and (T_J, δ_J) the decomposition of J associated with y. Within these decompositions there is an edge $e_H \in E(T_H)$ incident with the leaf that is mapped to $v_{\overline{Z}}$ and analogously there is an edge $e_J \in E(T_J)$ to the leaf mapped to v_Z. Let H' be obtained from J by reversing the tight cut contraction of v_Z. By our choices this means, that H' contains a tight cut $\partial_{H'}(Z')$ such that the two tight cut contractions are exactly H and J. In order to construct a perfect matching decomposition $(T_{H'}, \delta_{H'})$ of H', we create a new tree $T_{H'}$ from T_H and T_J by identifying the edges e_H and e_J as the new edge $e_{H,J}$. In addition we define the new mapping without v_Z and $v_{\overline{Z}}$ as follows.

$$\delta_{H'} \colon \mathsf{L}(T_{H'}) \to V(H'), \delta_{H'}(v) := \begin{cases} \delta_H(v), & v \in V(H) \setminus \{v_{\overline{Z}}\} \\ \delta_J(v), & v \in V(J) \setminus \{v_Z\} \end{cases}$$

As $\partial_{H'}(Z)$ is a tight cut, $\partial_{H'}(e_{H,J})$ has matching porosity one. Let $e \in E(T_{H'}) \setminus \{e_{H,J}\}$. Suppose, there is a perfect matching $M \in \mathcal{M}(H')$ such that $|\partial_{H'}(e) \cap M| \geq \max\{\mathrm{pmw}(H), \mathrm{pmw}(J)\} + 1$. Without loss of generality, assume that $e \in E(T_H)$. Then, by construction of $T_{H'}$ there is exactly one shore of $\partial_{H'}(e)$, say X, that contains the vertices of J. Again, since $\partial_{H'}(Z)$ is tight, there is exactly one edge of M with exactly one endpoint in $V(J) \setminus \{v_Z\}$, let v be its endpoint in $V(H) \setminus \{v_{\overline{Z}}\}$. Now, consider $M' := (M \cap E(H)) \cup \{vv_{\overline{Z}}\}$ and note that M' is a perfect matching of H. Moreover, $|M' \cap \partial_H(e)| = |\partial_H(e) \cap M| \geq \mathrm{pmw}(H) + 1$. This yields a contradiction to the definition of perfect matching width. Hence, $\mathrm{mp}(\partial_{H'}(e)) \leq \max\{\mathrm{pmw}(H), \mathrm{pmw}(J)\}$ for all $e \in E(T_{H'})$. Now, we can delete x from F and associate H' with y. This yields a new tree F', which is smaller than F and still meets all of our assumptions. Hence, we can continue the process until the new tree F' does not contain any edges. At this point the graph associated with the sole vertex of F' will be G itself, so we have constructed a decomposition for G with perfect matching width at most the maximum over the decompositions we started with. $\square$

Before we continue towards bounding the perfect matching width of matching minors, we have to discuss conformal subgraphs. These provide a lower bound on the perfect matching width of a graph and therefore are a first step in that direction. Let T be a subcubic tree. We can obtain a cubic tree T' from T by iteratively choosing a degree two vertex and contracting one of its two incident edges. The tree T' is, up to isomorphism, uniquely determined by T and we call T' the tree obtained by *trimming* T. Note that $\mathsf{L}(T) = \mathsf{L}(T')$.

Lemma 5.1.15 (C*)**.** Let G be a graph with a perfect matching and $H \subseteq G$ a conformal subgraph of G. Then $\operatorname{pmw}(H) \leq \operatorname{pmw}(G)$.

Proof. Let (T, δ) be an optimal perfect matching decomposition of G and
$$L_{\overline{H}} \coloneqq \{\ell \in \mathsf{L}(T) \mid \delta(\ell) \in V(G) \setminus V(H)\}.$$
Then, $T - L_{\overline{H}}$ is a subcubic tree. Now, remove from $T - L_{\overline{H}}$ iteratively all vertices that became leaves and thus are not mapped to any vertex by δ. We call the resulting tree T''. Let T' be the tree obtained by trimming T''. We define $\delta' \colon \mathsf{L}(T') \to V(H)$ by restricting δ to $\mathsf{L}(T')$ and claim that (T', δ') is a perfect matching decomposition of H of width at most $\operatorname{pmw}(G)$.

Suppose, there is an edge $e \in E(T')$ that corresponds to a cut $\partial_H(X')$ in H and a perfect matching $M' \in \mathcal{M}(H)$ such that $|\partial_H(X') \cap M'| \geq \operatorname{pmw}(G) + 1$. Then, by construction $e \in E(T)$ and thus e corresponds to a cut $\partial_G(X)$ in G as well. Moreover, $X' \subseteq X$ and $V(H) \setminus X' \subseteq V(G) \setminus X$. Since H is a conformal subgraph of G, there is a perfect matching $M \in \mathcal{M}(G)$ with $M' \subseteq M$ and thus $|\partial_G(X) \cap M| \geq |\partial_G(X') \cap M| \geq \operatorname{pmw}(G) + 1$. Hence, $\operatorname{width}(T, \delta) \geq \operatorname{pmw}(G) + 1$ which contradicts (T, δ) to be an optimal perfect matching decomposition of G. $\square$

Let G be a graph with a perfect matching M and consider an M-decomposition (T, δ) of width $k \in \mathbb{N}$. Handling a single tight cut contraction suffices, since the M-decompositions we obtain for the two tight cut contractions can be seen to be M'-decompositions again where M' is the restriction of M to the two contractions. This allows us to apply induction and reduce the initial matching covered graph G all the way down to its bricks and braces.

Key to obtaining an M-decomposition for a tight cut contraction of G from an M-decomposition (T, δ) of G is the decision where in the trimmed version of the decomposition tree to attach a new leaf for the contraction vertex. If there is an edge in T that separates the vast majority of the vertices of one of the tight cut shores from the vertices of the other, this decision is not too complicated to make. But if such an edge does not exists, or in other words (T, δ) does not distinguish between the two shores of our tight cut, it is much harder to decide. In Proposition 5.1.14 we have seen that there always exist perfect matching decompositions with edges reflecting the tight cuts. However, these decompositions are not necessarily optimal and at this point we are not able to provide a bound on the approximation they provide.

Our decision on the decomposition to place the contraction vertex is based on some implications of Lemma 5.1.10. If $\partial_G(Z)$ is a non-trivial tight cut of G, then $|Z|$ is odd and thus for all $X \subseteq V(G)$ the cut $\partial_G(X)$ of G has exactly one shore that contains an odd number of vertices of Z. If $|X|$ is even, this shore also contains an odd number of vertices of $\overline{Z}$. This observation leads us to the following lemma.

Note that any cut induced by an inner edge of an M-decomposition is even, since both shores are M-conformal.

Lemma 5.1.16 (C*)**.** Let G be a matching covered graph, $X \subseteq V(G)$ be even, and $\partial_G(Z)$ a non-trivial tight cut of G, as well as v_Z the contraction vertex in $G_Z \coloneqq G/(Z \to v_Z)$. If $|X \cap Z|$ is odd, then $\mathrm{mp}(\partial_{G_Z}((X \setminus Z) \cup \{v_Z\})) \leq \mathrm{mp}(\partial_G(X))$.

Proof. Suppose $\mathrm{mp}(\partial_G(X)) < \mathrm{mp}(\partial_{G_Z}((X \setminus Z) \cup \{v_Z\}))$. Let $M' \in \mathcal{M}(G_Z)$ be a perfect matching maximising $\partial_{G_Z}((X \setminus Z) \cup \{v_Z\})$. Then M' contains exactly one edge incident with v_Z. Thus, by assumption, $\mathrm{mp}(\partial_{G_Z}((X \setminus Z) \cup \{v_Z\})) = \mathrm{mp}(\partial_G(X)) + 1$. Since $\partial_G(X)$ is of even porosity, $\operatorname{mp}(\partial_{G_Z}((X \setminus Z) \cup \{v_Z\}))$ is odd. But with $|X \cap Z|$ being odd, $|(X \cap Z) \cup \{v_Z\}\,|$ must be even. This is a contradiction to Lemma 5.1.10. $\square$

If (T, δ) is an M-decomposition of G, then, as we have seen in the proof of Lemma 5.1.16, the only cuts whose matching porosity can exceed the width of (T, δ) by placing the contraction vertex and 'keeping' the rest of the decomposition as it is, are those of matching porosity exactly $\operatorname{width}(T, \delta)$.

For each of those cuts we need to indicate which of its two shores contains an odd number of vertices of a tight cut shore. To this end, we define the following orientation of the edges of T. Our definition does not require (T, δ) to be an M-decomposition. However, in case it is, we are able to make further observations

Let G be a matching covered graph, $\partial_G(Z)$ a non-trivial tight cut of G and (T, δ) a perfect matching decomposition of G. We define the *Z-orientation* $\vec{T_Z}$ of T as the orientation of the edges of T, such that for every edge $t_1t_2 \in E(\operatorname{spine}(T))$, $(t_1, t_2) \in E(\vec{T_Z})$ if and only if $|\delta(T_{t_2}) \cap Z|$ is odd. Additionally, every edge $t\ell \in E(T)$, where ℓ is a leaf, is oriented away from ℓ, that is $(\ell, t) \in E(\vec{T_Z})$. Note that the Z-orientation of the edge t_1t_2 is well defined since $|Z|$ is odd (see Figure 5.2 for an example). If there is a vertex $t \in V(\vec{T_Z})$ such that at least two of its incident edges are outgoing edges, we call t an *inconsistency*.

The idea is that $\vec{T_Z}$ should tell us where to put the contraction vertex in order to obtain a decomposition of the tight cut contraction of G obtained by contracting Z. However, this only works if T_Z has no inconsistencies.

Lemma 5.1.17 (C*)**.** Let G be a matching covered graph, $\partial_G(Z)$ be a non-trivial tight cut in G and (T, δ) a perfect matching decomposition of G. If $\vec{T_Z}$ has an inconsistency $t \in V(\vec{T_Z})$, then all three edges incident with t are outgoing.

Proof. Let $t \in V(\vec{T_Z})$ be an inconsistency of $\vec{T_Z}$ with incident edges e_1, e_2 and e_3 such that T_i is the component of $T - e_i$ that does not contain t for every $i \in [1, 3]$. Suppose e_1 and e_2 are outgoing edges and e_3 is incoming for t.

Then, by the definition of $\vec{T_Z}$, the following holds:

i) $|\delta(T_1 + T_2) \cap Z|$ is odd and $|\delta(T_3) \cap Z|$ is even,
ii) $|\delta(T_1) \cap Z|$ is odd and $|\delta(T_2 + T_3) \cap Z|$ is even, and

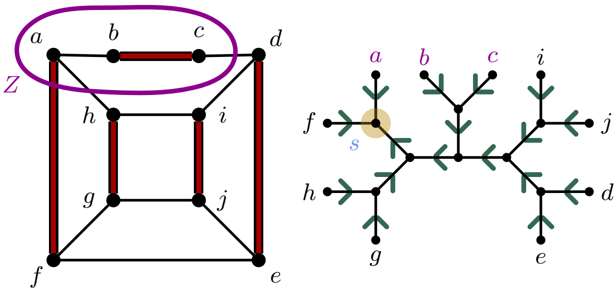


Figure 5.2.: A matching covered graph G with a non-trivial tight cut $\partial_G(Z)$, a perfect matching M, and an M-decomposition (T, δ) of width four. The arrows in T are the edges forming the Z-orientation of T, note that it is free of inconsistencies and has a unique sink s.

iii) $|\delta(T_2) \cap Z|$ is odd and $|\delta(T_1 + T_3) \cap Z|$ is even.

Since T_1, T_2 and T_3 are pairwise disjoint, these statements are clearly contradictory and thus, e_3 cannot be an incoming edge of t. $\square$

If a Z-orientation does not have any inconsistencies, there exists a unique sink vertex s in $\vec{T_Z}$. Additionally, s is adjacent to a leaf $t \in V(T)$ and $\delta(t) \in Z$ (see Lemma 5.1.18).

So, to obtain a perfect matching decomposition of the tight cut contraction obtained from G by contracting Z into a single vertex v_Z, we forget all vertices of Z, delete the corresponding leaves from T (except for t) and map t to the contraction vertex v_Z. Finally, we trim this new tree. Not only does this yield a perfect matching decomposition, the width of this new decomposition is at most the width of the original decomposition. If our decomposition was an M-decomposition in the first place, the result is even stronger.

Lemma 5.1.18 (C*)**.** Let G be a matching covered graph with a perfect matching M, $\partial_G(Z)$ a non-trivial tight cut in G, and (T, δ) an M-decomposition of G. Then, $\vec{T_Z}$ is free of inconsistencies and has a unique sink that is adjacent to two leaves.

Proof. As Z induces a non-trivial tight cut, there is a unique edge $xy \in M$ with $x \in Z$ and $y \in \overline{Z}$. All other vertices of Z are matched within Z. In the M-decomposition (T, δ) for every $t_1t_2 \in E(T - \mathsf{L}(T))$ the unique subtree T_{t_i} with $|\delta(T_{t_i}) \cap Z|$ being odd is exactly the one that contains x. Therefore, in $\vec{T_Z}$ every inner edge is oriented towards the subtree that contains x and thus there cannot be an inconsistency, as δ is a bijection and the tree containing x is well defined for every inner edge.

Moreover, let $t \in V(\vec{T_Z})$ such that t is adjacent to two leaves ℓ_1 and ℓ_2 where $\delta(\ell_1) = x$. Note that the number of leaves of T equals the number of vertices of G, which is always even. Then, for every vertex $t' \in V(\vec{T_Z}) \setminus \{t, \ell_1, \ell_2\}$ there is a directed path in $\vec{T_Z}$ from t' to t. By the definition of Z-orientations, $(\ell_1, t), (\ell_2, t) \in E(\vec{T_Z})$ which implies that t is a sink of $\vec{T_Z}$ and no vertex apart from t can be a sink. ☐

Note that, in the proof above, $\delta(\ell_2) = y$ and thus, in the decomposition for the tight cut contraction we construct from (T, δ), the contraction vertex and y are again siblings. So, if we start out with an M-decomposition of a matching covered graph G, then the Z-orientations of said decomposition behave exactly as intended. This allows us to obtain new decompositions for tight cut contractions of somewhat controlled width and thus yields the main result of this section.

Proposition 5.1.19 (C*)**.** Let G be a matching covered graph, $\partial_G(Z)$ a non-trivial tight cut in G, $M \in \mathcal{M}(G)$, and (T, δ) an M-decomposition of G of width k. Moreover, let $G_Z := G/(\overline{Z} \to v_{\overline{Z}})$. Then, there is an $M|_{G_Z}$-decomposition of G_Z of width at most k.

Proof. We consider the Z-orientation $\vec{T_Z}$ of T. By Lemma 5.1.18, $\vec{T_Z}$ is free of inconsistencies and has a unique sink s. Moreover, as we have seen, s is adjacent to two leaves t_x and t_y of T such that $\delta(t_x) = x \in Z$, $\delta(t_y) = y \in \overline{Z}$ and $xy \in M$ is the unique edge of M in $\partial_G(Z)$.

We construct a perfect matching decomposition (T', δ') for G_Z. To this end, let $L_{\overline{Z}} := \big\{t \in \mathsf{L}(T) \mid \delta(t) \in \overline{Z} \setminus \{y\}\big\}$ and T' be the cubic tree obtained from $T - L_{\overline{Z}}$ by trimming. Then $\mathsf{L}(T') = \mathsf{L}(T) \setminus L_{\overline{Z}}$ and for every $t \in \mathsf{L}(T')$ and every inner edge $t_1t_2 \in E(T')$, t is a leaf of the tree T'_{t_i} if and only if t is a leaf of the subtree T_{t_i}. Therefore, every bipartition of $\mathsf{L}(T)$ induced by an inner edge of T' is also induced by an edge in T.

To obtain δ' from δ we do not change anything for Z but just replace y by $v_{\overline{Z}}$. So for all $t \in \mathsf{L}(T')$ let

$$\delta'(t) := \begin{cases} v_{\overline{Z}}, & \text{if } \delta(t) = y, \text{ and} \\ \delta(t), & \text{otherwise.} \end{cases}$$

The restriction $M|_{G_Z}$ of M to G_Z contains all edges with both endpoints in Z and additionally the edge $xv_{\overline{Z}}$, so by construction, (T', δ') it is an $M|_{G_Z}$-decomposition of G_Z.

Now, let $t_1t_2 \in E(T')$ be an inner edge and $\partial_G(t_1t_2)$ the cut induced by t_1t_2 in G via (T, δ). Then, $\partial_G(t_1t_2)$ has a unique shore $Y \subseteq V(G)$ that contains y and, as (T, δ) is an M-decomposition, $|Y|$ is even. Moreover, the cut $\partial_{G_Z}(t_1t_2)$ induced by t_1t_2 in G_Z via (T', δ') has a shore $Y' = (Y \setminus Z) \cup \{v_{\overline{Z}}\}$. As $|Y \cap Z|$ is odd by choice of Y, Lemma 5.1.16 gives us $\mathrm{mp}(\partial_{G_Z}(Y')) \leq \mathrm{mp}(\partial_G(Y)) \leq k$ and thus concludes the proof. $\square$

Since Proposition 5.1.19 provides an $M|_{G_Z}$-decomposition of the tight cut contraction G_Z, we can now choose a new tight cut in G_Z and continue with a new iteration of the tight cut decomposition procedure. So finally, we reach decompositions of the bricks and braces of G whose width is still bounded by the width of the original M-decomposition of G. By then applying Theorem 5.1.13 we obtain the following corollary.

Corollary 5.1.20 (C*)**.** Let G be a graph with a perfect matching and H a brick or brace of G. Then $\mathrm{pmw}(H) \leq 2\,\mathrm{pmw}(G)$.

By iteratively contracting tight cuts we cannot significantly increase the perfect matching width. As bicontractions are a special case of tight cut contractions and by Lemma 5.1.15 the width of a conformal subgraph of G is bounded by the width of G itself, we obtain a similar corollary for the matching minors of G.

Corollary 5.1.21 (C*)**.** Let G be a graph with a perfect matching and H be a matching minor of G. Then $\mathrm{pmw}(H) \leq 2\,\mathrm{pmw}(G)$.

Moreover, if we consider the M-width of a matching covered graph G, we obtain an even stronger result which concludes this section.

Corollary 5.1.22 (C*)**.** Let G be a graph with a perfect matching M, and H a brick, brace or a matching minor obtained by a series of bicontractions from an M-conformal subgraph of G. Then $M|_H\text{-}\,\mathrm{pmw}(H) \leq M\text{-}\,\mathrm{pmw}(G)$.

5.2. Braces of Perfect Matching Width 2

The only matching covered graph of perfect matching width one is K_2. Apart from this every perfect matching decomposition of a matching covered graph contains a vertex that is adjacent to two leaves (which, by definition, are mapped to two distinct vertices of G) and, as G is matching covered, there is a perfect matching which does not match these vertices with each other. Therefore, the cut in G induced by the non-leaf edge of said vertex in the decomposition has matching porosity two. So two is a natural lower bound on the perfect matching width of braces. One approach to width parameters can be to investigate the structure of of graphs of small width. Since by Proposition 5.1.14 the perfect matching width of a graph is bounded from above by the width of its bricks and braces, studying the structure of braces of perfect matching width two appears to be a good starting point towards a better understanding of the parameter itself. We present two possible characterisations of perfect matching width two braces, one in terms of edge-maximal graphs similar to the k-tree characterisation of treewidth k graphs (see [Arn85] for an overview on this topic) and the other one in terms of elimination orderings, which again resembles similar results on treewidth.

We start out with some core observations on the type of decomposition trees we have to expect for braces of perfect matching width two.

Lemma 5.2.1 (C*)**.** Let G be a brick or brace of perfect matching width two and (T, δ) be an optimal perfect matching decomposition. Then, $\mathrm{spine}(T)$ is cubic.

Proof. By Corollary 5.1.8, it suffices to show that T is free of odd edges. Suppose T has an odd edge t_1t_2, then $X_i \coloneqq \delta(T_{t_i})$ contains an odd number of vertices for $i \in [1, 2]$. Then Lemma 5.1.10 implies that $\mathrm{mp}(\partial_G(X_1))$ is odd. As the width of (T, δ) is 2 and t_1t_2 is an inner edge of T, $|X_1| \geq 3$, $|X_2| \geq 3$ and $\mathrm{mp}(\partial_G(X_1)) = 1$. Thus $\partial_G(X_1)$ must be a non-trivial tight cut of G contradicting G being a brick or a brace. $\square$

If G is a graph with a perfect matching and $X \subseteq V(B)$ is a set such that $\mathrm{mp}(\partial_G(X)) = 2$, then for every perfect matching M of G we must have $|\partial_G(X)| \in \{0, 2\}$. Since we are interested in braces, it seems natural to ask for the structure of non-trivial cuts in highly extendible bipartite graphs as one might find some connection to proper k-tight cuts and the relation between minority and majority as seen in Lemma 3.1.58.

Lemma 5.2.2 (X*). Let $k \in \mathbb{N}$ be a positive integer, B be a k-extendible and bipartite graph, and $X \subseteq V(G)$ such that $\mathrm{mp}(\partial_B(X)) = k$ and $k+2 \leq |X| \leq |V(B)| - (k+2)$. Then $\mathrm{imbalance}(X) = k$.

Proof. Any perfect matching can have at most $|\operatorname{Min}(X)|$ many edges in X and thus $\mathrm{imbalance}(X) \leq \mathrm{mp}(\partial_B(X)) = k$. Moreover, by Lemma 5.1.10 $k \equiv |X| \equiv \mathrm{imbalance}(X) \pmod 2$. We set $k' := \mathrm{imbalance}(X) \leq k-2$. Without loss of generality we may assume $\operatorname{Maj}(X) \subseteq V_2$. We consider two special subgraphs of B. Let $B_1 := B[\operatorname{Min}(X) \cup (V_2 \setminus \operatorname{Maj}(X))]$ be the bipartite graph on exactly the edges of $\partial_B(X)$ with endpoints in $\operatorname{Min}(X)$ and $B_2 := B[(V_1 \setminus \operatorname{Min}(X)) \cup \operatorname{Maj}(X)]$ the graph an the remaining edges of $\partial_B(X)$. Suppose $\nu(B_1) \geq \frac{k-k'}{2} + 1$ and let F be a matching of size $\frac{k-k'}{2} + 1$ in B_1. Now $|F| \leq k$ and so, by Theorem 3.1.69, there is a perfect matching M_F of B with $F \subseteq M_F$. As $\mathrm{mp}(\partial_B(X)) = k$, at most $k - \frac{k-k'}{2} - 1 = \frac{k+k'}{2} - 1$ edges of $M_F \cap \partial_B(X)$ have an endpoint in $\operatorname{Maj}(X)$. Thus we obtain

$$\begin{aligned}
|\operatorname{Maj}(X) \setminus V(\partial_B(X) \cap M_F)| &\geq |\operatorname{Maj}(X)| - \frac{k+k'}{2} + 1 \\
&= |\operatorname{Maj}(X)| - \frac{k}{2} - \frac{k'}{2} + 1 \\
&= |\operatorname{Min}(X)| + k' - \frac{k}{2} - \frac{k'}{2} + 1 \\
&> |\operatorname{Min}(X)| - \frac{k-k'}{2} - 1 \\
&\geq |\operatorname{Min}(X) \setminus V(\partial_B(X) \cap M_F)|.
\end{aligned}$$

And so $\mathrm{imbalance}(X \setminus V(\partial_B(X) \cap M_F)) \geq 1$ contradicting M_F being a perfect matching, thus implying $\nu(B_1) \leq \frac{k-k'}{2}$. With similar arguments on B_2 we obtain $\nu(B_2) \leq \frac{k+k'}{2}$.

Now, since $\nu(B_1) = \tau(B_1)$ and $\nu(B_1) = \tau(B_1)$ by König's Theorem, we can find a set $T \subseteq V(B)$ of size at most $\frac{k-k'}{2} + \frac{k+k'}{2} = k$ such that $T \cap e \neq \emptyset$ for all $e \in \partial_B(X)$. We assumed $k+2 \leq |X| \leq |V(B)| - k - 2$ and so both $X \setminus T$ and $\overline{X} \setminus T$ are non-empty. Hence T is a separator of order at most k contradicting Theorem 3.1.67 and thus we must have $\mathrm{imbalance}(X) = k$. $\square$

For braces of perfect matching width two, the important case is $k = 2$ in Lemma 5.2.2. Using this we can now prove that there are no degree-3-vertices in the spine of the spine of a width-2-decomposition of a brace. This means that any optimal perfect matching decomposition of a brace B with $\text{pmw}(B) = 2$ has a linear structure.

Proposition 5.2.3 (C*)**.** Let B be a brace of perfect matching width two and (T, δ) a perfect matching decomposition of minimum width for B. Then, spine(spine(T)) is a path.

Proof. Suppose there is a vertex $t \in V(\text{spine}(\text{spine}(T)))$ with three neighbours t_1, t_2 and t_3. By Lemma 5.2.1, spine(T) is cubic and so every t_i is adjacent to exactly two vertices of the spine of T apart from t. Moreover, each of these neighbours again has exactly two neighbours distinct from t_i in T. Let T_i be the component of $T - tt_i$ for $i \in [1, 3]$ that does not contain t and let $X_i := \delta(T_i)$. The above observations imply $|X_i| \geq 4$ for all $i \in [1, 3]$. As T is free of odd edges by Corollary 5.1.8 and Lemma 5.2.1, $\text{mp}(X_i) = 2$ and so Lemma 5.2.2 yields $\text{imbalance}(X_i) = 2$.

Without loss of generality we can assume that two of the three sets have an excess in V_1 while the last one, say X_3, has an excess in V_2. This holds as the case where the excesses of all three sets are of the same colour implies $\text{imbalance}(V(B)) = 6$, a direct contradiction to the existence of a perfect matching in B. However, even under this assumption we still obtain $|V_1| = |V_2| + 2$ and thus, $V(G)$ is not balanced. Since B has a perfect matching, this is impossible and thus, spine(spine(T)) cannot have a vertex of degree three. □

Lemma 5.2.2 establishes the distribution of the two colours V_1 and V_2 in any set of matching porosity k of sufficient size in a k-extendible brace. Suppose we have a k-extendible brace for $k \geq 2$ that has a perfect matching decomposition (T, δ) such that spine(spine(T)) is a path. Next we observe how these nested sets of matching porosity k behave. More precisely, suppose we are given two edges e_x and e_y of spine(spine(T)) and shores X and Y of their cuts $\partial_B(e_x)$ and $\partial_B(e_y)$ respectively, if $X \subseteq Y$, then the majority of X is the majority of Y.

Lemma 5.2.4 (X*)**.** Let $k \geq 2$ be an integer and B be a k-extendible bipartite graph with $|V(B)| \geq 2k + 4$, such that there exists perfect matching decomposition (T, δ) of width k for B with spine(spine(T)) being a path. Furthermore let e_1, e_2 be two adjacent edges of spine(spine(T)) such that $\partial_B(e_i)$ has a shore X_i with $X_1 \subseteq X_2$ and $\mathrm{mp}(\partial_B(X_1)) = \mathrm{mp}(\partial_B(X_2)) = k$. Then $|X_1 \cap V_i| + 1 = |X_2 \cap V_i|$ for both $i \in [1, 2]$.

Proof. First of all note that $|X_2| - |X_1| = 2$ since spine(spine(T)) is a path and the parity of the sets must be preserved by Lemma 5.1.10. Now suppose $|X_1 \cap V_1| + 2 = |X_2 \cap V_1|$ which implies $X_2 \setminus X_1 \subseteq V_1$. By Lemma 5.2.2 we know that $\mathrm{imbalance}(X_i) = \mathrm{mp}(\partial_B(X_i)) = k$, or $|X_i| = k$, or $|\overline{X_i}| = k$ for both $i \in [1, 2]$. As $\mathrm{mp}(\partial_B(X_2)) = k$ we may rule out $|\overline{X_1}| = k$, similarly we obtain $|X_2| \geq k+2$. Suppose $\mathrm{imbalance}(X_1) \leq k-2$ and $|X_1| = k$. If $|\overline{X_2}| = k$, then $|V(B)| = |X_1| + 2 + |\overline{X_2}| = 2k + 2$ which contradicts $|V(B)| \geq 2k+4$. So $\mathrm{imbalance}(X_2) = k$. As $\mathrm{imbalance}(X_1) \leq k - 2$ and $|X_2| = |X_1| + 2$ it follows that $\mathrm{imbalance}(X_1) = k - 2$ and $\mathrm{Maj}(X_1) \subseteq V_1$. From this observation we also get $\mathrm{Maj}(X_2) \subseteq V_1$. We consider the bipartite graphs $B_1 \coloneqq B[\mathrm{Maj}(X_2) \cup \mathrm{Maj}(\overline{X_2})]$ and $B_2 \coloneqq B[\mathrm{Min}(X_2) \cup \mathrm{Min}(\overline{X_2})]$ where the edge set of B_1 consists exactly of those edges in $\partial_B(X_2)$ incident with vertices from $\mathrm{Maj}(X_2)$ and the edge set of B_1 consists of the remaining edges in the cut. Under our current assumptions we have $|\mathrm{Min}(X_2)| = |\mathrm{Min}(X_1)| = 2$ and thus $\nu(B_2) \leq 2$. Suppose there is a matching F of size two in B_2. Then, by Theorem 3.1.66, F is extendible and thus B has a perfect matching M_F containing F. But now $X_2 \setminus V(F)$ still has k vertices of V_1 and no vertex of V_2 left and thus $|M_F \cap \partial_B(X_2)| = |X_2| = k + 2$ which is a contradiction. So $\nu(B_2) \leq 1$. With this we can also conclude $\nu(B_1) \leq k - 1$ since a matching of size k in B_1 would force the existence of a matching of size two in B_1 with another application of the k-extendibility of B. Now we can find a vertex cover in $B_1 \cup B_2$ of size k and so we can hit all edges in $\partial_B(X_2)$ with at most k vertices. By our assumptions $|X_2| = k + 2$ and $|\overline{X_2}| \geq k + 2$ and thus we have found a separator of order k in B which contradicts Theorem 3.1.67. So if $|X_1| = k$, then $\mathrm{imbalance}(X_1) = k$ and thus X_1 is monochromatic, moreover, as $\mathrm{imbalance}(X_2) = k$, $X_2 \setminus X_1$ must contain vertices from both colour classes. So we can assume $|X_1| \geq k + 2$. Then by Lemma 5.2.2 $\mathrm{imbalance}(X_1) = k$. If $|\overline{X_2}| \geq k + 2$ we have $\mathrm{imbalance}(X_1) = \mathrm{imbalance}(X_2)$ and the assertion follows. So suppose $|\overline{X_2}| = k$. But then $|\overline{X_1}| = k + 2$ and from the arguments above we can derive that $\overline{X_2}$ is monochromatic, again implying the assertion. $\square$

For $k = 2$ in the above lemma, the two extreme cases, namely $|X| = 4$ and $|X| = |V(B)| - 4$ play an important role. In case that $|X| = 4$, Lemma 5.2.2 requires X to contain exactly one vertex of one of the two colour classes and three vertices of the other. The case $|X| = |V(B)| - 4$ is similar as $|\overline{X}| = 4$. Hence in the first case $B[X]$ is a claw[2], and in the second case $B[\overline{X}]$ is a claw.

In general, whenever we have two consecutive cuts with both shores bigger than $k + 2$ in a perfect matching decomposition (T, δ) of a k-extendable brace where spine(spine(T)) is a path, the two laminar shores differ by exactly one vertex from each colour class and thus, for general k we can expect to find a star with $k+1$ leaves at both 'ends' of the decomposition.

Lemma 5.2.5 (X*)**.** Let $k \geq 2$ be an integer and B be a k-extendible bipartite graph with $|V(B)| \geq 2k + 4$, such that there exists a perfect matching decomposition (T, δ) of width k for B with spine(spine(T)) being a path. Then T contains two edges e_1 and e_2 where $\partial_B(e_i)$ has a shore X_i of size $k + 2$ satisfying the following conditions.

i) $e_1 = e_2$ if and only if $|V(B)| = 2k + 4$,
ii) $X_1 \cap X_2 = \emptyset$,
iii) if $\mathrm{Maj}(X_1) \subseteq V_1$, then $\mathrm{Min}(X_2) \subseteq V_1$ and vice versa, and
iv) $B[X_i]$ is a star such that its central vertex has no neighbour in $\overline{X_i}$ for both $i \in [1, 2]$.

Proof. Let p_s and p_t be the two endpoints of the path spine(spine(T)). We order the edges of spine(spine(T)) according to their appearance when traversing along spine(spine(T)) from p_s to p_t. So $E(\text{spine}(\text{spine}(T))) = \{e'_1, \ldots, e'_\ell\}$ where p_s is incident with e'_1 and p_t is incident with e'_ℓ. Let $j \in [1, \ell]$ be the smallest number such that $\mathrm{mp}(\partial_B(e'_j)) = k$ and let $X'_1 \coloneqq \delta(T_1)$ where T_1 is the subtree of $T - e'_j$ containing p_s. We claim that $|X'_1| = k$ and $X'_1 \subseteq V_i$ for some $i \in [1, 2]$.

Suppose $|X'_1| \geq k + 2$. Note that $|X'_1| = k + 1$ is impossible due to Lemma 5.1.10. Then by Lemma 5.2.2 we must have $\mathrm{imbalance}(X'_1) = k$. Consider the shore X''_1 of $\partial_B(e'_{j-1})$ with $X''_1 \subseteq X'_1$. By choice of j, $\mathrm{mp}(\partial_B(e'_{j-1})) \leq k - 1$ and thus $\mathrm{imbalance}(X''_1) \leq k - 1$. Still $|X''_1| \geq k$ since $|X'_1| - |X''_1| \leq 2$ and as $\mathrm{mp}(\partial_B(X'_1)) = k$, $|\overline{X''_1}| \geq k$. If $\partial_B(X''_1)$ were to contain a matching F of size k, then $\mathrm{mp}(\partial_B(X''_1)) = k$ since F

[2]The graph $K_{1,3}$ is called a *claw*.

would be extendible. So $\partial_B(X_1'')$ cannot contain a matching of size k and thus, by König's Theorem, there exists a set $S \subseteq V(B)$ of size at most $k-1$ covering all edges in $\partial_B(X_1'')$. This however would be a separator of size $k-1$ in B, contradicting Theorem 3.1.67. Either way we reach a contradiction.

Thus $|X_1'| = k$. Now let $e_1 := e'_{j+1}$ and X_1 be the shore of $\partial_B(e_1)$ containing X_1'. Then $|X_1| \in [k+1, k+2]$ and so by Lemma 5.1.10 $|X_1| = k+2$. We also obtain $\mathrm{mp}(\partial_B(X_1)) = k = \mathrm{imbalance}(X_1)$. Moreover, by Lemma 5.2.4, $|(X_1 \setminus X_1') \cap V_1| = |(X_1 \setminus X_1') \cap V_2| = 1$. Thus our claim holds true and $X_1' \subseteq V_i$ for some $i \in [1,2]$, without loss of generality we assume $i = 1$.

Now let $b \in X_1$ be the sole vertex of $V_2 \cap X_1$. If b has a neighbour $a \in \overline{X_1}$, then ab must be contained in a perfect matching of B and thus $\mathrm{mp}(\partial_B(X_1)) = k+2$ which is impossible. Therefore $\mathrm{N}_B(b) \subseteq X_1$ and as B is $(k+1)$-connected, $\mathrm{N}_B(b) = X_1 \setminus \{b\}$. Hence $B[X_1]$ is a star.

In order to obtain e_2 and X_2 let $j' \in [1, \ell]$ be the largest number such that $\mathrm{mp}(\partial_B(e_{j'})) = k$ and then let $e_2 := e'_{j-1}$ and X_2 be the shore of $\partial_B(e_2)$ disjoint from X_1. By evoking arguments analogous to the discussion of X_1 one can reach the conclusion that $B[X_2]$ is again a star whose centre has no neighbour in $\overline{X_2}$, and $|X_2| = k+2$.

Now, since $|X_1| = |X_2| = k+2$ and by choice of e_1 and e_2 we have $e_1 = e_2$ if and only if $X_1 \cup X_2 = V(B)$ which concludes the proof. □

Let B be a k-extendible brace, where $k \geq 2$, that has a perfect matching decomposition (T, δ) of width k for B such that spine(spine(T)) is a path. Starting with the outermost shores of cuts we found in the previous lemma, we can show that the whole decomposition (T, δ) has a special structure. We are going to do this by induction and the following lemma works as the base.

Lemma 5.2.6 (X*)**.** Let $k \geq 2$ be an integer and B be a k-extendible bipartite graph with $|V(B)| \geq 2k+6$, such that there exists perfect matching decomposition (T, δ) of width k for B with spine(spine(T)) being a path. Let e_1 be an edge of spine(spine(T)) such that $\partial_B(e_1)$ has a shore X_1 with $|X_1| = k+2$, $\mathrm{mp}(\partial_B(X_1)) = k$, and $B[X_1]$ is a star whose central vertex has no neighbour in $\overline{X_1}$. Then there exists an edge $e \in E(T)$ incident with e_1 where $\partial_B(e)$ has a shore $X \subseteq V(B)$ satisfying

i) $X_1 \subseteq X$,
ii) $|X| = k + 4$, and
iii) $\mathrm{N}_B(\mathrm{Min}(X)) \subseteq \mathrm{Maj}(X)$.

Proof. Since $|V(B)| \geq 2k + 6$ we can definitely find an edge $e \in \mathrm{spine}(\mathrm{spine}(T))$ such that $\partial_B(e)$ has a shore X with $X_1 \subseteq X$. Moreover, as G is k-extendible and (T, δ) is of width k we must have $\mathrm{mp}(\partial_B(X)) = k$ as otherwise we could find a small separator in B. So with Lemma 5.1.10 we obtain $|X| = |X_1| + 2 = k + 4$. Additionally, with Lemma 5.2.4 it follows that X differs from X_1 by exactly one vertex of each colour.

Without loss of generality we can assume $\mathrm{Min}(X) \subseteq V_1$. There is a unique vertex $a_1 \in X_1 \cap V_1$ and a unique vertex $a \in (X \setminus X_1) \cap V_1$. By Lemma 5.2.5 a_1 does not have a neighbour in $\overline{X_1}$. Now suppose a has a neighbour $b \in \overline{X}$. Then there is a perfect matching M of G containing ab, but then $M \cap \partial_B(X)$ must also contain edges for the $k + 1$ other vertices of X that cannot be matched to a_1 and so $\mathrm{mp}(\partial_B(X)) \geq k + 2$. Therefore $\mathrm{N}_B(\mathrm{Min}(X)) \subseteq X$ and we are done. □

Proposition 5.2.7 (X*). Let $k \geq 2$ be an integer and B be a k-extendible bipartite graph such that there exists perfect matching decomposition (T, δ) of width k for B with $\mathrm{spine}(\mathrm{spine}(T))$ being a path. Then for all $e \in \mathrm{spine}(\mathrm{spine}(T))$ with $\mathrm{mp}(\partial_B(e)) = k$, any shore X of $\partial_B(e)$ satisfies

i) $\mathrm{imbalance}(X) = k$, and
ii) $\mathrm{N}_B(\mathrm{Min}(X)) \subseteq X$.

Proof. By Lemma 5.2.5 there are edges $e_1, e_2 \in \mathrm{spine}(\mathrm{spine}(T))$ with shores $X_1, X_2 \subseteq V(B)$ of their corresponding cuts such that

- $\mathrm{imbalance}(X_i) = k$,
- $\mathrm{N}_B(\mathrm{Min}(X_i)) \subseteq X_i$,
- $|X_i| = k + 2$, and
- $X_1 \cap X_2 = \emptyset$

for both $i \in [1, 2]$. If $|V(B)| = 2k + 4$, then, again by Lemma 5.2.5, $e_1 = e_2$ and this is the only edge we have to consider, so here we are done.

So we can assume B to have at least $2k + 6$ vertices. Let $P' \subseteq \mathrm{spine}(\mathrm{spine}(T))$ be the unique path with e_1 and e_2 as its two end-edges and let P be the path obtained from P' by deleting its endpoints. Now $E(P)$ is exactly the set of edges of $\mathrm{spine}(\mathrm{spine}(T))$ apart from e_1 and

e_2, that induce cuts of matching porosity k. Note that $E(P) \neq \emptyset$ since $|V(G)| \geq 2k+6$. For all $e \in E(P)$ let X_e be the shore of $\partial_B(e)$ with $X_1 \subseteq X_e$.

We show the assertion by induction over ℓ where $|X_e| = k+2+2\ell$. The case $\ell = 1$ follows directly from Lemma 5.2.6 and corresponds to the edge of P sharing an endpoint of e_1. So now let $e \in E(P)$ be any edge that does not share an endpoint with e_1. By definition P shares an endpoint t with e_1 and so there is an edge $e' \in E(P)$ that shares an endpoint with e and is closer to t than e. Now $|X_e| = |X_{e'}|+2$ by Lemma 5.2.4 and by our induction hypothesis $\text{imbalance}(X_{e'}) = k$ and $\mathrm{N}_B(\mathrm{Min}(X_{e'})) \subseteq X_e$. From Lemma 5.2.4 we also get that the difference between X_e and $X_{e'}$ is exactly one vertex of every colour, so there is a unique vertex a in $(X_e \setminus X_{e'}) \cap V_1$. Let us assume without loss of generality that $\mathrm{Min}(X_e) \subseteq V_1$ and suppose a has a neighbour b in $\overline{X_e}$. Then there is a perfect matching M of B containing ab and $M \cap \partial_B(X_e)$ must contain at least $k+2$ edges since $\mathrm{Min}(X_{e'}) \subseteq V_1$ follows from our assumption. Thus $\mathrm{N}_B(\mathrm{Min}(X_e)) \subseteq X_e$ and we are done. □

By applying the above findings to braces of perfect matching width two, we obtain the following.

Corollary 5.2.8 (C*)**.** Let B be a brace of perfect matching width two, (T, δ) be an optimal perfect matching decomposition of G, $e \in E(\text{spine}(\text{spine}(T)))$ and X a shore of $\partial_B(e)$. Then no vertex of the minority of X has a neighbour in $\overline{X}$.

So, given a perfect matching decomposition (T, δ) of width two for a brace B we know that spine(spine(T)) is a path and one of its endpoints can be identified with a claw in B. Moreover, if the central vertex of said claw is a vertex of V_1, then spine(spine(T)) induces a linear ordering of V_1 which is uniquely determined by (T, δ) except for the order of the last three vertices. Let $a \in V_1$ be any vertex in V_1 and $X_a \subseteq V_1$ be the set of vertices smaller or equal to a in the ordering induced by (T, δ), then Corollary 5.2.8 together with Lemma 5.2.2 implies $|X_a| + 2 = |\,\mathrm{N}_B(X_a)|$. Inspired by this observation, we present a definition for elimination orderings in bipartite matching covered graphs.

Definition 5.2.9 (Matching Elimination Width)**.** Let B be a bipartite matching covered graph and $\Lambda(V_i)$ be the set of all linear orderings of

V_i for $i \in [1,2]$. Let $\lambda \in \Lambda(V_i)$. For every $v \in V_i$ we define the set of *reachable* vertices in V_{3-i} as

$$\mathrm{Reach}[B,\lambda,v] := \mathrm{N}_B(\mathrm{Prec}[B,\lambda,v]), \text{ where}$$
$$\mathrm{Prec}[B,\lambda,v] := \left\{v' \in V_i \mid \lambda(v') \leq \lambda(v)\right\}.$$

We also call these the *reachability-set* and the *predecessor-set* respectively. The *width* of such an ordering is given by

$$\mathrm{width}(\lambda) := \max_{v \in V_i}(|\mathrm{Reach}[B,\lambda,v]| - |\mathrm{Prec}[B,\lambda,v]|).$$

Now the *matching elimination width* of B (with respect to V_i) is defined as

$$\mathrm{meow}_i(B) := \min_{\lambda \in \Lambda(V_i)} \mathrm{width}(\lambda).$$

Please note that by Theorem 3.1.69 $|\mathrm{Reach}[B,\lambda,v]| - |\mathrm{Prec}[B,\lambda,v]| \geq 0$ for all $\lambda \in \Lambda(V_i)$ and all $v \in V_i$. Moreover, if v is not the largest vertex of λ, then $|\mathrm{Reach}[B,\lambda,v]| - |\mathrm{Prec}[B,\lambda,v]| \geq 1$ as B is matching covered.

What follows is a characterisation of braces of perfect matching width two in terms of their matching elimination width. To be more precise, we show that an ordering of the vertices in V_1 of width two can be used to construct a perfect matching decomposition (T,δ) of width two such that spine(spine(T)) is a path. Also, any linear ordering of V_1 obtained from such a path in a perfect matching decomposition (T,δ) of width two provides an ordering of V_1 of width two.

Theorem 5.2.10 (C*)**.** Let B be a brace on at least 6 vertices. Then $\mathrm{pmw}(B) = 2$ if and only if $\mathrm{meow}_1(B) = 2$.

Proof. First, let (T,δ) be a perfect matching decomposition for B of width two. Then, by Proposition 5.2.3, spine(spine(T)) is a path. Let $n := |V_1|$, then $|V(B)| = 2n$ and T has $2n$ leaves. So by Observation 5.1.6, spine(T) has $2n-2$ vertices and as spine(T) has a leaf for every two vertices of B, $|\mathsf{L}(\mathrm{spine}(T))| = n$.

Thus, spine(spine(T)) has $n-2$ vertices, let $t_1, \ldots, t_{n-2}$ be its vertices ordered by occurrence and t_1 being the endpoint that, by Lemma 5.2.5, corresponds to a claw in B whose central vertex is $v_1 \in V_1$. We define a bijective function $\lambda^{-1} \colon [1,n] \to V_1$ whose inverse provides the desired ordering. We set $\lambda^{-1}(1) := v_1$.

For each $i \in [1, n-3]$ let $X_i := \delta(T_i)$ where T_i is the subtree of $T - t_i t_{i+1}$ that contains t_1. By our definition of v_1 and t_1, $X_1 \cap V_1 = \{v_1\}$. Now, consider $i \in [2, n-3]$. Clearly $X_j \subseteq X_i$ for all $j < i$ and by Lemma 5.2.4, $X_i \setminus X_{i-1}$ contains exactly two vertices, one being $u_i \in V_2$ and the other one being $v_i \in V_1$. Set $\lambda^{-1}(i) := v_i$. At last let $\{v_{n-2}, v_{n-1}, v_n\} = \overline{X_{n-3}} \cap V_1$ where the order of these three vertices is chosen arbitrarily and set $\lambda^{-1}(j) := v_j$ for all $j \in [n-2, n]$.

Now, $\lambda = (\lambda^{-1})^{-1}$ is a linear ordering of V_1. Note that $\text{meow}_1(G) \geq 2$ due to Theorem 3.1.69. Hence it is only left to show that $\text{width}(\lambda) = 2$.

Let $v \in V_1$ be chosen arbitrarily. If $v \in \{v_{n-2}, v_{n-1}, v_n\}$ we have nothing to show, so suppose $v = v_i$ for some $i \in [1, n-3]$. Then, $\text{Reach}[B, \lambda, v] = X_i \cap V_2$ and $\text{Prec}[B, \lambda, v] = X_i \cap V_1 = \{v_1, \ldots, v_i\}$. Lemma 5.2.2 yields $\text{imbalance}(X_i) = 2$ and as $\{v_1\}$ is the minority of X_1, we obtain that V_1 contains the minority of X_i from Lemma 5.2.4. Therefore, $|\text{Reach}[B, \lambda, v] - \text{Prec}[B, \lambda, v]| = 2$. As i was chosen arbitrarily, $\text{width}(\lambda) = 2$ and thus $\text{meow}_1(G) = 2$.

Second, for the reverse direction, let λ be a linear ordering of V_1 of width two, and let $n := |V_1|$. Since B is a brace, $|\text{Reach}[B, \lambda, v]| - |\text{Prec}[B, \lambda, v]| \geq 2$ for all $v \in V_1$ with $\lambda(v) \leq n-2$. Let $X_1 := \{v_1\} \cup \text{N}_B(v_1)$ and for all $i \in [1, n-3]$ let $X_i := X_{i-1} \cup \{v_i\} \cup \text{N}_B(v_i)$ and then let $X_{n-2} := X_{n-3} \cup \{v_{n-2}, v_{n-1}, v_n\} \cup \text{N}_B(\{v_{n-2}, v_{n-1}, v_n\})$. We claim that $\text{mp}(\partial_B(X_i)) = 2$ for all $i \in [1, n-2]$ and $|X_j| - |X_{j-1}| = 2$ for all $j \in [2, n-2]$ as well as $|X_1| = |X_{n-2} \setminus X_{n-3}| = 4$.

By construction, for all $i \in [1, n-2]$, $\text{N}_B(V_1 \cap X_i) \subseteq X_i$ and thus $\text{mp}(\partial_B(X_i)) = |X_i| - 2|V_1 \cap X_i| = |V_2 \cap X_i| - |V_1 \cap X_i| = 2$, where the last equality follows from the width of λ. Now, consider $j \in [1, n-3]$. By definition, $|X_j \cap V_1| - |X_{j-1} \cap V_1| = 1$ and, as we have seen above, $|V_2 \cap X_j| - |V_1 \cap X_j| = |V_2 \cap X_{j-1}| - |V_1 \cap X_{j-1}|$ hence, $|X_j \cap V_2| - |X_{j-1} \cap V_2| = 1$ as well. At last, clearly $|X_1| = 4$ by definition and the width of λ. Moreover $|V_2 \cap X_j| - |V_1 \cap X_j| = 2$ and thus $|X_{n-3} \cap V_2| - |X_{n-3} \cap V_1| = 2$ implying $|X_{n-3} \cap V_2| = n-1$, so $|\overline{X_{n-2}}| = 4$.

We now use the X_i to construct a perfect matching decomposition of width two for B. The idea is simple, we introduce a path on $n-2$ vertices $t_1, \ldots, t_{n-2}$ and identify X_i with t_i for all i. We construct a tree T by first, introducing two new leaf neighbours for t_1 and t_{n-2} and one new leaf neighbour for each t_j with $j \in [2, n-3]$ and second, introducing two leaf neighbours again for every leaf added in the first step. This results in the two endpoints of our original path being identified with four new leaves each, while every internal vertex of the path is identified with two leaves of the new tree T. We start creating δ by mapping the four leaves identified with t_1 to the vertices of X_1 and the four leaves identified with t_{n-2} to the vertices of $\overline{X_{n-3}}$. By our observations above, for each $j \in [2, n-3]$, $|X_j| - |X_{j-1}| = 2$ and so for each such j we can map the two leaves of T identified with t_j to the two vertices in $X_j \setminus X_{j-1}$. The result is a perfect matching decomposition (T, δ) of B and, since $\mathrm{mp}(\partial_B(X_i)) = 2$ for all $i \in [1, n-2]$, it is of width two. This completes our proof. $\square$

Let B be a brace of perfect matching width two and λ a linear ordering of V_1 such that $\mathrm{width}(\lambda) = 2$. Suppose for some $v \in V_1$ there is a $u \in \mathrm{Reach}[B, \lambda, v]$ with $uv \notin E(B)$, then λ is also a width-2-ordering of $B + uv$. Using this observation, we can add edges to our brace until we reach a brace B' such that $\mathrm{meow}_1(B' + uv) > \mathrm{meow}_1(B') = 2$ for every edge uv with $v \in V_1$, $u \in V_2$ and $uv \notin E(B')$.

By following this idea of constructing an edge-maximal brace of perfect matching width two we obtain a special kind of bipartite graphs. We call a brace $L_n = B$ a *bipartite ladder* of *order* n if $V_1 = \{v_1, \ldots, v_n\}$, $V_2 = \{u_1, \ldots, u_n\}$ and $E(B) = E_1 \cup E_2 \cup E_3$ where

i) $E_1 \coloneqq \{v_i u_j \mid \text{for all } 1 \leq j \leq i \leq n\}$,
ii) $E_2 \coloneqq \{v_i u_{i+1} \mid \text{for all } 1 \leq i \leq n-1\}$, and
iii) $E_3 \coloneqq \{v_i u_{i+2} \mid \text{for all } 1 \leq i \leq n-2\}$.

The graphs L_1, which is a single edge and L_2 which is isomorphic to C_4 are not very interesting due to their size. For $n \geq 3$ however these graphs grow more complex, see Figure 5.3 for an illustration on L_3, L_4 and L_5.

The following corollary is a nice consequence of Theorem 3.1.69.

Corollary 5.2.11 (C*)**.** Let B be a brace and $v_1 \in V_1$, $v_2 \in V_2$ such that $v_1 v_2 \notin E(B)$. Then $B + v_1 v_2$ is a brace.

This corollary allows the construction of edge-maximal braces of width two we are aiming for. We conclude this section with a second characterisation of perfect matching width two braces, this time in terms of edge-maximal supergraphs.

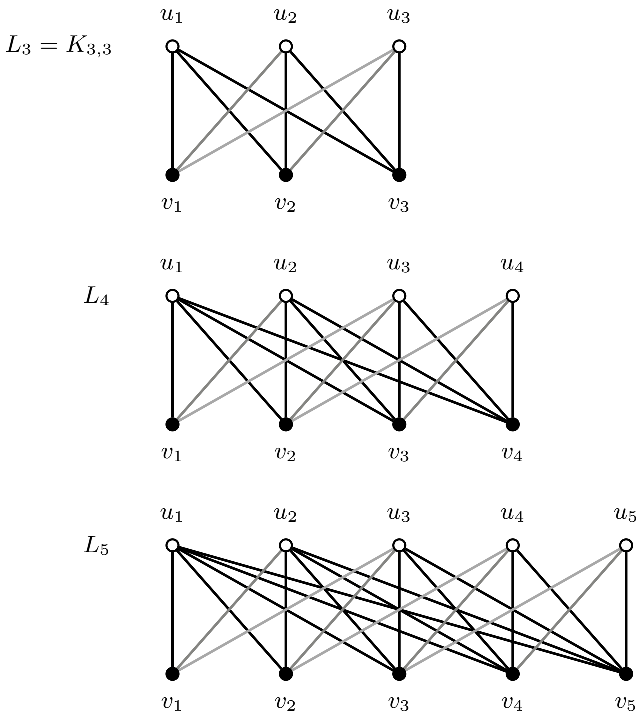


Figure 5.3.: The bipartite ladders of order 3, 4, and 5.

Theorem 5.2.12 (C*)**.** Let B be brace with $|V_1| = n$. Then, $\mathrm{pmw}(B) = 2$ if and only if $B \subseteq L_n$.

Proof. We start by proving that every conformal subgraph of L_n is of perfect matching width 2 or isomorphic to K_2. To do so, by Lemma 5.1.15, it suffices to show $\mathrm{pmw}(L_n) = 2$ for all $n \in \mathbb{N}$ with $n \geq 2$. The definition of L_n directly provides an ordering λ of $V_1 = \{v_1, \dots, v_n\}$ with $\lambda(v_i) = i$. We prove that $\mathrm{width}(\lambda) = 2$. Let $i \in [1, n-3]$ be arbitrary. By definition, $\mathrm{N}_{L_n}(v_i) = \{u_1, \dots, u_{i+2}\} \subseteq \mathrm{Reach}[B, \lambda, v_i]$. Moreover, as $\mathrm{N}_{L_n}(v_j) \subseteq \mathrm{N}_{L_n}(v_i)$ for all $j \leq i$, $\mathrm{N}_{L_n}(v_i) = \mathrm{Reach}[B, \lambda, v_i]$. Therefore, $|\mathrm{Reach}[B, \lambda, v_i]| - |\mathrm{Prec}[B, \lambda, v_i]| = 2$ for all $i \in [1, n-2]$ and thus, $\mathrm{width}(\lambda) = 2$. By Theorem 5.2.10 the assertion follows.

Now, let B be a brace of perfect matching width two. Then, there is an ordering λ of V_1 of width two by Theorem 5.2.10. Let us number the vertices of V_1 according to λ, so for all $i \in [1, n]$ let $v_i \coloneqq \lambda^{-1}(i)$. We construct a numbering of the vertices in V_2 as follows. Let $\mathrm{N}_B(v_1) = \{u_1, u_2, u_3\}$ be numbered arbitrarily. The size of the neighbourhood of a_1 follows immediately from the width of λ and the fact that B is a brace. Now, as a consequence of Lemma 5.2.4, for every $i \in [1, n-2]$, $\mathrm{Reach}[B, \lambda, v_i] \setminus \mathrm{Reach}[B, \lambda, v_{i-1}]$ contains exactly one vertex, which is in V_2. Let u_{i+2} be this vertex. Now, $\mathrm{N}_B(v_i) \subseteq \mathrm{Reach}[B, \lambda, v_i]$ for all $i \in [1, n]$ and thus B does not contain an edge that does not obey the definition of a bipartite ladder with respect to the orderings of V_1 and V_2 as obtained above. If there are indices $i \in [1, n]$ and $j \in [1, n]$ such that $v_i u_j \notin E(B)$, but $j \leq i + 2$, then we simply add the edge $v_i u_j$ to B. By Corollary 5.2.11 $B + v_i u_j$ is still a brace and by choice of i and j, adding this edge does not change the predecessor- and reachability-sets of any vertices in V_1, hence λ is still an ordering of width two for $G + v_i u_j$. Thus, we can keep adding edges in this fashion until we do not find such a pair of indices any more. In that case let B' be the newly obtained brace. By construction, B' is isomorphic to L_n and thus B is a conformal subgraph of L_n. □

5.2.1. Computing Perfect Matching Decompositions of Width 2

The complexity of recognising bipartite graphs of perfect matching width at most k and to construct a bounded width decomposition as a witness, plays a major part in the question whether perfect matching width can be used for algorithmic applications at all. In this section we establish a polynomial time algorithm to compute an optimal perfect matching decomposition of a brace of perfect matching width two.

In order to achieve this, we use the fact that, due to Theorem 5.2.10, we can find a matching elimination ordering if B has perfect matching width two. Key to the construction of this ordering is Lemma 5.2.5, which tells us that we have to start with a vertex that, together with its neighbourhood, induces a claw, which means a vertex of degree three. So in particular any bipartite matching covered graph that does not have a degree three vertex can be ruled out as a candidate for perfect matching width two. Next, we use Corollary 5.2.8 and Lemma 5.2.4, so in each step after choosing the claw we have to choose one additional vertex such that its neighbourhood contains at most one vertex that is not already in the neighbourhood of the previously chosen vertices. If at some point we are not able to find another vertex meeting these requirements, we either chose the wrong claw and have to start over, or $\mathrm{pmw}(B) \geq 3$. Certainly there are only so many different degree three vertices in B and so we can simply try them all.

Lemma 5.2.13 (C*)**.** Let B be a brace. Then *Algorithm* 1 computes an ordering λ of width two from input B and 1 if and only if $\mathrm{pmw}(B) = 2$.

Proof. First, suppose Algorithm 1 returns an ordering λ for the input B and 1. Then, we can consider the sets $\mathrm{Prec}[B, \lambda, v]$ and $\mathrm{Reach}[B, \lambda, v]$. Proving $|\,\mathrm{Reach}[B, \lambda, \lambda^{-1}(j)]| - |\,\mathrm{Prec}[B, \lambda, \lambda^{-1}(j)]| \leq 2$ for all $j \in [1, |V_1|]$ by induction shows $\mathrm{width}(\lambda) = 2$, as $2 \leq \mathrm{width}(\lambda)$ since B is a brace. If $j \in \{1, |V_1| - 1, |V_1|\}$, there is nothing to show. So, suppose $j \in [2, |V_1| - 3]$ and let $v \coloneqq \lambda^{-1}(j)$. That is, v is chosen in the iteration for $i = j$ in step 9. Let P_v and U_v be the sets P and U during this step of the algorithm. The set P_v contains all vertices that were previously chosen by Algorithm 1 and thus are smaller than v with respect to λ. Hence $\mathrm{Prec}[B, \lambda, v] = P_v \cup \{v\}$ and $\mathrm{Prec}[B, \lambda, \lambda^{-1}(j-1)] = P_v$. With v being chosen at step j, we know $|\,\mathrm{N}_B(v) \setminus \mathrm{N}_B(P_v)| \leq 1$. Therefore,

$$\begin{aligned} |\,\mathrm{Reach}[B, \lambda, v]| - |\,\mathrm{Prec}[B, \lambda, v]| &= |\,\mathrm{N}_B(P_v \cup \{v\})| - |P_v \cup \{v\}\,| \\ &\leq |\,\mathrm{N}_B(P_v)| + 1 - (|P_v| + 1) \\ &\leq |P_v| + 3 - (|P_v| + 1) = 2. \end{aligned}$$

Hence, by Theorem 5.2.10, $\mathrm{width}(\lambda) = 2$ and therefore $\mathrm{pmw}(B) = 2$. Second, suppose $\mathrm{pmw}(B) = 2$. By Theorem 5.2.10, there exists an ordering σ of V_1 with $\mathrm{width}(\sigma) = 2$. We have already seen that, if Algorithm 1 returns an ordering λ, it will be of width two. So what remains to show is that the algorithm returns an ordering.

Suppose it does not. Let $v_1 \coloneqq \lambda^{-1}(1)$. Since Algorithm 1 only terminates without returning an ordering when it looped through all elements for the choice in step 3, it reaches the point where it chooses v_1. Now, Algorithm 1 can choose the next element in step 10 fulfilling the demand in step 11 according to the ordering λ. Since it does not end up returning an ordering it eventually differs from any optimal ordering and then reaches the point $k \in [2, |V_1|]$ at which there is no element to choose in step 10 fulfilling the demand in step 11. Let $v_1 \ldots, v_k$ be elements of V_1 that Algorithm 1 ordered this way so far before it gets stuck. Let σ be chosen among all width-two-orderings of V_1 maximising $j \in [1, k-1]$ such that $\sigma^{-1}(i) = v_i$ for all $1 \leq i < j$ and $\sigma^{-1}(j) \neq v_j$. We refer to the elements after v_j in σ by $y_h \coloneqq \sigma^{-1}(h)$ for all $h \in [j+1, |V_1|]$. By the definition of the algorithm, $|\,\mathrm{N}_B(v_j) \setminus \mathrm{N}_B(v_1, \ldots, v_{j-1})| \leq 1$. Let σ' be the ordering obtained from σ by inserting v_j at the position j instead of its position $j + x$ in σ. So σ' contains the elements of V_1 in the order $v_1, \ldots, v_j, y_{j+1}, \ldots, y_{j+x-1}, y_{j+x+1}, \ldots, y_{|V_1|}$.

Suppose, $\mathrm{width}(\sigma') \geq 3$. There is a vertex $y_{h'}$ with $h' \in [j+1, j+x-1]$ such that

$$|\,\mathrm{Reach}[B, \sigma', y_{h'}]| - |\,\mathrm{Prec}[B, \sigma', y_{h'}]| \geq 3.$$

But $|\,\mathrm{Prec}[B, \sigma', y_{h'}]| = |\,\mathrm{Prec}[B, \sigma, y_{h'}]| + 1$ and with

$$|\,\mathrm{N}_B(v_j) \setminus \mathrm{N}_B(v_1, \ldots, v_{j-1})| \leq 1$$

we obtain $|\,\mathrm{Reach}[B, \sigma', y_{h'}]| \leq |\,\mathrm{Reach}[B, \sigma, y_{h'}]| + 1$. Thus,

$$|\,\mathrm{Reach}[B, \sigma, y_{h'}]| - |\,\mathrm{Prec}[B, \sigma, y_{h'}]| \geq 3,$$

which contradicts σ being of width two. Hence $\mathrm{width}(\sigma') = 2$. However, this is a contradiction to the choice of σ as σ' now coincides on the first j positions with the choice of Algorithm 1. Thus, the algorithm does not get stuck once it chose the right claw and therefore, Algorithm 1 returns an ordering. □

So Algorithm 1 produces an elimination ordering of width two if and only if the brace B that was given as input is of perfect matching width two. We just have to translate this ordering into a perfect matching decomposition and are done. In the second part of the proof of Theorem 5.2.10 a procedure is given to obtain a perfect matching decomposition of B from a matching elimination ordering of width two. Since all sets necessary for the construction of this decomposition can be computed from the ordering by iterating over edges and vertices of B at most once, this procedure can be done in polynomial time and thus, we obtain the following result which concludes this section.

Theorem 5.2.14 (C*)**.** Let B be a brace. There is a polynomial time algorithm that computes a perfect matching decomposition of width two if and only if $\mathrm{pmw}(B) = 2$.

Algorithm 1 Compute Width-Two-Ordering

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procedure ORDER(brace B, $i \in [1, 2]$)
    $\lambda^{-1} \leftarrow \emptyset$
    for all $v \in V_i$ **do**
        $\lambda^{-1} \leftarrow \emptyset$
        if $|\mathrm{N}_B(v)| = 3$ **then**
            $\lambda^{-1}(1) \leftarrow v$
            $U \leftarrow V_i \setminus \{v\}$
            $P \leftarrow \{v\}$
            for all $i \in [2, |V_i|]$ **do**
                for all $v' \in U$ **do**
                    if $|\mathrm{N}_B(v') \setminus \mathrm{N}_B(P)| \leq 1$ **then**
                        $\lambda^{-1}(i) \leftarrow v'$
                        $P \leftarrow P \cup \{v'\}$
                        $U \leftarrow U \setminus \{v'\}$
                        break
                if $\lambda^{-1}(i) = \emptyset$ **then**
                    break
            if $\lambda^{-1}(|V_i|) \neq \emptyset$ **then return** λ
    return $\mathrm{pmw}(B) \geq 3$.

5.2.2. Bipartite Matching Covered Graphs of M-Width 2

Section 5.2 provides a complete characterisation of braces of perfect matching width two. However, we are not able to lift this result to all bipartite matching covered graphs, since we do not know whether the braces of a matching covered bipartite graph of perfect matching width two are also of perfect matching width two themselves. To be more precise, for a matching covered bipartite graph B with $\mathrm{pmw}(B) = 2$, the best we know about any brace H of it is $\mathrm{pmw}(H) \in [2, 4]$ by Corollary 5.1.20. We can however consider the M-perfect matching width instead since here Proposition 5.1.19 implies that $\mathrm{pmw}(B)$ bounds $\mathrm{pmw}(H)$. Indeed, by Corollary 5.1.22, B has M-width two if and only if all of its braces H have $M|_H$-width two.

In this section we present a full characterisation of the braces of M-width two and thus, provide a description of all matching covered bipartite graphs that have a perfect matching M such that their M-width is 2.

Key to this characterisation is the observation that, given a brace B, $2 \leq \mathrm{pmw}(B) \leq M\text{-}\mathrm{pmw}(B)$ for all $M \in \mathcal{M}(B)$. So, if $M\text{-}\mathrm{pmw}(B) = 2$ for some M, then any optimal M-decomposition of B will also be an optimal perfect matching decomposition of G. Therefore we can apply the results from Section 5.2. This immediately implies a rather strict bound on the number of vertices, which in turn narrows down the braces of M-width two to exactly two, namely $K_{3,3}$ and C_4.

Proposition 5.2.15 (C*)**.** Let B be a brace, then the following statements are equivalent.

i) $M\text{-}\mathrm{pmw}(B) = 2$ for an $M \in \mathcal{M}(B)$,
ii) $M\text{-}\mathrm{pmw}(B) = 2$ for all $M \in \mathcal{M}(B)$, and
iii) B is isomorphic to C_4 or $K_{3,3}$.

Proof. In order to prove this statement, we first deduce item iii) from item i) and then observe that we can find the same type of decomposition for every $M \in \mathcal{M}(B)$ which then implies item ii).

Let B be a brace and $M \in \mathcal{M}(B)$ such that $M\text{-}\mathrm{pmw}(B) = 2$, then $\mathrm{pmw}(B) = 2$ as well. Let (T, δ) be an optimal M-decomposition for B, then it also is an optimal perfect matching decomposition of B. Now suppose $|V(B)| \geq 8$. Then by Lemma 5.2.5, there is an edge

$e \in E(\text{spine}(\text{spine}(T)))$ such that $\partial_B(e)$ has a shore X of size 4 that induces a claw in B. In particular, $\text{imbalance}(X) = 2$ and thus X is not M-cornformal. This is a contradiction to the definition of M-decompositions as e is an inner edge of T. So $|V(G)| \leq 6$. On at most 6 vertices there are only two braces: C_4 and $K_{3,3}$.

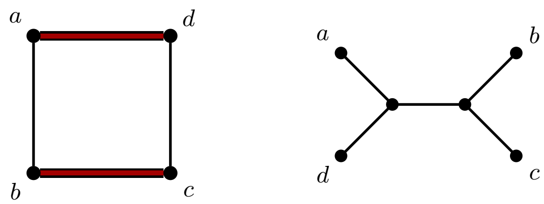


Figure 5.4.: The brace C_4 together with a perfect matching M and an M-decomposition (T, δ) of width two.

First, consider C_4. Let $M \in \mathcal{M}(C_4)$ be a perfect matching. Then, $V(C_4) = \{a, b, c, d\}$ and without loss of generality $M = \{ad, bc\}$. As C_4 is a cycle, the only other perfect matching of C_4 is $E(C_4) \setminus M = \{ab, cd\}$. We construct a perfect matching decomposition (T, δ) as follows. Take two vertices t_1 and t_2 joined by an edge. We create a cubic tree T by adding two leaves t_i^1 and t_i^2 as new neighbours to each of the t_i for $i \in [1, 2]$. Then, let $\delta(t_1^1) := a$, $\delta(t_1^2) := d$, $\delta(t_2^1) := b$ and $\delta(t_2^2) := c$ (see Figure 5.4). Now, (T, δ) is an M-decomposition of C_4 and the matching porosity of every cut induced by an edge of T is either one or two. Note that for the other perfect matching of C_4 we just have to adapt the mapping δ such that for each $i \in [1, 2]$ the leaves t_i^1 and t_i^2 are mapped to the endpoints of the same edge and thus $M\text{-}\operatorname{pmw}(C_4) = 2$ for all $M \in \mathcal{M}(C_4)$.

Second consider $K_{3,3}$ and let $V_1 = \{a, b, c\}$ and $V_2 = \{d, e, f\}$ and $M = \{af, be, cd\}$ a perfect matching of $K_{3,3}$. We again construct an M-decomposition (T, δ) of our brace. This time consider a claw on the vertices $\{t, t_1, t_2, t_3\}$ such that t is the central vertex. For each $i \in [1, 3]$ we introduce two new neighbours t_i^1 and t_i^2 to t_i which will be the leaves of our cubic tree T. Then let $\delta(t_1^1) := a$ and $\delta(t_1^2) := f$. For the remaining two edges of M proceed analogously by choosing an $i \in [2, 3]$ for each of the remaining edges and then mapping the leaves t_i^1 and t_i^2 to the endpoints of the chosen edge. Now, (T, δ) is an M-decomposition of $K_{3,3}$

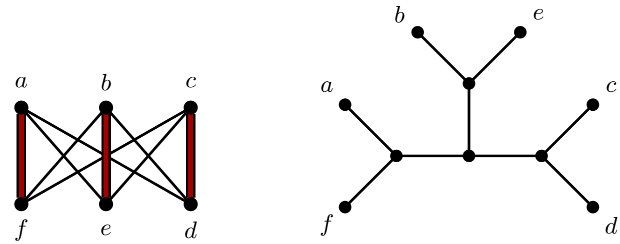


Figure 5.5.: The brace $K_{3,3}$ together with a perfect matching M and an M-decomposition (T, δ) of width two.

and for every inner edge e of T the cut induced by e has a shore of size two, hence $\mathrm{width}(T, \delta) = 2$ (see Figure 5.5 for an illustration). Again, we can adapt the same strategy for every perfect matching $M' \in \mathcal{M}(B)$ and thus $M\text{-}\operatorname{pmw}(B) = 2$ for all $M \in \mathcal{M}(B)$.

We have seen that for each of the braces C_4 and $K_{3,3}$ the M-width equals two for all perfect matchings M. So, in particular there exists such a matching and thus, item ii) implies item i) again and the proof is complete. $\square$

With Proposition 5.2.15 we are able to deduce a similar theorem for general bipartite matching covered graphs of M-width two.

Theorem 5.2.16 (C*)**.** Let B be a bipartite matching covered graph, then the following statements are equivalent.

i) $M\text{-}\operatorname{pmw}(B) = 2$ for an $M \in \mathcal{M}(B)$,
ii) $M\text{-}\operatorname{pmw}(B) = 2$ for all $M \in \mathcal{M}(B)$, and
iii) Every brace of B is either isomorphic to C_4 or to $K_{3,3}$.

Proof. By Theorem 3.1.61, every two tight cut decomposition procedures of a matching covered graph produce the same list of bricks and braces. This implies that any two maximal families of pairwise laminar non-trivial tight cuts in a matching covered graph have the same size. We are going to use this observation as a tool for induction.

Let B be a bipartite matching covered graph and $\mathcal{F}_B$ a maximal family of pairwise laminar tight cuts in B. We prove the equivalence of items i) to iii) by induction over $|\mathcal{F}_B|$.

The base case with $|\mathcal{F}_B| = 0$ is the case where B is a brace and thus follows from Proposition 5.2.15.

Assume $|\mathcal{F}_B| \geq 1$ and let $\partial_B(Z)$ be any non-trivial tight cut in $\mathcal{F}_B$. Let $B_Z \coloneqq B/(Z \to v_Z)$, and $B_{\overline{Z}} \coloneqq B/(\overline{Z} \to v_{\overline{Z}})$. By induction hypothesis, the three statements are equivalent for both B_Z and $B_{\overline{Z}}$.

Assume M-$\operatorname{pmw}(B) = 2$ for an $M \in \mathcal{M}(B)$ (item i)), then by Corollary 5.1.22 $M|_{B_Z}$-$\operatorname{pmw}(B_Z) = M|_{B_{\overline{Z}}}$-$\operatorname{pmw}(B_{\overline{Z}}) = 2$ and thus, the braces of both B_Z and $B_{\overline{Z}}$ are isomorphic to C_4 or $K_{3,3}$. Since the braces of B are exactly the union of the braces of B_Z and $B_{\overline{Z}}$, item iii) holds for B as well.

Next, assume that item iii) holds for B. Pick any matching $M' \in \mathcal{M}(B)$, then by induction hypothesis $M'|_{B_Z}$-$\operatorname{pmw}(B_Z) = M'|_{B_{\overline{Z}}}$-$\operatorname{pmw}(B_{\overline{Z}}) = 2$. Let $e_Z \in M'|_{B_Z}$ and $e_{\overline{Z}} \in M'|_{B_{\overline{Z}}}$ be the two edges covering v_Z and $v_{\overline{Z}}$ in the respective contractions for the respective reductions of M'. Let u_X be the endpoint of e_X that is not v_X for both $X \in \{Z, \overline{Z}\}$. Moreover, let (T_X, δ_X) be an optimal $M'|_X$-decomposition of B_X for both $X \in \{Z, \overline{Z}\}$. In T_Z there is a vertex t_Z that is adjacent to the two leaves of T_Z that are mapped to v_Z and u_Z, let $t_{\overline{Z}}$ be chosen analogously. Observe, that $M' = ((M'|_{B_Z} \cup M'|_{B_{\overline{Z}}}) \setminus \{e_Z, e_{\overline{Z}}\}) \cup \{u_Z u_{\overline{Z}}\}$. We construct an M'-decomposition (T', δ') as follows. Let T'_X be obtained from T_X be deleting the two leaves adjacent to t_X for both $X \in \{Z, \overline{Z}\}$. Then, let T'' be the tree obtained from T'_Z and $T'_{\overline{Z}}$ by identifying t_Z and $t_{\overline{Z}}$, call the new vertex t. At last, let T' be the tree obtained from T'' by adding a new vertex t', the edge tt' and two new leaves t_1 and t_2 adjacent to the new vertex t'. Then, T' is a cubic tree and $|\mathsf{L}(T')| = |V(B)|$. In the next step we define $\delta' : \mathsf{L}(T') \to V(B)$ as follows:

$$\delta'(\ell) \coloneqq \begin{cases} \delta_Z(\ell), & \text{if } \ell \in \mathsf{L}(T_Z) \setminus \left\{\delta_Z^{-1}(v_Z)\right\}, \\ \delta_{\overline{Z}}(\ell), & \text{if } \ell \in \mathsf{L}(T_{\overline{Z}}) \setminus \left\{\delta_{\overline{Z}}^{-1}(v_{\overline{Z}})\right\}, \\ u_{\overline{Z}}, & \text{if } \ell = t_1, \text{ and} \\ u_Z, & \text{if } \ell = t_2. \end{cases}$$

Now, (T', δ') is an M'-decomposition of B. Moreover, let $e \in E(T')$ be an inner edge of T', then either e is an inner edge of T_Z or $T_{\overline{Z}}$ and by construction of T' and the fact that $\partial_B(Z)$ is tight, $\operatorname{mp}(\partial_B(e)) = 2$, or $e = tt'$. In the later case, $\partial_B(e)$ has a shore of size two and thus $\mathrm{mp}(\partial_B(e)) = 2$. Therefore, $\mathrm{width}(T', \delta') = 2$ and so M'-pmw$(B) = 2$ for all $M' \in \mathcal{M}(B)$, that is item ii) holds. Since item ii) implies item i), we are done. □

So, in order to recognise a bipartite matching covered graph B of M-width two, one just needs to check whether B has a brace not isomorphic to C_4 or $K_{3,3}$. Lovász has shown that the tight cut decomposition of a matching covered graph can be computed in polynomial time (see [Lov87]) and thus, Theorem 5.2.16 implies a polynomial recognition algorithm for bipartite matching covered graphs of M-width two. Moreover, the proof of Theorem 5.2.16 is constructive and can be used to obtain an M-decomposition of width two for any $M \in \mathcal{M}(B)$, given a bipartite matching covered graph B of M-width two, from the decompositions of its braces. As these braces are only C_4 and $K_{3,3}$, whose optimal M-decompositions are given in the proof of Proposition 5.2.15, we obtain the following corollary.

Corollary 5.2.17 (C*)**.** Let B be a bipartite matching covered graph and $M \in \mathcal{M}(B)$. Then, we can compute in polynomial time either an M-decomposition of width two, or a brace of B that is neither isomorphic to C_4, nor to $K_{3,3}$.

5.2.3. Linear Perfect Matching Width

We have seen that for any brace B of perfect matching width two an optimal perfect matching decomposition (T, δ) has a particular shape, namely spine(spine(T)) is a path. Moreover, by applying Lemma 3.1.58 we can see that any cut induced by an edge of spine(spine(T)) is indeed a 2-tight cut. At the core of our characterisation of braces of perfect matching width two sits Proposition 5.2.7 which is in fact not limited to the case $k = 2$, it only needs the brace under consideration to be k-extendible and to have a decomposition whose spine of the spine is a path.

In Chapter 2 we discussed several versions of the cops & robber game for undirected graphs and digraphs. Some possible combinations were left out here. On undirected graphs, when considering just the non-monotone game with an invisible robber, one obtains a width parameter that is very similar to treewidth, with one major difference: The decomposition resembles a path instead of a tree. Consequently, the width parameter is called *pathwidth*. Similarly one can consider the weak variant of the directed cops & robber game and make the robber invisible. This time the outcome is a parameter similar to DAG-width with the difference, that the decomposition resembles a directed path and so the parameter is called *directed pathwidth*. Notice that the game equivalent to directed pathwidth is the weak variant, while the game for directed treewidth is the strong variant. So what about the strong variant of the directed cops & robber game where the robber is invisible? Apart from the following conjecture, there does seem to exist much research on the topic.

Conjecture 5.2.18 (Barát's Conjecture, [Bar06])**.** There exists a constant $c \in \mathbb{R}$ such that for all digraphs D

$$c \cdot \text{cops}_{w,iv}(D) \leq \text{cops}_{s,iv}(D) \leq \text{cops}_{w,iv}(D).$$

No type of bound seems to be known so far. Similar to how the two directed parameters differ, we now define two 'linear' or 'pathlike' variants of perfect matchings width inspired by our results on braces of perfect matching width two.

Definition 5.2.19 (Linear Perfect Matching Width)**.** Let G be a graph with a perfect matching. A perfect matching decomposition (T, δ) of G is said to be *linear* if $\text{spine}(\text{spine}(T))$ is a path. The *linear perfect matching width* of G, denoted by $\text{lpmw}(G)$, is defined as the minimum width over all linear perfect matching decompositions of G.

Definition 5.2.20 (Strict Linear Perfect Matching Width)**.** Let B be a bipartite graph with a perfect matching. A linear perfect matching decomposition (T, δ) of G is said to be *strict* if for all $e \in E(\text{spine}(\text{spine}(T)))$ the cut $\partial_B(e)$ is a generalised tight cut. The *strict linear perfect matching width* of B, denoted by $\text{slpmw}(G)$, is defined as the minimum width over all strict perfect matching decompositions of B.

