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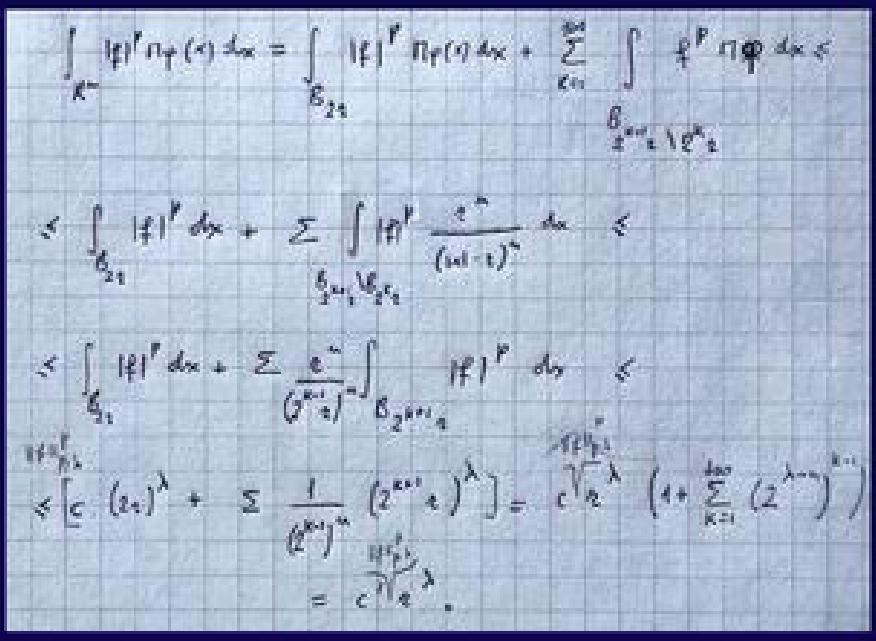
Morrey Spaces

Introduction and Applications to Integral Operators and PDE's

Volume I



Yoshihiro Sawano
Giuseppe Di Fazio
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Morrey Spaces

Introduction and Applications to Integral Operators and PDE's, Volume I

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Preface

Morrey spaces were introduced by Charles Morrey, who considered an estimate of the difference of functions in 1938 in order to investigate the local behavior of solutions to second order elliptic partial differential equations [323]. The technique is very useful in many areas in mathematics, in particular in harmonic analysis, potential theory, partial differential equations and mathematical physics. Our emphasis in this book is mainly on harmonic analysis. We consider Morrey spaces in connection with potential theory.

We describe some milestone results in this book. Chiarenza and Frasca showed the boundedness of the Hardy–Littlewood maximal operator, singular integral operators and fractional integral operators in Morrey spaces [75]. The boundedness of fractional maximal operators and fractional integral operators is originally due to Adams [2]. Di Fazio and Ragusa generalized some results on commutators [123] due to Coifman, Rochberg and Weiss [84] and due to Chanillo in [66]. Mizuhara and Nakai considered generalized Morrey spaces. They showed the boundedness of the Hardy–Littlewood maximal operator, singular integral operators and fractional integral operators [316, 327]. Independently Guliyev considered generalized weak Morrey spaces. Sawano and Tanaka considered Morrey spaces for general Radon measures [394]. Komori-Furuya and Shirai considered weighted Morrey spaces as a special case of this general framework of Radon measures [254]. The framework by Komori-Furuya and Shirai was carried over to the one in generalized weighted Morrey spaces by Guliyev and Mustafayev [168]. Independently of Komori-Furuya and Shirai, Samko considered the weighted Morrey spaces [376, 377]. There are many works on weighted Morrey spaces under the condition similar to the one for Lebesgue spaces. However, as is pointed out by Tanaka [425] and Duoandiotxea and Rosental [106], as well as Sawano and Nakamura [334, 335], weighted Morrey spaces go beyond weighted Lebesgue spaces in the sense that some weights which do not satisfy the Muckenhoupt condition make the operators bounded.

Morrey-type spaces originated from the idea developed by Guliyev and Burenkov [45]. Morrey-type spaces became a prototype of real interpolation. Burenkov and Nursultanov showed that real interpolation yields general local Morrey-type spaces [49].

Interpolation of Morrey spaces is one of the difficult problems in harmonic analysis. Roughly there are two interpolation methods, the real interpolation and the complex interpolation. We still have open problems in both

interpolations of Morrey spaces. Stampaccia obtained a partial result in 1964 [413]. Cobos et. al also showed a result on the complex interpolation functor in 1998 [80]. Blasco, Ruiz and Vega provided a prominent counterexample on the interpolation of Morrey spaces [37]. Despite all the recent progress in the area of interpolation of Morrey spaces, there are still many important open problems. In the last decade more and more works on interpolation of Morrey spaces emerged. Lemarié-Rieusset pointed out that Morrey spaces are not closed under the first complex interpolation [265, 266]. Later on Yang, Yuan and Lu described complex interpolation of Morrey spaces [294]. Their description was solidified by Sawano and Hakim [179]. Recently Sawano and Mastyło gave an explicit formula of the complex interpolation of Morrey spaces [305]. However, to the best knowledge of the authors, we have no general information on what this complex interpolation spaces look like.

Another important aspect is that Morrey spaces arise naturally when we consider the multiplier space from a Besov space to Lebesgue spaces [263].

One of the important aims in this book is to discuss the current state of the art and perspectives of developments of this theory of Morrey spaces. We want to provide a shelf to contain all these developments. We did our best to include as many references as we can. However, by no means do we hope that this book is merely a database. We strongly believe that there is always room in discussing function spaces no matter how big the theory becomes. What is important is that we can handle (unknown) sets of functions in a reasonable manner. Probably, knowing all that is written in this book is insufficient to launch a new study on function spaces. Nor do we need to read it completely. We wrote this book to be able to get a smooth access to all of these topics. We chose a topic from which the readers can learn to handle function spaces. In the long study of Morrey spaces, some incorrect propositions were believed to be true. For example, including Sawano, many people believed that Morrey spaces are closed under the first complex interpolation, which was disproved by Lemarié-Riuesset. We would like to explain why such a mistake happened for example.

In the 1980's and 1990's, we obtained a number of results on the boundedness of classical operators in Morrey spaces and related function spaces. Meanwhile around the beginning of the 21st century there were new active developments in this area. They resulted in necessary and sufficient conditions in many cases.

We explain the organization of the first half of this book, which consists of 10 chapters.

Chapter 1 starts with defining some fundamental function spaces such as Lebesgue spaces, Morrey spaces, Lorentz spaces and Orlicz spaces together with related function spaces. We suppose that we are familiar with Lebesgue spaces. See [156] for example. Our main function spaces, Morrey spaces, will appear in Chapter 1 so as not to get lost regarding what we are discussing. One of the important purposes of introducing various function spaces is that we can grasp as many functions as we can. The function spaces we will take

up in these chapters will play such a role. We can say that this note expands the editorial [386].

Chapters 2 and 3 are of preliminary nature, which is interesting in its own right. Chapter 2 collects some auxiliary observations in functional analysis. Chapter 3 takes up polynomials. The facts that we prove in Chapters 2 and 3 will appear not so many times. The facts on functional analysis in this book are quite fundamental and can be found in some textbooks such as [310].

Chapters 4, 5, 6 and 7 are also of preliminary nature. However, unlike Chapters 2 and 3, we will frequently use what we prove in Chapters 4, 5, 6 and 7. This part largely overlaps the existing textbooks such as [146, 176, 156, 157, 415, 416, 382]. Although a thorough explanation of fundamental facts in harmonic analysis makes the book thicker, we decided to include it to the minimum so as to be self-contained.

When we consider some problems in mathematics, it is not sufficient to be able to handle functions themselves. Usually, functions we will consider are transformed by some mappings described by means of integrals. So, we introduce some important operators here. Chapter 4 deals with the fundamental operators in many branches of mathematics. We consider the Hardy–Littlewood maximal operator, singular integral operators fractional integral operators and fractional maximal operators. Chapter 5 includes commutators generated by these operators.

The theory of Morrey spaces can be staged on metric measure spaces; it is enough to have a distance function and a Borel measure to develop a theory of Morrey spaces. So, we work in general metric measure spaces in Chapter 6. As a special case of metric measure spaces, Chapter 7 deals with weighted function spaces. Chapter 7 contains the solution of the A_2 conjecture.

We embark on the study of Morrey spaces, especially operators acting on Morrey spaces in Chapters 8, 9 and 10. In some sense, Morrey spaces are introduced so as to cover what Lebesgue spaces cannot do. We seek to have a better understanding of mathematical transforms by the use of Morrey spaces. For example, the fractional integral operator is given by $f \mapsto |\cdot|^{-n+\alpha} * f$. Here $0 < \alpha < n$. The well-known theorem by Hardy–Littlewood–Sobolev says that this transform maps L^p to L^q under a certain condition. Since the fractional integral operator is the convolution operator, the Young inequality will be a useful tool. However, the kernel $|\cdot|^{-n+\alpha}$ never belongs to L^p for any $0 < p \leq \infty$. We expect Morrey spaces to explain why this convolution mapping is bounded. In many cases we resort to the density argument in functional analysis when we define operators acting on function spaces. Chapter 8 precisely discuss whether this is possible in Morrey spaces. For example, $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, the space of all compactly supported essentially bounded functions, is not dense in Morrey spaces. Many other plausible candidates fail to be dense too. Another way of investigating the boundedness of operators in function spaces is to employ duality. Chapter 9 invesitigates duality. Unfortunately, we do not have any description of the dual space of Morrey spaces. However, Morrey spaces are realized as the dual spaces of certain function spaces. Based

on what we investigated in Chapters 8 and 9, Chapter 10 investigates the boundedness of operators in Morrey spaces. The Hardy–Littlewood maximal operator can be defined for any measurable functions since it is defined via the integral for non-negative measurable functions. So, with ease we can show that the Hardy–Littlewood maximal operator is bounded on Morrey spaces. Almost the same applies to fractional maximal operators and fractional integral operators since the integral kernels of these operators are non-negative. However, we have to be careful when handling singular integral operators and commutators generated by singular integral operators. As we will see in Chapter 8, Morrey spaces do not have $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ as a dense subspace. So, we cannot resort to the density argument. Instead, we can use duality results obtained in Chapter 9. An alternative way to define singular integral operators is to investigate carefully the size of the integrals.

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Notation in this book

Sets and set functions

(1) We write $\#A$ to denote the cardinality of a set A.

(2) The metric open ball defined by ℓ^2 is usually called a *ball*. We denote by $B(x,r)$ the *ball centered at x of radius r*. Namely, we write

$$B(x,r) \equiv \{y \in \mathbb{R}^n \,:\, \|x-y\| < r\}$$

when $x \in \mathbb{R}^n$ and $r > 0$. Given a ball B, we denote by $c(B)$ its *center* and by $r(B)$ its *radius*. We write $B(r)$ instead of $B(o,r)$, where $o \equiv (0,0,\ldots,0)$.

(3) By a "cube" we mean a compact cube whose edges are parallel to the *coordinate axes*. The metric closed ball defined by ℓ^∞ is called a *cube*. If a cube has center x and radius r, we denote it by $Q(x,r)$. Namely, we write

$$Q(x,r) \equiv \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \,:\, \max_{j=1,2,\ldots,n} |x_j - y_j| \le r \right\}$$

when $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. For a cube Q or a right-open cube Q, define $\mathcal{Q}(Q)$ to be the set of all cubes of the same type contained in Q and $\mathcal{Q}^\sharp(Q)$ to be the set of all cubes of the same type containing Q. From the definition of $Q(x,r)$, its volume is $(2r)^n$. We write $Q(r)$ instead of $Q(o,r)$. Given a cube Q, we denote by $c(Q)$ the *center of Q* and by $\ell(Q)$ the *sidelength of Q*: $\ell(Q) = |Q|^{1/n}$, where $|Q|$ denotes the volume of the cube Q. The symbol $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^n)$ denotes the set of all cubes.

(4) Given $Q \in \mathcal{Q}$ and $k > 0$, $k\,Q$ means the *cube concentric to Q with sidelength $k\,\ell(Q)$*. Given a ball B and $k > 0$, we denote by $k\,B$ the *ball concentric to B with radius $k\,r(B)$*.

(5) For $\nu \in \mathbb{Z}$ and $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, we define $Q_{\nu m} \equiv \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu} \right)$. Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ the set of such cubes.

The elements in $\mathcal{D}(\mathbb{R}^n)$ are called *dyadic cubes*.

(6) Let E be a measurable set. Then we denote its *indicator function* by χ_E. If E has positive measure and E is integrable over f, Then denote by $m_E(f)$ the *average of* f over E. $|E|$ denotes the *volume of* E.

(7) If we are working on $\mathbb{R}^n$, then $\mathcal{B}$ denotes the set of all balls in $\mathbb{R}^n$, while $\mathcal{Q}$ denotes the set of all cubes in $\mathbb{R}^n$. Be careful because $\mathcal{B}$ can be used for a different purpose: When we are working on a measure space $(X, \mathcal{B}, \mu)$, then $\mathcal{B}$ stands for the set of all Borel sets.

(8) The symbol 2^X denotes the set of all subsets in X.

(9) A tacit understanding is that by a "cube" we mean a closed cube whose edges are parallel to the coordinate axes. However, we say that (right-open) dyadic cubes are also cubes.

(10) We define the *upper half space* $\mathbb{R}^n_+$ and the *lower half space* $\mathbb{R}^n_-$ by

$$\mathbb{R}^n_\pm \equiv \{(x', x_n) \in \mathbb{R}^n \,:\, \pm x_n > 0\}. \tag{0.1}$$

Numbers

(1) Let $a \in \mathbb{R}$. Then write $a_+ \equiv \max(a, 0)$ and $a_- = a \wedge 0 \equiv \min(a, 0)$. Correspondingly, given an $\mathbb{R}$-valued function f, f_+ and f_- are functions given by $f_+(x) \equiv \max(f(x), 0)$ and $f_-(x) \equiv \min(f(x), 0)$, respectively.

(2) Let $a, b \in \mathbb{R}$. Then write $a \wedge b \equiv \min(a, b)$. Correspondingly, given $\mathbb{R}$-valued functions f, g, $f \wedge g$ are functions given by $f \wedge g(x) \equiv \min(f(x), g(x))$.

(3) The constants C and c denote positive constants that may change from one occurrence to another. Because the two constants c can be different, the inequality $0 < 2c < c$ is by no means a contradiction. When we add a subscript, for example, this means that the constant c depends upon the parameter. It can happen that the constants with subscript differ according to the above rule. In particular, we prefer to use c_n for various constants that depend on n, when we do not want to specify its precise value.

(4) Let $A, B \geq 0$. Then $A \lesssim B$ and $B \gtrsim A$ mean that there exists a constant $C > 0$ such that $A \leq CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously, while $A \simeq B$ means that there exists a constant $C > 0$ such that $A = CB$.

(5) When we need to emphasize or keep in mind that the constant C depends on the parameters α, β, γ etc:

  (1) Instead of $A \lesssim B$, we write $A \lesssim_{\alpha,\beta,\gamma,\ldots} B$.

(2) Instead of $A \gtrsim B$, we write $A \gtrsim_{\alpha,\beta,\gamma,\dots} B$.

(3) Instead of $A \sim B$, we write $A \sim_{\alpha,\beta,\gamma,\dots} B$.

(4) Instead of $A \simeq B$, we write $A \simeq_{\alpha,\beta,\gamma,\dots} B$.

(*a*) We define $\mathbb{N} \equiv \{1, 2, \dots\}$, $\mathbb{Z} \equiv \{0, \pm 1, \pm 2, \dots\}$ and $\mathbb{N}_0 \equiv \{0, 1, \dots\}$.

(*b*) For $a \in \mathbb{R}^n$, we write $\langle a \rangle \equiv \sqrt{1 + |a|^2}$.

(*c*) We occasionally identify $\mathbb{R}^{m+n}$ with $\mathbb{R}^m \times \mathbb{R}^n$.

Function spaces

(1) We use $\cdot$ for functions; $f = f(\cdot)$.

(2) The function spaces are tacitly on $\mathbb{R}^n$ unless otherwise stated.

(3) Let $\mathcal{X}$ be a Banach space. We denote its norm by $\|\cdot\|_{\mathcal{X}}$.

(4) Let Ω be an open set in $\mathbb{R}^n$. Then $C_{\rm c}^\infty(\Omega)$ denotes the set of smooth functions with compact support in Ω.

(5) Let $1 \le j \le n$. The symbol x_j denotes not only the j-th coordinate but also the function $x = (x_1, x_2, \dots, x_n) \mapsto x_j$.

(6) The space $L^2(\mathbb{R}^n)$ is the Hilbert space of square integrable functions on $\mathbb{R}^n$ whose inner product is given by

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g}(x)\mathrm{d}x. \tag{0.2}$$

(7) Let E be a measurable set and f be a measurable function with respect to the Lebesgue measure. Then write $m_E(f) \equiv \dfrac{1}{|E|}\displaystyle\int_E f$.

(8) Let $0 < \eta < \infty$, E be a measurable set, and f be a positive measurable function with respect to the Lebesgue measure. Then write $m_E^{(\eta)}(f) \equiv m_E(f^\eta)^{\frac{1}{\eta}}$.

(9) Let $0 < \eta < \infty$. We define the *powered Hardy–Littlewood maximal operator* $M^{(\eta)}$ by

$$M^{(\eta)} f(x) \equiv \sup_{R>0} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|^\eta \mathrm{d}y \right)^{\frac{1}{\eta}}.$$

(10) The space $\mathrm{C}(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$.

(11) The space $\mathrm{BC}(\mathbb{R}^n)$ denotes the set of all bounded continuous functions on $\mathbb{R}^n$.

(12) Occasionally we identify the value of functions with functions. For example $\sin x$ denotes the function on $\mathbb{R}$ defined by $x \mapsto \sin x$.

(13) Given a Banach space $\mathcal{X}$, we denote by $\mathcal{X}^*$ its dual space.

(14) Let μ be a measure on a measure space $(X, \mathcal{B}, \mu)$. Given a μ-measurable set A with positive μ-measure and a function f, we write

$$m_Q(f) \equiv \frac{1}{\mu(A)} \int_A f(x) \mathrm{d}\mu(x).$$

Let $0 < \eta < \infty$. Then define $m_Q^{(\eta)}(f) \equiv m_Q(f^\eta)^{\frac{1}{\eta}}$ whenever f is positive.

(15) If notational confusion seems likely, Then we use $[\quad]$ to denote $Mf(x) = M[f](x)$, $T\varphi(\xi) = T[\varphi](\xi)$, etc.

(16) For $j = 1, 2, \ldots, n$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$ stands for the partial derivative. The symbol Δ stands for the Laplacian $\sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2}$.

(17) We denote the $L^p(\mathbb{R}^n)$-norm by $\|\cdot\|_{L^p}$. For other function spaces such as Hölder's continuous function space $\mathcal{C}^\gamma(\mathbb{R}^n)$ of order $\gamma > 0$, we use $\|\cdot\|_{\mathcal{C}^\gamma}$ to stress the function spaces.

(18) When we consider function spaces on a domain Ω, we denote by $\mathcal{C}^\gamma(\Omega)$ the *Hölder continuous function space of order* γ.

(19) A quasi-norm over a linear space $\mathcal{X}$ enjoys positivity, homogeneity, and quasi-triangle inequality: for some $\alpha \geq 1$, $\|f + g\|_{\mathcal{X}} \leq \alpha(\, \|f\|_{\mathcal{X}} + \|g\|_{\mathcal{X}}\,)$ $(f, g \in \mathcal{X})$. However, to simplify, we frequently omit the word "quasi". Likewise, we abbreviate the word "quasi-Banach space" to Banach space.

(20) The Kronecker delta function defined on a set X is given by $\delta_{jk} \equiv \begin{cases} 1 & (j = k), \\ 0 & (j \neq k). \end{cases}$ for $j, k \in X$.