We briefly revisit the idea of strict linear perfect matching width in Section 5.3 and Chapter 6. For now let us close this section with another open problem. For bipartite graphs $\text{lpmw}(B) \leq \text{slpmw}(B)$ follows immediately.

Question 5.2.21. Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\operatorname{slpmw}(B) \leq f(\operatorname{lpmw}(B))$ for all bipartite graphs B with perfect matchings?

In the following section we will see that a positive answer to Conjecture 5.2.18 implies a positive answer for Question 5.2.21, while the reverse is true only if f is a linear function.

5.3. Duality and Directed Treewidth

We have seen in Section 3.2 that there is a close connection between bipartite graphs with perfect matchings and digraphs. Knowing about this connection, Norine wrote in his thesis, that the bipartite version of Conjecture 5.1.3 should follow from the Directed Grid Theorem. However, at first glance there does not seem to be a straightforward connection between directed treewidth and perfect matching width. In some special cases we can already draw such a connection. For braces of perfect matching width two, Lemma 3.2.13 provides a direct translation of an optimal perfect matching decomposition into a directed path decomposition. But generally we cannot expect the cuts that show up in perfect matching decompositions to be generalised tight cuts. Hence, to draw the connection between directed treewidth and perfect matching width we need a more general approach to translate cuts of bounded matching porosity into some kind of separation in an M-direction.

5.3.1. Edge Cuts and Separators

Inspecting the guard sets of a directed tree decomposition more closely reveals that guards are also not supposed to block all directed paths that go in one direction. Instead, the guards are meant to make sure that no strong component of D avoids the guards and contains vertices from below and above the guarded edge in the directed tree decomposition. So one could say that we only need to destroy the matching connectivity between the two shores of a bounded matching porosity cut in B by deleting a conformal set of small size.

So, given a cut $\partial_B(X)$ in a bipartite graph B with a perfect matching M we are interested in a set $F \subseteq M$ that meets all M-conformal cycles with vertices in X and $\overline{X}$, and whose size is bounded in some function of $\mathrm{mp}(\partial_B(X))$.

Definition 5.3.1 (Guarding Set)**.** Let B be a bipartite graph with a perfect matching M, and $X \subseteq V(B)$. An M-conformal cycle C is said to *cross* the cut $\partial_B(X)$ if $E(C) \cap \partial_B(X) \neq \emptyset$. A set $F \subseteq M$ is a *hitting set* for a family $\mathcal{C}$ of M-conformal cycles if $F \cap E(C) \neq \emptyset$ for all $C \in \mathcal{C}$. Moreover, F is called a *guard* for $\partial_B(X)$ if $\partial_B(X) \cap M \subseteq F$ and F is a hitting set for the family of all M-conformal cycles that cross $\partial_B(X)$.

This section is dedicated to the proof of the following theorem.

Theorem 5.3.2 (D*)**.** Let B be a bipartite graph with a perfect matching M, and $X \subseteq V(B)$ a set of vertices with $\mathrm{mp}(\partial_B(X)) = k$, then there exists a guard $F \subseteq M$ of $\partial_B(X)$ with $|F| \leq 2k + k^2$.

We do this by establishing a sequence of small lemmas regarding a construction we present and refine in what follows.

Lemma 5.3.3 (D*)**.** Let B be a bipartite graph with a perfect matching M, and $X \subseteq V(B)$. If $F \subseteq M$ is a guard for $\partial_B(X)$, then no elementary component of $B - V(F)$ can contain vertices of both $X \setminus V(F)$ and of $\overline{X} \setminus V(F)$.

Proof. Suppose there exists an elementary component K with vertices in both $X \setminus V(F)$ and $\overline{X} \setminus V(F)$. Then K must have at least four vertices since otherwise, K would be isomorphic to K_2 and its single edge would have to be an edge of $M \cap \partial_B(X) \subseteq F$. Now let $e \in E(K) \cap \partial_B(X)$ be an edge of K (and observe that, by definition, $e \notin M$). Also, let M' be a perfect matching of K containing e. Since $F \subseteq M$ and K is an elementary component of $B - V(F)$, $M_K := M \cap E(K)$ is a perfect matching of K. Moreover, $e \notin M_K$. If we consider the subgraph K' of K consisting solely of the edges of M' and M_K, every component either is an M-conformal cycle, or isomorphic to K_2. Since $e \notin M_K$, the endpoints of e are covered by distinct edges of M, and thus the component of K' containing e must be an M-conformal cycle that crosses $\partial_B(X)$ and avoids F. However, such a cycle cannot exist by definition. $\square$

An important observation one can make in the proof of the lemma above is that, if one were to delete the vertices of a set $F \subseteq M$ and there still is an elementary component with vertices on both sides of $\partial_B(X)$, then there exists a cycle C that is M-conformal in B, which still has edges in $\partial_B(X)$. That means, if $F = \partial_B(X)$, then there exists the perfect matching $M' \coloneqq M \Delta E(C)$ with $|\partial_B(X) \cap M'| \geq |\partial_B(X) \cap M| + 2$. We make this observation more exact in the following lemma.

Lemma 5.3.4 (D*)**.** Let B be a bipartite graph with a perfect matching M, $X \subseteq V(B)$ such that $\operatorname{mp}(\partial_B(X)) = k$, and $|M \cap \partial_B(X)| = k$. Then $M \cap \partial_B(X)$ is a guard for $\partial_B(X)$.

Proof. Suppose there is an M-conformal cycle C avoiding $F \coloneqq M \cap \partial_B(X)$ but crossing $\partial_B(X)$. Let $W \coloneqq E(C) \cap \partial_B(X)$, then clearly $W \cap M = \emptyset$. However, since C avoids F we can define a new perfect matching M' of B as $M' \coloneqq M \Delta E(C)$. Then $W \cup F \subseteq M'$ and thus $|M' \cap \partial_B(X)| > k = \operatorname{mp}(\partial_B(X))$ which is a contradiction. Hence no M-alternating cycle that crosses $\partial_B(X)$ can avoid F. $\square$

For the next part, we need some additional notation. Let B be a bipartite graph with a perfect matching M, $X \subseteq V(B)$ be an M-conformal set of vertices and $M' \neq M$ another perfect matching of B. Then let us denote for every $W \subseteq M'$ by $F_{M',M}(W)$ the set of edges of M that match the vertices in $V(W)$. Note that $|F_{M',M}(W)| \leq 2|W|$. Let H be an elementary component of $B - W$, the *M-box* of H is the set $\square_{H,M} \coloneqq V(H) \setminus V(F_{M',M}(W))$.

The following observation is an immediate consequence of the definition of $\leq_2$, Theorem 3.1.24, and $V(W) \subseteq V(F_{M',M}(W))$.

Observation 5.3.5 (D*)**.** Let B be a bipartite graph with perfect matchings M and M', let $W \subseteq M'$ and H_1 and H_2 be two distinct elementary components of $B - V(W)$. Then, if $H_1 \leq_2 H_2$, there is no internally M-conformal path P in $B - V(F_{M',M}(W))$ such that P starts in a vertex of $V_2 \cap \square_{H_1,M}$, ends in a vertex of $V_1 \cap \square_{H_2,M}$, and is otherwise disjoint from $\square_{H_1,M} \cup \square_{H_2,M}$.

We fix the following for the upcoming lemmata.

Let B be a bipartite graph with a perfect matching M. Let $X \subseteq V(B)$ be an M-conformal set and let M' be a perfect matching with $|\partial_B(X) \cap M'| = \operatorname{mp}(\partial_B(X)) = k$ as well as $W \coloneqq \partial_B(X) \cap M'$. Let λ be a linearisation of the partial order $\leq_2$ of elementary components of $B - V(W)$ and let us number the elementary components $H_1, \ldots, H_\ell$ of $B - V(W)$ such that $\lambda(H_i) = i$ for all $i \in [1, \ell]$.

A set $I \subseteq [1, \ell]$ is a *dangerous configuration* if there exists an M-conformal cycle C of $B - V(F_{M',M}(W))$ such that

i) $V(C) \subseteq \bigcup_{i \in I} \square_{H_i,M}$,
ii) $\square_{H_i,M} \cap V(C) \neq \emptyset$ for all $i \in I$, and
iii) there are $i, j \in I$ such that $\square_{H_i,M} \subseteq X$ and $\square_{H_j,M} \subseteq \overline{X}$.

If I is a dangerous configuration, we call $i_I := \max I$ the *endpoint* of I, the cycle C is a *base cycle* of I.

Lemma 5.3.6 (D*)**.** Let $B' := B - V(F_{M',M}(W))$. There exists an M-conformal cycle C in B' that crosses $\partial_{B'}(X \setminus V(F_{M',M}(W)))$ if and only if there exists a dangerous configuration I with C as a base cycle.

Proof. The reverse direction follows immediately from the definition of dangerous configurations. If I is dangerous with base cycle C, then C contains vertices of both X and $\overline{X}$ and thus crosses $\partial_{B'}(X \setminus V(F_{M',M}(W)))$. Hence it suffices to prove the forward direction. So let C be an M-conformal cycle in B' that crosses $\partial_{B'}(X \setminus V(F_{M',M}(W)))$. Now let $I := \{i \in [1, \ell] \mid \square_{H_i,M} \cap V(C) \neq \emptyset\}$. Then $V(C) \subseteq \bigcup_{i \in I} \square_{H_i,M}$ and clearly, the second requirement is met by the definition of I. At last we know that C crosses $\partial_{B'}(X \setminus V(F_{M',M}(W)))$. By lemmata 5.3.3 and 5.3.4 there cannot exist $j \in [1, \ell]$ such that $V(H_j) \cap X \neq \emptyset$ and $V(H_j) \cap \overline{X} \neq \emptyset$ at the same time. Hence there must be $i, j \in I$ such that $\square_{H_i,M} \subseteq V(H_i) \subseteq X$ and $\square_{H_j,M} \subseteq V(H_j) \subseteq \overline{X}$. $\square$

In the fixed setting we are working on, let $i \in [1, \ell - 1]$ be any number. We associate a specific edge cut in B with i and λ as follows:

$$\partial_\lambda(H_i) := \partial_B\Big(\bigcup_{j=1}^{i} V(H_j) \cup V_1(W)\Big).$$

Lemma 5.3.7 (D*)**.** For all $i \in [1, \ell - 1]$ and all perfect matchings M'' of B, we have $|\partial_\lambda(H_i) \cap M''| = k$.

Proof. Let $i \in [1, \ell - 1]$ be arbitrary. By definition of the partial order $\leq_2$ of the H_j no vertex $v \in \bigcup_{j=1}^{i} V_2(H_j)$ can have a neighbour in $\bigcup_{j=i+1}^{\ell} V(H_j)$.

So the only neighbours v can have outside of $\bigcup_{j=1}^{i} V(H_j)$ must be vertices of $V_1(W)$. Hence every perfect matching M'' of B must match v to a vertex within $\bigcup_{j=1}^{i} V(H_j) \cup V_1(W)$ and thus it must have exactly

$$|\bigcup_{j=1}^{i} V_1(H_j) \cup V_1(W)| - |\bigcup_{j=1}^{i} V_2(H_j)| = |V_1(W)| = k$$

many edges in $\partial_\lambda(H_i)$. □

Lemma 5.3.8 (D*)**.** Let $I \subseteq [1, \ell]$ be a dangerous configuration and let C be a base cycle of I. Moreover, let $i \coloneqq \min I$ and $i \leq j < i_I$, then $\partial_\lambda(H_j) \cap E(C) \cap M \neq \emptyset$.

Proof. Clearly, with C being a cycle, $|E(C) \cap \partial_\lambda(H_j)| \geq 2$. Now suppose $\partial_\lambda(H_j) \cap E(C) \cap M = \emptyset$ and let $e_1, e_2 \in E(C) \cap \partial_\lambda(H_j)$ be two distinct edges with $e_p = u_p v_p$ such that $v_p \in \bigcup_{j'=1}^{j} V(H_j') \cup V_1(W)$ for each $p \in [1, 2]$. Let us further choose e_1 and e_2 such that there is a subpath P of C from v_1 to v_2 that avoids u_1 and u_2 and does not contain an edge of $\partial_\lambda(H_j)$. To find P move along C starting in v_1 and away from u_1 until the first time we reach an endpoint of another edge in $\partial_\lambda(H_j)$, this will be v_2. By our assumption $\{e_1, e_2\} \cap M = \emptyset$ and thus, with C being M-conformal, $E(P) \cap M$ must be a perfect matching of P. Hence P must have an even number of vertices in particular and thus is of odd length. But by Observation 5.3.5 $v_1, v_2 \in V_1$ and thus P is an odd length path joining two vertices of V_1. With B being bipartite this is impossible and our claim follows. □

We are finally ready to prove the main result of this section.

Proof of Theorem 5.3.2. First let $F_{-1} \coloneqq M \cap \partial_B(X)$, $B_0 \coloneqq B - V(F_{-1})$, $X_0 \coloneqq X \setminus V(F_{-1})$, and $M_0 \coloneqq M \setminus F_{-1}$. Clearly, every M_0 conformal cycle in B_0 is also an M-conformal cycle in B that avoids F_{-1}. Each such cycle that crosses $\partial_{B_0}(X_0)$ also crosses $\partial_B(X)$, and every conformal set in B_0 is also conformal in B. Moreover, $\mathrm{mp}(\partial_{B_0}(X_0)) = k_0 \coloneqq k - |F_{-1}|$ and X_0 is M_0-conformal. Now let M' be a perfect matching of B_0 with $|M' \cap \partial_{B_0}(X_0)| = k_0$ and let $F_0 \coloneqq F_{M',M_0}(\partial_{B_0}(X_0) \cap M')$. Then $|F_0| \leq 2k_0$ since every edge of F_0 covers an endpoint of an edge in $\partial_{B_0}(X_0) \cap M'$ and there are $2k_0$ such endpoints.

Let λ be a linearisation of the partial order $\leq_2$ of the elementary components of $B_0 - V(\partial_{B_0}(X_0) \cap M')$. Let us number the elementary components of $B_0 - V(\partial_{B_0}(X_0) \cap M')$ $H_1, \dots, H_\ell$ such that $\lambda(H_i) = i$ for all elementary components. We can now choose $i_1 \in [1, \ell]$ to be the smallest number such that there is a dangerous configuration I_1 with $i_1 = i_{I_1}$. Then for every dangerous configuration I with the smallest element $i \leq i_1$ we must have $i_I \geq i_1$. Hence each base cycle of such a configuration must have an edge in $F_1 := (\partial_\lambda(H_{i_1 - 1}) \cap M_0) \cup (\partial_\lambda(H_{i_1}) \cap M_0)$ by Lemma 5.3.8. Indeed, every M_0-conformal cycle that crosses $\partial_{B_0}(X_0)$, avoids F_0, and has vertices in $\bigcup_{j=1}^{i_1} \square_{H_j, M_0}$ is met by F_1 by Lemma 5.3.6. Moreover, by Lemma 5.3.7 we have $|F_1| \leq 2k_0$.

Now suppose the sets $F_1, \dots, F_{p-1} \subseteq M_0$ with $|F_j| \leq 2k_0$ for all $j \in [1, p-1]$ have already been constructed together with pairwise disjoint dangerous configurations $I_1, \dots, I_{p-1}$. Moreover let us assume that $1 \leq j < j' \leq p-1$ implies $i_{I_j} < h$ where h is the smallest member of $I_{j'}$ and $\bigcup_{j=1}^{p-1} F_j$ meets all base cycles of dangerous configurations I for which some $i'' \in I$ exists with $i'' < i_{I_{p-1}}$. Let $i_p \in [i_{p-1} + 1, \ell]$ be the smallest number such that there is a dangerous configuration I_p with base cycle C that avoids $\bigcup_{j=1}^{p-1} F_j$. This means I_p must be disjoint from $\bigcup_{j=1}^{p-1} I_j$. Let $F_p := (\partial_\lambda(H_{i_p - 1}) \cap M_0) \cup (\partial_\lambda(H_{i_p}) \cap M_0)$. By lemmata 5.3.6 and 5.3.8, $\bigcup_{j=1}^{p} F_j$ meets all $\partial_{B_0}(X_0)$ crossing M_0-conformal cycles that avoid F_0 and have a vertex in $\bigcup_{j=1}^{i_p} \square_{H_j, M_0}$. With Lemma 5.3.7 we also have $|F_p| \leq 2k_0$.

With B being finite and thus ℓ being a natural number there must be some q such that we cannot find an i_{q+1} as above. Suppose $q > \frac{k_0}{2}$. Clearly every I_j, $j \in [1, q]$ has a base cycle C_j that is M_0 conformal and crosses $\partial_{B_0}(X_0)$. However, with X_0 being M_0-conformal, C_j must have at least two edges in $\partial_{B_0}(X_0)$ that do not belong to M_0. Since $I_1, \dots, I_q$ are pairwise disjoint, also the $C_1, \dots, C_q$ are also pairwise disjoint. So we construct the following perfect matching of B_0:

$$M'' := M_0 \Delta \bigcup_{j=1}^{q} E(C_j).$$

Then $|\partial_{B_0}(X_0) \cap M''| \geq \sum_{j=1}^{q} |\partial_{B_0}(X_0) \cap E(C_j) \setminus M_0| \geq 2q > 2\frac{k_0}{2} = \mathrm{mp}(\partial_{B_0}(X_0))$ which is impossible. Hence our process must stop after $q \leq \left\lfloor \frac{k_0}{2} \right\rfloor$ many steps.

In total we get a set $F \coloneqq F_{-1} \cup F_0 \cup \bigcup_{j=1}^{q} F_j$ that meets all M-conformal cycles crossing $\partial_B(X)$ and satisfying $\partial_B(X) \cap M \subseteq F$. So F is a guard of $\partial_B(X)$. Moreover, we have

$$\begin{aligned}
|F| &\leq |F_{-1}| + |F_0| + \sum_{j=1}^{q} |F_j| \\
&\leq k - k_0 + 2k_0 + \frac{k_0}{2} 2k_0 \\
&\leq k + k_0 + k_0^2 \\
&\leq 2k + k^2.
\end{aligned}$$

□

5.3.2. Cyclewidth and Directed Branchwidth

How can we translate results on matching porosity into the setting of digraphs using our notion of M-directions? To answer this question let us consider some graph G with a perfect matching M. Note that G is not necessarily bipartite as this general idea can be applied to any graph with a perfect matching. Now let $X \subseteq V(G)$ be an M-conformal set with $\mathrm{mp}(\partial_G(X)) = k$ for some $k \in \mathbb{N}$. Since X is M-conformal, $\partial_G(X) \cap M = \emptyset$. Let $M' \in \mathcal{M}(G)$ be a perfect matching with $|M' \cap \partial_G(X)| = k$ and consider the graph $G' \coloneqq G[M' \cup M]$ that only consists of edges from $M' \cup M$. By Observation 3.1.32 every component of G' either is an M-M'-conformal cycle or isomorphic to K_2. Moreover, no edge of $M' \cap \partial_G(X)$ can belong to a component isomorphic to K_2 in G' and thus each of these edges must be contained in an M-M'-conformal cycle. Let $\mathcal{C}$ be the collection of all components of G' that have an edge in $\partial_G(X)$. Then we have $\bigcup_{C \in \mathcal{C}} E(C) \cap \partial_G(X) \subseteq M'$ and in particular $|\bigcup_{C \in \mathcal{C}} E(C) \cap \partial_G(X)| = k$. Hence if the matching porosity of $\partial_G(X)$ is k we find a family of pairwise disjoint M-conformal cycles in G that share k edges with $\partial_G(X)$ in total. Now let $Y \subseteq V(G)$ be another M-conformal set, and let $\mathcal{C}$ be a family of pairwise disjoint M-conformal cycles in G. Suppose $|\bigcup_{C \in \mathcal{C}} E(C) \cap \partial_G(Y)| = k'$ for some $k' \in \mathbb{N}$. Let us denote by $E(\mathcal{C})$ the set $\bigcup_{C \in \mathcal{C}} E(C)$, and let $M'' \coloneqq M \Delta E(\mathcal{C})$. Then M'' is a perfect matching of G and $|\partial_G(Y) \cap M''| = k'$ which implies $\mathrm{mp}(\partial_G(Y)) \geq k'$. So if we can find a family of pairwise disjoint M-conformal cycles in G, then the number of edges this family has in our cut is a lower bound on its matching porosity. Let ζ be an M-signing of G, then there is a bijection between the M-conformal cycles in G and the directed cycles in $\mathcal{D}_{\pm}(G, M, \zeta)$. This leads us to the following definitions and observation.

Definition 5.3.9 (Cycle Porosity)**.** Let (G, σ) be a bidirected graph, and $X \subseteq V(G)$. The *cycle porosity* of the cut $\partial_G(X)$ is defined as

$$\operatorname{cp}(\partial_G(X)) \coloneqq \max_{\substack{\mathcal{C} \text{ family of} \\ \text{pairwise disjoint} \\ \text{directed cycles}}} |E(\mathcal{C}) \cap \partial_G(X)|.$$

In a slight abuse of notation, we use cycle porosity also for digraphs as those are a special case of bidirected graphs. Let G be a graph with a perfect matching M, and $X \subseteq V(G)$ be an M-conformal set. We denote by $M(X)$ the set of edges of M with both endpoints in X.

Observation 5.3.10 (X^*)**.** Let G be a graph with a perfect matching M, $X \subseteq V(G)$ be an M-conformal set, and ζ be an M-signing of G. Then $\operatorname{mp}(\partial_G(X)) = \operatorname{cp}(\partial_{\mathcal{D}_{\pm}(G,M,\zeta)}(M(X)))$.

Similar to our definition of perfect matching width we can now use cycle porosity to define a branchwidth parameter for bidirected graphs.

Definition 5.3.11 (Cycle Width)**.** Let (G, σ) be a bidirected graph. A *cycle decomposition* of (G, σ) is a cp-branch decomposition (T, δ) over $V(G)$ where T is a cubic tree and $\delta \colon \mathsf{L}(T) \to V(G)$ is a bijection.

The *width* of a cycle decomposition (T, δ) is defined as **half**[3] of its cp-width, and the *cycle width* of (G, σ), denoted by $\operatorname{cycw}(G, \sigma)$, is the minimum width over all cycle decompositions for (G, σ).

Please note that the cycle porosity of a cut is preserved under switchings. As before, in a slight abuse of notation, we may use cycle width directly on digraphs instead of digraphic bidirected graphs.

Our goal is to use the idea of cycle width to deduce the bipartite version of Conjecture 5.1.3 from the Directed Grid Theorem. In a first step we establish a relation between the cycle width of digraphs and directed treewidth.

Theorem 5.3.12 (D^*)**.** Let D be a digraph. Then $\operatorname{cycw}(D) - 1 \leq \operatorname{dtw}(D) \leq 18 \operatorname{cycw}(D)^2 + 36 \operatorname{cycw}(D) + 9$.

[3]Note that the cycle porosity of any cut is even, we add the factor $\frac{1}{2}$ to the definition to be more in line with other width parameters by allowing any value from $\mathbb{N}$.

Then we establish a connection between the perfect matching width of a graph G with a perfect matching M and its M-bidirections.

Theorem 5.3.13 (X*). Let G be a graph with a perfect matching M and an M-signing ζ. Then $\frac{1}{2}\operatorname{pmw}(G) \leq \operatorname{cycw}(\mathcal{D}_{\pm}(G, M, \zeta)) \leq \operatorname{pmw}(G)$.

The bipartite version of Conjecture 5.1.3 then follows from the Directed Grid Theorem with an application of Lemma 3.2.17.

Cycle Width and Directed Treewidth To establish Theorem 5.3.12 we need to show two directions. First we prove that cycle width is bounded from above by a function in the directed treewidth. For this, we construct a cycle decomposition from a directed tree decomposition in two steps. First, we push all vertices contained in bags of inner vertices of the arborescence into leaf bags. Second, we transform the result into a cubic tree.

Definition 5.3.14 (Leaf Directed Tree Decomposition). A relaxed directed tree decomposition (T, β, γ) of a digraph D is called a *leaf directed tree decomposition* if $\beta(t) = \emptyset$ for all $t \in V(T) \setminus \mathsf{L}(T)$.

So first, we show that a directed tree decomposition can be turned into a leaf decomposition without changing its width.

Lemma 5.3.15 (B*). Let (T, β, γ) be a directed tree decomposition of a digraph D. There is a linear time algorithm that computes a leaf directed tree decomposition of D of the same width.

Proof. For every inner vertex $t \in V(T)$ such that $\beta(t) \neq \emptyset$ we add a new leaf t' adjacent to t (and no other vertices of T) and thus obtain a new tree T'. The new bags are defined by $\beta' \coloneqq V(T') \to 2^{V(D)}$ with $\beta'(t) \coloneqq \beta(t)$ for $t \in \mathsf{L}(T)$, and $\beta'(t) \coloneqq \emptyset$ and $\beta'(t') \coloneqq \beta(t)$ for $t \in V(T) \setminus \mathsf{L}(T)$. The new guards are defined by

$$\gamma'(e) \coloneqq \begin{cases} \gamma(e), & \text{if } e \in E(T) \\ \beta(t'), & \text{if } e = (t, t') \text{ for some } t \in V(T) \setminus \mathsf{L}(T), \text{ and } t' \in \mathsf{L}(T) \end{cases}$$

We prove that (T', β', γ') is a relaxed directed tree decomposition. The bags given by β' still provide a near partition of $V(D)$. For all edges $e = (t, t') \in E(T)$ it is still the case that $\gamma(e)$ strongly guards $\beta(T'_{t'})$ and for the new edges this condition is obvious. Finally, if Γ' is defined for

(T', β', γ') as Γ for (T, β, γ), then $\Gamma'(t) = \Gamma(t)$ for all $t \in V(T)$ and for all t' we have $\Gamma'(t') \subseteq \Gamma(t)$. Hence, the width of the decomposition did not change. Note that the new decomposition can be computed in linear time. □

So whenever we are given a directed tree decomposition of a digraph D, we can manipulate it such that exactly the leaf-bags are non-empty. This is still not enough since a cycle decomposition requires every leaf to be mapped to exactly one vertex, meaning that every bag has to be of size at most one, and also the decomposition tree itself has to be cubic. The following lemma shows how a leaf directed tree decomposition can be further manipulated to meet the above requirements, again in polynomial time and without changing the width. We call a (relaxed) directed tree decomposition *cubic* (*subcubic*) if the underlying tree of its arborescence is cubic (subcubic).

Lemma 5.3.16 (B*)**.** Let D be a digraph and $k \in \mathbb{N}$. If the directed treewidth of is at most k, then there is a cubic leaf directed tree decomposition of width k for D.

Proof. Let D be a digraph and (T'', β'', γ'') a directed tree decomposition of width k. Then, due to Lemma 5.3.15, there exists a leaf directed tree decomposition (T, β, γ) of D of the same width. Algorithm 2 takes this as input and transforms it into a subcubic leaf directed tree decomposition (T', β', γ').

The resulting tree T' is subcubic and only the leaf bags contain vertices. Now we want to check whether the output of the algorithm again yields a proper directed tree decomposition of the desired width.

In the first part we split the bag of each leaf up into bags of single vertices which are added as new children. We show that such a split of a leaf ℓ does not destroy the properties of the directed tree decomposition. The new vertices ℓ_v obtain bags of size one. The new edge (ℓ, ℓ_v) obtains the guard $\beta(\ell) \cup \gamma(x, \ell_v)$.

Due to $|\beta(\ell) \cup \gamma(x, \ell_v) \cup \{v\}\,| = |\beta(\ell) \cup \gamma(x, \ell)| \leq k + 1$, the width of the new decomposition is still at most k.

Algorithm 2 cubify a leaf decomposition

procedure CUBIFY$((T, \beta, \gamma))$
    $T' \leftarrow T$, $\beta' \leftarrow \beta$, $\gamma' \leftarrow \gamma$
    for all $\ell \in \mathsf{L}(T)$ **do**
        $x \leftarrow$ parent of ℓ
        for all $v \in \beta(\ell)$ **do**
            introduce new vertex ℓ_v
            $T' \leftarrow T' + (\ell, \ell_v)$
            $\beta'(\ell_v) \leftarrow \{v\}$
            $\gamma'(\ell, \ell_v) \leftarrow \beta(\ell) \cup \gamma(x, \ell)$
        $\beta'(\ell) \leftarrow \emptyset$
    while T' not subcubic **do**
        let $t \in V(T)$ with $\deg_D(t) = d + 1 > 3$
        $x \leftarrow$ parent of t
        let $c_1, \dots, c_d$ be the children of t in topological order of their bags
        introduce new vertices $t_1, \dots, t_{d-1}$ with empty bags
        $T' \leftarrow T' - t + \{t_1, \dots, t_{d-1}\} + \{(t_i, c_i) \mid i \in [1, d-1]\} + \{(t_i, t_{i+1}) \mid i \in [1, d-2]\} + (t_{d-1}, c_d) + (x, t_1)$
        $\gamma'(t_i, c_i) \leftarrow \gamma(t, c_i)$ for all $i \in [1, d-1]$
        $\gamma'(t_{d-1}, c_d) \leftarrow \gamma(t, c_d)$
        $\gamma'(t_i, t_{i+1}), \gamma'(x, t_1) \leftarrow \gamma(x, t)$
    return (T', β', γ')

The guard of the edge going to ℓ_v contains v, therefore every walk in D starting and ending at v intersects the guard. So after the first part of the algorithm (T', β', γ') is still a proper directed tree decomposition.

In the second part we split high degree vertices into paths. For every vertex t of (total) degree $d + 1 > 3$ we introduce $d - 1$ new vertices $t_1, \dots, t_{d-1}$. Let $c_1, \dots, c_d$ be the children of t. We can assume without loss of generality that the children are ordered by the topological order of their bags. That is if $i < j$, then every path from $\beta(T_{c_j})$ to $\beta(T_{c_i})$ intersects $\Gamma(v)$. The subtrees rooted at the children stay intact and are attached differently to the subtree above $T - T_t$. To do this we first remove t from T obtaining subtrees T_r containing the root and the parent x of t as a leaf, and T_{c_i} for every child of t. We now add the new vertices as follows. The former parent x builds a path with the new vertices t_i in increasing order. Then every t_i is mapped to the corresponding c_i, leaving c_d which is also mapped to t_{d-1}, which only has two neighbours so far, since its the last on the path.

For all the subtrees that stay the same during the construction it is clear that no walk can leave them and come back without intersecting a guard. But we introduce new subtrees that contain several child-subtrees of t. Let T'_{t_i} be such a subtree. Assume there is a walk W in D starting and ending in $\beta(T_{t_i})$ containing a vertex from $\beta(T - T_{t_i})$ but no vertex of $\gamma(t)$, which is the guard for the edge towards t_i. There are two possibilities. Either W contains a vertex of $T' - T_{t_1} = T - T_t$, which directly yields a contradiction to (T, β, γ) being a proper directed tree decomposition. Or W contains a vertex from $\beta(T_{t_j})$ for some $j < i$. But this would imply that there is a path from $\beta(T_{c_i})$ to $\beta(T_{c_j})$, which contradicts the topological ordering.

Thus, the output of the algorithm is again a proper directed tree decomposition. $\square$

So given a directed tree decomposition of a digraph D, we can transform it into a subcubic leaf directed tree decomposition (T, β, γ) of D in linear time without changing its width. It remains to show that if we forget about the orientation of the edges of the arborescence T, (T, β) defines a cycle decomposition of bounded width.

Proposition 5.3.17 (B*)**.** Let D be a digraph. Then $\operatorname{cycw}(D) \leq \operatorname{dtw}(D) + 1$ and a cycle decomposition of D of width at most k can be computed from a directed tree decomposition of D of width k in polynomial time.

Proof. Let $k \coloneqq \operatorname{dtw}(D)$. Due to Lemma 5.3.16 there exists a subcubic leaf directed tree decomposition (T, β, γ) of D of width k such that every leaf bag contains at most one vertex. We want to show that (T, β) yields a cycle decomposition of width at most $k + 1$. The function β already provides a bijection between $\mathsf{L}(T)$ and $V(D)$. So next we show that every edge $e \in E(T)$ satisfies $\operatorname{cp}(\partial_D(e)) \leq 2|\gamma(e)|$. Afterwards we make the subcubic decomposition cubic.

Let $\mathcal{C}$ be a minimal family of pairwise disjoint directed cycles in D and let $e \in E(T)$. We show that $|\partial_D(e) \cap E(\mathcal{C})| \leq 2|\gamma(e)|$. Let $X_1, X_2 \subseteq V(D)$ be the two shores of the cut $\partial_D(e)$ such that $X_1 = \beta(T')$ where $T' \subseteq T$ is the subtree of T not containing the root. Furthermore, let $Y_1 \subseteq V(\mathcal{C}) \cap X_1$ be the vertices of the cycles in $\mathcal{C}$ incident with an edge of $\partial_D(e) \cap E(\mathcal{C})$.

Let $\mathcal{W}$ be the collection of directed walks $W_{v,w}$ from a vertex $v \in Y_1$ to some $w \in Y_1$ such that

i) $W_{v,w}$ is a subwalk of some cycle in $\mathcal{C}$, and
ii) $\emptyset \neq V(W_{v_1}) \setminus \{v_1, v_2\} \subseteq X_2$.

In other words, $\mathcal{W}$ is the set of walks (paths or cycles) starting in some $v \in Y_1$ and going along the cycle in $\mathcal{C}$ that contains v and ending in the first vertex in X_1 after leaving it from v_1. Clearly, the walks in $\mathcal{W}$ are not necessarily vertex disjoint as the paths may share common endpoints in Y_1.

Let $W_1, W_2 \in \mathcal{W}$ be two walks with $V(W_1) \cap V(W_2) \neq \emptyset$, then there is a cycle $C \in \mathcal{C}$ such that both W_1 and W_2 are subwalks of C. Hence $|V(W_1) \cap V(W_2)| \leq 2$. As $\gamma(e)$ is a guard in the relaxed directed tree decomposition, it must contain a vertex of every walk in $\mathcal{W}$. Every vertex can guard at most two paths, hence $\mathrm{cp}(\partial_D(e)) = |\mathcal{W}| \leq 2|\gamma(e)| \leq 2(k+1)$.

Now note that there still might be vertices of degree two in T. Since they are not leaves, β does not map them to any vertex of $V(G)$. Therefore, the two edges incident to a vertex of degree two induce the same cut and we can contract one of them to reduce the number of vertices of degree two. Let (T', β) be the decomposition obtained in this way, then (T', β) is cubic and all cuts induced by edges still have porosity at most $2k+2$. Thus, (T', β) is a cycle decomposition for D of width at most $k+1$. $\square$

An interesting property of cycle width, especially in comparison with directed treewidth, is that it is monotone under butterfly minors. This means in particular that one can expect to find nice characterisations for the class of digraphs of cycle width at most k at least for small values of k.

Theorem 5.3.18 (B*)**.** Let D be a digraph and D' a butterfly minor of D. Then $\mathrm{cycw}(D') \leq \mathrm{cycw}(D)$.

Proof. We first note that the cycle width is closed under taking subgraphs. To see this let $D' \subseteq D$ be a subgraph of D and (T, φ) a cycle decomposition of D. We delete every leaf that corresponds to a vertex in $V(D) \setminus$

$V(D')$ and eliminate vertices of degree two if needed as in the proof of Proposition 5.3.17 to obtain a new cycle decomposition (T', φ') of D' whose maximal cycle porosity is at most the maximal cycle porosity of T, φ. So $\mathrm{cycw}(D') \leq \mathrm{cycw}(D)$.

Next, we want to show that butterfly contracting an edge in D does not increase the cycle width. Let $D' \coloneqq D/e$ for some edge $e = (u, v) \in E(D)$. Since e is butterfly contractible it is the only outgoing edge from u or the only ingoing edge of v. We assume the former case first. Note that every cycle containing u also contains v.

We obtain (T', φ') from (T, φ) by deleting the leaf ℓ of T mapped to u and contracting one of the two edges in T incident with the unique neighbour of ℓ in T in order to obtain the cubic tree T'. Let $x_{u,v}$ be the contraction vertex, we set $\phi'(\phi^{-1}(v)) \coloneqq x_{u,v}$ while leaving the mapping of the other leaves intact.

All cuts in the decomposition for which u and v lie on the same shore do not change their porosity. So consider a cut $\partial_D(X)$, $v \in X$, induced by (T, φ) that separates u and v, then (T', φ') induces a cut $\partial_{D/e}(X')$ where $X' = (X \setminus \{v\}) \cup \{x_{u,v}\}$ and $V(D/e) \setminus X' = V(D) \setminus (X \cup \{u\})$.

Suppose there is a family of pairwise disjoint directed cycles $\mathcal{C}$ in D that contains a cycle C with $u \in V(C)$ and satisfies $|\partial_D(X) \cap E(\mathcal{C})| = \mathrm{cp}(\partial_D(X))$ as well as $\partial_D(X) \cap E(C) \neq \emptyset$. Let C' be the cycle in D/e obtained from C after the contraction of e.

Let $\mathcal{C}'$ be a family of directed cycles in D/e. At most one cycle C contains $x_{u,v}$. If C also exists in D, then $\mathcal{C}'$ is a family of cycles in D as well. Otherwise, as (u, v) is the only edge leaving u, the predecessor of $x_{u,c}$ on C has an edge to u in D. Thus we can construct a cycle C' in D from C by replacing the edge $(y, x_{u,v})$ by a path yuv. Then C' crosses $\partial_D(X)$ at least as often as C and the number of edges of $(\mathcal{C} \setminus (C)) \cup \{C'\}$ in the cut is at least as high as the number of edges of $\mathcal{C}$ in $\partial_D(X)$.

For handling the case that the edge e is the only ingoing edge of v, we claim that the cycle width does not change if we reverse all directions of the edges in the graph. Our claim holds, since we still get exactly the same cycles just with reversed direction and they still cross the same cuts. Therefore the decomposition stays exactly the same with the same porosities for all cuts. By these arguments $\mathrm{cycw}(D') \leq \mathrm{cycw}(D)$ holds for every butterfly minor D' of D. $\square$

With this we have established one of the two inequalities from Theorem 5.3.12. For the reverse we need a way to show that any digraph of bounded cycle width also has bounded directed treewidth. To achieve this goal we utilise Theorem 5.3.2 and the cops & robber game.

Lemma 5.3.19 (D*)**.** Let D be a digraph and $X \subseteq V(D)$. If $\mathrm{cp}(\partial_D(X)) = k$, then there is a hitting set of size at most $2k^2 + 4k$ for all directed cycles crossing $\partial_D(X)$.

Proof. With D being a digraph there exists a bipartite graph B and a perfect matching M such that $D = \mathcal{D}(B, M)$. For every $v \in V(D)$ let us identify the edge $e_v \in M$ that uniquely corresponds to v in B. Then X naturally corresponds to the set $Y \coloneqq V(\{e_v \mid v \in X\}) \subseteq V(B)$. By calling upon Theorem 5.3.2, it now suffices to show $\mathrm{mp}(\partial_B(Y)) \leq 2k$ in order to prove our claim. In fact $\mathrm{mp}(\partial_B(Y)) = 2k$ follows immediately from Observation 5.3.10 and thus we are done. $\square$

Please note that the proofs of Theorem 5.3.2 and Lemma 5.3.19 are constructive in the sense that the construction of the separators and the digraphs D_i can be done in polynomial time. This yields the following algorithmic result.

Corollary 5.3.20 (D*)**.** Let D be a digraph and $X \subseteq V(D)$. There exists a polynomial time algorithm that finds a hitting set S for all directed cycles that cross $\partial_D(X)$ of size at most $2\,\mathrm{cp}(\partial_D(X))^2 + 4\,\mathrm{cp}(\partial_D(X))$.

To show that there exists an upper bound of directed tree width in terms of cycle width as desired we introduce an intermediate width parameter which bridges the gap between directed tree width and cycle width.

Definition 5.3.21 (Thickness)**.** Let (G, σ) be a bidirected graph and $X \subseteq V(G)$. The *thickness* of X, denoted by $\mathrm{thick}(X)$, is the size of a minimum hitting set for all directed cycles on (G, σ) that cross $\partial_G(X)$.

Definition 5.3.22 (Bidirected branchwidth)**.** Let (G, σ) be a bidirected graph. A *bidirected branch decomposition* of (G, σ) is a thick-branch decomposition (T, δ) over $V(G)$ where T is a cubic tree and $\delta\colon \mathsf{L}(T) \to V(G)$ is a bijection.

The *width* of a bidirected branch decomposition (T, δ) is defined as its thick-width, and the *bidirected branchwidth* of (G, σ), denoted by $\mathrm{bbw}(G, \sigma)$, is the minimum width over all bidirected branch decompositions for (G, σ).

Let D be a digraph. A *directed branch decomposition* of D is a bidirected branch decomposition of D and the *directed branchwidth* of D, denoted by $\mathrm{dbw}(D)$, is defined as its bidirected branchwidth.

If k cops have a winning strategy in the strong variant of the directed cops and robber game, then Theorem 2.3.15 implies that the directed treewidth cannot be larger than $3k - 2$. We show that a directed branch decomposition of small width can be used to find a winning strategy for a bounded number of cops.

Lemma 5.3.23 (D^*)**.** If D is a digraph with a directed branch decomposition of width k, then $\mathrm{cops}_s(D) \leq 3k$.

Proof. We may assume D to be strongly connected. Let (T, δ) be a directed branch decomposition of width k for D and let $\ell \in V(T)$ be an arbitrary leaf with neighbour t_0 in T. Then $\mathrm{thick}(\ell t_0) = 1$ since $\delta(\ell)$ hits all directed cycles containing $\delta(\ell)$. For every edge $e \in E(T)$ let us denote by S_e a minimum hitting set for the directed cycles crossing e. Since $\mathrm{width}((T, \delta)) = k$ we know $|S_e| \leq k$ for all $e \in E(T)$.

Now let us place a cop on $\delta(\ell)$ as well as on every vertex in $\bigcup_{t \in \mathrm{N}_T(t_0) \setminus \{\ell\}} S_{t_0 t}$ and denote by X_0 the set of all vertices occupied by cops this way. In total, we now have placed at most $3k$ cops. For every $t \in \mathrm{N}_T(t_0) \setminus \{\ell\}$ let T_t be the subtree of $T - t_0 t$ containing t, then no strong component of $D - X_0$ can contain vertices from $\delta(T_t)$ and $\delta(T_{t'})$ simultaneously, if $t \neq t' \in \mathrm{N}_T(t_0) \setminus \{\ell\}$. Hence there must be a $t \in \mathrm{N}_T(t_0) \setminus \{\ell\}$ such that the robber has chosen a strong component of $D - X_0$ contained in $\delta(T_t)$. We now derive a new cop-position $X_1 \coloneqq \bigcup_{dt \in E(T)} S_{dt}$. Since $S_{t_0 t} \subseteq X_0 \cap X_1$, the robber cannot leave $\delta(T_t)$. We set $t_1 \coloneqq t$.

Now suppose we are in the following position: There is an edge $t_{i-1} t_i$, the current cop position $X_i = \bigcup_{t_i t' \in E(T)} S_{t_i t'}$, and the robber component R_i is contained in $\delta(T_{t_i})$. By definition of X_i there cannot be distinct $t, t' \in \mathrm{N}_T(t_i) \setminus \{t_{i-1}\}$ such that R_i has vertices of both $\delta(T_t)$ and $\delta(T_{t'})$, so we may assume $R_i \subseteq \delta(T_t)$. We set $t_{i+1} \coloneqq t$. If t_{i+1} is a leaf of T, we set $X_{i+1} \coloneqq S_{t_i t_{i+1}} \cup \{\delta(t_{i+1})\}$. Since $S_{t_i t_{i+1}} \subseteq X_i \cap X_{i+1}$ the robber cannot leave $\delta(t_{i+1})$ and thus we have captured her. Otherwise t_{i+1} is not a leaf and we set $X_{i+1} \coloneqq \bigcup_{t_{i+1}t' \in E(T)} S_{t_{i+1}t'}$. With X_{i+1} being the new cop position and $S_{t_i t_{i+1}} \subseteq X_i \cap X_{i+1}$ the new robber component R_{i+1} must be contained in $\delta(T_{t'})$ for some $t' \in \mathrm{N}_T(t_{i+1}) \setminus \{t_i\}$. Thus we can continue with the process. Since T is finite, we will eventually catch the robber, and by definition, we have $|X_i| \leq 3k$ for all i. $\square$

In light of Lemma 5.3.16 it is straight forward to see $\mathrm{dbw}(D) - 1 \leq \mathrm{dtw}(D)$. Thus by combining lemmata 5.3.16 and 5.3.23 and Theorem 2.3.15 one obtains the following relation between directed branchwidth and directed treewidth.

Proposition 5.3.24 (D*). Let D be a digraph. Then $\frac{1}{9}\,\mathrm{dtw}(D) - 1 \leq \mathrm{dbw}(D) \leq \mathrm{dtw}(D) + 1$.

With Lemma 5.3.19 at our disposal the following corollary is straight forward to prove.

Corollary 5.3.25 (D*). Let D be a digraph with $\mathrm{cycw}(D) = k$, then $\mathrm{dbw}(D) \leq 2k^2 + 4k$.

Combining propositions 5.3.17 and 5.3.24 and Corollary 5.3.25 now yields Theorem 5.3.12.

Cycle Width and Perfect Matching Width Towards a proof of Theorem 5.3.13 we aim to utilise Observation 5.3.10. Since the latter is a statement on M-conformal sets it makes sense to also bring back our findings on M-width, especially Theorem 5.1.13. The following lemma connects the M-width of G and the cycle width of any M-bidirection of G.

Lemma 5.3.26 (X*). Let G be a graph with a perfect matching M, and let ζ be an M-signing of G. Then $M\text{-}\mathrm{pmw}(G) = 2\,\mathrm{cycw}(\mathcal{D}_{\pm}(G, M, \zeta))$.

Proof. We first prove that $M\text{-}\mathrm{pmw}(G) \geq 2\,\mathrm{cycw}(\mathcal{D}_{\pm}(G, M, \zeta))$. Assume $M\text{-}\mathrm{pmw}(G) = k$ for some $k \in \mathbb{N}$. Then there is a perfect matching decomposition (T, δ) of width k such that all shores of the cuts induced by inner edges are M-conformal. We will construct a cycle decomposition (T', φ) of $(H, \sigma) \coloneqq \mathcal{D}_{\pm}(G, M, \zeta)$ from the perfect matching decomposition

(T, δ). In (T, δ) the leaves containing two vertices matched by M share a neighbour. We define $T' \coloneqq T - \mathsf{L}(T)$. Recall that matching edges become vertices in $\mathcal{D}_{\pm}(G, M, \zeta)$. For $xy \in M$ let t_{xy} be the common neighbour of $\delta^{-1}(x)$ and $\delta^{-1}(y)$. We define $\varphi(t_{xy}) \coloneqq xy$.

Now assume this decomposition has an edge $e \in T'$ that induces a cut $\partial_H(X)$ of cycle porosity more than $2k$. Then by Observation 5.3.10 we must have $\mathrm{mp}(\partial_G(e)) > 2k$ as well, contradicting our assumption. Therefore (T', φ) is a cycle decomposition of (H, σ) of width at most k.

Now we prove that $M\text{-}\,\mathrm{pmw}(G) \leq 2\,\mathrm{cycw}(H, \sigma)$. Let $\mathrm{cycw}(H, \sigma) = k$ for some $k \in \mathbb{N}$. Then there is a cycle decomposition (T, φ) of (H, σ) with width k. We will construct a perfect matching decomposition (T', δ) of G from the cycle decomposition. The construction mirrors the first part of the proof. For every leaf in T we introduce two new child vertices that are mapped to the two endpoints of the matching edge which is contracted into a vertex of (H, σ). Formally, $V(T') \coloneqq V(T) \cup \{t_i \mid t \in \mathsf{L}(T), i \in \{\ell, r\}\}$ and $E(T') \coloneqq E(T) \cup \{tt_i \mid t \in \mathsf{L}(T), i \in \{\ell, r\}\}$, where all t_r and t_ℓ are new vertices. Now for all $t \in \mathsf{L}(T)$, if $\varphi(t)$ is the vertex $xy \in M$ of H, then $\delta(t_\ell) \coloneqq x$ and $\delta(t_r) \coloneqq y$. Since now all pairs of vertices that are matched by M have a common parent vertex in T', the shores of the cuts induced by inner edges are M-conformal. Therefore the width of (T', δ) yields an upper bound on $M\text{-}\,\mathrm{pmw}(G)$ and by Observation 5.3.10 this upper bound is exactly $2k$ as desired. $\square$

Theorem 5.3.13 now follows by applying Theorem 5.1.13 to Lemma 5.3.26. Indeed, for bipartite graphs and digraphs we obtain the following relation between perfect matching width and directed tree width.

Theorem 5.3.27 (X*)**.** Let B be a bipartite graph with a perfect matching M. Then

$$\mathrm{pmw}(B) - 1 \leq \mathrm{dtw}(\mathcal{D}(B, M)) \leq 72\,\mathrm{pmw}(B)^2 + 144\,\mathrm{pmw}(B) + 9.$$

5.3.3. The Bipartite Matching Grid Theorem

With Theorem 5.3.27 finally all pieces are in place to prove the bipartite version of Conjecture 5.1.3. To do this let us first define a variant of the grid we are looking for in the conjecture.

Definition 5.3.28 (Cylindrical Matching Grid)**.** The *cylindrical matching grid* CG_k of order k is defined as follows. Let $C_1, \dots, C_k$ be k vertex disjoint cycles of length $4k$. For every $i \in [1,k]$ let $C_i = (v_1^i, v_2^i, \dots, v_{4k}^i)$, $V_1^i \coloneqq \left\{v_j^i \mid j \in \{1,3,5,\dots,4k-1\}\right\}$, $V_2^i \coloneqq V(C_i) \setminus V_1^i$, and $M_i \coloneqq \left\{v_j^i v_{j+1}^i \mid v_j^i \in V_1^i\right\}$. Then CG_k is the graph obtained from the union of the C_i by adding

$$\left\{v_j^i v_{j+1}^{i+1} \mid i \in [1,k-1] \text{ and } j \in \{1,5,9,\dots,4k-3\}\right\}, \text{ and}$$

$$\left\{v_j^i v_{j+1}^{i-1} \mid i \in [2,k] \text{ and } j \in \{3,7,11,\dots,4k-1\}\right\}$$

to the edge set. We call $M \coloneqq \bigcup_{i=1}^k M_i$ the *canonical matching* of CG_k. See Figure 5.6 for an illustration.

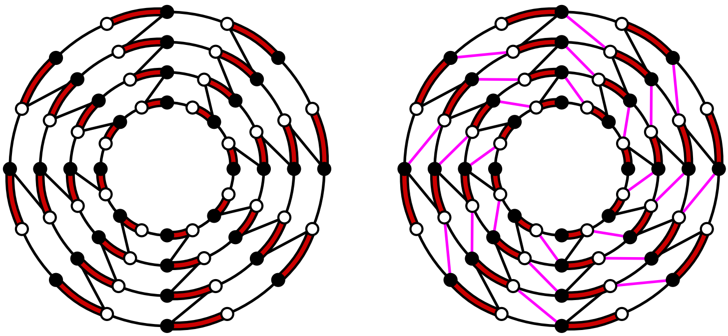


Figure 5.6.: The cylindrical matching grid of order 4 with the canonical matching on the left and an internal quadrangulation on the right.

Let $k \in \mathbb{N}$ be some positive integer and M be the canonical perfect matching of CG_k. The reason why we call M the canonical matching becomes apparent when considering the M-direction of CG_k. By choice of M, each of the k concentric cycles C_i, $i \in [1,k]$, becomes a directed cycle of length $2k$ in $\mathcal{D}(CG_k, M)$ and all of these cycles go in the same direction in a planar embedding of $\mathcal{D}(CG_k, M)$. Moreover, there exist $2k$ pairwise disjoint paths that alternate between going from C_1 to C_k and reverse, these become directed paths in $\mathcal{D}(CG_k, M)$. Overall this means $\mathcal{D}(CG_k, M)$ is exactly the cylindrical grid of order k from the Directed

Grid Theorem. Hence CG_k is exactly the split of the cylindrical grid of order k.

Theorem 5.3.29 (B*)**.** There exists a function $\mathsf{g}_{\mathrm{cyl}} \colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every bipartite graph B with a perfect matching M either $\mathrm{pmw}(B) \leq \mathsf{g}_{\mathrm{cyl}}(k)$ or B contains CG_k as an M-minor such that $M|_{CG_k}$ is the canonical matching of CG_k.

Proof. We set $\mathsf{g}_{\mathrm{cyl}}(k) \coloneqq 2\mathsf{g}_{\mathrm{dir}}(k) + 2$ and assume $\mathrm{pmw}(B) > \mathsf{g}_{\mathrm{cyl}}(k)$. By Theorem 5.3.27 this means

$$\mathrm{dtw}(\mathcal{D}(B, M)) > \frac{1}{2}\mathsf{g}_{\mathrm{cyl}}(k) - 1 = \mathsf{g}_{\mathrm{dir}}(k).$$

By Theorem 2.3.22, contains the cylindrical grid of order k as a butterfly minor. Hence by Lemma 3.2.17, B must contain CG_k as an M-minor and, moreover, $M|_{CG_k}$ must be the canonical matching of CG_k. $\square$

While Theorem 5.3.29 is already a powerful theorem, we are not quite there with our proof of the bipartite part of Conjecture 5.1.3. The next step towards the conjecture is to further refine our cylinder from above.

Let CG_k be the cylindrical matching grid of order k. The canonical *internal quadrangulation* $CG_k^{\square}$ of CG_k is defined as the graph obtained from the cylindrical grid by adding the following edges.

$$\left\{ v_j^i v_{j+1}^{i+1} \mid i \in [1, k-1] \text{ and } j \in \{2, 4, \ldots, 4k\} \right\}$$

See Figure 5.6 for an illustration.

Lemma 5.3.30 (D*)**.** Let $k \in \mathbb{N}$ be a positive integer. The cylindrical matching grid CG_{3k} contains $CG_k^{\square}$ as a matching minor.

Proof. We describe how to create a model $\mu \colon CG_k^{\square} \to CG_{3k}$.

First let $i \in \{3\ell - 1 \mid 1 \leq \ell \leq k\}$ and $j \in [1, k]$, we define models for four vertices $a^i_{j,\mathrm{down}}$, $b^i_{j,\mathrm{up}}$, $a^i_{j,\mathrm{up}}$, and $b^i_{j,\mathrm{down}}$. See Figure 5.7 for an illustration on how the models of these vertices will be arranged in CG_{3k}.

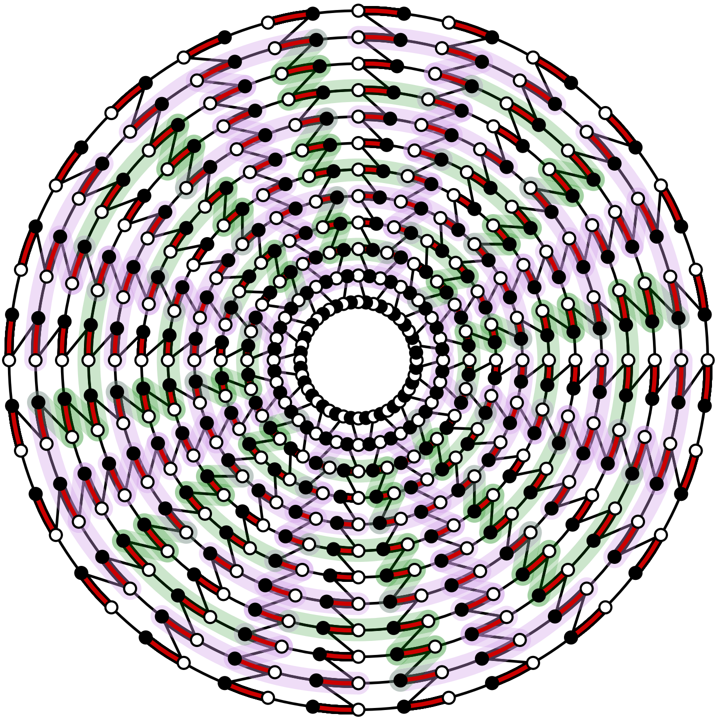


Figure 5.7.: The canonical internal quadrangulation of CG_4 as a matching minor of CG_{12}.

$$\begin{aligned}
\mu(a^i_{j,\mathrm{down}}) \coloneqq & (v^i_{1+12(j-1)}, v^i_{2+12(j-1)}, v^i_{3+12(j-1)}) \\
& \cup (v^{i-1}_{3+12(j-1)}, v^{i-1}_{4+12(j-1)}, v^i_{3+12(j-1)}) \\
\mu(b^i_{j,\mathrm{up}}) \coloneqq & (v^i_{4+12(j-1)}, v^i_{5+12(j-1)}, v^i_{6+12(j-1)}) \\
& \cup (v^i_{4+12(j-1)}, v^{i+1}_{3+12(j-1)}, v^{i+1}_{4+12(j-1)}) \\
\mu(a^i_{j,\mathrm{up}}) \coloneqq & (v^i_{7+12(j-1)}, v^i_{8+12(j-1)}, v^i_{9+12(j-1)}) \\
& \cup (v^i_{9+12(j-1)}, v^{i+1}_{10+12(j-1)}, v^{i+1}_{9+12(j-1)}) \\
\mu(b^i_{j,\mathrm{down}}) \coloneqq & (v^i_{10+12(j-1)}, v^i_{11+12(j-1)}, v^i_{12+12(j-1)}) \\
& \cup (v^{i-1}_{10+12(j-1)}, v^{i-1}_{9+12(j-1)}, v^{i-1}_{10+12(j-1)})
\end{aligned}$$

As a next step we add models for the edges of the concentric cycles of $CG^{\square}_k$, here we identify $b^i_{0,\mathrm{down}}$ with $b^i_{k,\mathrm{down}}$ and v^i_0 with v^i_{12k}.

$$\begin{aligned}
\mu(b^i_{j-1,\mathrm{down}} a^i_{j,\mathrm{down}}) \coloneqq & v^i_{12+12(j-2)} v^i_{1+12(j-1)} \\
\mu(a^i_{j,\mathrm{down}} b^i_{j,\mathrm{up}}) \coloneqq & v^i_{3+12(j-1)} v^i_{4+12(j-1)} \\
\mu(b^i_{j,\mathrm{up}} a^i_{j,\mathrm{up}}) \coloneqq & v^i_{6+12(j-1)} v^i_{7+12(j-1)} \\
\mu(a^i_{j,\mathrm{up}} b^i_{j,\mathrm{down}}) \coloneqq & v^i_{9+12(j-1)} v^i_{10+12(j-1)}
\end{aligned}$$

This in particular means that $C_i \subseteq \mu(CG^{\square}_k)$ for all $i \in \{3\ell - 1 \mid \ell \in [1,k]\}$. Next we connect the cycles, so let $i \in \{3\ell - 1 \mid \ell \in [1, k-1]\}$.

$$\begin{aligned}
\mu(b^i_{j,\mathrm{up}} a^{i+3}_{j,\mathrm{down}}) \coloneqq & v^{i+1}_{4+12(j-1)} v^{i+2}_{3+12(j-1)} \\
\mu(a^i_{j,\mathrm{up}} b^{i+3}_{j,\mathrm{down}}) \coloneqq & v^{i+1}_{9+12(j-1)} v^{i+2}_{10+12(j-1)}
\end{aligned}$$

Compare figures 5.7 and 5.8 to see how our model μ so far describes CG_k as a matching minor of CG_{3k}.

In the next step we describe how to build the models for the new edges that we need to add to our CG_k to form the canonical inner quadrangulation. We identify $b^i_{0,\mathrm{down}}$ and $b^i_{k,\mathrm{down}}$.

$$\begin{aligned}
& \mu(b^i_{j,\mathrm{up}} a^{i+3}_{j,\mathrm{up}}) \coloneqq \\
& \quad (v^{i+1}_{4+12(j-1)}, \dots, v^{i+1}_{8+12(j-1)}, v^{i+2}_{7+12(j-1)}, v^{i+2}_{8+12(j-1)}, v^{i+3}_{7+12(j-1)}) \\
& \mu(b^i_{j-1,\mathrm{down}} a^{i+3}_{j,\mathrm{down}}) \coloneqq \\
& \quad (v^i_{12+12(j-2)}, v^{i+1}_{11+12(j-2)}, v^{i+1}_{12+12(j-2)}, v^{i+2}_{11+12(j-2)}, \dots, v^{i+2}_{3+12(j-1)})
\end{aligned}$$

For an illustration compare Figure 5.9. Moreover, from the construction, it is clear that the model of every edge of $CG_k^{\square}$ is an internally M-conformal path, where M is the canonical matching of CG_{3k}. Also, the model of every vertex is a barycentric tree with exactly one exposed vertex. So in total μ is a matching minor model of $CG_k^{\square}$ in CG_{3k}. $\square$

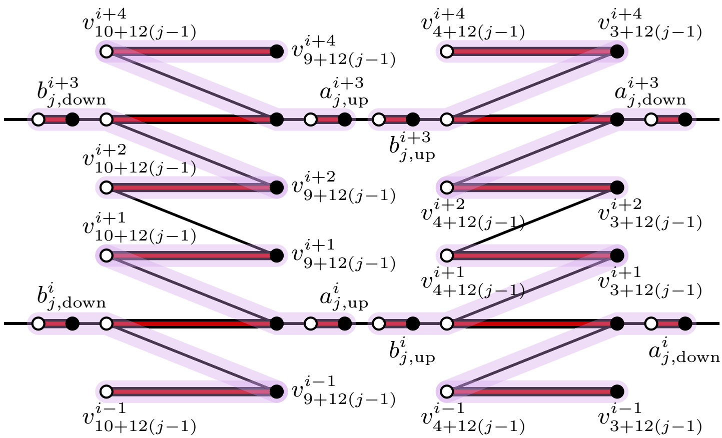


Figure 5.8.: The situation of the up- and down- vertices in the model of $CG_k^{\square}$ and the models of the edges of its CG_k-subgraph.

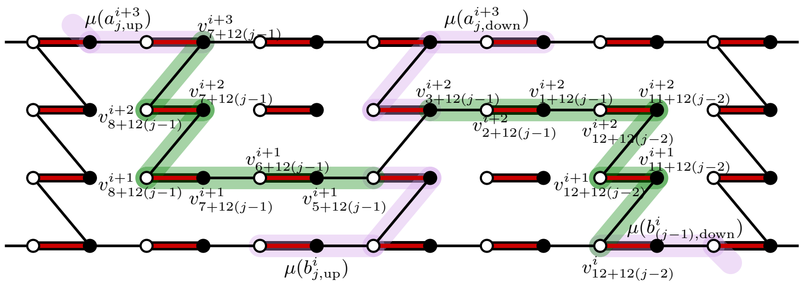


Figure 5.9.: The models for the new edges forming the inner quadrangulation of CG_k.

At last we find the $2k \times 2k$-grid as a matching minor in an inner quadrangulation of a cylindrical matching grid of appropriate size.

Lemma 5.3.31. If $k \in \mathbb{N}$ is even, $CG_k^{\square}$ contains the $k \times k$-grid as a matching minor.

Proof. Let M be the canonical perfect matching of $CG_k^{\square}$ and let V_1, V_2 be the two colour classes of $CK_k^{\square}$ such that $v_1^1 \in V_1$. We create a new perfect matching M' for $CG_k^{\square}$ by "switching" M along every second of the concentric cycles. Formally let

$$M' := (M \setminus \bigcup_{i=2,\text{ even}}^{k} E(C_i)) \cup \bigcup_{i=2,\text{ even}}^{k} (E(C_i) \setminus M).$$

In the following, we describe how to construct a matching minor model of the $k \times k$-grid iteratively from a C_4 in $CG_k^{\square}$ by extending the model by small building blocks. A *piece* is one of the following three configurations:

- A *base piece* $B_{i,j}$ starting on the vertex $v_j^i \in V_2$. It consists of the two paths
$$(v_j^i, v_{j+1}^i, v_{j+2}^i, v_{j+3}^i, v_{j+4}^i) \text{ and}$$
$$(v_{j+1}^{i+1}, v_{j+2}^{i+1}, v_{j+3}^{i+1}, v_{j+4}^{i+1}, v_{j+5}^{i+1})$$
together with the edges $v_j^i v_{j+1}^{i+1}$, $v_{j+3}^i v_{j+4}^{i+1}$, and $v_{j+4}^i v_{j+5}^{i+1}$.
- A *width piece* $W_{i,j}$ starting on the vertex $v_j^i \in V_1$. It consists of the three paths
$$(v_j^i, v_{j+1}^i, v_{j+2}^i, v_{j+3}^i, v_{j+4}^i),$$
$$(v_{j+1}^{i+1}, v_{j+2}^{i+1}, v_{j+3}^{i+1}, v_{j+4}^{i+1}, v_{j+5}^{i+1}), \text{ and}$$
$$(v_{j+2}^{i+2}, v_{j+3}^{i+2}, v_{j+4}^{i+2}, v_{j+5}^{i+2}, v_{j+6}^{i+2})$$
together with the edges $v_j^i v_{j+1}^{i+1}$, $v_{j+1}^i v_{j+2}^{i+1}$, $v_{j+4}^i v_{j+5}^{i+1}$, $v_{j+1}^{i+1} v_{j+2}^{i+2}$, $v_{j+4}^{i+1} v_{j+5}^{i+2}$, and $v_{j+5}^{i+1} v_{j+6}^{i+2}$.
- And a *height piece* $H_{i,j}$ starting on the vertex $v_j^i \in V_2$. It consists of the three paths
$$(v_j^i, v_{j+1}^i, v_{j+2}^i, v_{j+3}^i, v_{j+4}^i, v_{j+5}^i, v_{j+6}^i, v_{j+7}^i),$$
$$(v_{j+1}^{i+1}, v_{j+2}^{i+1}, v_{j+3}^{i+1}, v_{j+4}^{i+1}, v_{j+5}^{i+1}, v_{j+6}^{i+1}, v_{j+7}^{i+1}, v_{j+8}^{i+1}), \text{ and}$$
$$(v_{j+4}^{i+2}, v_{j+5}^{i+2}, v_{j+6}^{i+2}, v_{j+7}^{i+2}, v_{j+8}^{i+2}, v_{j+9}^{i+2})$$
together with the edges $v_j^i v_{j+1}^{i+1}$, $v_{j+3}^i v_{j+4}^{i+1}$, $v_{j+4}^i v_{j+5}^{i+1}$, $v_{j+7}^i v_{j+8}^{i+1}$, $v_{j+3}^{i+1} v_{j+4}^{i+2}$, $v_{j+4}^{i+1} v_{j+5}^{i+2}$, $v_{j+7}^{i+1} v_{j+8}^{i+2}$, and $v_{j+8}^{i+1} v_{j+9}^{i+2}$.

As a first step, we show how to create a model of the 4×4-grid from a specific C_4 in $CG_4^\square$, see Figure 5.10 for an illustration.

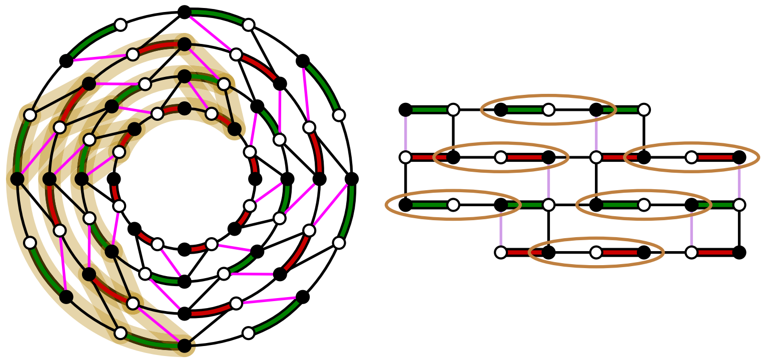


Figure 5.10.: A model of the 4×4-grid in $CG_4^\square$ (on the left) and a close-up of the model (on the right), in which the white vertices of degree two that should be bicontracted are encircled.

Let us choose as the C_4 the one induced by $\{v_1^1, v_2^1, v_2^2, v_3^2\}$. Then take the height piece $H_{2,2}$ and the base piece $B_{1,2}$ and let G_4 be the graph obtained by the union of $H_{2,2}$, $B_{1,2}$, and the C_4 chosen above. The vertex v_j^i with the largest j in G_4 is v_{11}^4 and thus, since each C_i has 16 vertices, none of the horizontal paths in G_4 closes a cycle. More over, if we now bicontract the vertices v_4^1, v_4^2, v_4^3, v_8^2, v_8^3, and v_8^4, we obtain exactly the 4×4-grid. Note that, by construction, G_4 is in fact M'-conformal in $CG_4^\square$ and thus we have found our desired matching minor.

So now assume that for some even k we have already constructed an M'-conformal graph G_k in $CG_{k+2}^\square$ by using our pieces and starting with the C_4 on the vertices $\{v_1^1, v_2^1, v_2^2, v_3^2\}$. In the last step of this proof, we show how to extend G_k to G_{k+2}, a bisubdivision of the $(k+2) \times (k+2)$-grid. Let $k = 2z$, we add the following pieces:

- the base piece $B_{1,4z-6}$ and the height piece $H_{2(z-1),4z-6}$,
- for every $i \in [1, z-2]$ the width piece $W_{2i,4(z+i)-7}$, and
- for every $j \in [1, z-2]$ the width piece $W_{2z-2,4(z+j)-3}$.

Since G_k is M'-conformal, the graph G_{k+2} obtained by adding the above pieces still is M'-conformal by construction. Consider the set

$$S_{k+2} \coloneqq \left\{v_i^1 \mid j \in [3, k+2],\ i \text{ odd}\right\} \cup \left\{v_{12}^j \mid j \in [2, k+2],\ j \text{ even}\right\},$$

Every vertex in S_{k+2} has degree two in G_{k+2} and thus is bicontractible. Bicontracting all vertices in S_{k+2} yields the desired $(k+2)$-$(k+2)$-grid. $\square$

We are finally ready to prove our grid theorem for bipartite graphs with perfect matchings.

Theorem 5.3.32 (D*). There exists a function $\mathsf{g}_{\mathrm{m}} \colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every bipartite graph B with a perfect matching either $\mathrm{pmw}(B) \leq \mathsf{g}_{\mathrm{cyl}}(k)$ or B contains the $2k \times 2k$-grid as a matching minor.

Proof. We set $\mathsf{g}_{\mathrm{m}}(k) \coloneqq \mathsf{g}_{\mathrm{cyl}}(6k)$ for every $k \in \mathbb{N}$. Suppose $\mathrm{pmw}(B) > \mathsf{g}_{\mathrm{m}}(k)$. Then by Theorem 5.3.29, B contains CG_{6k} as a matching minor. Using Lemma 5.3.30 yields that B contains $CG_{2k}^{\square}$ as a matching minor, and finally Lemma 5.3.31 lets us find the $2k \times 2k$-grid as a matching minor within $CG_{2k}^{\square}$. This completes the proof. $\square$

5.4. The Disjoint Paths Problem on Bipartite Graphs of Bounded Width

The most prominent algorithmic application of directed treewidth is an XP-algorithm for the directed k-disjoint paths problem. One way of generalising this algorithm to perfect matching width and bipartite graphs with perfect matchings would be to test for every $M \in \mathcal{M}(B)$ whether the desired paths exist. By Theorem 5.3.27 each of these instances would have bounded directed treewidth and thus the above algorithm would be applicable. The problem with this idea is that there might be an exponential number of perfect matchings in B, so this approach is not sufficient to solve the bipartite k-DAPP. To cope with this, we present an approach in this section that incorporates the issue of considering all perfect matchings of B into a dynamic programming algorithm inspired by the one from [JRST01] .

Theorem 5.4.1 (D*). Let B be a bipartite graph with a perfect matching, $k \in \mathbb{N}$ a positive integer and $\mathcal{I}$ a family of k terminal pairs. There exists an algorithm that decides in time $|V(B)|^{\mathcal{O}(k+\mathrm{pmw}(B)^2)}$ the k-DAPP with input $\mathcal{I}$ on B.

By utilising Theorem 5.4.1, we are able to give at least a partial solution to the open problem of matching minor containment in the form of a parametrised algorithm for bipartite graphs of bounded perfect matching width.

Theorem 5.4.2 (D*)**.** Let H be a fixed bipartite matching covered graph and B a bipartite graph with a perfect matching. There exists an algorithm with running time $|V(B)|^{\mathcal{O}(|V(H)|^2+\mathrm{pmw}(B)^2)}$ that decides whether B contains H as a matching minor.

Let B be a bipartite graph with a perfect matching and $\mathcal{I} = \{(s_1,t_1),\dots,(s_k,t_k)\}$ a family of terminal pairs. Let M be a perfect matching of B and $\mathcal{P} = \{P_1,\dots,P_k\}$ a family of internally disjoint and internally M-conformal paths in B such that P_i has endpoints s_i and t_i for every $i \in [1,k]$. We call $(M,\mathcal{P})$ a *solution* for $\mathcal{I}$. Let $W \subseteq E(B)$ be a matching. A solution $(M,\mathcal{P})$ for $\mathcal{I}$ in B *extends* W if $W \subseteq M$ and every terminal is matched by some edge in W.

A problem that needs to be addressed before we go any further is that our terminals are not necessarily distinct. In some cases this might lead to problems for the way our algorithm works. Before we continue, let us discuss how we get around this issue. Notice the following: Let $x \in V(B)$, be a vertex that occurs in at least one pair of $\mathcal{I}$ and let us denote the total number of occurrences of x as a terminal by $\mathrm{multi}(a)$. Then in every solution $(M,\mathcal{P})$, every path $P \in \mathcal{P}$ that connects x to some other terminal must end in an edge that is not contained in M and connects x to some neighbour x' of x. Moreover, the edge of M covering x cannot be contained in any $P \in \mathcal{P}$. Hence we may pick a collection of $\mathrm{multi}(x)$ many neighbours of x, select an extendible matching W' that covers the selected vertices, but not x, and now for each of these edges pick the endpoint not adjacent to x. Each of these picked vertices belong to the same colour class as x. For our graph B let $V_1' \subseteq V_1 \setminus \{s_1,\dots,s_k\}$ and $V_2' \subseteq V_2 \setminus \{t_1,\dots,t_k\}$ be selections of such vertices together with the extendible set of all matching edges W' covering these new vertices. Note that W' must be chosen such that $W \cup W'$ is extendible. For every $(s_i,t_i) \in \mathcal{I}$ now select a vertex $s_i' \in V_1'$ and $t_i' \in V_2'$ that is a neighbour of

s_i, t_i respectively. Then we have formed a distinct family $\mathcal{I}'$ of k terminal pairs and therefore we may now consider an instance of the bipartite k-matching linkage problem instead.

Let us now formalise the above discussion. Given a family of terminal pairs $\mathcal{I} = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ and an extendible set W such that all terminals are matched by W and every edge of W matches a terminal, we call a pair $(\mathcal{I}', W')$ a $(\mathcal{I}, W)$-*proxy* if

i) $\mathcal{I}' = \{(s'_1, t'_1), \ldots, (s'_k, t'_k)\}$ is a family of k terminal pairs where $s_i \neq s_j$ and $t_i \neq t_j$ for every choice of distinct values for $i, j \in [1, k]$ (we call such a family *distinct*),
ii) $W' \cup W$ is extendible, every terminal of $\mathcal{I}'$ is matched by some edge of W', every edge of W' matches a terminal of $\mathcal{I}'$, and $W \cap W' = \emptyset$, and
iii) for every $i \in [1, k]$, if $s'_i v \in W'$, then v is a neighbour of s_i and if $vt'_i \in W'$, then v is a neighbour of t_i.

It might happen, that s_i and t_i of the original instance are already adjacent, in such cases, we might have to consider additional cases of smaller instances, where the edge $s_i t_i$ is already one of the paths in a possible solution. Indeed, without loss of generality, we may always assume $s_i t_i$ to be part of our solution and thus the terminal pair (s_i, t_i) does not need to be considered. Hence we may assume all terminal pairs to be non-adjacent.

A perfect matching decomposition (T, δ) is *safe* for W and $\mathcal{I}$ if every W-extending solution $\mathcal{P}$ for $\mathcal{I}$ satisfies the following inequality for every $e \in E(T)$:

$$|\partial_B(e) \cap \bigcup_{P \in \mathcal{P}} E(P)| \leq 2\,\mathrm{width}(T, \delta).$$

The high level strategy of our algorithm is as follows:

- We choose an extendible matching $W \subseteq E(B)$ of size at most $2k$ such that all terminals of $\mathcal{I}$ are covered.
- Next choose a $(\mathcal{I}, W)$-proxy $(\mathcal{I}', W')$.
- Then we compute a perfect matching decomposition (T, δ) for $B - V(W)$ that is safe for $(\mathcal{I}', W')$ and its width is bounded in a function of $\mathrm{pmw}(B)$ and k.
- We apply dynamic programming on (T, δ) in order to either find a solution that extends W' or refute the existence of such a solution.

- Finally, if for some W and some $(\mathcal{I}, W)$-proxy we find a solution we extend it to a solution for $\mathcal{I}$ and W and return "Yes", otherwise we return "No".

Let $|V(B)| = n$. If $f_1(\mathrm{pmw}(B), k, n)$ describes the time needed to compute (T, δ) and $f_2(\mathrm{pmw}(B), k, n)$ describes the time necessary for the dynamic programming on (T, δ), the overall running time of our algorithm can then be expressed by $\mathcal{O}(n^{2k} \cdot n^{4k} \cdot f_1(\mathrm{pmw}(B), k, n) \cdot f_2(\mathrm{pmw}(B), k, n))$ where the constants only depend on k and $\mathrm{pmw}(B)$.

Once the functions f_1 and f_2 are established, Theorem 5.4.1 follows as an immediate consequence of the high-level approach described above. Indeed, please note that, by slightly modifying the proofs below, one can obtain the following more general result:

Corollary 5.4.3 (D*)**.** Let B be a bipartite graph with a perfect matching, $k \in \mathbb{N}$ an integer, $\mathcal{I}$ a family of k terminal pairs, and $F \subseteq E(B)$ an extendible set. There exists an algorithm that decides in time $|V(B)|^{\mathcal{O}(k+\mathrm{pmw}(B))^2}$ whether there exists an F-extending solution for $\mathcal{I}$ or not.

As an immediate consequence, if $D = \mathcal{D}(B, M)$ is some digraph, by choosing $F = M$ Corollary 5.4.3 together with Theorem 5.3.27 implies the original result on the directed disjoint path problem for digraphs of bounded directed treewidth in [JRST01].

5.4.1. Computing a Perfect Matching Decomposition

Some preliminary results are needed. For one, we need to be able to check for given $W \subseteq E(B)$ whether there exists a perfect matching extending W. This boils down to checking if $B - V(W)$ has a perfect matching. And second, we must be able to compute a perfect matching decomposition of bounded width.

The first problem can be solved in polynomial time by Edmonds' famous Blossom Algorithm [Edm65], or, since we work on bipartite graphs, by the Hungarian Method [Kuh55], so this part will not be much of a concern to us.

For the second part, we make use of Theorem 2.3.18.

Let B be a bipartite graph with a perfect matching M and $D \coloneqq \mathcal{D}(B, M)$. In light of Theorem 2.3.18, it would be enough to compute a perfect matching decomposition of bounded width for B from a directed tree decomposition of bounded width for D which we already know how to do in polynomial time by Proposition 5.3.17 and the proof of Lemma 5.3.26. We would like to maintain a bit of this niceness in the perfect matching decomposition we produce.

Let B be a bipartite graph with a perfect matching. A perfect matching decomposition (T, δ) of width w is *nice* if T is rooted at some vertex $r \in V(T)$ and

i) $V(T - r)$ can be partitioned into four sets of vertices:
   - The leaves, $\mathsf{L}(T)$, which are the vertices of degree one.
   - The *basic* vertices, $\mathrm{basic}(T)$, which are those vertices $t \in V(T)$ whose successors are leaves of T.
   - The *joins*, $\mathrm{join}(T)$, which are the vertices $t \in V(T)$ with two distinct successors t_1 and t_2 such that there is no edge from $V_2 \cap \delta(T_{t_1})$ to $V_1 \cap \delta(T_{t_2})$, and $B[\delta(T_{t_1})]$ is elementary.
   - The *guards*, $\mathrm{guard}(T)$, which are the vertices $t \in V(T)$ satisfying one of the following properties:
     - $|\delta(T_t)| \leq 2k$ and $\delta(T_t)$ is conformal (Type 1), or
     - t has two distinct successors t_1 and t_2 such that t_1 is a guard of Type 1 and t_2 either is a join, or $B[\delta(T_{t_2})]$ is conformal and elementary. (Type 2)

ii) if r is not a leaf of T for every successor t of r one of the following holds:
   - t is a guard of Type 1, or
   - t either is a join, or $B[\delta(T_t)]$ is conformal and elementary, and

   the successors of r of this type can be sorted as $t_1, \ldots, t_h$, $h \leq 3$ such that if $1 \leq i < j \leq h$, then there is no edge from $V_1 \cap \delta(T_{t_j})$ to $V_2 \cap \delta(T_{t_i})$.