(21) When two normed spaces X, Y are isomorphic with equivalence of norms, we write $X \approx Y$.

(22) For subsets A, B of a linear space V and $v \in V$, define the Minkovski sum by

$$v + A \equiv \{v + a \,:\, a \in A\}, \quad A + B \equiv \{\, a + b \,:\, a \in A,\, b \in B\}.$$

There is an exception: If $k > 0$ and Q is a right-open/compact cube, we denote by kQ the k-times expansion of Q, that is, kQ is a cube concentric to Q and $|kQ| = k^n |Q|$.

(23) When two topological spaces $\mathcal{X}, \mathcal{Y}$ are homeomorphic, we write $\mathcal{X} \approx \mathcal{Y}$.

(24) When A and B are sets, $A \subset B$ stands for the inclusion of sets. If, in addition, both A and B are topological spaces, and if the natural embedding mapping $A \to B$ is continuous, we write $A \hookrightarrow B$ in the sense of continuous embedding. If A and B are quasi-normed spaces with the embedding constant $k > 0$, then $A \overset{k}{\hookrightarrow} B$.

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Chapter 1

Banach function lattices

This is a preliminary part of this book. We recall some fundamental inequalities as well as important theorems. The starting point is the review of the theory of Lebesgue spaces in Section 1.1. Historically, as early as 1925 people came to realize Lebesgue spaces themselves are insufficient to describe the boundedness properties of operators. For example, Kolmogorov noticed that L^1 is not sufficient for the Hilbert transform. So we are interested in the tools to overcome the problem. Roughly speaking, Banach function spaces, Banach lattices are notions to indicate that we have a certain linear space of functions together with a norm.

Since Morrey spaces are main function spaces in this book, we take up Morrey spaces in Section 1.2. Having defined Morrey spaces and related function spaces in Section 1.2, Section 1.3 considers local Morrey spaces, which are closely related to Morrey spaces. However, local Morrey spaces differ seriously from Morrey spaces. We will consider different function spaces (Banach lattices) in Sections 1.4 and 1.5. Although we consider Morrey spaces in this book, we will need some other related function spaces to consider Morrey spaces deeply and to consider the boundedness properties of operators. We consider distribution functions of measurable functions in Section 1.4 to replace Lebesgue spaces with other function spaces, while as another method of replacing Lebesgue spaces we deal with Orlicz spaces in Section 1.5. Section 1.6, which is different from other sections but still fundamental, introduces Sobolev spaces and Hölder–Zygmund spaces, although they are not Banach lattices.

1.1 L^p spaces

Here we recall some fundamental properties on integration theory including Lebesgue spaces. This will be useful when we consider Morrey spaces since Lebesgue spaces are realized as a special case of Morrey spaces. Conversely in this book it will be important what happens when Morrey spaces differ from Lebesgue spaces. Section 1.1.1 deals with measure spaces. Section 1.1.2 handles integration theorems. Section 1.1.3 considers Fubini's theorem. Theorem 5 is a fundamental tool to express the Lebesgue norm of functions. The Lebesgue space L^p initially appears in Section 1.1.3. Among other theorems,

we postpone the proof of Theorem 12 till Section 4.1. Many of the results in Section 1.1 are standard; see some textbooks on integration theory.

1.1.1 Measure space

Here, we collect some terminologies needed in this book. We state the theorems without any detailed proof.

Let us start with σ-algebras.

Definition 1 (σ-algebra)**.** Let X be a set. A non-void family $\mathcal{B} \subset 2^X$ is a *σ-algebra* if the following properties hold:

(1) $\mathcal{B}$ is closed under complement. That is, $A \in \mathcal{B} \Longrightarrow A^{\mathrm{c}} \in \mathcal{B}$.

(2) $\mathcal{B}$ is closed under countable union. Namely, $\bigcup_{j=1}^{\infty} A_j \in \mathcal{B}$, whenever $A_1, A_2, \ldots, A_j, \ldots \in \mathcal{B}$.

We adopt the following standard definition of measures:

Definition 2 (Measure)**.** Let $\mathcal{B}$ be a σ-algebra over X. A mapping $\mu : \mathcal{B} \to [0, \infty]$ is a *measure* if the following two properties hold.

(1) $\mu(\emptyset) = 0$.

(2) If $A_1, A_2, \ldots, A_j, \ldots \in \mathcal{B}$ are disjoint, that is, $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$\sum_{j=1}^{\infty} \mu(A_j) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right).$$

Let $L^0(\mu)$ be the space of all measurable functions. When we consider the case of Euclidean space, $L^0(\mathbb{R}^n)$ denotes the set of all Lebesgue measurable functions. It may happen that the value of $f \in L^0(\mathbb{R}^n)$ lies in $\mathbb{C}$ or $[0, \infty]$. We need other terminologies for later consideration.

Definition 3. A *regular measure* μ means that μ has the inner and outer regularities. To be precise, if μ is a regular Borel measure (Radon measure), then for each Borel set E of finite measure, we have the *inner regularity*

$$\mu(E) = \sup\{\mu(K) : K \text{ runs over all compact subsets of } E\}$$

and the *outer regularity*

$$\mu(E) = \inf\{\mu(U) : U \text{ runs over all open sets containing } E\}. \tag{1.1}$$

See [191, p. 3] or [319, Theorem 2.1] for example.

1.1.2 Integration theorems

As for convergence theorems, we assume that the readers are familiar with Theorems 1 and 2 below. Let $\mathbb{M}^+(\mathbb{R}^n)$ be the cone of all non-negative Lebesgue measurable functions and $\mathbb{M}^+(\mu)$ be the cone of all non-negative μ-measurable functions defined on a measure space $(X, \mathcal{B}, \mu)$. In particular, we define

$$\mathbb{M}^+(\mathbb{R}^n) \equiv \{f : \mathbb{R}^n \to [0, \infty] \ : \ f \text{ is measurable } \}.$$

Theorem 1. *Suppose that $(X, \mathcal{B}, \mu)$ is a measure space. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $\overline{\mathbb{R}}$-valued μ-measurable functions.*

(1) (Monotone convergence theorem.) *Suppose that $\{f_j\}_{j=1}^{\infty} \subset L^0(\mu)$ is an increasing sequence: We have $0 \le f_j \le f_{j+1}$, $\mu-$a.e.. Then*

$$\lim_{j\to\infty} \int_X f_j(x)\mathrm{d}\mu(x) = \int_X \lim_{j\to\infty} f_j(x)\mathrm{d}\mu(x). \tag{1.2}$$

(2) (Fatou's lemma.) *For $\{f_j\}_{j=1}^{\infty} \subset \mathbb{M}^+(\mu)$, we have*

$$\int_X \liminf_{j\to\infty} f_j(x)\mathrm{d}\mu(x) \le \liminf_{j\to\infty} \int_X f_j(x)\mathrm{d}\mu(x).$$

(3) (Dominated convergence theorem, Lebesgue's convergence theorem.) *Suppose that $\{f_j\}_{j=1}^{\infty}$ converges μ-almost everywhere to f. Assume further that there exists $g \in L^1(\mu)$ such that $|f_j| \le g$ a.e. for all $j \in \mathbb{N}$. Then $\lim_{j\to\infty} \int_X f_j(x)\mathrm{d}\mu(x) = \int_X f(x)\mathrm{d}\mu(x)$.*

See [236, Chapter 13] or [319, Chapter 1], for example.

Theorem 2 (Change of integration and differentiation)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, and we suppose that a function $f : X \times (a, b) \to \mathbb{C}$ satisfies the following:*

(1) *For each $t \in (a, b)$, $f(\cdot, t) \in L^1(\mu)$.*

(2) *For μ-almost all $x \in X$ the function $t \mapsto f(x, t)$ is differentiable for all $t \in (a, b)$.*

(3) *There exists $g \in L^1(\mu)$ such that $\left|\dfrac{\partial f}{\partial t}(x, t)\right| \le g(x)$ for all $t \in (a, b)$ and for μ-almost every $x \in X$.*

Then $\dfrac{d}{dt} \displaystyle\int_X f(x, t)\mathrm{d}\mu(x) = \int_X \frac{\partial f}{\partial t}(x, t)\mathrm{d}\mu(x).$

1.1.3 Fubini theorem and Lebesgue spaces

Fubini's theorem is one of the important tools in this book. We now consider the product of measures. We omit the proof here; see [236, Chapter 16] or [319, §1.8] for example.

Definition 4 ($(\mu\otimes\nu)^*$)**.** Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{B}',\nu)$ be a couple of measure spaces. Then define an outer measure $(\mu\otimes\nu)^*$ on $X\times Y$ by $(\mu\otimes\nu)^*(\emptyset)=0$ and

$$(\mu\otimes\nu)^*(A)=\inf\left\{\sum_{j=1}^{\infty}\mu(E_j)\nu(F_j)\ :\ A\subset\bigcup_{j=1}^{\infty}E_j\times F_j\right\},\quad A\in 2^{X\times Y}\setminus\{\emptyset\}.$$

We say that $E\subset X\times Y$ is $(\mu\otimes\nu)^*$-measurable, if for all $G\subset X\times Y$, $(\mu\otimes\nu)^*(G)=(\mu\otimes\nu)^*(E\cap G)+(\mu\otimes\nu)^*(E^{\mathrm{c}}\cap G)$.

The following theorems are fundamental, whose proof we do not recall.

Theorem 3. *The set of the form $E\times F$ with $E\in\mathcal{M}$ and $F\in\mathcal{N}$ is $(\mu\otimes\nu)^*$-measurable and $(\mu\otimes\nu)^*(E\times F)=\mu(E)\nu(F)$. Here it will be understood that $0\times\infty=\infty\times 0=0$.*

Consequently, we are led to the measure $\mu\otimes\nu$: If E is $(\mu\otimes\nu)^*$-measurable, then we write $\mu\otimes\nu(E)=(\mu\otimes\nu)^*(E)$.

Let $(X,\mathcal{B},\mu)$ be a measure space. Recall that a measure μ is *σ-finite*, if X can be partitioned into a countable collection of disjoint subsets with finite μ-measure.

From Definition 4, we can create another measure $\mu\otimes\nu$. About this new measure, we can say the following:

Theorem 4 (Fubini's theorem)**.** *Suppose that $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{N}',\nu)$ are σ-finite measure spaces. Then*

$$\begin{aligned}\iint_{X\times Y}f(x,y)\mathrm{d}\mu\otimes\nu(x,y)&=\int_X\left(\int_Y f(x,y)\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x)\\&=\int_Y\left(\int_X f(x,y)\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y),\end{aligned}$$

provided f is a non-negative $(\mathcal{M}\otimes\mathcal{N})$-measurable function or $f\in L^1(\mathcal{M}\otimes\mathcal{N})$.

We move on to the Lebesgue spaces. Let $(X,\mathcal{B},\mu)$ be a measure space because it will be necessary to state our results in full generality although we mainly work on Euclidean spaces. One of the simplest ways to measure the size of functions is the use of Lebesgue spaces. What is important here is that we consider the case of $0<p<1$. Sometimes considering the case of $0<p<1$ will be useful. For example, when we consider the product $f\cdot g$ of L^1 functions f and g, we notice that $f\cdot g$ is nothing but $L^{\frac{1}{2}}$ functions. Once again, since we use Lebesgue spaces on general measure spaces for later consideration, we state our results in full generality.

Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $0 < p \leq \infty$. We tolerate the case $p = \infty$. The space $L^p(\mu)$ denotes the set of all $f \in L^0(\mu)$ for which the quantity $\|f\|_{L^p(\mu)} \equiv \left(\int_X |f(x)|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}}$ is finite if $p < \infty$ and $\|f\|_{L^\infty(\mu)} \equiv \mathrm{ess.sup}_{x \in X} |f(x)|$ is finite if $p = \infty$. As usual if $(X, \mathcal{B}, \mu)$ is the Lebesgue measure, then we write $L^p(\mathbb{R}^n) = L^p(\mu)$. Define

$$L^p(\mu) \equiv \{ f \in L^0(\mu) \, : \, \|f\|_{L^p(\mu)} < \infty \} / \sim .$$

Here, the equivalence relation $\sim$ is defined by $f \sim g \Longleftrightarrow f = g$ a.e.. Below omit this equivalence in defining function spaces.

Example 1. Let $\gamma \in \mathbb{R}$ and $0 < p < \infty$. We define

$$f_\gamma(x) \equiv |x|^\gamma \chi_{B(1)}(x), \quad g_\gamma(x) \equiv |x|^\gamma \chi_{B(1)^c}(x) \quad (x \in \mathbb{R}^n). \tag{1.3}$$

(1) $f_\gamma \in L^p(\mathbb{R}^n)$ if and only if $\gamma > -\dfrac{n}{p}$.

(2) $g_\gamma \in L^p(\mathbb{R}^n)$ if and only if $\gamma < -\dfrac{n}{p}$.

(3) For any $\gamma \in \mathbb{R}$, $f_\gamma + g_\gamma = |\cdot|^{-\frac{n}{p}}$ never belongs to $L^p(\mathbb{R}^n)$.

It is noteworthy that the functions do not belong to $L^p(\mathbb{R}^n)$ in the border case $\gamma = -\dfrac{n}{p}$.

In this book, by using other function spaces, we consider this limit situation.

When we consider the integral, it matters how large the level set is. See Theorem 5 below. This motivates us to give the following definition.

Definition 5 (Distribution)**.** Let $f : X \to \mathbb{C}$ be a measurable function. Then the *distribution function* $\lambda_f : [0, \infty) \to [0, \infty]$ is a function defined by

$$\lambda_f(t) = \lambda_{f,\mu}(t) \equiv \mu\{ x \in X \, : \, |f(x)| > t \} \quad (t \geq 0).$$

The so called "Layer Cake Formula" below will be important when we analyze the Lebesgue norm of functions.

Theorem 5 (Layer-Cake Formula)**.** *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $0 < p < \infty$ and f be a μ-measurable function. Then we have*

$$\|f\|_{L^p(\mu)} = \left(p \int_0^\infty t^{p-1} \lambda_{f,\mu}(t) \mathrm{d}t \right)^{\frac{1}{p}}. \tag{1.4}$$

More generally, if $\Phi : [0, \infty) \to [0, \infty)$ is a C^1-increasing function such that $\Phi(0) = 0$, then

$$\int_X \Phi(|f(x)|) \mathrm{d}\mu(x) = \int_0^\infty \Phi'(t) \lambda_{f,\mu}(t) \mathrm{d}t. \tag{1.5}$$

Proof We concentrate on (1.4): The proof of (1.5) is merely a slight adaptation of (1.4). We observe

$$|f(x)|^p = \int_0^{|f(x)|} p\, t^{p-1} \mathrm{d}t = \int_0^{\infty} p\, t^{p-1} \chi_{(t,\infty)}(|f(x)|) \mathrm{d}t$$

for $x \in X$. By Fubini's theorem (Theorem 4) for non-negative functions, we calculate that

$$\begin{aligned}
(\|f\|_{L^p(\mu)})^p &= \int_X |f(x)|^p \mathrm{d}\mu(x) \\
&= \int_0^{\infty} \left(\int_X p\, t^{p-1} \chi_{(t,\infty)}(|f(x)|) \mathrm{d}\mu(x) \right) \mathrm{d}t \\
&= \int_0^{\infty} p\, t^{p-1} \left(\int_X \chi_{(t,\infty)}(|f(x)|) \mathrm{d}\mu(x) \right) \mathrm{d}t \\
&= \int_0^{\infty} p\, t^{p-1} \lambda_{f,\mu}(t) \mathrm{d}t.
\end{aligned}$$

By taking the p-th root, we obtain (1.4).

We move on to the norm inequalities of Lebesgue spaces. Let $(X, \mathcal{B}, \mu)$ be a measure space. Also let $1 \le p \le \infty$, and define the *conjugate of* p by $p' \equiv \dfrac{p}{p-1}$, if $1 \le p < \infty$. If $p = \infty$, then define $p' = 1$. Note that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, whenever $1 \le p \le \infty$. A direct calculation shows that $(p')' = p$. The Hölder inequality is fundamental throughout this book. In fact, it occasionally matters how we use it.

Theorem 6 (Hölder's inequality)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $1 \le p \le \infty$. Then $\|f \cdot g\|_{L^1(\mu)} \le \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\mu)}$ for all $f, g \in L^0(\mu)$.*

See [236, Theorem 17.3] or [319, Theorem 12.3], for example.

According to Minkowski's inequality, which we recall below, we see that $L^p(\mu)$ is a normed space. Recall that a linear space $\mathcal{X}$ is a *normed space*, if it comes with a function $\|\cdot\|_{\mathcal{X}} : \mathcal{X} \to [0, \infty)$, which satisfies the following property. In this case $\|\cdot\|_{\mathcal{X}} : \mathcal{X} \to [0, \infty)$ is a *norm* on a linear space $\mathcal{X}$.

(1) Let $x \in \mathcal{X}$. Then $\|x\|_{\mathcal{X}} = 0$ implies $x = 0$.

(2) $\|a \cdot x\|_{\mathcal{X}} = |a| \cdot \|x\|_{\mathcal{X}}$ for all $a \in \mathbb{K}$ and $x \in \mathcal{X}$.

(3) Let $x, y \in \mathcal{X}$. Then the triangle inequality $\|x + y\|_{\mathcal{X}} \le \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{X}}$ holds.

We now recall Minkowski's inequality; see [236, Theorem 17.4] or [319, Theorem 12.4], for example.

Theorem 7 (Minkowski's inequality)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $1 \le p \le \infty$. Then $\|f + g\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}$ for all $f, g \in L^0(\mu)$.*

When $0 < p < 1$, the triangle inequality fails to be true. Instead, we use the following p-convexity or p-triangle inequality:

Proposition 8 (p-convexity)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $0 < p \le 1$. Then $\|f+g\|_{L^p(\mu)}{}^p \le \|f\|_{L^p(\mu)}{}^p + \|g\|_{L^p(\mu)}{}^p$ for all $f, g \in L^0(\mu)$.*

The proof, which uses Exercise 6, is left for the readers.

Duality is an important tool to estimate the size of functions. Let $1 \le p < \infty$ and $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Then we can describe the dual of $L^p(\mu)$ by way of $L^{p'}(\mu)$ as follows:

Theorem 9 (Duality $L^p(\mu)$-$L^{p'}(\mu)$)**.** *Let $1 \le p < \infty$ and $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Then $L^p(\mu)^* = L^{p'}(\mu)$ with coincidence of norms. Speaking precisely, let $f \in L^{p'}(\mu)$ and define $F_f \in L^p(\mu)^*$ by*

$$g \in L^p(\mu) \mapsto \int_X f(x)g(x)\mathrm{d}\mu(x) \in \mathbb{C}.$$

Then $f \in L^{p'}(\mu) \mapsto F_f \in L^p(\mu)^*$ is well defined and $\|f\|_{L^{p'}(\mu)} = \|F_f\|_{L^p(\mu)^*}$. Furthermore if $F : L^p(\mu) \to \mathbb{K}$ is a continuous linear functional, i.e., there exists $M > 0$ such that $|F(h)| \le M\|h\|_{L^p(\mu)}$ for all $h \in L^p(\mu)$, then F is realized as $F = F_f$ with some $f \in L^{p'}(\mu)$.

Proof We do not recall the proof of this fundamental theorem. For the proof, see [81, Example 3.5.4] and [319, Theorem 12.7], for example.

In addition to Theorem 9, it is important to write the Lebesgue norm in the dual form.

Theorem 10. *Let $1 \le p \le \infty$ and $(X, \mathcal{B}, \mu)$ be a measure space. Then for all $f \in L^0(\mu)$, $\|f\|_{L^p(\mu)} = \sup\{\|f \cdot g\|_{L^1(\mu)} : \|g\|_{L^{p'}(\mu)} = 1\}$.*

Proof Again we do not recall the proof of this fundamental theorem. For the proof see [258, Theorem 3.1.6], for example.