Given a distinct set $\mathcal{I}$ of terminal pairs for the k-DAPP on B and an extendible set $W \subseteq E(B)$ matching all terminals such that if $e \in W$, then an endpoint of e is a terminal, we call a perfect matching decomposition (T, δ) a *$(\mathcal{I}, W)$-decomposition* for B, if it is nice and safe for $\mathcal{I}$ and W.

In the following we describe how to obtain a $(\mathcal{I}, W)$-decomposition for a bipartite graph B with a perfect matching. As a base of our algorithm, we are going to use Theorem 2.3.18 and then manipulate the obtained decomposition in order to create a nice perfect matching decomposition. A tuple (T, β, γ) is called a *proto-directed tree decomposition* for the digraph D if it satisfies all the conditions of a directed tree decomposition except that we allow empty bags and still every vertex of D must be contained in exactly one bag of (T, β, γ).

A proto-directed tree decomposition (T, β, γ) of width w for a digraph D is *prepared* if

i) T is subcubic,
ii) if $t \in V(T)$ has a unique successor t', then $\beta(T_{t'})$ induces a strong component of $D - \gamma(t, t')$ or contains at most $w + 1$ vertices, and
iii) if $t \in V(T)$ has two distinct successors t_1 and t_2, then
   - $\beta(T_{t_1})$ either contains at most $w + 1$ vertices and $\beta(T_{t_2})$ also either has at most $w+1$ vertices or induces a strongly connected subgraph of $D - \gamma(t, t_2)$, or
   - $\beta(T_{t_1})$ induces a strongly connected subgraph of $D - \gamma(t, t_1)$ and there is no directed edge with tail in $\beta(T_{t_2})$ and head in $\beta(T_{t_1})$ in D.

Lemma 5.4.4 (D*)**.** Let D be a digraph and $\mathrm{dtw}(D) \leq w$. There exists an algorithm with running time $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$ that computes a prepared proto-directed tree-decomposition of width at most $3w - 2$ for D.

Proof. Let (T_0, γ_0, β_0) be the nice directed tree-decomposition obtained via the algorithm in Theorem 2.3.18. Let us call a proto-directed tree-decomposition where every vertex of degree at most three satisfies the axioms of a prepared proto-directed tree decomposition and every other vertex satisfies the axioms of a nice directed tree decomposition *almost prepared*. Clearly (T_0, γ_0, β_0) is almost prepared. Now let (T_j, γ_j, β_j) be an almost prepared proto-directed tree-decomposition.

Pick any vertex $t \in V(T)$ of degree more than three and let $t_1, \ldots, t_\ell$ be its successors. Then $\beta_j(T_{j,t_i})$ induces a strong component of $D - \gamma_j(t, t_i)$ for all $i \in [1, \ell]$. Indeed, $\beta_j(T_{j,t_i})$ induces a strong component of $D - \Gamma_j(t)$ for all $i \in [1, \ell]$, without loss of generality let us assume that the t_i are numbered in such a way that for all $1 \leq i < k \leq \ell$ there is no directed edge with tail in $\beta_j(T_{j,t_k})$ and head in $\beta_j(T_{j,t_i})$. We define a proto-directed tree decomposition $(T_{j+1}, \beta_{j+1}, \gamma_{j+1})$ as follows. Let T_{j+1} be the arborescence obtained from T_j by introducing a new vertex t', the edge (t, t') and replacing (t, t_i) by (t', t_i) for all $i \in [2, \ell]$. Then $\beta_{j+1}(t'') := \beta_j(t'')$ for all $t'' \in V(T_j)$ and $\beta_{j+1}(t') := \emptyset$. Moreover, let $\gamma_{j+1}(e) := \gamma_j(e)$ for all $e \in E(T_j) \setminus \{(t, t_2), \ldots, (t, t_\ell)\}$, $\gamma_{j+1}(t, t') := \gamma_j(d, t) \cup \beta_j(t)$, where (d, t) is the unique ingoing edge at t in T_j, and $\gamma_{j+1}(t', t_i) := \gamma_j(t, t_i)$ for all $i \in [2, \ell]$. Clearly $\mathrm{width}(T_{j+1}, \beta_{j+1}, \gamma_{j+1}) \leq \mathrm{width}(T_j, \beta_j, \gamma_j)$, so we just need to show that $(T_{j+1}, \gamma_{j+1}, \beta_{j+1})$ is indeed a proto-directed tree decomposition. To be more precise, we only need to show that $\gamma_{j+1}(t, t')$ is a valid guard for $\beta_{j+1}(T_{j+1,t'})$. Let P be any directed walk starting and ending on a vertex of $\beta_{j+1}(T_{j+1,t'})$ while containing a vertex of $D - \beta_{j+1}(T_{j+1,t'})$. If P lies in $\beta_{j+1}(T_{j+1,t})$, then P must contain a vertex of $\beta_{j+1}(t)$ since there is no edge from $\beta_{j+1}(T_{j+1,t'})$ to $\beta_{j+1}(T_{j+1,t_1})$ by construction. So if P avoids $\beta_{j+1}(t)$, then P must contain a vertex of $D - \beta_{j+1}(T_{j+1,t})$ and thus, it must contain a vertex of $\gamma_{j+1}(d, t)$.

Then $(T_{i+1}, \gamma_{i+1}, \beta_{i+1})$ is almost prepared and has less vertices of degree at least four that (T_i, γ_i, β_i). In fact, after at most $|V(T_0)|$ steps we have obtained a prepared proto-directed tree decomposition. $\square$

Given a bipartite graph B with a perfect matching, $\mathcal{I} = \{(s_1, t_1), \ldots, (s_k, s_k)\}$ a distinct family of terminal pairs, and an extendible $W \subseteq E(B)$ matching all terminals, we call a set $F \subseteq \binom{V(B)}{2}$ a *W-completion*, if for every W-extending solution $(M, \mathcal{P})$, the graph induced by the edge set

$$(F \cup W \cup \bigcup_{P \in \mathcal{P}} E(P)) \setminus \{xy \in W \mid x = s_i \text{ and } y = t_i \text{ for some } i \in [1, k]\}$$

consists exclusively of M-alternating cycles. Please note that, by definition, $|F| \leq k$ for all W-completing F.

Lemma 5.4.5 (D*)**.** Let B be a bipartite graph with a perfect matching, $\mathcal{I} = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ a family of distinct terminal pairs, and $W \subseteq E(B)$ an extendible set covering all terminals such that if $e \in W$, then an endpoint of e is a terminal. Then a W-completion F can be found in linear time.

Proof. We obtain F as follows: Initialise F and U with $\emptyset$. Pick some $s_i \in V(\mathcal{I}) \setminus U$ and add it to U, let $e \in W$ be the edge of W covering s_i and let x be its other endpoint. Next we have to consider several cases. If $x = t_i$ we just add t_i to U and continue with a new $s_{i'} \in V(\mathcal{I}) \setminus U$, in this case nothing else must be done. If $x = t_j$ for some $j \in [1,k] \setminus \{i\}$ add s_j and t_j to U, select the edge $e \in W$ to be the edge covering s_j, let x be its other endpoint. In case s_j was already in U before, it must be equal to the original $s_{i'}$ this cycle of the process was started with. Then every W-extending solution clearly closes an alternating cycle and we may proceed with a new $s_i \in V(\mathcal{I}) \setminus U$. Otherwise reiterate the process with s_j in the role of s_i, in this case, we are still within the same cycle. And at last, if $x \notin V(\mathcal{I})$ we may consider t_i and the edge $e' \in W$ covering t_i, let y be its other endpoint.

The process here is rather similar to what came before. Clearly $y \neq s_i$ and, moreover, $y \notin U$. If however $y = s_j$ for some $i \in [1,k] \setminus \{i\}$, then add s_j and t_j to U, set $e' \in W$ to be the edge covering t_j and reiterate the process with t_j in the role of t_i. If, on the other hand, $y \notin V(\mathcal{I})$, let $s_{i'}$ be the vertex this cycle was started with. Now any solution, together with the edges of W, produces a path P' with endpoints x and y that can be made into an alternating cycle by adding the edge xy to B if it does not already exist. Hence we add xy to F and proceed with the next $s_i \in V(\mathcal{I}) \setminus U$. Once $U = V(\mathcal{I})$ our set F is W-completing by the discussion above. □

Adding a W-completion F to our graph B should not change its perfect matching width by too much.

Observation 5.4.6. Let D be a digraph and $F \subseteq E(\overline{D})$ a set of edges not in D. Then $\operatorname{dtw}(D+F) \leq \operatorname{dtw}(D) + |F|$.

Proof. Let (T, β, γ) be a directed tree decomposition for D of optimal width and let $S \subseteq V(D)$ be the set of tails of the edges in F. Now add S to every guard of (T, β, γ). Clearly this increases the width of our decomposition by at most $|S| \leq |F|$ and the result is a directed tree decomposition for $D + F$. □

Lemma 5.4.7 (D*)**.** Let B be a bipartite graph, $\mathcal{I} = \{(s_1,t_1),\dots,(s_k,t_k)\}$ a distinct family of terminal pairs, and $W \subseteq E(B)$ an extendible set covering all terminals such that if $e \in W$, then an endpoint of e is a terminal. Let $\operatorname{pmw}(B) \leq w$ and $n := |V(B)|$. There exists an algorithm with running time $2^{\mathcal{O}((w^2+k)\log(w^2+k))}n^{\mathcal{O}(1)}$ that produces a $(\mathcal{I}, W)$-decomposition of width at most $432w^2 + 864w + 22 + 6k$ for B.

Proof. Let us first compute a perfect matching M extending W and a W-completing set F. Clearly both can be done in polynomial time. Now let $D \coloneqq \mathcal{D}(B, M)$, $D' \coloneqq \mathcal{D}(B+F, M)$, and $F' \coloneqq E(D') \setminus E(D)$. Then F' corresponds to the edges in F added to B. By Theorem 5.3.27 we obtain $\mathrm{dtw}(D) \leq 72\,\mathrm{pmw}(B)^2 + 144\,\mathrm{pmw}(B) + 9$. With Observation 5.4.6 this means $\mathrm{dtw}(D+F) \leq 72\,\mathrm{pmw}(B)^2 + 144\,\mathrm{pmw}(B) + 9 + k$. Observe that $D + F' = D'$. Now let (T, β, γ) be the prepared proto-directed tree decomposition of width at most $216w^2 + 432w + 10 + 3k$ obtained from Lemma 5.4.4.

In the next step, we show how to obtain a nice perfect matching decomposition for $B + F$ from (T, β, γ). First of all, since $D' = \mathcal{D}(B+F, M)$, every vertex v of D' corresponds to an edge e_v of M, let us denote the endpoint of e_v in V_1 by a_v and the other endpoint by b_v. Since T is an arborescence, it already is rooted at some vertex, say r. In what follows, we explain how to manipulate this tree T and how we define a bijection δ' step by step in order to create a cycle decomposition for D' of bounded width. This cycle decomposition is then translated into a perfect matching decomposition with the required properties.

Let $t \in V(T)$ be any vertex with $\beta(t) \neq \emptyset$ and let d be its predecessor. We only discuss the case in which t is not the root, but the other case can be solved in a similar way. There are three possible cases, depending on the number of successors t has in T.

Case 1: Vertex t is a leaf in T.

In this case, let us construct a rooted cubic tree T' with root t and otherwise disjoint from T such that T' has exactly $|\beta(t)|$ many leaves. Add T' to T and extend the bijection δ' such that the restriction of δ' to the leaves of T' is a bijection between said leaves and $\beta(t)$. Then the edge (d, t) induces an edge cut $\partial_D(X)$ with $X = \beta(t)$, and thus $\mathrm{cp}(\partial_D(X)) \leq |\beta(t)| \leq 216w^2 + 432w + 11 + 3k$ and every edge of T' induces an edge cut with even smaller cycle porosity. Mark every non-leaf vertex in T' as a guard. This mark does not hold a special significance for this decomposition, but will be used in the second half of this proof to show that we can construct a nice perfect matching decomposition.

Case 2: Vertex t has a unique successor d' in T.

Let T' be a rooted cubic tree with root t', completely disjoint from T and exactly $|\beta(t)|$ leaves, add T' to T together with the edge (t, t') and extend

δ' for the leaves of T' as above. With the same arguments we obtain bounds on the cycle porosity of every edge of T' and the edge (t, t'). Mark every non-leaf vertex in T' as a guard.

Case 3: Vertex t has two successors in T.

Here we first subdivide the edge (d, t), i.e. we replace it by the directed path (d, t', t) where t' is a vertex newly introduced to T. Then we create a rooted cubic tree T' with exactly $|\beta(t)|$ leaves, rooted at t'' that is disjoint from the modified tree T and introduce the edge (t', t''). Afterwards, we extend δ' to the leaves of T' as before and again obtain bounds on the cycle porosity of the edge cuts induced by the edges of T' and (t', t''). Mark t' and every non-leaf vertex in T' as guards.

Let (T', δ') be the cycle decomposition for D' obtained by applying the above constructions to all vertices of (T, β, γ) with non-empty bags. Since (T, β, γ) was a proto-directed tree decomposition of width at most $216w^2 + 432w + 10 + 3k$ it is straight forward to prove that all edges of T' that were not discussed in the construction induce, with respect to δ', edge cuts of cycle porosity at most $432w^2 + 864w + 22 + 6k$ in D'. Hence $\text{width}(T', \delta') \leq 432w^2 + 864w + 22 + 6k$.

Now let us create a new rooted tree T'' from T' by introducing for every leaf t of T' two new successors t_{V_1} and t_{V_2} and defining $\delta(t_{V_1}) \coloneqq a_{\delta'^{-1}(t)}$ and $\delta(t_{V_2}) \coloneqq b_{\delta'^{-1}(t)}$. The result is a perfect matching decomposition (T'', δ) for $B+F$. The bound $\text{width}(t, \delta) \leq 432w^2+864w+22+6k$ follows from Observation 5.3.10.

Additionally, since we started out with a prepared proto-directed tree decomposition, it is relatively straight forward to check that (T'', δ) is nice. For the sake of completion we discuss this in the following paragraph.

Let $t \in V(T'')$ be any non-root vertex.

Case A: Vertex t is a leaf.

Here we are done immediately since $t \in \mathsf{L}(T'')$.

Case B: Vertex t is adjacent to a leaf.

In this case, by construction of T'' from T', t must have exactly two successor which both are leaves and thus $t \in \text{basic}(T'')$.

Case C: Vertex t is not adjacent to a leaf, but has been marked as a guard in the construction of (T', δ').

First let us assume $t \in V(T)$, in this case, t must have been a leaf of T and thus $\beta(t)$ directly corresponds to $\delta(T''_t)$ and t is indeed a guard of T''. Otherwise, t must have been introduced during the construction of T' from T and thus there must be a vertex $d \in V(T)$ with $\beta(d) \neq \emptyset$ that is responsible for the introduction of t. In case d has a unique successor in T, t belongs to the newly introduced rooted cubic tree and thus $\delta(T''_t)$ has at most $2w$ vertices and is conformal. Thus t is indeed a guard. Hence we may assume d to have two successors in T. Then we subdivided the incoming edge at d with a new vertex, say d' and added a rooted cubic tree R with root d'' as a new successor of d'. If $t \in V(R)$ we are done by the same argument as above. If $t = d'$, then the successors of d' are d and d''. We have already seen that d'' is a guard of T and since (T, β, γ) is a prepared proto-directed tree decomposition, d must be a join as d has two successors t_1 and t_2 satisfying the appropriate requirements.

Case D: Vertex t is a vertex of the original T but has not been marked as a guard during the construction of T'.

This means in particular that t is not a leaf of T and does not have a unique successor. Indeed, in this case, t must have exactly two successors t_1 and t_2. We may assume t_1 and t_2 to be ordered such that there is no edge from $\beta(T_{t_2})$ to $\beta(T_{t_1})$. Let t'_1 and t'_2 be the two successors of t in T'', then it follows that there is no edge from $V_2 \cap \delta(T''_{t_2})$ to $V_1 \cap \delta(T''_{t_1})$. Moreover, with the same argument $\beta(T_{t_1})$ is strongly connected and thus $\delta(T''_{t_1})$ is elementary. Hence t is a join.

This completes the argument and thus (T'', δ) is nice. What is left to show is that (T'', δ) is safe for $\mathcal{I}$ and W. Note that the width of (T'', δ) cannot increase by deleting F and thus it also is a perfect matching decomposition for B with the same bound on its width.

We claim that (T'', δ) is safe for W and $\mathcal{I}$. Suppose there exists a solution $M', \mathcal{P}$ and an edge $e \in E(T'')$ such that

$$|\partial_B(e) \cap \bigcup_{P \in \mathcal{P}} E(P)| > 864w^2 + 1728w + 44 + 12k.$$

With F being W-completing, $F \cup W \cup \bigcup_{P \in \mathcal{P}} E(P)$ induces a family $\mathcal{C}$ of pairwise disjoint M'-conformal cycles in $B + F$. Then let $\mathcal{C}'$ be the collection of all cycles in $\mathcal{C}$ with edges in $\partial_B(e)$. Let $E(\mathcal{C}') \coloneqq \bigcup_{C \in \mathcal{C}'} E(C)$. Since $\text{width}(T'', \delta) \leq 432w^2 + 864w + 22 + 6k$, at most $432w^2 + 864w + 22 + 6k$ of the edges in $E(\mathcal{C}') \cap \partial_B(e)$ can belong to M'. Hence $|(E(\mathcal{C}) \cap \partial_B(e)) \setminus$

$M'| > 432w^2 + 864w + 22 + 6k$. Consider the perfect matching $M'' := M'\Delta E(\mathcal{C}')$ of $B + F$. Note that this is the point where we need $\mathcal{I}$ to be a distinct family. Then $|\partial_B(e) \cap M''| > 432w^2 + 864w + 22 + 6k$ contradicting our assumption. So our claim follows. $\square$

Note that in case $t = 0$, Lemma 5.4.7 implies the existence of an FPT-approximation algorithm that produces a nice perfect matching decomposition for bipartite graphs B with a perfect matching.

5.4.2. The Dynamic Programming for Bipartite k-DAPP

With Lemma 5.4.7 we have fixed

$$f_1(\mathrm{pmw}(B), k, |V(B)|) := 2^{\mathcal{O}((\mathrm{pmw}(B)^2+k)\log(\mathrm{pmw}(B)^2+k))}|V(B)|^{\mathcal{O}(1)},$$

so from now on we will only be concerned with the dynamic programming on $(\mathcal{I}, W)$-decompositions.

In most parts, we lean on the algorithm for the directed disjoint paths problem developed by Johnson et al. for digraphs of bounded directed treewidth [JRST01]. However, we face several challenges here. The first one is that we cannot assume that there is no perfect matching M for which some internally M-conformal path P exists with $|\partial_B(e) \cap E(P)| \gg 2\,\mathrm{width}(T, \delta)$. The only thing we can be sure of is that no such path can be part of our solution. Second, while we are exclusively interested in perfect matchings of B that extend W, there might still be an exponential number of them, and thus we must store additional information in order to cope with this fact.

Let B be a bipartite graph with a perfect matching, $\mathcal{I}$ a distinct family of k terminal pairs for the k-DAPP in B, $W \subseteq E(B)$ an extendible set matching all terminals such that if $e \in W$, then an endpoint of e is a terminal, and F a W-completion. A subgraph L of B is called a *linkage* if there exists a perfect matching M and a family of pairwise internally disjoint internally M-conformal paths $\mathcal{P}$ such that $L = \bigcup_{P\in\mathcal{P}} P$, and L has exactly $|\mathcal{P}|$ components. A linkage L is a $(\mathcal{I}, W)$-*linkage* if there exists a solution $(M, \mathcal{P})$ for $\mathcal{I}$ in B extending W such that $L = \bigcup_{P\in\mathcal{P}} P$. Please note that for a $(\mathcal{I}, W)$-linkage L the corresponding W-extending solution $(M, \mathcal{P})$ is uniquely determined apart from the edges of $M \cap (E(B) \setminus \bigcup_{P\in\mathcal{P}} E(P))$. A *part* of L is a subgraph $L' \subseteq L$ such that some path $P \in \mathcal{P}$ exists with

$L' \subseteq P$. Let $X \subseteq V(B)$, a *part of* L *in* X is a component of $L[X \cap V(L)]$, we denote the set of all parts of L in X by $\mathrm{parts}_L(X)$.

We say that a linkage L in G is (k,w)-*limited* in X and G, for some integer w, if for every set $Y \subseteq X$ with $\mathrm{mp}(\partial_B(Y)) \leq w$ we have $|\,\mathrm{parts}_L(Y)| \leq k + w$.

Lemma 5.4.8 (D*)**.** Let B be a bipartite graph with a perfect matching, $\mathcal{I}$ a distinct family of k terminal pairs, $W \subseteq E(B)$ an extendible set covering all terminals such that if $e \in W$, then an endpoint of e is a terminal, and F a W-completion. Let $X \subseteq V(B)$ and L a $(\mathcal{I}, W)$-linkage in B as well as w a positive integer, then L is (k,w)-limited in X and $B+F$.

Proof. The proof is similar to the safety-part in the proof of Lemma 5.4.7. Let $Y \subseteq X$ be any set with $\mathrm{mp}(\partial_{B+F}(Y)) \leq w$ and suppose $|\,\mathrm{parts}_L(Y)| \geq k + w + 1$. Let L' be any component of L and consider $\ell' := |\partial_{B+F}(Y) \cap E(L')|$. If $\ell' \geq 1$, then

$$\left\lceil \frac{\ell'}{2} \right\rceil \leq |\,\mathrm{parts}_{L'}(Y)| \leq 1 + \frac{\ell'}{2}.$$

Hence we obtain the following:

$$\begin{aligned} k + w + 1 \leq |\,\mathrm{parts}_L(Y)| &= \sum_{L' \text{ component of } L} |\,\mathrm{parts}_{L'}(Y)| \\ &\leq \sum_{L' \text{ component of } L} 1 + \frac{|\partial_{B+F}(Y) \cap E(L')|}{2} \\ &= k + \frac{|\partial_{B+F}(Y) \cap E(L)|}{2} \end{aligned}$$

Therefore $2w + 2 \leq |\partial_{B+F}(Y) \cap E(L)|$ and thus, with F being W-completing, there must exist a W-extending perfect matching M of B such that L is a family of internally M-conformal paths and thus, $L + F$ is a family of M-alternating cycles. Hence there exists a perfect matching of $B + F$ with at least $2w + 2$ edges in $\partial_{B+F}(Y)$ contradicting the choice of Y. $\square$

Since we will be working on a $(\mathcal{I}, W)$-decomposition of bounded width, from now on the case where linkages are not (k,w)-limited will be ignored, as it wont occur in our algorithm.

Let B be a bipartite graph with a perfect matching, $W \subseteq E(B)$ an extendible set, $k, w \in \mathbb{N}$ two integers, $X \subseteq V(B)$, and $U \subseteq \partial_B(X)$ a set such that $W \cup U$ is extendible and $W \cap \partial_B(X) \subseteq U$. A *$(k,w)$-$U$-itinerary for X* is a mapping f_U that assigns every tuple $(\ell, \mathcal{J}, J)$, where

- $\ell \in [1, |X|]$ is an integer,
- $\mathcal{J}$ is a distinct family of $j \in [0, k+w]$ terminal pairs from $X \setminus V(U \setminus J)$, and
- $J \subseteq E(B)$ is a matching covering all terminals of $\mathcal{J}$ and every edge of J covers some terminal of $\mathcal{J}$ such that $W \cup U \cup J$ is extendible, and $J \cap \partial_B(X) = U \cap \partial_B(X)$,

a value 0 or 1 such that the following is guaranteed:

i) If $f_U(\ell, \mathcal{J}, J) = 0$, then there exists no $(\mathcal{J}, J)$-linkage L in $B[X \setminus V(U \setminus J)]$ with $|V(L)| = \ell$ such that a J-extending solution $M, \mathcal{Q}$ exists with $W \cup J \cup U \subseteq M$, which is (k,w)-limited in X.
ii) If $f_U(\ell, \mathcal{J}, J) = 1$, then there exists a $(\mathcal{J}, J)$-linkage L in $B[X \setminus V(U \setminus J)]$ with $|V(L)| = \ell$ such that a J-extending solution $M, \mathcal{Q}$ exists with $W \cup J \cup U \subseteq M$.

Lemma 5.4.9 (D*)**.** Let B be a bipartite graph with a perfect matching, $W \subseteq E(B)$ an extendible set, and $k, w \in \mathbb{N}$ two integers. Furthermore let $X, Y \subseteq V(B)$ be two disjoint subsets such that there is no edge between $V_1 \cap Y$ and $V_2 \cap X$ and let $U \subseteq \partial_B(X \cup Y)$ be an extendible set with $W \cap \partial_B(X \cup Y) \subseteq U$. Assume that for every $Z \in \{X, Y\}$ and every extendible $U_Z \subseteq \partial_B(Z)$ with $|U_Z| \leq w$ and $W \cap \partial_B(Z) \subseteq U_Z$ we are given a (k,w)-U_Z-itinerary $f^Z_{U_Z}$. Then there exists an algorithm with running time $\mathcal{O}((k+w)!(2k+3w)^{4(k+w)}|X \cup Y|^{4k+12w+2})$ that produces a (k,w)-U-itinerary for $X \cup Y$.

Proof. Let $\ell \in [1, |X \cup Y|]$ and $j \in [0, k+w]$, let $\mathcal{J} = \{(s_1, t_1), \dots, (s_j, t_j)\}$ be a distinct set of $j \in [0, k+w]$ terminal pairs in $X \cup Y$ and $J \subseteq E(B)$ an extendible set such that every edge in J covers a terminal of $\mathcal{J}$ and every terminal is covered by some edge in J. We need to choose some additional sets of edges before we can start, in order to determine the value of $f_U(\ell, \mathcal{J}, J)$. We iterate over all choices of sets R, H, R_X, and R_Y satisfying the following requirements.

i) $R \subseteq \partial_B(X) \cap \partial_B(Y)$ such that
    - $W \cap \partial_B(X) \cap \partial_B(Y) \subseteq R$,

- $|(U \cap \partial_B(X)) \cup R| \leq w$ and $|(U \cap \partial_B(Y)) \cup R| \leq w$, and
- $W \cup U \cup J \cup R$ is extendible.

ii) $H \subseteq \partial_B(X) \cap \partial_B(Y)$ such that
- H is a matching of size at most w,
- no edge of H is incident with an edge of $U \cup R$, and
- the set of endpoints of the edges in H in $Z \in \{X, Y\}$ is denoted by Z_H.

iii) For $Z \in \{X, Y\}$, $R_Z \subseteq E(B[Z])$ such that
- every vertex in Z_H is covered by some edge of R_Z,
- every edge of R_Z covers a vertex in Z_H, and
- $W \cup U \cup J \cup R \cup R_Z$ is extendible.

In what follows let R, H, and the R_Z be fixed. For $Z \in \{X, Y\}$ let $U_Z \coloneqq R \cup (U \cap \partial_B(Z))$ and note that, by choice and our assumption, we are given a (k, w)-U_Z-itinerary $f^Z_{U_Z}$ for Z. Let us denote the set of endpoints in V_2 of the edges in R_X by V_{2,R_X} and the set of endpoints in V_1 of the edges in R_Y by V_{1,R_Y}. There may exist some paths that belong to a linkage we are interested in which start in a vertex of $V_1 \cap Y$ and end in a vertex of $V_2 \cap X$. However, each such path must necessarily use an edge of R and in total, since we are still only interested in (k, w)-limited linkages, we cannot cross the cut between X and Y too often. Still, we need to address this problem by possibly considering additional terminals not belonging to those we were given by $\mathcal{J}$. We approach the problem of merging the two itineraries with respect to the chosen sets above by constructing an auxiliary digraph $D_{W,U,X,Y}[\mathcal{J}, J, R, H, R_X, R_Y]$ of constant size.

For the vertices of $D_{W,U,X,Y}[\mathcal{J}, J, R, H, R_X, R_Y]$ we define the following sets:

$$V_X \coloneqq \{s_i \in X \mid i \in [1, j]\} \cup \{t_i \in X \mid i \in [1, j]\} \cup V_{2,R_X} \cup \{v_e \mid e \in R\}$$

$$V_Y \coloneqq \{s_i \in Y \mid i \in [1, j]\} \cup \{t_i \in Y \mid i \in [1, j]\} \cup V_{1,R_Y} \cup \{v_e \mid e \in R\}.$$

And for the edges let

$$\begin{aligned} E_X \coloneqq &\{(s_i, v) \mid s_i \in V_X \text{ and } v \in V_{2,R_X}\} \cup \{(v_e, t_i) \mid t_i \in V_X \text{ and } e \in R\} \\ &\cup \{(v_e, u) \mid e \in R \text{ and } u \in V_{2,R_X}\}, \text{ and} \\ E_Y \coloneqq &\{(s_i, v_e) \mid s_i \in V_Y \text{ and } e \in R\} \cup \{(v, t_i) \mid t_i \in V_Y \text{ and } v \in V_{1,R_Y}\} \\ &\cup \{(u, v_e) \mid e \in R \text{ and } u \in V_{1,R_Y}\}. \end{aligned}$$

Then

$$D_{W,U,X,Y}[\mathcal{J},J,R,H,R_X,R_Y] \coloneqq (V_X,E_X)\cup(V_Y,E_Y) \\ +\{(u,v)\mid uw\in R_X,\ wz\in H, \text{ and } zv\in R_Y\}\,.$$

Let L be a directed $\mathcal{J}$-linkage in $D_{W,U,X,Y}[\mathcal{J},J,R,H,R_X,R_Y]$ such that L has at most $t+w$ components in (V_Z,E_Z) for both $Z\in\{X,Y\}$. Then from L we can derive two instances of the linkage problem for the matching case, one in $B[X]$ and the other in $B[Y]$, namely $\mathcal{J}_{L,X}\coloneqq E(L)\cap E_X$ and $\mathcal{J}_{L,Y}\coloneqq E(L)\cap E_Y$. Additionally we define for $Z\in\{X,Y\}$

$$U_{L,Z} \coloneqq (\partial_B(Z)\cap U)\cup R, \text{ and}$$
$$J_{L,Z} \coloneqq \{e\in J\mid e\in E(B[Z])\cup\partial_B(Z)\} \\ \cup\{e\in R_Z\mid e \text{ covers a terminal in } \mathcal{J}_{L,Z}\}\,.$$

If there now exist integers ℓ_1 and ℓ_2 with $\ell=\ell_1+\ell_2$ such that

$$f^X_{U_{L,X}}(\ell_1,\mathcal{J}_{L,X},J_{L,X}) = f^Y_{U_{L,Y}}(\ell_1,\mathcal{J}_{L,Y},J_{L,Y}) = 1,$$

the two solutions in $B[X]$ and $B[Y]$ can be combined and we may set $f_U(\ell,\mathcal{J},J)\coloneqq 1$.

In total, since we iterate over all possible choices and combinations, this process correctly computes a $(k+w)$-U-itinerary for $X\cup Y$. The running time follows from the number of possible choices we need to consider and the size and construction of $D_{W,U,X,Y}[\mathcal{J},J,R,H,R_X,R_Y]$. Please note that the bound given in the statement of the lemma is probably not optimal, but it suffices for our purposes. □

Lemma 5.4.9 describes how to merge partial solutions at join-vertices of a $(\mathcal{I},W)$-decomposition, once a set U has been fixed. The next lemma addresses the same problem at guard-vertices. Indeed for our purposes, it suffices to only consider guard- and join-vertices in a bottom-up fashion in order to find the desired solution.

Lemma 5.4.10 (D*)**.** Let B be a bipartite graph with a perfect matching, $W\subseteq E(B)$ an extendible set, and $k,w\in\mathbb{N}$ two integers. Furthermore let $X,Y\subseteq V(B)$ be two disjoint subsets such that $\mathrm{mp}(\partial_B(X))\leq w$ and $|Y|\leq w$, and let $U\subseteq\partial_B(X\cup Y)$ be an extendible set with $W\cap\partial_B(X\cup Y)\subseteq U$. Assume that for every extendible $U_X\subseteq\partial_B(X)$ with $|U_X|\leq w$ and $W\cap\partial_B(X)\subseteq U_X$ we are given a (k,w)-U_X-itinerary $f^X_{U_X}$. Then there exists an algorithm with running time $\mathcal{O}((w+k)!w^{\frac{1}{2}w}(|X|+w)^{4k+10w+2})$ that produces a (k,w)-U-itinerary for $X\cup Y$.

Proof. Let $\ell \in [1, |X \cup Y|]$, $j \in [0, k+w]$, and $\mathcal{J} = \{(s_1, t_1), \ldots, (s_j, t_j)\}$ be a distinct set of $j \in [0, k+w]$ terminal pairs in $X \cup Y$ and $J \subseteq E(B)$ an extendible set such that every edge in J covers a terminal of $\mathcal{J}$ and every terminal is covered by some edge in J. Next we iterate over all possible choices for the sets R, H, and R_X defined analogously to those in the proof of Lemma 5.4.9. Additionally, we iterate over all possible choices of $W \cup U \cup J \cup R \cup R_X$-extending perfect matchings M_{J,R,R_X} of the graph $B_Y \coloneqq B[Y \cup V(W) \cup V(U) \cup V(J) \cup V(R) \cup V(R_X)]$. Since $|Y| \leq w$ there are at most $w^{\frac{w}{2}}$ such perfect matchings of B_Y. We need six additional vertex sets in order to construct another auxiliary digraph that will be used similarly to the one in Lemma 5.4.9. However, since we do not know which colour the endpoints of an edge in $\partial_B(X) \cap \partial_B(Y)$ have, with respect to the shore we are interested in, the construction is slightly more complicated.

i) $V_{1,R} \coloneqq \bigcup_{e \in R} e \cap V_1 \cap X$
ii) $V_{2,R} \coloneqq \bigcup_{e \in R} e \cap V_2 \cap X$
iii) $V_{1,H} \coloneqq \bigcup_{e \in R_X} e \cap V_1 \setminus \bigcup_{e \in H} e$
iv) $V_{2,H} \coloneqq \bigcup_{e \in R_X} e \cap V_2 \setminus \bigcup_{e \in H} e$
v) $V_{1,X} \coloneqq \{s_i \in X \mid (s_i, t_i) \in \mathcal{J}\}$
vi) $V_{2,X} \coloneqq \{t_i \in X \mid (s_i, t_i) \in \mathcal{J}\}$

As a first component we need the digraph $D_1 \coloneqq \mathcal{D}(B_Y, M_{J,R,R_X})$. Second let

$$V_X \coloneqq \left\{u' \mid u \in V_{1,R} \cup V_{1,H} \cup V_{1,X}\right\} \cup \left\{v' \mid v \in V_{2,R} \cup V_{2,H} \cup V_{2,X}\right\},$$

$$E_X \coloneqq \left\{(u', v') \mid u \in V_{1,R} \cup V_{1,H} \cup V_{1,X} \text{ and } v \in V_{2,R} \cup V_{2,H} \cup V_{2,X}\right\}, \text{ and}$$

$$\begin{aligned} E' \coloneqq &\left\{(u', v_e) \mid u \in V_{1,H} \cup V_{1,R},\ u \in e \in R \cup R_X, \text{ and } v_e \in V(D_1)\right\} \\ &\cup \left\{(v_e, v') \mid v \in V_{2,H} \cup V_{2,R},\ v \in e \in R \cup R_X, \text{ and } v_e \in V(D_1)\right\}. \end{aligned}$$

In total these definitions give rise to the digraph

$$D_{W,U,X,Y}[\mathcal{J}, J, R, H, R_X, M_{J,R,R_X}] \coloneqq (V_X, E_X) \cup D_1 + E'.$$

Now let

$$\begin{aligned} \mathcal{J}' \coloneqq \{(u_i, w_i) \mid\ & u_i = s_i \text{ if } s_i \in Y, \text{ else, } u_i = s_i'; \\ & w_i = t_i \text{ if } t_i \in Y \text{ else, } w_i = t_i'\}. \end{aligned}$$

For every $e \in J$ we identify any endpoint of J that is a terminal of $\mathcal{J}'$ with the vertex $v_e \in V(D_1)$. Then, by construction, every solution $\mathcal{P}$, M for $\mathcal{J}'$, W in $B[X \cup Y]$ such that M extends M_{J,R,R_X} naturally corresponds to a family of pairwise internally disjoint directed paths in

$D_{W,U,X,Y}[\mathcal{J}, J, R, H, R_X, M_{J,R,R_X}]$ that links $\mathcal{J}'$. On the other hand, let $\mathcal{P}$ be a family of pairwise internally disjoint directed paths linking $\mathcal{J}'$, such that the following requirements are met:

i) Let $Q := \bigcup_{P \in \mathcal{P}} P$, then the total number, over all paths $P \in \mathcal{P}$, of subgraphs of Q that are a maximal directed subpaths of $P[V_X \cup \{v_e \mid e \in R_X\}]$ does not exceed $k + w$.

ii) If $v \in V(P)$ such that $v = v_e$ for some $e \in M_{J,R,R_X}$ and there is some $u' \in V_X$ such that $u \in e$, then u' does not occur in any other path of $\mathcal{P}$. Similarly, if $u' \in V(P) \cap V_X$ such that some $e \in M_{J,R,R_X}$ exists with $u \in e$, then v_e does not occur in any path of $\mathcal{P}$ besides possibly P.

Let P' be a subgraph of Q that is a maximal directed subpath of $P[V_X \cup \{v_e \mid e \in R_X\}]$ for some $P \in \mathcal{P}$ and let $u'_{P'}$ be the starting point of P and $v'_{P'}$ its end. We define a terminal pair (u, v) in $B[X]$ as follows:

- If $u_{P'} \in V_{1,R} \cup V_{1,H} \cup V_{1,X}$ set $u := u_{P'}$, otherwise there must be some $e \in R_X$ such that $u'_{P'} = v_e$. In this case let u_e be the endpoint of e in V_1 and set $u := u_e$.
- Similarly, if $v_{P'} \in V_{2,R} \cup V_{2,H} \cup V_{2,X}$ set $v = v_{P'}$, otherwise there must be some $e \in R_X$ such that $v'_{P'} = v_e$. In this case let u'_e be the endpoint of e in V_2 and set $v := u'_e$.

Let $\mathcal{J}_{\mathcal{P}}$ be the collection of all terminal pairs (u, v) defined as above. Then no vertex of X occurs in two different terminal pairs of $\mathcal{J}_{\mathcal{P}}$ and every terminal is covered by an edge of $J \cup R_X$. We define two additional sets as before:

$$\begin{aligned} U_{\mathcal{P},X} :=& (\partial_B(X) \cap U) \cup R, \text{ and} \\ J_{\mathcal{P},X} :=& \{e \in J \mid e \in E(B[X]) \cup \partial_B(X)\} \\ & \cup \{e \in R_X \mid e \text{ covers a terminal in } \mathcal{J}_{L,X}\}. \end{aligned}$$

If there now exist integers ℓ_1 and ℓ_2 with $\ell = \ell_1 + \ell_2$ such that

$\ell_2 = 2|V(Q) \cap \{v_e \mid e \in M_{J,R,R_X} \setminus R_X\}|$, and $f^X_{U_{\mathcal{P},X}}(\ell_1, \mathcal{J}_{\mathcal{P}}, J_{\mathcal{P},X}) = 1$,

We can combine the parts of $\mathcal{P}$ in $D_1[\{v_e \mid e \in M_{J,R,R_X} \setminus R_X\}]$ and a solution for $\mathcal{J}_{\mathcal{P}}$ to obtain a solution for $X \cup Y$ and U. Hence we may set $f_U(\ell, \mathcal{J}, J) := 1$. By iterating over all possible choices for the various sets we are sure to produce a complete (k, w)-U-itinerary for $X \cup Y$. $\square$

Using lemmata 5.4.9 and 5.4.10, we are now able to merge partial solutions at all join- and guard vertices. For basic vertices obtaining partial solutions is straight forward, since we may only choose the edges of the perfect matchings covering the two singular vertices that lie in the two subtrees beneath. In order to obtain a (k, w)-U-itinerary for every possible U, we just have to call the corresponding merge operation for every possible choice of U. At any given time there are $\mathcal{O}(|V(B)|^w)$ such choices, which overall implies the following:

Corollary 5.4.11 (D*)**.** Let B be a bipartite graph with a perfect matching, $\mathcal{I}$ a distinct set of k terminal pairs, W an extendible set covering all terminals such that every edge in W covers a terminal and (T, δ) a $(\mathcal{I}, W)$-decomposition of width w for B. There exists an algorithm that decides in time $\mathcal{O}(|V(B)|^{4k+13w+3})$ whether there exists a solution for $\mathcal{I}$, W or not.

Corollary 5.4.11, together with the approximation factor for our $(\mathcal{I}, W)$-decomposition from Lemma 5.4.7, fixes

$$f_2(\mathrm{pmw}(B), k, |V(B)|) \coloneqq \mathcal{O}(|V(B)|^{5616\,\mathrm{pmw}(B)+11232\,\mathrm{pmw}(B)^2+189+73k}).$$

Together with our previous results this completes the proof of Theorem 5.4.1.

Deciding Matching Minor Containment The importance of the disjoint paths problem in the Graph Minors series by Robertson and Seymour is due to the fact that checking for minor containment can be reduced to certain instances of the disjoint paths problem. For bipartite graphs with perfect matchings, this is also true.

Proof of Theorem 5.4.2. By Corollary 3.1.105, if H is a matching minor of B, then there exists a perfect matching M of B such that there exists an M-model of H in B. Every vertex $v \in V(H)$ is represented by a barycentric tree in B, and it is not hard to see that we may always choose such a barycentric tree such that the number of vertices of degree at least 3 is at most $\deg_H(v)$. Let $\mu\colon H \to B$ be such a model. By Lemma 3.1.104 we may further assume that M corresponds to a perfect matching M_H of H and $\mu(uv)$ is internally M-conformal if and only if $uv \notin M_H$. Moreover, if $uv \in M_H$, then $\mu(uv)$ is M conformal. Hence for every v, it suffices to guess the at most $\deg_H(v)$ many edges of M and ask for pairwise internally disjoint internally M-conformal paths connecting them in an appropriate way. Additionally, we need an internally M-conformal path representing every $uv \in E(H) \setminus M_H$ and for each of those, we need to find an edge of M for each of the two endpoints. Since we also guessed the edges of M covering the only vertex of $\mu(x)$ not covered by $E(\mu(x)) \cap M$ for both $x \in \{u, v\}$ if $uv \in M_H$, the endpoints of these two edges, not belonging to their respective vertex models must also be linked by paths. Since it is not feasible to check for M-models of H for every perfect matching M of B, we instead check all possible choices of extendible sets F of size at most $2|E(H)| + \sum_{v \in V(H)} \deg_H(v) = 4|E(H)|$. In fact, since we also do not know which edge of our set F belongs to the model of which vertex or edge of H, we also need to try all possible configurations. But this only worsens our running time by a factor depending exclusively on the size of F. Hence in total we need to call the algorithm from Theorem 5.4.1 at most $\mathcal{O}(|V(B)|^{4|E(H)|})$ times with $k \leq 4|E(H)| \leq 4|V(H)|^2$, and thus our claim follows. $\square$

5.5. Perfect Matching Width and Treewidth

A natural question for any new width parameter is how it compares to other, already known parameters. We have already seen a way to relate the perfect matching width of bipartite graphs and directed treewidth. However, to apply our findings the graph itself has to be transformed. In the first part of this short section we discuss the relation between the (undirected) treewidth of G and its perfect matching width. To do this we use a parameter introduced by Vatshelle [Vat12] which is already known to be equivalent to treewidth but is much closer to perfect matching width in spirit.

Definition 5.5.1 (Maximum Matching on Edge Cuts)**.** Let G be a graph and $X \subseteq V(G)$. We denote by $\mathrm{mm}(X)$ the number $\nu(G[\partial_G(X)])$ which is the maximum number of pairwise disjoint edges in $\partial_G(X)$.

Definition 5.5.2 (Maximum Matching Width)**.** Let G be a graph. A *maximum matching decomposition* of G is an mm-branch decomposition (T, δ) over $V(G)$ where T is a cubic tree and $\delta\colon \mathsf{L}(T) \to V(G)$ is a bijection.

The *width* of a maximum matching decomposition (T, δ) is defined as its mm-width, and the *maximum matching width* of G, denoted by $\mathrm{mmw}(G)$, is the minimum width over all maximum matching decompositions for G.

Theorem 5.5.3 ([Vat12, JST18])**.** Let G be a graph. Then $\mathrm{mmw}(G) \leq \mathrm{tw}(G) + 1 \leq 3\,\mathrm{mmw}(G)$.

With this it is straight forward to bound the perfect matching width of a graph G with a perfect matching in terms of its treewidth.

Proposition 5.5.4 (X^*)**.** Let G be a graph with a perfect matching. Then $\mathrm{pmw}(G) \leq \mathrm{tw}(G) + 1$.

Proof. By Theorem 5.5.3 we have $\mathrm{mmw}(G) \leq \mathrm{tw}(G) + 1$, so there exists a maximum matching decomposition (T, δ) for G of width at most $\mathrm{tw}(G)+1$. Now let M be any perfect matching of G and $t_1t_2 \in E(T)$. Note that $M \cap \partial_G(\delta(T_{t_1}))$ is a matching, hence $|M \cap \partial_G(\delta(T_{t_1}))| \leq \mathrm{mm}(\delta(T_{t_1})) \leq \mathrm{tw}(G)+1$. Indeed, as M was chosen arbitrarily we have $\mathrm{mp}(\partial_G(\delta(T_{t_1}))) \leq \mathrm{tw}(G) + 1$ and thus the mp-width of (T, δ) is at most $\mathrm{tw}(G) + 1$ and our claim follows. $\square$

While treewidth gives us an upper bound on the perfect matching width of G, the reverse is not true in general. With these findings we close this chapter.

Proposition 5.5.5 (X^*)**.** For every $k \in \mathbb{N}$ with $k \geq 2$ there exists a brace B_k with $\mathrm{pmw}(B_k) = 2$ and $\mathrm{tw}(B_k) \geq k$.

Proof. First note that for every $t \in \mathbb{N}$, $\mathrm{tw}(K_{t+1}) = t$. So if we can show that B_k contains K_{k+1} as a minor we have proven $\mathrm{tw}(B_k) \geq k$. Let B_k be the bipartite ladder L_k of order k. Then, by Theorem 5.2.12, $\mathrm{pmw}(B_k) = 2$ for all k. Now let $k \geq 2$ be chosen arbitrarily. We choose the perfect matching $M \coloneqq \{u_iv_i \mid i \in [1, k]\}$. By definition of L_k we know u_iv_j for every $i \in [1, k]$ and every $j \in [i, k]$, so by contracting all edges in M we obtain a graph on k vertices with $\binom{k}{2}$ edges. So B_k/M is isomorphic to K_{k+1} and we are done. $\square$

Chapter 6.

Tangles and a Unified Decomposition

In the Graph Minor Project, tangles are introduced as a concept dual to treewidth. That is, tangles can be seen as 'areas of high connectivity', or better, areas which cannot be cut through with small separators. Consequently, a graph with a large tangle cannot be of high treewidth, since a tree decomposition of small width would require to cut through the respective 'area of high connectivity' identified by the tangle. This base idea helped to expand the idea of bounded width tree decompositions to tree decompositions where only the size of the intersection between neighbouring bags is bounded, while the bags themselves might be large, but we can make some statements of their general structure. In Theorem 2.2.8 we have seen an example of such a decomposition. Here the intersections of neighbouring bags, sometimes called the *adhesion sets*, have size at most three, and the bags either contain $\mathcal{M}_8$ or planar graphs of arbitrary size.

In this chapter we show how tangles can be generalised to bipartite graphs with perfect matchings. Essentially this can be done by applying a sequence of lemmas to the directed counterparts of these concepts. Still, there is some deeper insight that can be gained from considering the setting of bipartite graphs with perfect matchings over the setting of digraphs. To give a better intuition on how the interplay of directed separations and general tight cuts functions with regards to width parameters, in Section 6.1 we adapt the proofs of Erde [Erd20] to introduce the notion of 'blockages' as a concept dual to strict linear perfect matching width. In Section 6.3 we show how to directly use the directed counterparts to obtain a notion of tangles dual to perfect matching width. This chapter's main theorem states that the tangle number of a bipartite matching covered graph B and the tangle number of any M-direction of B are at most a small constant factor apart. More specifically, every tangle in an M-direction induces a tangle of at least half the order in B and every tangle in B induces a tangle in every M-direction of B whose order only differs by a small constant factor.

6.1. Blockages and Strict Linear Perfect Matching Width

Strict linear perfect matching width already yields relatively structured decompositions, but for our purposes this is not enough. Let B be a bipartite graph with a perfect matching and $X, Y \subseteq V(B)$ two sets that induce generalised tight cuts. In the following we say that X and Y *cross* if $\partial_B(X)$ and $\partial_B(Y)$ cross. They are said to be *laminar* if $X \subseteq Y$, $X \subseteq \overline{Y}$, $Y \subseteq X$, or $Y \subseteq \overline{X}$. Note that, in case X and Y cross and $\operatorname{Maj}(X) \cup \operatorname{Maj}(Y) \subseteq V_i$ for some $i \in [1,2]$, the two sets $X \cap Y$ and $X \cup Y$ induce generalised tight cuts as well. The matter becomes more complicated once $\operatorname{Maj}(X) \cup \operatorname{Maj}(Y) \nsubseteq V_i$ for any $i \in [1,2]$. So we would like to have a guarantee that once we have fixed $i \in [1,2]$ all pairwise laminar sets that occur as shores of the cuts induced by the edges of the spine of the spine of a linear decomposition have the same majority. This leads to a more refined version of strict linear perfect matching width.

Definition 6.1.1 (Ordered Linear Perfect Matching Width)**.** Let B be a bipartite graph with a perfect matching. A linear perfect matching decomposition (T, δ) of G is said to be *ordered* if for all $e \in E(\operatorname{spine}(\operatorname{spine}(T)))$ the cut $\partial_B(e)$ is a generalised tight cut and for all edges $t_1t_2, t_2t_3 \in E(\operatorname{spine}(\operatorname{spine}(T)))$ we have $\operatorname{Maj}(\delta(T_{t_1})) \subseteq \operatorname{Maj}(\delta(T_{t_2}))$ where T_{t_i} is the component of $T - t_it_{i+1}$ that contains t_i for both $i \in [1,2]$. The *ordered linear perfect matching width* of B, denoted by $\operatorname{olpmw}(G)$, is defined as the minimum width over all strict perfect matching decompositions of B.

It is straight forward to check that $\operatorname{slpmw}(B) \leq \operatorname{olpmw}(B)$ for all bipartite graphs with perfect matchings. The following subsection is concerned with establishing an upper bound on $\operatorname{olpmw}(B)$ in terms of the strict linear perfect matching width of B.

6.1.1. Ordered Linear Perfect Matching Width

First we introduce an abstraction of linear perfect matching decompositions based on the work of Erde [Erd20] and Diestel and Oum [DO17, Die18, DO19].

Definition 6.1.2 (The Family of Generalised Tight Cuts)**.** Let B be a bipartite matching covered graph. We denote by $\mathcal{T}(B)$ the family of all sets $X \subseteq V(B)$ that induce a generalised tight cut. For $i \in [1,2]$, $\mathcal{T}^i(B)$ is the family of all sets $X \subseteq V(B)$ that induce a generalised tight cut and satisfy $\mathrm{Maj}(X) \subseteq V_i$. Let $k \in \mathbb{N}$, then $\mathcal{T}_k(B)$ is the family of sets $X \subseteq V(B)$ with $\mathrm{mp}(\partial_B(X)) \leq k$, and $\mathcal{T}_k^i(B) \subseteq \mathcal{T}_k(B)$ contains exactly those X from $\mathcal{T}_k(B)$ whose majority belongs to V_i.

Definition 6.1.3 ($\mathcal{T}$-Paths)**.** Let B be a bipartite matching covered graph and $\mathcal{T}' \subseteq \mathcal{T}(B)$. A $\mathcal{T}'$-path is a tuple (P, α) where $P = (t_0, t_1, \dots, t_{p+1})$ is a path and $\alpha\colon E(P) \to \mathcal{T}'$ such that for all $i \in [0, p-1]$ we have $\alpha(t_it_{i+1}) \subseteq \alpha(t_{i+1}t_{i+2})$.
If $\mathcal{T}' \subseteq \mathcal{T}^i(B)$ for some $i \in [1,2]$ we say that (P, α) is ordered.
The *width* of a $\mathcal{T}'$-path (P, α), denoted by $\mathrm{width}(P, \alpha)$, is the largest integer $k \in \mathbb{N}$ such that there exists $i \in [0, p]$ for which $\alpha(t_it_{i+1})$ induces a proper k-tight cut.
For every $i \in [0, p]$ let $X_i \coloneqq \alpha(t_it_{i+1})$, and let us denote by $\mathcal{T}[P, \alpha]$ the collection of all X_i. The *internal accuracy* of (P, α) is defined as

$$\text{int-acc}(P, \alpha) \coloneqq \max\left\{|X_{i+1} \setminus X_i| - 1 \mid i \in [0, p-1]\right\},$$

while its *accuracy* is the value

$$\mathrm{acc}(P, \alpha) \coloneqq \max\Big(\left\{|X_0| - \mathrm{mp}(\partial_B(X_0)) + 1, |\overline{X_{p-1}}| - \mathrm{mp}(\partial_B(X_p)) + 1\right\} \cup \left\{\text{int-acc}(P, \alpha)\right\}\Big)$$

The $\mathcal{T}'$-path (P, α) is *almost accurate* if $\text{int-acc}(P, \alpha) \leq 1$. Similarly, it is *accurate* if $\mathrm{acc}(P, \alpha) = 1$.

Note that in case (P, α) is an accurate $\mathcal{T}(B)$-path, X_0 and $\overline{X_p}$ are monochromatic.

Lemma 6.1.4 (X^*)**.** Let B a bipartite matching covered graph with at least six vertices. Then $\mathrm{slpmw}(B) \leq k$ if and only if there exists an accurate $\mathcal{T}(B)$-path of width at most k.

Proof. Let (T, δ) be a strict linear perfect matching decomposition (T, δ) for B where every $e \in E(\text{spine}(\text{spine}(T)))$ induces a cut of matching porosity at most k, and let $P' \coloneqq \text{spine}(\text{spine}(T))$. Moreover, let $q+2$ be the number of vertices in P' and $P' = (t_0, t_1, \dots, t_{q+1})$. For each $i \in [0, q]$ let T_i be the subtree of $T - t_i t_{i+1}$ that contains t_i. We set $\alpha'(t_i t_{i+1}) \coloneqq \delta(T_i)$ for every $i \in [0, q]$. Since (T, δ) is a strict linear perfect matching decomposition, $\partial_B(e)$ is a generalised tight cut for every $e \in E(P')$ and thus $\alpha(e) \in \mathcal{T}(B)$ for every $e \in E(P')$. Notice that for every $i \in [0, q-1]$ we have $|\delta(T_{i+1} \setminus \delta(T_i))| \leq 2$ since (T, δ) is a linear perfect matching decomposition. Additionally note that every cut induced by an edge of T has matching porosity at most k. Hence (P', α') is an almost accurate $\mathcal{T}(B)$-path of width at most k. Now consider the sets X_0 and X_q for (P', α'). Both of them have size at most four. In case both are monochromatic we have $|X_i| - \text{mp}(\partial_B(X_i)) + 1 = 1$ for each $i \in \{1, q\}$ and thus (P', α') is accurate and we are done. Suppose this is not the case. Let $i \in \{1, q\}$ be chosen such that X_i is not monochromatic. We claim that $B[X_i]$ is a subgraph of a claw. Suppose it is not, then X_i must have two vertices from each colour class. Since X_i induces a generalised tight cut this means that $\partial_B(X_i) = \emptyset$. But with $|X_i| \leq 4$ and $|V(B)| \geq 6$ and B being connected this is impossible and our claim follows. Let $v_i \in X_i$ be the unique minority vertex in X_i and $u_i \in X_i \setminus \{v_i\}$ be any other vertex. If X_0 is not monochromatic we introduce the vertex t_{-1} and the edge $t_{-1}t_0$ to P' and define the set $X_{-1} \coloneqq X_0 \setminus \{v_0, u_0\}$. Additionally, if X_q is not monochromatic we introduce the vertex t_{q+2} and the edge $t_{q+1}t_{q+2}$ to P' and define the set $X_{q+1} \coloneqq X_q$ we then delete v_q and u_q from X_q. Let P be the resulting path and α obtained by augmenting α' with $\alpha(t_{-1}t_0) \coloneqq X_{-1}$ in case X_0 is not monochromatic, and $\alpha(t_{q+1}t_{q+2}) \coloneqq X_{q+1}$ in case X_q is not monochromatic. The resulting (P, α) is now an accurate $\mathcal{T}(B)$-path of width at most k as desired.

For the reverse let (P, α) be an accurate $\mathcal{T}(B)$-path of width at most k. We consider the sets $X_i \in \mathcal{T}[P, \alpha]$. We are going to iteratively grow $P = (t_0, t_1, \dots, t_{p+1})$ into a cubic tree T and define $\delta \colon \mathsf{L}(T) \to V(B)$ on its leaves. Please note that we may assume the X_i to be distinct as otherwise we simply shorten P and remove the duplicates.

First consider X_0. As X_0 is monochromatic we may proceed as follows: Let P_0 be a path on $|X_0| - 1$ vertices such that P_0 and P share exactly the vertex t_0. For every internal vertex $t \in V(P_0)$ add a vertex d_t and the edge td_t. Add the vertex d_0 and the edge t_0d_0. At last let t' be the other endpoint of P_0. We add two vertices d_1 and d_2 and the edges $t'd_1$, $t'd_2$. The result is a subcubic tree, call it T_0', with exactly $|X_0|$ leaves. We define δ to be a bijection between the leaves of T_0' and X_0. At last let T_0 be the tree obtained from P by adding T_0'.

Let, for some $i \in [0, p-1]$, T_i be the tree grown from P so far such that δ covers all vertices in X_i. Suppose i is chosen maximal with this property. Then, as (P, α) is accurate, $|X_{i+1} \setminus X_i| \leq 2$. In case $X_{i+1} \setminus X_i = \{v\}$ we introduce a new vertex d as a leaf adjacent to t_{i+1} and map d via δ to v. Otherwise $X_{i+1} = \{v_1, v_2\}$ and we first introduce a new neighbour d for t_{i+1} and then add leaves d_1 and d_2 adjacent to d. Set $\delta(d_i) := v_i$ for both $i \in [1, 2]$. Let T_{i+1} be the newly constructed tree.

Finally we need to consider $\overline{X_p}$ which again is monochromatic. Here we may proceed analogously to X_0 and finally obtain a linear perfect matching decomposition (T, δ) for B. Moreover, every edge e of $\text{spine}(\text{spine}(T))$ is an edge of P, P_0, or the path P_p obtained for $\overline{X_p}$, and the cut induced by this edge equals the cut induced by $\alpha(e)$, or one of its shores is monochromatic and of size at most k. Hence the strictness of (T, δ) follows from (P, α) being a $\mathcal{T}(B)$-path. Similarly, every $e \in E(\text{spine}(\text{spine}(T)))$ must induce a cut of matching porosity at most k by the width of (P, α). □

If we require both the strict linear perfect matching decomposition and the $\mathcal{T}(B)$-path to be ordered, following the arguments in the proof of Lemma 6.1.4 we also obtain the next result.

Lemma 6.1.5 (X*). Let B a bipartite matching covered graph on at least six vertices. Then $\text{olpmw}(B) \leq k$ if and only if there exists an ordered and accurate $\mathcal{T}(B)$-path of width at most k.

Let us examine accurate $\mathcal{T}(B)$-paths that are not ordered more closely. Let P be a path, we say that the edges $e, e' \in E(P)$ are *consecutive* if they share exactly one endpoint.

Lemma 6.1.6 (X*). Let B be a matching covered bipartite graph and (P, α) an accurate $\mathcal{T}(B)$-path of width k and $e, e' \in E(P)$ be consecutive edges such that $\alpha(e) \subseteq \alpha(e')$, and $\text{Maj}(\alpha(e)) \subseteq \text{Min}(\alpha(e')) \subseteq V_i$ for some $i \in [1, 2]$. Then $\alpha(e)$ induces a tight cut.

Proof. Without loss of generality let us assume $\mathrm{Maj}(\alpha(e)) \subseteq V_1$. Let $X_f := \alpha(f)$ for both $f \in \{e, e'\}$. We first examine $Y := X_{e'} \setminus X_e$.

Suppose there exists some $v \in Y \cap V_1$, then $v \in \mathrm{Min}(\overline{X_e})$ and by Lemma 3.1.58 v cannot have a neighbour in X_e. Since $v \in \mathrm{Min}(X_{e'})$, all neighbours of v must be in Y. As $|Y| \leq 2$ since (P, α) is accurate this means that v has at most one neighbour which is impossible with B being matching covered and Theorem 3.1.67. So $Y \subseteq V_2$. Moreover, notice that by Lemma 3.1.58 no vertex of X_e can have a neighbour in $\overline{X_{e'}}$.

Suppose there is a perfect matching M of B with $|M \cap \partial_B(X_e)| \geq 2$. Since $|Y| \leq 2$, $\mathrm{N}_B(X_e \cap V_1) \subseteq X_{e'}$, and $\mathrm{N}_B(\mathrm{Min}(X_e)) \subseteq X_e$ we also know $|M \cap \partial_B(X_e)| \leq 2$. So we may assume that equality holds. In this case however $M \cap \partial_B(X_{e'}) = 0$ and thus $X_{e'}$ is balanced. This means $\overline{X_{e'}}$ is empty which contradicts the definition of accurate $\mathcal{T}(B)$-paths. Hence $|M \cap \partial_B(X_e)| = 1$. Consequently, $|X_e|$ is odd and thus it induces a tight cut. □

We need a special operation on $\mathcal{T}(B)$-paths to transform any accurate $\mathcal{T}(B)$-path into an ordered one. The idea here is to 'flip' a segment for which the sets mapped by α have an undesired majority. If $\mathrm{Maj}(X) \subseteq V_2$, then $\mathrm{Maj}(\overline{X}) \subseteq V_1$ while $\partial_B(X) = \partial_B(\overline{X})$. So hopefully our flipping operation does not change the width too much.

Let B be a matching covered bipartite graph and (P, α) be a $\mathcal{T}(B)$-path. Moreover, let Q be a subpath of P. The *flip* of Q in (P, α) is the $\mathcal{T}(B)$-path (P, α') obtained as follows. Let $Q = (q_0, q_1, \ldots, q_{p+1})$, and let X_i be the set $\alpha(q_i q_{i+1})$ for every $i \in [0, p]$, and $X_{-1} := \emptyset$. Finally let e_0 be the edge of P, if it exists, incident with q_0 but not with q_1. We set $Y_{-1} := \alpha(e_0)$ and in case e_0 does not exist we set $Y_{-1} := \emptyset$. For every $e \in E(P) \setminus E(Q)$ we set $\alpha'(e) := \alpha(e)$, and for every $i \in [0, p]$ we set $Y_i := Y_{i-1} \cup (X_{p-i} \setminus X_{p-1-i})$. Then $\alpha'(q_i q_{i+1}) := Y_i$ for all $i \in [0, p]$.

Theorem 6.1.7 (X*)**.** Let B be a bipartite matching covered graph. Then $\mathrm{slpmw}(B) \leq \mathrm{olpmw}(B) \leq \mathrm{slpmw}(B) + 2$.

Proof. By Lemma 6.1.4 there exists an accurate $\mathcal{T}(B)$-path (P, α) of width $\mathrm{slpmw}(B)$. Let $e_0 \in E(P)$ be an end-edge of P such that $\alpha(e_0)$ contains at most four vertices. Without loss of generality let us assume $\mathrm{Maj}(\alpha(e_0)) \subseteq V_1$. We call a subpath $Q = (q_0, q_1, \ldots, q_{p+1})$ of P a *segment* if it is a maximal subpath of P where $\mathrm{Maj}(\alpha(q_iq_{i+1})) \subseteq V_2$ for every $i \in [0, p-2]$ and $\mathrm{Maj}(\alpha(q_pq_{p+1})) \subseteq V_1$. By e_Q we denote the unique edge of P incident with q_0 but not with q_1. Note that e_Q must exist and with $Y_Q := \alpha(e_Q)$ we have $\mathrm{Maj}(Y_Q) \subseteq V_1$ by the maximality of Q. For any edge $e \in E(P)$ we say that e is *right of* Q if it belongs to the component of $P - q_pq_{p+1}$ that does not contain e_Q, and it is *left of* Q if $e \in E(P) \setminus E(Q)$ and e is not right of Q. If P has no segment, it is already an ordered $\mathcal{T}(B)$-path, so we may assume that P has at least one segment. In this case let Q be the segment of P closest to e_0 and let (P, α') be the flip of Q in (P, α). Note that for every edge $e \in E(P)$ right of Q we have $\alpha'(e) = \alpha(e)$. Hence it suffices to show that $\alpha'(q_iq_{i+1})$ is a generalised tight cut of matching porosity at most $\mathrm{slpmw}(B) + 2$ and that (P, α') has less segments than (P, α). The claim then follows from the fact that we did not change any cut that was not induced by an edge of Q and that any two segments of (P, α) have disjoint edge sets. Consider the set $Y_0 = \alpha(e_Q) \cup (X_p \setminus X_{p-1})$. Since $\mathrm{Maj}(X_{p-1}) \subseteq \mathrm{Min}(X_p) \subseteq V_2$, by Lemma 6.1.6 we know that X_{p-1} induces a tight cut. Moreover, from the proof of Lemma 6.1.6 we can deduce that $X_p \setminus X_{p-1} \subseteq V_1$. Hence Y_0 induces a generalised tight cut, $\mathrm{Maj}(Y_0) \subseteq V_1$, and, as $|X_p \setminus X_{p-1}| \leq 2$ and $\mathrm{mp}(\partial_B(\alpha(e_Q))) \leq \mathrm{slpmw}(B)$, we have $\mathrm{mp}(\partial_B(Y_0)) \leq \mathrm{slpmw}(B) + 2$. Let $i \in [1, p]$ and assume that for every $j \in [0, i-1]$, Y_j induces a generalised tight cut, $\mathrm{Maj}(Y_j) \subseteq V_1$, and $\mathrm{mp}(\partial_B(Y_j)) \leq \mathrm{slpmw}(B) + 2$. If $i = p$ then $Y_i = X_p$ and since, by the definition of segments, $\mathrm{Maj}(X_p)$ and X_p is a set that belongs to an accurate $\mathcal{T}(B)$-path of width $\mathrm{slpmw}(B)$ our claim follows. Consequently we may assume $i \leq p - 1$. Since X_{p-1-i} induces a generalised tight cut and $\mathrm{Maj}(X_{p-1-i}) \subseteq V_2$ we know $\mathrm{Maj}(\overline{X_{p-1-i}}) \subseteq V_1$. Moreover, with X_p inducing a tight cut and having its majority in V_1 we know that $\mathrm{Maj}(\overline{X_{p-1-i}} \setminus \overline{X_p}) \subseteq V_1$. To see this consider the graph $B/(\overline{X_p} \rightarrow v)$ and the set $(\overline{X_{p-1-i}} \setminus \overline{X_p}) \cup \{v\}$. Next we need to show that $\mathrm{N}_B(Y_i \cap V_2) \subseteq Y_i$. Let $u \in Y_i \cap V_2$ be a vertex with a neighbour v in $\overline{Y_i}$. Then $u \in Y_i \setminus \alpha(e_Q)$. Let $j \in [0, i]$ be the smallest number such that $u \in Y_j$. As X_p has its majority in V_1 and induces a tight cut, v must belong to $X_{p-1-j} \setminus \alpha(e_Q)$. But then there is a vertex in the minority of X_{p-1-j} with a neighbour in $\overline{X_{p-1-j}}$ contradicting the fact that X_{p-1-j} induces a generalised tight cut and Lemma 3.1.58. Hence Y_i induces a generalised tight cut in B.

So it is left to show $\mathrm{mp}(\partial_B(Y_i)) \leq \mathrm{slpmw}(B) + 2$. Note that $\partial_B(Y_i) \subseteq \partial_B(\alpha(e_Q)) \cup \partial_B(X_p) \cup \partial_B(X_{p-1-i})$. As $\partial_B(\alpha(e_Q))$ and $\partial_B(X_{p-1-i})$ both must be tight cuts by Lemma 6.1.6, and $\mathrm{mp}(\partial_B(X_{p-1-i})) \leq \mathrm{slpmw}(B)$, no perfect matching can have more than $\mathrm{slpmw}(B) + 2$ edges in $\partial_B(\alpha(e_Q)) \cup \partial_B(X_p) \cup \partial_B(X_{p-1-i})$. Therefore $\mathrm{mp}(\partial_B(Y_i)) \leq \mathrm{slpmw}(B) + 2$ and we are done. Our claim follows by induction. Clearly in (P, α') no segment can contain an edge of Q or left of Q. Hence every edge that belongs to such a segment must lie right of Q and must also be a segment of (P, α). Thus we have reduced the number of segments and the assertion follows. $\square$

Shifting $\mathcal{T}$-Paths As we have seen in the previous subsection, strict linear perfect matching width and ordered linear perfect matching width differ only by a small additive constant. Hence any notion that is dual to ordered linear perfect matching width must also be dual to the less restrictive strict variant. The next step towards such a dual notion is to adapt some of the tools from [Erd20] for the matching setting. Let B be a matching covered bipartite graph and (P, α) be an ordered $\mathcal{T}(B)$-path with edges e_0 and e_p such that $\alpha(e_0) \subseteq \alpha(e_0)$. We say that e_0 is the *initial edge* and $X_0 \coloneqq \alpha(e_p)$ is the *initial set*, while e_p is the *terminal edge* and $X_p \coloneqq \alpha(e_p)$ is the *terminal set* of (P, α). Let $P = (t_0, \ldots, t_{p+1})$. We fix our notation from earlier and set $X_i \coloneqq \alpha(t_i t_{i+1})$ for all $i \in [0, p]$. Suppose $\mathrm{Maj}(X_0) \subseteq V_2$, then $\mathrm{Maj}(\overline{X_p}) \subseteq V_1$. Hence by 'reversing' the path P and replacing every set X_i with its complement we obtain a new ordered $\mathcal{T}(B)$-path (P, α') of the same width and accuracy where $\mathrm{Maj}(X_0') \subseteq V_1$. This means we may always assume $\mathrm{Maj}(X_0) \subseteq V_1$ when we consider an ordered $\mathcal{T}(B)$-path.

Definition 6.1.8 (Up- and Down-Shifts)**.** Let (P, α) be as above and $i \in [0, p]$ as well as $Y \in \mathcal{T}(B)$ with $X_i \subseteq Y$ and $\mathrm{Maj}(Y) \subseteq V_1$. The *up-shift* of (P, α) *onto* Y *at* X_i is the $\mathcal{T}(B)$-path (P', α') where $P' = t_i P$ and $\alpha'(t_j t_{j+1}) \coloneqq X_j \cup Y$ for all $j \in [i, p]$. By Lemma 3.1.59 it follows immediately that (P', α') is an ordered $\mathcal{T}(B)$-path. Moreover, Y is the initial set of (P', α'). Next let $i \in [0, p]$ and $Y \subseteq \mathcal{T}(B)$ such that $Y \subseteq X_i$ and $\mathrm{Maj}(Y) \subseteq V_1$. We define the *down-shift* of (P, α) *onto* Y *at* X_i as the $\mathcal{T}(B)$-path (P', α) where $P' \coloneqq P t_{i+1}$ and $\alpha'(t_j t_{j+1}) \coloneqq Y \cap X_j$ for all $j \in [0, i+1]$. Again it is straightforward to see that (P', α') is an ordered $\mathcal{T}(B)$-path with terminal set Y.

If (P,α) is a $\mathcal{T}(B)$-path of width k, then it is in fact a $\mathcal{T}_k(B)$-path. Let (P',α') be an up- or down-shift of (P,α). While (P',α') still is a $\mathcal{T}(B)$-path, it is not necessarily still a $\mathcal{T}_k(B)$-path as the matching porosity of some of the induced cuts might increase. Still if $X,Y \in \mathcal{T}(B)$ satisfy $\mathrm{Maj}(X) \cup \mathrm{Maj}(Y) \subseteq V_i$ for some $i \in [1,2]$, Lemma 3.1.59 provides the following equality:

$$\mathrm{mp}(\partial_B(X)) + \mathrm{mp}(\partial_B(Y)) = \mathrm{mp}(\partial_B(X \cap Y)) + \mathrm{mp}(\partial_B(X \cup Y)).$$

From now an, if we are given two sets $X,Y \in \mathcal{T}(B)$ we say that X and Y are *aligned* if $\mathrm{Maj}(X) \cup \mathrm{Maj}(Y) \subseteq V_i$ for some $i \in [1,2]$. Note that, if C and D are aligned we implicitly mean $C,D \in \mathcal{T}(B)$.

Let $X,Y \in \mathcal{T}(B)$ be aligned with $X \subseteq Y$, we set

$$\mathrm{sep}(X,Y) \coloneqq \min\left\{\mathrm{mp}(\partial_B(Z)) \mid X \subseteq Z \subseteq Y \text{ and } X, Y, \text{ and } Z \text{ are aligned}\right\}.$$

We say that a set $Z \subseteq V(B)$ is *up-linked* to $X \in \mathcal{T}(B)$ if X and Z are aligned, $X \subseteq Z$, and $\mathrm{mp}(\partial_B(Z)) = \mathrm{sep}(X,Z)$. A set $Z \subseteq V(B)$ is *down-linked* to $X \in \mathcal{T}(B)$ if X and Z are aligned, $Z \subseteq X$, and $\mathrm{mp}(\partial_B(Z)) = \mathrm{sep}(Z,X)$.

Lemma 6.1.9 (X*, see Lemma 10 from [Erd20])**.** Let B be a bipartite matching covered graph and (P,α) a $\mathcal{T}_k(B)$-path with $P = (t_0,\dots,t_{p+1})$. If $Z \in \mathcal{T}(B)$ is up-linked to X_i for some $i \in [0,p]$, then the up-shift of (P,α) onto Z at X_i is a $\mathcal{T}_k(B)$-path, and if Z is down-linked to X_i for some $i \in [0,p]$, then the down-shift of (P,α) onto Z at X_i is a $\mathcal{T}_k(B)$-path

Proof. We only consider the first case as the second one follows along similar lines. So let (P',α') be the up-shift of (P,α) onto Z at X_i. We already know that (P',α') is a $\mathcal{T}(B)$-path, hence it suffices to show $\mathrm{width}(P',\alpha') \leq \mathrm{width}(P,\alpha) \leq k$. Since Z is up-linked to X_i we know $\mathrm{mp}(\partial_B(Z)) \leq \mathrm{mp}(\partial_B(X_i))$. Let $j \in [i,p]$, we aim to prove $\mathrm{mp}(\partial_B(Z \cup X_j)) \leq \mathrm{width}(P,\alpha)$. Note that with $X_i \subseteq X_j$ and $X_i \subseteq Z$ we have $X_i \subseteq X_j \cap Z \subseteq Z$. Consequently, since Z is up-linked to X_i, $\mathrm{mp}(\partial_B(X_j \cap Z)) \geq \mathrm{mp}(\partial_B(Z))$ and thus, by Lemma 3.1.59, we must have

$$\mathrm{mp}(\partial_B(X_j \cup Z)) \leq \mathrm{mp}(\partial_B(X_j)) \leq \mathrm{width}(P,\alpha) \leq k.$$

As j was chosen arbitrarily the claim follows. □

So the width of a $\mathcal{T}(B)$-path can be preserved by certain shifts, but what about its accuracy or at least its internal accuracy? Indeed, since every up- or down shift of a $\mathcal{T}(B)$-path onto some set Z has Z as its initial or terminal set, and, as Z can be of arbitrary size, we will probably not be able to preserve the accuracy. So let us settle for the internal accuracy.

Lemma 6.1.10 (X*, see Lemma 11 from [Erd20])**.** Let B be a bipartite matching covered graph and (P, α) a $\mathcal{T}_k(B)$-path with $P = (t_0, \dots, t_{p+1})$. If $Z \in \mathcal{T}(B)$ is up-linked to X_i for some $i \in [0, p]$, then the up-shift of (P, α) onto Z at X_i is of the same internal accuracy as (P, α), and $|\overline{X_p \cup Z}| \leq |\overline{X_p}|$.
Similarly, if Z is down-linked to X_i for some $i \in [0, p]$, then the down-shift of (P, α) onto Z at X_i is of the same internal accuracy as (P, α), and $|X_0 \cap Z| \leq |X_0|$.

Proof. As we did for Lemma 6.1.9, we only prove the first part of the assertion as the second one can be derived analogously. Let (P', α') be the up-shift of (P, α) onto Z at X_i, then $P' = t_i P$. The bound $|\overline{X_p \cup Z}| \leq |\overline{X_p}|$ on the terminal set follows immediately, so we only have to show $|(X_{j+1} \cup Z) \setminus (X_j \cup Z)| \leq |X_{j+1} \setminus X_j|$ for all $j \in [i, p]$. Clearly this is true and thus we are done. ☐

6.1.2. ω-Blockages and $\mathcal{T}_k$-Paths of Accuracy ω

For $\mathcal{T}(B)$-paths we are suddenly confronted with two parameters one might strife to optimise: the width and the accuracy. The width describes the bound on the matching porosity of the cuts involved, while the accuracy describes how large the 'chunks' of our graph B are that are separated by those cuts. In an ordered perfect matching decomposition of optimal width both of these parameters need to be minimised. The version of blockages we present in this subsection, derived from the definition of diblockages in [Erd20], aims to act as a general concept of duality for both of these parameters.

In the most general sense, a tangle is an orientation of the 'separations' in some abstract system of separations. That is, each separation has two sides, one from which it points away – commonly called the *small side* – and one it points towards – which then is called the *large side*. In our case the 'separations' are generalised tight cuts which can be represented by the sets in $\mathcal{T}(B)$.

Definition 6.1.11 (Orientations of Generalised Tight Cuts)**.** Let B be a matching covered bipartite graph and $k \in \mathbb{N}$ be a positive integer. A *partial orientation* of $\mathcal{T}_k(B)$ is a set $\mathfrak{O} \subseteq \mathcal{T}_k(B)$ such that if $X \in \mathfrak{O}$, then $\overline{X} \notin \mathfrak{O}$. If $X \in \mathfrak{O}$, we call X the *small side* of $\partial_B(X)$.

A partial orientation $\mathfrak{O}$ of $\mathcal{T}_k(B)$ is an *orientation* if for every $X \in \mathcal{T}_k(B)$ we either have $X \in \mathfrak{O}$ or $\overline{X} \in \mathfrak{O}$.

Given a partial orientation $\mathfrak{P}$ of $\mathcal{T}_k(B)$ we write $\mathcal{T}_{\mathfrak{P}}$ for the set

$$\mathcal{T}_k(B, \mathfrak{P}) \coloneqq \left\{ X \in \mathcal{T}_k(B) \mid \text{neither } X \in \mathfrak{P}\text{, nor } \overline{X} \in \mathfrak{P} \right\}.$$

To be able to work with orientations of $\mathcal{T}_k(B)$ in a meaningful way we need to impose some additional requirements.

Definition 6.1.12 (Consistent Orientations)**.** Let B be a matching covered bipartite graph, $k \in \mathbb{N}$ be a positive integer, and $\mathfrak{P}$ a partial orientation of $\mathcal{T}_k(B)$. We say that $\mathfrak{P}$ is *consistent* if $X \in \mathfrak{P}$ and $Y \in \mathcal{T}_k(B)$ such that X and Y are aligned and $Y \subseteq X$ implies $Y \in \mathfrak{P}$.

Note that our definition of consistency also implies that in case $\overline{X} \in \mathfrak{P}$ and $Y \in \mathcal{T}_k(B)$ such that X and Y are aligned and $X \subseteq Y$, then $\overline{Y} \in \mathfrak{P}$.

At last let $\mathfrak{P}$ be a partial orientation of $\mathcal{T}_k(B)$. An orientation $\mathfrak{O}$ of $\mathcal{T}_k(B)$ *extends* $\mathfrak{P}$ if $\mathfrak{P} \subseteq \mathfrak{O}$. Let $\omega \in \mathbb{N}$ be a positive integer. We fix a root for our orientations in the sense that we require all generalised tight cuts with an almost trivial, with respect to ω, shore to be oriented away from said shore.

$$\mathfrak{R}_\omega \coloneqq \{X \in \mathcal{T}_k(G) \mid |X| \leq \omega + \mathrm{mp}(\partial_B(X) - 1)\}.$$

To make sure $\mathfrak{R}_\omega$ is well defined as a partial orientation of $\mathcal{T}_k(B)$, we require our bipartite matching covered graph B to have at least $2\omega + 2k$ vertices.

Definition 6.1.13 (ω-Blockage)**.** Let $\omega, k \in \mathbb{N}$ be two positive integers, and let B be a matching covered bipartite graph with $|V(B)| \geq 2\omega + 2k$. An ω-*blockage* of *order* k is an orientation $\mathfrak{B}$ of $\mathcal{T}_k(B)$ such that

i) $\mathfrak{B}$ extends $\mathfrak{R}_\omega$,
ii) $\mathfrak{B}$ is consistent, and
iii) if $X, Y \in \mathfrak{B}$ such that X and $\overline{Y}$ are aligned, and $X \subseteq \overline{Y}$, then $|\overline{X} \cap \overline{Y}| \geq \omega + 2$.

A 1-blockage of order k is called a *blockage* of order k. The blockage number of a bipartite matching covered graph B, denoted by block(B), is the largest integer k such that B has a blockage of order k.

Our goal is to show a duality between the existence of an ordered $\mathcal{T}_k(B)$-path of accuracy ω and that of an ω-blockage of order k. For this we need one additional definition to bridge between $\mathcal{T}_k(B)$-paths that respect certain partial orientations of $\mathcal{T}_k(B)$ and ω-blockages.

Let $\mathfrak{P}$ be a partial orientation of $\mathcal{T}_k(B)$ and $\omega \in \mathbb{N}$ a positive integer. We say that a $\mathcal{T}_k(B)$-path (P, α) with $P = (t_0, t_1, \dots, t_{p+1})$ is $(\omega, \mathfrak{P})$-*admissible* if

i) (P, α) is ordered,
ii) int-acc$(P, \alpha) \leq \omega$,
iii) $X_0 \in \mathfrak{P} \cup \mathfrak{R}_\omega$, and
iv) $\overline{X_p} \in \mathfrak{P} \cup \mathfrak{R}_\omega$.

Note that an $(\omega, \mathfrak{R}_\omega)$-admissible $\mathcal{T}_k(B)$-path (P, α) must be of accuracy at most ω.

Theorem 6.1.14 (X*, see Theorem 12, [Erd20])**.** Let $k, \omega \in \mathbb{N}$ be positive integers, and B be a matching covered bipartite graph on at least $2\omega + 2k$ vertices. Then exactly one of the following holds:

i) either B has an ordered $\mathcal{T}_k(B)$-path of accuracy ω, or
ii) there exists an ω-blockage of $\mathcal{T}_k(B)$.

Proof. As in the proof of Theorem 12 from [Erd20] we prove a slightly stronger statement. We show that for every consistent partial orientation $\mathfrak{P}$ of $\mathcal{T}_k(B)$ which extends $\mathfrak{R}_\omega$ exactly one of the following holds:

- either there exists an $(\omega, \mathfrak{P})$-admissible $\mathcal{T}_k(B)$-path, or
- there is an ω-blockage of $\mathcal{T}_k(B)$ which extends $\mathfrak{P}$.

Since an $(\omega, \mathfrak{R}_\omega)$-admissible $\mathcal{T}_k(B)$-path (P, α) must be of accuracy at most ω the statement of the theorem follows from our new claim.

We start out by showing that both statements cannot be true at the same time. Towards a contradiction let us assume that there exists an $(\omega, \mathfrak{P})$-admissible $\mathcal{T}_k(B)$-path (P, α) with $P = (t_0, t_1, \dots, t_{p+1})$, and an ω-blockage $\mathfrak{B}$ extending $\mathfrak{P}$. As $\mathfrak{B}$ extends $\mathfrak{P}$ and (P, α) is $(\omega, \mathfrak{P})$-admissible, we have $X_0, \overline{X_p} \in \mathfrak{P} \cup \mathfrak{R}_\omega \subseteq \mathfrak{B}$. Let $i \in [1, p]$ be chosen maximal with the property $X_i \in \mathfrak{B}$, then we must have $i \leq p-1$ as $X_p \notin \mathfrak{B}$. Hence $\overline{X_{i+1}} \in \mathfrak{B}$ and since $X_i \subseteq X_{i+1}$ we must have

$$|X_{i+1} \setminus X_i| = |\overline{X_i} \cap X_{i+1}| = |\overline{X_i} \cap \overline{\overline{X_{i+1}}}| \geq \omega + 2.$$

This however means int-acc$(P, \alpha) \geq \omega + 1$, contradicting our assumption that (P, α) is $(\omega, \mathfrak{P})$-admissible.

We prove the claim by induction on $|\mathcal{T}_k(B, \mathfrak{P})|$. So first let us assume $|\mathcal{T}_k(B, \mathfrak{P})| = 0$. In this case $\mathfrak{P}$ is a consistent orientation of $\mathcal{T}_k(B)$ and thus if it is an ω-blockage of $\mathcal{T}_k(B)$ we are done. Consequently we may assume $\mathfrak{P}$ to not be an ω-blockage. Since $\mathfrak{R}_\omega \subseteq \mathfrak{R}$ there must exist $X, Y \in \mathfrak{P}$ with X and $\overline{Y}$ being aligned, $X \subseteq \overline{Y}$, and $|\overline{Y} \setminus X| = |\overline{X} \cap \overline{Y}| \leq \omega + 1$. Let $P = (t_0, t_1, t_2)$, $\alpha(t_0 t_1) := X$, and $\alpha(t_1 t_2) := \overline{Y}$, then (P, α) is indeed an $(\omega, \mathfrak{P})$-admissible path and we are done.

Therefore we may assume $|\mathcal{T}_k(B, \mathfrak{P})| \geq 1$. Let us also assume that there is no ω-blockage $\mathfrak{B}$ of $\mathcal{T}_k(B)$ that extends $\mathfrak{P}$. Now there must exist some $A \in \mathcal{T}_k(B)$ such that neither $A \in \mathfrak{P}$ nor $\overline{A} \in \mathfrak{P}$. Let us choose $C_1, C_2 \in \mathcal{T}_k(B, \mathfrak{P})$ such that A, C_1, and C_2 are pairwise aligned, $C_1 \subseteq A$ is minimal, and $C_2 \supseteq A$ is maximal.

Let $\mathfrak{P}_1 := \mathfrak{P} \cup \{C_1\}$, and $\mathfrak{P}_2 := \mathfrak{P} \cup \left\{\overline{C_2}\right\}$, then $\mathfrak{P}_i$ must be a consistent partial orientation of $\mathcal{T}_k(B)$ for both $i \in [1, 2]$. Indeed, by the minimality of C_1 we know that any $D \in \mathcal{T}_k(B)$ which is aligned with C_1 and satisfies $D \subset C_1$ must either belong to $\mathfrak{P}$, or $\overline{D}$ belongs to $\mathfrak{P}$. Since $C_1 \in \mathcal{T}_k(B, \mathfrak{P})$ and $\overline{C_1} \subset \overline{D}$, the consistency of $\mathfrak{P}$ implies $\overline{D} \notin \mathfrak{P}$ and thus $D \in \mathfrak{P}$. Hence $\mathfrak{P}_1$ is consistent. With an analogous argument one can deduce $\overline{D} \in \mathfrak{P}$ for all $D \in \mathcal{T}_k(B)$ which are aligned with C_2 and satisfy $C_2 \subset D$. With this also the consistency of $\mathfrak{P}_2$ is assured. So we may apply our induction hypothesis to both $\mathfrak{P}_1$ and $\mathfrak{P}_2$ as $|\mathcal{T}_k(B, \mathfrak{P}_i)| = |\mathcal{T}_k(B, \mathfrak{P})| - 1$ for both $i \in [1, 2]$.

Every ω-blockage of $\mathcal{T}_k(B)$ that extends $\mathfrak{P}_i$ for some $i \in [1, 2]$ also extends $\mathfrak{P}$ and thus we may assume that there is an $(\omega, \mathfrak{P}_i)$-admissible $\mathcal{T}_k(B)$-path (P_i, α_i) for each $i \in [1, 2]$. If any of these also is $(\omega, \mathfrak{P})$-admissible we are done immediately, so suppose neither of them is. The only way this is possible however is if C_1 is the initial set of (P_1, α_1), and C_2 is the terminal set of (P_2, α_2). The only other option would be that C_1 is the terminal set of (P_1, α_1) which, by the $(\omega, \mathfrak{P}_1)$-admissibility of (P_1, α), would imply $\overline{C_1} \in \mathfrak{P}_1 \cup \mathfrak{R}_\omega$ contradicting the definition of partial orientations. A similar argument can be made for C_2. Let $Z \in \mathcal{T}(B)$ be chosen to be aligned with C_1 and C_2 such that $C_1 \subseteq Z \subseteq C_2$, and $\mathrm{mp}(\partial_B(Z)) = \mathrm{sep}(C_1, C_2)$. Then Z is up-linked to C_1 and down-linked to C_2.

Let (P_1', α_1') be the up-shift of (P_1, α_1) onto Z at C_1, and (P_2', α_2') be the down-shift of (P_2, α_2) onto Z at C_2. Consequently, Z is the initial set of (P_1', α_1') and the terminal set of (P_2', α_2'). Let $P_i' = (t_0^i, t_1^i, \ldots, t_{p_i+1}^i)$ for both i. We create a new path

$$\hat{P} := (t_0^2, t_1^2, \ldots, t_{p_2}^2 = t_0^1, t_{p_2+1}^2 = t_1^1, t_2^1, \ldots, t_{p_1+1}^1),$$

and for each $e \in E(\hat{P})$ we set

$$\hat{\alpha}(e) := \begin{cases} \alpha_2(e), & \text{if } e \in E(P_2)\text{, or} \\ \alpha_1(e), & \text{otherwise.} \end{cases}$$

Note that by this construction we have $E(P_1) \cap E(P_2) \cap E(\hat{P}) = (t_{p_2}^2 t_{p_2+1}^2) = \{t_0^1 t_1^1\}$, $\alpha_1(t_0^1 t_1^1) = Z = \alpha_2(t_{p_2}^2 t_{p_2+1}^2)$, and $\hat{\alpha}(t_{p_2}^2 t_{p_2+1}^2) = Z$ as well.

We claim that $(\hat{P}, \hat{\alpha})$ is an $(\omega, \mathfrak{P})$-admissible $\mathcal{T}_k(B)$-path. Each (P_i, α_i) is an ordered $\mathcal{T}_k(B)$-path and since Z is up-linked to C_1 and down-linked to C_2, so is $(\hat{P}, \hat{\alpha})$ by Lemma 6.1.9. For each $i \in [1, 2]$ and $j \in [0, p_i]$ let us identify by X_j^i the set $\alpha_i(t_j^i t_{j+1}^i)$. Since each (P_i, α_i) is $(\omega, \mathfrak{P}_i)$-admissible we have $|X_{j+1}^i \setminus X_j^i| \leq \omega + 1$ for every $i \in [1, 2]$ and $j \in [0, p_i - 1]$. Hence by Lemma 6.1.10 we obtain $\text{int-acc}(\hat{P}, \hat{\alpha}) \leq \omega$.

So we are left with checking the initial set $\hat{X}_0 := X_0^2 \cap Z$ and the terminal set $\hat{X}_{\hat{p}} := X_{p_1}^1 \cup Z$ of $(\hat{P}, \hat{\alpha})$. Note that $X_0^2 \in \mathfrak{P} \cup \mathfrak{R}_\omega$ since (P_2, α_2) is $(\omega, \mathfrak{P}_2)$-admissible. Then, as $X_0^2 \cap Z \subseteq X_0^2$ we must also have $\hat{X}_0 \in \mathfrak{P}$. Similarly we have $\overline{X_{p_1}^1} \in \mathfrak{P}$ and with $X_{p_1}^1 \subseteq X_{p_1}^1 \cup Z$ it follows that $\overline{\hat{X}_{\hat{p}}} \subseteq \overline{X_{p_1}^1}$ which implies $\overline{\hat{X}_{\hat{p}}} \in \mathfrak{P}$ by the consistency of $\mathfrak{P}$. $\square$

By setting $\omega := 1$ we can combine Theorem 6.1.14 and Lemma 6.1.5 to obtain the following duality theorem on ordered linear perfect matching width.

Theorem 6.1.15 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, and B be a bipartite matching covered graph with at least $2k + 2$ vertices. Then $\mathrm{olpmw}(B) \leq k$ if and only if B has no blockage of order k.

6.2. Generalised Tight Cuts and Directed Separations

While some of the analogues are not necessarily straight forward to find, in general our definition of ω-blockages and the proofs towards Theorems 6.1.14 and 6.1.15 come from Erde's investigation of directed pathwidth. His ω-*diblockages* however are orientations of directed separations in digraphs rather than generalised tight cuts. By using lemmata 3.2.11 and 3.2.13 we are able switch back and forth between these two concepts, so an interesting question to ask would be:

Is there a direct way to obtain Theorem 6.1.15 from the findings of Erde [Erd20]?

Diblockages Indeed, if there is a way this would also immediately give a link between $\mathrm{olpmw}(B)$ and the directed pathwidth of an M-direction of B. Let us first introduce some of the notation of Erde for the setting of digraphs.

Let D be a digraph and (A, B), (E, F) be two directed separations in D. We write $(A, B) \leq (E, F)$ if $A \subseteq E$ and $B \supseteq F$. Note that the symbol '$\leq$' encodes a more restricted version of two directed separations being laminar. Let us denote by $\vec{\mathcal{S}}(D)$ the set of all directed separations in D, and for $k \in \mathbb{N}$ let $\vec{\mathcal{S}}_k(D)$ be the set of all directed separations[1] in D that are of order at most k.

While for generalised tight cuts we were able to only consider one of the two shores and therefore identify the generalised tight cut and its small or big side by a single set, the same is no longer true for directed separations in digraphs. Several different notations have been proposed to handle the problem of orienting a directed separation, which is already implicitly oriented by the direction the edges are allowed to bypass the separator. For our purposes it suffices to stick to Erde's notation. That is, an orientation of some set $\mathcal{S} \subseteq \vec{\mathcal{S}}(D)$ will be defined as a bipartitioned set $\mathfrak{O} = \mathfrak{O}^+ \cup \mathfrak{O}^-$ where the membership of $S \in \mathcal{S}$ to $\mathfrak{O}^+$ or $\mathfrak{O}^-$ determines which of the two entries is considered to be the 'large' side of S.

[1] Actually, Erde defined $\vec{\mathcal{S}}_k(D)$ to be the set of directed separations of order at most $k-1$ to be more in line with the definition of directed pathwidth where 1 is subtracted from $\mathrm{cops}_{w,iv}(D)$. However, our definition works better in the context of this thesis.

Definition 6.2.1 (Orientation of Directed Separations)**.** Let D be a digraph and $k \in \mathbb{N}$. A *partial orientation* of $\vec{\mathcal{S}}_k(D)$ is a pair $\mathfrak{P} = (\mathfrak{P}^+, \mathfrak{P}^-)$ of disjoint subsets of $\vec{\mathcal{S}}_k(D)$.

A partial orientation $\mathfrak{O} = (\mathfrak{O}^+, \mathfrak{O}^-)$ is an *orientation* of $\vec{\mathcal{S}}_k(D)$ if $\mathfrak{O}^+ \cup \mathfrak{O}^- = \vec{\mathcal{S}}_k(D)$.

We say that a partial orientation $\mathfrak{P} = (\mathfrak{P}^+, \mathfrak{P}^-)$ of $\vec{\mathcal{S}}_k(D)$ is *consistent* if it satisfies the following two requirements:

i) if $(A, B) \in \mathfrak{P}^+$ and $(A, B) \geq (E, F) \in \vec{\mathcal{S}}_k(D)$, then $(E, F) \in \mathfrak{P}^+$, and

ii) if $(A, B) \in \mathfrak{P}^-$ and $(A, B) \leq (E, F) \in \vec{\mathcal{S}}_k(D)$, then $(E, F) \in \mathfrak{P}^-$.

As a first step let us prove that the notions of partial orientations, orientations, and consistency are preserved by taking the split of a digraph D and taking the M-direction of a bipartite graph B.

Lemma 6.2.2 (X^*)**.** Let $k \in \mathbb{N}$ be a positive integer, B be a matching covered bipartite graph with a perfect matching M, and let $D \coloneqq \mathcal{D}(B, M)$. Then

$$\{X \in \mathcal{T}_k(B) \mid \mathrm{Maj}(X) \subseteq V_1\} = \left\{\mathcal{D}(E, F, M) \mid (E, F) \in \vec{\mathcal{S}}_k(D)\right\}, \text{ and}$$
$$\vec{\mathcal{S}}_k(D) = \{\mathcal{S}(X) \mid X \in \mathcal{T}_k(B) \text{ and } \mathrm{Maj}(X) \subseteq V_1\}.$$

Proof. Let (E, F) be a directed separation of order at most k in D. Then, by Lemma 3.2.11 $X \coloneqq \mathcal{S}(E, F)$ is a generalised tight cut with $\mathrm{mp}(\partial_B(X)) = |E \cap F| \leq k$ and $\mathrm{Maj}(X) \subseteq V_1$. Hence $X \in \{X \in \mathcal{T}_k(B) \mid \mathrm{Maj}(X) \subseteq V_1\}$. On the other hand let $X \in \mathcal{T}_k(B)$ such that $\mathrm{Maj}(X) \subseteq V_1$, then $(E, F) \coloneqq \mathcal{D}(X, M)$ is a directed separation of order $\mathrm{mp}(\partial_B(X))$ by Lemma 3.2.13. □

Lemma 6.2.3 (X^*)**.** Let $k \in \mathbb{N}$ be a positive integer, B be a matching covered bipartite graph with a perfect matching M, and let $D \coloneqq \mathcal{D}(B, M)$. Moreover, let $\mathfrak{P}_1 \subseteq \mathcal{T}_k(B)$ and $\mathfrak{P}_2 = (\mathfrak{P}_2^+, \mathfrak{P}_2^-)$ where $\mathfrak{P}_2^+$ and $\mathfrak{P}_2^-$ are disjoint subsets of $\vec{\mathcal{S}}_k(D)$. At last let

$$\mathfrak{P}_1' \coloneqq (\mathfrak{P}_1'^+, \mathfrak{P}_1'^-) \text{ where}$$
$$\mathfrak{P}_1'^+ \coloneqq \{\mathcal{D}(X, M) \mid X \in \mathfrak{P}_1 \text{ and } \mathrm{Maj}(X) \subseteq V_1\},$$
$$\mathfrak{P}_1'^- \coloneqq \left\{\mathcal{D}(\overline{X}, M) \mid X \in \mathfrak{P}_1 \text{ and } \mathrm{Maj}(X) \subseteq V_2\right\}, \text{ and}$$
$$\mathfrak{P}_2' \coloneqq \left\{\mathcal{S}(E, F), \overline{\mathcal{S}(G, H)} \mid (E, F) \in \mathfrak{P}_2^+ \text{ and } (G, H) \in \mathfrak{P}_2^-\right\}.$$

Then the following statements are true:

i) $\mathfrak{P}_1$ is a partial orientation of $\mathcal{T}_k(B)$ if and only if $\mathfrak{P}_1'$ is a partial orientation of $\vec{\mathcal{S}}_k(D)$.

ii) $\mathfrak{P}_1$ is an orientation of $\mathcal{T}_k(B)$ if and only if $\mathfrak{P}_1'$ is an orientation of $\vec{\mathcal{S}}_k(D)$.

iii) $\mathfrak{P}_1$ is a consistent partial orientation of $\mathcal{T}_k(B)$ if and only if $\mathfrak{P}_1'$ is a consistent partial orientation of $\vec{\mathcal{S}}_k(D)$.

iv) $\mathfrak{P}_2$ is a partial orientation of $\vec{\mathcal{S}}_k(D)$ if and only if $\mathfrak{P}_2'$ is a partial orientation of $\mathcal{T}_k(B)$.

v) $\mathfrak{P}_2$ is an orientation of $\vec{\mathcal{S}}_k(D)$ if and only if $\mathfrak{P}_2'$ is an orientation of $\mathcal{T}_k(B)$.

vi) $\mathfrak{P}_2$ is a consistent partial orientation of $\vec{\mathcal{S}}_k(D)$ if and only if $\mathfrak{P}_2'$ is a consistent partial orientation of $\mathcal{T}_k(B)$.

Proof. We only show the first three equivalences, the last three, so (iv) to (vi), follow along similar arguments by applying Lemma 6.2.2.

Towards (i) let us first assume $\mathfrak{P}_1$ to be a partial orientation of $\mathcal{T}_k(B)$. Note that $\mathfrak{P}_1'^{+} \cup \mathfrak{P}_1'^{-} \subseteq \vec{\mathcal{S}}_k(D)$ follows from Lemma 6.2.2. Suppose there exists some $(X, Y) \in \mathfrak{P}_1'^{+} \cap \mathfrak{P}_1'^{-}$. Then we must have $\mathrm{Maj}(\mathcal{S}(X, Y)) \subseteq V_1$ and thus $\mathcal{S}(X, Y) \in \mathfrak{P}_1$, but also $\mathrm{Maj}(\mathcal{S}(X, Y)) \subseteq V_2$ and thus $\overline{\mathcal{S}(X, Y)} \in \mathfrak{P}_1$. This however contradicts our assumption that $\mathfrak{P}_1$ is a partial orientation of $\mathcal{T}_k(B)$.

For the reverse we assume $\mathfrak{P}_1'$ to be a partial orientation of $\vec{\mathcal{S}}_k(D)$. Again by Lemma 6.2.2 we know that $\mathfrak{P}_1 \subseteq \mathcal{T}_k(B)$. Suppose there exists $X \in \mathcal{T}_k(B)$ with $X, \overline{X} \in \mathfrak{P}_1$. Without loss of generality we may assume $\mathrm{Maj}(X) \subseteq V_1$ and then $\mathcal{D}(X, M) \in \mathfrak{P}_1'^{+}$. As $\mathrm{Maj}(\overline{X}) \subseteq V_2$ we must have $\mathcal{D}(\overline{\overline{X}}, M) = \mathcal{D}(X, M) \in \mathfrak{P}_1^{-}$ which again contradicts our assumption. Thus $\mathfrak{P}_1$ must be a partial orientation of $\mathcal{T}_k(B)$.

The equivalence (ii) follows immediately from (i) and Lemma 6.2.2.

Towards (iii) we know by (i) that both $\mathfrak{P}_1$ and $\mathfrak{P}_1'$ are partial orientation of their respective sets of separations. Hence all we need to show is the consistency. Let us first assume $\mathfrak{P}_1$ to be consistent. Consider $(E, F) \in \mathfrak{P}_1'^{+}$ and $(G, H) \in \vec{\mathcal{S}}_k(D)$ and let us assume $(E, F) \geq (G, H)$. In this case we have $G \subseteq E$ and thus $\mathcal{S}(G, H) \subseteq \mathcal{S}(E, F)$. As $\mathcal{S}(E, F) \in \mathfrak{P}_1$ and $\mathfrak{P}_1$ is consistent we must have $\mathcal{S}(G, H) \in \mathfrak{P}_1$ as well. Consequently we find $(G, H) \in \mathfrak{P}_1'^{+}$. So now consider $(E, F) \in \mathfrak{P}_1'^{-}$ and $(E, F) \leq$

$(G,H) \in \vec{\mathcal{S}}_k(D)$. Then $E \subseteq G$ implying $\mathcal{S}(E,F) \subseteq \mathcal{S}(G,H)$ and therefore $\overline{\mathcal{S}(E,F)} \supseteq \overline{\mathcal{S}(G,H)}$. As $\overline{\mathcal{S}(E,F)} \in \mathfrak{P}_1$ we must, by consistency, also have $\overline{\mathcal{S}(G,H)} \in \mathfrak{P}_1$ and thus $(G,H) \in \mathfrak{P}_1'^{-}$. Therefore $\mathfrak{P}_1'$ is consistent.

For the reverse let us now assume $\mathfrak{P}_1'$ to be consistent and let $X, Y \in \mathcal{T}_k(B)$ be aligned with $X \subseteq Y$ and $Y \in \mathfrak{P}_1$. Suppose $\mathrm{Maj}(Y) \subseteq V_1$, then $\mathcal{D}(X,M) \leq \mathcal{D}(Y,M)$ and $\mathcal{D}(Y,M) \in \mathfrak{P}_1'^{+}$. As $\mathfrak{P}_1'$ is consistent this means $\mathcal{D}(X,M) \in \mathfrak{P}_1'$ as well. So we may assume $\mathrm{Maj}(Y) \subseteq V_1$ which means $\mathcal{D}(\overline{Y},M) \in \mathfrak{P}_1'^{-}$. Moreover, as $\overline{Y} \subseteq \overline{X}$ we also have $\mathcal{D}(\overline{Y},M) \leq \mathcal{D}(\overline{X},M)$ and therefore $\mathcal{D}(\overline{X},M) \in \mathfrak{P}_1'^{-}$. Hence $X \in \mathfrak{P}_1$ and thus $\mathfrak{P}_1$ is consistent. $\square$

Our condition for ω-blockages of $\mathcal{T}_k(B)$ requires the intersection of two large sides to still be somewhat big. But in the original definition regarding Theorem 2.2.21 instead of the large sides, the interaction of the small sides mattered. Next we are interested in how exactly these two conditions relate for the case of two separations that meet the requirements of blockages. To fully illustrate these requirements, let us first introduce ω-diblockages. Similar to the matching setting we require a certain minimal set of orientations. More precisely, if one of the two parts in a directed separation is too small we want this side to be the small side in every orientation. Let $\omega, k \in \mathbb{N}$ be two positive integers with $\omega \geq k$. We define $\mathfrak{R}_\omega = (\mathfrak{R}_\omega^+, \mathfrak{R}_\omega^-)$ by

$$\mathfrak{R}_\omega^+ := \left\{ (E,F) \in \vec{\mathcal{S}}_k(D) \mid |E| \leq \omega \right\}$$
$$\mathfrak{R}_\omega^- := \left\{ (E,F) \in \vec{\mathcal{S}}_k(D) \mid |F| \leq \omega \right\}.$$

Similar to before, we need to require our digraph D to have a minimum number of vertices, this time at least $2\omega - k + 2$, to ensure $\mathfrak{R}_\omega^+ \cap \mathfrak{R}_\omega^- = \emptyset$. As before, an orientation $\mathfrak{O} = (\mathfrak{O}^+, \mathfrak{O}^-)$ of $\vec{\mathcal{S}}_k(D)$ *extends* a partial orientation $\mathfrak{P} = (\mathfrak{P}^+, \mathfrak{P}^-)$ of $\vec{\mathcal{S}}_k(D)$ if $\mathfrak{P}^+ \subseteq \mathfrak{O}^+$ and $\mathfrak{P}^- \subseteq \mathfrak{O}^-$.

Definition 6.2.4 (ω-Diblockage)**.** Let $\omega, k \in \mathbb{N}$ be two positive integers satisfying $\omega \geq k$, and D be a digraph with at least $2\omega - k + 2$ vertices. An *ω-diblockage* of $\vec{\mathcal{S}}_k(D)$ is an orientation $\mathfrak{O} = (\mathfrak{O}^+, \mathfrak{O}^-)$ of $\vec{\mathcal{S}}_k(D)$ such that

i) $\mathfrak{O}$ extends $\mathfrak{R}_\omega$,
ii) $\mathfrak{O}$ is consistent, and
iii) if $(E,F) \in \mathfrak{O}^+$ and $(E,F) \leq (G,H) \in \mathfrak{O}^-$, then $|F \cap G| \geq \omega + 1$.

In case $\omega = k$, we call $\mathfrak{O}$ a *diblockage of order* k. We denote by $\mathrm{block}(D)$ the largest integer k such that D has a diblockage of order k.

So next let us imagine that instead of the large sides we are interested in the interaction of the small sides. For this let $\mathfrak{P} = (\mathfrak{P}^+, \mathfrak{P}^-)$ be a partial orientation of $\vec{\mathcal{S}}_k(D)$ and let $(E, F) \in \mathfrak{P}^+ \cup \mathfrak{P}^-$. We denote the small side of (E, F) as

$$\mathrm{small}(E,F) := \begin{cases} E, & (E,F) \in \mathfrak{P}^+ \\ F, & (E,F) \in \mathfrak{P}^- \end{cases},$$

while the large side of (E, F) is denoted by

$$\mathrm{large}(E,F) := \begin{cases} E, & (E,F) \in \mathfrak{P}^- \\ F, & (E,F) \in \mathfrak{P}^+ \end{cases}.$$

Definition 6.2.5 (Diclogging)**.** Let $k \in \mathbb{N}$ be a positive integer and D be a digraph. An orientation $\mathfrak{O} = (\mathfrak{O}^+, \mathfrak{O}^-)$ of $\vec{\mathcal{S}}_k(D)$ is a *diclogging of order* k if for all laminar separations $(E, F), (G, H) \in \mathfrak{O}^+ \cup \mathfrak{O}^-$ we have $\mathrm{small}(E, F) \cup \mathrm{small}(G, H) \neq V(D)$. We denote by $\mathrm{clogg}(D)$ the largest integer k such that D has a diclogging of order k.

Lemma 6.2.6 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, D be a digraph, and $\mathfrak{O} = (\mathfrak{O}^+, \mathfrak{O}^-)$ be a diclogging of order k. Then $\mathfrak{O}$ extends $\mathfrak{R}_k$ and is consistent.

Proof. For the consistency let $(E, F) \in \mathfrak{O}^+$, and $(E, F) \geq (G, H) \in \vec{\mathcal{S}}_k(D)$. Suppose $(G, H) \in \mathfrak{O}^-$. By definition of '$\leq$' we have $G \subseteq E$ and $F \subseteq H$ and thus $E \cup H = V(D)$ contradicting our definition of order k dicloggings. Hence we must have $(G, H) \in \mathfrak{O}^+$.

Similarly let $(E, F) \in \mathfrak{O}^-$ and $(E, F) \leq (G, H) \in \vec{\mathcal{S}}_k(D)$, and suppose $(G, H) \in \mathfrak{O}^+$. By definition we have $E \subseteq G$ and $H \subseteq F$ and thus $F \cup G = V(D)$. Consequently we must have $(G, H) \in \mathfrak{O}^-$.

So $\mathfrak{O}$ is indeed consistent. What is left is to show that it also extends $\mathfrak{R}_k$. Let $(E, F) \in \mathfrak{R}_k^+$ and suppose $(E, F) \in \mathfrak{O}^-$. As by definition we must have $|E| \leq k$, $(E, V(D)) \in \vec{\mathcal{S}}_k(D)$. Moreover, $F \leq V(D)$ and thus $(E, V(D)) \leq (E, F)$. Note that $(E, V(D)) \in \mathfrak{O}^-$ is impossible as $(E, V(D)) \leq (E, V(D))$ and $\mathrm{small}((E, V(D))) = V(D)$. Hence we must have $(E, V(D)) \in \mathfrak{O}^+$ which in turn means $\mathrm{small}(E, V(D)) \cup \mathrm{small}(E, F) = E \cup F = V(D)$. Consequently $(E, F) \in \mathfrak{O}^-$ is impossible and we must have $\mathfrak{R}_k^+ \subseteq \mathfrak{O}^+$.

With similar arguments one can see $\mathfrak{R}_k^- \subseteq \mathfrak{O}^-$ and thus $\mathfrak{O}$ indeed extends $\mathfrak{R}_k$. $\square$

Next let us relate the existence of a diblockage and the existence of a diclogging of D.

Lemma 6.2.7 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, D be a digraph, and $\mathfrak{B} = (\mathfrak{B}^+, \mathfrak{B}^-)$ be a diblockage of order k. Then $\mathfrak{O} = (\mathfrak{B}^+ \cap \vec{\mathcal{S}}_{\lfloor \frac{k}{2} \rfloor}(D), \mathfrak{B}^- \cap \vec{\mathcal{S}}_{\lfloor \frac{k}{2} \rfloor}(D))$ is a diclogging of order $\lfloor \frac{k}{2} \rfloor$.

Proof. Let $(E, F), (G, H) \in \vec{\mathcal{S}}_{\lfloor \frac{k}{2} \rfloor}(D)$ be two directed separations with $(E, F) \leq (G, H)$. We have to show that $\mathrm{small}(E, F) \cup \mathrm{small}(G, H) \neq V(D)$. To do this, we consider the possible cases in which direction (E, F) and (G, H) can be oriented.

First assume $(G, H) \in \mathfrak{B}^+$, then, by consistency of $\mathfrak{B}$, we must also have $(E, F) \in \mathfrak{B}^+$. So $\mathrm{small}(E, F) \cup \mathrm{small}(G, H) = E \cup G = G$. Note that in case $G = V(D)$ we would necessarily have $|H| \leq k$ and thus $(G, H) \in \mathfrak{R}_k^- \subseteq \mathfrak{B}^-$. Hence $G \neq V(D)$.

A similar argument can be made if $(E, F) \in \mathfrak{B}^-$ and thus we may assume $(E, F) \in \mathfrak{B}^+$ and $(G, H) \in \mathfrak{B}^-$. Hence we must have $|F \cap G| \geq k+1$ since $\mathfrak{B}$ is a diblockage of order k. If the small sides, E and H, were to cover the whole digraph we would have $(F \setminus E) \cap (G \setminus H) = \emptyset$. For this however consider

$$\begin{aligned}
|(F \setminus E) \cap (G \setminus H)| &\geq |F \cap G| - |E \cap F| - |G \cap H| \\
&\geq k + 1 - \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor \\
&\geq k + 1 - k \\
&\geq 1.
\end{aligned}$$

Therefore $\mathfrak{O}$ is in fact a diclogging of order $\lfloor \frac{k}{2} \rfloor$. $\square$

Lemma 6.2.8 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, D be a digraph, and $\mathfrak{O} = (\mathfrak{O}^+, \mathfrak{O}^-)$ be a diclogging of order k. Then $\mathfrak{O}$ is a diblockage of order k.

Proof. Let $(E, F), (G, H) \in \vec{\mathcal{S}}_k(D)$ be chosen such that $(E, F) \in \mathfrak{O}^+$, $(G, H) \in \mathfrak{O}^-$, and $(E, F) \leq (G, H)$. Suppose $|F \cap G| \leq k$. Then, since

$E \subseteq G$ and $H \subseteq F$ we have $E \cap F \subseteq F \cap G$ and we have $G \cap H \subseteq F \cap G$. Hence $(G, F) \in \vec{\mathcal{S}}_k(D)$. Moreover $(E, F) \leq (G, F) \leq (G, H)$ and $(G, F) \in \mathfrak{O}^+ \cup \mathfrak{O}^-$. Suppose $G = \text{small}(G, F)$, then $G \cup H = V(D)$ contradicting $\mathfrak{O}$ to be a diclogging since $H = \text{small}(G, H)$. Hence we may assume $\text{small}(G, F) = F$, but with $\text{small}(E, F) = E$ we have $E \cup F = V(D)$ for two small sides. Thus both orientations of (G, F) are impossible and therefore $|F \cap G| \geq k + 1$. Hence $\mathfrak{O}$ is indeed a diblockage of order k by Lemma 6.2.6. □

Corollary 6.2.9 (X*)**.** Let D be a digraph. Then $\text{clogg}(D) \leq \text{block}(D) \leq 2\,\text{clogg}(D)$.

Blockages The previous paragraph introduced the directed analogue of blockages together with a variant that considers the small sides rather than the large sides of the separation. In a second step we establish a similar link between small and large sides for the matching setting. However, we need to translate the idea of 'two small sides spanning the whole graph' into the setting of perfect matchings. Here, since we deal with edges cuts instead of separations and, in some way, with matching edges instead of vertices, instead of covering the whole graph with a set of vertices it might be more natural to ask for the edges of any perfect matching to have at least one endpoint in one of the small sides. This translates into the following definition.

Let G be a graph with a perfect matching M, and let $X \subseteq V(G)$. We denote by $M_G(X)$ the set $M \cap E(G[X])$.

Definition 6.2.10 (Clogging)**.** Let $k \in \mathbb{N}$ be a positive integer and B be a matching covered bipartite graph. An orientation $\mathfrak{O}$ of $\mathcal{T}_k(B)$ is a *clogging of order* k if for all sets $X, Y \in \mathfrak{O}$ with $\partial_B(X)$ and $\partial_B(Y)$ being laminar, and all perfect matchings $M \in \mathcal{M}(B)$ we have

$$((\partial_B(X) \cup \partial_B(Y)) \cap M) \cup M_B(X \cup Y) \neq M.$$

We denote by $\text{clogg}(B)$ the largest integer k such that B has a clogging of order k.

Similar to the undirected case we first have to show that any clogging of order k is consistent and extends $\mathfrak{R}_k$, then, in a second step, we show how to translate between a clogging of B and a diclogging of an M-direction of B.

Lemma 6.2.11 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, B a matching covered bipartite graph, and $\mathfrak{O}$ a clogging of order k. Then $\mathfrak{O}$ extends $\mathfrak{R}_k$ and is consistent.

Proof. For the consistency let $X \in \mathfrak{O}$ and $Y \in \mathcal{T}_k(B)$ such that X and Y are aligned and $Y \subseteq X$. Suppose $\overline{Y} \in \mathfrak{O}$. Since $Y \subseteq X$ we have $X \cup \overline{Y} = V(B)$ and thus $M_B(X \cup \overline{Y}) = M$ for all perfect matchings M of B. As this clearly contradicts the definition of a clogging we must have $Y \in \mathfrak{O}$ and thus $\mathfrak{O}$ is consistent.

Next let $X \in \mathcal{T}_k(B)$ such that $|X| \leq \mathrm{mp}(\partial_B(X))$ and suppose $\overline{X} \in \mathfrak{O}$. There must exist a perfect matching M with an edge $e \in M$ such that $e \subseteq X$ since otherwise we would have an immediate contradiction. Consider $Y := X \cap V_1$. Since $|X| \leq \mathrm{mp}(\partial_B(X)) \leq k$ we have $|Y| = \mathrm{mp}(\partial_B(Y)) \leq k$ and $Y \in \mathcal{T}_k(B)$. Suppose $\overline{Y} \in \mathfrak{O}$, then $(M' \cap \partial_B(Y)) \cup M'_B(\overline{Y}) = M'$ for all $M' \in \mathcal{M}(B)$. Hence we may assume $Y \in \mathfrak{O}$. However, $E(B[X]) \subseteq \partial_B(Y)$ and thus $(M' \cap (\partial_B(Y) \cup \partial_B(X))) \cup M'_B(\overline{X} \cup Y) = M'$ for all $M' \in \mathcal{M}(B)$. So $\mathfrak{O}$ must extend $\mathfrak{R}_k$. □

Proposition 6.2.12 (X*)**.** Let B be a matching covered bipartite graph. Then $\mathrm{clogg}(B) \leq \mathrm{block}(B) \leq 2\,\mathrm{clogg}(B)$.

Proof. We start by showing that every blockage contains a clogging of at least half its order. Let $k := \mathrm{block}(B)$, let $\mathfrak{B}$ be a blockage of order k for B, and let $X, Y \in \mathcal{T}_{\lfloor \frac{k}{2} \rfloor}(B)$ such that X and Y are aligned and $X \subseteq Y$. We discuss all possible orientations of X and Y.

First assume $Y \in \mathfrak{B}$. By Lemma 6.2.11 $\mathfrak{B}$ is consistent and thus this means $X \in \mathfrak{B}$ as well. Then $(M \cap (\partial_B(X)) \cup \partial_B(Y)) \cup M_B(X \cup Y) = (M \cap \partial_B(Y)) \cup M_B(Y)$ for all $M \in \mathcal{M}(B)$. Suppose $(M' \cap \partial_B(Y)) \cup M'_B(Y) = M'$ for some $M' \in \mathcal{M}(B)$, then $|\overline{Y}| \leq \lfloor \frac{k}{2} \rfloor$. Moreover, by Lemma 3.1.58 there must exist $i \in [1, 2]$ such that $\overline{Y} \subseteq V_i$. Hence $|\overline{Y}| \leq \mathrm{mp}(\partial_B(\overline{Y}))$ and as $\mathfrak{B}$ is consistent by Lemma 6.2.11 we must have $\overline{Y} \in \mathfrak{B}$. Hence $M' \cap \partial_B(Y) \cup M'_B(Y) \neq M'$ for all $M' \in \mathcal{M}(B)$.

Next assume $\overline{X} \in \mathfrak{B}$. Since $X \subseteq Y$ we have $\overline{Y} \subseteq \overline{X}$ and because X and Y are aligned, so are $\overline{Y}$ and $\overline{X}$. Hence this case is analogue to the previous one and thus we may close it.

Therefore we may assume $X, \overline{Y} \in \mathfrak{B}$ and thus $|\overline{X} \cap Y| \geq 3$ as $\mathfrak{B}$ is a blockage. Now suppose there is some $M \in \mathcal{M}(G)$ such that $(M \cap$

$(\partial_B(X) \cup \partial_B(Y))) \cup M_B(X \cup \overline{Y}) = M$. Without loss of generality we may assume $\mathrm{Maj}(X) \subseteq V_1$. Now we consider two cases, namely $\overline{X} \cap Y \cap V_2 = \emptyset$ and $\overline{X} \cap Y \cap V_2 \neq \emptyset$.

First assume $\overline{X} \cap Y \cap V_2 = \emptyset$ and let $a \in \overline{X} \cap Y$ be any vertex. By Lemma 3.1.58 for every vertex in $\overline{X} \cap Y$ there must be an edge in $\partial_B(Y) \cap M$ that matches it and thus $|\overline{X} \cap Y| \leq \lfloor \frac{k}{2} \rfloor$. Let $\overline{X} \cap Y = \{a_1, \ldots, a_\ell\}$, and consider $Z_i \coloneqq Y \setminus \{a_{i+1}, \ldots, a_\ell\}$ for all $i \in [1, \ell-1]$. Then $X \subset Z_i \subset Y$, and Z_i is aligned with X and Y. Moreover, $\mathrm{mp}(\partial_B(Z_i)) \leq \mathrm{mp}(\partial_B(X)) + \mathrm{mp}(\partial_B(Y)) - 1$ and $\mathrm{N}_B(Z_i \cap V_2) = \mathrm{N}_B(X \cap V_2) \subseteq Z_i$ and thus $Z_i \in \mathcal{T}_k(B)$ for all $i \in [1, \ell-1]$. We show by induction on $\ell - i$ that $\overline{Z_i} \in \mathfrak{B}$. Suppose $Z_{\ell-1} \in \mathfrak{B}$, then $|\overline{Z_{\ell-1}} \cap Y| = |\{a_\ell\}| = 1 < 3$ which contradicts the fact that $\mathfrak{B}$ is a blockage. Hence $\overline{Z_{\ell-1}} \in \mathfrak{B}$. Let $i \in [1, \ell-2]$, then $Z_{i+1} \in \mathfrak{B}$ by induction, and $|\overline{Z_i} \cap Z_{i+1}| = |\{a_{i+1}\}| = 1 < 3$, hence $\overline{Z_i} \in \mathfrak{B}$. In particular this means $\overline{Z_1} \in \mathfrak{B}$. Now consider $|\overline{X} \cap Z_1| = |\{a_1\}| = 1 < 3$. With this we must have $\overline{X} \in \mathfrak{B}$ which contradicts our assumption. Hence $(M \cap (\partial_B(X) \cup \partial_B(Y))) \cup M_B(C \cup \overline{Y}) \neq M$ for all $M \in \mathcal{M}(G)$.

So next assume $\overline{X} \cap Y \cap V_2 \neq \emptyset$. As before we know that every vertex of $\overline{X} \cap V_2$ must be matched by an edge of $M \cap (\partial_B(X) \cup \partial_B(Y))$, thus $|\overline{X} \cap Y| \leq k$. Let $\overline{X} \cap Y \cap V_1 = \{a_1, \ldots, a_p\}$, $\overline{X} \cap Y \cap V_2 = \{a_{p+1}, \ldots, a_q\}$, and for every $i \in [1, q-1]$ let $Z_i \coloneqq Y \setminus \{a_{i+1}, \ldots, a_q\}$. Note that $Z_i \in \mathcal{T}_k(B)$ for every $i \in [1, q-1]$ as before. Suppose $\overline{Z_{p-1}} \in \mathfrak{B}$, then we are in the situation of the previous case with the sets X and Z_{p-1} and reach a contradiction. So we may assume $Z_{p-1} \in \mathfrak{B}$. Next we show that $\overline{Z_j} \in \mathfrak{B}$ for all $j \in [p, q-1]$ by induction on $q - j$. This is done as before: in case $Z_{q-1} \in \mathfrak{B}$ we would have $|\overline{Z_{q-1}} \cap Y| = |\{a_q\}| = 1 < 3$ and thus $\overline{Z_{q-1}} \in \mathfrak{B}$. So we may assume $j \in [p, q-2]$ and $\overline{Z_{j+1}} \in \mathfrak{B}$ and a similar argument shows that $\overline{Z_j} \in \mathfrak{B}$ as well. Finally this means $Z_{p-1} \in \mathfrak{B}$ and $\overline{Z_p} \in \mathfrak{B}$. But $|\overline{Z_{p-1}} \cap Z_p| = |\{a_p\}| = 1 < 3$ which contradicts the assumption that $\mathfrak{B}$ is a blockage of order k. Consequently we again have $M \cap (\partial_B(X) \cup \partial_B(Y)) \cup M_B(X \cup \overline{Y}) \neq M$ for all $M \in \mathcal{M}(G)$ and with that we may close the case.

For the other inequality let us assume $\mathfrak{O}$ is a clogging of order k for B. By Lemma 6.2.11 $\mathfrak{O}$ is consistent and extends $\mathfrak{R}_k$. Let $X, Y \in \mathfrak{O}$ be chosen such that X and $\overline{Y}$ are aligned, and $X \subseteq \overline{Y}$. We need to show that $|\overline{X} \cap \overline{Y}| \geq 3$. Suppose $|\overline{X} \cap \overline{Y}| \leq 2$. Since $\mathfrak{O}$ is a clogging we know that $M_B(\overline{X} \cap \overline{Y}) \neq \emptyset$ for any perfect matching M of B and by our previous assumption there exists a unique edge $ab \in M$ such that $\overline{X} \cap \overline{Y} = \{a, b\}$. Let us assume without loss of generality that $\mathrm{Maj}(X) \subseteq V_1$ and $b \in V_2$. Since B is matching covered, b must have another neighbour besides a, let c be such a neighbour of b. Then $c \notin \overline{X}$ since $\mathrm{Maj}(\overline{Y}) \subseteq V_1$ and thus, by Lemma 3.1.58, every neighbour of b must be in $\overline{Y}$. Moreover, $\overline{Y} \cap \overline{X} = \{a, b\}$. Let M' be a perfect matching of B with $bc \in M'$. There must exist some vertex d in B such that $ad \in M'$ and $d \notin \overline{X} \cap \overline{Y}$. This however means $(M' \cap (\partial_B(X) \cup \partial_B(Y))) \cup M'_B(X \cup Y) = M'$ which contradicts $\mathfrak{O}$ being a clogging. Hence we must have $|\overline{X} \cap \overline{Y}| \geq 3$ and thus $\mathfrak{O}$ is a blockage. $\square$

Translating Cloggings and Dicloggings For the next step let us first generalise our notions of splits and M-directions of directed separations and generalised tight cuts to orientations. Let $k \in \mathbb{N}$ be a positive integer, D a strongly connected digraph, B a matching covered bipartite graph, $\mathfrak{O}_1 = (\mathfrak{O}_1^+, \mathfrak{O}_1^-)$ an orientation of $\vec{\mathcal{S}}_k(D)$, and $\mathfrak{O}_2$ an orientation of $\mathcal{T}_k(B)$. Let $p \in [1, k]$ be an integer, we denote by $\mathfrak{O}_1|_p$ the *suborientation* $(\mathfrak{O}_1^+ \cap \vec{\mathcal{S}}_p(D), \mathfrak{O}_1^- \cap \vec{\mathcal{S}}_p(D))$, and by $\mathfrak{O}_2|_p$ the *suborientation* $\mathfrak{O}_2 \cap \mathcal{T}_p(B)$. The *split* of $\mathfrak{O}_1$ is an orientation of $\mathcal{T}_k(\mathcal{S}(D))$ defined as follows:

$$\mathcal{S}(\mathfrak{O}_1) \coloneqq \left\{\mathcal{S}(E, F) \mid (E, F) \in \mathfrak{O}_1^+\right\} \cup \left\{\overline{\mathcal{S}(E, F)} \mid (E, F) \in \mathfrak{O}_1^-\right\}.$$

The *M-direction* of $\mathfrak{O}_2$ is an orientation of $\vec{\mathcal{S}}_k(\mathcal{D}(B, M))$ defined as follows:

$$\mathcal{D}(\mathfrak{O}_2, M) \coloneqq (\mathcal{D}(\mathfrak{O}_2, M)^+, \mathcal{D}(\mathfrak{O}_2, M)^-), \text{ where}$$
$$\mathcal{D}(\mathfrak{O}_2, M)^+ \coloneqq \{\mathcal{D}(X, M) \mid X \in \mathfrak{O}_2 \text{ and } \mathrm{Maj}(X) \subseteq V_1\}, \text{ and}$$
$$\mathcal{D}(\mathfrak{O}_2, M)^- \coloneqq \left\{\mathcal{D}(\overline{X}, M) \mid X \in \mathfrak{O}_2 \text{ and } \mathrm{Maj}(X) \subseteq V_2\right\}.$$

Note that $\mathcal{S}(\mathcal{D}(\mathfrak{O}_2, M)) = \mathfrak{O}_2$ and $\mathcal{D}(\mathcal{S}(\mathfrak{O}_1), M') = \mathfrak{O}_1$ where M' is the perfect matching of $\mathcal{S}(D)$ for which $\mathcal{D}(\mathcal{S}(D), M') = D$. Towards the main result of this section we prove that the M-direction of an order k clogging is a diclogging of the same order in the M-direction of B. For this we first need to show that the property of being laminar is preserved by these operations.

Lemma 6.2.13 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, B a matching covered bipartite graph with a perfect matching M, $D \coloneqq \mathcal{D}(B, M)$, and $X, Y \in \mathcal{T}_k(B)$ two aligned sets with $\mathrm{Maj}(X) \subseteq V_1$. Then $\mathcal{D}(X, M)$ and $\mathcal{D}(Y, M)$ are laminar if and only if X and Y are, moreover $\mathcal{D}(X, M) \leq \mathcal{D}(Y, M)$ if and only if $X \subseteq Y$.

Proof. For each $Z \in \{X, Y\}$ let $(E_Z, F_Z) := \mathcal{D}(Z, M)$.

Let us first assume X and Y are laminar. We show the claim via a case distinction over which of the four sets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$, and $\overline{X} \cap \overline{Y}$ is empty.

Case $X \cap Y = \emptyset$: This means $(M \cap (\partial_B(X) \cup \partial_B(Y))) \cup M_B(X \cup Y) = \emptyset$. Since X and Y are aligned and their majority lies in V_1, we immediately obtain $E_X \cap E_Y = \emptyset$.

Case $X \cap \overline{Y} = \emptyset$: Then in particular $(M \cap \partial_B(X \cap \overline{Y})) \cup M_B(X \cup \overline{Y}) = \emptyset$ and thus $E_X \cap F_Y \subseteq F_X \cap E_Y$. This is due to the fact that every edge of M in $(E_X \cap F_Y) \cap (F_X \cap E_Y)$ is an edge with one endpoint in $X \cap Y$ and the other one in $\overline{X} \cap \overline{Y}$. Consequently we have $(E_X \cap F_Y) \setminus (F_X \cap E_Y) = \emptyset$ and thus are done.

Case $\overline{X} \cap Y = \emptyset$: By applying the arguments from the previous case we get $(F_X \cap E_Y) \setminus (E_X \cap F_Y) = \emptyset$.

Case $\overline{X} \cap \overline{Y} = \emptyset$: This case is completely analogue to the case $X \cap Y = \emptyset$.

The reverse direction can be seen by analogue arguments along a similar case distinction and thus we omit it here. In particular one can observe that from $X \subseteq Y$ it follows immediately that $(M \cap \partial_B(X)) \cup M_B(X) \subseteq (M \cap \partial_B(Y)) \cup M_B(Y)$ and $(M \cap \partial_B(\overline{Y})) \cup M_B(\overline{Y}) \subseteq (M \cap \partial_B(\overline{X})) \cup M_B(\overline{X})$, and thus $(E_X, F_X) \leq (E_Y, F_Y)$. Again the reverse direction can be seen analogously. □

Lemma 6.2.14 (X*). Let $k \in \mathbb{N}$ be a positive integer, B a matching covered bipartite graph with a perfect matching M, $D := \mathcal{D}(B, M)$, and $\mathfrak{O}$ a clogging of $\mathcal{T}_k(B)$. Then $\mathcal{D}(\mathfrak{O}, M)$ is a diclogging of $\vec{\mathcal{S}}_k(D)$.

Proof. First note that $\{\mathcal{D}(X, M) \mid X \in \mathcal{T}_k(B) \text{ and } \mathrm{Maj}(X) \subseteq V_1\} = \vec{\mathcal{S}}_k(D)$ by lemmata 3.2.11 and 3.2.13. Hence $\mathcal{D}(\mathfrak{O}, M)$ is indeed an orientation of $\vec{\mathcal{S}}_k(D)$ as we have seen in Lemma 6.2.3.

Now let $(E, F), (G, H) \in \mathcal{D}(\mathfrak{O}, M)^+ \cup \mathcal{D}(\mathfrak{O}, M)^-$ be two laminar separations and note that $\mathcal{S}(E, F)$ and $\mathcal{S}(G, H)$ must induce laminar cuts in B by Lemma 6.2.13. We consider the possible cases of how these are oriented.

Case $(E, F), (G, H) \in \mathcal{D}(\mathfrak{O}, M)^+$: Then $\mathcal{S}(E, F), \mathcal{S}(G, H) \in \mathfrak{O}$ and thus $(M \cap \partial_B(\mathcal{S}(E, F))) \cup M_B(\mathcal{S}(E, F)) \cup (M \cap \partial_B(\mathcal{S}(G, H))) \cup M_B(\mathcal{S}(G, H)) \neq$

M and therefore there must be some $e \in M$ such that $e \notin E \cap G$. Consequently $E \cap G \neq V(D)$.

Case $(E,F) \in \mathcal{D}(\mathfrak{O},M)^+$, $(G,H) \in \mathcal{D}(\mathfrak{O},M)^-$: Here we have $\mathcal{S}(E,F), \overline{\mathcal{S}(G,H)} \in \mathfrak{O}$ and thus $(M \cap \partial_B(\mathcal{S}(E,F))) \cup M_B(\mathcal{S}(E,F)) \cup (M \cap \partial_B(\overline{\mathcal{S}(G,H)})) \cup M_B(\overline{\mathcal{S}(G,H)}) \neq M$ and therefore there must be some $e \in M$ such that $e \notin E \cap H$. Consequently $E \cap H \neq V(D)$.

Case $(E,F) \in \mathcal{D}(\mathfrak{O},M)^-$, $(G,H) \in \mathcal{D}(\mathfrak{O},M)^+$: Then $\overline{\mathcal{S}(E,F)}, \mathcal{S}(G,H) \in \mathfrak{O}$ and thus $(M \cap \partial_B(\overline{\mathcal{S}(E,F)})) \cup M_B(\overline{\mathcal{S}(E,F)}) \cup (M \cap \partial_B(\mathcal{S}(G,H))) \cup M_B(\mathcal{S}(G,H)) \neq M$ and therefore there must be some $e \in M$ such that $e \notin F \cap G$. Consequently $F \cap G \neq V(D)$.

Case $(E,F),(G,H) \in \mathcal{D}(\mathfrak{O},M)^-$: Then $\overline{\mathcal{S}(E,F)}, \overline{\mathcal{S}(G,H)} \in \mathfrak{O}$ and thus $(M \cap \partial_B(\overline{\mathcal{S}(E,F)})) \cup M_B(\overline{\mathcal{S}(E,F)}) \cup (M \cap \partial_B(\overline{\mathcal{S}(G,H)})) \cup M_B(\overline{\mathcal{S}(G,H)}) \neq M$ and therefore there must be some $e \in M$ such that $e \notin F \cap H$. Consequently $F \cap H \neq V(D)$. And thus $\mathcal{D}(\mathfrak{O},M)$ is indeed a diclogging of $\vec{\mathcal{S}}_k(D)$. □

Reversing Lemma 6.2.14 sadly is not as straight forward. The problem here is that from $\mathfrak{O}$ being a diclogging of some M-direction D of B we know that two small sides of laminar separations cannot cover all of D and thus not all edges of M. However, there might still exist another perfect matching M' of B for which this is not true. To make sure that all perfect matchings of B are satisfied we have to pay a small constant factor in our translation.

Lemma 6.2.15 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, B a matching covered bipartite graph with a perfect matching M, $D := \mathcal{D}(B,M)$, and $\mathfrak{O} = (\mathfrak{O}^+, \mathfrak{O}^-)$ a diclogging of $\vec{\mathcal{S}}_k(D)$. Then $\mathcal{S}(\mathfrak{O}|_{\lfloor \frac{k}{2} \rfloor})$ is a clogging of $\mathcal{T}_{\lfloor \frac{k}{2} \rfloor}(B)$.

Proof. Let $X, Y \in \mathcal{S}(\mathfrak{O}|_{\lfloor \frac{k}{2} \rfloor})$ be two sets such that $\partial_B(X)$ and $\partial_B(Y)$ are laminar. Without loss of generality let us assume $\mathrm{Maj}(X) \subseteq V_1$. Since $\mathfrak{O}$ is a diclogging of $\vec{\mathcal{S}}_k(D)$ we know $((\partial_B(X) \cup \partial_B(Y)) \cap M) \cup M_B(X \cup Y) \neq M$. Suppose there is $M' \in \mathcal{M}(B)$ such that $((\partial_B(X) \cup \partial_B(Y)) \cap M') \cup M'_B(X \cup Y) \cap M' = M'$. Then $M'_B(\overline{X} \cap \overline{Y}) = \emptyset$, but since M has an edge in $B[\overline{X} \cap \overline{Y}]$ we know $\overline{X} \cap \overline{Y} \cap V_i \neq \emptyset$ for both $i \in [1,2]$. From this we can deduce that neither $X \subseteq Y$ nor $Y \subseteq X$ since in both cases the edge cut around the larger set would contain edges of M' incident with vertices of both colour classes in $\overline{X} \cap \overline{Y}$ which contradicts Lemma 3.1.58. Hence either $\overline{X} \subseteq Y$ or $X \subseteq \overline{Y}$. It suffices to only consider the second case as the first one can be resolved analogously. Let $\overline{X} \cap \overline{Y} \cap V_1 = \{a_1, \dots, a_p\}$, and $\overline{X} \cap \overline{Y} \cap V_2 = \{a_{p+1}, \dots, a_q\}$. For every $i \in [1, q-1]$ let $Z_i := X \cup \{a_1, \dots, a_i\}$ and note that $Z_i \in \mathcal{T}_k(B)$ for all $i \in Z_i$. Moreover, $\mathrm{Maj}(Z_i) \subseteq V_1$ for all $i \in [1, p]$ and thus $\mathcal{D}(Z_i, M) \in \vec{\mathcal{S}}_k(D)$. Indeed, $\mathcal{D}(X, M) \leq \mathcal{D}(Z_i, M) \leq \mathcal{D}(Z_{i+1}, M)$ for all $i \in [1, p-1]$. We also set $Z_0 := X$ and $Z_q := \overline{Y}$. First assume $\mathrm{Maj}(\overline{Y}) \subseteq V_1$. Then also $\mathrm{Maj}(Z_i) \subseteq V_1$ for all $i \in [p, q-1]$ and thus $\mathcal{D}(Z_i, M) \in \vec{\mathcal{S}}_k(D)$, and $\mathcal{D}(X, M) \leq \mathcal{D}(Z_i, M) \leq \mathcal{D}(Z_{i+1}, M)$ for all $i \in [p, q-2]$. Let $i \in [0, q]$ be the largest integer such that $Z_i \in \mathcal{S}(\mathfrak{O})$, and let $j := i + 1$ which implies $\overline{Z_j} \in \mathcal{S}(\mathfrak{O})$. Since $X = Z_0, Y = \overline{Z_q} \in \mathcal{S}(\mathfrak{O})$, i and j are well defined. Then we have $|\overline{Z_i} \cap Z_j| = |\{a_j\}| = 1$ and, moreover, for every perfect matching $M'' \in \mathcal{M}(B)$ we have $(M'' \cap (\partial_B(Z_i) \cup \partial_B(Z_j))) \cup M''_B(Z_i \cup \overline{Z_j}) = M''$. This means $\mathcal{D}(Z_i, M) \leq \mathcal{D}(Z_j, M)$, $\mathcal{D}(Z_i, M) \in \mathfrak{O}^-$, $\mathcal{D}(Z_j, M) \in \mathfrak{O}^+$, and $\mathrm{small}(\mathcal{D}(Z_i, M)) \cup \mathrm{small}(\mathcal{D}(Z_j, M)) = M = V(D)$. As this is a contradiction to $\mathfrak{O}$ being an order k diclogging we may close this case. Next assume $\mathrm{Maj}(\overline{Y}) \subseteq V_2$. Suppose there exists some $a \in \overline{X} \cap \overline{Y} \cap V_1$. Then a cannot have a neighbour in X and also it cannot have a neighbour in Y by Lemma 3.1.58. Hence a perfect matching such as M' cannot exist. So we may assume $\overline{X} \cap \overline{Y} \cap V_1 = \emptyset$. Consequently, $M''_B(\overline{X} \cap \overline{Y}) = \emptyset$ for all $M'' \in \mathcal{M}(B)$ including M. As this contradicts our assumption above this case cannot occur and we are done. $\square$

In total we obtain the following two corollaries.

Corollary 6.2.16 (X*)**.** Let B be a bipartite matching covered graph with a perfect matching M. Then $\mathrm{clogg}(B) \leq \mathrm{clogg}(\mathcal{D}(B, M)) \leq 2\,\mathrm{clogg}(B)$.

Corollary 6.2.17 (X*)**.** Let B be a bipartite matching covered graph with a perfect matching M. Then $\frac{1}{2}\,\mathrm{block}(B) \leq \mathrm{block}(\mathcal{D}(B, M)) \leq 2\,\mathrm{block}(B)$.

Please note that the bounds in Corollary 6.2.17 are probably not tight. When inspecting ordered linear perfect matching decompositions more closely one can observe that these can be translated into a directed path decomposition of almost the same width immediately. Similarly, any directed path decomposition can be transformed into an ordered linear perfect matching decomposition of almost the same width. Indeed, with 'almost the same width' we mean that these two parameters only differ by a small additive constant.

6.3. Tangles and a Unified Width Measure

In the previous sections we were almost exclusively concerned with orientations of directed separations and generalised tight cuts that form dual notions to 'linear' width parameters. However, in light of the bigger picture, it would be nice to have a matching theoretic analogue of Theorem 2.2.21 that allows us to decompose a bipartite matching covered graph in a tree like fashion while identifying and possibly distinguishing the different areas of 'high connectivity'. To do this, we take a slightly different approach.

First we introduce our notion of tangles for bipartite matching covered graphs. Then we compare it with the notion of directed tangles from [GKK+20] and show that similar methods as in the case of cloggings can be used to create a tangle from a directed tangle and vice versa. In a third step we apply some of the results of [GKK+20] to immediately deduce a duality result and obtain the aforementioned decomposition.

Matching Tangles We start with a definition of tangles appropriate for the setting of bipartite matching covered graphs.

Definition 6.3.1 (Tangle)**.** Let $k \in \mathbb{N}$ be a positive integer and B be a matching covered bipartite graph. An orientation $\mathfrak{O}$ of $\mathcal{T}_k(B)$ is a *tangle of order* k if for all sets $X, Y, Z \in \mathfrak{O}$, and all perfect matchings $M \in \mathcal{M}(B)$ we have

$$((\partial_B(X) \cup \partial_B(Y) \cup \partial_B(Z)) \cap M) \cup M_B(X \cup Y \cup Z) \neq M.$$

We denote by tangle(B) the largest integer k such that B has a tangle of order k.

We denote by $\mathsf{Tangles}(B)$ the family of all tangles of B. Let $k \in \mathbb{N}$ be a positive integer and $\mathsf{T} \subseteq \mathsf{Tangles}(B)$. We denote by $\mathsf{T}|_k$ the set $\{\mathfrak{T}|_k \mid \mathfrak{T} \in \mathsf{T}\}$, where $\mathfrak{T}|_k := \{X \in \mathfrak{T} \mid \mathrm{mp}(\partial_B(X)) \leq k\}$.

Definition 6.3.2 (Distinguishing Tangles)**.** Let B be a matching covered graph and $\mathfrak{T}, \mathfrak{T}' \in \mathsf{Tangles}(B)$.

A set $X \in \mathcal{T}_{\mathrm{tangle}(B)}(B)$ *distinguishes* $\mathfrak{T}$ and $\mathfrak{T}'$ if $X \in \mathfrak{T} \setminus \mathfrak{T}'$ or $X \in \mathfrak{T}' \setminus \mathfrak{T}$. We say that $\mathfrak{T}$ and $\mathfrak{T}'$ are *indistinguishable* if $\mathfrak{T} \subseteq \mathfrak{T}'$ or $\mathfrak{T}' \subseteq \mathfrak{T}$. Otherwise they are *distinguishable*. We say that X is a $(\mathfrak{T}, \mathfrak{T}')$*-distinguisher* if it distinguishes $\mathfrak{T}$ and $\mathfrak{T}'$, and the *order* of a $(\mathfrak{T}, \mathfrak{T}')$-distinguisher X is $\mathrm{mp}(\partial_B(X))$.

We say that $\mathfrak{T}$ and $\mathfrak{T}'$ are *h-distinguishable*, for some $h \in \mathbb{N}$, if there is a set $X \in \mathcal{T}_h(B)$ that distinguishes $\mathfrak{T}$ and $\mathfrak{T}'$. Moreover, $\mathfrak{T}$ and $\mathfrak{T}'$ are *h-indistinguishable* if no set in $\mathcal{T}_h(B)$ distinguishes $\mathfrak{T}$ and $\mathfrak{T}'$.

Let $\mathsf{T} \subseteq \mathsf{Tangles}(B)$. Our goal is to show that for every such T we can find a decomposition of B that arranges the vertices of B in a tree, while distinguishing, in some sense, all tangles in T. For this we have to overcome several hurdles. First we need to show that the existence of high order tangles is indeed dual to B having small perfect matching width. Second, our notion of perfect matching decompositions is not ideal to describe the tree structures that arise from families of tangles, so we need a notion of decomposition that is slightly more relaxed. At last we need to bring the concepts of tree-like decompositions of B and tangles together into one unified decomposition. To do this, as mentioned above, we make use of the fact that recently such a goal was achieved for digraphs with regards to directed treewidth by Giannopoulou et al. in [GKK+20]. So our approach is similar to the one for cloggings: we simply show that tangles and their directed cousins can be translated into one another by paying only a small factor. Before we start our discussion with a more appropriate way of decomposing graphs with perfect matchings, we need some preliminary observations.

Let $\mathfrak{T}$ be a tangle in a bipartite graph with a perfect matching. Note that the definition of tangles does not require the three sets to be pairwise distinct, hence it follows for every pair $X, Y \in \mathfrak{T}$ that
$$((\partial_B(X) \cup \partial_B(Y)) \cap M) \cup M_B(X \cup Y) \neq M.$$
From this we may immediately derive the following:

Observation 6.3.3 (X*)**.** Let B be a bipartite graph with a perfect matching and $\mathfrak{T}$ a tangle in B. Then $\mathfrak{T}$ is a clogging in B.

Indeed, by Lemma 6.2.11 this means that every tangle must be consistent.

Observation 6.3.4 (X*)**.** Let $k \in \mathbb{N}$ be a positive integer, B be a bipartite graph with a perfect matching, and $\mathfrak{T}$ a tangle of $\mathcal{T}_k(B)$. Then $\mathfrak{T}$ is consistent and $\mathfrak{T}$ extends $\mathfrak{R}_k$.

Another Width Parameter for Graphs with Perfect Matchings An important example of tangles in a bipartite matching covered graph B are the braces of B which are not C_4 or $K_{3,3}$. Let J be a brace of B, then every tight cut of B has a shore that contains at least $|V(J)| - 1$ vertices of J. Consider $\mathcal{T}_1(B)$ and define an orientation $\mathfrak{O}$ such that $X \in \mathfrak{O}$ if and only if $X \in \mathcal{T}_1(B)$, and $|X \cap V(J)| \leq 1$. Since J is neither C_4, nor $K_{3,3}$, J has at least eight vertices and thus for any three sets $X, Y, Z \in \mathfrak{O}$ we have $M_B(\overline{X \cup Y \cup Z}) \neq \emptyset$ for every $M \in \mathcal{M}(B)$. Hence $\mathfrak{O}$ is indeed a tangle in B. So every brace J of B, that is not C_4 or $K_{3,3}$, defines a tangle of $\mathcal{T}_1(B)$. Similarly, one can observe that every tangle of $\mathcal{T}_1(B)$ produces a brace of B. Please note the similarity between this observation and Theorem 5.2.16. However, if we consider a tight cut decomposition of B with its tree structure, then some of the braces we encounter possibly need more than three tight cut contractions. Hence the tree that corresponds to this tight cut decomposition has vertices of degree larger than three. This means that cubic trees, and thus perfect matching decompositions, might not necessarily be the best possible choice to describe and distinguish the tangles in B. Towards a decomposition of B that interacts with its tangles in a structured way, we use this subsection to introduce a width parameter whose decomposition resembles more closely the properties of a (directed) tree decomposition, allows any tree as its decomposition tree, and still is equivalent to perfect matching width.

If e is a (directed) edge and v a vertex, we write $v \sim e$ if v is an endpoint of e.

Definition 6.3.5 (Matching Treewidth)**.** Let G be a graph with a perfect matching. A *matching tree decomposition* of G is a tuple (T, β) such that T is a tree, and $\beta \colon V(T) \to 2^{V(G)}$, where $\beta(t)$ is called a *bag*, and $\{\beta(t) \mid t \in V(T)\}$ is a near partition[2].

For every edge $t_1t_2 \in E(T)$ we write T_{t_i} for the component of $T - t_1t_2$ that contains t_i for $i \in [1, 2]$. We write $\beta(T_{t_i})$ for the set $\bigcup_{t \in V(T_{t_i})} \beta(t)$.

[2]That is, any two distinct bags are disjoint and we allow empty bags.

Then $\partial_B(\beta(T_{t_1})) = \partial_B(\beta(T_{t_2}))$ and thus, we may identify this edge cut with the edge t_1t_2. Thus we may write $\partial_B(t_1t_2) := \partial_B(\beta(T_{t_1}))$.

For every $M \in \mathcal{M}(G)$ we define the *M-bags* β_M derived from β as

$$\beta_M(t) := M_B(\beta(t)) \cup \bigcup_{t \sim e \in E(T)} (\partial_B(e) \cap M).$$

The *M-width* of (T, β) is now defined as

$$M\text{-}\operatorname{width}(T, \beta) := \max_{t \in V(T)} |\beta_M(t)|,$$

while the *width* of (T, β) is defined as

$$\operatorname{width}(T, \beta) := \max_{M \in \mathcal{M}(G)} M\text{-}\operatorname{width}(T, \beta).$$

The *edge width* of (T, β) is $\max_{e \in E(T)} \operatorname{mp}(\partial_B(e))$.

Finally we define the *matching treewidth* of G, denoted by $\operatorname{mtw}(G)$, as the minimum width over all matching tree decompositions of G.

Please note that every matching covered graph G has a matching tree decomposition (T, β) of edge width one, such that for every $e \in E(T)$, $\partial_B(e)$ is a non-trivial tight cut, and the graph obtained by contracting the shore of every $\partial_B(e)$ that does not contain $\beta(t)$ for every $y \sim e \in E(T)$ and a fixed $t \in V(T)$ is a brace or brick of G. Indeed, this means the tight cut decomposition of G can be represented by a matching tree decomposition of bounded width.

Let us relate perfect matching width and matching treewidth.

Theorem 6.3.6 (X*)**.** Let G be a graph with a perfect matching. Then $\operatorname{pmw}(G) \leq \operatorname{mtw}(G) \leq \frac{3}{2} \operatorname{pmw}(G)$.

Proof. Let (T, δ) be a perfect matching decomposition of G of minimum width. We simply define $\beta(t) := \{\delta(t)\}$ for all leaves t of T and set $\beta(t) := \emptyset$ for all inner vertices of T. Then (T, β) is a matching tree decomposition of G.

Let $\ell \in \mathsf{L}(T)$ be any leaf, then $|\beta(\ell)| = 1$ and thus $\delta(\ell)$ must be matched by an edge of $\partial_G(\beta(\ell))$, and thus $|\beta_M(\ell)| = 1$ for all $M \in \mathcal{M}(G)$. Now let $t \in V(T)$ be an internal vertex of T, and let $e_1, e_3, e_3 \in E(T)$ be the three incident edges at t. Note that $\beta(t) = \emptyset$ and thus $\beta_M(t) = \bigcup_{i=1}^{3} \partial_G(e_i)$. Moreover, for every $e \in \partial_G(e_i)$ for some $i \in [1,2]$ there is $j \in [1,3] \setminus \{i\}$ such that $e \in \partial_G(e_j)$. Hence

$$|\beta_M(t)| \leq \frac{3}{2} \sum_{i=1}^{3} |\partial_B(e_i) \cap M| \leq \frac{3}{2} \operatorname{pmw}(G),$$

and thus $\operatorname{width}(T, \beta) \leq \frac{3}{2} \operatorname{width}(T, \delta) = \frac{3}{2} \operatorname{pmw}(G)$.

Now, to prove $\operatorname{pmw}(G) \leq \operatorname{mtw}(G)$ let (T, β) be a matching tree decomposition for G of minimum width. First note that $\operatorname{mp}(\partial_G(\beta(t))) \leq \operatorname{mtw}(G)$ for every $t \in V(T)$. In general, T is not a cubic tree, so first let us show that we can transform (T, β) into a matching tree decomposition (T', β') of the same width where T' is subcubic. We do this by induction over $n = |\operatorname{spine}(T)| - |\{t \in V(T) \mid \deg_T(t) \geq 4\}|$. Suppose $n = 0$, then every inner vertex of T must have degree at least four. Let $t \in V(\operatorname{spine}(T))$ be any vertex and $t_1, \dots, t_p$ be its neighbours where $p \geq 4$. We introduce a path $P_t = (d_1, \dots, d_{p-2})$ such that $V(P_t) \cap V(T) = \emptyset$, and set $\beta'(d_1) \coloneqq \beta(t)$, and $\beta'(d_i) = \emptyset$ for all $i \in [2, p-2]$. Then we introduce the edges $t_1 d_1$, $t_1 d_2$, $t_{p-1} d_{p-2}$, $t_p d_{p-2}$, and $t_j p_{j-1}$ for all $j \in [3, p-2]$, and finally we delete the vertex t. Let (T', β') be the resulting decomposition, where β' is defined as β on all vertices that do not belong to P_t. Then $\partial_G(e)$ has not changed for all $e \in E(T') \cap E(T)$. Moreover, $\partial_B(t_i d_j) = \partial_B(t_i t)$ for all $i \in [1, p]$ and j chosen accordingly. So we only need to show $|\beta_M(d_i)| \leq \operatorname{mtw}(G)$ for all $i \in [1, p-2]$. For every M and every edge $e \in E(P_t)$ we have $\partial_G(e) \cap M \subseteq \beta_M(t)$, and thus for every $d \in V(P_t)$ we have $\beta'_M(d) \subseteq \beta_M(t)$. Hence our claim follows. So we may assume $n \geq 1$. By picking any vertex $t \in V(T)$ with degree at least four and applying the same construction as above we obtain a decomposition (T', β') of the same width with $|\operatorname{spine}(T')| - |\{t \in V(T') \mid \deg_{T'}(t) \geq 4\}| = n + 1$ and thus we may assume that, for our decomposition (T, β), T is indeed a subcubic tree.

Next we want to push all vertices from the bags in (T, β) to the leaves of T and make sure that every leaf-bag contains exactly one vertex. Note that we may assume $\beta(\ell) \neq \emptyset$ for every $\ell \in \mathsf{L}(T)$.

In a first step we make sure that exactly the leaf-bags are non-empty. Let $t \in V(T)$ be any vertex with a non-empty bag that is not a leaf. If t has only two neighbours we introduce the vertex d_0 adjacent to t and set $\beta'(t) = \emptyset$, $\beta'(d_0) = \beta(t)$. By arguments similar to those above one can see that this does not change the width of the decomposition. Assume t has three distinct neighbours t_1, t_2, t_3. Then introduce d', d'', and d_0 with the edges $d'd''$ and $d''d_0$, introduce the edges t_1d', t_2d', and t_3d'', and delete t. We set $\beta'(d_0) := \beta(t)$, and $\beta'(d') = \beta'(d'') := \emptyset$. It again is straight forward to check that the resulting decomposition has still width $\mathrm{mtw}(G)$. By iterating over all internal vertices of T we make sure that all internal bags are empty.

So we may assume that for our optimal decomposition (T, β), the tree T is subcubic and all internal bags are empty. Suppose T has a vertex t of degree two, then $\beta(t) = \emptyset$ as t is not a leaf and thus we may contract one of its incident edges to remove the degree two vertex. Hence we may assume that T is cubic.

At last, for every $\ell \in \mathsf{L}(T)$ let L_ℓ be a cubic tree with $|\beta(\ell)| + 1$ leaves such that ℓ is a leaf of L_ℓ. Then add all L_ℓ to T. Let δ be a bijection between $\bigcup_{\ell \in \mathsf{L}(T)} (\mathsf{L}(L_\ell) \setminus \{\ell\})$ and $V(G)$ such that for every $\ell \in \mathsf{L}(T)$, $\delta|_{\mathsf{L}(L_\ell - \ell)}$ is a bijection between $\mathsf{L}(L_\ell - \ell)$ and $\beta(\ell)$. Let (T', δ) be the resulting perfect matching decomposition of G. Note that $\mathrm{mp}(\partial_G(e)) \leq \mathrm{mtw}(G)$ for all edges $e \in E(T') \cap E(T)$. Let M be a perfect matching of G, $\ell \in \mathsf{L}(T)$, and $e \in E(L_\ell)$. Then we must have $\partial_G(e) \cap M \subseteq \beta_M(\ell) \leq \mathrm{mtw}(G)$ and thus $\mathrm{width}(T', \delta) \leq \mathrm{mtw}(G)$. $\square$

Hence for bipartite graphs we may apply Theorem 5.3.27 to obtain the following.

Corollary 6.3.7 (X*)**.** Let B be a bipartite graph with a perfect matching M. Then

$$\frac{2}{3}\,\mathrm{mtw}(B) - 1 \leq \mathrm{dtw}(\mathcal{D}(B, M)) \leq 72\,\mathrm{mtw}(B)^2 + 144\,\mathrm{mtw}(B) + 9.$$

6.3.1. Applying Structural Digraph Theory

In what follows we introduce the necessary definitions for the directed version of tangles and their corresponding decompositions. We show how ‘ditangles’ and tangles of bipartite graphs with perfect matchings are related by generalising the proof we used to translate between cloggings and dicloggings, and finally deduce the desired results from their directed counterparts.

Definition 6.3.8 (Ditangle)**.** Let D be a digraph and $k \in \mathbb{N}$ be a positive integer. An orientation $\mathfrak{D} = (\mathfrak{D}^+, \mathfrak{D}^-)$ of $\vec{\mathcal{S}}_k(D)$ is a *ditangle* of *order* k if for all $S_1, S_2, S_3 \in \mathfrak{D}^+ \cup \mathfrak{D}^-$ we have

$$\mathrm{small}(S_1) \cup \mathrm{small}(S_2) \cup \mathrm{small}(S_3) \neq V(D).$$

We denote by $\mathrm{tangle}(D)$ the largest integer $h \in \mathbb{N}$ such that D has a ditangle of order h.

By combining several known inequalities regarding dual concepts of directed treewidth one can obtain a duality theorem for directed treewidth and the existence of ditangles.

Theorem 6.3.9 ([GKK+20])**.** Let D be a digraph. Then $\frac{1}{18}\,\mathrm{dtw}(D) - 1 \leq \mathrm{tangle}(D) \leq \mathrm{dtw}(D) + 1$.

Of Tangles and Ditangles To relate tangles and ditangles, and therefore obtain a duality result for tangles and perfect matching width (matching treewidth) we need to be able to translate between ditangles and tangles.

Lemma 6.3.10 (X*)**.** Let B be a bipartite graph with a perfect matching M, $k \in \mathbb{N}$ be a positive integer, and $\mathfrak{T}$ a tangle of order k in B. Then $\mathcal{D}(\mathfrak{T}, M)$ is a ditangle of order k in $\mathcal{D}(B, M)$.