Local integrability will be important in this book. This will be made clearer after we define Morrey spaces.

Definition 6 ($L^p_{\mathrm{loc}}(\mathbb{R}^n)$)**.** Let $1 \le p \le \infty$. The space $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ collects all $f \in L^0(\mathbb{R}^n)$ such that $f \in L^p(K)$ for each compact set K, or equivalently, $\chi_B \cdot f \in L^p(\mathbb{R}^n)$ for each ball B in $\mathbb{R}^n$. The topology of $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is generated by the functional $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \mapsto \int_K |f(y)|^p \mathrm{d}y$, where K moves over all compact sets.

Example 2. Let $\gamma \in \mathbb{R}$ and $0 < p < \infty$. We define $f_\gamma(x) \equiv |x|^\gamma$ for $x \in \mathbb{R}^n$. Since $f_\gamma \in L^p(\mathbb{R}^n)$ if and only if $\gamma > -\dfrac{n}{p}$, it follows that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if $\gamma > -\dfrac{n}{p}$. To compensate for the failure in the case of $\gamma = -\dfrac{n}{p}$, we will define the *weak Lebesgue space* $\mathrm{W}L^p(\mathbb{R}^n)$ as the set of all $g \in L^0(\mathbb{R}^n)$ for which $\|g\|_{\mathrm{W}L^p} \equiv \sup_{\lambda>0} \lambda \|\chi_{(\lambda,\infty]}(|g|)\|_{L^p}$ is finite. See Definition 7 below.

By combining Hölder's inequality with Definition 6, we can locally embed Lebesgue spaces at the cost of the integrability index.

Corollary 11. *Let $0 < q < p \leq \infty$. Then $L^p(\mathbb{R}^n) \hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^n)$.*

Although we will get out of the framework of Lebesgue spaces, we still often have some local integrability. In this case, it is important to consider the average. For a measurable set E and $f \in L^1(E)$ over which f is integrable, we define $m_E(f)$ to be the average of f over E:

$$m_E(f) \equiv \frac{1}{|E|}\int_E f(y)\mathrm{d}y. \tag{1.6}$$

Since the Lebesgue differentiation theorem will be important, we state it without the proof. Keeping the elementary inequalities in mind, we will apply the boundedness of the Hardy–Littlewood maximal operator to establish the Lebesgue differentiation theorem. The Lebesgue differentiation theorem is important because we can "visualize" the value $f(x)$ through integrals. Since we often handle integral operators, we have much information on the quantity described by integrals. So, the Lebesgue differentiation theorem is important.

Theorem 12 (Lebesgue differentiation theorem)**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\lim_{r\downarrow 0} m_{B(x,r)}(|f - f(x)|) = 0 \tag{1.7}$$

for almost all $x \in \mathbb{R}^n$. A point x for which (1.7) *holds is called a Lebesgue point of f.*

We will prove Theorem 12 after we prove the boundedness of the Hardy-Littlewood maximal operator: We postpone its proof.

The function $|\cdot|^{-\frac{n}{p}}$ is not a member in $L^p(\mathbb{R}^n)$. Hence, we are interested in function spaces close to $L^p(\mathbb{R}^n)$ which contain $|\cdot|^{-\frac{n}{p}}$. We will see that the weak $L^p(\mathbb{R}^n)$-space (the weak Lebesgue spaces, the Marcinkiewicz space) will serve this purpose.

Definition 7 (Marcinkiewicz space, weak Lebesgue space)**.**

(1) Let $0 < p < \infty$. Then the *weak $L^p(\mu)$-space* (or the *Marcinkiewicz space*) $\mathrm{W}L^p(\mu)$ is the space of all functions $f : X \to \mathbb{C}$ which satisfies $\|f\|_{\mathrm{W}L^p(\mu)} \equiv \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty$ if $p < \infty$.

(2) Define $\mathrm{W}L^\infty(\mu) \equiv L^\infty(\mu)$.

(3) As usual if $(X, \mathcal{B}, \mu)$ is the Lebesgue measure, then we write $L^p(\mathbb{R}^n) = L^p(\mu)$ and $\|\cdot\|_{\mathrm{W}L^p} = \|\cdot\|_{\mathrm{W}L^p(\mu)}$.

Example 3. Let $\gamma \in \mathbb{R}$ and $0 < p < \infty$. Define f_γ and g_γ as in Example 1. Then $f_\gamma \in \mathrm{W}L^p(\mathbb{R}^n)$ if and only if $\gamma \geq -\dfrac{n}{p}$ and $g_\gamma \in \mathrm{W}L^p(\mathbb{R}^n)$ if and only if

$\gamma \le -\dfrac{n}{p}$, while $f_\gamma \in L^p(\mathbb{R}^n)$ if and only if $\gamma > -\dfrac{n}{p}$ and $g_\gamma \in L^p(\mathbb{R}^n)$ if and only if $\gamma < -\dfrac{n}{p}$. Note $f_{-\frac{n}{p}} + g_{-\frac{n}{p}} = |\cdot|^{-\frac{n}{p}} \in \mathrm{W}L^p(\mathbb{R}^n)$.

We will investigate an embedding relation for decreasing functions. Here, recall that a continuous (Borel) measure is a measure μ which does not charge any point in the space; $\mu(\{a\}) = 0$ for any point a. Here and below $w \in \mathcal{B}(I)$ means that w is Borel measurable. Here, we state our result in full generality using continuous measures. We use Lemma 13 to prove Theorem 315. Here and below denote by $\mathbb{M}^{\downarrow}(0,\infty)$, the set of all non-negative decreasing functions on $(0,\infty)$.

Lemma 13. *Let $0 < p \le q < \infty$, and let μ, ν be continuous Borel measures on $(0,\infty)$. If $\mathcal{B} \equiv \sup\limits_{s>0} \dfrac{\sqrt[q]{\mu(0,s)}}{\sqrt[p]{\nu(0,s)}} < \infty$, then $\|f\|_{L^q(\mu)} \le \mathcal{B}\|f\|_{L^p(\nu)}$ for all $f \in \mathbb{M}^{\downarrow}(0,\infty)$.*

Proof We may approximate f with simple decreasing functions. Let $g \equiv f^p$. Then for the function g of the form $g = \sum\limits_{k=1}^{K} a_k \chi_{(0,b_k)}$, with $a_1, a_2, \ldots, a_K > a_0 = 0$ and $b_0 = 0 < b_1 < b_2 < \cdots < b_K$, we have

$$\|g\|_{L^{\frac{q}{p}}(\mu)} \le \sum_{k=1}^{K} \left(\int_0^\infty a_k{}^{\frac{q}{p}} \chi_{(0,b_k)}(t) \mathrm{d}\mu(t) \right)^{\frac{p}{q}}$$

since $q \ge p$. Again since $q \ge p$, we have

$$\|g\|_{L^{\frac{q}{p}}(\mu)} \le \mathcal{B}^p \sum_{k=1}^{K} a_k \nu(0,b_k) = \mathcal{B}^p \|g\|_{L^1(\nu)}.$$

So the proof is complete.

Let $\mathbb{M}^{+}(0,\infty)$ denote the set of all non-negative measurable functions in $(0,\infty)$. Likewise, it is not so hard to show that

$$\sup_{t>0} f(t)w(t) \lesssim \left(\int_0^\infty f(t)^p v(t) \mathrm{d}t \right)^{\frac{1}{p}} \tag{1.8}$$

for all $f \in \mathbb{M}^{\downarrow}(0,\infty)$ and for all $0 < p \le 1$, as long as $v, w \in \mathbb{M}^{+}(0,\infty)$ satisfy

$$\sup_{a>0} \left(\frac{1}{v(0,a)} \sup_{t\in(0,a)} w(t) \right) = \sup_{a>0} \left(\frac{1}{\|v\|_{L^1(0,a)}} \sup_{t\in(0,a)} w(t) \right) < \infty,$$

since we can reduce the matters to the case $p = 1$.

Lemma 14. *Let $-\infty < a < b < \infty$ and $\theta \in (0,1)$. For all $f \in \mathbb{M}^{\uparrow}(0,\infty)$, we have* $\left(\int_a^b g(t)\mathrm{d}t\right)^{\theta} \le \theta \int_a^b (t-a)^{\theta-1} g(t)^{\theta}\mathrm{d}t$.

Proof Assume for the time being that g is a function of the form:

$$g = \sum_{j=1}^{N} \lambda_j \chi_{(a_{j-1},a_j]}$$

with $a_0 = a < a_1 < \cdots < a_N = b$ and $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_N$, then

$$\left(\int_a^b g(t)\mathrm{d}t\right)^{\theta} = \left(\sum_{j=1}^{N} \lambda_j (a_j - a_{j-1})\right)^{\theta}$$

and

$$\theta \int_a^b (t-a)^{\theta-1} g(t)^{\theta}\, d\theta = \sum_{j=1}^{N} {\lambda_j}^{\theta}((a_j-a)^{\theta} - (a_{j-1}-a)^{\theta}).$$

To prove the conclusion, we induct on N by handling λ_N and $b = a_N$ as if they were variables as well. If $N = 1$, there is nothing to prove. Hence, assume that the assertion is true for $N-1$ and suppose that we have f described above. By translation, we may assume $a = 0$. Let $B \equiv \sum_{j=1}^{N-1} \lambda_j (a_j - a_{j-1}) \ge 0$. Since ${a_N}^{\theta} \le {a_{N-1}}^{\theta} + (a_N - a_{N-1})^{\theta}$ and $0 < \theta \le 1$, we have

$$\begin{aligned}
&\frac{d}{\mathrm{d}\lambda_N}\left(\sum_{j=1}^{N} {\lambda_j}^{\theta}({a_j}^{\theta} - {a_{j-1}}^{\theta}) - (\lambda_N(a_N - a_{N-1}) + B)^{\theta}\right)\\
&= \theta({a_N}^{\theta} - {a_{N-1}}^{\theta}){\lambda_N}^{\theta-1} - \theta(a_N - a_{N-1})(\lambda_N(a_N - a_{N-1}) + B)^{\theta-1}\\
&\ge \theta({a_N}^{\theta} - {a_{N-1}}^{\theta}){\lambda_N}^{\theta-1} - \theta(a_N - a_{N-1})(\lambda_N(a_N - a_{N-1}))^{\theta-1}\\
&\ge 0.
\end{aligned}$$

Thus, assuming that the result is true for $N-1$, we see that

$$\left(\int_a^b g(t)\mathrm{d}t\right)^{\theta} \le \theta \int_a^b (t-a)^{\theta-1} g(t)^{\theta}\mathrm{d}t.$$

In the general case, we decompose and approximate g by using functions of the form $\sum_{j=1}^{\infty} \lambda_j \chi_{(0,a_j)}$.

We end this section with a variant of Lebesgue spaces called mixed Lebesgue spaces. Let $0 < p_1, p_2, \ldots, p_n \le \infty$ be constants. Write $\mathbf{p} \equiv$

$(p_1, p_2, \dots, p_n)$. Then define the mixed Lebesgue norm $\|\cdot\|_{L^{\mathbf{p}}}$ by

$$\|f\|_{L^{\mathbf{p}}} \equiv \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \dots, x_n)|^{p_1} \mathrm{d}x_1 \right)^{\frac{p_2}{p_1}} \mathrm{d}x_2 \right)^{\frac{p_3}{p_2}} \cdots \mathrm{d}x_n \right)^{\frac{1}{p_n}}.$$

A natural modification for x_j is made when $p_j = \infty$, $j = 1, 2, \dots, n$. We define the *mixed Lebesgue space* $L^{\mathbf{p}}(\mathbb{R}^n)$ to be the set of all $f \in L^0(\mathbb{R}^n)$ with $\|f\|_{L^{\mathbf{p}}} < \infty$.

1.1.4 Exercises

Exercise 1. Let $f : X \to \mathbb{C}$ be a complex-valued measurable function. Denote $h_+ = \max(h, 0)$ and $h_- = \max(-h, 0)$ for a real-valued measurable function h. Show that the following are equivalent:

(1) $f \in L^1(\mu)$.

(2) $\mathrm{Re}(f) \in L^1(\mu)$ and $\mathrm{Im}(f) \in L^1(\mu)$.

(3) $\mathrm{Re}(f)_\pm \in L^1(\mu)$ and $\mathrm{Im}(f)_\pm \in L^1(\mu)$.

Exercise 2.

(1) Choose functions integrable on $[0, \infty)$ with respect to the Lebesgue measure among the functions listed below.

(2) Do the same thing replacing $[0, \infty)$ with $[1, \infty)$.

(3) Do the same thing replacing $[0, \infty)$ with $[0, 1)$.

$$(A)\, e^{-x} \quad (B)\, x^\alpha\, (\alpha < -1) \quad (C)\, x^{-1} \quad (D)\, x^\alpha\, (-1 < \alpha < 0) \quad (E)\, 1$$

$$(F)\, x^\alpha\, (\alpha > 0) \quad (G)\, xe^{-x} \quad (H)\, x^4 \sin x \quad (I)\, x^k e^{-\sqrt{x}} \quad (J)\, \frac{e^{-x}}{x-1}$$

$$(K)\, \frac{\log(x+1)^{\frac{1}{2}}}{x^{\frac{4}{3}}} \quad (L)\, \frac{e^{-x^2}}{\sqrt{|x-2|}}.$$

Exercise 3. Find the necessary and sufficient condition for $a, b \in \mathbb{R}$ to satisfy $\displaystyle\int_{\mathbb{R}^n} \frac{|x|^a}{1+|x|^b} \mathrm{d}x < \infty$.

Exercise 4. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $f \in L^0(\mu)$. Show that $\lim_{p \to \infty} \|f\|_{L^p(\mu)} = \|f\|_{L^\infty(\mu)}$ and that

$$\lim_{p \downarrow 0} \mu(X)^{-\frac{1}{p}} \|f\|_{L^p} = \exp\left(\frac{1}{\mu(\mu)} \int_X \log|f(x)| \mathrm{d}\mu(x) \right).$$

Exercise 5. Calculate the Lebesgue integral $\int_e^{e^2} \frac{\log x}{x} \mathrm{d}x$.

Exercise 6.

(1) Show that $(a+b)^p \leq a^p + b^p$ for all $a, b > 0$ and $0 < p \leq 1$.

(2) Prove Proposition 8.

Exercise 7. Show the following scaling law in Lebesgue spaces $L^p(\mathbb{R}^n)$: For all $f \in L^p(\mathbb{R}^n)$ and $t > 0$ $\|f(t\cdot)\|_{L^p} = t^{-\frac{n}{p}} \|f\|_{L^p}$.

Exercise 8. Show that the Lebesgue space $L^p([0,1])$ with $0 < p < 1$ is not a normed space. What is the property of the normed space that fails?

Exercise 9. Suppose that $h : \mathbb{R} \to \mathbb{C}$ is a continuous function such that $\lim_{|t|\to\infty} h(t) = A$ exists.

(1) Show that $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(t)\mathrm{d}t = A$.

(2) Show that $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\log \frac{T}{|t|}\right) h(t)\mathrm{d}t = A$. Hint: Let $\mathbb{M}^+(\mathbb{R})$ denote the set of all non-negative measurable functions defined on $\mathbb{R}$. Justify that we can assume that $h \in \mathbb{M}^+(\mathbb{R})$ is an even function. Then use

$$\begin{aligned}\frac{1}{2T} \int_{-T}^{T} \left(\log \frac{T}{|t|}\right) h(t)\mathrm{d}t &= \frac{1}{T} \int_0^T \left(\log \frac{T}{t}\right) h(t)\mathrm{d}t \\ &= \frac{1}{T} \int_0^T \left(\int_t^T \frac{\mathrm{d}s}{s}\right) h(t)\mathrm{d}t \\ &= \frac{1}{T} \int_0^T \left(\frac{1}{s} \int_0^s h(t)\mathrm{d}t\right) \mathrm{d}s,\end{aligned}$$

where for the last equality we have used Fubini's theorem.

Exercise 10. Let $\mathbb{M}^+(\mu)$ stand for the set of all non-negative μ-measurable functions. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f \in \mathbb{M}^+(\mu)$. Define inductively

$$f_0 \equiv f, \quad f_{j+1} \equiv f_j - \frac{1}{j+1} \chi_{(\frac{1}{j+1}, \infty)}(f_j).$$

Then show that $\lim_{j\to\infty} f_j = 0$. Consequently, we have a decomposition $f = \sum_{j=0}^{\infty} \frac{1}{j+1} \chi_{E_j}$ for some measurable set E_j.

1.2 Morrey spaces

The aim of this book is investigating Morrey spaces keeping in mind that Morrey spaces cover Lebesgue spaces and hence that Morrey spaces are

expected to enjoy nicer properties than Lebesgue spaces. Here, we define Morrey spaces in Section 1.2.1 and give some functions in Morrey spaces in Section 1.2.2.

Section 1.2.3 also considers some examples of functions in Morrey spaces by paying attention to the parameters in Morrey spaces. Since Morrey spaces have two parameters, we expect that Morrey spaces with different parameters are connected with each other. We check that Morrey spaces are nested in Section 1.2.4. Section 1.2.5 is a different nature from the previous sections. We consider weak Morrey spaces as an analogue of Morrey spaces. We handle weak function spaces. Weak Lebesgue spaces appeared in Example 2. We expand the idea there. As shown, weak Morrey spaces are used to compensate for the failure of the boundedness of operators. We return to Morrey spaces in Section 1.2.6. There are many function spaces other than Morrey spaces. Some of them are Banach lattices. The notion of Banach lattices is used to cover many (solid) function spaces. One of the other useful tools to classify function spaces is the use of Banach function spaces. However, as it turns out, Morrey spaces are not Banach function spaces in general. Hence, we propose a notion which substitutes for Banach function spaces. We actually define ball Banach function spaces in Section 1.2.6. Once again a different function in Morrey spaces appears there.

1.2.1 Morrey norms

Section 1.2.1 defines and investigates (classical) Morrey spaces. There are several notation to denote Morrey spaces but we use the letter $\mathcal{M}$ instead of $\mathcal{L}$ consistently in this book.

Morrey spaces are the most fundamental in this book. Having defined Lebesgue spaces, we now turn to Morrey spaces. Let us define the norm. We write $\mathbb{R}^{n+1}_+ \equiv \mathbb{R}^n \times (0,\infty)$.

Definition 8 ($\mathcal{M}^p_q(\mathbb{R}^n)$)**.** Let $0 < q \le p \le \infty$. For an $L^q_{\mathrm{loc}}(\mathbb{R}^n)$-function f, its (classical) Morrey norm is defined by

$$\|f\|_{\mathcal{M}^p_q} = \|f\|^{\mathcal{B}}_{\mathcal{M}^p_q} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}. \tag{1.1}$$

If there is need to stress the dimension we work in, we write $\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$ instead of $\|f\|_{\mathcal{M}^p_q}$. The (classical) Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for which the norm $\|f\|_{\mathcal{M}^p_q}$ is finite.

If $0 < q \le p = \infty$, then the Lebesgue differentiation theorem shows that $\mathcal{M}^p_q(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$, so we exclude this case.

In other words, $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if there exists $D \ge 0$ such that $\|f\|_{L^q(B(x,r))} \le D|B(x,r)|^{\frac{1}{p}-\frac{1}{q}}$ for all $x \in \mathbb{R}^n$ and $r > 0$. The minimal value of D in this inequality is $\|f\|_{\mathcal{M}^p_q}$.

Since $f \cdot g \in \mathcal{M}^p_q(\mathbb{R}^n)$ for any $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, $\mathcal{M}^p_q(\mathbb{R}^n)$ is not a space of functions possessing any kind of common smoothness of any order.

Before we go further, a couple of remarks on the definition and the notation of the Morrey norm may be in order. For $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ define $Q_{jk} \in \mathcal{Q}$ by

$$Q_{jk} \equiv \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_n}{2^j}, \frac{k_n+1}{2^j}\right) = \prod_{l=1}^{n} \left[\frac{k_l}{2^j}, \frac{k_l+1}{2^j}\right).$$

A dyadic cube is a set of the form Q_{jk} for some $j \in \mathbb{Z}, k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Finally, $\mathcal{D}(\mathbb{R}^n)$ stands for the set of all dyadic cubes.

Remark 1.

(1) Define (temporarily) the Morrey norms $\|f\|_{\mathcal{M}^p_q}^{\mathcal{B}_0}$ and $\|f\|_{\mathcal{M}^p_q}^{\mathcal{Q}}$ of $f \in L^0(\mathbb{R}^n)$ by

$$\|f\|_{\mathcal{M}^p_q}^{\mathcal{B}_0} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} r^{\frac{n}{p}-\frac{n}{q}} \left(\int_{B(x,r)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}} \tag{1.2}$$

and

$$\|f\|_{\mathcal{M}^p_q}^{\mathcal{Q}} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |Q(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q(x,r)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}. \tag{1.3}$$

The *dyadic Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}^{\mathcal{D}}$ is defined by

$$\|f\|_{\mathcal{M}^p_q}^{\mathcal{D}} \equiv \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

Then $\|f\|_{\mathcal{M}^p_q} \sim \|f\|_{\mathcal{M}^p_q}^{\mathcal{D}}$ for all $f \in L^0(\mathbb{R}^n)$. Then $\|f\|_{\mathcal{M}^p_q} \sim_{n,p,q} \|f\|_{\mathcal{M}^p_q}^{\mathcal{B}_0} \sim_{n,p,q} \|f\|_{\mathcal{M}^p_q}^{\mathcal{Q}} \sim_{n,p,q} \|f\|_{\mathcal{M}^p_q}^{\mathcal{D}}$ for all $f \in L^0(\mathbb{R}^n)$. This means that we can define the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ by using the norms $\|\cdot\|_{\mathcal{M}^p_q}^{\mathcal{B}_0}$, $\|\cdot\|_{\mathcal{M}^p_q}^{\mathcal{Q}}$ $\|\cdot\|_{\mathcal{M}^p_q}^{\mathcal{D}}$: The resulting space will be the same. We sometimes identify all of them; the superindexes $\mathcal{B}, \mathcal{B}_0, \mathcal{D}$ are sometimes omitted. See Exercise 13.

(2) In this book, all norms $\|\cdot\|_{\mathcal{M}^p_q}$, $\|\cdot\|_{\mathcal{M}^p_q}^{\mathcal{B}_0}$ and $\|\cdot\|_{\mathcal{M}^p_q}^{\mathcal{Q}}$ are used. Usually there will be no confusion.

(3) There are some traditions of how to express Morrey norms. Some prefer to use the notation:

$$\|f\|_{\mathcal{L}^q_\lambda} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} \left(\frac{1}{r^\lambda}\int_{B(x,r)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

Here, $0 < q < \infty$ and $0 < \lambda < n$. See the classical but important work of Peetre [353]. By letting

$$\lambda \equiv \left(1 - \frac{q}{p}\right) n \tag{1.4}$$

we obtain $\|f\|_{\mathcal{L}^q_\lambda} = \|f\|_{\mathcal{M}^p_q} = \|f\|^{\mathcal{B}_0}_{\mathcal{M}^p_q}$. However, in this book one prefers to use the notation $\mathcal{M}^p_q$.

As is seen from this remark, it is impossible to find the precise value of the norm $\|f\|_{\mathcal{M}^p_q}$ for a given f. Consequently, in many cases we give up calculating $\|f\|_{\mathcal{M}^p_q}$: We often content ourselves with estimating $\|f\|_{\mathcal{M}^p_q}$.

The case where $0 < p < q < \infty$ is excluded because this case boils down to a triviality.

Proposition 15. *The space $\mathcal{M}^p_q(\mathbb{R}^n)$ is trivial in the following sense*: *Let $x_0 \in \mathbb{R}^n$ be fixed. If $f \in L^0(\mathbb{R}^n)$ satisfies*

$$\mathcal{A} \equiv \sup_{r>0} |B(x_0, r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x_0,r)} |f(y)|^q \mathrm{d}y \right)^{\frac{1}{q}} < \infty$$

for some $p < q$, then $f(x) = 0$ for almost all $x \in \mathbb{R}^n$.

In this negative assertion, we can freeze the point x_0 instead of allowing it to vary among all points in $\mathbb{R}^n$.

Proof From the definition of $\mathcal{A}$,

$$\left(\int_{B(x_0,r)} |f(y)|^q \mathrm{d}y \right)^{\frac{1}{q}} \le |B(x_0, r)|^{\frac{1}{q}-\frac{1}{p}} \mathcal{A}.$$

Letting $r \to \infty$, we have $\|f\|_{L^q} = 0$. Thus, $f(x) = 0$ for almost all $x \in \mathbb{R}^n$.

A quasi-norm over a linear space $\mathcal{X}$ enjoys positivity, homogeneity, and quasi-triangle inequality: for some $\alpha \ge 1$, $\|f+g\|_{\mathcal{X}} \le \alpha(\|f\|_{\mathcal{X}} + \|g\|_{\mathcal{X}})$ $(f, g \in \mathcal{X})$. We explain what happens if p and/or q is less than 1.

Remark 2. It can happen that p and/or q is less than 1. In this case, we formally define the space $\mathcal{M}^p_q(\mathbb{R}^n)$ by (1.1) to obtain a quasi-Banach space, where the triangle inequality is replaced by the quasi-triangle inequality;

$$\|f + g\|_{\mathcal{M}^p_q} \le 2^{\max(\frac{1}{q}-1,0)} (\|f\|_{\mathcal{M}^p_q} + \|g\|_{\mathcal{M}^p_q}) \tag{1.5}$$

for all $f, g \in L^0(\mathbb{R}^n)$. Note that (1.5) follows from the triangle inequality immediately when $1 \le q < \infty$. Meanwhile, when $0 < q < 1$, (1.5) follows from the inequality $(a + b)^q \le 2^{\max(q-1,0)}(a^q + b^q)$ for all $a, b \ge 0$.

As the following theorem shows, we can say that Lebesgue spaces are realized as a special case of Morrey spaces.

Theorem 16. *For $0 < p < \infty$, $\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with coincidence of norms.*

Proof Let $f \in L^0(\mathbb{R}^n)$. We write down the definition of the norm $\|f\|_{\mathcal{M}^p_p}$;

$$\|f\|_{\mathcal{M}^p_p} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{p}} \|f\|_{L^p(B(x,r))} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} \|f\|_{L^p(B(x,r))}.$$

Then, we have

$$\|f\|_{\mathcal{M}^p_p} \leq \left(\int_{\mathbb{R}^n} |f(y)|^p \mathrm{d}y\right)^{\frac{1}{p}} = \|f\|_{L^p}. \tag{1.6}$$

Meanwhile, by the monotone convergence theorem, for any $x \in \mathbb{R}^n$, we have

$$\|f\|_{L^p} = \lim_{m\to\infty} \|f\|_{L^p(B(x,m))} \leq \sup_{r>0} \|f\|_{L^p(B(x,r))} = \|f\|_{\mathcal{M}^p_p}. \tag{1.7}$$

Combining (1.6) and (1.7), we obtain the desired result.

A cone in a linear space is a subset closed under addition and multiplication by positive scalars. We remark that when $q = 1$, we can consider measures instead of functions as follows:

Definition 9 ($\mathbb{M}^p_1(\mathbb{R}^n)$)**.** The space $\mathbb{M}^p_1(\mathbb{R}^n)$ consists of all Radon measures μ on $\mathbb{R}^n$ satisfying $\|\mu\|_{\mathbb{M}^p_1} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} r^{\frac{n}{p}-n}|\mu|(B(x,r)) < \infty$, where $|\mu|$ denotes the total variation of μ.

Here, we content ourselves with examples.

Example 4.

(1) Let $1 \leq p < \infty$. Then $\mathcal{M}^p_1(\mathbb{R}^n) \hookrightarrow \mathbb{M}^p_1(\mathbb{R}^n)$.

(2) Let μ be the Hausdorff measure on $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. More precisely, for a Borel measurable set $E \subset \mathbb{R}^n$ we let

$$\mu(E) = \int_{\mathbb{R}^{n-1}} \chi_E(x', 0)\mathrm{d}x'.$$

Then $\mu \in \mathbb{M}^{\frac{n}{n-1}}_1(\mathbb{R}^n)$.

1.2.2 Examples of functions in Morrey spaces

Our main concern is what type of functions does belong to Morrey spaces. Here, we will collect some examples of Morrey functions. As we saw in Theorem 16, the Morrey space $\mathcal{M}^p_p(\mathbb{R}^n)$ with $0 < p < \infty$ is nothing but $L^p(\mathbb{R}^n)$. Hence, we may ask ourselves how the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $0 < q < p < \infty$ differs from $L^p(\mathbb{R}^n)$. One of the big gaps between them is that $L^p(\mathbb{R}^n)$ requires only one cube "$\mathbb{R}^n$", while we need to consider all the cubes in $\mathbb{R}^n$ in the definition of the norm in the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $0 < q < p < \infty$. This difference actually yields many strange phenomena below.

We calculate the Morrey norm of the indicator functions of the balls.

Example 5. Let $0 < q \le p < \infty$. Let B be an open ball. Then

$$\|\chi_B\|_{\mathcal{M}^p_q} = \|\chi_B\|_{L^p}. \tag{1.8}$$

In fact, it is easy to see that

$$\|\chi_B\|_{\mathcal{M}^p_q} \le \|\chi_B\|_{L^p} \tag{1.9}$$

from Theorem 16 and Hölder's inequality. (See also Theorem 19 to follow.) If we write out the norm $\|\chi_B\|_{\mathcal{M}^p_q}$ in full, then

$$\|\chi_B\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |\chi_B(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

We can calculate and evaluate the integral precisely. The result is:

$$\|\chi_B\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} |B(x,r)\cap B|^{\frac{1}{q}} \ge |B|^{\frac{1}{p}} = \|\chi_B\|_{L^p}. \tag{1.10}$$

Combining (1.9) and (1.10), we obtain (1.8).

A major problem concerning simple/fundamental function spaces $\mathrm{BC}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ is that neither of them contains $|\cdot|^{-\alpha}$ for any $\alpha \in \mathbb{R}$. However, this simplest function appears everywhere but to handle this function, we need to restrict its domain. The Morrey space $\mathcal{M}^{\frac{n}{\alpha}}_q(\mathbb{R}^n)$ with $1 < q < \frac{n}{\alpha}$ contains $|\cdot|^{-\alpha}$. Here and below, $B(r)$ abbreviates $B(x,r)$ with x the origin.

Example 6. The Morrey space $\mathcal{M}^{\frac{n}{\alpha}}_q(\mathbb{R}^n)$ with $1 < q < \frac{n}{\alpha}$ contains $f_\alpha(x) \equiv |x|^{-\alpha}$, $x \in \mathbb{R}^n$. To check this, we observe that

$$\sup_{x\in\mathbb{R}^n} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f_\alpha\|_{L^q(B(x,r))} = |B(r)|^{\frac{1}{p}-\frac{1}{q}} \|f_\alpha\|_{L^q(B(r))}. \tag{1.11}$$

As a corollary of this example, we have the following relation:

Example 7. Let $0 < q < p < \infty$, and let $\alpha, \beta \in \mathbb{R}$.

(1) Let $f(x) = f_\alpha(x) \equiv |x|^\alpha$, $x \in \mathbb{R}^n$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha = -\dfrac{n}{p}$.

(2) Let $g \equiv f\chi_{B(1)}$. Then $g \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha \ge -\dfrac{n}{p}$. Likewise let $h \equiv f\chi_{\mathbb{R}^n\setminus B(1)}$. Then $h \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha \le -\dfrac{n}{p}$.

(3) Let $k \equiv f_\alpha\chi_{B(1)} + f_\beta\chi_{\mathbb{R}^n\setminus B(1)}$. From (2), we see that $k \in \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $\alpha \ge -\dfrac{n}{p} \ge \beta$.

We present an example of functions to detect the difference of q in $\mathcal{M}^p_q(\mathbb{R}^n)$.

Example 8 (Toothbrush)**.** Let $0 < q < p < \infty$. We choose a very small number α so that $\left(\frac{\alpha}{N}\right)^{\frac{1}{p}} = \alpha^{\frac{1}{q}}$ for each $N \in \mathbb{N}$. We divide equally $[0,1]^n$ into N^n cubes to have $Q_1^N, Q_2^N, \dots, Q_{N^n}^N$. We will consider $f = f_N \equiv \sum_{j=1}^{N^n} \chi_{\alpha Q_j^N} \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. We claim that $\|f\|_{\mathcal{M}^p_q} \sim \|f\|_{L^q}$. To this end we use

$$\|f\|_{\mathcal{M}^p_q} \equiv \|f\|^{\mathcal{Q}}_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |Q(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q(x,r)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

Since f is supported on $[0,1]^n$, we may assume that $Q(x,r)$ runs over all cubes contained in $[0,1]^n$. We first note that the supremum is almost attained by letting $r = 2\alpha$: If we choose $x \in \mathbb{R}^n$ suitably, then we have $|Q(x,r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q(x,r))} \lesssim |Q(r)|^{\frac{1}{p}} \simeq \|\chi_{\alpha Q_j}\|_{L^p} = \|f\|_{L^q}$. If $2r < \alpha$ instead, then we have $|Q(x,r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q(x,r))} \le |Q(x,r)|^{\frac{1}{p}} < \|f\|_{L^q}$ no matter where x is. Thus, to show the result, we have only to consider the cubes $Q(x,r)$ with $\alpha \le 2r \le 1$. If $Q(x,r)$ intersects Q_j^N and Q_k^N with $1 \le j < k \le N^n$, then there exists $R \in \mathcal{Q}$ such that $R \cap \mathrm{supp}(f)$ is realized as the union $\{Q_j^N\}_{j\in J}$ for some $J \subset \{1,2,\dots,N^n\}$, that R contains $Q(x,r)$ and that $|R| \lesssim |Q(x,r)|$. Hence, by translating R if necessary, we may assume that $R = [0, kN^{-1}]^n$ for some $k = 1,2,\dots,N$. For this R, we can readily check that $|Q(x,r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q(x,r))} \lesssim \|f\|_{L^q}$.

We consider another example whose motivation comes from the ternary Cantor set. The parameter q seems to reflect the local and geometric structure as the next example shows:

Example 9 (Self-similar decreasing sequence for $\mathcal{M}^p_q(\mathbb{R}^n)$)**.** We let $p > q > 0$ and $R > 1$ be fixed so that

$$(1+R)^{-\frac{1}{p}} = 2^{\frac{1}{q}}(1+R)^{-\frac{1}{q}}. \tag{1.12}$$

It is noteworthy that the case where $R = 2$ corresponds to the ternary Cantor set. For a vector $\varepsilon \in \{0,1\}^n$, we define an affine transformation T_ε by

$$T_\varepsilon(x) \equiv \frac{1}{1+R}x + \frac{R}{1+R}\varepsilon \quad (x \in \mathbb{R}^n).$$

Let $E_0 \equiv [0,1]^n$ and $E_{j,0} \equiv [0,(1+R)^{-j}]^n$. Suppose that we have defined $E_0, E_1, E_2, \dots, E_j$, $j \in \mathbb{N}$. Define $E_{j+1} \equiv \bigcup_{\varepsilon\in\{0,1\}^n} T_\varepsilon(E_j)$. Then we will show that

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim (1+R)^{-j\frac{n}{p}} = \|\chi_{E_{j,0}}\|_{\mathcal{M}^p_q} = \|\chi_{E_{j,0}}\|_{L^p} = \|\chi_{E_j}\|_{L^q}, \tag{1.13}$$

where the implicit constants in $\sim$ are independent of j but can depend on p and q and the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ used here is defined by (1.3). A direct calculation shows $\|\chi_{E_j}\|_{\mathcal{M}^p_q} \geq \|\chi_{E_{j,0}}\|_{\mathcal{M}^p_q} = (1+R)^{-j\frac{n}{p}} = \|\chi_{E_{j,0}}\|_{L^p} = \|\chi_{E_j}\|_{L^q}$. Thus, we need to show $\|\chi_{E_j}\|_{\mathcal{M}^p_q} \lesssim \|\chi_{E_{j,0}}\|_{L^p}$. Let us calculate $\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim \sup_{S\in\mathcal{Q}} |S|^{\frac{1}{p}-\frac{1}{q}}|S\cap E_j|^{\frac{1}{q}}$, where $\mathcal{Q}$ denotes the set of all cubes. Fix $j\in\mathbb{N}$. Let us temporarily say that $Q\in\mathcal{Q}$ is wasteful, if Q does not contribute to the supremum, that is, if there exists $S\in\mathcal{Q}$ such that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q\cap E_j|^{\frac{1}{q}} < |S|^{\frac{1}{p}-\frac{1}{q}}|S\cap E_j|^{\frac{1}{q}}.$$

Thus, by definition, if $\ell(Q)>1$, then

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q\cap E_j|^{\frac{1}{q}} < |E_j|^{\frac{1}{q}} = |[0,1]^n|^{\frac{1}{p}-\frac{1}{q}}|[0,1]^n\cap E_j|^{\frac{1}{q}}.$$

In addition, if the side-length of Q is less than $(1+R)^{-j}$, then it is wasteful. Indeed, then the equality

$$\begin{aligned}
&\sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q\cap E_j|^{\frac{1}{q}} \,:\, Q\in\mathcal{Q}, |Q|\leq(1+R)^{-jn}\right\}\\
&=\sup\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q\cap E_j|^{\frac{1}{q}} \,:\, Q\in\mathcal{Q}, Q\subset E_j\}\\
&=\sup\left\{|Q|^{\frac{1}{p}} \,:\, Q\in\mathcal{Q},\, Q\subset E_j\right\}\\
&=|E_{j,0}|^{\frac{1}{p}}
\end{aligned}$$

holds. Here, we obtain the first equality by translating Q so that Q is included in $E_{j,0}$. This calculation shows that $|Q|^{\frac{1}{p}-\frac{1}{q}}|Q\cap E_j|^{\frac{1}{q}} < |E_{j,0}|^{\frac{1}{p}}$ for any such cube. Thus, if the cube Q is not wasteful, then $(1+R)^{-j}\leq\ell(Q)\leq 1$. Thus, there exists $k=1,2,\dots,n$ such that $(1+R)^{-kn}\leq|Q|\leq(1+R)^{-(k-1)n}$. In this case, since any connected component Z satisfies $(1+R)^{-kn}=|Z|\leq|Q|\leq(1+R)^{-(k-1)n}$, $3Q$ contains a connected component of E_k. Hence, it follows that

$$\begin{aligned}
&\|\chi_{E_j}\|_{\mathcal{M}^p_q}\\
&\sim\sup\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q\cap E_j|^{\frac{1}{q}} \,:\, Q \text{ contains a connected component of } E_j\ \}.
\end{aligned}$$

Let S be a cube which contains a connected component of E_j and is not wasteful. By symmetry, we may assume $S=I\times I\times\cdots\times I$ for some interval I. We define

$$S^*\equiv\operatorname{co}\left(\bigcup\{W \,:\, W \text{ is a connected component of } E_j \text{ intersecting } S\}\right),$$

where $\operatorname{co}(A)$ stands for the smallest convex set containing a set A. Then a geometric observation shows that S^* engulfs k^n connected component of E_j for some $1\leq k\leq 2^j$. Take an integer l so that $2^{l-1}\leq k\leq 2^l$. Then

$$|S^*\cap E_j| = k^n(1+R)^{-jn},\quad |S^*|\sim(1+R)^{-jn+ln}.$$

Consequently, from (1.12) we have

$$\begin{aligned}|S^*|^{\frac{1}{p}-\frac{1}{q}}|S^*\cap E_j|^{\frac{1}{q}} &\sim 2^{\frac{ln}{q}}(1+R)^{-j\frac{n}{q}}(1+R)^{(-j+l)(\frac{n}{p}-\frac{n}{q})}\\ &= 2^{\frac{ln}{q}}(1+R)^{-j\frac{n}{p}+l\left(\frac{n}{p}-\frac{n}{q}\right)}\\ &= 2^{\frac{ln}{q}}(1+R)^{l\left(\frac{n}{p}-\frac{n}{q}\right)}(1+R)^{-j\frac{n}{p}}\\ &= (1+R)^{-j\frac{n}{p}}.\end{aligned}$$

Thus, we obtain (1.13).