Proof. Let $S_1, S_2, S_3 \in \mathcal{D}(\mathfrak{T}, M)^+ \cup \mathcal{D}(\mathfrak{T}, M)^-$ and suppose $\mathrm{small}(S_1) \cup \mathrm{small}(S_2) \cup \mathrm{small}(S_3) = V(\mathcal{D}(B, M)) = M$. For each $i \in [1, 3]$ let $X_i \in \mathfrak{T}$ such that $\mathcal{D}(X_i, M) = S_i$ in case $\mathrm{Maj}(X_i) \subseteq V_1$, and $\mathcal{D}(\overline{X_i}, M) = S_i$ otherwise. Then we must have

$$\bigcup_{i=1}^{3} M_B(X_i) \cup \partial_B(X_i) \cap M = \bigcup_{i=1}^{3} \mathrm{small}(S_i) = V(\mathcal{D}(B, M)) = M.$$

This however contradicts our assumption that $\mathfrak{T}$ is a tangle in B and thus we must have $\mathrm{small}(S_1) \cup \mathrm{small}(S_2) \cup \mathrm{small}(S_3) \neq V(\mathcal{D}(B, M))$ for all choices of S_1, S_2, S_3. Hence $\mathcal{D}(\mathfrak{T}, M)$ is a ditangle of order k in $\mathcal{D}(B, M)$. □

Lemma 6.3.11 (X*)**.** Let B be a bipartite graph with a perfect matching M, $D \coloneqq \mathcal{D}(B, M)$, $k \in \mathbb{N}$ a positive integer, and $\mathfrak{T} = (\mathfrak{T}^+, \mathfrak{T}^-)$ a ditangle of order k in D. Then $\mathcal{S}(\mathfrak{T}|_{\lfloor \frac{k}{3} \rfloor})$ is a tangle of order $\lfloor \frac{k}{3} \rfloor$ in B.

Proof. Let $X_1, X_2, X_3 \in \mathcal{S}(\mathfrak{T}|_{\lfloor \frac{k}{3} \rfloor})$. Towards a contradiction we suppose there exists $N \in \mathcal{M}(B)$ such that $\bigcup_{i=1}^{3} N_B(X_i) \cup \partial_B(X_i) \cap N = N$. For each $i \in [1,3]$ let $S_i := \mathcal{D}(X_i, M)$ in case $\mathrm{Maj}(X_i) \subseteq V_1$, otherwise let $S_i := \mathcal{D}(\overline{X_i}, M)$. Moreover, for every $i \in [1,3]$ let $S_i = (G_i, H_i)$.

First let us assume $X_i \subseteq V_j$ for one $j \in [1,2]$ and all $i \in [1,3]$. Then we must have that $|\overline{X_1 \cup X_2 \cup X_3}| \leq k$ and $\overline{X_1 \cup X_2 \cup X_3} \subseteq V_{3-j}$. This however means that $|\overline{\bigcup_{i=1}^{3} \mathrm{small}(S_i)}| \leq k$. Hence $S := (V(D), \overline{\bigcup_{i=1}^{3} \mathrm{small}(S_i)}) \in \mathfrak{R}_k^- \subseteq \mathfrak{T}^-$ by Lemma 6.2.6 and the fact that $\mathfrak{T}$ must also be a diclogging of order k. Without loss of generality let us assume $j = 1$, the other case follows analogously. Then we know $\mathrm{small}(S_i) = G_i$ for all $i \in [1,3]$. As $S_i \in \vec{\mathcal{S}}_{\lfloor \frac{k}{3} \rfloor}(D)$ we know that $(G_p \cup G_q, H_p \cap H_q) \in \vec{\mathcal{S}}_k(D)$ for all $p, q \in [1,3]$. Suppose $(G_p \cup G_q, H_p \cap H_q) \in \mathfrak{T}^-$, then $\mathrm{small}(G_p \cup G_q, H_p \cap H_q) = H_p \cap Hq$ and thus we would have with $G_p \cup G_q \cup H_p \cap H_q = V(D)$ three small sides that cover all of D. As $\mathfrak{T}$ is a ditangle this is impossible and thus $(G_p \cup G_q, H_p \cap H_q) \in \mathfrak{T}^+$ for all $p, q \in [1,3]$. Hence $G_1 \cup G_2$ and G_3 each are a small side of some directed separation in $\mathfrak{T}$. We also know from our discussion above that $\overline{G_1 \cup G_1 \cup G_3}$ is a small side of some directed separation in $\mathfrak{T}$. All three of these sets together however make $V(D)$ which again contradicts $\mathfrak{T}$ being a ditangle. Hence there must exist some $h \in [1,3]$ such that $\mathrm{Maj}(X_h) \cup \mathrm{Maj}(X_i) \not\subseteq V_j$ for every $i \in [1,3] \setminus \{h\}$ and every $j \in [1,2]$.

Without loss of generality let us assume $\mathrm{Maj}(X_1) \subseteq V_1$ and $\mathrm{Maj}(X_2) \cup \mathrm{Maj}(X_3) \subseteq V_2$. Now $\overline{X_1 \cup X_2 \cup X_3}$ contains at most $\lfloor \frac{k}{3} \rfloor$ vertices from V_2 and at most $\lfloor \frac{2k}{3} \rfloor$ vertices from V_1. Let $\overline{X_1 \cup X_2 \cup X_3} \cap V_1 = \{a_1, \dots, a_p\}$ and $\overline{X_1 \cup X_2 \cup X_3} \cap V_2 = \{a_{p+1}, \dots, a_q\}$ where $p \leq \lfloor \frac{2k}{3} \rfloor$ and $q \leq k$. Let $h \in [1,p]$ be the largest integer such that $(E_1, F_1) := \mathcal{D}(X_1 \cup \{a_1, \dots, a_h\}, M) \in \mathfrak{T}^+$. Suppose $h \leq p-1$. Then let $(E_2, F_2) := \mathcal{D}(X_1 \cup \{a_1, \dots, a_{h+1}\}, M)$, and $(E_3, F_3) := \mathcal{D}(\{a_{h+1}\}, M)$. By construction we know $(E_i, F_i) \in \vec{\mathcal{S}}_k(D)$, $\mathrm{small}(E_1, F_1) = E_1$, $\mathrm{small}(E_2, F_2) = F_2$, and $\mathrm{small}(E_3, F_3) = E_3$. Moreover $\overline{E_1} \subseteq F_2 \cup E_3$ and thus $M = V(D) = E_1 \cup F_2 \cup E_3$. Since this is impossible with $\mathfrak{T}$ being a ditangle of order k, we must have $h = p$. Note that $\overline{(X_1 \cup \{a_1, \dots, a_p\}) \cup X_2 \cup X_3} = \{a_{p+1}, \dots, a_q\} \subseteq V_2$. Hence for every perfect matching $N \in \mathcal{M}(B)$ we have $N_B((X_1 \cup \{a_1, \dots, a_p\}) \cup X_2 \cup X_3) \cup N \cap \partial_B(\{a_{p+1}, \dots, a_q\}) = N$.

This means, as $\mathrm{small}(S_i) = H_i$ for $i \in [2,3]$, that $E_1 \cup H_2 \cup H_3 = V(D)$ and thus we have reached a contradiction.
Therefore we obtain $\bigcup_{i=1}^{3} N_B(X_i) \cup \partial_B(X_i) \cap N \neq N$ for all $N \in \mathcal{M}(B)$ and thus $\mathcal{S}(\mathfrak{T}|_{\lfloor \frac{k}{3} \rfloor})$ is a tangle in B. $\square$

So to summarize our findings we may combine the two lemmas above.

Theorem 6.3.12 (X*)**.** Let B be a bipartite graph with a perfect matching M. Then $\mathrm{tangle}(B) \leq \mathrm{tangle}(\mathcal{D}(B, M)) \leq 3\,\mathrm{tangle}(B)$

In light of Theorem 6.3.9 this also shows that tangles are indeed related to perfect matching width.

Tree Labellings The next step towards an actual tangle-based structure theorem for bipartite graphs with perfect matchings is to establish a way to describe any given set of tangles in a tree like way.

Definition 6.3.13 (Distinguishing Ditangles)**.** Let D be a digraph and $\mathfrak{T} = (\mathfrak{T}^+, \mathfrak{T}^-)$, $\mathfrak{T}' = (\mathfrak{T}'^+, \mathfrak{T}'^-)$ be tangles in D. We say that $\mathfrak{T}$ and $\mathfrak{T}'$ *indistinguishable* if $\mathfrak{T}^+ \subseteq \mathfrak{T}'^+$ and $\mathfrak{T}^- \subseteq \mathfrak{T}'^-$, or $\mathfrak{T}'^+ \subseteq \mathfrak{T}^+$ and $\mathfrak{T}'^- \subseteq \mathfrak{T}^-$. A directed separation $S \in \vec{\mathcal{S}}_{\mathrm{tangle}(D)}(D)$ *distinguishes* $\mathfrak{T}$ and $\mathfrak{T}'$, or is a $(\mathfrak{T}, \mathfrak{T}')$*-distinguisher* if $S \in \mathfrak{T}^+ \cap \mathfrak{T}'^-$ or $S \in \mathfrak{T}^- \cap \mathfrak{T}'^+$. The *order* of a $(\mathfrak{T}, \mathfrak{T}')$-distinguisher $S = (X, Y)$ is $|X \cap Y|$. We say that $\mathfrak{T}$ and $\mathfrak{T}'$ are *h-distinguishable* for some $h \in \mathbb{N}$, if there is a $(\mathfrak{T}, \mathfrak{T}')$-distinguisher of order h. We also say that $\mathfrak{T}$ and $\mathfrak{T}'$ are *h-indistinguishable* if there is no $(\mathfrak{T}, \mathfrak{T}')$-distinguisher in $\vec{\mathcal{S}}_h(D)$.
Finally let S and P be sets of ditangles in D, then a (S, P)*-distinguisher* is a separation $S \in \vec{\mathcal{S}}_{\mathrm{tangle}(D)}(D)$ such that S is a $(\mathfrak{S}, \mathfrak{P})$-distinguisher for every choice of $\mathfrak{S} \in \mathcal{S}$ and $\mathfrak{P} \in \mathcal{P}$.

In general, the separations induced by the edges of a directed tree decomposition are not directed separations. Indeed, there exist examples that show that such separations are impossible to achieve within some bound of directed treewidth in some classes of digraphs. Hence, to obtain a decomposition of a digraph D that reflects the tangles of D and, at least somehow, the structure captured by a directed tree decomposition, some intermediate steps are necessary. A first one is to find some tree-like representation of a set of ditangles $\mathcal{D}$ in D. This is done via so called *tree-labellings*.

Definition 6.3.14 (Ditangle Tree-Labelling)**.** Let D be a digraph and $\mathcal{F}$ be a family of ditangles in D. A F-*tree-labelling* is a triple (T, β, γ), where T is a tree, and $\beta\colon V(T) \to \mathsf{F}$ and $\gamma\colon E(T) \to \vec{\mathcal{S}}_{\mathrm{tangle}(D)}(D)$ are functions such that

i) β is a bijection,
ii) if $t, t' \in V(T)$ are distinct and P is the unique path in T between t and t', then for every edge $e \in E(P)$ with $\mathrm{ord}(\gamma(e)) = \min_{e' \in E(P)} \mathrm{ord}(\gamma(e'))$, the separation $\gamma(e)$ is a $(\beta(t), \beta(t'))$-distinguisher of minimum order, and
iii) for every $e = t_1t_2 \in E(T)$ there are $d_i \in V(T_{t_i})$ for both $i \in [1, 2]$ such that $\gamma(e)$ is a $(\beta(d_1), \beta(d_2))$-distinguisher of minimum order.

Theorem 6.3.15 ([GKK+20])**.** Let D be a digraph and F be a family of pairwise distinguishable ditangles in D. Then there exists an F-tree-labelling.

We can now define a matching theoretic version of tree-labellings and then combine Lemma 6.3.10 and Theorem 6.3.15 to obtain a version of Theorem 6.3.15 appropriate for bipartite graphs with perfect matchings.

Definition 6.3.16 (Tangle Tree-Labelling)**.** Let B be a bipartite graph with a perfect matching and F be a family of tangles in B. A F-*tree-labelling* is a triple (T, β, γ), where T is a tree, and $\beta\colon V(T) \to \mathsf{F}$ and $\gamma\colon E(T) \to \vec{\mathcal{S}}_{\mathrm{tangle}(B)}(B)$ are functions such that

i) β is a bijection,
ii) if $t, t' \in V(T)$ are distinct and P is the unique path in T between t and t', then for every edge $e \in E(P)$ with $\mathrm{mp}(\partial_B(\gamma(e))) = \min_{e' \in E(P)} \mathrm{mp}(\partial_B(\gamma(e')))$, the set $\gamma(e)$ is a $(\beta(t), \beta(t'))$-distinguisher of minimum order, and
iii) for every $e = t_1t_2 \in E(T)$ there are $d_i \in V(T_{t_i})$ for both $i \in [1, 2]$ such that $\gamma(e)$ is a $(\beta(d_1), \beta(d_2))$-distinguisher of minimum order.

Proposition 6.3.17 (X*)**.** Let B be a bipartite graph with a perfect matching and F be a family of pairwise distinguishable tangles in B. Then there exists an F-tree-labelling.

Proof. Let $M \in \mathcal{M}(B)$ be any perfect matching and $D := \mathcal{D}(B, M)$. Moreover, let

$$\mathsf{Q} := \left\{ \mathcal{D}(\mathfrak{T}, M) = (\mathcal{D}(\mathfrak{T}, M)^+, \mathcal{D}(\mathfrak{T}, M)^-) \mid \mathfrak{T} \in \mathsf{F} \right\}.$$

Then by Lemma 6.3.10, Q is a family of ditangles in D. Suppose there are $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathsf{Q}$ such that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are distinct and yet indistinguishable. Without loss of generality let us assume $\mathfrak{D}_1^+ \subseteq \mathfrak{D}_2^+$ and $\mathfrak{D}_1^- \subseteq \mathfrak{D}_2^-$. This however means that there are $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathsf{F}$ such that $\mathcal{D}(\mathfrak{T}_i, M) = \mathfrak{D}_i$ for both $i \in [1, 2]$. Moreover, we must have $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$ and thus we found two tangles in F that are indistinguishable. As this would contradict our choice of F, the ditangles in Q must be pairwise distinguishable. By Theorem 6.3.15 there exists a Q-tree-labelling (T, β, γ). Note that for every $\mathfrak{T} \in \mathsf{Q}$, $\mathcal{S}(\mathfrak{T}) \in \mathsf{F}$. For every $t \in V(T)$ let $\beta'(t) \colon \mathcal{S}(\beta(t))$, and for every $e \in E(T)$ let $\gamma'(e) := \mathcal{S}(\gamma(e))$. It is straight forward to see that (T, β', γ') is an F-tree-labelling. $\square$

A Tangle-Tree Decomposition Let B be a bipartite graph with a perfect matching and F be a family of pairwise distinguishable tangles in B. As mentioned above, one cannot expect to find an F-tree-labelling where all corresponding cuts are pairwise laminar. What we can, however, is finding a matching tree decomposition (T, β) of bounded edge-width, whose decomposition tree resembles an F-tree-labelling. Moreover, let us suppose that k bounds the maximum matching porosity of a cut induced by some set in some tangle in F. If there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ that bounds the edge width of this matching tree decomposition we aim for by $f(k)$, then Theorem 5.3.2 guarantees us that we can find, for every edge $e \in E(T)$ and every perfect matching $M \in \mathcal{M}(B)$, a set $F \subseteq M$ of size at most $f(k)^2 + 2f(k) + k$ that guards the cut induced by e, and contains all edges M has in the generalised tight cut associated with e in the F-tree-labelling.

Definition 6.3.18 (Tangle Tree Decomposition)**.** Let B be a bipartite graph with a perfect matching, and let T be a family of pairwise distinguishable tangles in B. A *matching tree decomposition for* T is a tuple (T, β, γ, τ), where T is a tree, τ is an injective map from T to $V(T)$, $\gamma \colon E(T) \to \mathcal{T}(B)$, and $\beta \colon V(T) \to 2^{V(B)}$, such that

i) $T' := T[\{\tau(\mathfrak{T}) \mid \mathfrak{T} \in \mathsf{T}\}]$ is a subtree of T, and $(T', \tau, \gamma|_{E(T')})$ is a T-tree-labelling,

ii) (T, β) is a matching tree decomposition of B, and
iii) for all $e \in E(T)$, $\partial_B(e)$ and $\partial_B(\gamma(e))$ are laminar.

We say that (T, β, γ, τ) *extends* the F-tree-labelling $(T', \tau, \gamma|_{E(T')})$. The *edge-width* of (T, β, γ, τ) is the edge-width of (T, β). Let $f\colon \mathbb{N} \to \mathbb{N}$ be any function. The edge-width of (T, β, γ, τ) is *f-bounded* in $(T', \tau, \gamma|_{E(T')})$ if $\operatorname{mp}(\partial_B(e)) \leq f(\operatorname{mp}(\partial_B(\gamma(e))))$ for all $e \in E(T')$.

A *tangle tree decomposition* for B is a matching tree decomposition for some family F of tangles in B.

Note that the cuts $\partial_B(e)$ induced by the edges of T in the definition above are indeed pairwise laminar. So in some sense the structure above can be understood as a way to slightly 'nudge' the generalised tight cuts that distinguish our tangles in such a way that we obtain a laminar family of cuts that roughly described the tree like structure of the areas of 'high connectivity'[3]. The price we pay for this 'nudging' is that the resulting cuts are no longer generalised tight cuts. Still, since each cut $\partial_B(e)$ is laminar with the generalised tight cut associated with e via γ it is possible to select any tangle $\mathfrak{T} \in \mathsf{T}$ and orient all edges of T towards the unique side that either contains the large side of $\gamma(e)$, or that does not contain its small side. Hence in a way, a tangle tree decomposition for a bipartite graph with a perfect matching can be seen as an approximation of its actual tangle structure where we allow any edge cut of bounded matching porosity instead of just generalised tight cuts.

We will show the following theorem for bipartite graphs with perfect matchings.

Theorem 6.3.19 (X*)**.** Let B be a bipartite graph with a perfect matching.

i) Let $k \in \mathbb{N}$ be a positive integer and let T be a family of pairwise distinguishable tangles of order at least k in B. Then there exists a matching tree decomposition for $\mathsf{T}|_k$ of edge-width at most $2k^2 + 4k$. More precisely, every $\mathsf{F}|_k$-tree-labelling can be extended to a matching tree decomposition of edge-width at most $2k^2 + 4k$.
ii) Let F be a family of pairwise distinguishable tangles in B. Then for every F-tree-labelling $\mathcal{L}$ there exists a matching tree decomposition that extends $\mathcal{L}$ and whose edge width is $2k^2 + 4k$-bounded in $\mathcal{L}$.

[3]If 'connectivity' is the right term for the matching setting.

As before, most of the proof of Theorem 6.3.19 consists of a translation between the setting of digraphs and the setting of bipartite graphs with prefect matchings. To achieve this we first need the corresponding notions for digraphs.

Definition 6.3.20 (Ditangle Tree Decomposition)**.** Let D be a digraph, and T be a family of pairwise distinguishable ditangles in D. A *directed tree decomposition for* T is a tuple $\mathcal{D} = (T, \beta, \gamma, \tau, \omega)$, where T is an arborescence, τ is an injective map from T to $V(T)$, $\beta \colon V(T) \to 2^{V(D)}$, $\gamma \colon E(T) \to \vec{\mathcal{S}}(D)$, and $\omega \colon E(T) \to 2^{V(D)}$, such that

i) $T' \coloneqq \mathrm{un}(T[\{\tau(\mathfrak{T}) \mid \mathfrak{T} \in \mathsf{T}\}])$ is a subtree of $\mathrm{un}(T)$, and $\mathcal{L} \coloneqq (T', \tau, \gamma|_{V(T')})$ is a T-tree-labelling,
ii) (T, β, ω) is a directed tree decomposition of D, and
iii) for all $e \in E(T)$ let $(G_e, H_e) \coloneqq \gamma(e)$, then $G_e \cap H_e \subseteq \omega(e)$.

We say that $\mathcal{D}$ *extends* $\mathcal{L}$. The *edge-width* of $\mathcal{D}$ is $\max_{e \in E(T)} |\omega(e)|$. We say that the edge-width of $\mathcal{D}$ is *f-bounded* in $\mathcal{L}$ for some function $F \colon \mathbb{N} \to \mathbb{N}$ if $|\omega(e)| \leq f(|\gamma(e)|)$ for all $e \in E(T')$ and, furthermore, if $e = (s,t) \in E(T) \setminus E(T[V(T')])$ then there is an edge $e' = (s, s') \in T[V(T')]$ with $|\omega(e)| < |\omega(e')|$.

A *ditangle tree decomposition* of D is a directed tree decomposition for some family F of pairwise distinguishable ditangles.

Theorem 6.3.21 ([GKK+20])**.** Let D be a digraph.

i) Let $k \in \mathbb{N}$ be a positive integer and let T be a family of pairwise distinguishable ditangles of order at least k in D. Then there exists a directed tree decomposition for $\mathsf{T}|_k$ of edge-width at most $k^2 + 2k$. More precisely, every $\mathsf{F}|_k$-tree-labelling can be extended to a directed tree decomposition of edge-width at most $k^2 + 2k$.
ii) Let F be a family of pairwise distinguishable ditangles in D. Then for every F-tree-labelling $\mathcal{L}$ there exists a directed tree decomposition that extends $\mathcal{L}$ and whose edge width is $k^2 + 2k$-bounded in $\mathcal{L}$.

By closely inspecting the proof of Theorem 6.3.21 one can obtain the following strengthening of the second result.

Lemma 6.3.22 ([GKK+20])**.** Let D be a digraph, T a family of pairwise distinguishable ditangles in D, (T', τ, γ') a F-tree-labelling. Then there exists a directed tree decomposition $(T, \beta, \gamma, \tau, \omega)$ for T that extends

(T', τ, γ') and whose edge-width is $k^2 + 2k$-bounded in (T', τ, γ') such that for every $e = (s,t) \in E(T)$ with $\gamma(e) = (X_1, X_2)$ there is $i \in [1,2]$ with $\beta(T_t) \subseteq X_i$.

Proof of Theorem 6.3.19. Since the first statement of the theorem follows immediately from the second one, it suffices to only prove (ii). So let B be a bipartite graph with a perfect matching M, and let T of pairwise distinguishable tangles in B. Similar to the proof of Proposition 6.3.17 we consider the digraph $D \coloneqq \mathcal{D}(B, M)$ together with the family $\mathsf{Q} \coloneqq \{\mathcal{D}(\mathfrak{T}, M) \mid \mathfrak{T} \in \mathsf{T}\}$ of pairwise distinguishable ditangles.

Let (T', τ, γ') be T-tree-labelling. At least one such tree-labelling must exist by Proposition 6.3.17, and by setting $\tau'(t) \coloneqq \mathcal{D}(\tau(t), M)$ and $\gamma''(e) \coloneqq \mathcal{D}(\gamma'(e), M)$ for all $t \in V(T')$ and $e \in E(T')$ we obtain an F-tree-labelling (T', τ', γ'').

Now we may use Lemma 6.3.22 on (T', τ', γ'') to obtain a directed tree decomposition $(T, \beta', \gamma''', \tau'', \omega')$ for F whose edge-width is k^2+2k-bounded in (T', τ', γ'') such that for every $e = (s,t) \in E(T)$ with $\gamma'''(e) = (X_1, X_2)$ there is $i \in [1,2]$ with $\beta'(T_t) \subseteq X_i$. Let $h \in \mathbb{N}$ be the smallest integer such that $\operatorname{ord}(S) \leq h$ for all $\mathfrak{T} = (\mathfrak{T}^+, \mathfrak{T}^-) \in \mathsf{F}$ and all $S \in \mathfrak{T}^+ \cup \mathfrak{T}^-$. Notice that, since $|\omega'(e)| \leq h^2 + 2h$, the cycle porosity of $\beta'(T_t)$ is at most $2h^2 + 4h$ for every $e = (s,t) \in E(T)$. That is, because by the definition of directed tree decompositions, every directed walk that leaves $\beta'(T_t)$ and then re-enters $\beta'(T_t)$ must contain a vertex of $\omega'(e)$.

Now let us consider $(\operatorname{un}(T), \beta, \gamma, \tau)$, where τ is as above, $\beta(t) \coloneqq V(\beta'(t)) \subseteq M$, and $\gamma(e) \coloneqq \mathcal{S}(\gamma'''(e))$. Then is it straight forward to check that $(\operatorname{un}(T), \beta, \gamma, \tau)$ is a matching tree decomposition extending (T', γ', τ) whose edge width is $2k^2 + 4k$-bounded in (T', γ', τ). $\square$

Similar to our findings in Section 6.2, the bounds obtained via the strategy of translating results from digraph theory to the setting of bipartite graphs with perfect matchings are probably not optimal. Indeed, it is possible that a closer examination of tangle tree decompositions might also help to improve the bounds for their digraphic analogues. However, when one does not care to much about the optimality of bounds, this chapter illustrates the ease with which structural findings from one setting can be translated into similar results in the other setting. Indeed, it is probably possible to obtain a ditangle tree decomposition for any M-direction of B

just from one tangle tree decomposition of B. In this sense, the notion of tangles appears to provide a unified way to describe the regions of high connectivity in any M-direction of B.

Chapter 7.

Excluding a Planar Matching Minor

So far we have investigated the properties of bipartite graphs with bounded perfect matching width as well as several dual notions. A natural next step in our pursuit of a matching theoretic minor theory would be to start describing the structure of classes of bipartite graphs that are defined by excluding a single matching covered bipartite graph as a matching minor by means of perfect matching width and its dual notions.

For any class of graphs $\mathcal{C}$ let us denote the class of graphs consisting of all graphs, that do not have any graph from $\mathcal{C}$ as a minor, by $\mathrm{Forbidden}(\mathcal{C})$. The first description of the class $\mathrm{Forbidden}(T)$, where T is a forest, was given in the first issue of the Graph Minors Project [RS83] which states that there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathrm{Forbidden}(T)$ has pathwidth at most $f(|V(T)|)$. We already stated a more general version of this in the form of Theorem 2.2.30. In this chapter we present a matching theoretic version of Theorem 2.2.30. For this we need two basic ingredients: On one side Theorem 5.3.32 provides us with large grid matching minors whenever the perfect matching width of a graph becomes too large. So if, on the other side, we have a matching theoretic analogue of Lemma 2.2.29 we will be able to prove the following theorem.

Theorem 7.0.1 (D*)**.** A proper matching minor closed class $\mathcal{G}$ of bipartite matching covered graphs has bounded perfect matching width if and only if it excludes a planar bipartite matching covered graph.

In Section 7.1 we present a proof of Theorem 7.0.1 based on the aforementioned matching theoretic analogue of Lemma 2.2.29. An important application of these findings in the Graph Minors Project was the characterisation of all graphs that have the Erdős-Pósa property for minors. In Section 7.3 we generalise this approach to show a similar result for matching minors in bipartite graphs. We show that every bipartite and planar matching covered graph has the matching Erdős-Pósa property for matching minors by adapting techniques from the original proofs for the setting of matching covered bipartite graphs. For the reverse however, the nature of the **matching** Erdős-Pósa property prevents us from doing the same. Instead, in Section 7.2 we use our insight on infinite anti-chains of butterfly minors gained from matching minors to present a version of Theorem 2.3.31 that interacts with anti-chains instead of a single graph. A nice pay off from this approach allows us to replace 'butterfly minor of the cylindrical grid' by a purely topological condition.

7.1. Planar Matching Covered Graphs in Grids

As explained above, the most important part towards Theorem 7.0.1 after the grid theorem itself is a matching theoretic version of Lemma 2.2.29. To achieve this goal, we make use of the iterative construction for bipartite matching covered graphs in the form of an ear decomposition as in Theorem 3.1.44. For this Please note that any ear we add to our graph is in fact an internally M-conformal path for some perfect matching M. Moreover, one can observe that any bipartite matching covered graph B has an ear decomposition and a perfect matching M, such that the conformal cycle B_2 obtained from K_2 by adding the first ear is M-conformal in B, and every B_i obtained from adding an additional ear P has the property that P is internally M-conformal.

Additionally, in case B is bipartite, matching covered, and planar, we can choose an ear decomposition as above in such a way that B_{i+1} can be drawn in the plane and the newly added ear is part of the boundary of a face.

Theorem 7.1.1 (D*)**.** For every planar bipartite matching covered graph H there exists a number ω_H such that H is a matching minor of the cylindrical matching grid of order ω_H.

Proof. Let B be a bipartite, matching covered and planar graph. Moreover, for any even k, let M be the residual perfect matching obtained by the strategy for finding the $k \times k$-grid as a matching minor of $CG_k^{\square}$ as described in the proof of Lemma 5.3.31.

We prove the claim by induction on the number of ears in an ear decomposition of B and strengthen it in the sense that we claim that there always exists an M-conformal matching minor model. As a base consider a single cycle of even length ℓ. Clearly each such cycle is actually contained as an M-conformal bisubdivision in the $\ell' \times \ell'$-grid, where ℓ' is the smallest natural number satisfying $\frac{\ell}{2} \leq \ell'$. So let $K_2 = B_1 \subset B_2 \subset \cdots \subset B_t$ be an ear decomposition of B. By the induction hypothesis, there exists an even number $\omega_{B_{t-1}}$ such that B_{t-1} is a matching minor of the $\omega_{B_{t-1}} \times \omega_{B_{t-1}}$-grid. Let μ' be an M-conformal matching minor model of B_{t-1} in said grid. Let P be the ear that, added to B_{t-1}, creates $B_t = B$. Then the canonical embedding of the grid in the plane induces an embedding of $\mu'(B_{t-1})$ in the plane and there exists a face f of said drawing that corresponds to the face of B_{t-1} in which P must be placed. Since P is non-empty, f must have more than four vertices, and thus there must exist a C_4 in the interior of f in the grid. Moreover, since $\mu'(B_{t-1})$ is M-conformal, there must exist such a C_4, say C, that does not contain a single edge of M.

We now draw two orthogonal lines through the centre of C, ℓ_1 in parallel to the columns of our grid and ℓ_2 in parallel to the rows of the grid. Each of the two ℓ_i can now be associated with an edge cut of the grid, containing only edges not in M. Together ℓ_1 and ℓ_2 partition the grid into four quadrants, see Figure 7.1 for an illustration. Let us say that the shores of ℓ_1 are $X_1 \cup X_2 \subseteq V(B)$ and $Y_1 \cup Y_2 \subseteq V(B)$, while the shores of ℓ_2 are $X_1 \cup Y_1$ and $X_2 \cup Y_2$. Please note that each of the X_i and Y_i is M-conformal. Moreover, let us fix X_1 to be the top left quadrant and Y_2 to be the bottom right one.

Now let H' be the $(\omega_{B_{t-1}} + p) \times (\omega_{B_{t-1}} + p)$-grid where $p = |V(P)|$, note that p is even, and let us map the vertices of the X_i and Y_i to the four corners of H', let X_i' and Y_i' be the corresponding vertex sets in H', let h be said mapping. Let us furthermore extend M to the corresponding perfect matching of H'. In order to extend μ' to a model of B_{t-1} in H' we need to replace the edges in the two cuts ℓ_1 and ℓ_2 by internally M-conformal paths connecting X_1 with X_2 and Y_1 with Y_2. In case μ' uses two vertical edges incident with the two endpoints of an edge of M, this might not be possible. To deal with this problem we apply a further blow-up to H', namely we double its width. Let H be the

$(\omega_{B_{t-1}} + p) \times (3\omega_{B_{t-1}} + p - 4)$-grid obtained from H' as follows. First, let $x \in \mathbb{N}$ be the number of columns in $H'[X_1]$. Then let Z_X be the $p \times x$-grid made up of the vertices in the columns of H' that connect $h(X_1)$ and $h(X_2)$. Let Z_Y be defined analogously, see Figure 7.1 for an illustration.

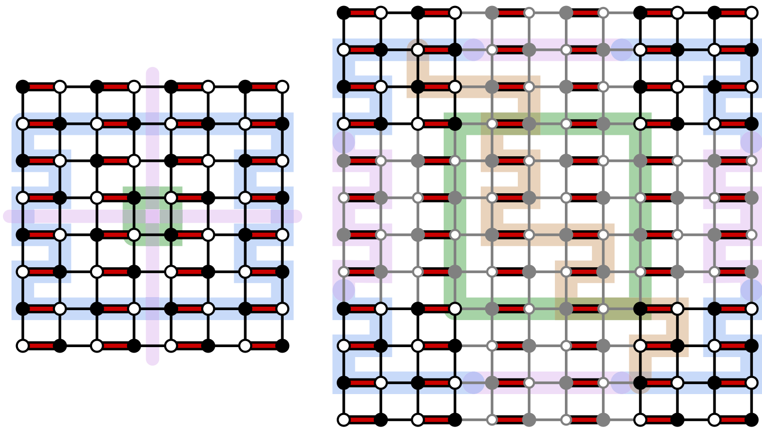


Figure 7.1.: Expanding a grid together with a model of an even cycle to add an ear. The small marked C_4 is replaced by a large grid which then is extended to make a new quadratic grid. Then the old model is extended by routing through the new part and lastly the ear is routed through the newly added central grid.

For every $W \in \{X, Y\}$ take $H'_W := H'[W_1 \cup W_W \cup W_2]$, then subdivide every horizontal edge and complete each thereby newly created column to a path. For every $v_{i,j}$ of H'_W, $1 \leq i \leq \omega_{B_{t-1}} + p$, $1 \leq j \leq x - 1$, we thereby created two new vertices $v^1_{i,j}$ and $v^2_{i,j}$ subdividing the edge $v_{i,j}v_{i,j+1}$. Similarly for every $1 \leq i \leq \omega_{B_{t-1}} + p$ and every $x + p + 1 \leq j \leq \omega_{B_{t-1}} + p - 1$. Let us again adapt M to be the canonical extension of the perfect matching we used for H'.

We now describe how to extend μ' to H. Let $v \in V(B_{t-1})$ and $u_{i',j'} \in V(\mu'(v))$, then let $h(u_{i',j'}) = v_{i,j}$. Every edge $u_{i'',j'}u_{i',j'} \in E(\mu'(v))$ with $i'' \in \{i' - 1, i' + 1\}$ is replaced by the edge $h(u_{i'',j'})h(u_{i',j'})$. We extend the model of v by the path $(v_{i,j}, v^1_{i,j}, v^2_{i,j})$

if $j \in \{n \mid 1 \leq n \leq x-1,$ or $x+p+1 \leq n \leq \omega_{B_{t-1}}+p-1\}$, and every edge $u_{i',j'}u_{i',j'+1} \in E(\mu'(v)) \setminus \ell_1$ is replaced by the edge $v^2_{i,j}h(u_{i',j'+1})$. If there is an edge $ab \in E(\mu'(v)) \cap \ell_1$, we replace this edge by the horizontal and internally M-conformal $h(a)$-$h(b)$-path in H. At last, an edge $ab \in E(\mu'(v)) \cap \ell_2$ is replaced by a vertical $h(a)$-$h(b)$-path in H. This path has to use vertices from at most two columns and may go to the left (in decreasing j direction) if and only if $h(a)$ and $h(b)$ are in the column x or $\omega_{B_{t-1}}+p$.

Now let $uw \in E(B_{t-1})$. If $u_{i'j'}u_{i',j'+1} \in E(\mu(uw)) \setminus \ell_1$, then let $v_{i,j} := h(u_{i'j'})$ and we replace the edge by the path $(v_{i,j}, v^1_{i,j}, v^2_{i,j}, v_{i,j+1})$ where $v_{i,j+1} = h(u_{i',j'+1})$. Edges $u_{i'j'}u_{i',j'+1} \in E(\mu(uw)) \cap \ell_1$ are replaced by the unique internally M-conformal horizontal $h(u_{i',j'})$-$h(u_{i',j'+1})$-path in H. An edge $ab \in E(\mu'(uw)) \cap \ell_2$ is replaced by a vertical $h(a)$-$h(b)$-path in H. This path has to use vertices from at most two columns and may go to the left (in decreasing j direction) if and only if $h(a)$ and $h(b)$ are in the column x or $\omega_{B_{t-1}}+p$. At last, a vertical edge $ab \in E(\mu(uw)) \setminus \ell_2$ will simply be replaced by $h(a)h(b)$.

In total let μ'' be the matching minor model of B_{t-1} constructed following the rules above. It is straight forward to check that $\mu''(B_{t-1})$ is M-conformal.

By construction there exists a $p \times p$-grid F in the face f' of $\mu''(B_{t-1})$ corresponding to the face f we chose in $\mu'(B_{t-1})$. As a last step, we have to add an internally M-conformal path P to our matching minor model in order to form a matching minor model μ of B. Let $a, b \in V(B_{t-1})$ be the endpoints of P, then both $\mu''(a)$ and $\mu''(b)$ must have an old vertex on f'. After possibly stretching the model of f' a bit we can find disjoint internally M-conformal paths from a and b to F, let a' and b' be their respective endpoints. Since F is a $p \times p$-grid we can easily find an internally M-conformal a'-b'-path P' within F. This path P' together with μ'' forms our desired matching minor model μ of B in H. At last note that H is a conformal subgraph of the $(3\omega_{B_{t-1}}+p-4) \times (3\omega_{B_{t-1}}+p-4)$-grid and thus we are done. $\square$

Please note that the function given in the proof above is exponential in the number of vertices of G and thus probably far from optimal. We immediately obtain an approximate description of the bipartite matching covered graphs that exclude a planar and bipartite matching covered graph H as a matching minor as stated in Theorem 7.0.1.

From Theorem 7.0.1 and Theorem 5.4.2 we can now also obtain a first positive algorithmic result regarding the complexity of matching minor testing in bipartite graphs with perfect matchings.

Corollary 7.1.2 (D*)**.** Let H be a bipartite, planar, and matching covered graph and B a bipartite graph with a perfect matching. There exists a constant $c = c(H)$ and an algorithm with running time $\mathcal{O}(|V(B)|^c)$ that decides whether B contains H as a matching minor.

7.2. Digraphs and Erdős-Pósa for Butterfly Minor Anti-Chains

Let us consider the more restrictive setting of digraphs. Here some interesting phenomena occur regarding the Erdős-Pósa property for butterfly minors and topological studies in general. As we have seen in Section 2.3.3, a strongly connected digraph H has the Erdős-Pósa property for butterfly minors if and only if it is a butterfly minor of the cylindrical grid. In some sense this is already a topological requirement as 'being a cylindrical grid minor' can be seen as some notion of embeddability in a very specialised surface. Moreover, planarity is indeed a necessary requirement for any strongly connected digraph H to be a minor of the cylindrical grid. But, as we will see in Chapter 8, the reverse is far away from being true. A remarkably simple example of a strongly connected planar digraph which is not a grid minor is the digraph depicted in Figure 7.2. The reason that this particular digraph is not contained in the cylindrical grid as a butterfly minor is, that in every planar embedding, the two concentric cycles, as depicted in the figure, must be oriented in opposing directions. A close inspection of the cylindrical grid reveals that there cannot be two such directed cycles.

Indeed, when inspecting the cylindrical grid more carefully one can observe that it has a planar embedding that has additional properties on top of just being planar.

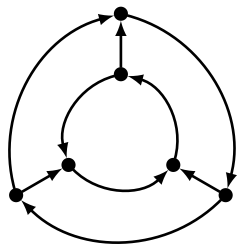


Figure 7.2.: A strongly connected and planar digraph that is **not** a butterfly minor of the cylindrical grid.

7.2.1. The Strong Genus of Digraphs

Let us first continue to talk about planar embeddings and a connection between embeddability of digraphs in the plane and planar bipartite graphs. Consider the cylindrical grid D of any order together with a planar embedding. Now zoom in on any vertex v and inspect an open disc ζ with v at its centre such that ζ does not contain any other vertex of D. Note that we can draw a curve γ through v connecting two points of the boundary of ζ such that every incoming edge of D incident with v lies on one side of γ, while every edge emanating from v lies on the other side of γ. Moreover, note that, by the definition of butterfly minors, every butterfly minor of D must also have a plane embedding with this property. With this we may rule out any planar digraph which does not have such an embedding as a candidate for being a butterfly minor of the cylindrical grid. See Figure 7.3 for a strongly connected planar digraph which does not have such an embedding. Moreover, notice that this particular digraph has exactly two butterfly contractible edges and by contracting both of them one obtains $\overleftrightarrow{K}_3$.

Let us formally introduce this concept. The definitions given here only scratch the surface of topological graph theory, see [Sta78, Arc96] for broader introduction and an overview of the topic.

Let G be a graph or digraph. Then G corresponds to a topological space called the *geometric realisation* of G. In this space the vertices are distinct points and the edges are subspaces homeomorphic to the closed interval

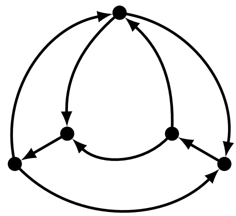


Figure 7.3.: A strongly connected and planar digraph that has no strong embedding.

$[0,1]$ over the real numbers[1] joining their endpoints. An *embedding* of G into some topological space X is a homeomorphism between the geometric realisation of G and a subspace of X. In a slight abuse of notation we use G for both the graph G and its geometric realisation. A *surface* is a compact Hausdorff topological space which is locally isomorphic to $\mathbb{R}^2$. There are two ways to construct these surfaces; either take a sphere and attach $n \in \mathbb{N}$ handles to it, or take a sphere and attach $m \in \mathbb{N}$ crosscaps. Let us denote by Σ_n the surfaces of the first kind and by $\tilde{\Sigma}_m$ the surfaces of the second kind.

Theorem 7.2.1 ([Bra21])**.** The surfaces in $\{\Sigma_n \mid n \in \mathbb{N}\}$ and $\left\{\tilde{\Sigma}_M \mid m \in \mathbb{N},\ m \geq 1\right\}$ are pairwise non-homeomorphic and every surface is homeomorphic to a member of one of these two families.

A surface Σ is *orientable* and of *orientable genus* n if it is homeomorphic to Σ_n, similarly, Σ is *non-orientable* and of *non orientable genus* m if it is homeomorphic to $\tilde{\Sigma}_m$.

A 2-*cell embedding* or *map* of a (di)graph G is an embedding in which every face is homeomorphic to an open disk. The *genus* of a (di)graph G is the smallest integer $g \in \mathbb{N}$ such that G can be embedded in Σ_g, and its *non-orientable genus* is the smallest integer $g' \in \mathbb{N}$ such that G can be embedded in $\tilde{\Sigma}_{g'}$. The *Euler genus* of G, denoted by $\text{genus}(G)$, is the smallest integer $h \in \mathbb{N}$ such that G can be embedded in $\Sigma_{\frac{h}{2}}$ or $\tilde{\Sigma}_h$.

Let G be a (di)graph embedded in a surface Σ, $v \in V(G)$ a vertex and $\zeta \subseteq \Sigma$ an open disc centred at v such that every edge of G incident with

[1] In this instance we do **not** use our definition of $[0,1]$ as the set $\{0,1\}$.

v contains exactly one point from the boundary β of ζ. Let $F \subseteq E(G)$ be the edges incident with v and $\{F_1, F_2\}$ be a bipartition of F. For each $f \in F$ let $p_f \in \beta$ be the point that f has on the boundary of ζ. We say that (F_1, F_2) is a *butterfly in* ζ if there exists a curve γ through v in Σ with both endpoints, x and y on β such that we can number the two internally disjoint curves $\beta_1 \subseteq \beta$ and $\beta_2 \subseteq \beta$ with endpoints x and y to obtain $\{p_f \mid f \in F_i\} \subseteq \beta_i$ for both $i \in [1, 2]$.

Definition 7.2.2 (Strong Embedding)**.** Let D be a digraph and Σ be a surface. An embedding $\mu\colon D \to \Sigma$ of D into Σ is *strong* if for every vertex $v \in V(D)$ there exist $r_v \in \mathbb{R}$ and an open disc $\zeta \subseteq \Sigma$ of radius r_v centred at v such

$$(\{(u,v) \mid (u,v) \in E(D)\}, \{(v,u) \mid (v,u) \in E(D)\})$$

is a butterfly in ζ.

The smallest integer $h \in \mathbb{N}$ such that D can be strongly embedded in $\Sigma_{\frac{h}{2}}$ or $\tilde{\Sigma}_h$ is called the *strong genus* of D. We denote the strong genus of D by $\mathrm{sgenus}(D)$. If $\mathrm{sgenus}(D) = 0$, D is said to be *strongly planar*.

Note that the strong genus of a digraph D is closed under vertex and edge deletion. Moreover, let $e = (u, v)$ be a butterfly contractible edge of D and assume D is strongly embedded into some surface Σ. By definition of butterfly minors (u, v) is the only outgoing edge of u, or the only incoming edge at v. In both cases, after adjusting the embedding of D into Σ for the digraph D' obtained from D by contracting e, the incoming and outgoing edges of the contraction vertex w still form a butterfly in some open disc in Σ centred at w. Hence we have the following observation.

Observation 7.2.3 (X^*)**.** Let D be a digraph and D' be a butterfly minor of D, then $\mathrm{sgenus}(D') \leq \mathrm{sgenus}(D)$.

As seen in Figure 7.3 we have $\mathrm{genus}(D) \leq \mathrm{sgenus}(D)$, but the reverse is not true. In fact, in Chapter 8 we will see that there does not exist a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{sgenus}(D) \leq f(\mathrm{genus}(D))$. The combination of these observations illustrates that the topology associated with butterfly minors should in fact consider the strong genus rather than the Euler genus of D. Indeed, the strong genus of digraphs is closely linked to the Euler genus of their splits.

Proposition 7.2.4 (X*)**.** Let B be a bipartite graph with a perfect matching M and $D \coloneqq \mathcal{D}(G, M)$. Then $\operatorname{genus}(B) = \operatorname{sgenus}(D)$.

Proof. First let $g \coloneqq \operatorname{genus}(B)$ and consider an embedding of B into a surface Σ such that $\Sigma = \Sigma_{2g}$ if B has an embedding in Σ_{2g}, and $\Sigma = \tilde{\Sigma}_g$ otherwise. Now contract the edges of M and let $ab \in M$ be any edge. Note that we may find an open disc $\zeta \subseteq \Sigma$ and a curve γ through a such that $\{\{ab\}, \{ax \mid ax \in E(B - b)\}\}$ is a butterfly in ζ. Indeed, the same holds true if we swap a and b. Hence after contracting ab into the vertex v_{ab}, $\{\{v_{ab}x \mid ax \in E(B - b)\}, \{v_{ab}x \mid bx \in E(B - a)\}\}$ is a butterfly in ζ and thus D has a strong embedding in Σ.

For the reverse let $h \coloneqq \operatorname{sgenus}(B)$ and consider a strong embedding of D into a surface Σ such that $\Sigma = \Sigma_{2h}$ if D has a strong embedding in Σ_{2h}, and $\Sigma = \tilde{\Sigma}_h$ otherwise. Consider $\mathcal{S}(D)$ and let M be the perfect matching of $\mathcal{S}(D)$ such that D is the M-direction of $\mathcal{S}(D)$. Adapt the embedding of D in Σ for $\mathcal{S}(D)$ by placing the two endpoints of each edge in M as close together as possible. Let $v \in V(D)$ be any vertex and $e_v = ab \in M$ the corresponding matching edge in $\mathcal{S}(D)$ with $a \in V_1$. Since, in our embedding of D in Σ, the out- and incoming edges at every vertex$v \in V(D)$ form a butterfly, this butterfly induces a bipartition of the edges of $\mathcal{S}(D)$ incident with the endpoints of e_v that resembles this butterfly. Hence the edge ab can be added to the embedding without producing a crossing. □

Hence we obtain the following immediate corollary which was implicitly stated in [RST99, GT11].

Corollary 7.2.5 ([RST99, GT11])**.** A digraph D is strongly planar if and only if $\mathcal{S}(D)$ is planar.

In light of Corollary 7.2.5 it comes at no surprise that there exists a deep connection between strongly planar digraphs and non-even digraphs. To state the corresponding result we need a digraphic version of the trisum operation.

Definition 7.2.6 (Small-Cycle-Sum)**.** Let D_0 be a digraph, let $u, v \in V(D_0)$, and let $(u, v), (v, u) \in E(D_0)$. Let D_1 and D_2 be such that $D_1 \cup D_2 = D_0$, $V(D_1) \cap V(D_2) = \{u, v\}$, $V(D_1) \setminus V(D_2) \neq \emptyset$, $V(D_2) \setminus$

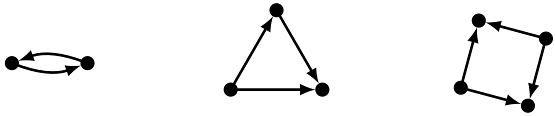


Figure 7.4.: The subgraphs necessary for the small-cycle-sum operation.

$V(D_1) \neq \emptyset$, and $E(D_1) \cap E(D_2)$. Let D be obtained from D_0 by deleting some (possibly neither) of the edges (u, v), (v, u). We say that D is a 2-*sum* of D_1 and D_2.

Let D_0 be a digraph, let $u, v, w \in V(D_0)$ and $(u, v), (w, v), (w, u) \in E(D_0)$, and assume that D_0 has a directed cycle containing the edge (w, v), but not the vertex u. Let D_1 and D_2' be such that $D_1 \cup D_2' \neq D_0$, $V(D_1) \cap V(D_2') = \{u, v, w\}$, $V(D_1) \setminus V(D_2') \neq \emptyset$, $V(D_2') \setminus V(D_1) \neq \emptyset$, and $E(D_1) \cap E(D_2') = \{(u, v), (w, v), (w, u)\}$. Let D_2' have no edge with tail v, and no edge with head w and note that this means that (w, v) is butterfly contractible in D_2', let D_2 be the digraph obtained from D_2' by contracting (w, v). Let D be obtained from D_0 by deleting some (possible none) of the edges $(u, v), (w, v), (w, u)$. We say that D is a 3-*sum* of D_1 and D_2.

Let D_0 be a digraph, let $x, y, u, v \in V(D_0)$ as well as $(x, y), (x, v), (u, y), (u, v)$, and assume that D_0 has a directed cycle containing precisely two of the edges $(x, y), (x, v), (u, y), (u, v)$. Let D_1 and D_2' be such that $D_1 \cup D_2' = D_0$, $V(D_1) \cap V(D_2') = \{x, y, u, v\}$, $V(D_1) \setminus V(D_2') \neq \emptyset$, $V(D_2') \setminus V(D_1) \neq \emptyset$, and $E(D_1) \cap E(D_2') = \{(x, y), (x, v), (u, y), (u, v)\}$. Let D_2' have no edge with tail y or v, and no edge with head x or u and note that this means that the edges (x, y) and (u, v) are butterfly contractible. Let D_2 be the digraph obtained from D_2' by contracting the edges (x, y) and (u, v). Finally, let D be obtained from D_0 by deleting some (possible none) of the edges $(x, y), (x, v), (u, y), (u, v)$. We say that D is a 4-*sum* of D_1 and D_2.

We say that a digraph D is a *small cycle sum* of two digraphs D_1 and D_2 if it is an i-sum of D_1 and D_2 for some $i \in [1, 3]$.

Theorem 7.2.7 ([RST99])**.** Let D be a strongly 2-connected digraph. Then D is non-even if and only if it can be obtained from a family of strongly 2-connected strongly planar digraphs and F_7 by repeated applications of the small-cycle-sum operation.

Indeed, since $\mathcal{S}(D)$ is isomorphic to an odd Möbius ladder for every $D \in \mathfrak{A}(\overleftrightarrow{K}_3)$, Theorem 3.3.4 allows us to understand Theorem 7.2.7, at least in some sense, as a digraphic version of Wagner's characterisation of K_5-minor free graphs, where $\overleftrightarrow{K}_3$ takes on the role of K_5. A natural question to ask would be, whether one can also find a digraphic version of Wagner's Theorem on planar graphs. To obtain a theorem resembling Theorem 2.2.3 we first need to have a look at the case of bipartite graphs with perfect matchings. One direction of the following theorem was proven in [RST99]. The reverse follows from Theorem 3.3.4 and Corollary 4.3.8.

Theorem 7.2.8 ([RST99])**.** A brace B is planar if and only if it does not contain $K_{3,3}$, the Heawood graph, and the Rotunda as a matching minor.

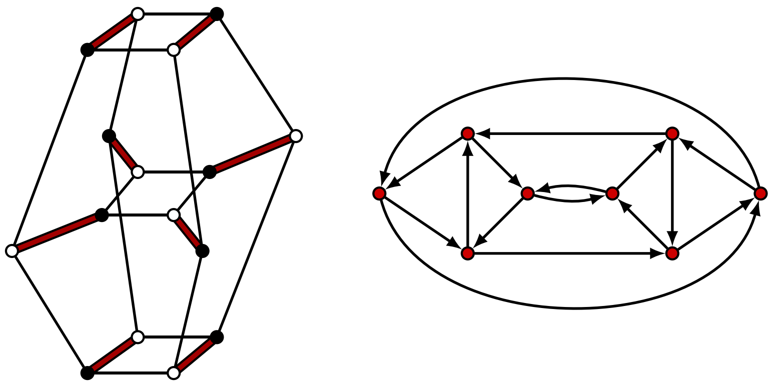


Figure 7.5.: The Rotunda R with a perfect matching M, and the M-direction $\vec{R} = \mathcal{D}(R, M)$ on the right.

By applying Lemma 3.2.25 and Proposition 7.2.4 to Theorem 7.2.8 we obtain the following characterisation of strongly 2-connected strongly planar digraphs.

Proposition 7.2.9 (X*)**.** Let D be a strongly 2-connected digraph. Then D is strongly planar if and only if it does not contain a digraph from $\mathfrak{A}(\overleftrightarrow{K}_3)$, $\mathfrak{A}(\vec{R})$, as defined in Figure 7.5, and $\mathfrak{A}(F_7)$ as a butterfly minor.

Interestingly, $\mathfrak{A}(\overleftrightarrow{K}_3)$ consists completely of planar digraphs, while $\mathfrak{A}(\vec{R})$ contains planar and non-planar digraphs. All of these characterisations heavily rely on 2-extendibility or strong 2-connectivity to make the trisum or small-cycle-sum operation work. Still one should expect that there is a characterisation of all planar matching covered graphs, and therefore all strongly connected strongly planar digraphs in terms of matching minors or butterfly minors respectively. The most promising way to achieve such a characterisation is probably to consider the setting of bipartite graphs with perfect matchings and then use Lemma 3.2.25 and Proposition 7.2.4 as we did above.

Strongly Planar Digraphs and Butterfly Minors of the Cylindrical Grid While the cylindrical grid generally is not strongly 2-connected, it can be observed, as described above, to be strongly planar. Still strong planarity does not seem to be enough as the digraph in Figure 7.2 is also strongly planar[2] but not a butterfly minor of the cylindrical grid. To get closer to a digraphic analogue of Lemma 2.2.29 we will use Theorem 7.1.1 and the notion of canonical anti-chains.

Theorem 7.2.10 (X^*)**.** Let D be a strongly connected digraph. Then D is strongly planar if and only if $\mathfrak{A}(D)$ contains a butterfly minor of the cylindrical grid.

Proof. Let us assume D to be strongly planar. Then $B \coloneqq \mathcal{S}(D)$ is planar and matching covered. By Theorem 7.1.1 B is a matching minor of the $\omega_B \times \omega_B$-grid. The $\omega_B \times \omega_B$-grid however is a matching minor of $CG_{3\omega_B}$ by Lemmata 5.3.30 and 5.3.31. Let G be the cylindrical grid of order $3\omega_B$, then $\mathcal{S}(G) = CG_{3\omega_B}$ and thus, by Lemma 3.2.25 G must contain a butterfly minor H which is a member of $\mathfrak{A}(D)$.

For the reverse direction let us assume there is $H \in \mathfrak{A}(D)$ such that H is a butterfly minor of the cylindrical grid. That means for some $k \in \mathbb{N}$, the cylindrical grid of order k, let us call it G, contains H as a butterfly minor. By Lemma 3.2.17 this means that $\mathcal{S}(G) = CG_k$ contains $\mathcal{S}(H)$ as a matching minor. As $\mathcal{S}(D)$ is a matching minor of $\mathcal{S}(H)$ and $\mathcal{S}(H)$ is a matching minor of a planar graph, $\mathcal{S}(D)$ must be planar and therefore D is strongly planar. $\square$

[2] In fact every subcubic digraph that is planar is necessarily strongly planar.

There is an immediate consequence of Theorem 7.2.10 which we state without proof. A proof for the matching theoretic analogue can be found in Section 7.3.

Corollary 7.2.11 (X*)**.** Let $\mathcal{D}$ be a proper butterfly minor closed class of digraphs. Then $\mathcal{D}$ has bounded directed treewidth if and only if there exists a strongly connected strongly planar digraph H such that no member of $\mathcal{D}$ contains a digraph from $\mathfrak{A}(H)$ as a butterfly minor.

7.2.2. Erdős-Pósa for Anti-Chains

With Theorem 7.2.10 we have an exact description of all strongly connected digraphs D for which $\mathfrak{A}(D)$ contains a butterfly minor of the cylindrical grid. Moreover, since recognising strongly planar digraphs is equivalent to recognising planar bipartite graphs with perfect matchings, we can recognise these digraphs in polynomial time. Let us define a generalised version of the Erdős-Pósa property for digraphs based on canonical anti-chains.

Definition 7.2.12 (Generalised Erdős-Pósa Property for Butterfly Minors)**.** Let H be a strongly connected digraph. We say that H has the *generalised Erdős-Pósa property for digraphs* if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, every digraph D either contains k pairwise disjoint subgraphs such that each of them has a butterfly minor isomorphic to some member of $\mathfrak{A}(H)$, or there exists a set $S \subseteq V(D)$ with $|S| \leq f(k)$ such that $D - S$ does not contain a digraph from $\mathfrak{A}(H)$ as a butterfly minor.

Our digraphic analogue of Theorem 2.2.32 is as follows. In the forward direction of the proof we use a generalised argument similar to the one used to proof the forward direction of Theorem 2.3.31, while for the reverse we also adapt the strategy from [AKKW16], this time we stick even closer to the original.

Theorem 7.2.13 (X*)**.** A strongly connected digraph D has the generalised Erdős-Pósa property for butterfly minors if and only if D is strongly planar.

Proof. Given a strongly connected strongly planar digraph D let us denote by ω_D the smallest integer w such that $\mathfrak{A}(D)$ contains a butterfly minor of the cylindrical grid of order w. Note that for any positive integer $k \in \mathbb{N}$ the cylindrical grid of order $k\omega_D$ contains k pairwise vertex disjoint subgraphs, all of which contain a digraph from $\mathfrak{A}(D)$ as a matching minor. Let us recursively define the function $f_D \colon \mathbb{N} \to \mathbb{N}$ for the generalised Erdős-Pósa property, where $f_D(0) := 0$, and for $k \geq 1$ let

$$f_D(k) := f_D(k-1) + \mathsf{g}_{\mathrm{dir}}(k\omega_D) + 1.$$

Now if $\mathrm{dtw}(D) \geq \mathsf{g}_{\mathrm{dir}}(k\omega_D) + 1$, then by Theorem 2.3.22 D contains the cylindrical grid of order $k\omega_D$ as a butterfly minor and thus, as discussed above, D contains k pairwise disjoint subgraphs, each of which contain a digraph from $\mathfrak{A}(D)$ as a butterfly minor. So we may assume D to have a directed tree decomposition (T, β, γ) of width at most $\mathsf{g}_{\mathrm{dir}}(k\omega_D)$. Let us choose $t \in V(T)$ such that $D[\beta(T_t)]$ contains a butterfly minor isomorphic to some member of $\mathfrak{A}(D)$, but for all $t' \in V(T_t)$ with $t \neq t'$, $D[\beta(T_{t'})]$ does not contain any digraph from $\mathfrak{A}(D)$ as a butterfly minor. If no such t exists, D does not contain a digraph from $\mathfrak{A}(D)$ as a butterfly minor and thus we are done immediately. Indeed, we may use this case as the base case $k = 0$ of our induction. Hence we may assume $k \geq 1$ and thus t exists. Then $|\beta(t)| \leq \mathrm{dtw}(D) + 1 \leq \mathsf{g}_{\mathrm{dir}}(k\omega_D) + 1$ and every butterfly minor of $D[\beta(t)]$ that belongs to $\mathfrak{A}(D)$ must contain a vertex of $\beta(t)$. By induction we either find $k-1$ pairwise vertex disjoint subgraph of $D - \beta(T_t)$ all of which have a member $\mathfrak{A}(D)$ as a butterfly minor, or there is a set S' of vertices with $|S'| \leq f_D(k-1)$ such that $D - \beta(t) - S'$ has no member of $\mathfrak{A}(D)$ has a butterfly minor. In the first case, all $k-1$ subgraphs are vertex disjoint from $D[\beta(T_t)]$ and thus we are done. Otherwise $|\beta(t) \cup S| \leq f_D(k)$, and we are also done. Therefore every strongly connected strongly planar digraph D has the generalised Erdős-Pósa property for butterfly minors.

For the reverse direction let H be a strongly connected digraph which is not strongly planar. For each $k \in \mathbb{N}$, $k \geq 1$, we construct a digraph $D_{H,k}$ which contains no two disjoint subgraphs that have a digraph from $\mathfrak{A}(H)$ as a butterfly minor, but where one must delete at least k vertices to remove all occurrences of members of $\mathfrak{A}(H)$ as butterfly minors in $D_{H,k}$. Since k is arbitrary, this proves that no non-strongly planar digraph can have the generalised Erdős-Pósa property for butterfly minors. Let G_k be the cylindrical grid of order k and let C_1 be the outer-most of its concentric cycles. Let us select $e_1 = (v_1^1, v_2^1)$, $e_2 = (v_3^1, v_4^1)$, $\ldots$, $(v_{2k-1}^1, v_{2k}^1) \in E(C_1)$, where we identify v_{2k}^1 and v_0^1. Then let $e = (u, v) \in E(H)$ be an arbitrary edge. We introduce k pairwise vertex disjoint copies $H_1, \ldots, H_k$ of H and denote the copy of (u, v) in H_i by (u_i, v_i) for all $i \in [1, k]$. Then $D_{H,k}$ is defined as the digraph obtained by deleting the edges (u_i, v_i) for every $i \in [1, k]$ and introducing the edges (u_i, v_{2i}^1) and (v_{2i-1}^1, v_i) for each $i \in [1, k]$. Again we identify v_{2k}^1 and v_0^1. See Figure 7.6 for an illustration

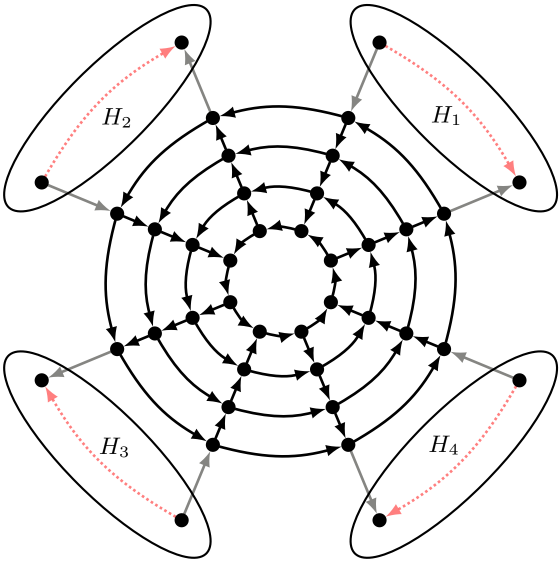


Figure 7.6.: A sketch of the construction of $D_{H,4}$ in the proof of Theorem 7.2.13.

First notice that any strongly connected subgraph K of $D_{H,k}$ such that K has a butterfly minor among $\mathfrak{A}(H)$ would need to contain a path from v_{2i}^1 to v_{2i-1}^1. To see this observe that any strongly connected subgraph K' of $D_{H,k}$ without such a path would either be a proper subgraph of H_i for some $i \in [1, k]$ and as $|V(J)| \geq |V(H)|$ and $|E(J)| \geq |E(H)|$ for all $J \in \mathfrak{A}(H)$ K' could not have a butterfly minor among $\mathfrak{A}(H)$, or K' would be a subgraph of G_k. But since H is strongly planar, $\mathfrak{A}(H)$ cannot contain a butterfly minor of the cylindrical grid by Theorem 7.2.10. Let P be a path from v^1_{2i} to v^1_{2i-1} as mentioned above. Note that for every $j \in [1,k] \setminus \{i\}$, $D_{H,k} - P$ does not contain a path from v^1_{2j} to v^1_{2j-1}. Hence $D_{H,k} - P$ does not have a butterfly minor among the graphs in $\mathfrak{A}(H)$ and thus $D_{H,k}$ cannot have two vertex disjoint subgraphs which each contain a butterfly minor from $\mathfrak{A}(H)$. On the other hand, let $S \subseteq V(D_{H,k})$ be a set of at most $k-1$ vertices. Then there must be some $i \in [1,k]$ such that S does not contain a vertex from H_i, and there is a directed path Q from v^1_{2i} to v^1_{2i-1} in G_k. Hence $H_i + (u_i, v^1_{2i}) + Q + (v^1_{2i-1}, v_i) - (u_i, v_i)$ is a subgraph of $D_{H,k} - S$ and it contains H as a butterfly minor and our proof is complete. $\square$

So while we now have a proof of Theorem 7.2.13 and a topological characterisation of all strongly connected digraphs that have the generalised Erdős-Pósa property for butterfly minors, it is not yet clear whether this result can be turned into an algorithm. If the directed treewidth of our input digraph D is larger than $f_H(k)$, which we can test in polynomial time by Theorem 2.3.18, we are done. Even though we do not exactly know how to construct the models within the cylindrical grid. But if the directed treewidth of D is smaller, we need to find the subtree T_t. To do so we would need to check for butterfly minor containment of some member of $\mathfrak{A}(H)$. By Theorem 2.3.29, if we fix a member $J \in \mathfrak{A}(H)$ we can test in polynomial time whether D contains J as a butterfly minor, but since $\mathfrak{A}(H)$ is potentially infinite this approach is not feasible.

However, in this second case we know that $\mathrm{pmw}(\mathcal{S}(D))$ is at most $f_H(k)$ by Theorem 5.3.27 and thus we can use Theorem 5.4.2 on any subgraph of D to check whether its split contains $\mathcal{S}(H)$ as a matching minor. By using Lemma 3.2.25 and iteratively reducing butterfly minors, we are able to find the desired vertex disjoint subgraphs that contain butterfly minors from $\mathfrak{A}(H)$, in polynomial time. Since the cylindrical grid of order ω_H has directed treewidth ω_H and thus its split has perfect matching width ω_H we can also use this approach to find k pairwise disjoint subgraphs of the cylindrical grid of order $k\omega_H$, all of which contain a member of $\mathfrak{A}(H)$ as a butterfly minor. This leads to the following corollaries.

Corollary 7.2.14 (X*)**.** Let H be a strongly connected strongly planar digraph. There exists a constant $\omega_H \in \mathbb{N}$ and an algorithm with running time $|V(D)|^{\mathcal{O}(|V(H)|^2+\omega_H^2)}$ that decides whether D contains a digraph from $\mathfrak{A}(H)$ as a butterfly minor.

Corollary 7.2.15 (X*)**.** Let H be a strongly connected strongly planar digraph and $k \in \mathbb{N}$ be a positive integer. There exists a constant $\omega_H \in \mathbb{N}$ and an algorithm with running time $|V(D)|^{\mathcal{O}(|V(H)|^2+k^2\omega_H^2)}$ that either finds k pairwise vertex disjoint subgraphs of D, each of which contain a digraph from $\mathfrak{A}(H)$ as a butterfly minor, or a set $S \subseteq V(D)$ with $|S| \leq f_H(k)$ such that $D - S$ has no butterfly minor isomorphic to a member of $\mathfrak{A}(H)$.

7.3. Matching Minors and the Erdős-Pósa Property

The primary goal of this chapter was the establishment of a matching theoretic analogue of Theorem 2.2.32. A first step towards this goal is of course a definition of the matching theoretic Erdős-Pósa property.

Definition 7.3.1 ((Bipartite) Erdős-Pósa Property for Matching Minors)**.** A (bipartite) matching covered graph H has the (bipartite) *Erdős-Pósa property for matching minors* if there exists a function $\varepsilon_H : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ any given (bipartite) matching covered graph G with a perfect matching $M \in \mathcal{M}(G)$ has k-pairwise disjoint M-conformal subgraphs, all of which contain H as a matching minor, or there exists an M-conformal set $S_H \subseteq V(G)$ with $|S_H| \leq \varepsilon_H(k)$ such that $G - S_H$ does not have H as a matching minor.

Please note that there is a subtle difference between the generalised Erdős-Pósa property for butterfly minors and the Erdős-Pósa property for matching minors. That is, in the directed case we are asking for some kind of hitting set for all subgraphs that contain a butterfly minor from the canonical anti-chain. In the matching setting however, deleting a conformal set in a graph with a perfect matching changes the perfect matchings themselves. So here is possible that $G - S$ still contains a subgraph that is a matching minor model of H, but the subgraph itself might not be conformal in $G - S$ any more. This property renders the matching version a bit more tricky to handle and marks the biggest novelty in the proof below.

A first step is to lift Theorem 7.1.1 to not only provide us with one H minor for some planar, bipartite and matching covered graph, but with k disjoint models. To see this simply observe that CG_{2k} has a perfect matching M and two vertex disjoint M-models of CG_k.

Corollary 7.3.2 (D*)**.** For every planar bipartite matching covered graph H there exists a number $\omega_{H,k}$ and a perfect matching M of the cylindrical matching grid of order $\omega_{H,k}$ such that $CG_{\omega_{H,k}}$ contains k pairwise vertex disjoint M-models of H.

For the remainder of this section let H be a fixed planar, bipartite, and matching covered graph. We define $f_H \coloneqq \mathbb{N} \to \mathbb{N}$ recursively as follows:

$$f_H(k) \coloneqq f_H(k-1) + 3(4\mathsf{g}_{\text{cyl}}(\omega_{H,k}) + 2\mathsf{g}_{\text{cyl}}(\omega_{H,k})^2).$$

Proposition 7.3.3 (D*)**.** Every planar bipartite matching covered graph H has the bipartite Erdős-Pósa property for matching minors.

Proof. We prove the claim by induction on k. For $k \leq 1$ there is nothing to show and thus we may assume $k \geq 2$. Let B be any bipartite graph with a perfect matching. If $\text{pmw}(B) > \mathsf{g}_{\text{cyl}}(\omega_{H,k})$, we are done immediately by the choice of $\omega_{H,k}$ and Theorem 5.3.29. Thus we may assume $\text{pmw}(B) \leq \mathsf{g}_{\text{cyl}}(\omega_{H,k})$. In this case let M be a perfect matching of B and (T, δ) be an M-decomposition of minimum width for B. Then $\text{width}(T, \delta) \leq 2\,\text{pmw}(B)$ by Theorem 5.1.13. Note that $B[\delta(t)]$ is an M-conformal subgraph of B for all inner edges $e \in E(T)$. We may assume that B contains a matching minor isomorphic to H since otherwise we would be done.

Now we replace some edges of T by directed edges with the colours blue and red. Let $t_1t_2 \in E(T)$ and if $i \in [1,2]$, let $j \in [1,2] \setminus \{i\}$. If for some $i \in [1,2]$ the graph $B[\delta(T_i)]$ contains a matching minor model of H that is conformal in B, delete t_1t_2 and proceed as follows:

- If $B[\delta(T_i)]$ has a matching minor isomorphic to H, introduce the blue edge (t_j, t_i).
- If the blue edge (t_j, t_i) does not exist, but $B[\delta(T_i)]$ contains a matching minor model of H that is conformal in B, introduce the red edge (t_j, t_i).

Afterwards every edge of T either still exists, is replaced by a single directed edge, two directed edges of the same colour, or two directed edges of different colour. Please note that, if (t_1, t_2) is a red edge and

$B_H \subseteq B[\delta(T_2)]$ a matching minor model of H that is conformal in B, then $\partial_B(t_1t_2) \cap M' \neq \emptyset$ for all perfect matchings M' of B for which B_H is M'-conformal. Moreover, at least one such perfect matching M' must exist.

Now, if the edge (t_1, t_2) exists and t_1 has another neighbour, say t_3, then (t_3, t_1) must also exist. Moreover, if (t_1, t_2) is blue, then (t_3, t_1) must also be blue. If (t_1, t_2) is red, then the colour of (t_3, t_1) is not determined by the colour of (t_1, t_2).

By Theorem 5.3.2 for every $e \in E(T)$ we can find a set $F \subseteq M$ with $|F| < 4\mathsf{g}_{\text{cyl}}(\omega_{H,k}) + 2\mathsf{g}_{\text{cyl}}(\omega_{H,k})^2$ that is a guard for $\partial_B(e)$. Let F_e denote such a guarding set for every $e \in E(T)$.

Suppose there exist $t_1, t_2 \in V(T)$ such that both edges, (t_1, t_2) and (t_2, t_1), exist and both are blue. In this case $F_{t_1t_2}$ is a guarding set for $\partial_B(t_1t_2)$ and for both $i \in \{1, 2\}$, $B_i \coloneqq B[\delta(T_i)]$ have a matching minor isomorphic to H. Hence we may apply our induction hypothesis to both B_i and are done.

Now suppose there exist $t_1, t_2 \in V(T)$ such that both edges, (t_1, t_2) and (t_2, t_1), exist and both are red. If M' is a perfect matching of B such that some $B[\delta(T_i)]$ contains an M'-conformal matching minor model of H, then $M' \cap \partial_B(t_1t_2) \neq \emptyset$. However, since $F_{t_1t_2}$ is a guarding set, $M'' \cap \partial_{B-V(F_{t_1t_2})}(\delta(t_1t_2) \setminus V(F_{t_1t_2})) = \emptyset$ for all perfect matchings of $B - V(F_{t_1t_2})$. Thus $B - F_{t_1t_2}$ does not have a matching minor isomorphic to H and we are done.

So we may assume that no edge of T is replaced by a monochromatic digon. In this case, there must be a vertex $t \in V(T)$ such that t has no outgoing blue edge. Let e_1, e_2, and e_3 be the edges incident with t in T. Then every matching minor model of H in B must either contain an edge of some $\partial_B(e_i)$, and thus must contain a conformal cycle crossing $\partial_B(e_i)$, or every perfect matching M' for which such a model is M'-conformal, has an edge in $\partial_B(e_i)$. In either case, $F \coloneqq F_{e_1} \cup F_{e_2} \cup F_{e_3}$ meets every M-conformal cycle that crosses $\partial_B(e_i)$ for all $i \in [1, 3]$. Hence in $B - V(F)$ no matching minor model of H exists. By setting $\varepsilon_H(k) \coloneqq 2f_H(k)$ our proof is complete. $\square$

Additionally, Corollary 7.1.2 provides us with the necessary tool to obtain an algorithmic version of Proposition 7.3.3.

Corollary 7.3.4 (D*)**.** Let H be a bipartite, planar, and matching covered graph, B a bipartite graph with a perfect matching, and $k \geq 1$ a positive integer. There exist constants $c_1 = c_1(H,k)$, $c_2 = c_2(H,k)$ and an algorithm with running time $\mathcal{O}(V(B)^{c_1})$ that either finds a perfect matching M of B and k pairwise vertex disjoint M-models of H in B, or a conformal set $S \subseteq V(B)$ of size at most c_2 such that $B - S$ does not have H as a matching minor.

As pointed out above, the matching version of the Erdős-Pósa property for minors does not necessarily ask for a hitting set as sometimes deleting a certain conformal set of vertices might destroy some perfect matchings in B and thereby render existing matching minor models non-conformal any more without actually hitting them. Due to this it is not obvious whether the standard approach to proving Theorem 2.2.32 can be applied for the reverse of Proposition 7.3.3. Instead we take a detour by utilising our findings in Section 7.2.2.

Theorem 7.3.5 (X*)**.** Let H be a matching covered bipartite graph. The following statements are equivalent:

i) H has the Erdős-Pósa property for matching minors
ii) $\mathcal{D}(H,M)$ has the generalised Erdős-Pósa property for butterfly minors for some $M \in \mathcal{M}(H)$, and
iii) $\mathcal{D}(H,M)$ has the generalised Erdős-Pósa property for butterfly minors for every $M \in \mathcal{M}(H)$.

Proof. To prove the assertion we take the following route: First we show that (i) implies (ii), then we deduce (iii) from (ii), and finally we show that (iii) implies (i) which completes the proof.

So let us assume H has the Erdős-Pósa property for matching minors and let $\varepsilon_H \colon \mathbb{N} \to \mathbb{N}$ be the associated function. Let us choose $M_H \in \mathcal{M}(H)$ and set $D_H \coloneqq \mathcal{D}(H, M_H)$. Then let D be any digraph, $B \coloneqq \mathcal{S}(D)$ and $M \in \mathcal{M}(B)$ such that $D = \mathcal{D}(B,M)$. Notice that D has k pairwise disjoint subgraphs, all of which contain some member of $\mathfrak{A}(D_H)$ as a butterfly minor, if and only if B has k pairwise disjoint M-conformal subgraphs all of which contain H as a matching minor. So in case D does not have k pairwise disjoint such digraphs, there must be an M-conformal set $S_H \subseteq V(B)$ with $|S_H| \leq \varepsilon_H(k)$ such that $B - S_H$ does not contain

H as a matching minor. Let $F \coloneqq M \cap E(B[S_H])$. Then, as S_H is M-conformal, we have $|F| \leq \frac{1}{2}\varepsilon_H(k)$, and by Lemma 3.2.25 $D - F$ does not contain any digraph from $\mathfrak{A}(D_H)$ as a butterfly minor. As our choice of D was arbitrary, we may set $f_{D_H} \coloneqq \frac{1}{2}\varepsilon_H$ and thus D_H has the generalised Erdős-Pósa property for butterfly minors.

Now let us assume there is $M_H \in \mathcal{M}(H)$ such that $D_H \coloneqq \mathcal{D}(H, M_H)$ has the generalised Erdős-Pósa property for butterfly minors. Let $M'_H \in \mathcal{M}(H) \setminus \{M_H\}$ and $D'_H \coloneqq \mathcal{D}(H, M'_H)$. Since the generalised Erdős-Pósa property for butterfly minors only cares about $\mathfrak{A}(D_H)$ and not necessarily about D_H itself, it suffices to show $\mathfrak{A}(D_H) = \mathfrak{A}(D'_H)$. Consider a digraph $J \in \mathfrak{A}(D'_H)$. Then every proper butterfly minor J' of J has the property that $\mathcal{S}(J')$ does not contain H as a matching minor, while $\mathcal{S}(J)$ does contain H as a matching minor. Therefore, J must be D_H-minimal and thus $J \in \mathfrak{A}(D_H)$. With the same argument one can also obtain $\mathfrak{A}(D_H) \subseteq \mathfrak{A}(D'_H)$ and our claim follows.

So at last we may assume $\mathcal{D}(H, M_H)$ has the generalised Erdős-Pósa property for butterfly minors for every $M_H \in \mathcal{M}(H)$ and let us fix any $M_H \in \mathcal{M}(H)$. Let $D_H \coloneqq \mathcal{D}(H, M_H)$, and let $f_{D_H} \colon \mathbb{N} \to \mathbb{N}$ be the function associated with the generalised Erdős-Pósa property for butterfly minors of D_H. Let B be any bipartite graph with a perfect matching M. As before, B contains k pairwise disjoint M-conformal subgraph, all of which have H as a matching minor, if and only if $D \coloneqq \mathcal{D}(B, M)$ contains k pairwise disjoint subgraphs, all of which have a butterfly minor from $\mathfrak{A}(D_H)$. So in case B does not have k such M-conformal subgraphs, then D does not have k such subgraphs either and thus there must exist a set $S_H \subseteq V(D)$ with $|S_H| \leq f_{D_H}(k)$ such that $D - S_H$ does not have any butterfly minor isomorphic to a member of $\mathfrak{A}(D_H)$. Note that $S_H \subseteq M$ and thus $|V(S_H)| \leq 2f_{D_H}(k)$, and $V(S_H)$ is an M-conformal set of vertices in B. Moreover, by Lemma 3.2.25 we know that $B - V(S_H)$ does not have H as a matching minor. So by setting $\varepsilon_H \coloneqq 2f_{D_H}$ we have found a function that witnesses the Erdős-Pósa property for matching minors of H. □

Now Theorem 7.3.5 allows us to use Theorem 7.2.13 in order to obtain the reverse direction of Proposition 7.3.3 and thus give a complete characterisation of all bipartite matching covered graphs that have the Erdős-Pósa property for matching minors. For this simply note that, if a bipartite and matching covered graph H has the Erdős-Pósa property for matching minors by Theorem 7.3.5, every M-direction D of H, where $M \in \mathcal{M}(H)$, has the generalised Erdős-Pósa property for butterfly minors. This however means that, by Theorem 7.2.13, D must be strongly planar, and thus, by Proposition 7.2.4, H must be planar. Hence we may close this chapter by stating the following theorem.

Theorem 7.3.6 (X^*)**.** Let H be a bipartite and matching covered graph. Then H has the Erdős-Pósa property for matching minors if and only if it is planar.

Chapter 8.

A Weak Structure Theorem

Bipartite matching covered graphs of small perfect matching width can be considered to be relatively well described by their perfect matching decompositions. As we have seen in Corollary 5.1.21, a matching minor of a bipartite graph of perfect matching width k cannot have perfect matching width more than $2k$. Moreover, on such a graph B we are able to solve the t-disjoint alternating paths problem in polynomial time by Theorem 5.4.1, and we are even able to determine, for a fixed bipartite matching covered graph H, whether B does contain H as a matching minor or not by Theorem 5.4.2. If the perfect matching width of B exceeds a certain value Theorem 5.3.29 yields the existence of a large planar matching minor. Indeed, even the areas that contain large grid matching minors or other matching minors of large perfect matching width can be arranged in a tree-like fashion, as seen in Theorem 6.3.19. By studying Theorem 5.3.29 more closely, one can obtain an approximate characterisation of all bipartite graphs with perfect matchings that exclude some bipartite, planar, and matching covered graph H as a matching minor, as stated in Theorem 7.0.1. Most of these results are obtained by a mixture of, sometimes previously established, results in structural digraph theory, such as Theorem 2.3.22 and Lemma 6.3.22, and the higher flexibility of the matching setting which essentially allows to change between equivalent digraphs whenever need arises. This second part has already produced some additional insight into the overall structure of digraphs, in particular when it comes to digraphs that exclude certain anti-chains of butterfly minors. As an example recall Theorem 7.3.5 which highlights the close relation between structural matching theory and structural digraph theory.

In most of the cases so far, we were concerned with the exclusion of some bipartite, planar, and matching covered graph H. But what can we say if H is not planar? This is largely where the input from structural digraph theory ceases to provide a deeper understanding, and where the direction of structural insight is indeed reversed. When excluding a single non-planar bipartite and matching covered graph as a matching minor, the structure that arises from excluding $K_{3,3}$ is probably the one that is best understood. Indeed, $K_{3,3}$ plays a major role in the bipartite Pfaffian recognition problem and several (computationally) equivalent problems on digraphs, as we described in Section 3.3. With Theorem 3.3.8 we have an example of a structure theorem that comes as close to similar results for (ordinary) minors, such as Wagner's Theorem on K_5-minor free graphs, as it might be possible within the realm of bipartite graphs with perfect matchings. Curiously, it was the discovery of Theorem 3.3.8 that lead to the first structure theorem of digraphs that exclude some strongly connected digraphs as a butterfly minor in form of Theorem 7.2.7. To this date, all known characterisations of classes of digraphs excluding some strongly connected butterfly minor are based on matching theory, and most proofs rely heavily on the matching theoretic machinery developed to deal with matching minors. One might ask what the reason for this could be, and the answer appears to be relatively straightforward: By Theorem 2.3.27 the 2-disjoint paths problem for digraphs is NP-complete, and thus it is unlikely that there is a directed version of the characterisation of 'flatness' as one can find in the form of Theorem 2.2.6. However, with Theorem 4.0.4, we have found a matching theoretic analogue, and especially by exploiting our finding on the existence of conformal bisubdivisions of $K_{3,3}$, we were able to turn this into the algorithmic counterpart Theorem 4.0.6 of Theorem 2.2.7. This means that whatever it is that is responsible for the problems one encounters when trying to work with butterfly minors on digraphs directly, this reason vanishes once one invokes the matching setting. Let H be some digraph. With the observation that a digraph D excludes the entire anti-chain $\mathfrak{A}(H)$ as butterfly minors if and only if $\mathcal{S}(D)$ does not contain $\mathcal{S}(H)$ as a matching minor, we have the perfect tool to synthesize a combined structure theory for digraphs and bipartite graphs with perfect matchings.

In this chapter, we combine all previous results into an approximate description of those bipartite graphs that exclude $K_{t,t}$ as a matching minor for some $t \in \mathbb{N}$ similar to Theorem 2.2.34 and Theorem 2.2.35. To do this, we once more combine the more structured digraphic setting, and make use of previously obtained results. This time we apply some lemmas used to prove a directed version of the Flat Wall Theorem and the increased flexibility of the matching setting to avoid the structural problems implied be results like Theorem 2.3.27. An important ingredient towards such a theorem is a notion of 'flatness' appropriate for the matching setting. Theorem 4.0.4 already hints at a possible notion of flatness, but the reductions used there are already too specialised. Indeed, not all crosses over conformal cycles are of a form that can be used to create larger matching minors. So Theorem 3.3.8 in its more general form might be a better tool for us.

Let B be a Pfaffian brace and H be a planar brace. We say that H is a *summand* of B if there exist planar braces $H_1, \dots, H_\ell$ such that B can be constructed from the H_i by repeated applications of the trisum operation, and $H = H_1$.

In contrast to the undirected setting, where it makes sense to speak about subgraphs as a means of reductions, in the setting of graphs with perfect matchings, we sometimes will have to perform tight cut contractions.

Let B and H be bipartite graphs with a perfect matching such that H has a single brace J that is not isomorphic to C_4. We say that H is a *J-expansion*. A brace G of B is said to be a *host of* H if G contains a conformal subgraph H' that is a J-expansion and can be obtained from H by repeated applications of tight cut contractions. The graph H' is called the *remnant of* H.

It makes sense for us to work with a cylindrical grid/wall rather than a square one. However, the only problem this brings is that any cylindrical wall has two faces which might be considered the natural 'outer face'. Indeed, we would like the inner-most and the outer-most cycle of the cylindrical wall to both bound faces in an appropriate reduction.

Let B, H, and J be bipartite graphs with perfect matchings such that H and J are conformal subgraphs of B. We say that H is *J-bound* if there exists a subgraph K of $B - J$ that is the union of elementary components of $B - J$ such that $K \cup J$ is matching covered, and H is a conformal subgraph of $K \cup J$. The graph $K \cup J$ is called a *J-base of H* in B.

Definition 8.0.1 (P-Flatness)**.** Let B be a bipartite graph with a perfect matching, and let H be a planar matching covered graph that is a J-expansion of some planar brace J. Moreover, let P be a collection of pairwise vertex disjoint faces of H such that P is a conformal subgraph of H. At last, let $A \subseteq V(B)$ be a conformal set. Then H is *P-flat* in B *with respect to A* if

i) H is a conformal subgraph of $B' := B - A$,
ii) some P-base of H in B' has a Pfaffian brace B'' that is a host of H, and
iii) B'' has a summand G that contains a remnant H' of H such that every remnant of a face from P within H' bounds a face of G.

The set A in the definition above is the *apex set*, similar to the one which occurs in the version of flatness used for the original Flat Wall Theorem. The set P takes on two roles at once, it mimics the separator of the separation (X, Y) in the original definition and also allows us to essentially prescribe which faces of H should take the role of the outer face. Since we do not require $B - A$ to be a brace or even matching covered, we need to remove all non-admissible edges and just take the matching covered subgraph that contains H. This is modelled by selecting a certain P-base of H in B'. Now that we have reduced $B - A$ to a matching covered graph B''', we must go one step further and get rid of the non-trivial tight cuts of B'''. This is done by selecting B'' to be a host of H in B'''. By requiring B'' to be Pfaffian, we ready ourselves for the final reduction. Indeed, since we insist G to be a summand of B'' this means B'' cannot be isomorphic to the Heawood graph by Theorem 3.3.8. Since G is a summand of B'', it must be a planar brace. To get to this point, some tight cut contractions could have been necessary, and thus we are only able to talk about a remnant H' of H, but since H was chosen to be a J-expansion of some planar brace J, this remnant is well defined. Similarly, the tight cut contractions could have shrunken some of the faces that were selected to form P, but we can still make out their remnants and thus (iii) resembles the third requirement of the original definition.

Next, we need a definition for our wall itself. Note that the cylindrical matching grid CG_k is already a cubic planar graph, and thus every bipartite graph with a perfect matching that has CG_k as a matching minor must also contain a conformal bisubdivision of CG_k. However, we need a slightly more restricted version of the cylindrical matching grid, since we want to apply some results that were originally proven for the cylindrical k-wall. Let M be the canonical matching of CG_{2k}, then CG_{2k} contains an M-conformal subgraph H such that $\mathcal{D}(H, M)$ is a cylindrical k-wall, and the split of a cylindrical k-wall clearly contains CG_k as a matching minor. Hence this restriction only costs us a factor of two in the function for the cylindrical grid.

Definition 8.0.2 (Matching Wall)**.** Let $k \in \mathbb{N}$ be a positive integer. The *elementary matching k-wall* with its *canonical matching* M is defined to be the bipartite graph W with perfect matching M such that $\mathcal{D}(W, M)$ is the elementary cylindrical k-wall. A *matching k-wall* W' is a bisubdivision of the elementary matching k-wall, and a perfect matching M' is its *canonical matching* if $\mathcal{D}(W', M')$ is a cylindrical k-wall.

The *perimeter* of W', denoted by $\mathsf{Per}(W')$, is the union of the outer-most and the inner-most M'-conformal cycle of W'.

Lemma 8.0.3 (G*)**.** There exists a function $\mathsf{w}_{\mathrm{m}} \colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every bipartite graph with a perfect matching M it holds that, if $\mathrm{pmw}(B) > \mathsf{w}_{\mathrm{m}}(k)$, then B contains a matching k-wall W as an M-conformal subgraph such that $M|_W$ is the canonical matching of W.

Consider the matching grid CG_k, and let X be a set that induces a non-trivial tight cut in CG_k. Notice that this means that either X or $\overline{X}$ must induce a subpath of length three on either the outer-most or the inner-most cycle of CG_k. This means there exists a unique brace J_k of CG_k that is not isomorphic to C_4. Now let W be a matching k-wall, then W contains CG_k as a subgraph, indeed, W contains a matching model of CG_k that contains all of W. Hence W is a J_k-expansion.

Let $k, t \in \mathbb{N}$ be positive integers and B be a bipartite graph with a conformal matching k-wall W, and let H be some bipartite matching covered graph. Let M be a perfect matching of B. We say that W *M-grasps* an H-matching minor if there exists a matching minor model $\mu \colon H \to B$ such that $\mu(H)$ is M-conformal, and for every $e \in E(H) \cap M|_H$ the M-conformal path $\mu(e)$ is completely contained in W. We say that W *grasps* an H-matching minor if there exists a perfect matching M of B such that W M-grasps H.

With this, we are ready to state our matching theoretic version of the Flat Wall Theorem.

Theorem 8.0.4 (G*)**.** Let $r, t \in \mathbb{N}$ be positive integers. There exist functions $\alpha\colon \mathbb{N} \to \mathbb{N}$ and $\rho\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every bipartite graph B with a perfect matching M the following is true: If W is an M-conformal matching $\rho(t, r)$-wall in B such that $M \cap E(W)$ is the canonical matching of W, then either

i) B has a $K_{t,t}$-matching minor[1] grasped by W, or
ii) there exist an M-conformal set $A \subseteq V(B)$ with $|A| \leq \alpha(t)$ and an M-conformal matching r-wall $W' \subseteq W - A$ such that W' is $\mathsf{Per}(W')$-flat in B with respect to A.

In the following, we sometimes say that a conformal matching k-wall W is *flat* in B, if k is large enough and the second part of the theorem above holds true for W.

A Weak Structure Theorem With Theorem 8.0.4 at hand, we can give an approximate characterisation of all bipartite graphs with perfect matchings that exclude $K_{t,t}$ as a matching minor for some $t \in \mathbb{N}$. This weak structure theorem is similar to Theorem 2.2.34 and in some sense can be seen as a generalisation of Theorem 3.3.8 in conjunction with Theorem 3.3.4.

Theorem 8.0.5 (G*)**.** Let $r, t \in \mathbb{N}$ be positive integers, α and ρ be the two functions from Theorem 8.0.4, and B be a bipartite graph with a perfect matching.

- If B has no $K_{t,t}$-matching minor, then for every conformal matching $\rho(t, r)$-wall W in B and every perfect matching M of B such that $M \cap E(W)$ is the canonical matching of W, there exist an M-conformal set $A \subseteq V(B)$ with $|A| \leq \alpha(t)$ and an M-conformal matching r-wall $W' \subseteq W - A$ such that W' is $\mathsf{Per}(W')$-flat in B with respect to A.
- Conversely, if $t \geq 2$ and $r \geq \sqrt{2\alpha(t)}$, and for every conformal matching $\rho(r, t)$-wall W in B and every perfect matching M of B such that $M \cap E(W)$ is the canonical perfect matching of W, there

[1] Please note that this matching minor is not necessarily M-conformal.

exist an M-conformal set $A \subseteq V(B)$ with $|A| \leq \alpha(t)$ and an M-conformal matching r-wall $W' \subseteq W - A$ such that W' is $\mathsf{Per}(W')$-flat in B with respect to A, then B has no matching minor isomorphic to $K_{t',t'}$, where $t' = 16\rho(t,r)^2$.

Proof. The first part of the theorem follows immediately from Theorem 8.0.4, since in case B does not have $K_{t,t}$ as a matching minor, the first part of Theorem 8.0.4 can never be true and thus every matching $\rho(t,r)$-wall must be flat in B.

For the reverse, note that an elementary matching $\rho(t,r)$-wall has exactly $16\rho(t,r)^2$ vertices. Now suppose B has a matching minor model $\mu\colon K_{t',t'} \rightarrow B$. Then there exists a perfect matching M such that μ is M-conformal. Indeed, $K_{t',t'}$ contains an $M|_{K_{t',t'}}$-conformal elementary matching $\rho(t,r)$-wall, and thus $\mu(K_{t',t'})$ contains an M-conformal matching $\rho(t,r)$-wall W. Indeed, for every vertex w of degree three in W there exists a unique vertex $u_w \in V(K_{t',t'})$ such that $w \in V(\mu(u_w))$, and in case $w \neq w'$ are both vertices of degree three in W, then $u_w \neq u_{w'}$. Moreover, if P is a path in W whose endpoints w and w' have degree three in W and all internal vertices are vertices of degree two in W, then $V(P) \subseteq V(\mu(u_w)) \cup V(\mu(u_{w'}))$.

By assumption there exist an M-conformal set $A \subseteq V(B)$ and an M-conformal matching r-wall $W' \subseteq W$ such that W' is $\mathsf{Per}(W')$-flat in B with respect to A. Now W' has $16r^2$ many vertices of degree three in W', $16r$ of which lie on $\mathsf{Per}(W')$. Since $r \geq \sqrt{2\alpha(t)}$, we have at least $32\alpha(t)$ many such degree three vertices. Thus, with $|A| \leq \alpha(t)$ and $t \geq 2$, there exist $w_1, \dots, w_6 \in V(W' - \mathsf{Per}(W'))$ such that $V(\mu(u_{w_i})) \cap A = \emptyset$ for all $i \in [1,6]$. This, however, means that for every $\mathsf{Per}(W')$-base H of W', every brace J of H that is a host of W' must contain $K_{3,3}$ as a matching minor and therefore no such J can be Pfaffian by Theorem 3.3.4. Hence W' cannot be $\mathsf{Per}(W')$-flat in B with respect to A and we have reached a contradiction. □

Organisation What remains is to prove Theorem 8.0.4. Our proof and the remainder of the chapter are organised as follows. In the current section we introduce the Directed Flat Wall Theorem as proposed in [GKKK20], we discuss the merits and the demerits of this theorem, and explain in more detail what challenges [GKKK20] faced and how they were handled.

The next three sections are dedicated to the proof of Theorem 8.0.4. Here we follow the known proofs of the (Directed) Flat Wall Theorem with some slight alterations. Specifically, given a bipartite graph B with a perfect matching M and a large M-conformal matching wall W we show that

i) If there are many pairwise disjoint internally M-conformal paths that are internally disjoint from W and have both of their endpoints in W but they are far apart from each other in W, we find $K_{t,t}$ as an M-minor grasped by W. This is done in Section 8.1 and we essentially adapt the tools introduced in [GKKK20] to achieve this.

ii) The second part consists of two steps at once, both of which can be solved by the same technique, but since they are still slightly different, we explain both:

   a) In case there are many pairwise disjoint internally M-conformal paths that are internally disjoint from W and have both of their endpoints in W and close together in W but not in the same cell, we can find many pairwise disjoint matching minor models of $K_{3,3}$, and those can be used to construct a matching minor model of $K_{t,t}$ which is grasped by W.

   b) Finally, we know that every internally M-conformal path that is internally disjoint from W must have both endpoints on the same cell of W. Hence we may associate with every cell of W a bipartite matching covered graph that is otherwise disconnected from W. If many of these cells are, essentially non-Pfaffian, we can again find many pairwise disjoint models of $K_{3,3}$ which then can be used to construct a matching minor model of $K_{t,t}$ that is grasped by W.

While the second and third steps are relatively similar to the proof of the undirected Flat Wall Theorem, they differ vastly from their directed analogues. Both steps, (ii) and (iii), are discussed in Section 8.3. The actual proof of Theorem 8.0.4 and thus the combination of all three steps is then done in Section 8.4.

We close this chapter with Section 8.5, where we use the established tools to give analogues of Theorem 8.0.4 and Theorem 8.0.5 for digraphs, where

$K_{t,t}$ is replaced by $\mathfrak{A}(\overleftrightarrow{K}_t)$, and the matching minor relation is replaced by the butterfly minor relation. We also use an old result of Thomassen to prove a new duality theorem for undirected treewidth.

The Directed Flat Wall Theorem Before we dive into the steps necessary to prove Theorem 8.0.4, let us quickly discuss the current state of art in terms of structure theorems for digraphs that exclude $\overleftrightarrow{K}_t$ as a butterfly minor. Clearly, the main difficulty one faces in the case of digraphs is the lack of a Two Paths Theorem and therefore the lack of an actual description of 'flatness' that resembles the flatness of undirected graphs. Instead, Giannopoulou et al. proposed the following relaxation of flatness. Let W be a cylindrical k-wall, for some $k \in \mathbb{N}$, with C_1 as its inner-most cycle and C_k as its outer-most cycle. The *perimeter* of W, denoted by $\mathsf{Per}(W)$, is $C_1 \cup C_k$.

Definition 8.0.6 (Almost-Flatness)**.** Let D be a digraph and $A \subseteq V(D)$ be a set of vertices. Let $d, t \geq 1$ be integers and let $W \subseteq D - A$ be a cylindrical wall. We say that W is *almost flat* in $D - A$ *with directed treewidth bounded by* d if the following holds.

i) There is a separation (X, Y) of D^2 such that $X \cap Y = A \cup \mathsf{Per}(W)$, $W - \mathsf{Per}(W) \subseteq Y$, and every vertex in Y reaches a vertex of $W - \mathsf{Per}(W)$ or is reachable from it.
ii) For every path Q in $D - A$ that is internally disjoint from $W - \mathsf{Per}(W)$, such that Q has end endpoint in $W - \mathsf{Per}(W)$ and the other in W there is a cell C of W such that the boundary of C contains both endpoints of Q. Furthermore, for every cell C of W, if Z is the set of vertices of a path P in $D - A$ with both endpoints on C and internally disjoint from W, then the components of $Z[\,]$, which are called *extensions*, have directed treewidth at most d.
iii) If T is a tile W of width five and c_1, c_2 are its two upper corners from left to right and d_1, d_2 are its two lower corners from left to right, then there are no two disjoint paths P_1, P_2 in $D - A - (W - T)$ connecting c_1 to d_2 and c_2 to d_2.

[2] Note that (X, Y) is indeed a separation, that is no edge of D has one endpoint in $X \setminus Y$ and the other in $Y \setminus X$. We emphasise this since we are in a digraph and (X, Y) could also denote a directed separation - which it does **not** here.

If W has at most t rows whose tiles do not satisfy property *iii)*, but W satisfies properties *i)* and *ii)* then we say that W is *t-barely flat*.

The idea behind bounding the directed tree width of the extensions is that Theorem 2.3.28 gives at least some handle on the existence of disjoint paths that use the extensions.

A *tournament* of size t is an orientation of the complete graph K_t, that is, a tournament is a subgraph of $\overleftrightarrow{K}_t$ obtained by deleting exactly one of the two edges (u, v) and (v, u) for every pair $u, v \in V(\overleftrightarrow{K}_t)$ of vertices.

There exist two versions of the Directed Flat Wall Theorem as proposed in [GKKK20]. The weaker version, in the sense that it excludes a tournament of size t as a butterfly minor instead of $\overleftrightarrow{K}_t$, roughly says that either the directed treewidth of D is bounded, or there **exists** a cylindrical k-wall W in a digraph D such that one can either find a tournament of size t as a butterfly minor grasped by W, or there exists a set $A \subseteq V(D)$ of bounded size and a reasonably big cylindrical wall $W' \subseteq W - A$ such that W' has the following properties:

i) $D - A - \mathsf{Per}(W)$ has a unique strong component K such that $W' \subseteq \mathsf{Per}(W) \cup K$, and
ii) W' is almost flat in $D - A$.

This structure theorem for digraphs excluding tournament butterfly minors has, in some sense, an even weaker sibling. In [Erd20], Erde proved that every digraph of large enough directed pathwidth, depending on some integer t, must contain every arborescence on t vertices as a butterfly minor. Both of these theorems are able to grasp some of the underlying structure of the digraphs that force their respective parameters to be large, but in both cases the butterfly minors they obtain are not necessarily strongly connected. When concerned with strongly connected butterfly minors, additional compromises are necessary.

Theorem 8.0.7 (Directed Flat Wall Theorem, [GKKK20])**.** There exist functions $a\colon \mathbb{N} \to \mathbb{N}$, $b\colon \mathbb{N} \to \mathbb{N}$, and $d\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all integers $t, r \in \mathbb{N}$ and every digraph D one of the following is true:

i) $\mathrm{dtw}(D) \leq d(t, r)$,
ii) D contains $\overleftrightarrow{K}_t$ as a butterfly minor, or

iii) there exist a set $A \subseteq V(D)$ with $|A| \leq a(t)$ and a cylindrical r-wall $W \subseteq D - A$ which is $b(t)$-barely flat in $G - D$ with directed treewidth bounded by $d(r, t)$.

The nature of the definition of being almost flat or b-barely flat, namely the part where the directed treewidth of the extensions needs to be bounded, greatly reduces the power of the Directed Flat Wall Theorem. Indeed, while the Flat Wall Theorem and Theorem 8.0.4 claim the existence of flat walls within **every** large enough wall, Theorem 8.0.7 can only ensure the existence of a somewhat-flat wall somewhere in the graph. This means that Theorem 8.0.7 in its current form cannot be used to obtain a directed analogue of the Weak Structure Theorem, a shortcoming which can be fixed by dropping the requirement for the extensions having bounded directed treewidth and thereby further weakening the theorem.

Still, apart from possible improvements on the functions a, b, and d, Theorem 8.0.7 is probably best possible for the purpose of capturing the structure of digraphs that exclude **exactly** $\overleftrightarrow{K}_t$ as a butterfly minor. Depending on the application however, it might be necessary to acknowledge that just excluding $\overleftrightarrow{K}_t$ as a butterfly minor still allows for other members of $\mathfrak{A}(\overleftrightarrow{K}_t)$ to be butterfly minors of D. Especially for topology this means that, by Proposition 7.2.4, Theorem 8.0.7 is not fit to capture the structure of digraphs of bounded strong genus.

8.1. Describing and Subdividing a Wall

Generally speaking, a *jump* is a path (undirected, directed, or internally M-conformal) that starts and ends on different vertices of a wall W (undirected, directed or matching), and whose endpoints belong to different cells of W. With *cell* we generally mean a face of the wall that is not bounded by a cycle of its perimeter. A jump is said to be *long* if its endpoints are relatively far apart from each other.

Let W be a matching wall with canonical matching M in a bipartite graph B. The goal of this section is to show that the existence of many pairwise disjoint long jumps that start from different cells implies the existence of a $K_{t,t}$ M-minor which is grasped by W. Notice that, by definition, $\mathcal{D}(W, M)$ is a cylindrical wall of the same order. Indeed, there exist many internally M-conformal paths that are long jumps over W in B if and only if there exist many directed paths that are long jumps over $\mathcal{D}(W, M)$ in $\mathcal{D}(B, M)$. Moreover, if we find $\overleftrightarrow{K}_t$ as a butterfly minor in $\mathcal{D}(B, M)$, then B contains $K_{t,t}$ as an M-minor as a consequence of Lemma 3.2.17. These observations allow us to use some of the tools developed in [GKKK20] in the directed setting to quickly obtain our desired result for the matching setting.

Definitions from the Directed Flat Wall Theorem To state one of the main tools we need from [GKKK20], we have to introduce some of the definitions and notation from the proof of the Directed Flat Wall Theorem. An important part of these definitions is a parametrisation of the cylindrical wall W that makes handling the large families of disjoint paths one needs to route through W a bit more approachable. An advantage, to some degree, of introducing the directed counterpart of this parametrisation first is, that we can lean on it when we need to introduce the matching theoretic analogues.

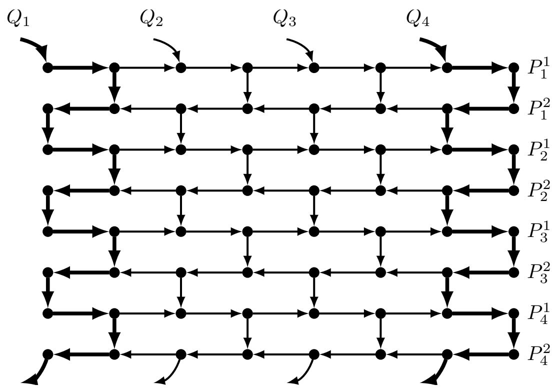


Figure 8.1.: The elementary cylindrical 4-wall. The thick edges of the cycles Q_1 and Q_4 mark its perimeter.

Suppose $\mathcal{P} = \{P_1, \dots, P_k\}$ is a family of pairwise disjoint directed paths, and Q is a directed path that meets all paths in $\mathcal{P}$ such that $P_i \cap Q$ is a directed path.

- We say that the paths $P_1, \dots, P_k$ *appear in this order on* Q if for all $i \in [1, k-1]$, $P_i \cap Q$ occurs on Q strictly before $P_{i+1} \cap Q$ with respect to the orientation of Q.
- In this case, for $i, j \in [1, k]$ with $i < j$, we denote by $Q[P_i, \dots, P_j]$ the minimal directed subpath of Q containing all vertices of $Q \cap P_\ell$ for $\ell \in [i, j]$.

It is convenient for us to imagine cylindrical walls, and similarly, as we will see later, their matching theoretic analogues, as illustrated in Figure 8.1. So we think of the concentric cycles as paths from the top to the bottom with an additional edge from the lowest row back to the top. The directed paths that alternately go in and out of the wall are then seen as the horizontal paths. Since the underlying undirected graph is planar and 3-connected, similar to other arguments before, Whitney's Theorem ensures a unique planar embedding. In particular, we will refer to the faces of the embedding as the faces of the graph itself. A face that is bounded by a cycle that is not the perimeter is called a *cell* of the wall.

Definition 8.1.1 (Vertical and Horizontal Paths)**.** Let $k \in \mathbb{N}$ be a positive integer and W be a cylindrical k-wall.

We denote the vertical paths of W by $Q_1, \dots, Q_k$, ordered from left to right. Let $\left\{P^i_j \mid i \in [1, 2],\ j \in [1, k]\right\}$ be the horizontal directed paths such that the paths P^1_j, $j \in [1, k]$, are oriented from left to right and the paths P^2_j, $j \in [1, k]$, are oriented from right to left such that P^i_j is above $P^{i'}_{j'}$ whenever $j < j'$ and P^1_j is above P^2_j for all $j \in [1, k]$. The top line is P^1_1. By $\hat{P}_j$ we denote the disjoint union of P^1_j and P^2_j for all $j \in [1, k]$.

Two horizontal paths P^i_j and $P^{i'}_{j'}$ are *consecutive* if $i \neq i'$, and $j' \in [j-1, j+1]$ or if $P^i_j + P^{i'}_{j'} = \hat{P}_j$. A family $\mathcal{P} \subseteq \left\{P^i_j \mid i \in [1, 2],\ j \in [1, k]\right\}$ is said to be *consecutive* if there do not exist paths $P_1, P_2 \in \mathcal{P}$, and $P_3 \in \left\{P^i_j \mid i \in [1, 2],\ j \in [1, k]\right\} \setminus \mathcal{P}$ such that there is no directed path from P_1 to P_2 in $W - P_3$. We extend our notation for $P^i_j[Q_p, \dots, Q_q]$ for $p < q$ in the natural way for $\mathcal{P}[Q_p, \dots, Q_q]$ and, in a slight abuse of notation, identify $\hat{P}_i$ and $\left\{P^1_i, P^2_i\right\}$.

For more convenience we write "Let $W = (Q_1, \dots, Q_k, \hat{P}_1, \dots, \hat{P}_k)$ be a cylindrical k-wall." to fix the embedding and naming of the vertical cycles and horizontal paths as explained above and depicted in Figure 8.1 for the case $k = 4$.

Definition 8.1.2 (W-Distance)**.** Let $k \in \mathbb{N}$ be a positive integer and $W = (Q_1, \ldots, Q_k, \hat{P}_1, \ldots, \hat{P}_k)$ be a cylindrical k-wall. Given two vertices $u, v \in V(W)$, we say that they have *W-distance* at least i if there exist i distinct vertical or i distinct horizontal paths whose removal separates u and v in W.

Definition 8.1.3 (Slice)**.** Let $k \in \mathbb{N}$ be a positive integer and $W = (Q_1, \ldots, Q_k, \hat{P}_1, \ldots, \hat{P}_k)$ be a cylindrical k-wall. A *slice* W' of W is a cylindrical wall containing the vertical paths $Q_i, \ldots, Q_{i+\ell}$ for all $i \in [1, k]$ and some $\ell \in [1, k-i]$, and the horizontal paths $P_1^1[Q_i, \ldots, Q_{i+\ell}], \ldots, P_k^2[Q_i, \ldots, Q_{i+\ell}]$. We say that W' is the *slice of W between Q_i and $Q_{i+\ell}$* and that it is of *width* $\ell + 1$.

Definition 8.1.4 (Strip)**.** Let $k \in \mathbb{N}$ be a positive integer and $W = (Q_1, \ldots, Q_k, \hat{P}_1, \ldots, \hat{P}_k)$ be a cylindrical k-wall. A *strip of height $j - i + 1$ between i and j* of W is the subgraph of W induced by the horizontal paths $\hat{P}_i, \ldots, \hat{P}_j$ for some $i < j \in [1, k]$ and the subpaths $Q_\ell[\hat{P}_i, \ldots, \hat{P}_j]$ for $\ell \in [1, k]$.

Definition 8.1.5 (Tiles)**.** Let $k \in \mathbb{N}$ be a positive integer and $W = (Q_1, \ldots, Q_k, \hat{P}_1, \ldots, \hat{P}_k)$ be a cylindrical k-wall. Let $i, j \in [1, k]$ and $d \in \mathbb{N}$ be positive integers. The *tile T of W at (i, j) of width d* is defined as the subgraph of W induced by

$$\bigcup_{\ell \in [i, i+2d+1]} Q_\ell[\hat{P}_j, \ldots, \hat{P}_{j+2d+1}] \cup \bigcup_{\ell \in [j, j+2d+1]} \hat{P}_\ell[Q_i, \ldots, Q_{i+2d+1}].$$

We call i the *column index* of the tile, j the *row index* of the tile, and say that the j-th row has a tile T if the row index of T is j. To make the notation a bit more compact we write $T_{i,j,d}$ for the tile of W at (i, j) of width w.

Since W is a cylindrical wall, there exist subgraphs of W that technically also form tiles, but that do not necessarily fit into our parametrisation of W. To overcome this, we agree for $\ell > k$ to set $P_\ell \coloneqq P_{((\ell-1) \mod k)+1}$. This means that tiles that start near the bottom are allowed to continue at the top. Indeed, the notions of top and bottom are only present because of the way we parametrised the wall, and thus even those tiles are well defined.

The *perimeter* of the tile T is $T \cap (Q_i \cup Q_{i+2d+1} \cup P^1_j \cup P^2_{j+2d+1})$. We call Q_i the *left path* of the perimeter, Q_{i+2d+1} its *right path*, P^1_j the *upper path* of the perimeter, and finally P^2_{j+2d+1} is its *lower path*.

The *corners* of a tile are the vertices $a, b, c, d \in V(T)$ where a, the *upper left corner*, is the common starting point of $T \cap Q_i$ and $T \cap P^1_j$, b, the *upper right corner*, is the end of $T \cap P^1_j$ and the starting point of Q_{i+2d+1}, c, the *lower left corner*, is the common end of $T \cap Q_i$ and $T \cap P^2_{j+2d+1}$, finally d, the *lower right corner*, is the end of $T \cap Q_{i+2d+1}$ and the starting point of $T \cap P^2_{j+2d+1}$.

The *centre* of T is the boundary of the unique cell C_T of W whose boundary consists of vertices from Q_{i+d+1}, Q_{i+d+2}, P^2_{j+d+1}, and P^1_{j+d+2}. All vertices of T which are not in the centre and not on the perimeter of T are called *internal*. See Figure 8.2 for an illustration of a tile.

Please note that by this definition, only cells that lie between P^1_i and P^2_i for some $i \in [1, k]$ can be centre of a tile. However, if we were to take the mirror image of our currently fixed embedding along a straight vertical line between $Q_{\lfloor \frac{k}{2} \rfloor}$ and $Q_{\lfloor \frac{k}{2} \rfloor + 1}$, we obtain a new embedding for which we then can reapply our parametrisation. By doing so, every path P^1_i now becomes a path $P^2_{i'}$, and P^2_i becomes $P^1_{i''}$ for $i, i', i'' \in [1, k]$. This means that we can define for every cell F of W a tile T_F such that F is the centre of T_F. The notion of tiles allows us to add another layer of parametrisation on top of a cylindrical wall. In some sense a tile can be seen as a generalisation of a cell that also contains a small acyclic wall inside to allow for additional routing. The next few definitions are used to add further details to how our walls are divided into different regions and how tiles are used to achieve this additional layer of parametrisation.

Definition 8.1.6 (Triadic Partitions)**.** Let $k \in \mathbb{N}$ be a positive integer and $W = (Q_1, \ldots, Q_{3k}, \hat{P}_1, \ldots, \hat{P}_{3k})$ be a cylindrical $3k$-wall. The *triadic partition* of W is the tuple

$$\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$$

such that for each $i \in [1, 3]$, W_i denotes the slice of W between $Q_{k(i-1)+1}$ and Q_{ik}, and W^i denotes the strip of W between the rows $k(i-1)+1$ and ik.

Definition 8.1.7 (Tiling)**.** Let $k \in \mathbb{N}$ be a positive integer and $W = (Q_1, \ldots, Q_{3k}, \hat{P}_1, \ldots, \hat{P}_{3k})$ be a cylindrical $3k$-wall with its triadic partition

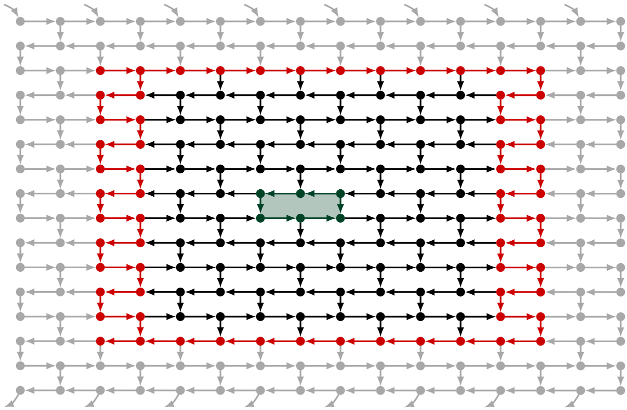


Figure 8.2.: The tile T at $(2,2)$ of width 2 in a cylindrical 8-wall. The green cell is the centre, the red paths are the perimeter of T, and the vertices that belong to T, but neither to the red paths, nor the green cell, are the internal vertices of T.

$\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$. A *tiling* is a family of pairwise disjoint tiles $\mathcal{T}$. Let W' be a slice of W, we say that $\mathcal{T}$ *covers* W' if every vertex $v \in V(W')$ with $\deg_W(v) = 3$ is contained in some tile of $\mathcal{T}$. The tiling $\mathcal{T}$ is said to *cover* $\mathcal{W}$ if it covers W_2. In most places we will use the following family of tilings: Let $f_w \colon \mathbb{N} \to \mathbb{N}$ be some function, $t \in \mathbb{N}$ a positive integer, and $\xi, \xi' \in [1, f_w(t)+1]$. We define the *column function* $\mathsf{c} \colon \mathbb{N} \to \mathbb{N}$ and the *row function* $\mathsf{r} \colon \mathbb{N} \to \mathbb{N}$ as follows[3]:

$$\mathsf{c}(p) \coloneqq (k+2-\xi) + (p-1)(2f_w(t)+1), \text{ and}$$
$$\mathsf{r}(q) \coloneqq \xi' + (q-1)(2f_w(t)+1).$$

We can now define our standard tiling for fixed f_w, ξ, and ξ'.

$$\mathcal{T}_{W,k,f_w(t),\xi,\xi'} \coloneqq \{T_{\mathsf{c}(p),\mathsf{r}(q),f_w(t)} \mid p \in [1, \left\lceil \frac{k+\xi-1}{2f_w(t)+1} \right\rceil + 1], \; q \in [1, \left\lceil \frac{3k-\xi'-1}{2f_w(t)+1} \right\rceil + 1]\}$$

Note that every tiling $\mathcal{T}_{W,k,f_w(t).\xi,\xi'}$ covers $\mathcal{W}$. Moreover, every cell of W_2 that lies between the two paths of $\hat{P}_i$ for some $i \in [1, 3k]$ is the centre of

[3] Please note that c and r do also depend on f_w, ξ, ξ', and k. However, it is more convenient to make these dependencies implicit.

some tile T' of some tiling $\mathcal{T}' \in \mathcal{T}_{W,k,f_w(t).\xi,\xi'}$. Hence if we perform the mirror-image operation as described after the definition of tiles, we are able to find in total $2(f_w(t)+1)^2$ many tilings that cover W_2, such that every cell of W_2 is the centre of some tile in one of these tilings.

We will use tilings in several different ways, and sometimes it is necessary to 'zoom out' of our current wall, i.e. to forget about some of the horizontal paths and vertical cycles in order to obtain a more streamlined version of our wall.

Definition 8.1.8 (Walls from a Tiling)**.** Let $k, d \in \mathbb{N}$ be positive integers, $W = (Q_1, \dots, Q_{3k}, \hat{P}_1, \dots, \hat{P}_{3k})$ be a cylindrical $3k$-wall with its triadic partition $\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$, and $\mathcal{T}$ be a tiling of width d that covers W_2. Moreover, let $\widetilde{W}$ be the some slice of W_2 and let I_Q be the largest set of integers such that Q_i contains vertices of a tile from $\mathcal{T}$ which intersects $\widetilde{W}$ for every $i \in I_Q$. Let $\widetilde{W}(\mathcal{T})$ be the union of the cycles Q_i, $i \in I_Q$, and the paths $P^i_j[\{Q_h \mid h \in I_Q\}]$ for every $(j,i) \in [1,3k] \times [1,2]$. We call $\widetilde{W}(\mathcal{T})$ the *extension of $\widetilde{W}$ that covers $\mathcal{T}$.*

Now, let $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$ be a four colouring of $\mathcal{T}$ and $i \in [1,4]$ be a fixed colour. Then let $J_Q \subseteq [1,3k]$ be the largest set of integers such that for all $j \in J_Q$ the vertical cycle Q_j of $\widetilde{W}(\mathcal{T})$ does not contain a vertex of some tile from $\mathcal{C}_i$. Similarly, let $J_P \subseteq [1,3k]$ be the largest set of integers such that for every $j \in J_P$, none of the two paths from $\hat{P}_j$ contains a vertex of a tile from $\mathcal{C}_i$.

By $\widetilde{W}[\mathcal{T}, i]$ we denote the subgraph of W induced by the union of the cycles Q_i, $i \in J_Q$, and the paths $P^i_j[\{Q_h \mid h \in J_Q\}]$ for every $(j,i) \in J_P$. We say that $\widetilde{W}[\mathcal{T}, i]$ is the ith $\mathcal{T}$-slice of $\widetilde{W}$.

Note that in $\widetilde{W}[\mathcal{T}, i]$, we essentially cut out the tiles of $\mathcal{C}_i$. This operation gives us a slice W' of some cylindrical wall for which the perimeter of every tile in $\mathcal{C}_i$ has become the perimeter of some cell. Next, we are going to find a tiling of W' such that every tile of $\mathcal{C}_i$ that belongs to $\widetilde{W}$ is captured by the centre of some tile in the new tiling.

Definition 8.1.9 (Tier II Tiling)**.** Let $t, k, k' \in \mathbb{N}$ be positive integers and $f\colon \mathbb{N} \to \mathbb{N}$ be some function where $k \geq k'$. Let W be a cylindrical $3k$-wall with its triadic partition $\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$, and $\mathcal{T} = \mathcal{T}_{W,k,f,\xi,\xi'}$ for some $\xi, \xi' \in [1, f(t)+1]$, as well as $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$ be a four colouring of $\mathcal{T}$ and $i \in [1,4]$ be a fixed colour. Moreover, let $\widetilde{W}$

be a slice of W_2 of width k' such that no tile of $\mathcal{C}_i$ contains a vertex of $\mathsf{Per}(\widetilde{W})$ and $\widetilde{\mathcal{T}}$ be the collection of all tiles from $\mathcal{T}$ that contain a vertex of $\widetilde{W}$.

The *tier II tiling for* $\widetilde{W}$ *and* i *obtained from* $\mathcal{T}$ is defined as the unique tiling $(\mathcal{T}, i, f)_{\mathrm{II}}[\widetilde{W}]$ of width f of $\widetilde{W}[\mathcal{T}, i]$ such that every $T \in \mathcal{C}_i \cap \widetilde{\mathcal{T}}$ is in the interior of the centre of some tile in $(\mathcal{T}, i, f)_{\mathrm{II}}[\widetilde{W}]$.

Since every tile in $\mathcal{T}$ consists of $2f(t) + 2$ path pairs, $(\mathcal{T}, i, f)_{\mathrm{II}}[\widetilde{W}]$ is well defined and does in fact cover all of $\widetilde{W}[\mathcal{T}, i, f]$.

At last, we need to introduce the notion of a wall grasping a butterfly minor. Let D and H be digraphs, and let W be a cylindrical wall in D. Let $B = \mathcal{S}(D)$, and M be the perfect matching of B such that $\mathcal{D}(B, M) = D$. We say that W *grasps* an H-butterfly minor of D if $\mathcal{S}(W)$ M-grasps a $\mathcal{S}(H)$-matching minor of B.

8.2. Step 1: Remove Long Jumps

As explained in the introduction of this chapter, there are two main steps to the proof of Theorem 8.0.4. This section is dedicated to prove that we can find $\overleftrightarrow{K}_t$ as a butterfly minor grasped by our cylindrical wall, or we find a vertex set A of bounded size and a large slice W' of our cylindrical wall such that there is no long jump over W' in $D - A$.

Long Jumps in Cylindrical Walls and $\overleftrightarrow{K}_t$-Butterfly Minors While Theorem 8.0.7 could supply us with an intermediate wall which we could then refine further, we aim for a more self-contained proof wherever possible and feasible. To accomplish this goal, we will use a single lemma from the original proof of Theorem 8.0.7, namely Lemma 8.2.2, together with a result on paths leaving and re-entering a fixed set of vertices. Indeed, it is necessary to further refine Lemma 8.2.2 for it to fit into the framework of our proof. We start by introducing the preliminary results. Let D be a digraph and $X \subseteq V(D)$. A *directed X-path* is a directed path P of length at least one that has both endpoints in X but is otherwise disjoint from X.

Theorem 8.2.1 ([GKKK20])**.** Let D be a digraph and $X \subseteq V(D)$. For all positive $k \in \mathbb{N}$, there are k pairwise vertex disjoint directed X-paths in D, or there exists a set $S \subseteq V(D)$ of size at most $2k$ such that every directed X-path in D contains a vertex of S.

Furthermore, there is a polynomial time algorithm which, given a digraph D and a set $X \subseteq V(D)$ as input, outputs k pairwise disjoint directed X-paths, or a set $S \subseteq V(D)$ of size at most $2k$ as above.

Let $k, w \in \mathbb{N}$ be positive integers, W be a cylindrical k-wall and W' be a slice of W. A directed $V(W')$-path P is called a *jump over* W' if $E(P) \cap E(W') = \emptyset$. We say that a directed $V(W')$-path P is a *w-long jump over* W' if for all $\xi, \xi' \in [1, w+1]$ the endpoints of P belong to distinct tiles T_1 and T_2 of the tiling $\mathcal{T}_{W,k,w,\xi,\xi'}$.

Lemma 8.2.2 ([GKKK20]). There exist functions $f_w \colon \mathbb{N} \to \mathbb{N}$, $f_P \colon \mathbb{N} \to \mathbb{N}$, and $f_W \colon \mathbb{N} \to \mathbb{N}$ such that for every $t \in \mathbb{N}$ the following holds: Let

- D be a digraph,
- W be a cylindrical $3k$-wall with $k \geq f_W(t)$ in D,
- $\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$ be the triadic partition of W, and
- $\mathcal{T} = \mathcal{T}_{W,k,f_w(t).\xi,\xi'}$ for some $\xi, \xi' \in [1, f_w(t)+1]$.

If there exists a subfamily $\mathcal{T}'$ of $\mathcal{T}$ and a family $\mathcal{J}$ of pairwise disjoint directed paths in D with the following properties:

i) Every member of $\mathcal{J}$ is internally disjoint from W but has both endpoints on W,
ii) $|\mathcal{T}'| = |\mathcal{J}| = f_P(t)$,
iii) for every $T_{\mathsf{c}(p),\mathsf{r}(q),f_w(t)} \neq T_{\mathsf{c}(p'),\mathsf{r}(q'),f_w(t)} \in \mathcal{T}'$ we have $\max\{|p-p'|, |q-q'|\} \geq 2$,
iv) there exists a bijection start: $\mathcal{T}' \to \mathcal{J}$ (end: $\mathcal{T}' \to \mathcal{J}$) such that the starting point (endpoint) of the path start(T) (end(T)) belongs to the centre of T,
v) $V(\text{start}(T)) \cap V(\mathcal{T}')$ ($V(\text{end}(T)) \cap V(\mathcal{T}')$) contains exactly the endpoint of start(T) (end(T)) where $V(\mathcal{T}') = \bigcup_{T' \in \mathcal{T}'} V(T')$, and finally
vi) the endpoints (starting points) of the paths in $\mathcal{J}$ are of mutual W-distance at least 4.

Then D has a $\overleftrightarrow{K}_t$-butterfly minor grasped by W.

By using the bounds obtained from the proofs in [GKKK20], we get the following rough estimates for the functions f_w, f_P, and f_W:

i) $f_w(t) = 2^8 t^{10}$,
ii) $f_P(t) = 2^7 t^8$, and
iii) $f_W(t) = 2^{32+t^{30}}$.

Refining Lemma 8.2.2 Lemma 8.2.2 is already a powerful tool. However, it is not straightforward how to obtain the apex set A just from its application. Hence in the following, we aim for further refinement and the existence of the set A.

Let $f_w \colon \mathbb{N} \to \mathbb{N}$ be some function, $t, k \in \mathbb{N}$ two positive integers, and $\xi, \xi' \in [1, f_w(t) + 1]$. Moreover, let $W = (Q_1, \dots, Q_{3k}, \hat{P}_1, \dots, \hat{P}_{3k})$ be a cylindrical $3k$-wall with its triadic partition $\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$ and $\mathcal{T} = \mathcal{T}_{W,k,f_w(t),\xi,\xi'}$. A *colouring* of $\mathcal{T}$ is a partition of $\mathcal{T}$ into four classes, namely $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, and $\mathcal{C}_4$ as follows: For every $i \in \left[1, \left\lceil \frac{k+\xi-1}{2f_w(t)+1} \right\rceil + 1\right]$ and every $j \in \left[1, \left\lceil \frac{3k-\xi'-1}{2f_w(t)+1} \right\rceil + 1\right]$ we assign to $T_{\mathsf{c}(i),\mathsf{r}(j),f_w(t)}$ the colour $(i \mod 2) + 2(j \mod 2) + 1$. This means that tiles where $\mathsf{c}(i)$ and $\mathsf{r}(j)$ are even get colour 1, the tiles where $\mathsf{r}(j)$ is even but $\mathsf{c}(i)$ is odd get 3, and so on. Hence every column is two-chromatic, every row is so as well, and between each pair of tiles from the same colour that share a row or a column, there is a tile of a different colour that separates those tiles in their respective row or column. Additionally, if T is some tile, then the eight tiles surrounding T are all of different colour than T itself.

Definition 8.2.3 (Auxiliary Digraph Type I)**.** Let $t, k, k', w \in \mathbb{N}$ be positive integers such that $k \geq k' \geq 2f_W(t) + 4f_P(t)(2w+1)$, $w \geq 2f_w(t)$, and $\xi, \xi' \in [1, w+1]$. Let D be a digraph containing a cylindrical $3k$-wall $W = (Q_1, \dots, Q_{3k}, \hat{P}_1, \dots, \hat{P}_{3k})$ with its triadic partition $\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$, and a tiling $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$. Let $i \in [1,4]$, $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$ be a four colouring of $\mathcal{T}$ and $W' \subseteq W$ be a slice of width k' of W_2. At last, let us denote by $\mathcal{T}'$ the family of tiles from $\mathcal{T}$ that share a vertex with W'. Similarly let $\mathcal{C}_i' := \mathcal{T}' \cap \mathcal{C}_i$. Then $D_i^1(W')$ is the digraph obtained from D by performing the following construction steps for every $T \in \mathcal{C}_i'$:

i) add new vertices x_T^{in} and x_T^{out},

ii) for every vertex u in the centre of T introduce the edges (u, x_T^{in}) and (x_T^{out}, u), and then
iii) delete all internal vertices of T.

We also need the following result: Let D be a digraph and $X, Y \subseteq V(D)$. A *half-integral X-Y-linkage* of order k is a family $\mathcal{P}$ of directed X-Y-paths such that every vertex of D is contained in at most two paths from $\mathcal{P}$. By $V(\mathcal{P})$ we denote the set $\bigcup_{P \in \mathcal{P}} V(P)$.

Theorem 8.2.4 ([GKKK20])**.** Let $k \in \mathbb{N}$ be a positive integer, D be a digraph, and $X, Y \subseteq V(D)$. If $\mathcal{P}$ is a half-integral X-Y-linkage of order $2k$ in D, then there exists a family $\mathcal{J}$ of pairwise disjoint X-Y-paths such that $V(\mathcal{J}) \subseteq V(\mathcal{P})$.

Lemma 8.2.5 (G*)**.** Let $t, k, k', w \in \mathbb{N}$ be positive integers such that $k \geq k' \geq 2f_W(t) + 2^{16} f_P(t) + 2$, $w \geq 2f_w(t) + 2^7 f_P(t)$, and $\xi, \xi' \in [1, w+1]$. Let D be a digraph containing a cylindrical $3k$-wall $W = (Q_1, \dots, Q_{3k}, \hat{P}_1, \dots, \hat{P}_{3k})$ with its triadic partition $\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$, and a tiling $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$. Let $i \in [1, 4]$, $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$ be a four colouring of $\mathcal{T}$ and $W' \subseteq W$ be a slice of width k' of W_2. Now let $\mathcal{T}'$ be the family of all tiles of $\mathcal{T}$ that are completely contained in W' and let $\widetilde{W}$ be the smallest slice of W that contains all tiles from $\mathcal{T}'$.

Consider the auxiliary digraph of type I $D_i^1(\widetilde{W})$ and let $\mathcal{C}_i'$ be as in the definition of $D_i^1(\widetilde{W})$. Define the sets

$$X_{\text{I}}^{\text{out}} := \left\{ x_T^{\text{out}} \mid T \in \mathcal{C}_i' \right\}, \text{ and}$$
$$X_{\text{I}}^{\text{in}} := \left\{ x_T^{\text{in}} \mid T \in \mathcal{C}_i' \right\}.$$

Additionally, we construct the set Y_{I} as follows: Let Q and Q' be the two cycles of $\mathsf{Per}(W_2)$. For every $j \in \left[1, \frac{3k}{4}\right]$, Y_{I} contains exactly one vertex of $Q \cap P_{4j}^1$, $Q \cap P_{4j+2}^2$, $Q' \cap P_{4j}^1$, and $Q' \cap P_{4j+2}^2$ each.

If there exists a family $\mathcal{L}$ of pairwise disjoint directed paths with $|\mathcal{L}| = 2^7 f_P(t)$ such that either

- $\mathcal{L}$ is a family of directed $X_{\text{I}}^{\text{out}}$-$Y_{\text{I}}$-paths, or
- $\mathcal{L}$ is a family of directed Y_{I}-X_{I}^{in}-paths,

then D has a $\overleftrightarrow{K}_t$-butterfly minor grasped by W.

Proof. The proof is divided into several steps and we start with a brief outline. Our goal is to construct a cylindrical wall $U \subseteq W$ of sufficient size, together with a family of $f_P(t)$ directed U-paths that meet the requirements of Lemma 8.2.2. If we are able to do this, then Lemma 8.2.2 yields the desired outcome.

Without loss of generality, let us assume $\mathcal{L}$ is a family of directed $X_{\mathrm{I}}^{\mathrm{out}}$-$Y_{\mathrm{I}}$-paths. The other case can be seen using similar arguments. Let L and P be directed paths. We say that P is a *long jump of* L if P is a w-long jump over W and $P \subseteq L$. We also say that P is a *jump of* L if P is a directed W-path. Towards our goal, we first show that we can use $\mathcal{L}$ to construct a half-integral $X_{\mathrm{I}}^{\mathrm{out}}$-$Y_{\mathrm{I}}$-linking $\mathcal{L}_1$ such that

i) $|\mathcal{L}_1| = 2^7 f_P(t)$,
ii) there exists a family $\mathcal{F} \subseteq \mathcal{T}'$ with $|\mathcal{F}| \leq 2^7 f_P(t)$, and
iii) for every $L \in \mathcal{L}_1$, every endpoint u of a jump of L with $u \in V(\widetilde{W})$ belongs to a tile from $\mathcal{C}_i' \cup \mathcal{F}$.

Once this is achieved, we use Theorem 8.2.4 to obtain a family $\mathcal{L}_2$ of pairwise disjoint directed $X_{\mathrm{I}}^{\mathrm{out}}$-$Y_{\mathrm{I}}$-paths of size $2^6 f_P(t)$ from $\mathcal{L}_1$. Afterwards, we remove the cycles and paths of $\widetilde{W}$ that meet tiles from $\mathcal{F}$ and obtain a new slice $\widetilde{W}'$ of some cylindrical wall. For this slice, we construct a tiling and a tier II tiling as well as a half-integral linking $\mathcal{L}_4$ of size $2^6 f_P(t)$ from $\mathcal{L}_3$ that connects the centres of some tiles in the tier II tiling to vertices of $\widetilde{W}'$ such that their endpoints are mutually far enough apart and every path in $\mathcal{L}_4$ is internally disjoint from the new wall. Another application of Theorem 8.2.4 then yields the family of long jumps necessary for an application of Lemma 8.2.2.

We start out with the construction of $\mathcal{L}_1$ and $\mathcal{F}$. For this, let $\mathcal{L}' := \mathcal{L}$, $\mathcal{L}_1 := \emptyset$, and $\mathcal{F} := \emptyset$. As long as $\mathcal{L}'$ is non-empty, perform the following actions:

Select some path $L \in \mathcal{L}$. In case L is internally disjoint from $\widetilde{W}$, add L to $\mathcal{L}_1$ and remove it from $\mathcal{L}'$. Otherwise let s_L be its starting point and let v_L be the first vertex of L, when traversing along L starting from s_L, that belongs to $\widetilde{W}$, but not to a tile from $\mathcal{C}_i'$.

i) If v_L does not belong to a tile from $\mathcal{F}$, let $T \in \mathcal{T} \setminus \mathcal{C}_i$ be the tile that contains v_L and add T to $\mathcal{F}$. Let R be a shortest directed path from v_L to Y_{I} in W such that R avoids all vertices of W that are contained in two different paths of $\mathcal{L}_1$ and that is internally disjoint from Lv_L. Now add Lv_LR to $\mathcal{L}_1$ and remove L from $\mathcal{L}'$. Note that such a path R must exist since the paths in $\mathcal{L}$ are pairwise disjoint, we never used T for such a re-routing before, and w and k' are chosen sufficiently large in proportion to $2^7 f_P(t)$. Also note that the path R is exactly the part, where we might go from integral to half-integral, but since our paths were pairwise disjoint to begin with, we can be sure that R does never meet a vertex contained in two distinct paths.

ii) So now suppose v_L belongs to a tile T from $\mathcal{F}$. Let us follow along v_LL until the first time we encounter a vertex u_L for which one of the following is true:
   a) u_L belongs to a tile T from $\mathcal{T} \setminus (\mathcal{C}_i \cup \mathcal{F})$, or
   b) every internal vertex of u_L belongs to $W - \widetilde{W}$ or to some tile from $\mathcal{C}_i \cup \mathcal{F}$.

   If a) is the case, repeat the instruction from i) but replace v_L by u_L. In this case T is added to $\mathcal{F}$. Otherwise b) must hold and here we may simply remove L from $\mathcal{L}'$ and add it to $\mathcal{L}_1$.

Now for every $L \in \mathcal{L}$ we added at most one tile to $\mathcal{F}$ and thus $|\mathcal{F}| \leq |\mathcal{L}|$. Moreover, from the construction it is clear that $\mathcal{L}_1$ is indeed a half-integral linkage from $X_{\mathrm{I}}^{\mathrm{out}}$ to Y_{I}. Also, please note that we may assume that every L meets each tile in $\mathcal{F}$ in at most $2^7 f_P(t) + 1$ horizontal path pairs and vertical cycles, since otherwise one could find a short cut through W itself.

Next, we may apply Theorem 8.2.4 to obtain a family $\mathcal{L}_2$ of pairwise disjoint directed $X_{\mathrm{I}}^{\mathrm{out}}$-$Y_{\mathrm{I}}$-paths with $V(\mathcal{L}_2) \subseteq V(\mathcal{L}_1)$ and $|\mathcal{L}_2| = 2^6 f_P(t)$. This completes the second step.

For the third step, let us consider $W'' := \widetilde{W}[\mathcal{T}, i]$ together with the tiling $\mathcal{T}'' := (\mathcal{T}, i, f)_{\mathrm{II}}[\widetilde{W}]$ and a four-colouring $\left\{\widetilde{\mathcal{C}}_1, \ldots, \widetilde{\mathcal{C}}_4\right\}$. Note that by choice of k' this means that W'' is a slice of width $k'' \geq 1 f_W(t) + 2^7 f_P(t)(2w+1)+1$ of some cylindrical $3k''$-wall that is completely contained in W. For each $L \in \mathcal{L}_2$ let $T_L^1 \in \mathcal{C}_i$ such that the starting point s_L of L belongs to T_L^1. Let $K_L^1 \in \mathcal{T}''$ be the tile whose centre is the perimeter of T_L^1. Choose any vertex s_L' of degree three in W'' that is not contained in any path of $\mathcal{L}_2$, and let R_L be a directed path from s_L to s_L' within T_L^1. Let $\mathcal{L}_3'$ be the resulting, and potentially now again half-integral, family of directed paths. Now there must exist $j \in [1,4]$ such that at least $2^4 f_P(t)$ of the paths from $\mathcal{L}_3'$ start at the centre of a tile from $\widetilde{\mathcal{C}}_j$. Let $\mathcal{L}_3'' \subseteq \mathcal{L}_3'$ be a family of exactly $2^4 f_P(t)$ such paths. Next let us consider the family $\mathcal{F}$. Let W''' be the subgraph of W'' induced by all vertical cycles and horizontal path pairs in W'' that do not contain a vertex of some tile in $\mathcal{F}$ that belongs to a path in $\mathcal{L}_3''$. Since $|\mathcal{F}| \leq 2^7 f_P(t)$ and each tile in $\mathcal{F}$ meets a path in $\mathcal{L}_3''$ in at most $2^7 f_P(t) + 1$ such cycles and pairs of horizontal paths, it follows that W''' is a slice of width $k''' \geq f_W(t) + 2$ of some cylindrical $3k'''$wall $W^* \subseteq W$. Moreover, W^* can be partitioned into three slices of width k''' as in its triadic partition, such that W''' is the slice in the middle. Let us rename the paths and cycles of W^* such that $W^* = (Q_1^*, \dots, Q_{3k'''}^*, \hat{P^*}_1, \dots, \hat{P^*}_{3k'''})$, and we construct the set Y^* as follows: Let Q^* and $'Q^*$ be the two cycles of $\mathsf{Per}(W''')$. For every $j \in \left[1, \frac{3k'''}{4}\right]$, Y^* contains exactly one vertex of $Q^* \cap P_{4j}^{*1}$, $Q \cap P_{4j+2}^{*2}$, $'Q^* \cap P_{4j}^{*1}$, and $'Q^* \cap P_{4j+2}^{*2}$ each. Let $L \in \mathcal{L}_3''$ be any path and t_L be the first vertex after its starting point L shares with either W''' or $W^* - W'''$. In case $t_L \in V(W''')$, simply add Lt_L to $\mathcal{L}_3'''$. Otherwise, let b_L be the endpoint of L in $W^* - W'''$. Then we can find a path R_L in W from b_L to a vertex t_L^* of Y^* such that t_L^* is of W^*-distance at least 4 to every endpoint of every path already in $\mathcal{L}_3'''$, R_L is internally disjoint from L, and R_L does not contain a vertex that is contained in two distinct paths from $\mathcal{L}_3'''$. Add LR_L to $\mathcal{L}_3'''$. Finally, $\mathcal{L}_3'''$ is a half-integral linkage from the set starting points S^* of the paths in $\mathcal{L}_3''$ to Y^* of size $2^4 f_P(t)$, and thus by Theorem 8.2.4 we can find a family $\mathcal{L}_4$ of pairwise disjoint directed paths from S^* to Y^* with $V(\mathcal{L}_4) \subseteq V(\mathcal{L}_3''')$ that is of size $2^3 f_P(t)$. It follows that all paths in $\mathcal{L}_4$ are internally disjoint from W'''. This concludes the fourth step.

Let us consider the tiles of $\widetilde{\mathcal{C}}_i$ whose centres contain a vertex of S^*. Since W''' might be a proper subgraph of W'', $\mathcal{T}''$ is not necessarily a tiling of W'''. Each such tile T, however, contains a tile T' of width $f_w(t)$ with the same centre. Since T can be surrounded by at most 8 tiles from $\mathcal{F}$ in W', we may find among the $2^3 f_P(t)$ many tiles T' a family $\mathcal{J}$ of $f_P(t)$ tiles that are pairwise disjoint and thus, since they all are constrcuted from the family $\widetilde{\mathcal{C}}_i$, they meet the distance requirements of the tiles in Lemma 8.2.2. Hence we may apply Lemma 8.2.2 and obtain a $\overleftrightarrow{K}_t$-butterfly minor grasped by W^*. Moreover, since $W^* \subseteq W$, this completes the proof of our lemma. □

With this we are able to handle all long jumps that attach to tiles of different colour. Using a second auxiliary digraph, we find a similar way to handle those long jumps over W that attach to tiles of the same colour by using Theorem 8.2.1.

Definition 8.2.6 (Auxiliary Digraph Type II)**.** Let $t, k, k', w \in \mathbb{N}$ be positive integers such that $k \geq k' \geq 2f_W(t)$, $w \geq 2f_w(t)$, and $\xi, \xi' \in [1, w+1]$. Let D be a digraph containing a cylindrical $3k$-wall $W = (Q_1, \dots, Q_{3k}, \hat{P}_1, \dots, \hat{P}_{3k})$ with its triadic partition $\mathcal{W} = (W, k, W_1, W_2, W_3, W^1, W^2, W^3)$, and a tiling $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$. Let $i \in [1,4]$, $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$ be a four colouring of $\mathcal{T}$ and $W' \subseteq W$ be a slice of width k' of W_2 such that no tile of $\mathcal{C}_i$ contains a vertex of the perimeter of W'. Then $D_i^2(W')$ is the digraph obtained from D by performing the following construction steps:

for every $T \in \mathcal{C}_i$, such that T contains a vertex of W', we do the following:

i) add a new vertex x_T, and
ii) for every vertex v that belongs to the interior or the centre of T, introduce the edges (x_T, v) and (v, x_T).

Once this is done, delete all vertices of W' that do not belong to tiles of $\mathcal{C}_i'$ Let X_{II}^i be the collection of all newly introduced vertices x_T.

Lemma 8.2.7 (G^*)**.** Let $t, k, k', w \in \mathbb{N}$ be positive integers, and $\xi, \xi' \in [1, w+1]$ where $w \geq 2f_w(t)$. Let D be a digraph containing a cylindrical $3k$-wall $W_0 = (Q_1, \dots, Q_{3k}, \hat{P}_1, \dots, \hat{P}_{3k})$, where $k \geq k' \geq 4f_W(t)^2$, with its triadic partition $\mathcal{W} = (W_0, k, W_1, W_2, W_3, W^1, W^2, W^3)$, a slice $W \subseteq W_2$ of width k', a tiling $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$, a four colouring $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$, and a fixed colour $i \in [1,4]$.

Then D has a $\overleftrightarrow{K}_t$-butterfly minor grasped by W_0, or there exists a set $\mathcal{Z}_{i,\xi,\xi'}^2 \subseteq \mathcal{T}$ with $|\mathcal{Z}_{i,\xi,\xi'}^2| \leq 8f_P(t)$ and a set $Z_{i,\xi,\xi'}^2 \subseteq V(D - W)$ with $|Z_{i,\xi,\xi'}^2| \leq 8f_P(t)$ such that every directed $V(W_0)$-path P in $D - Z_{i,\xi,\xi'}^2$ whose endpoints belong to different tiles of $\mathcal{C}_i$ contains a vertex of some tile in $\mathcal{Z}_{i,\xi,\xi'}^2$.

Proof. Let W' be the largest slice of W such that no tile of $\mathcal{C}_i$ contains a vertex of $\mathsf{Per}(W')$. Let us consider the auxiliary digraph $D_i^2(W')$ with the set X_{II}^i of newly added vertices. By applying Theorem 8.2.1 to the set X_{II}^i in $D_i^2(W')$, we either find a set Z of size at most $8f_P(t)$ that hits all directed X^i_{II}-paths, or there exists a family $\mathcal{J}'$ of $4f_P(t)$ pairwise disjoint directed X^i_{II}-paths in $D^2_i(W')$.

Let us assume the latter. Then, by construction of $D^2_i(W')$, no path in $\mathcal{J}'$ contains a vertex of W_2. Back in the digraph D, let us consider the tier II tiling $\mathcal{T}'' \coloneqq (\mathcal{T}, i, w)_{\mathrm{II}}[W']$ of width w of $W'' \coloneqq W'[\mathcal{T}, i]$. Notice that, by choice of k, W'' still contains a cylindrical $f_W(t)$-wall W''' such that the perimeter of every tile $T \in \mathcal{C}_i$, for which x_T is an endpoint of some path in $\mathcal{J}'$, bounds a cell of W'''. Let $\mathcal{T}''$ be a tiling of W''' such that the perimeter of every $T \in \mathcal{C}_i$, for which x_T is an endpoint of a path in $\mathcal{J}'$, is the centre of some tile in $\mathcal{T}''$. We now consider a four colouring $\{\mathcal{C}'_1, \dots, \mathcal{C}'_4\}$ of $\mathcal{T}''$. Then there must exist $j \in [1, 4]$ and a family $\mathcal{J}''$ of size $f_P(t)$ such that the starting points of every path in $\mathcal{J}''$ belongs to a tile of $\mathcal{C}_i$ whose perimeter is the centre of a tile in $\mathcal{C}'_j$. For every $J'' \in \mathcal{J}''$ let $T_1, T_2 \in \mathcal{C}_i$ be the two tiles such that J' is a directed x_{T_1}-x_{T_2}-path. We can now find a directed path J that starts on the perimeter of of T_1, ends on the perimeter of T_2, and is internally disjoint from W'''. Hence we find a family $\mathcal{J}$ of pairwise disjoint directed W'''-paths whose endpoints all lie on the centres of distinct tiles of $\mathcal{T}''$ and that all start at the centres of tiles from $\mathcal{C}'_j$. So we may apply Lemma 8.2.2 to find a $\overleftrightarrow{K}_t$-butterfly minor grasped by W'''. Moreover, with $W''' \subseteq W \subseteq W_0$, we have found a $\overleftrightarrow{K}_t$-butterfly minor grasped by W_0. Therefore we may assume that we find a set Z of size at most $8f_P(t)$ that hits all directed X^i_{II}-paths. Let $Z^2_{i,\xi,\xi'} \coloneqq Z \cap V(D)$, and $\mathcal{Z}^2_{i,\xi,\xi'} \coloneqq \{T \in \mathcal{T} \mid x_T \in Z\}$. Since $|Z| \leq 2f_P(t)$, the bounds on the two sets follow immediately. Moreover, since Z meets every directed X^i_{II}-path in D', every directed path with endpoints in distinct tiles of $\mathcal{C}_i$ which is otherwise disjoint from W_0 must contain a vertex from $Z^2_{i,\xi,\xi'}$ or meet a tile from $\mathcal{Z}^2_{i,\xi,\xi'}$. $\square$

8.3. Step 2: Crosses and Bipartite Non-Pfaffian Graphs

In the previous section, we have taken care of long jumps over our wall. The proofs presented there are closely related to those necessary for the Directed Flat Wall Theorem. This section is dedicated to the second step: removing local crosses from our wall. Since the lack of a Two Paths Theorem for digraphs makes dealing with crosses pretty difficult, this second part differs wildly from the directed case.

While most of the routing necessary for the results in this chapter will happen in the digraphic setting mostly for convenience, matching theory will come in at several places. The two most common techniques we are going to apply are changing the digraph we are working on by 'flipping' the perfect matching along some directed cycles and tight cut contractions to remove large quantities of the (di)graphs we work on without changing their overall structure in an intrusive way. An important part of this technique is the fact that we can use the 'flipping' of the perfect matching along a directed cycle to essentially[4] reverse its direction. Let us briefly investigate the operation of switching the perfect matching M along a horizontal cycle of a matching wall and its effect on the resulting M'-direction. Let $k \in \mathbb{N}$ be a positive integer and W be a cylindrical $4k$ wall, as well as W' be a slice of width $2k$ of W. Let us number the vertical cycles W' inherits from W as $Q_1, \dots, Q_{2k}$ and let us write $\hat{P}_1, \dots, \hat{P}_{4k}$ for the subpaths of the vertical paths of W that still are present in W'.

Then $B_{W'} \coloneqq \mathcal{S}(W')$ is a slice of width k of a matching $2k$-wall, and we may assume M to be its canonical matching. Hence $\mathcal{D}(B_{W'}, M) = W'$. For each $i \in [1, 2k]$ let C_i be the M-conformal cycle of $B_{W'}$ such that $\mathcal{D}(C_i, M) = Q_i$.

We define the *mixed matching* of $B_{W'}$, denoted by $\text{Mix}(W')$, as

$$\text{Mix}(W') \coloneqq M \Delta \bigcup_{\substack{i \in [1,2k] \\ i \text{ odd}}} E(C_i).$$

Thus $\text{Mix}(W')$ is obtained from M by switching M along every second vertical cycle starting with the first. Let us denote by $\mathsf{ab}(W')$ the digraph $\mathcal{D}(B_{W'}, \text{Mix}(W'))$. We say that $\mathsf{a}(W')$ and $\mathsf{b}(W')$ are the *twin walls* of W' as illustrated in Figure 8.3.

For the construction of $\mathsf{a}(W')$ and $\mathsf{b}(W')$, W' must necessarily be of even width. However, $\mathsf{ab}(W')$ can be constructed for any slice, so in slight abuse of notation we will use $\mathsf{ab}(W')$ to describe the digraph obtained from $\mathcal{S}(W')$ by switching the canonical matching along every second vertical cycle starting with the first.

Note that $\mathsf{ab}(W')$ contains two cylindrical k-walls, one using the now flipped versions of the odd Q_i, denoted by $\mathsf{a}(W')$, and the other using the still intact even Q_i, which is denoted by $\mathsf{b}(W')$. Moreover, if we

[4]Note that the operation does more to the digraph than 'just' reversing the direction of a cycle, it also affects all paths that enter or leave the cycle.

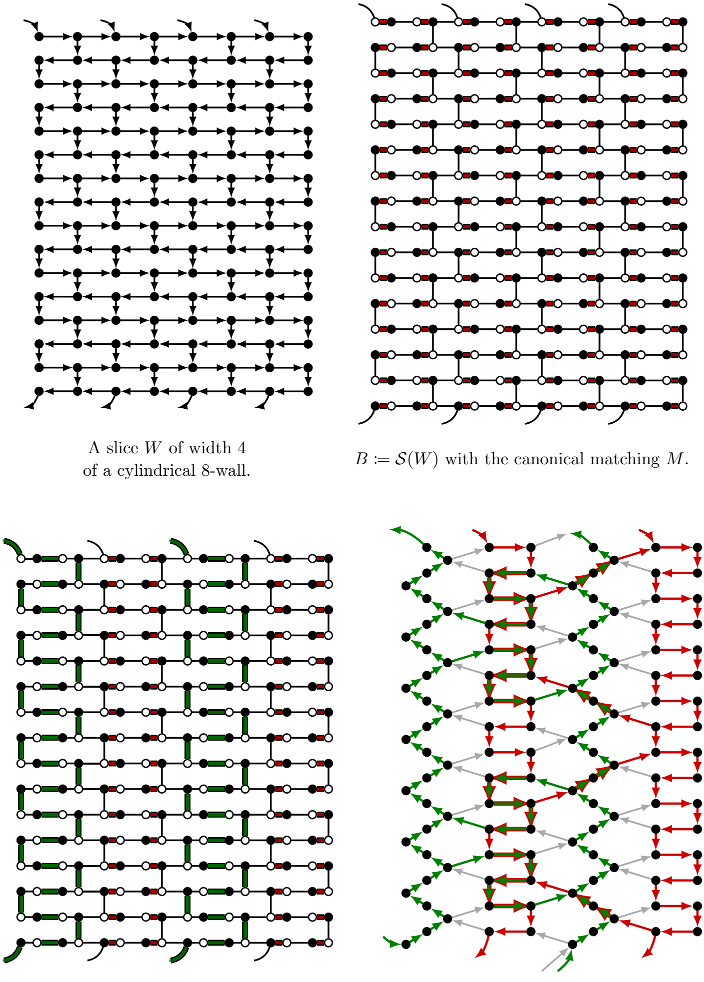


Figure 8.3.: The ‘flipping’ of a cylindrical 4-wall into two overlapping cylindrical 2-walls that are oppositely oriented.

start out with our embedding of W and keep this embedding during all transformations, the vertical cycles of $\mathsf{a}(W')$ go towards the top, while the vertical cycles of $\mathsf{b}(W')$ still go to the bottom. See Figure 8.3 for an illustration. Moreover, every path connecting one of the vertical cycles of $\mathsf{c}(W')$ to another one, for some $\mathsf{c} \in \{\mathsf{a}, \mathsf{b}\}$, must necessarily visit all vertical cycles of its twin that lie in between.

This construction allows us to move up- and downwards almost arbitrarily in a sufficiently large wall. This is an advantage which we already used in the proof of Theorem 7.1.1. In a slice W' of the cylindrical wall, every directed cycle must visit all of the vertical paths, but in $\mathsf{ab}(W')$ we are able to find directed cycles locally, which means there are strongly connected subgraphs of $\mathsf{ab}(W')$ which lie within some tiles of W'.

Non-Pfaffian Cells and Tiles Both techniques, changing the perfect matching and tight cut contractions, are of immediate importance for the first step towards controlling the crossings over our wall.

Let $w \in \mathbb{N}$ be some positive integer. In the following we will say that a slice W' of some cylindrical wall W in some digraph D is *clean* if there is no w-long jump over W' in D. We say that the slice W' is *proper* if it does not contain the perimeter of W.

A first step is to localise crossings to be able to use them for routing. Ideally, we want our crossings to occur 'over', or, more precisely, within the 'attachment' of a single cell in our wall. However, this is not necessarily possible, as there might still exist short jumps even over a clean slice. So instead, we consider crossings over tiles and then use a tier II tiling to force these crossings into a single cell. For this, we need the following:

- We need a proper definition of an 'attachment',
- there needs to be a way to use crossings, or in other words conformal bisubdivisions of $K_{3,3}$ as seen in Proposition 4.0.8, even if these are not M-conformal, and
- we need to show that the existence of a short jump with both endpoints in a tile immediately forces the existence of a conformal $K_{3,3}$ subdivision within the attachment of the tile.

It is important to note that, since we are interested in conformal bisubdivisions of $K_{3,3}$, it suffices to work on a brace. We require some additional observations on matching walls in bipartite graphs with perfect matchings and their braces.

Lemma 8.3.1 (G*)**.** Let $k \in \mathbb{N}$ be positive integer with $k \geq 2$, W be a matching $k+2$-wall with canonical matching M, G_k be a subdivision the cylindrical grid of order k, and W' be the unique proper slice of width k of W. Then the following statements are true:

i) W is an expansion of a brace J,
ii) J contains a remnant H of W', and
iii) H is a cubic graph that is an expansion of $\mathcal{S}(G_k)$.

Proof. We start by proving that W has a unique brace that is not isomorphic to C_4. To do this, let $X \subseteq V(W)$ be any set that induces a non-trivial tight cut in W. Moreover, let us assume that $\overline{X}$ contains more degree-3-vertices of W than X, while, in case both shores contain the same number of such vertices, let X be chosen arbitrarily. Note that X cannot contain vertices from two distinct vertical cycles of W, since we may switch their matchings independently and thus could always force at least two matching edges to lie in $\partial_W(X)$. Similarly, for every pair P_1, P_2 of horizontal paths, we can always find an M-conformal cycle C in W that contains both P_1 and P_2. Hence X cannot contain vertices from both P_1 and P_2, as otherwise it would either contain vertices from two distinct vertical cycles, or we could switch the matching along C to force at least two edges of the new matching into $\partial_W(X)$. Hence $W[X]$ must be one of three things:

i) an induced subpath of some vertical cycle of W,
ii) an induced subpath of some horizontal path of W, or
iii) a subdivided star that contains exactly one degree-3-vertex s of W and s lies at the centre of the star.

In the first two cases, contracting $\overline{X}$ clearly results in a cycle of even length and thus all of its braces are isomorphic to C_4. In the last case observe that, in order for X to induce a tight cut, the leaf vertices of the star must all have the same colour by Lemma 3.1.58 and thus, since we are forced to have an imbalance of exactly one, the centre of the star must be part of the majority as well.

So when contracting $\overline{X}$, this case yields a bisubdivision of the graph in Figure 8.4. It is straightforward to see that every brace of this graph

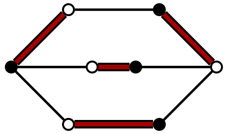


Figure 8.4.: The tight cut contraction of a star.

must be isomorphic to C_4. As we have seen, one of the two tight cut contractions of any chosen non-trivial tight cut in W yields only braces which are isomorphic to C_4. We have also seen that no such shore can contain more then one degree-3-vertex of W, and thus the other shore must still contain a remnant of some brace J which contains all other degree-3-vertices. Hence W contains a brace J which has at least as many degree-3-vertices as W has, and this brace J must be unique. Our claim now follows from Theorem 3.1.61.

Next, observe that there are exactly two vertical cycles C_1 and C_2 in W which contain degree-3-vertices that are linked by paths of even length along C_i, whose internal vertices are all of degree two in W. These two cycles are in fact exactly those which form the perimeter of W. Hence the second and third point from the assertion follow immediately. □

Definition 8.3.2 (Attachment)**.** Let B be a bipartite graph with a perfect matching M, $D \coloneqq \mathcal{D}(B, M)$, and W be a proper slice of some cylindrical wall W' in D. Let B' be a host of $\mathcal{S}(W')$ in B, and let $\widetilde{W}$ be the remnant of $\mathcal{S}(W)$ in B'.