We will present a similar example using dyadic cubes.

Example 10 (Dyadic rearrangement)**.** Let $0<q\le p<\infty$. Fix $j\in\mathbb{N}$. We define

$$a_0\equiv N(j)\equiv[2^{jn(1-\frac{q}{p})}],\quad a_{k+1}\equiv 1+[2^{-n}a_k],\quad (k=0,1,\dots).$$

Note that $a_{k+1}\le 1+2^{-n}a_k$ for each k and hence

$$a_k\le 2^{-nk}\left(a_0-\frac{1}{1-2^{-n}}\right)+\frac{1}{1-2^{-n}}.$$

We place disjoint cubes $Q_1,Q_2,\dots,Q_{N(j)}\in\mathcal{D}_j([0,1)^n)$ so that each $Q\in\mathcal{D}_k([0,1)^n)$ contains at most a_k cubes. Thus, writing $E_j\equiv Q_1\cup Q_2\cup\cdots\cup Q_{N(j)}$, we obtain

$$\begin{aligned}|Q|^{\frac{1}{p}-\frac{1}{q}}\|2^{\frac{jn}{p}}\chi_{E_j}\|_{L^q(Q)} &\le 2^{\frac{jn}{p}}(2^{-kn})^{\frac{1}{p}-\frac{1}{q}}(2^{-jn}a_k)^{\frac{1}{q}}\\ &\lesssim 2^{\frac{jn}{p}}(2^{-kn})^{\frac{1}{p}-\frac{1}{q}}(\max(2^{-jn},2^{-jn-kn+jn(1-\frac{q}{p})}))^{\frac{1}{q}}\\ &\lesssim 1\end{aligned}$$

for all $Q\in\mathcal{D}_k(\mathbb{R}^n)$. However

$$\|2^{\frac{jn}{p}}\chi_{Q_1\cup Q_2\cup\cdots\cup Q_{N(j)}}\|_{L^r(Q)}=2^{\frac{jn}{p}-\frac{jn}{q}}[2^{jn(1-\frac{q}{p})}]^{\frac{1}{r}}\to\infty$$

if $q<r\le p$.

The following is a sort of normalization: It can be arranged that $E_j\subset[0,2^{-j})^n$ for each $j\in\mathbb{N}$.

Hence, we can say that the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ reflects local regularity of functions more precisely than the Lebesgue norm $\|\cdot\|_{L^p}$.

A chain of equalities in (1.13) is the motivation of choosing R in (1.12).

Based on Example 9, we present another example. Although this is just a matter of scaling, we sometimes find it convenient to use increasing sequences.

Example 11 (Self-similar increasing sequence for $\mathcal{M}^p_q(\mathbb{R}^n)$)**.** Let $R>1$ solve $(1+R)^{\frac{n}{p}-\frac{n}{q}}2^{\frac{n}{q}}=1$. In Example 9, if one defines $F_j\equiv\{x\in\mathbb{R}^n\,:\,(1+R)^{-j}x\in$

$E_j\}$, then $\{F_j\}_{j_1}^{\infty}$ is an increasing sequence of sets and each F_j is made up of a disjoint union of cubes of length 1. It is noteworthy that

$$F_j \equiv \left\{ y + \sum_{k=1}^{j} R^k a_k \,:\, \{a_k\}_{k=1}^{j} \in \{0,1\}^j, y \in [0,1]^n \right\}.$$

If we denote by $Q(F_j)$, the set of all connected components of F_j, then $\sharp Q(F_j) = 2^{jn}$. By the scaling law of $\mathcal{M}^p_q(\mathbb{R}^n)$, we have $\|\chi_{F_j}\|_{\mathcal{M}^p_q} \sim 1$. If we let $F \equiv \{y + Ra_1 + R^2 a_2 + \cdots : \{a_j\}_{j=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \cap \ell^1(\mathbb{N}), y \in [0,1]^n\}$. Then we have $F_j = F \cap [0,(1+R)^j]^n$ for all $j \in \mathbb{N}$. It is noteworthy that $\|\chi_Q\|_{\mathcal{M}^p_q} \sim \|\chi_{F_j}\|_{\mathcal{M}^p_q} \sim 1$ for any $Q \in Q(F)$, where $Q(F) = \bigcup_{j=1}^{\infty} Q(F_j)$.

Example 12. Let $0 < q < p < \infty$. Set

$$G \equiv \bigcup_{k=1}^{\infty} (k - 1 + k^{\frac{p}{p-q}},\, k + k^{\frac{p}{p-q}}). \tag{1.14}$$

We show that $\chi_G \in \mathcal{M}^p_q(\mathbb{R})(\cap L^\infty(\mathbb{R}))$. In fact, keeping in mind that $G \subset (0,\infty)$, we observe

$$\begin{aligned}
\|\chi_G\|_{\mathcal{M}^p_q} &= \sup\left\{ (b-a)^{\frac{1}{p}-\frac{1}{q}} |(a,b) \cap G|^{\frac{1}{q}} \,:\, 0 < a < b < \infty \right\} \\
&\le \sup\left\{ (b-a)^{\frac{1}{p}-\frac{1}{q}} |(a,b) \cap G|^{\frac{1}{q}} \,:\, a,b \in \mathbb{R}, a < b < a+2 \right\} \\
&\quad + \sup\left\{ (b-a)^{\frac{1}{p}-\frac{1}{q}} |(a,b) \cap G|^{\frac{1}{q}} \,:\, a,b \in \mathbb{R}, 0 < a < b-2 < \infty \right\} \\
&\lesssim 1 + \sup\left\{ (b-a)^{\frac{1}{p}-\frac{1}{q}} |(a,b) \cap G|^{\frac{1}{q}} \,:\, a,b \in \mathbb{R}, 0 < a < b-2 < \infty \right\}.
\end{aligned}$$

Here, we used a geometric observation that $(a, a+2)$ can intersect at most 2 connected components of G.

Let $a_k \equiv k + k^{\frac{p}{p-q}}$ and $b_l \equiv l - 1 + l^{\frac{p}{p-q}}$ for $(k,l) \in \mathbb{N}_0 \times \mathbb{N}$ with $k \le l$. Given $a, b \in (0,\infty)$ with $a+2 < b$, we choose $k \in \mathbb{N}$ and $l \in \mathbb{N}$ so that $a_{k-1} < a \le a_k$ and that $b_l < b \le b_{l+1}$. If $b \le b_{k+1}$, we note that

$$(b-a)^{\frac{1}{p}-\frac{1}{q}} |(b-a) \cap G|^{\frac{1}{q}} \le (b-a)^{\frac{1}{p}-\frac{1}{q}} \le 1.$$

Hence, we are interested mainly in the case of $b > b_{k+1}$. Then $k < l$. Note that

$$(a,b) \subset (a_{k-1}, b_{l+1}),\ b_{l+1} - a_{k-1} \le 3(b-a),\ |G \cap (a_{k-1}, b_{l+1})| = l - k + 1.$$

Thus, it follows that

$$\begin{aligned}
\|\chi_G\|_{\mathcal{M}^p_q} &\lesssim 1 + \sup\left\{ (b_l - a_k)^{\frac{1}{p}-\frac{1}{q}} |(a_k, b_l) \cap G|^{\frac{1}{q}} \,:\, k,l \in \mathbb{N}, k \le l l \right\} \qquad (1.15) \\
&\le 1 + \sup\left\{ (l^{\frac{p}{p-q}} - k^{\frac{p}{p-q}} + 1)^{\frac{1}{p}-\frac{1}{q}} (l-k)^{\frac{1}{q}} \,:\, k,l \in \mathbb{N}, 2k \le l \right\} \\
&\quad + \sup\left\{ (l^{\frac{p}{p-q}} - k^{\frac{p}{p-q}} + 1)^{\frac{1}{p}-\frac{1}{q}} (l-k)^{\frac{1}{q}} \,:\, k,l \in \mathbb{N}, k < l < 2k \right\}.
\end{aligned}$$

It is easy to see that

$$l^{\frac{p}{p-q}} - k^{\frac{p}{p-q}} + 1 \sim l^{\frac{p}{p-q}}, \quad l - k \sim l \tag{1.16}$$

when $l \geq 2k$. Also, when $k < l < 2k$,

$$l^{\frac{p}{p-q}} - k^{\frac{p}{p-q}} + 1 \sim l^{\frac{p}{p-q}}(l-k). \tag{1.17}$$

Hence, inserting (1.16) and (1.17) into the above expression, we conclude $f \in \mathcal{M}^p_q(\mathbb{R})$.

Here and below for functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$, defined on different sets X and Y, respectively, we define their tensor product $f \otimes g : X \times Y \to \mathbb{C}$ by $f \otimes g(x,y) = f(x)g(y)$ for $(x,y) \in X \times Y$. Likewise we define the n-fold tensor product $\otimes^n f : X^n \to \mathbb{C}$.

Morrey spaces in higher dimensions can describe more precisely the singularity of lower dimensions. The next example is an attempt at this approach.

Example 13. Let $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Define $g \equiv f \otimes 1$. That is,

$$g(x,y) = f(x) \quad (x \in \mathbb{R}^n, y \in \mathbb{R}^m).$$

Choose $v > 0$ so that

$$\frac{n+m}{n}\left(\frac{1}{v} - \frac{1}{q}\right) + \frac{m}{qn} = \frac{1}{p} - \frac{1}{q}$$

Then $g \in \mathcal{M}^v_q(\mathbb{R}^{n+m})$. In fact, let $r > 0$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$. We calculate

$$\begin{aligned}
&|(x_0,y_0) + [-r,r]^{n+m}|^{\frac{1}{v}-\frac{1}{q}} \|g\|_{L^q((x_0,y_0)+[-r,r]^{n+m})} \\
&= \left(|[-r,r]^n|^{\frac{1}{v}-\frac{1}{q}}\right)^{\frac{n+m}{n}} \times |Q(r)|^{\frac{m}{qn}} \left(\int_{[-r,r]^n} |f(x)|^q \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}},
\end{aligned}$$

which shows $g \in \mathcal{M}^v_q(\mathbb{R}^{n+m})$.

The next example solidifies the above idea more precisely.

Example 14. Let $n \geq 2$. Define

$$f(x_1,x_2,\ldots,x_n) \equiv \frac{\chi_{[-1,1]}(2x_1)}{|x_1|(\log|x_1|)^2} \quad ((x_1,x_2,\ldots,x_n) \in \mathbb{R}^n).$$

Then $f \in \mathcal{M}^n_1(\mathbb{R}^n) \setminus \bigcup_{p>1} L^p_{\mathrm{loc}}(\mathbb{R}^n)$. In fact, $\displaystyle\int_0^1 \frac{\mathrm{d}t}{t^p|\log t|^{2p}} = \infty$ for any $p > 1$. Meanwhile, $\|f\|_{\mathcal{M}^n_1} = \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{|x_1|(\log|x_1|)^2}\mathrm{d}x_1 < \infty$.

We can parametrize the above example as follows:

Example 15. Let $0 < q < p < \infty$, $N \in \mathbb{N}$, and let $a \in \left(-\dfrac{q}{p-q}, -\dfrac{q}{p}\right)$.

(1) We have $\|f_N\|_{\mathcal{M}^p_q} \sim N^{\frac{n}{p}+\frac{an}{N}} = |[0,N]^n|^{\frac{1}{p}-\frac{1}{q}}\|f_N\|_{L^q([0,N]^n)}$ for $f_N^0 \equiv \sum_{j=1}^{N} \chi_{(j-1,j-1+j^a)}$ and $f_N = \otimes^N f_0^N$. See Example 12.

(2) We set $f_0^N \equiv \sum_{N=1}^{\infty} N^{-\frac{1}{p}-\frac{a}{q}} f_N(\cdot - N!)$ and $f_N \equiv \otimes^n f_N^0 = f_N^0 \otimes f_N^0 \otimes \cdots \otimes f_N^0$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

We end this section with other functions defined on the real line.

Example 16. Let $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Define $f_0^N \equiv \sum_{l=1}^{N} \chi_{(l,l+l^{-\varepsilon})}$. We have $|[0,N+1]|^{\frac{1}{p}-\frac{1}{q}}\|f_0^N\|_{L^q([0,N+1])} \sim N^{\frac{1}{p}-\frac{\varepsilon}{q}}$, since $\int_0^{N+1} f_0^N(t)\mathrm{d}t \sim N^{1-\varepsilon}$. Consequently, $\|f_0^N\|_{\mathcal{M}^p_q(\mathbb{R})} \sim \max(1, N^{\frac{1}{p}-\frac{\varepsilon}{q}})$.

1.2.3 The role of the parameters

As we have been seeing, Morrey spaces have two parameters p and q. Here, we will ask ourselves what are their roles. One of the intuitive but important obsevations on the parameter q is that q plays the role of local integrability because $\mathcal{M}^p_q(\mathbb{R}^n)$ is a subset of $L^q_{\mathrm{loc}}(\mathbb{R}^n)$. This observation will be made clearer in Theorem 18. The parameter p appears as the dilation index as we will see in Theorem 17 below. To this end we next show the scaling law in Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ for all $0 < q \le p < \infty$;

Theorem 17. *For all $0 < q \le p < \infty$, we have a scaling law: $\|f(t\cdot)\|_{\mathcal{M}^p_q} = t^{-\frac{n}{p}}\|f\|_{\mathcal{M}^p_q}$ for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $t > 0$.*

Proof We write out the norm in full:

$$\|f(t\cdot)\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{n}{p}-\frac{n}{q}} \left(\int_{B(x,r)} |f(ty)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

Changing variables, we have

$$\|f(t\cdot)\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}} \left(\frac{1}{|B(tx,tr)|}\int_{B(tx,tr)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

Since $t^n|B(x,r)| = |B(tx,tr)|$ for any balls $B(x,r)$, we have the desired result.

As we discussed before, the parameter q seems to serve to describe the local integrability of functions. We will clarify here the role of the parameter q. Denote by $L^0_{\mathrm{c}}(\mathbb{R}^n)$ the set of all compactly supported measurable functions.

Theorem 18 (The local integrability of q). *Let $0 < q < r < p < \infty$. Then $\mathcal{M}^p_q(\mathbb{R}^n) \cap L^0_{\rm c}(\mathbb{R}^n) \setminus L^r_{\rm loc}(\mathbb{R}^n) \neq \emptyset$.*

Proof We use the function $f \equiv f_N$ in Example 8. With α fixed, we have

$$\liminf_{N\to\infty} \frac{1}{N} \log\left(\frac{\|f_N\|_{L^r}}{\|f_N\|_{L^q}}\right) = \liminf_{N\to\infty} \frac{1}{N} \log\left(\frac{\|f_N\|_{L^r}}{\|f_N\|_{\mathcal{M}^p_q}}\right) > 1.$$

Thus, $f \equiv \displaystyle\sum_{N=1}^{\infty} \frac{1}{N^2\|f_N\|_{\mathcal{M}^p_q}} f_N \in \mathcal{M}^p_q(\mathbb{R}^n) \cap L^0_{\rm c}(\mathbb{R}^n) \setminus L^r_{\rm loc}(\mathbb{R}^n)$.

1.2.4 Inclusions in Morrey spaces

Here, we are concerned with the relation of two different Morrey spaces or of a Morrey space and a different function space. We ask ourselves about the embedding between two different Morrey spaces. The first easy but important thing to notice is that Morrey spaces are nested as is seen from Hölder's inequality:

Theorem 19. *For all $f \in L^0(\mathbb{R}^n)$ and $0 < q_1 \le q_0 \le p < \infty$, $\|f\|_{\mathcal{M}^p_{q_1}} \le \|f\|_{\mathcal{M}^p_{q_0}}$. In particular, $\mathcal{M}^p_{q_0}(\mathbb{R}^n) \overset{1}{\hookrightarrow} \mathcal{M}^p_{q_1}(\mathbb{R}^n)$.*

Proof Using (1.1), we write out the norms in full:

$$\|f\|_{\mathcal{M}^p_{q_1}} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q_1}} \left(\int_{B(x,r)} |f(y)|^{q_1}\mathrm{d}y\right)^{\frac{1}{q_1}} \tag{1.18}$$

$$\|f\|_{\mathcal{M}^p_{q_0}} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q_0}} \left(\int_{B(x,r)} |f(y)|^{q_0}\mathrm{d}y\right)^{\frac{1}{q_0}} \tag{1.19}$$

By Hölder's inequality (for probability measures), we have

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^{q_1}\mathrm{d}y\right)^{\frac{1}{q_1}} \le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^{q_0}\mathrm{d}y\right)^{\frac{1}{q_0}}. \tag{1.20}$$

Thus, by inserting inequality (1.20) into (1.18) and (1.19), we obtain the desired result.

The following theorem indicates that the embedding $\mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$ is not dense unlike Lebesgue spaces over a set of finite measure.

Theorem 20. *Let $1 < q < \tilde{q} < p$. Then $\mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$.*

This result in particular shows that $L^\infty_{\rm c}(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$.

Proof Let F be the set defined in Example 11. The function χ_F belongs to $\mathcal{M}^p_q(\mathbb{R}^n)$ thanks to Example 11.

We prove that χ_F is not in the closure of $\mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$ by showing $f \notin \mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$ if $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ satisfies $\|2\chi_F - f\|_{\mathcal{M}^p_q} < 1$. Indeed, if K is one of the connected components, then $\|f\|_{L^{\tilde{q}}(K)} \geq \|f\|_{L^q(K)} > 1$ since $1 > \|2\chi_F - f\|_{\mathcal{M}^p_q} \geq \|2 - f\|_{L^q(K)} \geq 2 - \|f\|_{L^q(K)}$. Thus, $\|f\|_{\mathcal{M}^p_{\tilde{q}}} \geq |[0, R^j]^n|^{\frac{1}{p}-\frac{1}{\tilde{q}}} \|f\|_{L^{\tilde{q}}([0,R^j]^n)} \geq R^{\frac{jn}{p}-\frac{jn}{\tilde{q}}} |F_j|^{\frac{1}{\tilde{q}}} = 2^{\frac{jn}{\tilde{q}}-\frac{jn}{q}} R^{\frac{jn}{q}-\frac{jn}{\tilde{q}}}$ for all $j \in \mathbb{N}$. Hence, $f \notin \mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$, since this is valid for all $j \in \mathbb{N}$.

The following proposition shows that $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \leq q < p < \infty$ is large unlike Lebesgue spaces. Denote by $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ the linear space of all functions which can be expressed as a sum of $L^1(\mathbb{R}^n)$-functions and $L^\infty(\mathbb{R}^n)$-functions. We discuss the structure of $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ as a normed space in Section 2.1.3.

Proposition 21. *Let $1 \leq q < p < \infty$. Then $\mathcal{M}^p_q(\mathbb{R}^n)$ is not included in $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$. Namely, there exists $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ which cannot be expressed as a sum of an integrable function and an essentially bounded function.*

Proof The function f in Example 21 below belongs to $f \in \mathcal{M}^p_q(\mathbb{R}^n) \setminus (L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n))$.

Recall that a quasi-metric space (X, d) is *separable* if there exists a countable set which is dense in X. Typically, (X, d) is a nondense quasi-metric measure space, if there exists an infinite set $E \subset X$ such that $\inf_{x,y \in E, x \neq y} d(x, y) > 0$. We will disprove that Morrey spaces are separable in general.