Now let $H \subseteq B$ and $M' \in \mathcal{M}(B)$ be chosen such that

i) $\mathcal{S}(W)$ is M'-conformal in B,
ii) $E(B - \mathcal{S}(W')) \cap M' \subseteq M$, and
iii) H is an induced M'-conformal subgraph of $\mathcal{S}(W')$ such that its outer face, i.e. the face of H that contains $\mathcal{S}(W') - H$ in the canonical embedding of $\mathcal{S}(W)$, is an M'-conformal cycle.

Let H' be the remnant of H in B'. The *attachment of H over W in B*, denoted by $\mathsf{Att}_{B,\mathcal{S}(W)}(H)$, is the elementary component of $B' - (\widetilde{W} - H')$ that contains the outer face of H'.

Let W be a matching wall and H be an induced M'-conformal subgraph of W whose outer face is an M-conformal cycle C where $M' \in \mathcal{M}(W)$.

Then C is conformal and separating in W and thus, by Lemma 4.1.2, H is matching covered. That means, for every $H \subseteq W$ for which an attachment exists, we know that H', as in the definition above, belongs to this attachment.

Since we are interested in tilings and their centres especially, we need to show that the attachment of a cell and the attachment of a tile are well defined objects. Moreover, we would like to know that the attachment of a tile contains the attachment of its centre. To this end, we introduce the following two lemmas.

Lemma 8.3.3 (G*)**.** Let $h, k, w \in \mathbb{N}$ be positive integers and W be a proper slice of width k of some matching h-wall where $h \geq k+2$. Let M be the canonical matching of W and $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$ for $\xi, \xi' \in [1, w+1]$. At last let $T \in \mathcal{T}$ be any tile that is completely contained in W and let Q be the vertical cycle of W that contains the leftmost part of $\mathsf{Per}(T)$. Then $M_T := M \Delta E(Q)$ is a perfect matching of W such that $\mathsf{Per}(T)$ contains an M_T-conformal cycle C_T and $T - (\mathsf{Per}(T) - C_T)$ is an M_T-conformal matching covered subgraph of W whose outer face is bounded by C_T.

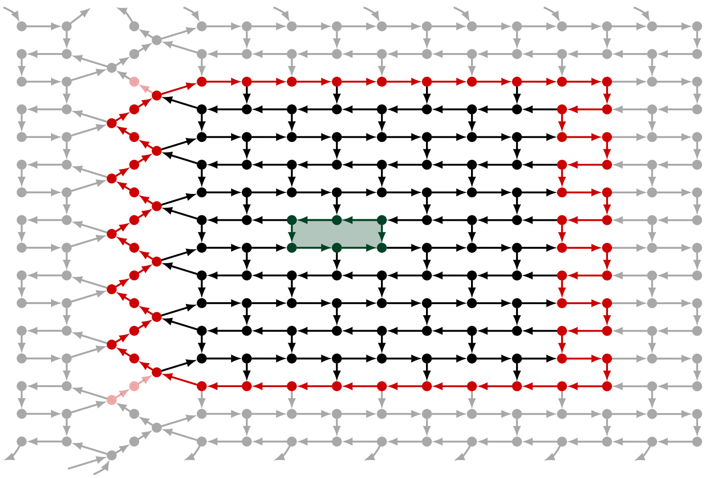


Figure 8.5.: The tile T in the M_T-direction of W.

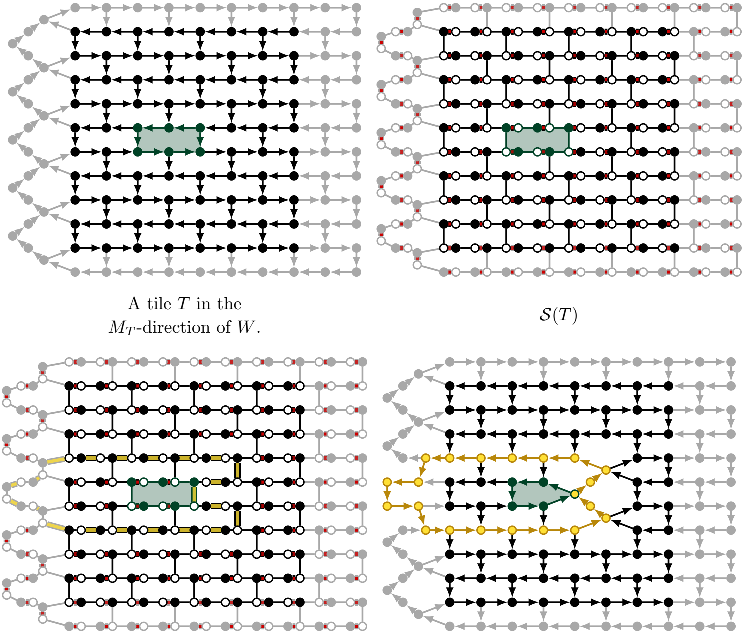
Proof. To see that the statement is true, observe that the boundary of T consists of a cycle and two vertices, one of them being the starting vertex s of the subpath of the upper path in $\mathsf{Per}(T)$, and the other one being the endpoint t of the subpath of the lower path in $\mathsf{Per}(T)$. By definition, Q is the left path of $\mathsf{Per}(T)$ and $L := Q \cap \mathsf{Per}(T)$. Then both s and t belong to L. See Figure 8.2 for an illustration. Now consider $L' := L - s - t$ and note that $\mathcal{S}(L')$ contains exactly two vertices which do not belong to the unique cycle of $\mathcal{S}(\mathsf{Per}(T))$. Let P be the path obtained from $\mathcal{S}(L')$ by removing these two vertices. Since we removed both endpoints of an M-conformal path, where M is the canonical matching of W, the result is an internally M-conformal path. Let C be the unique cycle of $\mathcal{S}(\mathsf{Per}(T))$, then, as $\mathsf{Per}(T) - Q$ is a directed path, $C - P$ is another internally M-conformal path. Now consider the perfect matching M_T. Then every internally M-conformal subpath of Q has now become an M_T-conformal path. So now $C - P$ is internally M_T-conformal, while P is M_T-conformal. Hence C is M_T-conformal as required. See Figure 8.5 for an illustration. □

Observation 8.3.4 (G*)**.** Let $h, k, w \in \mathbb{N}$ be positive integers and W be a proper slice of width k of some matching h-wall where $h \geq k + 2$. Let M be the canonical matching of W, and let C be a cell of W. Then C is a conformal cycle of W.

Proof. As W is a slice of a matching wall, it is planar and matching covered. Since C bounds a face of W, our claim follows immediately from Lemma 4.0.1. □

Hence the attachments of tiles and their centres are indeed well defined. Next, we want to see that the attachment of the centre of a tile is contained in the attachment of the tile itself.

Lemma 8.3.5 (G*)**.** Let $h, k, w \in \mathbb{N}$ be positive integers and W be a proper and clean slice of width k of some matching h-wall in a bipartite matching covered graph B, where $h \geq k+2$. Let M be a perfect matching of W that contains the canonical matching of W and $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$ for $\xi, \xi' \in [1, w + 1]$. At last let $T \in \mathcal{T}$ be any tile that is completely contained in W, and let C_T be the cycle that bounds the centre of T. Then $\mathsf{Att}_{B,W}(C_T) \subseteq \mathsf{Att}_{B,W}(T)$.



A tile T in the M_T-direction of W.

$\mathcal{S}(T)$

$\mathcal{S}(T)$ with the perfect matching M_T^*.

$\mathcal{D}(\mathcal{S}(T), M_T^*)$

Figure 8.6.: The tile T in the M_T^*-direction of W and its split.

Proof. By Lemma 8.3.3 we already know that T is an M_T-conformal and matching covered subgraph of W. As C_T bounds a face of T, which is matching covered, Lemma 4.0.1 still guarantees us the existence of a perfect matching M_T^* of B such that T and C_T are M_T^*-conformal and the M_T^*-direction of T is strongly connected. Let A_T^* be the M_T^*-direction of the attachment of T, while $A_{C_T}^*$ is the M_T^*-direction of the attachment of C_T. Then for every vertex v from $A_{C_T}^*$ there exists a directed path from v to $\mathcal{D}(C_T, M_T^*)$ and a directed path from $\mathcal{D}(C_T, M_T^*)$ to v, since $A_{C_T}^*$ must be strongly connected by the definition of attachments. See Figure 8.6 for an example of the construction of M_T^*. This, however, means that $\mathcal{S}(\mathcal{D}(T, M_T^*) \cup A_{C_T}^*)$ must be matching covered. Let B' be the host of W in B, and let $\widetilde{W}$, $\widetilde{T}$ be the remnants of W and T in B'. Then $\mathsf{Att}_{B,W}(C_T)$

must be contained in the elementary component of $B' - (\widetilde{W} - \widetilde{T})$ that contains $\widetilde{T}$ and thus we are done. $\square$

Suppose C_T is the centre of some tile T in a slice W as above. If $\mathsf{Att}_{B,W}(C_T)$ is non-Pfaffian it must contain a conformal bisubdivision of $K_{3,3}$. However, to be able to use Proposition 4.0.8 and Lemma 4.0.9, we need to show that these conformal bisubdivisions of $K_{3,3}$ cannot be separated from the remnant $\widetilde{T}$ of T by a non-trivial tight cut in $\mathsf{Att}_{B,W}(T)$.

Lemma 8.3.6 (G*)**.** Let $h, k, w \in \mathbb{N}$ be positive integers and W be a proper and clean slice of width k of some matching h-wall in a bipartite matching covered graph B, where $h \geq k+2$. Let M be a perfect matching of W that contains the canonical matching of W and $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$ for $\xi, \xi' \in [1, w+1]$. At last, let $T \in \mathcal{T}$ be any tile that is completely contained in W, and let C_T be the cycle that bounds the centre of T. Then every non-trivial tight cut in $\mathsf{Att}_{B,W}(T)$ has a shore X such that $V(\mathsf{Att}_{B,W}(C_T)) \subseteq X$, and $\mathsf{Att}_{B,W}(T)/(\overline{X} \to v_{\overline{X}})$ has a brace that is the host of T in $\mathsf{Att}_{B,W}(T)$.

Proof. Let B' be the host of W in B and let $\widetilde{W}$, $\widetilde{T}$, and $\widetilde{C_T}$ be the remnants of W, T, and C_T in B' respectively. Now let $A := \mathsf{Att}_{B,W}(T)$ and consider a non-trivial tight cut $\partial_A(X)$ in A. Note that neither X nor $\overline{X}$ can contain vertices purely from $A - \widetilde{T}$, since otherwise $\partial_{B'}(X)$ would be a non-trivial tight cut in B'. Indeed, X or $\overline{X}$ must contain vertices of the remnant of $\mathsf{Per}(T)$ in B'. Suppose $\overline{X}$ is that shore. Let us call a cell of T an *inner cell* if it does not contain vertices of the perimeter of T. If $\overline{X}$ contains vertices of the remnant of an inner cell of T, then there exist two disjoint conformal cycles of T, both of which have remnants in B' with edges in $\partial_{B'}(X)$, and thus $\partial_B(X)$ cannot be a tight cut. At last, suppose X contains vertices of $\mathsf{Att}_{B,W}(C_T)$, then we may find an M_T-conformal cycle C' in $\mathsf{Att}_{B,W}(C_T)$ that contains vertices of $\widetilde{C_T}$ and vertices of $\overline{X}$, but which is disjoint from the remnant of $\mathsf{Per}(T)$. This again yields a contradiction to $\partial_{B'}(X)$ being tight and thus our claim follows. $\square$

The last remaining piece before we can attempt to create the desired crossings is: What if the attachment of every cell is Pfaffian, but there still is a short jump with both endpoints in the interior of T?

Lemma 8.3.7 (G*)**.** Let $h, k, w \in \mathbb{N}$ be positive integers and W be a proper and clean slice of width k of some matching h-wall in a bipartite matching covered graph B, where $h \geq k+2$ and $w \geq 2$. Let M be a perfect matching of W that contains the canonical matching of W and $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$ for $\xi, \xi' \in [1, w+1]$. At last, let $T \in \mathcal{T}$ be any tile that is completely contained in W and J be a short jump in $\mathcal{D}(B, M)$ over $\mathcal{D}(W, M)$ with both endpoints in the interior of $\mathcal{D}(T, M)$. Then, if B' is the host of W in T and $\widetilde{T}$ is the remnant of T in B', T' contains a conformal bisubdivision H of $K_{3,3}$ such that all six degree-3-vertices of H belong to $\widetilde{T}$.

Proof. Consider the perfect matching $\widetilde{M} := M_T|_{\widetilde{T}}$ of $\widetilde{T}$. First of all note that, since J is a short jump in $\mathcal{D}(B, M)$ over $\mathcal{D}(W, M)$, and by Lemma 8.3.1 all faces of W are preserved in $\widetilde{W}$, which is the remnant of W in B', it corresponds to an internally $\widetilde{M}$-conformal path $\widetilde{J}$ with endpoints $a_1 \in V_1 \cap V(\widetilde{T})$ and $b_1' \in V_2 \cap V(\widetilde{T})$ such that a_1 and b_1' do not belong to the same face of $\widetilde{T}$, and $\widetilde{J}$ is internally disjoint from $\widetilde{T}$. Let a_1' and b_1 be chosen such that $a_1b_1, a_1'b_1' \in \widetilde{M}$. Since $\widetilde{T}$ is matching covered there exist $a_2 \in V_2 \cap V(\mathsf{Per}(\widetilde{T}))$ and $b_3 \in V_1 \cap V(\mathsf{Per}(\widetilde{T}))$ such that

i) $a_2b_2, a_3b_3 \in \widetilde{M}$ are distinct,
ii) the internally $\widetilde{M}$-conformal subpath P_1 of $\mathsf{Per}(\widetilde{T})$ with endpoints a_2 and b_3 is non-trivial, and
iii) $P_2 := \mathsf{Per}(\widetilde{T}) - P_1$ is as short as possible such that there exist disjoint internally $\widetilde{M}$-conformal paths L_a, L_b where L_a connects b_1 to a_2 and L_b connects a_1' to b_3.

Now, since $w \geq 2$, there exist b_4 and a_5 on P_1 such that the path R connecting b_4 and a_5 in P_1 is $\widetilde{M}$-conformal, and there are internally $\widetilde{M}$-conformal and disjoint paths Q_a and Q_b in $\widetilde{T}$ that satisfy:

i) Q_a connects a_1 and b_4,
ii) Q_b connects b_1 and a_5, and
iii) Q_a and Q_b avoid L_a and L_b.

By the choices of these paths, we have found in total nine pairwise internally disjoint paths such that

- a_1 and b_1 are joined by a $\widetilde{M}$-conformal path; that is the edge a_1b_1,
- a_2 and b_3 are joined by a $\widetilde{M}$-conformal path; namely the path P_2 together with the edges a_2b_2 and b_3a_3,

- a_5 and b_4 are joined by the $\widetilde{M}$-conformal path R,
- a_1 is joined by the internally $\widetilde{M}$-conformal path $\widetilde{J}b_1'a_1'L_b$ to b_3 and by the internally $\widetilde{M}$-conformal path Q_a to b_4,
- b_1 is joined by the internally $\widetilde{M}$-conformal path L_a to a_2, and by Q_b to a_5, and
- a_5 is joined to b_3 by an internally $\widetilde{M}$-conformal subpath of P_1, while b_4 and a_2 are joined by the remaining internally $\widetilde{M}$-conformal subpath of P_1 that does not contain a_5.

Hence overall, we obtain an $\widetilde{M}$-conformal bisubdivision of $K_{3,3}$. See Figure 8.7 for an illustration.

Another, less constructive, way to see that there must exist a conformal $K_{3,3}$ bisubdivision is to observe that $\widetilde{T}$ together with $\widetilde{J}$ is matching covered and non-planar. Moreover, similar to the proofs of Lemma 8.3.1 and Lemma 8.3.6, any non-trivial tight cut must either sit within a bisubdivided edge of T, form a star around a vertex of degree at least three, or consist entirely of internal vertices of $\widetilde{J}$. Hence $\widetilde{T} \cup \widetilde{J}$ is an expansion of a non-planar brace B'' that closely resembles $\widetilde{T}$ with a single short jump over it. Thus B'' is not the Heawood graph and therefore it must contain a conformal bisubdivision of $K_{3,3}$ by Theorem 3.3.8. □

A Tier II Cross By now we know that non-Pfaffian attachments of cells and short jumps force conformal $K_{3,3}$-bisubdivisions to exist. Next we show how to use this knowledge to create a perfect matching M' such that the M'-direction of our digraph contains two crossing paths within the centre of a tile. More precisely, we show how to create a cross over a single tile and then place a tier II tile around it to obtain a tile with a cross in its centre. Hence it suffices to show that we can locally manipulate the perfect matching of the attachment of a cell in order to create a cross over the cell, if we know that the attachment of the centre contains a structure similar to a tile itself.

Given the centre C of a tile T in a matching wall W with canonical matching M, we are particularly interested in the following four vertices: There are exactly two vertical cycles of W that meet C, let us call them Q_j and Q_{j+1}, where Q_j lies left of Q_{j+1} in the canonical embedding. Let $U_i := Q_i \cap C$ for each $i \in [j, j+1]$. Then U_j is M-conformal, while U_{j+1}

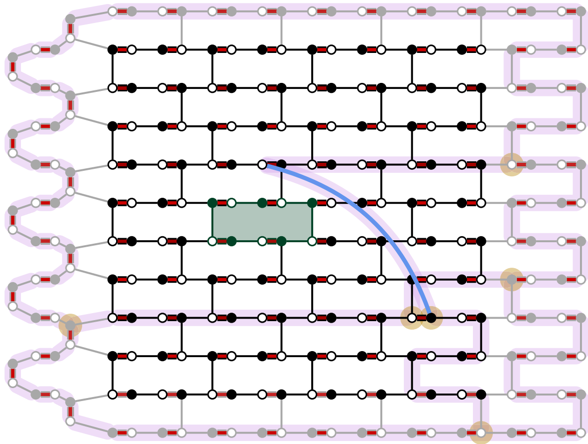


Figure 8.7.: An M_T-conformal bisubdivision of $K_{3,3}$ constructed using a short jump.

is internally M-conformal. Each of the U_i has exactly one endpoint in the upper path of C and one endpoint in the lower path of C. Moreover, let

- $b_T^1 \in V_2$ be the endpoint of U_j on the upper path of C,
- $a_T^2 \in V_1$ be the endpoint of U_j on the lower path of C,
- $a_T^1 \in V_1$ be the endpoint of U_{j+1} on the upper path of C, and
- $b_T^2 \in V_2$ be the endpoint of U_{j+1} on the lower path of C.

Additionally there exist c_T^1 and c_T^2 on U_{j+1} such that $a_T^1 c_T^2 \in M$, and $b_T^2 c_T^1 \in M$. From the definition it follows that $c_T^1, c_T^2 \notin V(C)$.

We say that a tile T is *non-Pfaffian* if either the attachment of the centre of T is a non-Pfaffian bipartite graph, or there exists a short jump over W with both endpoints on the interior of T.

Lemma 8.3.8 (G*)**.** Let $h, k, w \in \mathbb{N}$ be positive integers and W be a proper and clean slice of width k of some matching h-wall in a bipartite matching covered graph B, where $h \geq k+2$. Let M be a perfect matching of W that contains the canonical matching of W and $\mathcal{T} = \mathcal{T}_{W,k,w,\xi,\xi'}$ for $\xi, \xi' \in [1, w+1]$ with a four colouring $\{\mathcal{C}_1, \ldots, \mathcal{C}_4\}$, and for some $i \in [1, 4]$ let $W' := W[\mathcal{T}, i]$. Consider $(\mathcal{T}, i, w)_{\text{II}}[W]$ and let us select $T \in \mathcal{C}_i$ as well as $T' \in (\mathcal{T}, i, w)_{\text{II}}[W']$ such that the centre of T' contains T in its interior in the canonical embedding of W. Let C be the cycle of B that is the centre of T' and let $I_{T'}$ be the union of C and the component of $W - C$ that contains T. For $i \in [1, 2]$ let $a^i_{T'}$ and $b^i_{T'}$ be defined as above.

If T is non-Pfaffian, there exist a perfect matching N_T of B such that $\{e \in M \mid e \cap V(W - I_{T'}) \neq \emptyset\} \subseteq N_T$, and there exist vertex disjoint paths R_1 and R_2 in B such that

i) R_1 and R_2 are internally vertex disjoint from W',
ii) R_1 and R_2 are fully contained in $\mathsf{Att}_{B,W'}(T')$
iii) both paths are N_T-alternating,
iv) R_1 has endpoints $a^1_{T'}$ and $a^2_{T'}$ and the edge of R_1 that is incident with $a^2_{T'}$ lies in N_T, and
v) R_2 has endpoints $b^1_{T'}$ and $b^2_{T'}$ and the edge of R_2 that is incident with $b^2_{T'}$ lies in N_T.

Proof. By Lemma 8.3.3 we know that $I_{T'}$ is matching covered. Moreover, since T is a conformal subgraph of $I_{T'}$, we must also have $\mathsf{Att}_{B,W}(T) \subseteq \mathsf{Att}_{B,W'}(T')$. Therefore $\mathsf{Att}_{B,W'}(T')$ is non-Pfaffian as well.

Let us add the edges $a^i_{T'}b^j_{T'}$ to $\mathsf{Att}_{B,W'}(T')$ for all $i, j \in [1, 2]$ and let G be the resulting bipartite graph. By Theorem 3.1.69, G is still matching covered.

We claim that the host B' of $I_{T'}$ in G is also non-Pfaffian, and that it contains four distinct vertices, each of them representing a vertex from $\{a^1_{T'}, a^2_{T'}, b^1_{T'}, b^2_{T'}\}$ in the following sense: We say that a vertex u of B' *represents* a vertex v of $I_{T'}$ if $u, v \in V_i$ for some $i \in [1, 2]$, and there exists a tight cut $\partial_B(X)$ in G such that $v \in X$ and u is the vertex of B' obtained from contracting X.

Note that in this case, every edge e incident with u in B' can be replaced by a path P_v, which has exactly one endpoint outside of X, in G that shares the same properties regarding any perfect matching M' of G as e does regarding the remainder of M' in B'. That is, if $e \in M'|_{B'}$, then P_v is M'-conformal, and otherwise P_v is internally M'-conformal.

Towards the validity of our claim first let C' be the centre of T and suppose $\mathsf{Att}_{B,W}(C')$ is non-Pfaffian. In this case observe that $I_{T'}$ indeed meets all requirements of a tile in W and thus we may call upon Lemma 8.3.6 to see that B' must be non-Pfaffian. If the attachment of the centre of T is Pfaffian, there must exist a short jump J over W with both endpoints in

T. Then, by Lemma 8.3.7 and its proof, one can see that $I_{T'} \cup J$ contains a conformal bisubdivision of $K_{3,3}$ that contains the cycle C. Hence again B' is non-Pfaffian.

Similar to the proof of Lemma 8.3.6 one can observe that every non-trivial tight cut in $I_{T'}$ must either be a cut around a bisubdivided claw, or along a bisubdivided edge of W. Moreover, in $\mathsf{Att}_{B,W'}(T')$ the only non-trivial tight cuts can occur on the outer face of T' since any other non-trivial tight cut would correspond to a non-trivial tight cut in the brace that was used to construct $\mathsf{Att}_{B,W'}(T')$. Note that every pair among the four vertices $\left\{a^1_{T'}, a^2_{T'}, b^1_{T'}, b^2_{T'}\right\}$ can be separated on C by two degree-3-vertices of $I_{T'}$ which are not in $\left\{a^1_{T'}, a^2_{T'}, b^1_{T'}, b^2_{T'}\right\}$. Additionally, for each $i \in [1,2]$, the two vertices in $\left\{a^1_{T'}, a^2_{T'}, b^1_{T'}, b^2_{T'}\right\} \cap V_i$ can be separated by $\left\{a^1_{T'}, a^2_{T'}, b^1_{T'}, b^2_{T'}\right\} \cap V_{3-i}$ on C. Hence we may assume for each $x \in \left\{a^1_{T'}, a^2_{T'}, b^1_{T'}, b^2_{T'}\right\}$ to have a vertex $u_x \in V(B')$ that represents x and that all of these vertices are pairwise distinct.

By our addition of the fresh edges to obtain G, we now have the edges $a^i_{T'}b^j_{T'}$ in B^+ for all $i,j \in [1,2]$. Let $\hat{C}$ be the four-cycle consisting exactly of these four edges, and let B^+ be the resulting graph. By Theorem 3.1.69, B^+ is still a brace. Hence we may use Proposition 4.0.8 to find a conformal bisubdivision H of $K_{3,3}$ in B^+ that contains $\hat{C}$ as a subgraph. Let N'' be the perfect matching of H that contains the edge $u_{a^1_{T'}}u_{b^2_{T'}}$ but not the edge $u_{a^2_{T'}}u_{b^1_{T'}}$. As H is a bisubdivision of a brace and $\hat{C}$ is a subgraph of H, N'' must exist. Now let N' be a perfect matching of B^+ that contains N''. As H is a bisubdivision of $K_{3,3}$ there exist paths R'_1 and R'_2 with the following properties in H:

i) R'_1 and R'_2 are vertex disjoint and N' alternating,
ii) R'_1 has endpoints $u_{a^1_{T'}}$ and $u_{a^2_{T'}}$, and the edge of R'_1 that is incident with $a^2_{T'}$ lies in N', and
iii) R'_2 has endpoints $u_{b^1_{T'}}$ and $b_{a^2_{T'}}$, and the edge of R'_2 that is incident with $b^2_{T'}$ lies in N'.

Let N be a perfect matching of $\mathsf{Att}_{B,W'}(T') + \hat{C}$ such that $N' = N|_{B''}$. The paths R'_1 and R'_2 can now be extended to the desired paths R_1 and R_2 in $\mathsf{Att}_{B,W'}(T') + \hat{C}$.

Note that the only edge of N that is not an edge of B is $a^1_{T'}b^2_{T'}$. Also the only edges of $E(W') \cap M$ with exactly one endpoint in $I_{T'}$ are the two edges $a^1_{T'}c^2_{T'}$ and $b^2_{T'}c^1_{T'}$. Hence

$$N_T \coloneqq (M \setminus N) \cap E(W') \cup (N \setminus \{a^1_{T'}b^2_{T'}\})$$

is a perfect matching of B with properties as required by the assertion. In particular, since the subpath of C that is parallel to the edge $a^1_{T'}b^2_{T'}$ is internally M-conformal, every edge of $M \cap W - I_{T'}$ belongs to N_T. $\square$

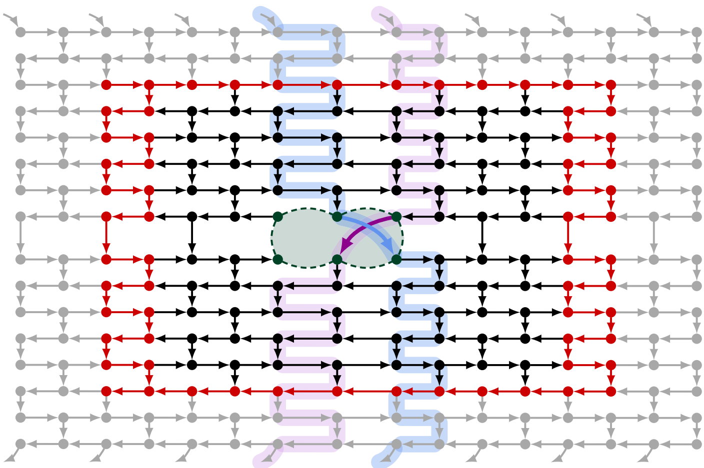


Figure 8.8.: The N_T-direction of the tile T', the cross obtained through Lemma 8.3.8, and two paths through the tile that use the cross. Note that we cannot guarantee anything specific about the matching within $I_{T'}$ besides to the matching edges that leave $I_{T'}$ and the crossing itself.

While we cannot really control how the perfect matching N_T changes the structure of $I_{T'}$ in the N_T-direction of B compared to its M-direction, we still know that most of our cylindrical wall is intact. Indeed, by our choices of the four vertices which are connected via the paths R_1 and R_2 and the fact that $c^2_T \in V_2$ and $c^1_T \in V_1$, we may follow along two parallel vertical cycles from the top of T', then move towards its centre, enter the two N_T-directions of R_1 and R_2, and switch between the two cycles.

See Figure 8.8 for an illustration. This means that we are able to switch the relative position of two disjoint paths that are routed through our cylindrical wall given that these paths enter T' from the top. This means that, given enough such tiles T' that are mutually far enough apart within W, we are able to re-order any given family of pairwise disjoint paths in any order we see fit. Of course within a cylindrical wall any directed path is only ever allowed to go left, right or downwards, but we may take some slice free of crossings and switch the perfect matching M along all vertical cycles of this slice. By doing so we create a perfect matching M', where we can have a large family of pairwise disjoint paths move 'upwards within W' without changing their relative positions to one another.

With this we have established all tools necessary for our proof of Theorem 8.0.4.

8.4. Proof of the Bipartite Flat Wall Theorem for Matching Minors

This section is entirely dedicated to the proof of Theorem 8.0.4. The proof goes through several phases and establishes a few subclaims along the way as follows:

- In the beginning we have a large matching wall W with its canonical matching M of B. From here on we mostly work in the digraphic setting and consider the cylindrical wall $U_0 := \mathcal{D}(W, M)$ in $D = \mathcal{D}(B, M)$. Note that, by Lemma 3.2.17, the existence of a $\overleftrightarrow{K}_t$-butterfly minor grasped by U_0 in D can be seen to imply the existence of a $K_{t,t}$-matching minor grasped by W in B.

  Phase I

  Phase I is an iterative process consisting of two steps which are applied to a slice U_i of U_0, where U_i is the end result of the ith round of Phase I:

Step I Here we simultaneously apply Lemma 8.2.5 for both parametrisations of U_0, every tiling defined by choices of $\xi, \xi' \in [1, w+1]$, and every colour class. This either results in a $\overleftrightarrow{K}_t$-butterfly minor grasped by U_0, or in bounded size sets F_i' and $\mathcal{F}_i'$ which are a set of vertices and a set of tiles respectively. The tiles in $\mathcal{F}'_i$ are considered *marked*. We then find a large slice U'_i of U_{i-1} without vertices of marked tiles.

Step II Next we apply Lemma 8.2.7 for all four colours of every tiling defined by $\xi, \xi' \in [1, w+1]$ and again either find a $\overleftrightarrow{K}_t$-butterfly minor grasped by U_0, or in bounded size sets F''_i and $\mathcal{F}''_i$, which are a set of vertices and a set of tiles respectively. We then find a slice U_i of U'_i of appropriate size that does not contain vertices of a marked tile.

So at the end of Phase I we have found bounded size sets F_I and $\mathcal{F}_I$ of vertices and tiles respectively which are now considered *marked*, and we have found a large slice U_I of U_0 which is free of marked vertices and tiles.

- We can now show that there either exists a $\overleftrightarrow{K}_t$-butterfly minor grasped by U_0, or there is no long jump over U_I in $D - F_I$. This is due to the fact that each long jump over U_I must meet some marked tiles of U_0. However, the number of pairwise distinct such tiles grows with every round and thus, after a certain threshold of such tiles has been surpassed, we can find many pairwise disjoint long jumps over U_0 within a single long jump over U_I, which implies the existence of a large clique butterfly minor.

  Phase II

  Now we know that only short jumps can exist over U_I in $D - F_I$. We divide U_I into $\mathcal{O}(t^2)$ slices, each of which can be partitioned into three appropriately sized slices. Then either one of these middle slices is already flat with respect to its perimeter, or we may apply Lemma 8.3.8 to locally change the perfect matching of a single tile for each of these middle slices and therefore we find $\mathcal{O}(t^2)$ disjoint crosses that are relatively far apart from each other. These can then be used to create $\overleftrightarrow{K}_t$.

So in Phase II we either find the flat wall as desired, or $\overleftrightarrow{K}_t$ in the M'-direction of our graph, where M' is the perfect matching obtained by creating the crossings through Lemma 8.3.8 and switching M along some of the vertical cycles of U_I. Phase II will be the only time we make active use of the matching setting.

Throughout the proof we will collect tiles from different tilings. In general we will say that a tile is *marked* if it either contains a vertex of some separator obtained from Theorem 2.3.8 and Lemma 8.2.5, or through Lemma 8.2.7, or one of these separators contains the fresh vertex created for it in the construction of some auxiliary graph (type I or II). This means that the overall number of tiles that are marked exceeds the size of the sets $\mathcal{F}_i$, still the number of columns that can contain marked tiles is bounded, which is enough for our purposes. A vertex of U is said to be *marked* if it belongs to a separator obtained from Theorem 2.3.8 and Lemma 8.2.5, or through Lemma 8.2.7. In each of the steps we introduce families of marked tiles and vertices and a slice of U is said to be *clear* if it does not contain vertices that are marked.

Proof of Theorem 8.0.4. Let $r, t \in \mathbb{N}$ be positive integers, B be a bipartite graph with a perfect matching M, $D \coloneqq \mathcal{D}(B, M)$, and W be an M-conformal matching $\rho(t,r)$-wall, where $\rho(t,r)$ will be determined throughout the proof. To do this we will introduce a constant d_i for each step, for which we will make more and more assumptions in the form of lower bounds. Let $U \coloneqq \mathcal{D}(W, M)$ be the M-direction of W, then U is a cylindrical $\rho(t,r)$-wall in D.

$$\text{Let us assume } \rho(t,r) \geq 3d_1.$$

Then let U be the cylindrical $3d_1$-wall $U = (Q_1, \dots, Q_{3k}, \hat{P}_1, \dots, \hat{P}_{3k})$ with its triadic partition $\mathcal{U} = (U, d_1, \widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3, \widetilde{U}^1, \widetilde{U}^2, \widetilde{U}^3)$. Throughout the proof let us fix

$$w \coloneqq 2f_w(t) + 2^7 f_P(t).$$

Recall that we may use Lemma 3.2.17 to transform a $\overleftrightarrow{K}_t$-butterfly minor grasped by U in D into a $K_{t,t}$-matching minor grasped by W in B. Hence, whenever we find such a butterfly minor in D we may consider the corresponding case to be closed.

Phase I Phase I is divided into $2^{11+8f_P(t)} f_P(t)$ *rounds*, each of which produces a slice U_i of $U_0 \coloneqq \widetilde{U}_2$ which is clean with respect to all tiles that have not been marked up to this point. We also obtain sets $F_i \subseteq V(D - \bigcup_{j=1}^{i-1} F_j)$ and $\mathcal{F}_i$ of marked vertices and tiles respectively for each round $i \in [1, 2^{11+8f_P(t)} f_P(t)]$, and in each round i we will work on the digraph $D_i \coloneqq D_{i-1} - F_i$. In this context, whenever we ask for a clean slice of the current slice U_{i-1} or U_i' we ask for a slice U' such that there do not exist $\xi, \xi' \in [1, w+1]$ whose corresponding tiling of U_0 has a tile

T, that is marked or contains a vertex of any separator set found so far, which satisfies $V(T) \cap V(U') \neq \emptyset$.

Each round is divided into two steps, *Step I* and *Step II*. Let $i \in [1, 2^{11+8f_P(t)} f_P(t)]$. From the previous round or the initial state, we may assume to have obtained the following sets and graphs, which serve as the *input* for round i:

- $F_{I,i-1} \subseteq V(D)$ such that $|F_{I,i-1}| \leq (i-1)(2^{11}(w+1)^2 f_P(t) + 2^5(w+1)^2 f_P(t))$,
- $D_{i-1} \coloneqq D - F_{I,i-1}$,
- $\mathcal{F}_{I,i-1} \subseteq \bigcup_{\xi,\xi' \in [1,w+1]} \mathcal{T}_{U_0,d_1,w,\xi,\xi'}$ of size at most $(i-1)(2^{11}(w+1)^2 f_P(t) + 2^5(w+1)^2 f_P(t))$, and
- a slice U_{i-1} of U_0 of width $((2^{12}(w+1)^3 f_P(t)+1)(2^6(w+1)^3 f_P(t)+1))^{2^{11+8f_P(t)} f_P(t)-i+1} d_2$ that is clean with respect to $F_{I,i-1}$ and $\mathcal{F}_{I,i-1}$ such that, in case $i-1 \geq 1$, every long jump over U_{i-1} in D_{i-1} contains a vertex of some tile in $\mathcal{F}_{I,j} \setminus \mathcal{F}_{I,j-1}$ for every $j \in [1, i-1]$.

For $i = 1$ the inputs are the graphs $D_0 \coloneqq D$, U_0, and two empty sets.

After round i is complete we require the following sets and graphs as its *output*:

- $F_{I,i} \subseteq V(D)$ such that $|F_{I,i}| \leq i(2^{11}(w+1)^2 f_P(t) + 2^5(w+1)^2 f_P(t))$,
- $D_i \coloneqq D - F_{I,i}$,
- $\mathcal{F}_{I,i} \subseteq \bigcup_{\xi,\xi' \in [1,w+1]} \mathcal{T}_{U_0,d_1,w,\xi,\xi'}$ of size at most $i(2^{11}(w+1)^2 f_P(t) + 2^5(w+1)^2 f_P(t))$, and
- a slice U_i of U_0 of width $((2^{12}(w+1)^3 f_P(t)+1)(2^6(w+1)^3 f_P(t)+1))^{2^{11+8f_P(t)} f_P(t)-i} d_2$ that is clean with respect to $F_{I,i}$ and $\mathcal{F}_{I,i}$ such that every long jump over U_i in D_i contains a vertex of some tile in $\mathcal{F}_{I,j} \setminus \mathcal{F}_{I,j-1}$ for every $j \in [1, i]$.

To be able to find a slice of width d_2 in the end we therefore must fix

$$d_1 \geq ((2^{12}(w+1)^3 f_P(t)+1)(2^6(w+1)^3 f_P(t)+1))^{2^{11+8f_P(t)} f_P(t)} d_2,$$

and we further assume $d_2 \geq 2^{16} f_W(t)^2$ to make sure we can apply Lemma 8.2.5 and Lemma 8.2.7 in every round. Note that this is not our final lower bound on d_2, just an intermediate assumption.

Next we describe the steps we perform in every round. Let $i \in [1, 2^{11+8f_P(t)} f_P(t)]$ and suppose we are given sets $F_{I,i-1}$, $\mathcal{F}_{I,i-1}$ and graphs D_{i-1}, U_{i-1} as required by the input conditions for round i.

Step I: Let $k_i := ((2^{12}(w+1)^3 f_P(t)+1)(2^6(w+1)^3 f_P(t)+1))^{2^{11+8f_P(t)}f_P(t)-i+1} d_2$. For each of the two possible parametrisations of U, we consider for every possible choice of $\xi, \xi' \in [1, w+1]$, the tiling $\mathcal{T} := \mathcal{T}_{U_{i-1},k_i,w,\xi,\xi'}$ together with its four colouring $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$. For each $j \in [1,4]$ we consider the smallest slice U' of U that contains all vertices which belong to some tile of $\mathcal{T}$. Consider the auxiliary digraph $D^1_j(U')$ of type I obtained from D_{i-1} and define the sets

$$X^{\text{out}}_{\text{I}} := \left\{x^{\text{out}}_T \mid T \in \mathcal{C}_j\right\}, \text{ and}$$
$$X^{\text{in}}_{\text{I}} := \left\{x^{\text{in}}_T \mid T \in \mathcal{C}_j\right\}.$$

Additionally we construct the set Y_{I} as follows: Let Q and Q' be the two cycles of $\mathsf{Per}(U_0)$. For every $j \in \left[1, \frac{3d_1}{4}\right]$, Y_{I} contains exactly one vertex of $Q \cap P^1_{4j}$, $Q \cap P^2_{4j+2}$, $Q' \cap P^1_{4j}$, and $Q' \cap P^2_{4j+2}$ each. Then remove all vertices of Y_{I} that do not belong to D_{i-1}. Note that, by choice of d_1 and the bound on $F_{I,i-1}$, this does not decrease the size of Y_{I} dramatically.

Then, if there is a family of $2^7 f_P(t)$ pairwise disjoint directed $X^{\text{out}}_{\text{I}}$-$Y_{\text{I}}$-paths in $D^1_j(U')$, Lemma 8.2.5 provides us with the existence of a $\overleftrightarrow{K}_t$-butterfly minor grasped by U and we are done. So we may assume that there does not exist such a family and thus we may find a set $Z_1 \subseteq V(D^1_j(U'))$ of size at most $2^7 f_P(t)$ that meets all such paths by Theorem 2.3.8. With a similar argument, we either find a $\overleftrightarrow{K}_t$-butterfly minor grasped by U, or a set $Z_2 \subseteq V(D^1_j(U'))$ of size at most $2^7 f_P(t)$ that meets all directed Y_{I}-X^{in}_{I}-paths in $D^1_j(U')$. Let $\pi \in [1,2]$ indicate which of the two parametrisations of U we are currently considering. Then we can define the following two sets:

$$Z_{\pi,\xi,\xi',j} := (Z_1 \cup Z_2) \cap V(D), \text{ and}$$
$$\mathcal{Z}_{\pi,\xi,\xi',j} := \left\{T \in \mathcal{T} \mid (V(T) \cup \left\{x^{\text{out}}_T, x^{\text{in}}_T\right\}) \cap (Z_1 \cup Z_2) \neq \emptyset\right\}.$$

Note that $\max\{|Z_{\pi,\xi,\xi',j}|, |\mathcal{Z}_{\pi,\xi,\xi',j}|\} \leq 2^8 f_P(t)$.

If we do not find a $\overleftrightarrow{K}_t$-butterfly minor grasped by U at any point, the sets $Z_{\pi,\xi,\xi',j}$ and $\mathcal{Z}_{\pi,\xi,\xi',j}$ are well defined for every possible choice of $\pi \in [1,2]$,

$\xi, \xi' \in [1, w+1]$, and $j \in [1,4]$. We collect these sets into the following to sets, which make up the first step towards the output of round i:

$$F'_{I,i} \coloneqq \bigcup_{\pi \in [1,2]} \bigcup_{\xi,\xi' \in [1,w+1]} \bigcup_{j \in [1,4]} Z_{\pi,\xi,\xi',j}, \text{ and}$$

$$\mathcal{F}'_{I,i} \coloneqq \bigcup_{\pi \in [1,2]} \bigcup_{\xi,\xi' \in [1,w+1]} \bigcup_{j \in [1,4]} \mathcal{Z}_{\pi,\xi,\xi',j}.$$

Consequently we have that $\max\left\{|F'_{I,i}|, |\mathcal{F}'_{I,i}|\right\} \leq 2^{11}(w+1)^2 f_P(t)$.

We may now find a clear slice $U'_i \subseteq U_{i-1}$ of width $(2^{12}(w+1)^3 f_P(t) + 1)^{2^{11+8f_P(t)} f_P(t)-i}(2^6(w+1)^3 f_P(t)+1)^{2^{11+8f_P(t)} f_P(t)-i+1} d_2$ which does not contain a marked vertex. Note that we lose the additional factor of $2(w+1)$ since we remove whole tiles of width w from U_{i-1}. Let $D'_i \coloneqq D_{i-1} - F'_{I,i}$. This concludes Step I of round i.

Claim 1. Every long jump J over U'_i in D'_i whose endpoints belong to tiles of different colour contains a vertex of a tile from $\mathcal{F}_{I,j} \setminus \mathcal{F}_{I,j-1}$ for every $j \in [1, i-1]$, and it contains a vertex of a tile from $\mathcal{F}'_{I,i}$.

Proof of Claim 1: Suppose J is also a long jump over U_{i-1} in D_{i-1}, then, as J still exists in $D'_{I,i}$, the tile in whose centre J starts, or the tile in whose centre J ends for some choices of $\pi \in [1,2]$, $\xi, \xi' \in [1, w+1]$, and $j \in [1,4]$, must be marked and therefore cannot belong to U'_i.

Hence J must contain some vertex of U_{i-1} as an internal vertex. Let T_s be the tile of U'_i in whose centre J starts, and let T be the first tile from the same tiling of U_{i-1}, that J meets after T_s. Let J' be the shortest subpath of J with endpoints in T_s and T. Then J' is a long jump over U_{i-1} in D_{i-1}. Therefore, by our assumptions on the input of the ith round of Phase I, the first part of our claim is satisfied. Moreover, if T has a different colour than T_s, T must be marked. So suppose T has the same colour as T_s. Nonetheless, since T_s and the tile T_t which contains the endpoint of J in the current tiling have different colours, J contains a directed subpath J'' which is a long jump over U_{i-1} and attaches to tiles of different colour. Hence our claim follows. ■

With this we are ready for Step II of round i.

Step II: For this step let $k_i \coloneqq (2^{12}(w+1)^3 f_P(t)+1)^{2^{11+8f_P(t)} f_P(t)-i}(2^6(w+1)^3 f_P(t)+1)^{2^{11+8f_P(t)} f_P(t)-i+1} d_2$. We are mainly concerned with the digraph D'_i. In Step II it suffices to fix one parametrisation of U since the construction of the type II auxiliary digraph leaves the complete interior of same-colour-tiles intact instead of only their centres. For every pair of $\xi, \xi' \in [1, w+1]$ let us consider the tiling $\mathcal{T} := \mathcal{T}_{U_i', k_i, w, \xi, \xi'}$ together with its four colouring $\{\mathcal{C}_1, \dots, \mathcal{C}_4\}$. Then, for every $j \in [1, 4]$ we call upon Lemma 8.2.7 which either provides us with a $\overleftrightarrow{K}_t$-butterfly minor grasped by U, and therefore closes the proof, or it produces two sets

$$\begin{aligned} Z^2_{\xi,\xi',j} \subseteq & V(D_i') \text{ of size at most } 2^3 f_P(t), \text{ and} \\ \mathcal{Z}^2_{\xi,\xi',j} \subseteq & \mathcal{T} \text{ of size at most } 2^3 f_P(t), \end{aligned}$$

such that every directed $V(U_i')$-path whose endpoints belong to different tiles of $\mathcal{C}_j$, contains a vertex of some tile in $\mathcal{Z}^2_{\xi,\xi',j}$. This allows us to form the sets for the second step of round i:

$$\begin{aligned} F''_{I,i} \coloneqq & \bigcup_{\xi,\xi' \in [1,w+1]} \bigcup_{j \in [1,4]} Z^2_{\xi,\xi',j}, \text{ and} \\ \mathcal{F}''_{I,i} \coloneqq & \bigcup_{\xi,\xi' \in [1,w+1]} \bigcup_{j \in [1,4]} \mathcal{Z}^2_{\xi,\xi',j}. \end{aligned}$$

As a result we obtain $\max\left\{|F''_{I,i}|, |\mathcal{F}''_{I,i}|\right\} \leq 2^5(w+1)^2 f_P(t)$, and we are able to produce the two sets which will be passed on to the next round.

$$\begin{aligned} F_{I,i} \coloneqq & F'_{I,i} \cup F''_{I,i} \cup F_{I,i-1}, \text{ and} \\ \mathcal{F}_{I,i} \coloneqq & \mathcal{F}'_{I,i} \cup \mathcal{F}''_{I,i} \cup \mathcal{F}_{I,i-1}. \end{aligned}$$

The bounds on $F_{I,i}$ and $\mathcal{F}_{I,i}$ follow immediately from the bounds on $F'_{I,i}$ and $F''_{I,i}$, $\mathcal{F}'_{I,i}$ and $\mathcal{F}''_{I,i}$, and the assumptions on the input of round i respectively.

The pigeon hole principle allows us to find a clear slice $U_i \subseteq U_i'$ of width $((2^{12}(w+1)^3 f_P(t)+1)(2^6(w+1)^3 f_P(t)+1))^{2^{11+8f_P(t)} f_P(t)-i} d_2$ which does not contain a marked vertex. Similar to Step I, we lose the additional factor of $2(w+1)$ since we remove whole tiles of width w from U_i'. Let $D_i \coloneqq D_i' - F''_{I,i}$. This concludes Step II of round i.

Claim 2. Every long jump over U_i in D_i contains a vertex of some tile in $\mathcal{F}_{I,j} \setminus \mathcal{F}_{I,j-1}$ for every $j \in [1, i]$.

Proof of Claim 2: Let J be a long jump over U_i in D_i, and let $\mathcal{T}$ be a tiling of U_0 defined by w and some $\xi, \xi' \in [1, w+1]$ such that J starts at the centre of some tile $T_{\mathsf{s}} \in \mathcal{T}$. Suppose all tiles of $\mathcal{T}$ which contain vertices of J belong to the same colour. Then J must have existed during the corresponding part of Step II of round i and thus either T_{s} or $T_{\mathsf{t}} \in \mathcal{T}$, which is the tile that contains the endpoint of J, must have been marked. Therefore J must contain at least one tile of a colour different from the one of T_{s}. Moreover, we may assume T_{s} and T_{t} to be of the same colour as otherwise we would be done by Claim 1. Next suppose J is also a long jump over U_{i-1}, then again J would have been considered during Step II as a long jump connecting two tiles of the same colour and thus T_{s} or T_{t} would have been marked. Therefore J must contain a vertex of some tile from U_{i-1}. Let J' be a shortest subpath from T_{s} to some tile T of U_{i-1}, then J' is a long jump over U_{i-1} and thus J contains a vertex of some tile of $\mathcal{F}_{I,j} \setminus \mathcal{F}_{I,j-1}$ for every $j \in [1, i-1]$ by our assumptions on the input of round i. If T has a different colour than T_{s}, then T would have been marked in Step I of round i, and if T shares the colour of T_{s} it must have been marked in Step II of round i. Either way our claim follows. ■

From Claim 2 it follows that we satisfy all requirements for the output of round i and thus round i is complete. We continue until we finish round $i = 2^{11+8f_P(t)} f_P(t)$ and obtain the following four objects as its output:

- a slice $U_I \coloneqq U_{2^{11+8f_P(t)} f_P(t)}$ of width d_2,
- a set $A \coloneqq F_{I,2^{11+8f_P(t)} f_P(t)}$ of size at most $t^{28} 2^{60+2^{10}t^8}$,
- a digraph $D_I \coloneqq D_{2^{11+8f_P(t)} f_P(t)} = D - A$, and
- a sequence $\mathcal{F}_{I,1} \subseteq \mathcal{F}_{I,2} \subseteq \cdots \subseteq \mathcal{F}_{I,2^{11+8f_P(t)} f_P(t)}$ such that every long jump over U_I in D_I contains a vertex of some tile in $\mathcal{F}_{I,i} \setminus \mathcal{F}_{I,i-1}$ for every $i \in [1, 2^{11+8f_P(t)} f_P(t)]$.

This brings us to the final claim of Phase I.

Claim 3. If there is a long jump over U_I in D_I, then there is a $\overleftrightarrow{K}_t$-butterfly minor grasped by U in D.

Proof of Claim 3: Let J be a long jump over U_I in D_I. We fix a parametrisation of U, $\xi, \xi' \in [1, w+1]$, and $c \in [1, 4]$ such that there exists a tile $T_{\mathsf{s}} \in \mathcal{T} \coloneqq \mathcal{T}_{U_0, d_1, w, \xi, \xi'}$ of colour c whose centre contains the starting point of J. Let $T_{\mathsf{t}} \in \mathcal{T}$ be the tile that contains the endpoint of J. As J is a long jump, note that $T_{\mathsf{s}} \neq T_{\mathsf{t}}$.

We now create a family $\mathcal{L}_0$ of $2^{9+8f_P(t)} f_P(t)$ pairwise disjoint subpaths of J with the following properties:

i) for every $L \in \mathcal{L}_0$, let $T_{L,1}$ and $T_{L,2}$ be the tiles of $\mathcal{T}$ that contain the starting point s_L and the endpoint t_L of L respectively, then there exist distinct $i_{L,1}, i_{L,2} \in [1, 2^{11+8f_P(t)} f_P(t)]$ such that s_L is a vertex of a tile from $\mathcal{F}_{I,i_{L,1}} \setminus \mathcal{F}_{I,i_{L,1}-1}$, and t_L is a vertex of some tile in $\mathcal{F}_{I,i_{L,2}} \setminus \mathcal{F}_{I,i_{L,2}-1}$, and

ii) if $L, L' \in \mathcal{L}_0$ are distinct, then $\{i_{L,1}, i_{L,2}\} \cap \{i_{L',1}, i_{L',2}\} = \emptyset$.

Let us initialise $\mathcal{I}_0 \coloneqq [1, 2^{11+8f_P(t)} f_P(t)]$ and for every subset $\mathcal{I}' \subseteq \mathcal{I}_0$, we define the family $\mathcal{F}_{\mathcal{I}'} \coloneqq \bigcup_{i \in \mathcal{I}'} \mathcal{F}_{I,i} \setminus \mathcal{F}_{I,i-1}$. Please note that every internal vertex of J that belongs to U must belong to some tile from $\mathcal{F}_{I,2^{11+8f_P(t)} f_P(t)}$, since otherwise we could find a directed path from the centre of T_s to the perimeter of U_0 contradicting the construction in Step I of Phase I, or we would have a directed path between two tiles of the same colour, where both of them are unmarked. This second outcome contradicts the construction in Step II of Phase I.

Consider the shortest subpath of J that shares its starting point with J and is a long jump over U. Let $T_{L_1,1}$ be the tile of $\mathcal{T}$ where this path ends and let s_{L_1} be the first vertex of J for which its successor along J does not belong to $T_{L_1,1}$. Note that there exists $i_{L_1,1} \in \mathcal{I}_0$ such that $T_{L_1,1} \in \mathcal{F}_{I,i_{L_1,1}} \setminus \mathcal{F}_{I,i_{L_1,1}-1}$ by the discussion above. Let L_1 be the shortest subpath of J that starts in s_{L_1} and ends in a vertex t_{L_1} for which $i_{L_1,2} \in \mathcal{I}_0 \setminus \{i_{L_1,1}\}$ exists such that t_{L_1} belongs to a tile from $\mathcal{F}_{I,i_{L_1,2}} \setminus \mathcal{F}_{I,i_{L_1,2}-1}$. Let $T_{L_1,2}$ be the tile of $\mathcal{T}$ that contains t_{L_1}. We add L_1 to $\mathcal{L}_0$ and set $\mathcal{I}_1 \coloneqq \mathcal{I}_0 \setminus \{i_{L_1,1}, i_{L_1,2}\}$. By our choice of $i_{L_1,1}$ and $i_{L_1,2}$, the path $t_{L_1}J$ still contains a vertex of a tile from $\mathcal{F}_{I,j} \setminus \mathcal{F}_{I,j-1}$ for every $j \in \mathcal{I}_1$.

Now let $q \in [2, 2^{9+8f_P(t)} f_P(t)]$ and assume that the paths $L_1, \ldots, L_{q-1}$ together with the tiles, indices and the set $\mathcal{I}_{q-1}$ have already been constructed. Follow along J, starting from $t_{L_{q-1}}$, until the next time we encounter the last vertex s_{L_q} of some tile from $\mathcal{F}_{\mathcal{I}_{q-1}}$ before J leaves said tile again. Let $T_{L_q,1} \in \mathcal{T}$ be the tile that contains s_{L_q}, and let $i_{L_q,1} \in \mathcal{I}_{q-1}$ be the integer such that s_{L_q} belongs to a tile of $\mathcal{F}_{I,i_{L_q,q}} \setminus \mathcal{F}_{I,i_{L_q,q}-1}$. Then let L_q be the shortest subpath of J that starts in s_{L_q} and ends in a vertex t_{L_q} which belongs to a tile from $\mathcal{F}_{I,i_{L_q,2}} \setminus \mathcal{F}_{I,i_{L_q,2}-1}$, where $i_{L_q,2} \in \mathcal{I}_{q-1} \setminus \{i_{L-q,1}\}$. We choose $T_{L_q,2} \in \mathcal{T}$ to be the tile that contains t_{L_q} and set $\mathcal{I}_q \coloneqq \mathcal{I}_{q-1} \setminus \left\{i_{L_q,1}, i_{L_q,2}\right\}$. As before notice that $t_{L_q}J$ still contains a vertex from some tile in $\mathcal{F}_{I,j} \setminus \mathcal{F}_{I,j-1}$ for every $j \in \mathcal{I}_q$. Add L_q to $\mathcal{L}_0$.

With every iteration we remove exactly two members from $\mathcal{I}$ and, as $|\mathcal{I}| = 2^{11+8f_P(t)}f_P(t)$, this means that by the time we reach some q for which $\mathcal{I}_q = \emptyset$, we have indeed constructed $2^{10+8f_P(t)}f_P(t)$ paths as required.

Now there must exist $c' \in [1,4]$ and a family $\mathcal{L}_1 \subseteq \mathcal{L}_0$ of size $2^{8+8f_P(t)}f_P(t)$ such that each path $L \in \mathcal{L}_1$ has at least one endpoint in $\mathcal{C}_{c'}$. And, in an immediate second step, we can find a family $\mathcal{L}_2 \subseteq \mathcal{L}_1$ of size $2^{7+8f_P(t)}f_P(t)$ such that every path in $\mathcal{L}_2$ starts in a tile of $\mathcal{C}_{c'}$, or every path in $\mathcal{L}_2$ ends in a tile of $\mathcal{C}_{c'}$. Without loss of generality we may assume that every path in $\mathcal{L}_2$ starts in a tile of $\mathcal{C}_{c'}$ since the other case follows with similar arguments.

Let $\widetilde{U}'$ be the smallest slice of U that contains all tiles from $\mathcal{C}_{c'}$, but no tile from $\mathcal{C}_{c'}$ meets the perimeter of $\widetilde{U}$. Then let $\widetilde{\mathcal{T}} \coloneqq (\mathcal{T}, c', w)_{\mathrm{II}}[\widetilde{U}']$ be the tier II tiling of $\widetilde{U} \coloneqq \widetilde{U}'[\mathcal{T}, c', w]$. Since the paths in $\mathcal{L}_2$ are pairwise disjoint, we can extend each $L \in \mathcal{L}_2$ such that it starts on the centre of the tile of $\widetilde{\mathcal{T}}$ which encloses its endpoint in U, while making sure that the resulting family of paths is still at least half-integral. Similarly, wherever necessary, we may extend the paths through U such that each of them also ends in a tile of $\widetilde{\mathcal{T}}$. Indeed, we can even guarantee that the endpoints of the resulting paths are mutually at $\widetilde{U}$-distance at least 4. Let $\mathcal{L}_3$ be the resulting half integral linkage.

Next consider the four colouring $\left\{\widetilde{\mathcal{C}}_1, \dots, \widetilde{\mathcal{C}}_4\right\}$ of $\widetilde{\mathcal{T}}$. Then there exists $\widetilde{c} \in [1,4]$ and a family $\mathcal{L}_4 \subseteq \mathcal{L}_3$ of size $2^{5+8f_P(t)}f_P(t)$ such that every path in $\mathcal{L}_4$ starts at the centre of some tile from $\mathcal{C}_{\widetilde{c}}$. It follows from the construction of $\mathcal{L}_0$ that no two paths in $\mathcal{L}_4$ start in the same tile.

By a similar argument, there exists a family $\mathcal{L}_5 \subseteq \mathcal{L}_4$ of size $2^{4+8f_P(t)}f_P(t)$ such that either none, or all paths in $\mathcal{L}_5$ end in tiles of $\widetilde{\mathcal{C}}_{\widetilde{c}}$.

In the first case we can extend every path in $\mathcal{L}_5$ towards the perimeter of $\widetilde{U}$ such that the resulting family $\mathcal{L}_6$ of paths remains at worst half-integral, and the endpoints of the resulting paths are mutually at $\widetilde{U}$-distance at least 4. Now Theorem 8.2.4 provides us with a family $\mathcal{L}_7$ of size $2^{3+8f_P(t)}f_P(t)$ such that $V(\mathcal{L}_7) \subseteq V(\mathcal{L}_6)$, and the paths in $\mathcal{L}_7$ are pairwise vertex disjoint. Hence Lemma 8.2.5 yields the existence of a $\overleftrightarrow{K}_t$-butterfly minor grasped by $\widetilde{U}$ and our claim follows.

In the second case we consider two subcases. Let $\mathcal{X}$ be the family of all tiles of $\widetilde{\mathcal{T}} \setminus \widetilde{\mathcal{C}}_{\widetilde{c}}$ which contain an internal vertex of some path in $\mathcal{L}_5$ but no endpoint of any path in $\mathcal{L}_5$.

Suppose $|\mathcal{X}| \geq 2^8 f_P(t)$ and recall that, if L and P are directed paths, we say that P is a *long jump of* L if P is a w-long jump over U and $P \subseteq L$. We also say that P is a *jump of* L, if P is a directed $V(U)$-path. Then we can use the technique from the first part of the proof of Lemma 8.2.5 to construct a half-integral family $\mathcal{L}_6$ such that

i) $|\mathcal{L}_6| = 2^{4+8f_P(t)} f_P(t)$, and
ii) for every $L \in \mathcal{L}_6$, every endpoint u of a jump of L with $u \in V(\widetilde{U})$ belongs to a tile from $\widetilde{\mathcal{C}}_{\widetilde{c}} \cup \mathcal{X}$.

We can then apply Theorem 8.2.4 to obtain a family $\mathcal{L}_7$ of size $2^{3+8f_P(t)} f_P(t)$ with $V(\mathcal{L}_7) \subseteq V(\mathcal{L}_6)$ such that the paths in $\mathcal{L}_7$ are pairwise disjoint and link the same two sets of vertices as the paths in $\mathcal{L}_6$ do. Finally Lemma 8.2.5 yields the existence of a $\overleftrightarrow{K}_t$-butterfly minor grasped by U.

So we may assume $|\mathcal{X}| < 2^8 f_P(t)$. In this case, we may find a subwall $\widetilde{U}'$ of $\widetilde{U}$ of order $d_1 - 2^8 f_P(t)(2w+1)$ that does not contain a vertex of any tile in $\mathcal{X}$ by removing, for every tile $T \in \mathcal{X}$, all edges and vertices of the horizontal cycles and vertical paths of T, which are not used by other cycles or paths. For each tile we remove during this procedure, we remove a row and a column of tiles and thereby might reduce the number of distinct tiles which contain starting vertices of paths in $\mathcal{L}_5$ by a factor of $\frac{1}{2}$. However, since $|\mathcal{X}| < 2^8 f_P(t)$, we can still find, after potentially expanding the start and end sections of some paths to again reach the slightly shifted perimeters of their tiles, a half-integral family $\mathcal{L}_6$ of size $2^4 f_P(t)$ of paths that start and end in tiles of $\widetilde{\mathcal{C}}_{\widetilde{c}}$ and that are otherwise disjoint from $\widetilde{U}'$. By using Theorem 8.2.4 we can transform this family into an integral family $\mathcal{L}_7$ of size $2^3 f_P(t)$ and thus an application of Lemma 8.2.7 yields a $\overleftrightarrow{K}_t$-butterfly minor grasped by U. ■

Concluding Phase I, Claim 3 either yields a $\overleftrightarrow{K}_t$-butterfly minor grasped by U and therefore closes the case, or U_I is in fact clean, meaning that U_I has no long jump in D_I. The set A is a set of vertices of D which means that it can be seen as a set of edges from M in B. Hence we may treat A as an M-conformal set of vertices in B of size at most $t^{28} 2^{61+2^{10}t^8}$. Consequently we may bound the function α from the statement of Theorem 8.0.4 as follows:

$$\alpha(t) \leq t^{28} 2^{61+2^{10}t^8}.$$

Phase II Until now all of our efforts were focused on the removal of long jumps, so the only jumps that remain must be short, i.e. their endpoints in U_I must have U_I-distance at most $w-1$. To meet the requirements from Phase I and have enough space left in U_I, let us make the following assumption:

$$d_2 \geq t^4(r + 2^{32+t^{30}}).$$

Let us partition U_I into $2t^4$ many slices as follows: First we can partition U_I into t^4 slices S_i of width $r + 2^{32+t^{30}}$. Each S_i is then partitioned into a slice H_i of width $r + 2^{15+t^30}$ that contains the left perimeter cycle of the of S_i, and a slice G_i of width $2^{31+t^{30}}$ containing the right cycle of the perimeter of S_i. For every $i \in [1, t^4]$ we may now further partition H_i. Let $N_{i,L} \subseteq H_i$ be the slice of width $2^{30+t^{30}}$ containing the left cycle of $\mathsf{Per}(H_i)$, let $N_{i,R}$ be the slice of width $2^{30+t^{30}}$ containing the right cycle of $\mathsf{Per}(H_i)$, and let $N_i := H_i - N_{L,i} - N_{R,i}$ be the remaining slice of width r.

Claim 4. If for all $i \in [1, t^4]$ the slice $\mathcal{S}(N_i)$ is **not** $\mathsf{Per}(\mathcal{S}(N_i))$-flat in B with respect to A, then there exists a $K_{t,t}$-matching minor grasped by W in B.

Proof of Claim 4: To start, notice that, since U_I is clean in D_I, every internally M-conformal path in $B - A$ whose endpoints belong to $\mathcal{S}(U_I)$ and which is internally disjoint from $\mathcal{S}(U_I)$ must correspond to a short jump over U_I in D_I. The goal of this proof is to adjust the perfect matching M of B in such a way that the M'-direction of the new perfect matching M' yields the desired $\overleftrightarrow{K}_t$-butterfly minor.

The Model of $\overleftrightarrow{K}_t$: Let us describe how the construction of the $\overleftrightarrow{K}_t$-butterfly minor works. We utilise essentially the same construction that was used in [GKKK20] to obtain Lemma 8.2.2.

Consider the slice $\widetilde{U}_1$ of width d_1 from the triadic partition of U. Partition $\widetilde{U}_1$ into $t+1$ slices of equal width named $U'_1, \dots, U'_{t+1}$. The first t of these slices will hold the *roots* of the t vertices of $\overleftrightarrow{K}_t$, where the root of the

ith vertex will be contained in U_i'. The root of the ith vertex will be a path V_i, subpath of some horizontal path of W_i' from left to right, with t incoming edges e_j^i and t outgoing edges f_j^i in such a way that

- the edges e_j^i and f_j^i, $j \in [1, t-1]$, belong to vertical cycles of U_i, $i \in [1, t]$,
- the heads of $e_1^i, e_2^i, \dots, e_t^i$ appear in the order listed when traversing along V_i,
- the tails of $f_1^i, f_2^i, \dots, f_t^i$ appear in the order listed when traversing along V_i, and
- the head of e_t^i appears on V_i before the tail of f_1^i when traversing along V_i.

We also require the slices U_i' to appear in the order $U_1', U_2', \dots, U_t'$ from left to right, and that all V_i belong to the same horizontal path. If we are able to find a family of pairwise disjoint paths, internally disjoint from the roots, such that for every pair of distinct $i, j \in [1, t]$ the head of f_j^i is linked to the tail of e_i^j, then the union of these paths together with the roots can be seen to form a butterfly minor model of $\overleftrightarrow{K}_t$. Indeed, each root can be contracted into a single vertex by the butterfly minor relation. If we do this for each of the t roots, then the resulting digraph is a subdivision of $\overleftrightarrow{K}_t$. For an illustration of a root see Figure 8.9.

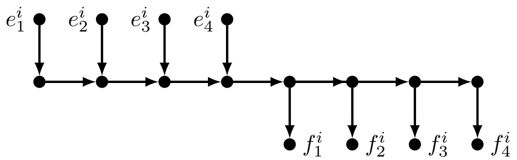


Figure 8.9.: The the root of the model of the ith vertex of $\overleftrightarrow{K}_4$.

Recall that a tile T of a matching wall W' is non-Pfaffian if either the attachment of the centre of T is a non-Pfaffian bipartite graph, or there exists a short jump over W' with both endpoints on the interior of T.

For every $i \in [1, t^4]$, let N_i' be the slice of width $4w + 2 + r$ of H_i, such that $N_i' - N_i$ consists of exactly two components which both are slices of width $2w + 1$. Since $\mathcal{S}(N_i)$ is not $\mathsf{Per}(\mathcal{S}(N_i))$-flat in B with respect to A, there must exist ξ, ξ' and tiling $\mathcal{T} = \mathcal{T}_{N_i', 4w+2+r, w, \xi, \xi'}$ containing a tile T_i such that $\mathcal{S}(T_i)$ is non-Pfaffian in $B - A$. Note that the centre of T_i

must lie within N_i, since otherwise we either could choose a different T_i, or $\mathcal{S}(N_i)$ would be $\mathsf{Per}(\mathcal{S}(T_i))$-flat in B with respect to A.

We may adjust the parametrisation of U and its embedding, such that the top most strip of height $16f_P(t) + 2w$ does not contain a vertex of T_i for any $i \in [1, t^4]$. This is to make sure that the top most strip of height $8f_P(t)$ is not used by any of the preliminary constructions below and can therefore later be used to construct our $\overleftrightarrow{K}_t$-butterfly minor.

Let U_I'' be the subgraph of U_I obtained as the union of all vertical cycles and horizontal paths of U_I which do not contain vertices of the T_i. Note that the resulting graph is again a slice of some wall. Then we may find a tile T_i' of width w in U_I'' for every $i \in [1, t^4]$, such that the interior of the centre of T_i' in U_I contains the entire tile T_i. Observe that, since the centres of the T_i lie within the N_i, the T_i' are completely contained in the N_i' respectively. Let $s_{i,L}$ be the top left, and $s_{i,R}$ be the top right vertex of the perimeter of T_i' as well as $t_{i,L}$ be the bottom left and $t_{i,R}$ be the bottom right vertex of the perimeter of T_i'. By Lemma 8.3.8 we can find a perfect matching M_i of $\mathcal{S}(N_i') \cup \mathsf{Att}_{B-A,\mathcal{S}(U_I)}(T_i)$ such that

- after slightly re-routing the horizontal paths of U_I'' in $\mathcal{D}(\mathcal{S}(D_I), M \Delta M_i)$, U_I'' still exists, and
- there are paths vertex disjoint directed paths L_i and R_i such that
  - L_i starts in $s_{i,L}$ and ends in $t_{i,R}$,
  - R_i starts in $s_{i,R}$ and ends in $t_{i,L}$, and
  - L_i and R_i are vertex disjoint from $U_I'' - \mathcal{D}(\mathcal{S}(T_i'), M_i)$.

Note that for $i \neq j \in [1, t^4]$, we know that M_i and M_j are disjoint. Moreover, $\mathcal{S}(N_i') \cup \mathsf{Att}_{B-A,\mathcal{S}(U_I)}(T_i)$ and $\mathcal{S}(N_j') \cup \mathsf{Att}_{B-A,\mathcal{S}(U_I)}(T_j)$ must be disjoint since otherwise we could find an internally M-conformal path P starting in $\mathcal{S}(T_i)$ and ending in $\mathcal{S}(T_j)$ which is internally disjoint from $\mathcal{S}(U_I)$. In D_I such a path P would correspond to a long jump over U_I by the construction of H_i and H_j, but by our assumption, there cannot exist such a long jump. Let $M' \coloneqq M \Delta \bigcup_{i=1}^{t^4} M_i$ and $D_{II}' \coloneqq \mathcal{D}(\mathcal{S}(D_I), M')$. Hence we can find a slice U_{II}' corresponding to the construction of U_I'' for all $i \in [1, t^4]$ simultaneously in D_{II}', and the paths L_i, R_i exist for all $i \in [1, t^4]$ such that for all $i \neq j \in [1, t^4]$, L_i and R_i are disjoint from L_j and R_j.

For every $i \in [1, t^4]$, let M_i' be the perfect matching of $\mathcal{S}(G_i)$ obtained by switching M along all horizontal cycles of $\mathcal{S}(G_i)$. At last let M'' be the perfect matching of $\mathcal{S}(U'_{t+1})$ obtained by switching M along all of its horizontal cycles. Then we may set $M_{II} := M'\Delta(\bigcup_{i=1}^{t^4} M'_i \cup M'')$ and consider $D_{II} := \mathcal{D}(B, M_{II})$. Let U_{II} be the union of $U - U_I$, the slices of $U_I I$ that correspond to the M_i-directions of the H_i together with the paths R_i and L_i, and the M_{II}-directions of the G_i. Note that U_{II} is indeed a subgraph of D_{II} that closely resembles a cylindrical wall, with a few holes, t^4 pairs of crosses over small areas over the wall, and some sections, corresponding the the G_i, in which the vertical cycles are oriented upwards instead of downwards. See Figure 8.10 for a rough sketch of the digraph U_{II}.

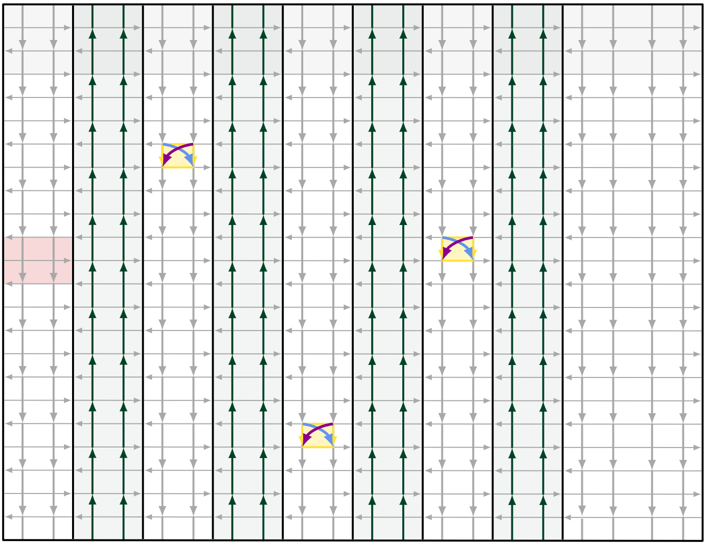


Figure 8.10.: A sketch of the digraph U_{II} with its crosses, the top strip of bounded height we aim to keep unused, and the section of the left slice which is used for the roots of $\overleftrightarrow{K}_t$ (highlighted in red).

As the final step of the proof we construct a $\overleftrightarrow{K}_t$-butterfly minor grasped by U_{II} in D_{II}. Since $\mathcal{S}(D_{II}) = B$, Lemma 3.2.17 can then be applied to find a $K_{t,t}$-matching minor grasped by W in B, which will conclude the proof of our claim.

Let us position the roots of the vertices for our $\overleftrightarrow{K}_t$ on the path $P^1_{\frac{3}{2}d_1}$. For each pair $(i,j) \in [1,t]^2$, we will construct a path $J_{i,j}$ starting at the head of f^j_i and eventually ending at the tail of e^i_j. Since the roots are ordered from left to right, we start out with a family of disjoint paths in U_{II} which are ordered from left to right as

$$J_{1,1}, J_{1,2}, \ldots, J_{2,1}, \ldots, J_{t-1,t}, J_{t,1}, \ldots, J_{t,t}$$

while moving downwards in the current embedding of U_{II}. We grow the paths $J_{i,j}$ until each of them ends on some $P^1_{h_1(i,j)}$ where $h_1(i,j) > h_1(i',j')$, if (i,j) is lexicographically smaller than (i',j'). Then we grow the paths along their respective $P^1_{h_1(i,j)}$, until each of them ends on a switched vertical cycle $Q_{h_2(i,j)}$ of U'_{t+1} where $h_2(i,j) > h_2(i',j')$, whenever (i,j) is lexicographically smaller then $h_2(i,j)$. Afterwards we move each path along its respective horizontal cycle upwards, until each $J_{i,j}$ reaches some $P^1_{h_3(i,j)}$, where $h_3(i',j') < h_3(i,j)$, if $J_{i,j}$ is further to the right than $J_{i',j'}$ while moving upwards, and each $P^1_{h_3(i,j)}$ is above the top path of T'_1, while avoiding the top strip of height $4f_P(t)$. This is possible by the choice of our parametrisation. Next move each $J_{i,j}$ towards the right along its $P^1_{h_3(i,j)}$ until the following requirements are met:

- the path $J_{1,1}$ ends on a horizontal cycle of $N_{L,1}$, which lies left of T'_1,
- the two paths $J_{1,2}$ and $J_{1,3}$ meet exactly the two horizontal cycles of N'_1 which contain the tail of L_1 and R_1 respectively, and
- the paths $J_{1,4}, \ldots, J_{t,t}$ end on horizontal cycles of $N_{R,1}$ such that $J_{i,j}$ ends on a cycle left of the cycle on which $J_{i',j'}$ ends, if $J_{i,j}$ passes T'_1 below $J_{i',j'}$.

Now, by growing the paths downwards, we may route $J_{1,2}$ and $J_{1,3}$ through the paths L_1 and R_1, thereby swapping their relative positions within the whole path family. We may then either grow the paths further downwards and then go right again, or use the switched horizontal cycles of G_1, to grow them back upwards within U_{II}. Either way, we are able to route the now neighbouring paths $J_{1,2}$ and $J_{1,4}$ (or $J_{2,1}$ in case $t = 3$) through T'_2. Since we have t^4 such crosses in total, we may repeat this process until, after passing through T'_{t^4}, the paths are in the following order when moving downwards in our current embedding of U_{II}:

$$J_{1,1}, J_{2,1}, J_{3,1}, \dots, J_{t,1}, J_{1,2}, \dots, J_{t,2}, \dots, J_{1,t}, \dots, J_{t,t}.$$

We can now use G_{t^4} to route all of these paths upwards and finally use the top strip of height $4f_P(t)$, which we kept untouched so far, to route the paths back to the roots of our vertices. With the changed order we are no able to grow each path $J_{i,j}$ such that it indeed ends on the tail of $e_{j,i}$, while making sure that our paths stay pairwise vertex disjoint. Hence we have found a $\overleftrightarrow{K}_t$-butterfly minor as desired and the proof of our claim is complete. ■

From Claim 4 it follows that, in case all t^4 slices $\mathcal{S}(N_i)$ are **not** $\mathsf{Per}(\mathcal{S}(N_i))$-flat in B with respect to A, then there exists a $K_{t,t}$-matching minor grasped by W in B. In this case we are done and thus we may assume that there exists some $i \in [1, t^4]$ such that $\mathcal{S}(N_i)$ is indeed $\mathsf{Per}(\mathcal{S}(N_i))$-flat in B with respect to A. Since N_i is of width r, $\mathcal{S}(N_i)$ contains a conformal matching r-wall $\widetilde{W}$ such that $\mathsf{Per}(\widetilde{W}) = \mathsf{Per}(\mathcal{S}(N_i))$ and thus our proof is complete. At last let us combine all assumptions on the d_j to obtain the following bound on $\rho(t,r)$:

$$\rho(t,r) \leq (2^{140}t^{76})^{2^{10}t^8+2^{12}}(r + 2^{32+t^{30}}).$$

□

Please note that many of the exponential parts of the bounds in α and ρ are due to specific difficulties encountered in the digraphic setting. Indeed, even the exponential lower bound on f_W necessary for Lemma 8.2.2 can probably be made polynomial by simply diving deeper into the increased freedom provided by the possibility of switching perfect matchings along directed cycles. At this point it is not clear whether the exponential part of Phase II in the proof of Theorem 8.0.4 can also be made polynomial. To do this, it is probably necessary to develop a technique for removing long jumps that does not rely on tilings.

8.5. Applications to Structural (Di)Graph Theory

In this section we explore the digraphic version of Theorem 8.0.4 as an alternative to Theorem 8.0.7. We believe that the results of the following two subsections further strengthen the importance of strong planarity and the strong genus of digraphs for structural digraph theory.

Indeed, an important result from the findings in this section will be the following.

Theorem 8.5.1 (G*)**.** Let D be a strongly connected digraph. Then $\mathfrak{A}(D)$ contains a planar digraph.

From this we can deduce that first for every g there exists a planar digraph of strong genus g, and second, that the notion of planarity in the theory of butterfly minors is probably not strong enough to play a role similar to the one planarity plays in undirected graph minor theory.

8.5.1. Another Flat Wall Theorem for Digraphs

An advantage of relying heavily on the digraphic setting for the proof of Theorem 8.0.4 is that we may first select a perfect matching M of our choosing and then just have to consider those M-conformal matching walls which have M as their canonical matching, and only M-conformal sets as our apex set. This means, we can directly translate our matching version of the flat wall theorem into the digraphic setting. To achieve this, two main factors need to be recalled.

First of all, what does it mean for a digraph D, if $\mathcal{S}(D)$ excludes $K_{t,t}$ as a matching minor, and similarly, what does it mean if $\mathcal{S}(D)$ does contain $K_{t,t}$ as a matching minor. In Section 3.2 we gave a preliminary answer to these questions in form of Lemma 3.2.25. So $\mathcal{S}(D)$ excludes $K_{t,t}$ as a matching minor if and only if D excludes the entire anti-chain $\mathfrak{A}(\overleftrightarrow{K}_t)$ as butterfly minors. Consequently, any result we can deduce from Theorem 8.0.4 will be a new result for the digraphic setting, distinct from Theorem 8.0.7 which explicitly does not take into account anti-chains of the butterfly minor relation. It also means that any digraphic analogue of Theorem 8.0.4 is a natural continuation of our findings from Chapter 7 such as Corollary 7.2.11 and Theorem 7.2.13, which act as natural counterparts to the respective results in the undirected setting.