Proposition 22. *Let $0 < q \leq p < \infty$.*

(1) *The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ does not have $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ as dense subspace.*

(2) *The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is not separable.*

Proof We recall the calculation for $f = f_{n/p}$ in Example 6.

(1) Observe that $|\cdot|^{-\frac{n}{p}}$ cannot be approximated by the functions in $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

(2) Let $f_j \equiv \chi_{B(2^j)} f$, $j \in \mathbb{N}$. Then $\|f_j - f_k\|_{\mathcal{M}^p_q} = \|f_1 - f_0\|_{\mathcal{M}^p_q} > 0$ for all $k, j \in \mathbb{N}$ with $k < j$.

1.2.5 Weak Morrey spaces

As we did for Lebesgue spaces, we can introduce weak Morrey spaces. The definition is as follows:

Definition 10 (Weak Morrey spaces)**.** Let $0 < q \leq p < \infty$. For $f \in L^0(\mathbb{R}^n)$ its weak Morrey norm is defined by $\|f\|_{\mathrm{W}\mathcal{M}^p_q} \equiv \sup_{\lambda > 0} \|\lambda \chi_{(\lambda,\infty]}(|f|)\|_{\mathcal{M}^p_q}$. The weak Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which the norm $\|f\|_{\mathrm{W}\mathcal{M}^p_q}$ is finite.

The following example explains the gap between $\mathcal{M}^p_q(\mathbb{R}^n)$ and $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$:

Example 17. Let $0 < q < p < \infty$. Let F be the set in Example 11. Define

$$f \equiv \sup_{k\in\mathbb{N}} \frac{\chi_F((1+R)^{-k}\cdot)}{\|\chi_F((1+R)^{-k}\cdot)\|_{\mathcal{M}^p_q}}.$$

Then for all $j \in \mathbb{N}$,

$$\begin{aligned}
&|[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{[0,(1+R)^j]^n} f(x)^q \mathrm{d}x\right)^{\frac{1}{q}} \\
&\geq |[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}} \left\{\int_{[0,(1+R)^j]^n} \left(\sup_{k\in\mathbb{N}\cap[1,j]} \frac{\chi_F((1+R)^{-k}x)}{\|\chi_F((1+R)^{-k}\cdot)\|_{\mathcal{M}^p_q}}\right)^q \mathrm{d}x\right\}^{\frac{1}{q}} \\
&\gtrsim j^{\frac{1}{q}}.
\end{aligned}$$

This disproves $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Meanwhile

$$\begin{aligned}
\|f\|_{\mathrm{W}\mathcal{M}^p_q} &= \sup_{\lambda>0} \lambda \|\chi_{(\lambda,\infty]}(|f|)\|_{\mathrm{W}\mathcal{M}^p_q} \\
&= \sup_{k\in\mathbb{N}} \frac{1}{\|\chi_F((1+R)^{-k}\cdot)\|_{\mathcal{M}^p_q}} \|\chi_{[1,\infty)}(\|\chi_F((1+R)^{-k}\cdot)\|_{\mathcal{M}^p_q}|f|)\|_{\mathrm{W}\mathcal{M}^p_q} \\
&= \sup_{k\in\mathbb{N}} \frac{1}{\|\chi_F((1+R)^{-k}\cdot)\|_{\mathcal{M}^p_q}} \|\chi_F((1+R)^{-k}\cdot))\|_{\mathrm{W}\mathcal{M}^p_q} \\
&= 1,
\end{aligned}$$

so that $f \in \mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$.

We can refine the embedding $\mathcal{M}^p_r(\mathbb{R}^n) \overset{1}{\hookrightarrow} \mathcal{M}^p_q(\mathbb{R}^n)$ for $0 < q < r \leq p < \infty$.

Lemma 23. *Let $0 < q < r \leq p < \infty$. Then $\mathrm{W}\mathcal{M}^p_r(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof Let $f \in \mathrm{W}\mathcal{M}^p_r(\mathbb{R}^n)$. Also let $Q \in \mathcal{Q}$. We need to estimate

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}.$$

We use Theorem 5 to have

$$\int_Q |f(x)|^q \mathrm{d}x = \int_0^\infty q\lambda^{q-1} |\{x \in Q : |f(x)| > \lambda\}| \mathrm{d}\lambda.$$

Since $|Q|^{\frac{1}{p}-\frac{1}{r}} |\{x \in Q : |f(x)| > \lambda\}|^{\frac{1}{r}} \leq \lambda^{-1} \|f\|_{\mathrm{W}\mathcal{M}^p_r}$, we obtain

$$\int_Q |f(x)|^q \mathrm{d}x \leq \int_0^\infty q\lambda^{q-1} \min(|Q|, \lambda^{-r} \|f\|_{\mathrm{W}\mathcal{M}^p_r}{}^r) \mathrm{d}\lambda.$$

If we calculate the right-hand side, we obtain $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

We compare weak Lebesgue spaces and Morrey spaces.

Theorem 24. *Let $0 < q < p < \infty$. Then $\mathrm{W}L^p(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof Let $f \in \mathrm{W}L^p(\mathbb{R}^n)$. Then for any cube Q,

$$\begin{aligned}|Q|^{\frac{q}{p}-1}\int_Q |f(x)|^q \mathrm{d}x &= |Q|^{\frac{q}{p}-1}\int_0^\infty q\lambda^{q-1}|\{x \in Q : |f(x)| > \lambda\}|\mathrm{d}\lambda \\ &\le |Q|^{\frac{q}{p}}\int_0^\infty q\lambda^{q-1}\min(1, (|Q|^{-\frac{1}{p}}\lambda^{-1}\|f\|_{\mathrm{W}L^p})^p)\mathrm{d}\lambda \\ &\simeq (\|f\|_{\mathrm{W}L^p})^q,\end{aligned}$$

as required.

1.2.6 Morrey spaces and ball Banach function spaces

We now consider Morrey spaces from the viewpoint of Banach function spaces. As a starting point, define Banach lattices.

Definition 11 (Banach lattice)**.** A *Banach lattice* on a measure space $(X, \mathcal{B}, \mu)$ is a Banach space $(E(\mu), \|\cdot\|)$ contained in $L^0(\mu)$ such that, for all $f, g \in E(\mu)$, the implication "$|f| \le |g| \to \|f\|_{E(\mu)} \le \|g\|_{E(\mu)}$" holds.

We recall that $\mathbb{M}^+(\mu)$ is the cone of all non-negative measurable functions on a measure space $(X, \mathcal{B}, \mu)$.

Definition 12 (Banach function norm)**.** A mapping $\rho : \mathbb{M}^+(\mu) \to [0, \infty]$ is called a *Banach function norm* if, for all $f, g, f_j, (j = 1, 2, 3, \ldots)$ in $\mathbb{M}^+(\mu)$, for all constants $a \ge 0$ and for all measurable subsets $E \in \mathcal{B}$, the following properties hold:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \le \rho(f) + \rho(g)$,

(P2) $\rho(g) \le \rho(f)$ if $0 \le g \le f$ a.e.,

(P3) the Fatou property; $\rho(f_j) \uparrow \rho(f)$ holds whenever $0 \le f_j \uparrow f$ a.e.,

(P4) $\rho(\chi_E) < \infty$ whenever $\mu(E) < \infty$,

(P5) whenever $\mu(E) < \infty$, $\|\chi_E f\|_{L^1(\mu)} \lesssim_E \rho(f)$ with the implicit constant depending on E and ρ but independent of f.

Quasi-Banach function norms are defined by replacing (P1) with a weaker requirement (P1)′ for some $\alpha \ge 1$.

(P1)′ $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \le \alpha(\rho(f) + \rho(g))$.

In this case, ρ is called a quasi-Banach function norm.

Here are examples of Banach function spaces.

Example 18. Let $1 \le p \le \infty$. Then $\ell^p(\mathbb{N})$ and $L^p(\mathbb{N})$ are Banach function spaces.

Example 19. Let Φ be a Young function such that $\Phi : [0, \infty) \to [0, \infty)$ is a convex homeomorphism. Suppose that we have a function $\Phi : [0, \infty) \to (0, \infty)$.

Then define the *Orlicz norm* by

$$\|f\|_{L^\Phi} \equiv \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \le 1\right\}$$

for $f \in L^0(\mathbb{R}^n)$. See also Definition 25 below.

The symbol $L^0(\mu)$ denotes the collection of all extended scalar-valued (real or complex) measurable functions over $(X, \mathcal{B}, \mu)$. As usual, any two functions coinciding a.e. will be identified.

Definition 13 (Banach function space)**.** Let ρ be a Banach function norm over a measure space $(X, \mathcal{B}, \mu)$. The collection $\mathcal{X} = \mathcal{X}(\rho)$ of all functions $f \in L^0(\mu)$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in \mathcal{X}$, define $\|f\|_{\mathcal{X}} \equiv \rho(|f|)$.

Example 20. We go back to Example 19. The space $L^\Phi(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which the norm $\|f\|_{L^\Phi}$ is finite. Then $L^\Phi(\mathbb{R}^n)$ is an example of Banach function spaces.

Unfortunately, unlike Orlicz spaces, Morrey spaces are not Banach function spaces. Let $1 \le q < p < \infty$. We aim here to prove that $\mathcal{M}^p_q(\mathbb{R}^n)$ is not a Banach function space. More precisely, we let $\rho(f) \equiv \|f\|_{\mathcal{M}^p_q}$ for $f \in \mathbb{M}^+(\mathbb{R}^n)$. Then ρ fails the property (P5); it can happen that $|E| < \infty$ and $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ while $\|f\|_{L^1(E)} = \infty$.

Example 21. Here, we exhibit an example showing that $\|\cdot\|_{\mathcal{M}^p_q}$ fails property (P5) when $1 < q < p < \infty$. For simplicity, we let $n = 1$ and $1 < q < 2 = p$; other cases are dealt with analogously. Let us consider the sequence

$$(a_1, a_2, \ldots) = \left(1, \frac{1}{4}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \ldots\right).$$

That is, $\{a_j\}_{j=1}^\infty$ is a decreasing sequence and 4^{-l} appears 2^l times for $l \in \mathbb{N}_0$.

We define $E \equiv \bigcup_{j=1}^\infty (10^j, 10^j + a_j) \subset \mathbb{R}$. Observe that

$$|E| = \sum_{j=1}^\infty a_j = 1 + \frac{1}{4} \times 2 + \frac{1}{16} \times 4 + \cdots = 2. \tag{1.21}$$

We also define a function f by $f \equiv \sum_{j=1}^\infty (a_j)^{-1/2} \chi_{(10^j, 10^j + a_j)}$. Then f belongs to $\mathcal{M}^p_q(\mathbb{R})$. In fact, since f is supported on $(0, \infty)$, we have

$$\|f\|_{\mathcal{M}^p_q} = \sup_{0<a<b<\infty} (b-a)^{\frac{1}{2}-\frac{1}{q}} \left(\sum_{j=1}^\infty a_j{}^{-\frac{q}{2}} |(a,b) \cap (10^j, 10^j + a_j)|\right)^{\frac{1}{q}}.$$

We distinguish two cases $2a < b$ and $2a > b$. For the latter case, there is at most one j such that $(a, b) \cap (10^j, 10^j + a_j) \neq \emptyset$. This implies

$$\sum_{j=1}^\infty a_j{}^{-\frac{q}{2}} |(a,b) \cap (10^j, 10^j + a_j)| \le 1.$$

Consequently

$$\begin{aligned}\|f\|_{\mathcal{M}^p_q} &\le 1+\sup_{0<2a<b<\infty}(b-a)^{\frac{1}{2}-\frac{1}{q}}\left(\sum_{j=1}^{\infty}\frac{1}{{a_j}^{\frac{q}{2}}}|(a,b)\cap(10^j,10^j+a_j)|\right)^{\frac{1}{q}}\\ &\le 1+2^{\frac{1}{q}-\frac{1}{2}}\sup_{0<b<\infty}b^{\frac{1}{2}-\frac{1}{q}}\left(\sum_{j=1}^{\infty}\frac{1}{{a_j}^{\frac{q}{2}}}|(0,b)\cap(10^j,10^j+a_j)|\right)^{\frac{1}{q}}\\ &\le 1+2^{\frac{1}{q}-\frac{1}{2}}\sup_{1\le b<\infty}b^{\frac{1}{2}-\frac{1}{q}}\left(\sum_{j=1}^{\infty}\chi_{(0,b+3)}(10^j+a_j)\right)^{\frac{1}{q}}\\ &\le 1+2^{\frac{1}{q}-\frac{1}{2}}\sup_{1\le b<\infty}b^{\frac{1}{2}-\frac{1}{q}}(\log_{10}(b+3))^{\frac{1}{q}}<\infty.\end{aligned}$$

Thus, we have

$$f\in\mathcal{M}^p_q(\mathbb{R}) \tag{1.22}$$

Meanwhile, a direct calculation shows that:

$$\int_E f(t)\mathrm{d}t=\sum_{j=1}^{\infty}(4^{-j})^{\frac{1}{2}}\cdot 2^j=\infty. \tag{1.23}$$

Thus from (1.21)–(1.23) we see that (P5) fails.

It should be noted that the function f above is not expressed as the sum of $L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$ since $(f-h)\chi_{\mathrm{supp}(f)}\notin L^1(\mathbb{R}^n)$ for any $h\in L^1(\mathbb{R}^n)$.

Motivated by Example 21, we present the following definition:

Definition 14 (Ball Banach function norm)**.** A mapping $\rho:\mathbb{M}^+(\mathbb{R}^n)\to[0,\infty]$ is called a *ball Banach function norm* (over $\mathbb{R}^n$) if, for all f,g,f_k $(k=1,2,3,\dots)$ in $\mathbb{M}^+(\mathbb{R}^n)$, for all constants $a\ge 0$ and for all balls B in $\mathbb{R}^n$, the following properties hold:

(P1) $\rho(f)=0\Leftrightarrow f=0$ a.e.; $\rho(af)=a\rho(f)$, $\rho(f+g)\le\rho(f)+\rho(g)$,

(P2) $\rho(g)\le\rho(f)$ if $0\le g\le f$ a.e.,

(P3) the Fatou property; $\rho(f_j)\uparrow\rho(f)$ holds whenever $0\le f_j\uparrow f$ a.e.,

(P4)′ $\rho(\chi_B)<\infty$,

(P5)′ $\|\chi_B f\|_{L^1}\lesssim_B\rho(f)$ with the implicit constant depending on B and ρ but independent of f.

Accordingly, the space generated by such ρ is called a ball Banach function space. The notions of quasi-ball Banach function norms and quasi-ball Banach function spaces are defined analogously by replacing (P1) with (P1)′ in Definition 12.

Example 22. When $1 \le q \le p < \infty$, then $\mathcal{M}^p_q(\mathbb{R}^n)$ is a ball Banach function space as is seen from $\|\chi_B\|_{\mathcal{M}^p_q} = |B|^{\frac{1}{p}}$ and $\|f\|_{L^1(B)} \le |B|^{1-\frac{1}{p}}\|f\|_{\mathcal{M}^p_q}$.

It is noteworthy that mixed Lebesgue spaces are not Banach function spaces but that they are ball Banach function spaces [470]. See Exercise 19.

1.2.7 Exercises

Exercise 11. Let $0 < q < u \le p < \infty$.

(1) [393, Proposition 2.3] Let F_j be the set in Example 11. Then show that $\|\chi_{F_j}\|_{\mathcal{M}^p_u} \ge \left(\frac{1+R}{2}\right)^{\frac{jn}{q}-\frac{jn}{u}}$ by considering the convex hull of F_j.

(2) [393, Proposition 2.5] Show that $\bigcup_{q<r\le p} \mathcal{M}^p_r(\mathbb{R}^n)$ is not closed in $\mathcal{M}^p_q(\mathbb{R}^n)$ by mimicking the proof of Theorem 20.

(2) [393, Proposition 3.1] Show that $\bigcup_{q<r\le p} \mathcal{M}^p_r(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$ by mimicking the proof of Theorem 20.

Exercise 12. [265, Lemma 2] Let $0 < q \le p < \infty$. Let E_j be the set constructed in Example 10, so that $|E_j| = 2^{-jn}[2^{jn(1-\frac{q}{p})}]$ and that each E_j contain $[0, 2^{-j})^n$. Then for a positive sequence $\{a_j\}_{j=1}^\infty$, the following are equivalent:

(1) $\sum_{j=1}^\infty a_j \chi_{E_j} \in L^q(\mathbb{R}^n)$.

(2) $\sum_{j=1}^\infty (2^{-\frac{jn}{p}} a_j)^q < \infty$.

Exercise 13. Let $f \in L^0(\mathbb{R}^n)$, and let $0 < q \le p < \infty$. Recall that the Morrey norm $\|f\|_{\mathcal{M}^p_q} = \|f\|^{\mathcal{B}}_{\mathcal{M}^p_q}$ is defined by Definition 8. Define

$$\|f\|^{\mathcal{Q}}_{\mathcal{M}^p_q} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |Q(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q(x,r)} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}. \tag{1.24}$$

Then show that $\|f\|_{\mathcal{M}^p_q} \sim \|f\|^{\mathcal{Q}}_{\mathcal{M}^p_q}$ for all $f \in L^0(\mathbb{R}^n)$.

Exercise 14.

(1) Which is true, $\mathcal{M}^4_2(\mathbb{R}^n) \subset L^4(\mathbb{R}^n)$, $\mathcal{M}^4_2(\mathbb{R}^n) \supset L^4(\mathbb{R}^n)$?

(2) Which is true for any $f \in L^0(\mathbb{R}^n)$, $\|f\|_{\mathcal{M}^4_2} \le \|f\|_4$, $\|f\|_{\mathcal{M}^4_2} \ge \|f\|_4$?

Exercise 15. Let $r > 1$ and $1 \le q \le p < \infty$. Calculate $\frac{\| |f|^r \|_{\mathcal{M}^p_q}}{(\|f\|_{\mathcal{M}^{pr}_{qr}})^r}$ for $f \in \mathcal{M}^{pr}_{qr}(\mathbb{R}^n) \setminus \{0\}$.

Exercise 16. We define the quantity $\|f\|_{\mathcal{M}^p_q}$ by the following formula:

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_I |I|^{\frac{1}{p}-\frac{1}{q}} \left(\int_I |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}$$

for $f \in L^0(\mathbb{R})$, where I runs over all intervals in $\mathbb{R}$.

(1) Let $f \in L^0(\mathbb{R})$. Write $\|f\|_{\mathcal{M}^4_2}$ out in full.

(2) Let $f_1 \equiv \chi_{[0,1]}$. Show that $\|f_1\|_{\mathcal{M}^4_2} = 1$.

(3) Let $f_2 \equiv \chi_{[0,1]\cup[1,2]}$. Show that $\|f_2\|_{\mathcal{M}^4_2} = \sqrt[4]{2}$.

(4) Let $f_3 \equiv \chi_{[0,1]\cup[2,3]}$. Show that $\|f_3\|_{\mathcal{M}^4_2} = \sqrt[4]{\frac{4}{3}}$.

(5) Let $f_4 \equiv \chi_{[0,1]\cup[3,4]}$. Show that $\|f_4\|_{\mathcal{M}^4_2} = 1$.

(6) Let $f_5 \equiv \chi_{[0,1]\cup[4,5]}$. Show that $\|f_5\|_{\mathcal{M}^4_2} = 1$.

Exercise 17. [18, Proposition] Let $1 \le q \le p < \infty$ and let $g \in L^q(0,1) \cap \mathbb{M}^{\downarrow}(0,1)$. Then by using the idea of the proof of Proposition 21, find a compactly supported function $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ such that $\lambda_f(t) \gtrsim \lambda_g(t)$ for all $t \ge 1$.

Exercise 18. Let $\rho : L^0(\mathbb{R}^n) \to [0,\infty]$ be a ball Banach function norm. Show that $\liminf_{k\to\infty} \rho(f_k) \ge \rho\left(\liminf_{k\to\infty} f_k\right)$ whenever $\{f_k\}_{k=1}^{\infty} \subset \mathbb{M}^+(\mathbb{R}^n)$.

Exercise 19. [470, Remark 7.21] Let $1 \le p_1, p_2, \ldots, p_n \le \infty$ be constants. Suppose that $p_j \ne p_k$ for some $1 \le j < k \le n$. Write $\mathbf{p} \equiv (p_1, p_2, \ldots, p_n)$. Show that the mixed Lebesgue space $L^{\mathbf{p}}(\mathbb{R}^n)$ is not a ball Banach function space. Hint: Suppose $n = 2$ for simplicity and $p_1 = 2$, $p_2 = 1$. In this case, consider $E \equiv \bigcup_{m=1}^{\infty} [m, m+m^{-1}] \times [m, m+m^{-1/2}]$.

1.3 Local Morrey spaces, B_σ-spaces, Herz spaces and Herz–Morrey spaces

Let $0 < q \le p < \infty$. Recall that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all functions $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{\mathcal{M}^p_q} \equiv \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q)}$ is finite. In Section 1.3.1, by introducing local Morrey spaces, we consider what happens if we assume that the center of cubes is the origin. Section 1.3.2 considers equivalent expressions to local Morrey spaces. We will briefly consider function spaces related to local Morrey spaces.

1.3.1 Local Morrey spaces

We will solidify our plan above; we will define local Morrey spaces by fixing the center of the cube in the definition of Morrey norms and then investigate the relation between Morrey spaces and local Morrey spaces. Local Morrey spaces have different names, which we will see in Section 1.3.1.

Definition 15 (Local Morrey norm)**.** Let $0 < q \le p \le \infty$. The *local Morrey norm* $\|\cdot\|_{\mathrm{L}\mathcal{M}^p_q}$ is defined by

$$\|f\|_{\mathrm{L}\mathcal{M}^p_q} \equiv \sup_{r>0} |Q(r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q(r))} \tag{1.1}$$

for $f \in L^0(\mathbb{R}^n)$. The *local Morrey space* $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{\mathrm{L}\mathcal{M}^p_q}$ is finite. Likewise, define the *weak local Morrey space* $\mathrm{WL}\mathcal{M}^p_q(\mathbb{R}^n)$ as the set of all $f \in L^0(\mathbb{R}^n)$ for which the norm $\|f\|_{\mathrm{WL}\mathcal{M}^p_q} = \sup_{\lambda>0} \lambda \|\chi_{(\lambda,\infty]}(|f|)\|_{\mathcal{M}^p_q}$ is finite.

Unlike Morrey spaces, we may tolerate the case where $0 < p, q \le \infty$ or $-\infty < p < 0 < q \le \infty$. However, we feel that it is enough to suppose $0 < q \le p < \infty$ for our main purpose.

Example 23. Let $0 < p, q \le \infty$ or $-\infty < p < 0 < q \le \infty$. Define the local Morrey norm formally by (1.1).

(1) Let $0 < q \le p < \infty$. Any element in $\mathcal{M}^p_q(\mathbb{R}^n)$ belongs to $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.

(2) Let $0 < q \le p < \infty$. Any $L^q(\mathbb{R}^n)$-function supported on a cube which does not contain the origin is an element in $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$. This is a crucial difference from the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$; this is a good contrast to Theorem 18. Let $f \in L^0(\mathbb{R}^n)$ be supported on $B(2) \setminus B(1)$, for example. Then $f \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $f \in L^q(\mathbb{R}^n)$. If $f \in L^q_{\mathrm{c}}(\mathbb{R}^n)$, $0 < q < \infty$, is supported away from the origin, then $f \in \mathrm{L}\mathcal{M}^\infty_q(\mathbb{R}^n)$.

(3) If $0 < p < q \le \infty$ or $-\infty < p < 0 < q \le \infty$, then $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n) = \{0\}$, which is similar to $\mathcal{M}^p_q(\mathbb{R}^n) = \{0\}$.

We collect some elementary inclusion properties of local Morrey spaces.

Proposition 25.

(1) *Let $0 < p < \infty$. Then $\mathrm{L}\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with coincidence of norms.*

(2) *If $0 < q_1 \le q_2 \le p < \infty$, then $\mathrm{L}\mathcal{M}^p_{q_2}(\mathbb{R}^n) \overset{1}{\hookrightarrow} \mathrm{L}\mathcal{M}^p_{q_1}(\mathbb{R}^n)$.*

(3) *Let $1 \le q \le p < \infty$. Then $\mathrm{WL}\mathcal{M}^p_q(\mathbb{R}^n)$ and $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ are complete. We have the following inclusions*:

$$\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n) \overset{1}{\hookrightarrow} \mathrm{WL}\mathcal{M}^p_q(\mathbb{R}^n), \quad \mathcal{M}^p_q(\mathbb{R}^n) \overset{1}{\hookrightarrow} \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n) \overset{1}{\hookrightarrow} \mathrm{WL}\mathcal{M}^p_q(\mathbb{R}^n).$$

(4) *Let* $1 \le q < p < \infty$. *Then* $|\cdot|^{-\frac{n}{p}} \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$

Proof Overall, the proofs are careful reexaminations of the case of Morrey spaces.

(1) Reexamine the proof of Theorem 16.

(2) Reexamine the proof of Theorem 19.

(3) Completeness of these spaces is clear because these spaces are continuously embedded into $\mathrm{W}L^q_{\mathrm{loc}}(\mathbb{R}^n)$. Embeddings follow from the definition of the norms.

(4) This is a simple consequence of the embedding $\mathcal{M}^p_q(\mathbb{R}^n) \overset{1}{\hookrightarrow} \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ and Example 6.

Some people call local Morrey spaces B_σ spaces. This simply reformulates the definition.

Definition 16. Let E be a ball quasi-Banach function space. Then, for $\sigma \in [0,\infty)$, we define the *E-based nonhomogeneous B_σ space* $B_\sigma(E)(\mathbb{R}^n)$ and the *E-based homogeneous B_σ space* $\dot{B}_\sigma(E)(\mathbb{R}^n)$ as the sets of $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{B_\sigma(E)} \equiv \sup_{r\ge 1} r^{-\sigma}\|f\chi_{Q(r)}\|_E < \infty$ and $\|f\|_{\dot{B}_\sigma(E)} \equiv \sup_{r>0} r^{-\sigma}\|f\chi_{Q(r)}\|_E < \infty$, respectively.

Example 24. As the candidate of $E(\mathbb{R}^n)$, we can list the following function spaces:

(1) $E = L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. In this case the corresponding function spaces $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$ are written $B_\sigma(L^p)(\mathbb{R}^n)$ and $\dot{B}_\sigma(L^p)(\mathbb{R}^n)$ and are called a *homogeneous B_σ-L^p space* and a *nonhomogeneous B_σ-L^p space*, respectively. It is noteworthy that

$$\dot{B}_{\frac{n}{q}-\frac{n}{p}}(L^q)(\mathbb{R}^n) \approx \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$$

with equivalence of norms for any $1 < q \le p < \infty$. We define

$$\|f\|_{\dot{B}^{p,\lambda}} \equiv \sup_{r>0} r^{-\lambda}\left(\frac{1}{|Q(r)|}\int_{Q(r)} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

and

$$\|f\|_{B^{p,\lambda}} \equiv \sup_{r\ge 1} r^{-\lambda}\left(\frac{1}{|Q(r)|}\int_{Q(r)} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

(2) $E = \mathrm{W}L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. In this case the corresponding function spaces $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$ are written $B_\sigma(\mathrm{W}L^p)(\mathbb{R}^n)$ and $\dot{B}_\sigma(\mathrm{W}L^p)(\mathbb{R}^n)$ and are called a homogeneous B_σ-weak L^p space and a nonhomogeneous B_σ-weak L^p space, respectively. It is noteworthy that

$$\dot{B}_{\frac{n}{q}-\frac{n}{p}}(\mathrm{W}L^q)(\mathbb{R}^n) \approx \mathrm{WL}\mathcal{M}^p_q(\mathbb{R}^n)$$

with equivalence of norms for any $1 < q \le p < \infty$.

(3) $E = \mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q \le p < \infty$. In this case the corresponding function spaces $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$ are written $B_\sigma(\mathcal{M}^p_q)(\mathbb{R}^n)$ and $\dot{B}_\sigma(\mathcal{M}^p_q)(\mathbb{R}^n)$ and are called a homogeneous B_σ-Morrey space and a nonhomogeneous B_σ-Morrey space, respectively.

(4) $E = \mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q \le p < \infty$. In this case the corresponding function spaces $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$ are written $B_\sigma(\mathrm{W}\mathcal{M}^p_q)(\mathbb{R}^n)$ and $\dot{B}_\sigma(\mathrm{W}\mathcal{M}^p_q)(\mathbb{R}^n)$ and are called a homogeneous B_σ-weak Morrey space and a nonhomogeneous B_σ-weak Morrey space, respectively.

Before we conclude this section, we give an example of an element in $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.

Example 25. Let $0 < q \le p < \infty$ and $0 < u < \infty$ and define

$$f(x) \equiv |x - 3\mathbf{e}_1|^{-\frac{n}{u}} \chi_{B(3\mathbf{e}_1,2)}(x) \quad (x \in \mathbb{R}^n).$$

Then f belongs to $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $u < q$.

1.3.2 Herz spaces and Herz–Morrey spaces

We also obtain a norm equivalent to the local Morrey norm $\|\cdot\|_{\mathrm{L}\mathcal{M}^p_q}$. In fact, we will show that Herz spaces are isomorphic to local Morrey spaces. To define Herz spaces, we set up notation. Write $Q_0 \equiv Q(1)$ and $C_j \equiv Q(2^j) \setminus Q(2^{j-1})$ for $j \in \mathbb{Z}$.

Here, we content ourselves with the definition of Herz spaces and a comparison with local Morrey spaces. To simplify the notation, we use the following norm: For a system $\{f_j\}_{j\in J} \subset L^0(\mathbb{R}^n)$ of functions, define

$$\|f_j\|_{\ell^q(L^p)} \equiv \|\{f_j\}_{j\in J}\|_{\ell^q(L^p)} = \left(\sum_{j\in J} \|f_j\|_{L^p}{}^q\right)^{\frac{1}{q}}.$$

The space $\ell^q(L^p, \mathbb{R}^n)$ is the set of all collections $\{f_j\}_{j=1}^\infty$ for which the quantity $\|\{f_j\}_{j\in J}\|_{\ell^q(L^p)}$ is finite.

Definition 17. Let $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$.

(1) The *nonhomogeneous Herz space* $K^\alpha_{pq}(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which the following norm $\|f\|_{K^\alpha_{pq}} \equiv \|\chi_{Q_0} f\|_{L^p} + \|\{2^{j\alpha}\chi_{C_j} f\}_{j=1}^\infty\|_{\ell^q(L^p)}$ is finite.

(2) The *homogeneous Herz space* $\dot{K}^\alpha_{pq}(\mathbb{R}^n)$ collects all $f \in L^0(\mathbb{R}^n)$ for which the following norm $\|f\|_{\dot{K}^\alpha_{pq}} \equiv \|\{2^{j\alpha}\chi_{C_j} f\}_{j=-\infty}^\infty\|_{\ell^q(L^p)}$ is finite.

Example 26. Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q < \infty$.

(1) $\chi_{[1,2]^n} \in \dot{K}^\alpha_{pq}(\mathbb{R}^n) \cap K^\alpha_{pq}(\mathbb{R}^n)$.

(2) $\chi_{B(1)} \in \dot{K}^{\alpha}_{pq}(\mathbb{R}^n)$ if and only if either one of the following conditions holds:

(i) $\alpha + \frac{n}{p} \geq 0$ and $0 < q < \infty$.
(ii) $\alpha + \frac{n}{p} = 0$ and $q = \infty$.

(2) $\chi_{B(1)} \in K^{\alpha}_{pq}(\mathbb{R}^n)$.

(3) $1 - \chi_{B(1)} \in \dot{K}^{\alpha}_{pq}(\mathbb{R}^n)$ if and only if either one of the following conditions holds:

(i) $\alpha + \frac{n}{p} \leq 0$ and $0 < q < \infty$.
(ii) $\alpha + \frac{n}{p} = 0$ and $q = \infty$.

(4) $1 \in \dot{K}^{\alpha}_{pq}(\mathbb{R}^n)$ if and only if $\alpha + \frac{n}{p} = 0$ and $q = \infty$.

(5) Let $\beta \in \mathbb{R}$. $|\cdot|^{\beta} \in \dot{K}^{\alpha}_{pq}(\mathbb{R}^n)$ if and only if $\alpha + \beta + \frac{n}{p} = 0$ and $q = \infty$.

One of the important things is that Herz spaces and local Morrey spaces are isomorphic as the following theorem shows. So, we can transplant mutually the results for these function spaces [126].

Theorem 26. *Let* $0 < q < p < \infty$. *Write* $\alpha \equiv \dfrac{n}{p} - \dfrac{n}{q} < 0$. *Then*

$$\dot{K}^{\alpha}_{p\infty}(\mathbb{R}^n) \approx \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$$

with equivalent norms.

Proof It is easy to see that $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n) \subset \dot{K}^{\alpha}_{p\infty}(\mathbb{R}^n)$. Let $f \in \dot{K}^{\alpha}_{p\infty}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f\|_{\mathrm{L}\mathcal{M}^p_q} &= \sup_{r>0} |Q(r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q(r))} \\
&\leq \sup_{r>0} \sum_{j=-\infty}^{0} |Q(r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q([-2^j r, 2^j r]^n \setminus [-2^{j-1} r, 2^{j-1} r]^n)} \\
&\leq \sum_{j=-\infty}^{0} 2^{j\frac{n}{q} - j\frac{n}{p}} \|f\|_{\dot{K}^{\alpha}_{p\infty}} \\
&\sim \|f\|_{\dot{K}^{\alpha}_{p\infty}}.
\end{aligned}$$

Motivated by Theorem 26, we define the *base* $\mathrm{L}\mathcal{Q}$ of cubes by

$$\mathrm{L}\mathcal{Q} \equiv \{Q \in \mathcal{D} : \mathrm{dist}_{\infty}(\{0\}, Q) = \ell(Q)\}. \tag{1.2}$$

A *local cube* is a cube in $\mathrm{L}\mathcal{Q}$. A direct consequence of Theorem 26 is that $\|f\|_{\mathrm{L}\mathcal{M}^p_q} \sim \sup_{Q \in \mathrm{L}\mathcal{Q}} u(Q)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^q}$ for any $f \in L^0(\mathbb{R}^n)$ whenever $1 < q < p < \infty$.

As a similar object, we consider Herz–Morrey spaces. Here, we content ourselves with the definition and some examples of functions in Herz–Morrey spaces.

Definition 18. Let $0 < p, q \le \infty$ and $\alpha, \lambda \in \mathbb{R}$. The *Herz–Morrey space* $MK^{\alpha,\lambda}_{pq}(\mathbb{R}^n)$ is defined as the set of all $f \in L^0(\mathbb{R}^n)$ for which the following norm

$$\|f\|_{MK^{\alpha,\lambda}_{pq}} \equiv \|\chi_{Q_0} f\|_{L^p} + \sup_{L\in\mathbb{N}_0} 2^{-L\lambda} \left(\sum_{j=1}^{L} (2^{j\alpha} \|\chi_{C_j} f\|_{L^p})^q \right)^{\frac{1}{q}}$$

is finite.

Example 27. Let $0 < p, q \le \infty$ and $\alpha, \lambda \in \mathbb{R}$. Let $f(x) \equiv |x|^\beta \chi_{\mathbb{R}^n \setminus B(1)}(x)$, $x \in \mathbb{R}^n$ with $\beta \in \mathbb{R}^n$. Then $2^{j\alpha} \|\chi_{C_j} f\|_{L^p} \sim 2^{j(\alpha+\beta+\frac{n}{p})}$.

(1) If $\alpha + \beta + \frac{n}{p} > 0$, then $f \in MK^{\alpha,\lambda}_{pq}(\mathbb{R}^n)$ if and only if $\lambda \ge \alpha + \beta + \frac{n}{p}$.

(2) If $\alpha + \beta + \frac{n}{p} = 0$, then $f \in MK^{\alpha,\lambda}_{pq}(\mathbb{R}^n)$ if and only if $\lambda > 0$.

(3) If $\alpha + \beta + \frac{n}{p} \le 0$, then $f \in MK^{\alpha,\lambda}_{pq}(\mathbb{R}^n)$ if and only if $\lambda \ge 0$.

1.3.3 Exercises

Exercise 20. Let $1 \le p \le \infty$ and $\sigma \ge 0$.

(1) Show that $\chi_{Q(1)} \in B_\sigma(L^p)(\mathbb{R}^n)$ for all $\sigma \ge 0$ and $1 \le p \le \infty$.

(2) Show that $\chi_{Q(1)} \in \dot{B}_\sigma(L^p)(\mathbb{R}^n)$ if and only if $\sigma \le \dfrac{n}{p}$.

(3) Show that $\chi_{[1,2]^n} \in B_\sigma(L^p)(\mathbb{R}^n)$ for all $\sigma \ge 0$ and $1 \le p \le \infty$.

(4) Show that $\chi_{[1,2]^n} \in \dot{B}_\sigma(L^p)(\mathbb{R}^n)$ for all $\sigma \ge 0$ and $1 \le p \le \infty$.

Exercise 21. Let $0 < q \le p < \infty$ and $f \in L^0(\mathbb{R}^n)$. Then show that $\|f(t\cdot)\|_{\mathrm{L}\mathcal{M}^p_q} = t^{-\frac{n}{p}} \|f(\cdot)\|_{\mathrm{L}\mathcal{M}^p_q}$ for all $t > 0$.

Exercise 22 (Diversity of local Morrey spaces)**.** Let $0 < q < r \le p < \infty$. Let F be the set defined in Example 11. Then using χ_F, show that $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ and $\mathrm{L}\mathcal{M}^p_r(\mathbb{R}^n)$ are different.

Exercise 23. Let $0 < q < p < \infty$, and let $f(x) = |x|^{-\frac{n}{p}}$.

(1) Show that $\|f\|_{\mathrm{L}\mathcal{M}^p_q} = \|f\|_{\mathcal{M}^p_q}$.

(2) Show that $\mathrm{dist}(f, L^\infty_{\mathrm{c}}(\mathbb{R}^n)) = \|f\|_{\mathrm{L}\mathcal{M}^p_q}$.

(3) Conclude that $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is not dense in $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.

Exercise 24. Let $0 < p, q \le \infty$ and $\alpha, \lambda \in \mathbb{R}$. Show that $MK^{\alpha,\lambda}_{pq}(\mathbb{R}^n)$ is a quasi-ball Banach function space.

Exercise 25. Let $0 < q < p < \infty$, and let $f(x) = |x|^{-\frac{n}{p}}$.

(1) Show that there exists an increasing sequence $\{R_j\}_{j=1}^{\infty}$ of positive real numbers such that $R_1 = 1$, $\|f\chi_{B(R_{j+1})\setminus B(R_j)}\|_{\mathrm{L}\mathcal{M}^p_q} \geq (1-2^{-j})\|f\|_{\mathrm{L}\mathcal{M}^p_q}$.

(2) Define

$$\Phi : \{a_j\}_{j=1}^{\infty} \in \ell^{\infty}(\mathbb{N}) \mapsto \sum_{j=1}^{\infty} a_j f\chi_{B(R_{j+1})\setminus B(R_j)} \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n).$$

Establish that $\|\Phi\|_{\ell^{\infty}(\mathbb{N})\to\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)} = 1$.

(3) Define

$$\Psi : \{a_j\}_{j=1}^{\infty} \in \ell^{\infty}(\mathbb{N}) \mapsto \sum_{j=1}^{\infty} a_j f\chi_{B(R_{j+1})\setminus B(R_j)} \in \mathcal{M}^p_q(\mathbb{R}^n).$$

Establish that $\|\Psi\|_{\ell^{\infty}(\mathbb{N})\to\mathcal{M}^p_q(\mathbb{R}^n)} = 1$.