Second is the question how to translate our notion of flatness to the digraphic setting. Since we have already seen the equivalence between the contraction of directed separations of order one and strongly 2-connected digraphs with tight cut contractions and braces, by recalling Theorem 7.2.13 and the small-cycle-sum operation we can deduce working analogues of the definitions necessary for H-flatness in bipartite graphs with perfect matchings.

Let D be a strongly 2-connected non-even digraph and let H be a strongly 2-connected strongly planar digraph. We say that H is a *summand* of D if there exist strongly planar strongly 2-connected digraphs $H_1, \dots, H_\ell$ such that D can be constructed from the H_i by means of small-cycle-sums, where $H = H_1$.

Let D and H be digraphs such that H has exactly one dibrace[5] J which is not isomorphic to $\overleftrightarrow{K}_2$. We say that H is a *J-expansion.* A dibrace G of D is said to be a *host* of H, if G contains a subgraph H' that is a J-expansion and can be obtained from H by means of directed tight cut contractions. The digraph H' is called the *remnant* of H.

Let D, H, and J be digraphs such that H and J are subgraphs of D. We say that H is *J-bound* if there exists a subgraph K of $D - J$ that is the union of strong components of $D - J$ such that $K \cup J$ is strongly connected and H is a subgraph of $K \cup J$. The digraph $K \cup J$ is called a *J-base* of H in D.

Definition 8.5.2 (P-Flatness for Digraphs)**.** Let D be a digraph and H be a strongly planar and strongly connected digraph that is a J-expansion of some strongly planar and strongly 2-connected digraph J. Moreover, let P be a collection of pairwise vertex disjoint faces of H which each are bound by a directed cycle. At last, let $A \subseteq V(D)$ be a set of vertices. Then H is *P-flat* in D *with respect to* A if

i) H is a subgraph of $D' \coloneqq D - A$,
ii) some P-base of H in D' has a non-even dibrace D'' that is a host of H, and
iii) D'' has a summand G that contains the remnant H' of H, such that every remnant of a face from P bounds a face of G.

This allows us to formulate the anti-chain version of the directed flat wall theorem.

[5] Recall that a dibrace of a digraph is a strongly 2-connected butterfly minor that can be obtained purely by means of directed tight cut contractions.

Theorem 8.5.3 (G*)**.** Let $r, t \in \mathbb{N}$ be positive integers. There exist functions $\vec{\alpha}\colon \mathbb{N} \to \mathbb{N}$ and $\vec{\rho}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every digraph the following is true: If W is a cylindrical $\vec{\rho}(t, r)$-wall in D, then either

i) There exists $H \in \mathfrak{A}(\overleftrightarrow{K}_t)$ such that D has an H-butterfly minor grasped by W, or
ii) there exists a set $A \subseteq V(D)$ with $|A| \leq \vec{\alpha}(t)$ and a cylindrical r-wall $W' \subseteq W - A$ such that W' is $\mathsf{Per}(W')$-flat in D with respect to A.

Proof. The theorem follows almost immediately from Theorem 8.0.4. Let D be a digraph, $t, r \in \mathbb{N}$ be positive integers, and W be a cylindrical $\rho(t, r)$-wall in D. Moreover, consider $B \coloneqq \mathcal{S}(D)$ together with the perfect matching M for which $\mathcal{D}(B, M) = D$ holds. Then $\mathcal{S}(W)$ is an M-conformal matching $\rho(t, r)$-wall such that M contains the canonical matching of $\mathcal{S}(W)$. So Theorem 8.0.4 leaves us with two cases. In the first case we find a $K_{t,t}$-matching minor in B grasped by $\mathcal{S}(W)$. It follows from Section 3.2 that this implies the existence of some $H \in \mathfrak{A}(\overleftrightarrow{K}_t)$, such that D contains an H-butterfly minor grasped by W. Otherwise there exist an M-conformal set $A' \subseteq V(B)$ of size at most $\alpha(t)$ and an M-conformal matching r-wall $U' \subseteq \mathcal{S}(W)$, such that U' is $\mathsf{Per}(U')$-flat in B with respect to A'. Since there is a bijection between the non-trivial tight cuts in B and the non-trivial directed separations of order one in D, we can choose A to be the set of all edges of M with both endpoints in A', and our claim follows. $\square$

Whilst a directed version of Theorem 8.0.4 can be obtained with a relatively straight forward argument, it is not clear whether one can adapt Theorem 8.0.5 into the digraphic setting. This is because the reverse direction of Theorem 8.0.5 requires an assumption on all perfect matchings, while in D we, a priori at least, can only talk about the unique perfect matching M which is used to obtain D from $\mathcal{S}(D)$ as its M-direction. Hence for a weak structure theorem for digraphs similar to Theorem 8.0.5 one first needs to find a way to express the structure of all perfect matching of $\mathcal{S}(D)$ in a concise way by just considering D. One direction this might go could be expanding Theorem 8.5.3 to all members of $\mathfrak{A}(W)$, where W is a cylindrical $\vec{\rho}(t, r)$-wall.

8.5.2. An Application to Treewidth of Undirected Graphs

Before the resolution of the Pfaffian Recognition Problem by McCuaig et al. only partial results were known. Among these is a result on symmetric digraphs[6] by Thomassen. This result shows an interesting feature of symmetric non-even digraphs.

Definition 8.5.4 (C_4-Cockade)**.** A graph G is a C_4*-cockade* if it can be constructed by means of 2-clique sums from a number of disjoint copies of C_4, see Figure 8.11 for some examples. A graph is a *partial* C_4*-cockade* if it is a subgraph of a C_4-cockade.

Theorem 8.5.5 ([Tho86])**.** A symmetric digraph $\overleftrightarrow{G}$ is non-even if and only if G is a partial C_4-cockade.

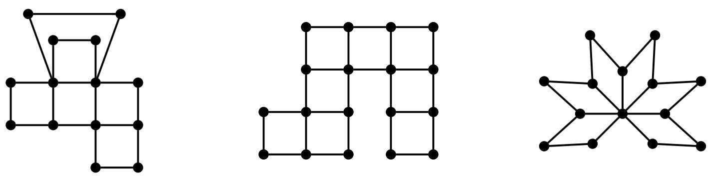


Figure 8.11.: Examples of C_4-cockades taken from [Tho86].

An immediate and easy to check corollary is the following:

Corollary 8.5.6 (G*)**.** Let G be a graph. If G is a partial C_4-cockade, then $\mathrm{tw}(G) \leq 2$.

By combining Theorem 8.5.5 with Corollary 8.5.6, Theorem 3.3.4, and Theorem 5.3.27 one obtains the following interesting relation between the existence of $K_{3,3}$ in $\mathcal{S}(\overleftrightarrow{G})$ and $\mathrm{tw}(G)$.

Corollary 8.5.7 (G*)**.** Let G be a graph. If $\mathcal{S}(\overleftrightarrow{G})$ contains $K_{3,3}$ as a matching minor then $\mathrm{tw}(G) \geq 2$, and if $\mathrm{tw}(G) \geq 3$, then $\mathcal{S}(\overleftrightarrow{G})$ contains $K_{3,3}$ as a matching minor.

[6] Recall that a digraph is symmetric if it can be obtained from an undirected graph by replacing every undirected edge by a digon.

So the existence of $K_{3,3}$ as a matching minor in $\mathcal{S}(\overleftrightarrow{G})$ is closely linked to the treewidth of G. The main result of this subsection is the following generalisation of Corollary 8.5.7.

Theorem 8.5.8 (G*)**.** There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $t \in \mathbb{N}$ and every graph G the following two statements hold:

i) If $\mathcal{S}(\overleftrightarrow{G})$ contains $K_{t,t}$ as a matching minor, then $\mathrm{tw}(G) \geq \frac{1}{2}t - 1$, and
ii) if $\mathrm{tw}(G) \geq f(t)$, then $\mathcal{S}(\overleftrightarrow{G})$ contains $K_{t,t}$ as a matching minor.

Towards Theorem 8.5.8 and Theorem 8.5.1 we start by showing that any large enough wall W contains $K_{t,t}$ as a matching minor in $\mathcal{S}(\overleftrightarrow{W})$. Since, by Corollary 2.2.28, every graph of large enough treewidth contains a large wall, Theorem 8.5.8 follows almost immediately. Similarly, as W is a planar graph, $\overleftrightarrow{W}$ is a planar digraph and with $K_{t,t}$ being a matching minor of $\mathcal{S}(\overleftrightarrow{K}_{t,t})$, Lemma 3.2.25 implies that $\overleftrightarrow{W}$ must contain a member of $\mathfrak{A}(\overleftrightarrow{K}_t)$ as a butterfly minor. Since all butterfly minors of $\overleftrightarrow{W}$ must be planar, Theorem 8.5.1 follows.

Lemma 8.5.9 (G*)**.** There exists a function $f_0\colon \mathbb{N} \to \mathbb{N}$ such that for every $t \in \mathbb{N}$, and every $f_0(t)$-wall W, $\mathcal{S}(\overleftrightarrow{W})$ contains $K_{t,t}$ as a matching minor.

Proof. First observe that every k-wall W_k has treewidth at least k. To see this, we may construct a bramble of order $k+1$ as follows: Let P be the right most path from top to bottom that contains two hub-vertices of every row of W_k except for the bottom one, from which it is disjoint. Next let B be the bottom row. At last, for each row i except the bottom one let T_i be the ith row except for the last two vertices together with the ith column, from left to right. Then each T_i has an edge to P and intersects B and each other T_j. However, no vertex is contained in more than two of these subgraphs, and P is disjoint from all of them. So a minimum hitting set for all of these subgraphs must contain a vertex from every row of W_k and an additional vertex to cover P which makes in total $k+1$ vertices.

Moreover, note that this is also a directed bramble in $\overleftrightarrow{W}_K$, hence $\mathrm{dtw}(\overleftrightarrow{W}_k) \geq k$ by Theorem 2.3.20. Consequently, for every $t \in \mathbb{N}$, we are guaranteed by Corollary 2.3.25 that every $\mathsf{g}_{\mathrm{dir}}(2t)$-wall W satisfies that $\overleftrightarrow{W}$ contains a cylindrical t-wall.

Now let W be a $\mathsf{g}_{\mathrm{dir}}(2\rho(t,2))$-wall. Then, as discussed above, $\overleftrightarrow{W}$ contains a cylindrical $\rho(t,2)$-wall U as a subgraph.

Suppose $\mathcal{S}(\overleftrightarrow{W})$ does not contain $K_{t,t}$ as a matching minor. Let $B := \mathcal{S}(\overleftrightarrow{W})$, and let M be the perfect matching of B such that $\mathcal{D}(B,M) = \overleftrightarrow{W}$. Then $\mathcal{S}(U)$ is an M-conformal subgraph of B. By Theorem 8.0.4 there must exist an M-conformal set $A \subseteq M$ of size at most $\alpha(t)$, such that $B - M$ contains an M-conformal matching 2-wall $U' \subseteq U$ which is $\mathsf{Per}(U')$-flat in B with respect to A. Let B'' be the Pfaffian brace that is the host of U' in a $\mathsf{Per}(U')$-base of $B - A$.

Note that being a brace means being strongly 2-connected in the setting of digraphs, which translates to 2-connectivity in the setting of undirected graphs. Moreover, a directed tight cut contraction in a symmetric digraphs is isomorphic to deleting all non-separator vertices of one of the shores. Hence $\mathrm{un}(\mathcal{D}(B'',M))$ is an actual subgraph of W. Let us call this subgraph H. For H we have the following informations:

i) H contains $\mathrm{un}(\mathcal{D}(U',M))$ as a subgraph, and
ii) $\overleftrightarrow{H}$ is non-even by Theorem 3.3.4.

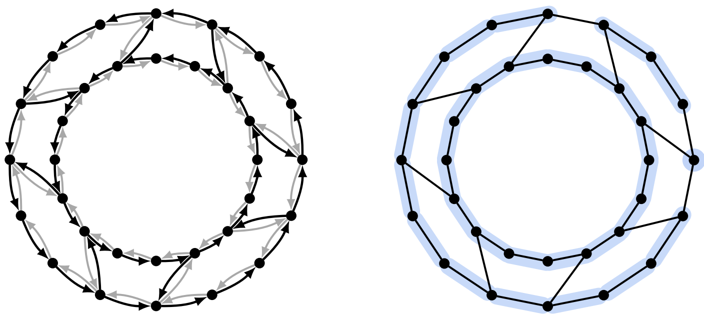


Figure 8.12.: An elementary cylindrical 2-wall as a subgraph of a symmetric digraph $\overleftrightarrow{G}$ (left) and a bramble of order 4 in G (right).

Consider Figure 8.12. Here we depict a cylindrical 2-wall Q as a subgraph of a symmetric digraph (on the left), and its underlying undirected graph (on the right). We consider the following family of subgraphs of Q: Let v_1 be a hub-vertex, i.e. a vertex of degree 3 in Q, on the outer cycle C' of Q, then let P be a shortest subpath of the outer cycle of Q such that one endpoint of P is a neighbour of v_1 and the other endpoint is a hub-vertex of Q. Next let $L \coloneqq C' - v_1 - P$, and at last let $C \coloneqq Q - v_1 - P - L$. The result are four pairwise disjoint connected subgraphs, one of them only consisting of v_1, of Q that are pairwise joined by an edge. Hence the collection of these four subgraphs forms a bramble of order four, implying $\mathrm{tw}(H) \geq 3$. By Corollary 8.5.7 this is a contradiction to $\overleftrightarrow{H}$ being non-even and therefore this contradicts Theorem 8.0.4 or $\mathcal{S}(\overleftrightarrow{W})$ contains $K_{t,t}$ as a matching minor. Thus we may assume the latter and the proof is complete. $\square$

As discussed above, Theorem 8.5.1 follows immediately. Hence all that is left to do is to prove Theorem 8.5.8, this concludes the chapter.

Proof of Theorem 8.5.8. First suppose $\mathcal{S}(\overleftrightarrow{G})$ contains $K_{t,t}$ as a matching minor. Then, by Corollary 5.1.21 we have $\mathrm{pmw}(\mathcal{S}(\overleftrightarrow{G})) \geq \frac{1}{2}t$ which in turn implies, by Theorem 5.3.27, that $\mathrm{dtw}(\overleftrightarrow{G}) \geq \frac{1}{2}t - 1$. Since $\mathrm{dtw}(\overleftrightarrow{G}) = \mathrm{tw}(G)$ the first part of our claim follows.

For the second part let f_0 be the function from Lemma 8.5.9, and let $f(t) \coloneqq \mathsf{g}_{\mathrm{undir}}(2f_0(t))$. Then, by Corollary 2.2.28, G contains an $f_0(t)$-wall as a subgraph, and by Lemma 8.5.9 this means that $\mathcal{S}(\overleftrightarrow{G})$ must contain $K_{t,t}$ as a matching minor. $\square$

Part III.

Conclusion

Chapter 9.

Concluding Remarks

While the main focus of Part II was on the establishment of a matching minor theory for bipartite graphs, our research into the topic raised other interesting questions, which are not directly linked to a general matching minor theory for bipartite graphs, or go beyond the bipartite setting.

In this chapter we introduce one additional field of research regarding bipartite graphs in the form of a generalisation of the chromatic number in Section 9.1, and state a conjecture relating the 'matching chromatic number' and the existence of certain matching minors for bipartite graphs.

Arguably the biggest question in structural matching theory right now is the complexity of the general Pfaffian recognition problem, or, to be more precise, its non-bipartite variant. In Section 5.1 we mentioned Norine's Pfaffian recognition algorithm for matching covered graphs of bounded perfect matching width. While the algorithm itself is polynomial, with the width of the perfect matching decomposition as a parameter, it still needs a bounded width decomposition as an input. One approach to tackle this more general problem could be to invoke the setting of bidirected graphs. As we have seen in Section 3.4, the bidirected setting and the non-bipartite setting of graphs with perfect matchings have many problems that make the application of established tools for minor theory hard, if not impossible. In Section 9.2 we try to explain the problems we encountered during our attempts at generalising directed treewidth to bidirected graphs.

By revisiting our list of questions from Section 3.5 and collecting the (partial) answers we were able to give in the previous chapters, we close this chapter with Section 9.3.

9.1. The Matching Chromatic Number

Let G be a graph. A *vertex colouring* of G is a function $c\colon V(G) \to \mathbb{N}$, it is a *$k$-colouring* for some integer k, if its image in $\mathbb{N}$ has size at most k. A k-colouring of G is *proper*, if for all $uv \in E(G)$ we have $c(u) \neq c(v)$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G has a proper k-colouring.

The study of the chromatic number and related problems, such as the famous Four Colour Problem, i.e. the question whether the chromatic number of a planar graph is at most four, is arguably one of the most prominent fields in graph theory. Indeed, the Four Colour Problem is closely tied into the study of planar graphs and thus, in light of Theorem 2.2.3 and Theorem 2.2.8, which were at least partially inspired by the Four Colour Problem, it can be seen as an important part of the origins of Graph Minor Theory as a whole. There exists a vast generalisation of the Four Colour Problem, which eventually became a theorem [AH76], in terms of the existence of complete minors known as Hadwiger's Conjecture.

Conjecture 9.1.1 (Hadwiger's Conjecture, [Had43])**.** Every graph G with $\chi(G) \geq t$ has K_t as a minor.

For $t \in [1,6]$ Hadwiger's Conjecture is known to be true [Wag37, Had43, RST93], while for $t \geq 7$ only approximative and probabilistic results are known.

There exists a natural generalisation of the chromatic number to digraphs. A *vertex colouring* of a digraph D is a function $c\colon \mathbb{N} \to \mathbb{N}$, it is a *$k$-colouring* for some integer k if its image in $\mathbb{N}$ has size at most k. A k-colouring of D is proper, if every directed cycle C in D contains two vertices u and v with $c(u) \neq c(v)$. The *dichromatic number* of D, denoted by $\vec{\chi}(D)$, is the smallest integer k such that D has a proper k-colouring.

As a directed version of the Four Colour Theorem, the Two Colour Conjecture posed by Erdős and Neumann-Lara and independently by Skrekovski still stands open.

Conjecture 9.1.2 (Two Colour Conjecture, [BFJ$^+$04, NL82])**.** Every oriented planar graph D satisfies $\vec{\chi}(D) \leq 2$.

While the Two Colour Conjecture itself is exclusively concerned with oriented graphs, and therefore explicitly does not care about butterfly minors, as those quickly yield digraphs with digons, recently Steiner, Garlet Milani, and the author were able to give a characterisation of of the class $\mathcal{F}_2$ defined as follows:

Let $k \in \mathbb{N}$ be an integer, then $\mathcal{F}_k$ is the largest class of digraphs such that for all $D \in \mathcal{F}_k$ we have

i) $\vec{\chi}(D) \leq k$, and
ii) every butterfly minor of D belongs to $\mathcal{F}_k$.

Theorem 9.1.3 ([MSW19])**.** The class $\mathcal{F}_2$ is exactly the class of non-even digraphs.

In light of Theorem 3.3.4 and Lemma 3.2.25, and by considering Hadwiger's Conjecture, an interesting reformulation of Theorem 9.1.3 is the following:

Theorem 9.1.4 (Theorem 9.1.3 reformulated)**.** The class $\mathcal{F}_2$ is exactly the class of digraphs excluding $\mathfrak{A}(\overleftrightarrow{K}_3)$ as butterfly minors.

The resemblance of Hadwiger's Conjecture in the statement of Theorem 9.1.4 is further emphasised by the following, more general, result.

Theorem 9.1.5 ([ACH$^+$16, GSS20])**.** There exists a function $\mathsf{c}\colon \mathbb{N} \to \mathbb{N}$ such that every digraph D with $\vec{\chi}(D) \geq \mathsf{c}(t)$ contains $\overleftrightarrow{K}_t$ as a subdivision.

However, the bound on $\mathsf{c}(t)$ is roughly $\mathcal{O}(4^{t^2})$ and thus far from the almost linear bound currently known for Hadwiger's Conjecture. When considering butterfly minors instead of subdivision, the bound can be improved to $t8^t$ [Ste20]. One reason for this large bound can possibly be found in only considering $\overleftrightarrow{K}_t$ itself instead of its canonical anti-chain. We conclude this section by introducing our matching variant of the dichromatic number and stating a version of Hadwiger's Conjecture appropriate for the setting of bipartite graphs with perfect matchings.

Definition 9.1.6 (Matching Chromatic Number)**.** Let G be a matching covered graph and $M \in \mathcal{M}(G)$. An *M-colouring* of G is a function $c\colon M \to \mathbb{N}$, it is a *k-M-colouring* if its image in $\mathbb{N}$ has size at most k. An M-K-colouring of G is *proper* if every M-conformal cycle C in G contains two edges $e_1, e_2 \in E(C) \cap M$ such that $c(e_1) \neq c(e_2)$.

The *M-chromatic number* of G, denoted by $\chi(G, M)$, is the smallest integer k such that G has a k-colouring. The *matching chromatic number* of G is then defined as

$$\chi_{\mathcal{M}}(G) \coloneqq \max_{M \in \mathcal{M}(G)} \chi(G, M).$$

Hence we may apply Theorem 3.3.4 and Theorem 9.1.3 to obtain the following two theorems.

Theorem 9.1.7 ([MSW19])**.** Every bipartite Pfaffian graph B satisfies $\chi_{\mathcal{M}}(B) \leq 2$.

Theorem 9.1.8 ([MSW19])**.** Every planar and bipartite graph B with a perfect matching satisfies $\chi_{\mathcal{M}}(B) \leq 2$.

While Theorem 9.1.7 implies that every bipartite matching covered graph with matching chromatic number at least 3 must contain $K_{3,3}$ as a matching minor, Theorem 9.1.8 can be seen as a matching theoretic analogue to the Four Colour Theorem itself. Hence we conjecture the following:

Conjecture 9.1.9 ([MSW19])**.** Every bipartite and matching covered graph B with $\chi_{\mathcal{M}}(B) \geq t$ contains $K_{t,t}$ as a matching minor.

An immediate question that might arise is, whether Conjecture 9.1.9 implies, or is implied by Hadwiger's Conjecture. Regarding Theorem 8.5.8, and especially Theorem 8.5.1, a connection between the two conjectures, however, appears to be unlikely.

9.2. Beyond Bipartite: Bidirected Graphs

Large parts of the matching minor theory for bipartite graphs with perfect matchings we established in part II are influenced or even directly derived from structural digraph theory. Interestingly, not many cases are known where it was the other way around, that is, where structural matching theory was used to obtain a result on digraphs. While such results definitely exist, see for example Theorem 7.2.7 from [RST99], or Theorem 7.2.13 and Theorem 8.5.3, they are rare and usually occur as a way to circumvent the problem of dealing with infinite anti-chains for butterfly minors. One reason for this phenomenon may be that it is usually easier to use the well established theory of digraphs to think about situations of bipartite graphs with a **fixed** perfect matching. Only where the interactions between several perfect matchings play a major role, the digraphic setting itself can be too restrictive to deal with the issues that occur. As an example for a situation where the interaction between different perfect matchings of the same bipartite graph played a major role recall Lemma 5.3.19.

Indeed, the intuition for the matching setting one can get from digraphs is so strong that even Theorem 7.2.7, while the theoreom itself is proven almost exclusively in terms of perfect matchings in bipartite graphs, was inspired by earlier works on digraphs such as [ST87]. Especially when it comes to width measures, different but equivalent concepts can be more useful than the actual decomposition itself depending on the problem at hand. Notions like brambles, the cops & robber game, and others can be defined in the matching setting as well, but they appear as much more intuitive when used for digraphs. Computationally the directed setting is especially relevant since the only algorithm for computing a small width perfect matching decomposition for a bipartite graph we currently know, apart from braces of perfect matching width two, essentially computes a directed tree decomposition and then transforms this into a cycle decomposition.

Arguably one of the biggest open problems presented throughout this thesis is Conjecture 5.1.3. At the beginning of Chapter 5 we proposed the following strategy towards solving Conjecture 5.1.3.

i) Solve Conjecture 5.1.3 for bipartite graphs, and then
ii) show that there is a function $g\colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, and every graph G with a perfect matching either $\operatorname{pmw}(G) \leq g(k)$, or G contains a conformal and bipartite subgraph H with $\operatorname{pmw}(H) \geq k$.

What is remarkable about this approach and our contribution to its success so far is, that the first part, i.e. the bipartite case, follows almost directly from Theorem 2.3.22. So here the slight simplification provided by the M-direction of bipartite graphs appears to provide the key advantage.

Therefore it might not be too far fetched to consider the more general case of bidirected graphs as a similar tool for handling the case of non-bipartite graphs with perfect matchings since many concepts we mostly used in the bipartite setting so far are preserved by the M-bidirection. Recall that Theorem 5.3.13 holds for bipartite and non-bipartite graphs alike. In light of Theorem 5.3.13 we may restate the approach to the solution of Conjecture 5.1.3 as follows:

i) Theorem 2.3.22 and Theorem 5.3.13 yield a grid theorem for digraphs of large cycle width.
ii) Show that there is a function $g' \colon \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, and every bidirected graph (G, σ) with $\mathrm{cycw}(G, \sigma) > g'(k)$, (G, σ) contains a digraphic subgraph (H, σ) such that $\mathrm{cycw}(H, \sigma) \geq k$.

While this must not be true for Conjecture 5.1.3 to hold, as technically it is a stronger statement to ask every perfect matching of $\mathcal{S}(G, \sigma)$ to witness high perfect matching width the same way, results like Theorem 5.1.13 let it appear at least somewhat plausible. Moreover, results like Theorem 3.3.15 give another strong incentive to at least try to pursue a better understanding of structural bidirected graph theory.

In this section we will probably not take big steps towards the resolution of the problem above, instead we try to give a rough overview about the current state of the, relatively unknown, field of structural bidirected graph theory. We introduce some of the more established notions and compare them with our findings from structural matching theory, to highlight some of the challenges one has to face to make any significant progress in this area, at least when moving towards a resolution of the above problem or the general recognition problem for non-even bidirected graphs is the goal.

9.2.1. Different Notions of 'Strong Connectivity'

In Section 3.2 we already hinted at the more than complex landscape of different notions of connectivity for bidirected graphs. Similar to how digraphs can be seen as a generalised version of undirected graphs, and how this generalisation gives rise to the difference between weak connectivity and strong connectivity, the step from digraphs to bidirected graphs is a similarly strong generalisation. So it can be expected that there are several descriptions of strong connectivity in digraphs, which become pairwise distinct once the step to bidirected graphs is made.

The basis for the different notions of strong connectivity in bidirected graphs lies in the difference between directed walks, trails and paths. In an undirected graph G there is a walk connecting two vertices u and v if and only if they are joined by a trail which is true if and only if they are joined by a path. In a digraph D however, direction must also play a role. Still there exists a directed walk from u to v if and only if there exists a directed trail from u to v, which in turn is true if and only if there is a directed path from u to v. These equivalences are no longer true once we consider bidirected paths in general.

Let us first recall the definition of directed walks, trails, and paths in bidirected graphs.

Definition 9.2.1 (Directed Walks, Trails, and Paths)**.** Let (G, σ) be a bidirected graph, and let

$$J = (v_0, e_1, v_1, e_2, \dots, v_{\ell-1}, e_\ell, v_\ell)$$

be a sequence where $v_i \in V(G)$ for all $i \in [0, \ell]$, and $e_j \in E(G)$ for all $j \in [1, \ell]$. We say that J is *directed* if for all $i \in [2, \ell]$ we have $\sigma(v_i, e_{i-1}) \neq \sigma(v_i, e_i)$. We mostly adapt terminology from the undirected case. So v_0 and v_ℓ are the *endpoints* of J, for $j \in [2, \ell - 1]$ the v_j are the *internal vertices* of J, ℓ is its *length*, and it is called a *walk* if it is a walk in G, a *trail* if it is a trail in G, and a *path* if it is a path in G.

An internal vertex v_i of J whose incident half-edges are of opposite sign is said to be *consistent* in J.

In case J is directed, and $v_\ell = v_{\ell-1}$, J is called a *sling around* v_0 in case J is a cycle in G. If additionally $\sigma(v_\ell, e_{\ell-1}) \neq \sigma(v_\ell, e_0)$, we say that J is a *closed directed walk*, *closed directed trail*, or *directed cycle*, if J is a closed walk, closed trail, or cycle in G respectively.

If J is a walk (trail, path, sling) and $\sigma(v_0, e_1) = \alpha$, $\sigma(v_\ell, e_\ell) = \beta$ we say that J is an (α, β)-walk (trail, path, sling) respectively.

Let $W = (v_0, e_1, v_1, e_2, \dots, v_{\ell-1}, e_\ell, v_\ell)$ be a trail from v_0 to v_ℓ. We denote by W^{-1} the *reversed trail* $(v_\ell, e_\ell, v_{\ell-1}, \dots, e_2, ev_1, e_1, v_0)$. If W is a path and $v_i \in V(W)$ is a vertex of the path we denote by v_iW the subpath of W starting at v_i and ending in v_ℓ, while Wv_i denotes the subpath of W starting in v_0 and ending in v_i. Moreover, if $W' = (v'_0, e'_1, v'_1, e'_2, \dots, v'_{\ell'-1}, e'_{\ell'}, v'_{\ell'})$ such that $v_i = v'_i$ and $v_iW' - v_i$ is disjoint from Wv_i, we denote the path $(v_0, e_1, v_1, e_2, \dots, e_{i-1}, v_i, e'_i, \dots, v'_{\ell'-1}, e'_{\ell'}, v'_{\ell'})$ by Wv_iW'.

As mentioned above, in bidirected graphs we cannot generally expect the existence of a directed trail or path purely because there exists a directed walk or trail from one vertex to another. In Figure 9.1 one can find two small examples to illustrate the phenomenon. Essentially, the requirement of the signs for two consecutive half edges, both incident with the same vertex, to be of opposing signs might stop us from following along a path directly. However, slings around internal vertices of a path that is not directed might allow us to make use of the paths as a walk or trail.

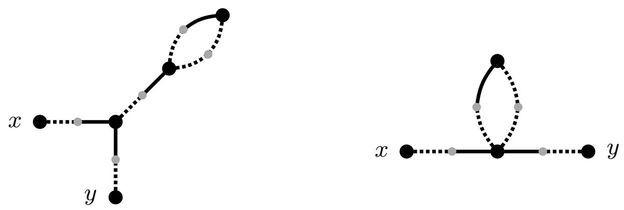


Figure 9.1.: Two bidirected graphs illustrating that two vertices (x and y) can be connected via a directed walk without being connected by a directed trail (on the left), and also two vertices may be connected by a directed trail while not being connected by a directed path (on the right).

In digraphs, strong connectivity can be defined via walks, trails, or paths equivalently. From our observations above, however, it becomes apparent that in general bidirected graphs we might get three completely different notions of connectivity.

Definition 9.2.2 (Strong Connectivity (Walks))**.** Let (G, σ) be a bidirected graph. Two vertices $u, v \in V(G)$ are *strongly connected by a walk* if there exist directed walks W_1 and W_2 in (G, σ) such that W_1 and W_2 have u and v as their endpoints, and, if e_i^x is the edge of W_i incident with $x \in [u, v]$, then $\sigma(e_x^1) \neq \sigma(e_x^2)$.

Theorem 9.2.3 ([AFN96])**.** The binary relation defined by the property of being strongly connected by a walk is an equivalence relation.

The advantages of walks as the basis of strong connectivity is, that any two directed walks that share an endpoint over which they have different signs, can be combined to form a new directed walk. This allows for great flexibility and gives rise to a nice overall structure in which the equivalence classes, or *strong walk components* of a bidirected graph (G, σ) are organised. Indeed, while it is impossible to fully retain the poset structure of strong components in digraphs, it was shown in [AFN96] that the strong walk components of a bidirected graph form a structure called a *signed poset*, see [Rei93] for more information on the topic.

For our purpose, however, walks are not restrictive enough. We are interested in a notion of connectivity that interacts well with cycle width, but there exist strongly walk connected bidirected graphs that do not contain a single directed cycle.

Another way to state that a digraph D is strongly connected is to say that every edge of D belongs to a directed cycle and D is weakly connected. If we exchange 'cycle' with 'closed trail' we obtain the notion of closed trail connectivity.

Definition 9.2.4 (Closed Trail Connectivity)**.** A bidirected graph (G, σ) is *closed trail connected* if G is connected, and for every edge $e \in E(G)$ there exists a closed directed trail T in (G, σ) with $e \in E(T)$.

Definition 9.2.5 (Strong Connectivity (Trails))**.** Let (G, σ) be a bidirected graph and $u, v \in V(G)$. We say that u and v are strongly connected by trails if there exist trails T_1 and T_2 from u to v such that T_1 is a $(+, \alpha)$-trail, and T_2 is a $(-, -\alpha)$-trail for some $\alpha \in \{+, -\}$.

Theorem 9.2.6 ([Kit17])**.** A bidirected graph (G, σ) is closed trail connected if and only if every pair of its vertices is strongly connected by trails. Moreover, the binary relation defined by the property of being strongly connected by a trail is an equivalence relation.

While the components defined by Theorem 9.2.6 in the natural way are no longer necessarily organised in a (signed) poset structure, there are still some interesting and non-trivial observations to be made. Similar to barriers in non-bipartite graphs with perfect matchings, the absence of directed (α, α)-trails between distinct vertices within a single component provides a partition of its vertex set [Kit17]. However, if we wanted to obtain a notion of strong connectivity for bidirected graphs that mirrors matching connectivity in the same way as it does in digraphs, we need to start considering directed paths and cycles.

9.2.2. Strongly Connected Bidirected Graphs

Let (G, σ) be a bidirected graph. We say that (G, σ) is *weakly connected*, if G is connected.

A pair of directed paths P_+ and P_- from u to v is called a *complementary pair* for u and v if P_+ is a $(+, \alpha)$-path, P_- is an $(-, -\alpha)$-path where $\alpha \in \{+, -\}$.

Definition 9.2.7 (Strong Connectivity)**.** A bidirected graph (G, σ) is *strongly connected*, or *strong*, if for every pair of distinct vertices $u, v \in V(G)$ there exists a complementary pair of paths (P_+, P_-) joining u and v such that any pair of vertices $u', v' \in V(P_+) \cup V(P_-)$ is joined by a complementary pair of paths. A *strong component* of (G, σ) is a maximal subgraph (H, σ) such that H is strongly connected.

Please note that this means that for any pair of vertices $u, v \in V(H)$ there must be a complementary pair of paths joining u and v that lies completely within H.

The last part of this definition is essential as there might exist vertices joined by a complementary pair of paths such that there is an internal vertex of (at least) one of the paths that does not have a complementary pair of paths to either of them. See Figure 9.2 for an example. This is undesirable as we would like our path pairs to exist within strong components and not having to rely on vertices outside.

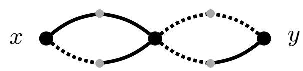


Figure 9.2.: Two vertices x and y joined by a complementary pair of paths while not being strongly connected.

At last we call u and v *truly connected* if for every possible choice of $\alpha, \beta \in \{+, -\}$ there exists a directed (α, β)-path from u to v. If all vertices of (G, σ) are pairwise truly connected, (G, σ) is said to be *truly connected*.

The main goal of this subsection is to establish the following theorem as a bidirected analogue of Theorem 3.1.37.

Theorem 9.2.8 (F*)**.** Let (G, σ) be a bidirected graph. Strong connectivity defines an equivalence relation on $V(G)$.

The overall strategy of the proof is the same as for its matching theoretic analogue. Indeed, once the next result has been established, the correspondence between directed cycles in an M-bidirection of a graph G with a perfect matching, and the M-conformal cycles in G yields Theorem 9.2.8 as a corollary.

Definition 9.2.9 (Circular Connectivity)**.** Let (G, σ) be a bidirected graph. An edge $e \in E(G)$ is called *circular* if there exists a directed cycle C in (G, σ) with $e \in E(C)$. The set of circular edges of (G, σ) is denoted by $\circlearrowright_{G,\sigma}$ and we set $\not\circlearrowright_{G,\sigma} := E(G) \setminus \circlearrowright_{G,\sigma}$. We set $\mathsf{cov}(G, \sigma) := (G - \not\circlearrowright_{G,\sigma}, \sigma)$ and call $\mathsf{cov}(G, \sigma)$ the *cover graph* of (G, σ). The bidirected graph (G, σ) is said to be *circularly connected* if $\mathsf{cov}(G, \sigma)$ is weakly connected, and any component of $\mathsf{cov}(G, \sigma)$ is a *circular component* of (G, σ). We call (G, σ) *totally cyclic* if $\not\circlearrowright_{G,\sigma} = \emptyset$, and *circular* if it is totally cyclic and weakly connected.

A directed (α, β)-path is said to be *circular* if all of its edges are circular. For a path P with endpoints u and v and an internal vertex $w \in V(P)$ we denote the edge of P incident with w that is closest to $x \in \{u, v\}$ by $e^P_{w,x}$. For $w \in \{u, v\}$ we write e^P_w for the unique edge on P incident with w.

Lemma 9.2.10 (F*)**.** A bidirected graph (G, σ) is circularly connected if and only if for every pair $u, v \in V(G)$ and every $\alpha \in \{+, -\}$ there exists some $\beta \in \{+, -\}$ such that there is a circular (α, β)-path from u to v.

Proof. The reverse direction follows immediately from the fact that any circular path in (G, σ) is a path in $\mathsf{cov}(G, \sigma)$, implying that $\mathsf{cov}(G, \sigma)$ must be connected.

For the forward direction we can assume (G, σ) to be circular. Towards a contradiction choose $u, v \in V(G)$ and $\alpha \in \{+, -\}$ such that u and v are in minimum distance to each other in (G, σ) with respect to the absence of any circular (α, β)-path from u to v for all $\beta \in \{+, -\}$. Let P' be any shortest u-v-path in (G, σ) and let $w \in V(P')$ such that $P' - v = P'w$. Hence $vw \in E(G)$. Moreover, by choice of P', we have $\mathrm{dist}((G, \sigma), u, w) < \mathrm{dist}((G, \sigma), u, v)$ and thus there is some $\gamma \in \{+, -\}$ such that a circular (α, γ)-path P from u to w exists. Clearly v cannot be a vertex of P since otherwise we would have a circular path from u to v starting with the sign α. Moreover, $\sigma(w, vw) = -\gamma$ would imply that we could just add the edge vw to P and, since $vw \in\circlearrowright_{G,\sigma}$, we again would have found a circular path from u to v starting on α. Since vw is circular, there must exist a directed cycle C containing vw. Let $x \in V(C) \cap V(P)$ be the vertex of C closest to u along P. Then C contains two circular x-v-paths P_+ and P_-, where the sign of the first edge of P_ε is ε for both $\varepsilon \in \{+, -\}$. Px must be internally disjoint from C. Should $u = x$ then this would mean that we have found our circular path in P_α. Otherwise let $\delta \coloneqq \sigma(x, e_x^{Px})$, then $PxP_{-\delta}$ is again a circular path from u to v starting on the sign α. Hence the pair u, v cannot exist in the first place and we are done. □

Observe that the required change of the sign at every internal vertex of a directed path or cycle simulates us moving through the contracted matching edge of the split. When entering a vertex with sign α in a bidirected graph, that means we enter the corresponding matching edge via its unique endpoint which got assigned the sign α. Since in a directed path we are only allowed to leave via a half edge with the opposite sign, so $-\alpha$, in the split we must necessarily traverse the matching edge itself to reach its other endpoint before we can move on. This observation means that Lemma 9.2.10 is indeed equivalent to Lemma 3.1.34 and thus a bidirected graph is circularly connected if and only if its split is elementary. Hence Theorem 9.2.8 is in fact equivalent to Theorem 3.1.37 which can be seen by translating back and forth between complementary pairs of directed paths in a bidirected graph (G, σ), and the complementary pairs of internally M-conformal paths in its split.

9.2.3. Bidirected Treewidth

As mentioned at the beginning of this section, structural digraph theory plays a huge role in the advancement of bipartite matching minor theory. From an algorithmic point of view, especially directed treewidth contributes a lot to this. So one could expect a similar route for the more general bidirected graphs. In this section we explore one possible, and relatively straight forward way to lift the notion of directed treewidth to the bidirected world.

Johnson et al. explained their main points that make a useful width parameter in [JRST01] based on what they had observed to be the most desirable features of treewidth. They state that the reasons behind the widespread attention treewidth had received can be condensed into the following four points:

i) Treewidth had served as a cornerstone of Graph Minors Theory,
ii) it can be used to prove theorems in structural graph theory,
iii) it has many algorithmic applications due to the fact that many NP-hard problems become tractible on graphs of bounded treewidth, and
iv) it has been successfully used in practical application.

They then introduce their notion of directed treewidth and try to give example results that should establish directed treewidth to be similarly powerful in the directed setting. Their reasoning was build upon the following points:

i) Directed treewidth corresponds to a directed version of the cops & robber game and there exist dual notions for directed treewidth that give insight to the structure of digraphs of large directed treewidth,
ii) directed treewidth can be approximated within a linear factor in polynomial time on digraphs of bounded directed treewidth, and
iii) the t-disjoint paths problem can be solved in polynomial time on digraphs of bounded directed treewidth.

In Part II we have demonstrated that perfect matching width also satisfies these requirements of a 'good' width parameter. Indeed, many, if not all, of the desirable properties of perfect matching width are closely linked to its relation with cycle width for digraphs and the usefulness of directed treewidth.

So far we were not able to find a notion of a 'bidirected treewidth' which corresponds to cycle width in the same way as directed treewidth does. We expect that it is possible to generalise the dynamic programming from Section 5.4 to cycle width, which thereby would give a second interesting algorithmic application to the notion, but still the problem of computing a bounded width decomposition remains unsolved. Hence, instead of doing that, we generalise directed treewidth to a, hopefully, appropriate way and show that there exists a version of the cops & robber game our bidirected treewidth corresponds to. We also show that there exists an

XP-approximation algorithm for a bounded width decomposition. Finally we end this section with a short discussion on why directed treewidth does not appear, at least a priori, to be equivalent to cycle width.

Let us start by introducing our version of the bidirected cops & robber game.

Definition 9.2.11 (Bidirected Cops & Robber Game)**.** Let (G, σ) be a digraph. A *play* of the cops & robber game on (G, σ) is a sequence

$$(C_0, R_0), (C_1, R_1), \dots, (C_{\ell-1}, R_{\ell-1}), (C_\ell, R_\ell)$$

such that $C_i \subseteq V(G)$, R_i is a strong component of $(G - C_i, \sigma)$ for all $i \in [0, \ell]$, and in $(G - (C_{i-1} \cap C_i), \sigma)$ there exists a strong component R with $V(R_{i-1}) \cup V(R_i) \subseteq V(R)$.

Every other definition regarding strategies, winning, monotonicity, inertness, and invisibility can be directly lifted from the definitions for undirected graphs.

The *cop number* of (G, σ), denoted by $\text{cops}(G, \sigma)$, is the smallest integer k such that there is a winning strategy for k cops in the bidirected cops & robber game.

To relate a winning strategy of k cops, or the lack thereof, to any kind of decomposition we need a more structural way to describe the existence of a winning strategy for the robber. Usually this is done by so called havens.

Definition 9.2.12 (Bidirected Haven)**.** Let (G, σ) be a bidirected graph, $k \in \mathbb{N}$, and $h \colon \binom{V(G)}{\leq k} \to 2^{V(G)}$. The function h is called a *bidirected haven of order k* or *bidirected k-haven* if for all $S \in \binom{V(G)}{\leq k}$ we have:

i) $h(S)$ is a strong component of $(G - S, \sigma)$, and
ii) $h(S) \subseteq h(S')$ for all $S' \subseteq S$.

Lemma 9.2.13 (F^*)**.** Let $k \in \mathbb{N}$ be a positive integer and (G, σ) be a strongly connected bidirected graph. Then the robber has a winning strategy for the relaxed cops & robber game on (G, σ) against k cops if and only if there exists a haven of order k for (G, σ).

Proof. Let w be a winning strategy for the robber against k cops in the bidirected cops & robber game. Since the robber wins if she is never caught we may assume that $w(C_1, C_2, R_1) = w(C_3, C_2, R_2)$ for all possible choices of $C_1, C_2, C_3 \in \binom{V(G)}{\leq k}$. This means that the robber only ever cares about the **new** cop position. For every $S \in \binom{V(G)}{\leq k}$ let H_S be the component appropriate for the version of cops & robber we are playing. So H_S is exactly the strong component $w(S)$ of $(G - S, \sigma)$. Note that for every choice of $C_1, C_2 \in \binom{V(G)}{\leq k}$ we must have that $H_{C_2} \subseteq H_{C_1 \cap C_2}$. Hence w already describes a bidirected haven of order k.

So we only need to show that we may assume $w(S)$ to be a strong component of $(G - S, \sigma)$ (a), and that our assumption from above (b) is also possible without loss of generality. Note that (a) holds immediately by definition. For (b) let us define $h(S) := w(\emptyset, S, (G, \sigma))$. We claim that h encodes a winning strategy against k cops in the following sense. If C_1 and C_2 are two consecutive cop positions and we start out on $h(C_1)$, then we first move to $h(C_1 \cap C_2)$, and then to $h(C_2)$. For this assume w to be as close as possible to the strategy obtained from h as described above. To see that we may move from $h(C_1)$ to $h(C_1 \cap C_2)$ suppose after $(C_1, h(C_1))$ the cops announce $C_1 \cap C_2$ as their new position. Then $w(C_1, C_1 \cap C_2, h(C_1)) = w(\emptyset, C_1 \cap C_2, (G, \sigma))$ as otherwise we could change the definition of $w(\emptyset, C_1 \cap C_2, (G, \sigma))$ to meet this equality and therefore contradict our choice of w. Similarly we must have $w(C_1 \cap C_2, C_2, h(C_1 \cap C_2)) = w(\emptyset, C_2, (G, \sigma))$ and our claim follows.

For the reverse define a strategy w for the robber from the bidirected haven h as described above. Note that by definition of bidirected havens this means that the robber can answer every cop-position for at most k cops with a new component that is reachable from her current one and thus she can never be caught. $\square$

We also need to describe a certain kind of linkedness of vertex sets in (G, σ). Since there is probably no bidirected version of Menger's Theorem we aim to describe this notion of linkedness in the sense that the vertices of our highly linked set cannot be spread somewhat evenly over many different strong components by deleting only a few number of vertices.

Definition 9.2.14 (k-Bilinked Set)**.** Let $k \in \mathbb{N}$ be a positive integer and (G, σ) be a bidirected graph. A set $X \subseteq V(G)$ is called *k-bilinked* if for every $S \in \binom{V(G)}{\leq k}$ there exists a strong component H of $(G - S, \sigma)$ such that $|V(H) \cap X| > \frac{|X|}{2}$.

Lemma 9.2.15 (F*)**.** Let $k \in \mathbb{N}$ be a positive integer, (G, σ) be a strongly connected bidirected graph, and $X \subseteq V(G)$. If X is a k-bilinked set, then (G, σ) has a haven of order k.

Proof. For each $S \in \binom{V(G)}{\leq k}$ let us denote by H_S the strong component of $(G - S, \sigma)$ that contains more than half of X. Note that for every $S' \subseteq S$ we must have $H_S \subseteq H_{S'}$ since strong components behave in a monotone way under taking subgraphs. That is, every strong component of $(G - S, \sigma)$ must be contained in a strong component of $(G - S', \sigma)$. Now let us define $h(S) := H_S$ for every $S \in \binom{V(G)}{\leq k}$. By our observation above it is straight forward to see that h is indeed a bidirected haven of order k. $\square$

We are ready to introduce our notion of bidirected treewidth. Together with the base definitions we define 'nice' versions of the corresponding decomposition that essentially represent robber monotone winning strategies for the cops. We show that the existence of a bounded width decomposition implies the existence of a winning strategy for the cops in the corresponding variant of the cops & robber game. Moreover, we show that the absence of a haven implies the existence of a nice decomposition of bounded width. For this we adapt the proof of Johnson et al. for the directed setting in [JRST01]. This proof also yields an XP-algorithm to compute this nice decomposition and thus allows us to approximate a small width decomposition in polynomial time for bidirected graphs of bounded bidirected treewidth.

Definition 9.2.16 (Biguard)**.** Let (G, σ) be a bidirected graph and $X, Y \subseteq V(G)$. The set Y is said to be a *biguard* of X if every directed cycle that has an edge in $\partial_G(X)$ contains a vertex of Y.

Definition 9.2.17 (Bidirected Treewidth)**.** Let (G, σ) be a bidirected graph. A *bidirected tree decomposition* for (G, σ) is a tuple (T, β, γ) where T is an arborescence, $\beta \colon V(T) \to 2^{V(G)}$ is a function such that $\{\beta(t) \mid t \in V(T)\}$ is a near partition[1] of $V(G)$, and $\gamma \colon E(T) \to 2^{V(G)}$ is another function satisfying the following requirement:

> For every $(d, t) \in E(T)$, $\gamma(d, t)$ is a biguard of $\beta(T_t) := \bigcup_{t' \in V(T_t)} \beta(t')$.

[1] Recall that this means we allow $\beta(t)$ to be empty.

Where T_t denotes the subarborescence of T with root t. For every $t \in V(T)$ we call $\beta(t)$ the *bag* of t and for every $e \in E(T)$, $\gamma(e)$ is the *guard* at e.
For every $t \in V(T)$ let $\Gamma(t) \coloneqq \beta(t) \cup \bigcup_{t \sim e} \gamma(e)$. The *width* of (T, β, γ) is defined as

$$\mathrm{width}(T, \beta, \gamma) \coloneqq \max_{t \in V(T)} |\Gamma(t)| - 1.$$

The *bidirected treewidth* of (G, σ), denoted by $\mathrm{btw}(G, \sigma)$ is the minimum width over all bidirected tree decompositions for (G, σ).

Note that, since for bidirected treewidth we are only interested in strong connectivity and edges that are contained in directed cycles, it follows immediately that the bidirected treewidth of a bidirected graph (G, σ) equals the maximum bidirected treewidth of its strong components. Moreover, if (G, σ) is strongly connected, then $\mathrm{btw}(G, \sigma) = \mathrm{btw}(\mathsf{cov}(G, \sigma))$. Therefore, from now on it suffices to consider circular bidirected graphs.

Let (G, σ) be a circular bidirected graph. A relaxed bidirected tree decomposition (T, β, γ) for D is *nice* if for every edge (d, t) the set $\beta(T_t)$ is strongly connected in $(G - \gamma(d, t), \sigma)$ and $\beta(T_t) \cap \gamma(d, t)$.

Lemma 9.2.18 (F*)**.** Let $k \in \mathbb{N}$ be an integer, (G, σ) be a bidirected graph, and (T, β, γ) be a bidirected tree decomposition of width k for (G, σ). Then there exists a winning strategy for $k+1$ cops in the (relaxed) cops & robber[2] game on (G, σ).

Proof. Let $r \in V(T)$ be the root of T and let $d_1, \dots, d_\ell$ be its children. Then we set $C_1 \coloneqq \Gamma(r)$ to be the first cop position. Note that for every strong component H of $(G - \Gamma(r), \sigma)$ there exists a unique $i \in [1, \ell]$ such that $V(H) \subseteq \beta(T_{d_i})$ since $\gamma(r, d_j)$ meets all directed cycle that cross over $\partial_G(\beta(T_{d_j}))$ for every $j \in [1, \ell]$.

Therefore there exists a unique integer $i \in [1, \ell]$ such that the robber position R_1 is contained in $\beta(T_{d_i})$, and the robber cannot leave $\beta(T_{d_i})$.

Now suppose there exists $t \in V(T)$ with children $t_1, \dots, t_q$ such that in round h the cop position is $\Gamma(t)$ and the robber position R_h is contained in t_j for some $j \in [1, q]$. Then $\gamma(t, t_j) \subseteq \Gamma(t) \cap \Gamma(t_j)$. Let us set $C_{h+1} \coloneqq \Gamma(t_j)$. Then the robber cannot leave $\beta(T_{t_j})$ and must choose her next position R_{h+1} such that there is some child d' of t_j with $R_{h+1} \subseteq \beta(T'_d)$. Since G

[2] The non-relaxed version of the game is its strong version.

and T are finite, eventually the vertex d' chosen by the robber is a leaf of T. At this point, the cops will capture her with their next move and thus, as by $\mathrm{width}(T, \beta, \gamma) = k$ we never used more than $k+1$ cops, we have a winning strategy for $k+1$ cops. $\square$

As an immediate consequence of Lemma 9.2.18 we obtain the following.

Corollary 9.2.19 (F*)**.** Let $k \in \mathbb{N}$ be a positive integer and (G, σ) be a bidirected graph. If (G, σ) contains a k-bilinked set or a bidirected haven of order k, then its bidirected treewidth is at least k.

Proof. By Lemma 9.2.15 the existence of a k-bilinked set implies the existence of a bidirected haven of order k. So we may assume that there is such a haven. Then Lemma 9.2.13 guarantees us a winning strategy for the robber against k cops. Thus the existence of a bidirected tree decomposition for (G, σ) of width at most $k-1$ would contradict Lemma 9.2.18. $\square$

With this we are ready to prove that the absence of a high order bidirected haven implies bounded bidirected treewidth.

Theorem 9.2.20 (F*)**.** Let (G, σ) be a circular bidirected graph and $k \in \mathbb{N}$ be a positive integer. Then either (G, σ) has a nice bidirected tree decomposition of width at most $3k+2$, or it contains a bidirected haven of order k.

Proof. We prove this claim by iteratively constructing a bidirected tree decomposition where we maintain several invariants for all non-leaf vertices of this decomposition. The claim is then proven by induction over the total number of vertices contained in leaf-bags that do not meet the requirements of our invariant.

Let us choose a bidirected tree decomposition (T, β, γ) for (G, σ) satisfying the following three conditions:

i) $|\Gamma(t)| \leq 3k+3$ for all $t \in V(T)$ that are not leaves of T,
ii) $|\gamma(e)| \leq 2k+1$ for every $e \in E(T)$, and
iii) (T, β, γ) is nice.

Such a decomposition does exist since we may select a trivial arborescence T with a single vertex r and set $\beta(r) \coloneqq V(G)$. Treat this initial step as the start of our induction.

For this initial step note that, in case $V(G)$ is k-bilinked we would have a bidirected haven of order k by Lemma 9.2.15 and thus we would be done. Hence we may assume that there exists a set $S \in \binom{V(G)}{k}$ such that no strong component of $(G - S, \sigma)$ contains more than half of the vertices in G. For every strong component K of $(G - S, \sigma)$ introduce a vertex d_K and the edge (r, d_K). Now set $\beta(r) \coloneqq S$, $\beta(d_K) \coloneqq V(K)$, and $\gamma(r, d_K) \coloneqq S$. Note that the result is indeed a nice bidirected tree decomposition with the desired properties.

Now let (T, β, γ) be a bidirected tree decomposition for (G, σ) satisfying the three conditions such that T has at least two vertices. In case we have $|\Gamma(t)| \leq 3k + 2$ for all vertices $t \in V(T)$ we are done. So we may assume that there exists a leaf $t \in V(T)$ with incoming edge e such that $|\Gamma(t)| > 3k + 2$, but still $|\gamma(e)| \leq 2k + 1$.

If $\gamma(e)$ is k-bilinked, then, by Lemma 9.2.15, we have a haven of order k and are done. Hence we can find a set $S' \in \binom{V(G)}{\leq k}$ such that every strong component of $(G - S', \sigma)$ contains at most k vertices of $\gamma(e)$.

Let u be some vertex of $\beta(T_t)$ and let $S \coloneqq S' \cup \{u\}$, then $|S| \leq k + 1$. Furthermore, let us set $\beta(t) \coloneqq S \cap \beta(T_t)$.

Now iterate over all strong components K of $G - S, \sigma$ that contain vertices of $\beta(T_t)$. Let K be such a component and let $F_K \coloneqq V(K) \cap \gamma(e)$. We claim that every strong component $K' \subseteq K$ of $(G - S - F_K, \sigma)$ is either completely contained in $\beta(T_t)$ or disjoint from it. To see this note that every directed cycle of $(G - S, \sigma)$ with vertices in K that contains edges of $\partial_G(\beta(T_t))$ must be met by a vertex of $\gamma(e)$, and, moreover, each such cycle must be completely contained in K by Lemma 9.2.10.

So for every strong component K of $(G - S, \sigma)$, and every strong component K' of $(G - S - F_K, \sigma)$ such that $K' \subseteq K \cap \beta(T_t)$ introduce a new child $d_{K'}$ of t together with the edge $(t, d_{K'})$. We set $\beta(d_{K'}) \coloneqq V(K')$, and $\gamma(e) \coloneqq S \cup F_K$. Since $|F_K| \leq \left\lfloor \frac{1}{2} |\gamma(e)| \right\rfloor \leq k$ we have $|S \cup F_K| \leq 2k + 1$. Hence the result is again a nice bidirected tree decomposition meeting all of our requirements. Moreover, as u does not belong to any K' constructed as above, the total number of vertices contained in leaf-bags that do not meet the requirements of our invariant has been reduced by at least one. Therefore our inductive step is complete and the claim follows. □

Thus bidirected treewidth appears to be a meaningful parameter for bidirected graphs that, as one can see from the cops & robber game, strictly generalises directed treewidth, while being essentially the same parameter on digraphs. In the following and last subsection we discuss, why bidirected treewidth might possibly not be equivalent to cycle width.

9.2.4. Barriers, Strong Components, and Cycle Porosity

To fully explain the problems one faces when confronted with the more complicated structure of strong components and separators in bidirected graphs as opposed to digraphs, we need to find adequate translations of certain concepts from structural matching theory to the bigraphic world. A major difference between digraphs and general bidirected graphs is the following: Let D be a strongly connected digraph and let P be a directed path with distinct endpoints, both of which belong to D, but no internal vertex of P is a vertex of D. Then $D + P$ must necessarily be strongly connected, since D must contain a directed path Q from the head of P to its tail and thus $P + Q$ is a directed cycle. In a strongly connected bidirected graph (G, σ), we also know that for every vertex $u \in V(G)$, every $\alpha \in \{+, -\}$, and every vertex $v \in V(G - u)$ there exists a directed (α, β)-path from u to v for some $\beta \in \{+, -\}$. However, since (G, σ) is not necessarily digraphic, β might not always be the sign we desire in any given situation. In fact, if we were to add a directed $(-\alpha, \beta)$-path P from u to v to (G, σ) such that no internal vertex of P belongs to G, then it is entirely possible that there does not exist a directed $(\alpha, -\beta)$-path from u to v and thus $(G + P, \sigma)$ is not necessarily strongly connected.

The Inner Structure of a Bidirected Graph The aim of this paragraph is to establish a decomposition of the vertex set of a strong bidirected graph into something we call *barriers*, resembling the barriers from matching theory. The matching theoretic counterpart of these barriers can be utilised to prove a bidirected analogue of the frame construction of Theorem 3.1.27 which shows that any bidirected graph can be build from strongly connected digraphic bidirected graphs and bidirected graphs that are truly connected.

Let (G,σ) be a bidirected graph, $u,v \in V(G)$ and $\alpha \in \{+,-\}$, we define the following binary relation. Let us write $u \overset{\alpha}{\leftrightarrow} v$ if and only if $u = v$, or u and v belong to the same strong component of (G,σ) and there is no (α,α)-path from u to v in (G,σ).

The following result is similar to Theorem 5.4 from [Kit17] and provides a key piece for our theory. The difference between our result and the one of Kita lies in the definition of strong connectivity the two results are based on. While strong connectivity divides the vertices of a bidirected graph into its strong components, every strong component itself has a more refined internal structure which, in some sense, can be seen as classes of lesser connectivity.

Proposition 9.2.21 (F*)**.** Let (G,σ) be a bidirected graph and $\alpha \in \{+,-\}$, then $\overset{\alpha}{\leftrightarrow}$ is an equivalence over $V(G)$.

Proof. Reflexivity and symmetry are immediate from the definition. Let $u,v,w \in V(G)$ be three distinct vertices such that $u \overset{\alpha}{\leftrightarrow} v$ and $v \overset{\alpha}{\leftrightarrow} w$ and suppose there exists a directed (α,α)-path P from u to w. By definition of $\overset{\alpha}{\leftrightarrow}$ all three vertices, u, v, and w, belong to the same strong component of (G,σ). Thus there exists a directed $(\alpha,-\alpha)$-path P' from v to u. If P and P' only meet in u, $P'uP$ is a directed (α,α) path from v to w which is impossible. Next suppose $w \in V(P')$, in this case let $\beta \in \{+,-\}$ such that $P'w$ is a directed (α,β)-path. By assumption $\beta \neq \alpha$ and thus $\beta = -\alpha$. Hence we can always find $x \in \{u,w\}$ such that P' is (or contains) a directed $(\alpha,-\alpha)$-path P'' from v to x with $V(P'') \cap (\{u,v\} \setminus \{x\}) = \emptyset$ and $V(P) \cap V(P'') \neq \emptyset$. Without loss of generality let us assume $x = u$. With the same arguments as before we immediately reach a contradiction if P'' is internally vertex disjoint from P. Let $y \in V(P'') \cap V(P)$ be the first vertex of P we meet when traversing along P'' starting on v. Note that $y = v$ is explicitly allowed. Moreover, let e_u be the edge on P incident with y that is closer to u on P and let e_w be the other edge of P incident with y. Then $\sigma(y,e_u) \neq \sigma(y,e_w)$ and thus there exists $z \in \{u,w\}$ such that $\alpha = \sigma(e_z)$. Now let P^z be the subpath of P with endpoints y and z, then $P''yP^z$ is a directed (α,α)-path from v to z. Such a path cannot exist and thus P itself cannot exist, completing our proof. ☐

For $\alpha \in \{+,-\}$ we denote the set of equivalence classes of $\overset{\alpha}{\leftrightarrow}$ in (G,σ) by $\mathcal{K}^\alpha(G,\sigma)$. We establish some basic results on the interaction of $\mathcal{K}^+(G,\sigma)$ and $\mathcal{K}^-(G,\sigma)$.

Lemma 9.2.22 (F*)**.** Let (G,σ) be a circular bidirected graph, $\alpha \in \{+,-\}$, $A \in \mathcal{K}^\alpha(G,\sigma)$, and $u \in V(G) \setminus A$. If there is some $v \in A$ such that there is no directed $(\alpha,-\alpha)$-path from v to u, then there exists no directed $(\alpha,-\alpha)$-path from A to u.

Proof. Suppose towards a contradiction there exist $v, w \in A$ such that there is no directed $(\alpha,-\alpha)$-path from v to u, but there exists a directed $(\alpha,-\alpha)$-path P from w to u. With (G,σ) being strong and $u \notin A$ there exists a directed (α,α)-path P' from v to u. If P and P' are disjoint except for u, $P'P$ is a directed (α,α)-path connecting v and w, contradicting $v, w \in A$. So let y be the first vertex of P that appears on P' when traversed from v to u. If $y = w$, then the last edge of $P'y$ must have the sign $-\alpha$ over y and thus $P'yP$ is a directed $(\alpha,-\alpha)$-path from v to u which cannot exist. Let e_v be the edge of $P'y$ incident with y, e_w the edge of P incident with y closest to w on P, and e_u the other edge of P incident with y. Then $\sigma(y,e_w) \neq \sigma(y,e_u)$ and, since there cannot be a directed (α,α)-path from v to w, $\sigma(y,e_w) = \sigma(y,e_v)$. Hence $P'yP$ is a directed $(\alpha,-\alpha)$-path from v to u which is a contradiction. □

For any bidirected graph (G,σ), $\alpha \in \{+,-\}$ and $A \in \mathcal{K}^\alpha(G,\sigma)$, let $\mathrm{B}^\alpha(A)$ be the set of all vertices $u \in V(G)$ such that u and A belong to the same strong component of (G,σ) and there exists no directed $(\alpha,-\alpha)$-path from A to u. The next observation follows immediately from the definition.

Observation 9.2.23 (F*)**.** Let (G,σ) be a bidirected graph, $\alpha \in \{+,-\}$, and $A \in \mathcal{K}^\alpha(G,\sigma)$, then $A \cap \mathrm{B}^\alpha(A) = \emptyset$.

Lemma 9.2.24 (F*)**.** Let (G,σ) be a bidirected graph, $\alpha \in \{+,-\}$, and $A \in \mathcal{K}^\alpha(G,\sigma)$ such that $\mathrm{B}^\alpha(A) \neq \emptyset$, then $\mathrm{B}^\alpha(A) \in \mathcal{K}^{-\alpha}(G,\sigma)$.

Proof. This proof consists of two parts. First we prove that there is no directed $(-\alpha,-\alpha)$-path between any two vertices of $\mathrm{B}^\alpha(A)$ and second we show that for every vertex $u \in \mathrm{B}^\alpha(A)$ and every $v \in V(D) \setminus \mathrm{B}^\alpha(A)$ there exists a directed $(-\alpha,-\alpha)$-path connecting u and v.

Let $u, w \in \mathrm{B}^\alpha(A)$ and suppose there exists a directed $(-\alpha, -\alpha)$-path P from u to w. Moreover let $v \in A$, then there exists a directed (α, α)-path P' from v to w. If P and P' are disjoint except for w, $P'P$ is a directed $(\alpha, -\alpha)$-path from v to u which cannot exist. Now let y be the first vertex of P that appears on P' when traversed from v to w. If $y = u$ then the sign of the last edge of $P'y$ over y must be α and thus $P'yP$ is a directed $(\alpha, -\alpha)$-path from v to w, contradicting the definition of $\mathrm{B}^\alpha(A)$. Let e_v be the edge of $P'y$ incident with y, e_w the edge of P incident with y closest to w on P, and e_u the other edge of P incident with y. Then $\sigma(y, e_w) \neq \sigma(y, e_u)$ and thus there must exist $x \in \{u, w\}$ such that $\sigma(y, e_v) \neq \sigma(y, e_x)$. Let P^x be the subpath of P from y to x, then $P'yP^x$ is a directed $(\alpha, -\alpha)$-path from v to x. Since the existence of such a path is a contradiction we are done with the first part.

Now let $u \in \mathrm{B}^\alpha(A)$, $v \in A$, and w be a vertex in the same strong component of (G, σ) such that $w \notin A \cup \mathrm{B}^\alpha(A)$. Suppose there exists no directed $(-\alpha, -\alpha)$-path connecting u and w. Since $w \notin \mathrm{B}^\alpha(A)$ we may choose v such that there exists a directed $(\alpha, -\alpha)$-path P from v to w. Moreover, with u and w being strongly connected, but not by a directed $(-\alpha, -\alpha)$-path, there must exist a directed $(-\alpha, \alpha)$-path P' from u to w. If P and P' are disjoint except for w, PwP' is a directed $(\alpha, -\alpha)$-path from v to u and thus cannot exist. Now let y be the first vertex of P that appears on P' when traversed from u to w. If $y = v$, then the sign of the last edge of P' over y must be $-\alpha$ and so $P'yP$ is a directed $(-\alpha, -\alpha)$-path connecting u and w contradicting our assumption. Let e_u be the edge of $P'y$ incident with y, e_v the edge of P incident with y that is closest to v on P and e_w the other edge of P incident with y. Then $\sigma(y, e_v) \neq \sigma(y, e_w)$ and either possibility for $\sigma(y, e_u)$ yields a contradiction. □

Corollary 9.2.25 (F*)**.** Let (G, σ) be a bidirected graph, $\alpha \in \{+, -\}$, and $A \in \mathcal{K}^\alpha(G, \sigma)$ such that $\mathrm{B}^\alpha(A) \neq \emptyset$, then $\mathrm{B}^{-\alpha}(\mathrm{B}^\alpha(A)) = A$.

Let S be any set and $\mathcal{X} := \{X_1, \dots, X_\ell\}$ a family of subsets of S. We call $\mathcal{X}$ a *near partition* of S if its members are pairwise disjoint and $S = \bigcup_{X \in \mathcal{X}} X$. Please note that this allows $\emptyset \in \mathcal{X}$ in contrast to the definition of partition.

A *signed partition* of a set U is a family $\mathcal{S}$ of tuples (X, Y) consisting of subsets $X, Y \subseteq U$ such that

i) $X \cap Y = \emptyset$ for all $(X, Y) \in \mathcal{S}$, and
ii) $\{X \mid (X, Y) \in \mathcal{S}\}$ and $\{Y \mid (X, Y) \in \mathcal{S}\}$ both form near partitions of U.

In particular this means that for every element $u \in U$ there exist unique and distinct tuples $(X^+, Y^+), (X^-, Y^-) \in \mathcal{S}$ such that $u \in X^+ \cap Y^-$.

So $\overset{+}{\leftrightarrow}$ and $\overset{-}{\leftrightarrow}$ together induce a signed partition on the vertex set of (G, σ). This particular signed partition is called the *Kita-decomposition* $\mathcal{K}(G, \alpha)$.

$$\mathcal{K}(G, \sigma) := \left\{(A, \mathrm{B}^+(A)) \mid A \in \mathcal{K}^+(G, \sigma)\right\} \\ \cup \left\{(\emptyset, B) \mid B \in \mathcal{K}^-(G, \sigma) \text{ and } \mathrm{B}^-(B) = \emptyset\right\}$$

We call an element of $\mathcal{K}(G, \sigma)$ a *barrier*.

Strong digraphic bidirected graphs can be characterised by the fact that their barriers take a very particular form.

Observation 9.2.26 (F*). A circular bidirected graph (G, σ) is digraphic if and only if $|\mathcal{K}(G, \sigma)| = 2$.

Bidirected Treewidth and Cycle Width Let D be a digraph, $S \subseteq V(G)$ a set of vertices, and K be a strong component of $D - S$. Then, by the discussion at the start of the subsection and our findings in the proof of Proposition 5.3.17 it follows that $\mathrm{cp}(\partial_D(V(K))) \leq 2|S|$.

Now let (G, σ) be a bidirected graph, $S \subseteq V(G)$ a set of vertices, and K be a strong component of $(G - S, \sigma)$. If there exists a directed path P of length at least three in $(G - S, \sigma)$ which is internally disjoint from K, but has both endpoints in K, then both endpoints must belong to the same barrier of K since otherwise there would exist a directed path within K which could close a directed cycle, thereby contradicting K being a strong component. So far we could not find a reason why the number of pairwise vertex disjoint such 'ear-paths' should be bounded and thus it is far from obvious whether there exists a relation between $|S|$ and $\mathrm{cp}(\partial_G(V(K)))$.

What can be said however is, that if there exists a function $g' \colon \mathbb{N} \to \mathbb{N}$ such that every bidirected graph (G, σ) with $\mathrm{cycw}(G, \sigma) \geq g'(k)$ contains a digraphic subgraph of cyclewidth at least k, then this subgraph must also have bidirected treewidth dependent on k. Therefore, it would follow, that the bidirected treewidth of a bidirected graph is bounded from below by a function of its cycle width.

Suppose there would also exist a generalisation of Theorem 5.3.2 for non-bipartite graphs, then one could use a strategy similar to the one for Corollary 5.3.25 to show that the cop number of a bidirected graph is bounded in a function of its cycle width. By Theorem 9.2.20 it would follow that bidirected treewidth is also bounded from above in a function of cycle width.

However, it is not clear whether a generalisation of Theorem 5.3.2 can exist, since the current proof relies heavily on properties specific to bipartite graphs. Either way, it appears that our knowledge of bidirected graphs in general is by far not deep enough to allow for an immediately successful approach towards the resolution of Conjecture 5.1.3. Nevertheless, it might be desirable to have a more thorough investigation of the structural properties of bidirected graphs, as they might still yield some additional insight to non-bipartite graphs with perfect matchings, and they also appear to be of independent interest.

9.3. Regarding Our List of Open Questions

In Section 3.5 we asked five questions as the main motivation for the research presented in this thesis. To conclude the last chapter, we revisit these five questions and gather our findings, summarising if and what progress has been made.

Question 3.5.1: The Pfaffian Recognition Problem The bipartite version of the Pfaffian recognition problem has been solved by McCuaig et al. in form of their structural characterisation of Pfaffian braces in Theorem 3.3.4. This solution, especially the use of four-cycle-sums as a matching version of clique sums appears to be an integral part of bipartite matching minor theory over all. As evidence for the importance of Theorem 3.3.4 recall our solution to the Two Paths Problem for bipartite matching covered graphs in Chapter 4, all key results such as the characterisation of the existence of a matching cross over a conformal cycle in Theorem 4.0.4, the fact that every four-cycle in a non-Pfaffian brace is contained in a conformal bisubdivision of $K_{3,3}$ as found in Proposition 4.0.8, and finally the algorithmic solution to the 2-MLP in Theorem 4.0.6 draw their power from Theorem 3.3.4. Indeed, one could think of all of these results as natural extensions of

Theorem 3.3.4, especially as there appears to be an important difference in quality between matching crosses over conformal cycles in Pfaffian braces, and those matching crosses in non-Pfaffian braces we are guaranteed by Lemma 4.0.10. Especially Lemma 4.0.9, which characterises the existence of conformal crosses, i.e. the kind of crosses which can actually be used for some sort of routing via internally M-conformal paths, underlines the significance of Pfaffian graphs and their structural properties. This significance is further elevated in Theorem 8.0.4 and Theorem 8.0.5 as it appears, that even excluding $K_{t,t}$ for $t \geq 4$ as a matching minor is, at least locally, only a matter of adding a bounded size apex set to a Pfaffian graph.

The fact that all of these concepts and results interact so tightly with the structure of bipartite Pfaffian braces and the solution to certain routing problems casts some shadow of doubt over the general Pfaffian Recognition Problem. While in bipartite graphs 2-MLP is a problem in P, in non-bipartite graphs even deciding whether there exist two internally disjoint alternating paths between two fixed vertices, a problem that is handled by Menger's Theorem in the bipartite case, has been observed to be computationally hard in Corollary 3.4.6, at least when being fixed to a single perfect matching.

In the bipartite case we have seen that there is a huge gap between the complexity of 2-MLP and the same question for a single perfect matching. So it might still be true that the problem, whether there **is** a perfect matching such that we can find the two disjoint paths in the non-bipartite setting is polynomial time solvable, proving this however might be extremely tricky and it seems to be unlikely that the tools known so far are enough to achieve a solution of the problem.

In Section 9.2 we made another attempt, following the route suggested by Norine, and took upon investigating the structural properties of general graphs with perfect matchings in a broader sense. However, first of all it appears to be unclear whether the setting of bidirected graphs yields enough insight to the problems to be of any use, and second, we might encounter a similar problem as pointed out above: what if 'there exists a perfect matching such that...', and 'for all perfect matchings it holds that...' lie further apart from each other than in the bipartite setting?

Over all, our understanding of the bipartite case of the Pfaffian Recognition Problem appears to have increased, while for the general version only new question marks seem to have appeared on the roadmap.

Question 3.5.2: The (Bipartite) Matching Minor Recognition Problem
At the beginning of the research towards this thesis only one algorithmic result regarding the testing for matching minor containment was known, namely Theorem 3.3.4 which implies Corollary 3.3.10. It can therefore be seen as a major leap forward that testing of matching minor containment for any fixed bipartite matching covered graph H in bipartite graphs with perfect matchings of bounded perfect matching width turns out to be polynomial time solvable as of Theorem 5.4.2. Indeed, by using our findings from Chapter 7 we were able to push our algorithm further and, by Corollary 7.1.2, add the whole class of planar, bipartite, and matching covered graphs to the list of graphs for which testing for matching minor containment in bipartite graphs with perfect matchings is possible in polynomial time.

For (ordinary) minors, the big break through that eventually lead to a polynomial time algorithm to test for the containment of a fixed minor H in Theorem 2.2.36 was the the resolution of the t-Disjoint Paths Problem for undirected graphs. To be more precise, Robertson and Seymour proved that there exists an algorithm whose exponential part in the running time solely depends on t and not on the size of the input graph. As explained in Section 2.2.3, essential for this algorithm was the discovery of the Flat Wall Theorem. In light of Theorem 8.0.4, a similar result for bipartite graphs with perfect matchings does not seem to be that far out of reach.

This brings us to the next question:

Question 3.5.4: The Bipartite t-Disjoint Alternating Paths Problem
The minor-testing algorithm of Robertson and Seymour, as well as Theorem 5.4.2 illustrate that testing for minors or matching minors can be seen as a special case of the version of the disjoint paths problem appropriate for the respective setting. Routing problems on their own are also among the more prominent applications of graph theory.

As mentioned in the beginning of Chapter 4, the generally hard Directed t-Disjoint Paths Problem can be seen as a specialised version of the

Bipartite t-Disjoint Alternating Paths Problem. A solution to t-MLP might therefore also yield a way to tackle a relaxed version of the Directed t-Disjoint Paths Problem. In light of results like the NP-hardness of the Directed 2-Disjoint Paths Problem as, seen in Theorem 2.3.27, it is therefore also mildly surprising that a theorem like Theorem 4.0.6 is possible. This result especially also raises the hope to find a general solution of the Bipartite t-DAPP, possibly similar to the solution for the t-DPP as found in the Matching Minor Project. For this, three different building blocks are necessary:

i) Solve the bipartite t-DAPP on bipartite graphs with perfect matchings of bounded perfect matching width,
ii) show that there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that the Bipartite t-DAPP becomes trivial if the input graph B contains $K_{f(t),f(t)}$ as a matching minor, and lastly
iii) use Theorem 8.0.4 to either solve the problem on bipartite graphs with large perfect matching width, or find a way to reduce the total number of vertices in such a graph, while preserving the possible existence of a solution to the given instance.

With Theorem 5.4.1, part (i) has already been taken care of. Towards part (ii), the importance of Lemma 4.0.9 for the solution of the 2-MLP might be an appropriate starting point for further research into this topic. At last, the mere existence of Theorem 8.0.4 gives a lot of hope towards a possible algorithm for the general problem. The main challenge here would be to ensure that removing a small number of vertices from within a flat matching wall does not change the overall matching structure of the given graph too much, as the flexibility of changing perfect matchings appears to make the difference between Theorem 2.3.27 and Theorem 4.0.6. A good starting point for the general problem could be to consider only planar bipartite and matching covered graphs. There already exists an algorithm [Sch94, CMPP13] for the Directed t-DPP on planar digraphs, this algorithm however appears to not be straightforward adaptable as it only considers one fixed perfect matching, and the total number of perfect matchings in a graph can be exponential. Hence even for the planar case, new techniques are required. Overall, however, a positive solution for the Biparite t-DAPP seems to be far more in reach than a solution to the general Pfaffian Recognition Problem.

Question 3.5.5: The Structure of Bipartite Matching Covered Graphs Excluding $K_{t,t}$ This question seems to have been answered in the most general way possible by Theorem 8.0.4 and Theorem 8.0.5. Of course the bounds found for the theorem in Chapter 8 are far from optimal or even desirable but this might be due to our heavy reliance on structural digraph theory. It is highly possible that we can find better bounds, or slightly different but still similarly strong theorems by putting more emphasis on the structure of the bipartite matching covered graph itself. Two other theorems that might require some optimisation are Theorem 5.3.29 which, again, makes heavy use of its digraphic counterpart, and Theorem 7.1.1 which also has an exponential bound by the way our proof works.

Apart from these optimisation problems, we can now replace Question 3.5.5 with another, more refined, question as follows:

Question 9.3.1. Let H be a brace. What is the structure of bipartite matching covered graphs that exclude H as a matching minor?

For $K_{3,3}$ this question was answered by Theorem 3.3.4. Moreover, in case H is planar, Theorem 7.0.1 provides a nice answer. But beyond this, Theorem 8.0.5 can only provide an incomplete picture and more research is necessary.

Question 3.5.6: Matching Minor Anti-Chains of Braces Our last question, which likely is also the deepest one, is also the one we have found the least answers to. While the findings of [RS86b] enabled Robertson and Seymour to show that any anti-chain for the graph minor relation that contains a planar graph must be finite. With Theorem 7.0.1 at hand, our hopes were high to replicate this success with regards to Question 3.5.6 and planar braces. However, we quickly discovered that the established methods for handling this kind of problems seemed not fit, or at least not strong enough, to handle the case of bipartite matching covered graphs. One reason for this might be found in the tight cut contraction, which replaces the otherwise relatively easy to handle cut vertices. Another problem might be the existence of many edges connecting the vertices of one colour class in a set of vertices to another set of vertices without increasing the 'connectivity' in a meaningful way. Moving forward, finding answers to this question will probably be the greatest challenge apart from generalisations of the questions above to the non-bipartite setting.

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Matching minors in bipartite graphs

In this thesis we adapt fundamental parts of the Graph Minors series of Robertson and Seymour for the study of matching minors and investigate a connection to the study of directed graphs. We develope matching theoretic to established results of graph minor theory:
We characterise the existence of a cross over a conformal cycle by means of a topological property. Furthermore, we develope a theory for perfect matching width, a width parameter for graphs with perfect matchings introduced by Norin. here we show that the disjoint alternating paths problem can be solved in polynomial time on graphs of bounded width. Moreover, we show that every bipartite graph with high perfect matching width must contain a large grid as a matching minor. Finally, we prove an analogue of the we known Flat Wall theorem and provide a qualitative description of all bipartite graphs which exclude a fixed matching minor.

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