1.4 Distributions and Lorentz spaces

To analyze operators appearing in many problems, the pointwise estimate often fails to work well. In view of the Layer-Cake formula theorem, it seems to make sense to analyze $\mu\{x \in X : |f(x)| > \lambda\}$ for each λ. Hence, we are interested in the size of this set.

Section 1.4 collects the way of measuring the size of functions using this type of sets in Section 1.4.2; Lorentz spaces are defined there. To this end we define distributions in Section 1.4.1. Keeping the structure of the (quasi-)norm defined in Section 1.4.2, we investigate the Hardy operator in Section 1.4.3. Since the decreasing rearrangement is a decreasing function, we collect some useful inequalities for monotone functions and apply it to Lorentz spaces in Section 1.4.4.

1.4.1 Distribution function

For any $0 < p < \infty$ $|\cdot|^{-\frac{n}{p}} \notin L^p(\mathbb{R}^n)$. On many occasions, this will be a problem because $|\cdot|^{-\frac{n}{p}}$ appears in many places. To overcome this problem, we seek to find function spaces. Lorentz spaces can be used for this purpose. We need the distribution function to define Lorentz spaces. Hence, we define the distribution function of measurable functions and then we collect some examples.

As a setup, recall the definition of $\lambda_f = \lambda_{f,\mu}$ for $f \in L^0(\mu)$ given in Definition 5 via some examples.

Example 28. Consider the Euclidean space $\mathbb{R}^n$ here. Let $\gamma \neq 0$. Let f_γ and g_γ be as in Example 1. For each $t > 0$, we have:

(1) (*i*) If $\gamma > 0$, then $\lambda_{f_\gamma}(t) = \max(v_n(1 - t^{n/\gamma}), 0)$.

(*ii*) If $\gamma < 0$, then $\lambda_{f_\gamma}(t) = \min(v_n, v_n t^{n/\gamma})$.

(2) (*i*) If $\gamma > 0$, then $\lambda_{g_\gamma}(t) = \infty$.

(*ii*) If $\gamma < 0$, then $\lambda_{g_\gamma}(t) = \max(v_n(t^{n/\gamma} - 1), 0)$

(3) $\lambda_{\chi_{[0,1]}}(t) = \chi_{[0,1)}(t)$.

Example 29. Let $E_1, E_2, \ldots, E_N$ be a sequence of disjoint measurable sets. Also let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N$. Then if we set $f \equiv \sum_{j=1}^{N} \alpha_j \chi_{E_j}$, then

$$f^* = \alpha_N \chi_{(0,\mu(E_N))} + \sum_{k=1}^{N-1} \alpha_k \chi_{[\mu(E_N \cup E_{N-1} \cup \cdots \cup E_{k+1}), \mu(E_N \cup E_{N-1} \cup \cdots \cup E_k))}.$$

Let I be an interval. By $\mathbb{M}^{\downarrow}(I)$ we mean the set of all non-negative non-increasing functions defined on I and by $\mathbb{M}^{\uparrow}(I)$ we mean the set of all non-negative nondecreasing functions defined on I. We collect a fundamental property of λ_f.

Theorem 27. *Let $f \in L^0(\mu)$. Then the distribution function $\lambda_f = \lambda_{f,\mu}$ belongs to $\mathbb{M}^{\downarrow}(0,\infty)$ and is continuous from the right.*

Proof It follows from Definition 5 that λ_f is decreasing. Let $t_0 \in [0,\infty)$. Then for any decreasing sequence $\{t_k\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty} t_k = t_0$,

$$\{x \in \mathbb{R}^n \,:\, |f(x)| > t_0\} = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^n \,:\, |f(x)| > t_k\}$$

and

$$\{x \in \mathbb{R}^n \,:\, |f(x)| > t_k\} \subset \{x \in \mathbb{R}^n \,:\, |f(x)| > t_{k+1}\} \quad (k \in \mathbb{N}).$$

Thus, we are in the position of using the monotone convergence theorem.

Although the function $f \mapsto \lambda_f(t)$ is not linear for fixed t, we still have the following substitute:

Lemma 28. *Let $f, g \in L^0(\mu)$. Then $\lambda_{f+g}(t_1 + t_2) \leq \lambda_f(t_1) + \lambda_g(t_2)$ for any $t_1, t_2 \geq 0$.*

Proof If $|f(x) + g(x)| > t_1 + t_2$, then $|f(x)| > t_1$ or $|g(x)| > t_2$. Thus $\{|f+g| > t_1 + t_2\} \subset \{|f| > t_1\} \cup \{|g| > t_2\}$. If $\lambda_{f+g}(t_1 + t_2) > \lambda_f(t_1) + \lambda_g(t_2)$, then there would exist $a_1, a_2 \in (0,\infty)$ such that $\lambda_{f+g}(t_1 + t_2) > a_1 + a_2$,

$\lambda_f(t_1) < a_1$ and $\lambda_f(t_2) < a_2$. This means that $\mu\{|f+g| > t_1+t_2\} > a_1+a_2$ and that $\mu\{|f| > t_1\} \le a_1$, $\mu\{|f| > t_2\} \le a_2$. This is a contradiction.

The distribution function is used to define Lorentz spaces. The definition is as follows:

Definition 19. Let $f \in L^0(\mathbb{R}^n)$. Then its *decreasing rearrangement* f^* is the function defined on $(0,\infty)$ by

$$f^*(t) \equiv \inf(\{s \in [0,\infty) : \lambda_f(s) \le t\} \cup \{\infty\}) \quad (t > 0).$$

We have two possibilities in the above since λ_f is monotone:

$$\{s \in [0,\infty) : \lambda_f(s) \le t\} \cup \{\infty\} = (f^*(t), \infty] \text{ or } [f^*(t), \infty].$$

It is also clear from the definition of distribution functions that $f^*(t) \le g^*(t)$ if $f, g \in L^0(\mathbb{R}^n)$ satisfy $|f| \le |g|$.

We consider a couple of examples.

Example 30. Let $\gamma < 0$. Let f_γ and g_γ be as in Example 1. Then

$$f_\gamma^*(t) = \chi_{[0,v_n)}(t)\left(\frac{v_n}{t}\right)^{\gamma/n} \quad (0 < t < \infty)$$

and

$$g_\gamma^*(t) = \left(\frac{t}{v_n}+1\right)^{\gamma/n} \quad (0 < t < \infty).$$

Example 31. Since $\lambda_{\chi_{[0,1]}} = \chi_{[0,1)}$,

$$\{s \in [0,\infty) : \lambda_{\chi_{[0,1]}}(s) \le t\} = \begin{cases} [1,\infty) & 0 < t < 1, \\ [0,\infty) & t \ge 1. \end{cases}$$

Thus $(\chi_{[0,1]})^* = \chi_{(0,1)}$.

Example 32. Let f be as in Example 32. Then for all $t > 0$,

$$\left(\sum_{j=1}^N \alpha_j \chi_{E_j}\right)^*(t)$$
$$= \alpha_N \chi_{(0,\mu(E_N))} + \sum_{k=1}^{N-1} \alpha_k \chi_{[\mu(E_N \cup E_{N-1} \cup \cdots \cup E_{k+1}), \mu(E_N \cup E_{N-1} \cup \cdots \cup E_k))}.$$

The following lemma shows the relation between f^* and λ_f:

Lemma 29. *Let $f \in L^0(\mu)$. Then for any $t > 0$, $\lambda_f(f^*(t)) \le t$, namely, $\mu\{x \in X : |f(x)| > f^*(t)\} \le t$.*

Proof We may assume that $f^*(t) < \infty$; otherwise the conclusion follows readily from $0 = \lambda_f(\infty) \le t$. Suppose $f^*(t) < \infty$. Then, since $f^*(t) = \inf\{s \in [0,\infty) : \lambda_f(s) \le t\}$, we have $\lambda_f(s) \le t$ for all $s > f^*(t)$. Therefore, by Theorem 27, $\lambda_f(f^*(t)) = \lim_{s \downarrow f^*(t)} \lambda_f(s) \le t$. Thus, the proof is complete.

An analogy to Lemma 29 holds. Note that the inequality reverses.

Lemma 30. *For any $t > 0$ and $f \in L^0(\mu)$, $\mu\{x \in X : |f(x)| \ge f^*(t)\} \ge t$.*

Proof Since $f^*(t) = \inf(\{s \in [0,\infty) : \lambda_f(s) \le t\} \cup \{\infty\})$, we have $\lambda_f(s) \ge t$ for all $s < f^*(t)$. Therefore

$$\mu\{x \in X : |f(x)| \ge f^*(t)\} = \lim_{s \uparrow f^*(t)} \mu\{x \in X : |f(x)| > s\} \ge t.$$

Thus, the proof is complete.

Lemmas 29 and 30 are important when we consider distributions of functions.

The following proposition together with Example 32 can be used to approximate the distribution functions.

Proposition 31. *Let $\{f_j\}_{j=1}^{\infty} \subset L^0(\mu)$ satisfy $0 \le f_j \le f_{j+1}$. Then $f \equiv \lim_{j\to\infty} f_j$ satisfies $f^*(t) = \lim_{j\to\infty} (f_j)^*(t)$ for all $t > 0$.*

Proof It is by the monotonicity clear that $f^*(t) \ge \lim_{j\to\infty} (f_j)^*(t)$. So, we need to establish that $f^*(t) \le \lim_{j\to\infty} (f_j)^*(t)$. Assume otherwise: $a = f^*(t) - \lim_{j\to\infty} (f_j)^*(t) > 0$. Then $f^*(t) - a \notin \{s \in [0,\infty) : \lambda_f(s) \le t\}$, or equivalently, $\lambda_f(f^*(t) - a) > t$. Consequently, there exists $b > 0$ such that $\lambda_f(f^*(t) - a) > b + t$. Thus, $\lambda_{f_j}(f^*(t) - a) > b + t$ by the monotone convergence theorem as long as $j \gg 1$. Consequently, $\lambda_{f_j}((f_j)^*(t)) > b + t$ for such j. This is a contradiction to Lemma 29.

Example 33. Let $f \in \mathbb{M}^{\downarrow}(0,\infty)$. Then we can approximate f almost everywhere with simple decreasing functions. Since f_j is a simple decreasing function, then $(f_j)^* = f_j$ as is seen from Example 32. Thanks to Proposition 31, $f^* = f$ almost everywhere.

In a special case like Example 30, we can regard λ_f as the inverse of f^*.

Proposition 32. *Let $f \in L^0(\mu)$. Assume that λ_f is positive, strictly decreasing and continuous. Then;*

(1) *$\lambda_f^{-1}(t) = f^*(t)$ for all $t > 0$,*

(2) *$\lambda_f(f^*(t)) = t$ for all $t > 0$.*

Proof

(1) This is clear since

$$f^*(t) = \inf(\{s \in [0,\infty) : \lambda_f(s) \le t\}) = \inf(\{s \in [0,\infty) : s \le \lambda_f^{-1}(t)\}).$$

(2) Simply observe that $\lambda_f(\lambda_f^{-1}(t)) = t$ and use (1).

In general, it may happen that $\lambda_f(f^*(t)) \neq t$ as the example of the function $f = \chi_{[0,2]}$ and the point $t = 1$ shows.

The distribution is important because we can express the L^p norm in terms of distributions.

Theorem 33. *For any $f \in L^0(\mu)$ and $0 < p \leq \infty$, $\|f\|_{L^p(\mu)} = \|f^*\|_{L^p(0,\infty)}$.*

Proof Simply consider the case where f is a simple function. In this case, the proof is direct.

As an application of Theorem 33, we justify the name of $\mathrm{W}L^p(\mu)$:

Lemma 34. *Let $0 < p < \infty$. Then $L^p(\mu) \overset{1}{\hookrightarrow} \mathrm{W}L^p(\mu)$.*

Proof Let $f \in L^p(\mu)$. According to Theorem 33, we have

$$\|f\|_{L^p} = \left(\int_0^\infty f^*(s)^p \mathrm{d}s\right)^{\frac{1}{p}} \geq \left(\int_0^t f^*(s)^p \mathrm{d}s\right)^{\frac{1}{p}}.$$

Since $f^* \in \mathbb{M}^{\downarrow}(0,\infty)$, we have

$$\|f\|_{L^p} \geq \left(\int_0^t f^*(t)^p \mathrm{d}s\right)^{\frac{1}{p}} = t^{\frac{1}{p}} f^*(t).$$

Hence, $\|f\|_{L^p} \geq \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) = \|f\|_{\mathrm{W}L^p}$. Thus, the proof is complete.

The following example is a useful application of Theorem 33.

Example 34. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let E be a measurable subset of X. Then $\|\chi_E f\|_{L^1(\mu)} \leq \displaystyle\int_0^{|E|} f^*(s)\mathrm{d}s$ for $g \in L^0(\mu)$. In fact,

$$\begin{aligned}
\int_E |f(x)|\mathrm{d}\mu(x) &= \int_X \chi_E(x)|f(x)|\mathrm{d}\mu(x) \\
&= \int_0^\infty (\chi_E f)^*(t)\mathrm{d}t \\
&= \int_0^{|E|} (\chi_E f)^*(t)\mathrm{d}t \\
&\leq \int_0^{|E|} f^*(s)\mathrm{d}s.
\end{aligned}$$

To conclude Section 1.4.1, we state an inequality connecting f^* and g^* with $(f+g)^*$.

Proposition 35. *For any $t_1, t_2 > 0$ and $f, g \in L^0(\mu)$, $(f+g)^*(t_1+t_2) \le f^*(t_1) + g^*(t_2)$. In particular, for $t > 0$*

$$(f+g)^*(t) \le f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right).$$

Proof By Lemma 29, $\lambda_f(f^*(t_1)) \le t_1$ and $\lambda_g(g^*(t_2)) \le t_2$. Thanks to Lemma 28, $\lambda_{f+g}(f^*(t_1) + g^*(t_2)) \le t_1 + t_2$. Hence

$$f^*(t_1) + g^*(t_2) \in \{s \in [0,\infty) : \lambda_{f+g}(s) \le t_1 + t_2\},$$

which implies $f^*(t_1) + g^*(t_2) \le (f+g)^*(t_1+t_2)$. Thus, the proof is complete.

1.4.2 Lorentz spaces

Section 1.4.2 works in a measure space $(X, \mathcal{B}, \mu)$ because we can readily pass our definitions and our results to $(X, \mathcal{B}, \mu)$. As a generalization of weak Lebesgue-spaces, we also define Lorentz spaces. We will see that Lorentz spaces generalize Lebesgue spaces.

Definition 20.

(1) Let $0 < p < \infty$ and $0 < q < \infty$. Then the *Lorentz space* $L^{p,q}(\mu)$ is the set of all $f \in L^0(\mu)$ for which the quasi-norm

$$\|f\|_{L^{p,q}(\mu)} \equiv \left\{ \int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}}$$

is finite.

(2) The space $L^{q,\infty}(\mu)$ stands for $L^\infty(\mu)$ for any $0 < q \le \infty$.

(3) If $0 < p < \infty$, then the *Lorentz space* $L^{p,\infty}(\mu)$ denotes the weak $L^p(\mu)$-space: $L^{p,\infty}(\mu) = \mathrm{W}L^p(\mu)$.

As usual if $(X, \mathcal{B}, \mu)$ is the Lebesgue measure, then we write $L^{p,q}(\mathbb{R}^n) = L^{p,q}(\mu)$ and $\|\cdot\|_{\mathrm{W}L^{p,q}} = \|\cdot\|_{\mathrm{W}L^{p,q}(\mu)}$.

Theorem 33 shows that $L^{p,p}(\mu) = L^p(\mu)$ with coincidence of norms. Indeed,

$$\|f\|_{L^{p,p}} = \left\{ \int_0^\infty (t^{\frac{1}{p}} f^*(t))^p \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{p}} = \left(\int_0^\infty f^*(t)^p \mathrm{d}t \right)^{\frac{1}{p}} = \|f\|_{L^p}.$$

When we consider integral inequalities in $(0,\infty)$, we frequently face the following norm:

Definition 21 ($\Phi_{\lambda,q}(0,\infty)$)**.**

(1) Let $\lambda \in \mathbb{R}$ and $0 < q < \infty$. Then $\Phi_{\lambda,q}(0,\infty)$ denotes the set of all $\varphi \in \mathbb{M}^+(0,\infty)$ for which

$$\|\varphi\|_{\Phi_{\lambda,q}(0,\infty)} \equiv \left\{ \int_0^\infty (t^{-\lambda}\varphi(t))^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}} < \infty.$$

(2) Let $\lambda \in \mathbb{R}$. Then $\Phi_{\lambda,\infty}(0,\infty)$ denotes the set of all $\varphi \in \mathbb{M}^+(0,\infty)$ for which $\|\varphi\|_{\Phi_{\lambda,\infty}(0,\infty)} \equiv \sup_{t>0} t^{-\lambda}\varphi(t) < \infty$.

In terms of Definition 21, $\|f\|_{L^{p,q}} = \|f^*\|_{\Phi_{p^{-1},q}(0,\infty)}$ for all $f \in L^0(\mathbb{R}^n)$.

Example 35. Let $\lambda, a \in \mathbb{R}$ and $0 < q < \infty$.

(1) $\chi_{[1,2]} \in \Phi_{\lambda,q}(0,\infty)$.

(2) $f(t) \equiv t^a, t > 0$ never belongs to $\Phi_{\lambda,q}(0,\infty)$.

(3) $g(t) \equiv e^{-t}$ belongs to $\Phi_{\lambda,\mu}(0,\infty)$ if and only if $\lambda < 0$.

(4) If $f \in \mathbb{M}^{\downarrow}(0,\infty)$, then $f^*(t) = f(t)$ for all $t > 0$, so that $\|f\|_{L^{p,q}(0,\infty)} = \left\{ \int_0^\infty (t^{\frac{1}{p}} f(t))^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}}$.

Example 36. Let $q < \infty$ and $\lambda < 0$. If $\varphi \in \mathbb{M}^{\downarrow}(0,\infty)$ and $\varphi(t_0) > 0$ for some $t_0 > 0$, then $\|\varphi\|_{\Phi_{\lambda,q}(0,\infty)} = \infty$. Indeed,

$$\|\varphi\|_{\Phi_{\lambda,q}(0,\infty)} \geq \varphi(t_0) \left(\int_{t_0}^\infty t^{-\lambda q - 1} \mathrm{d}t \right)^{\frac{1}{q}} = \infty.$$

So far we do not go into the detail of the theory of Lorentz spaces because Lorentz spaces are not the main objects of this book. However, in the context of real interpolation, Lorentz spaces will appear in this book; see Chapter 18.

1.4.3 Hardy operators and Hardy's inequality

The *Hardy operator*, which is defined by

$$\mathcal{H}f(t) \equiv \frac{1}{t} \int_0^t f(s)\mathrm{d}s \tag{1.1}$$

for suitable $f \in L^0(0,\infty)$, is used to control functions defined on $(0,\infty)$. The Hardy operator will play the role of a toy model of the Hardy–Littlewood maximal operator, which we take up later. This operator will be used to consider the function spaces of local type in this book. Hardy's inequality is a useful inequality that can be used to control some integrals.

The multiplication by monomials in the set $\Phi_{\lambda,q}(0,\infty)$ is described as follows:

Example 37. Let $\lambda, a \in \mathbb{R}$ and $0 < q \leq \infty$, and let $\varphi \in \Phi_{\lambda,q}(0,\infty)$. Then

$$\|\cdot^a \varphi\|_{\Phi_{\lambda,q}(0,\infty)} = \|\varphi\|_{\Phi_{\lambda-a,q}(0,\infty)} = \left(\int_0^\infty \tau^{(\lambda-a)q} \varphi(\tau)^q \frac{\mathrm{d}\tau}{\tau} \right)^{\frac{1}{q}}.$$

Similarly, we have the dilation law.

Example 38. Let $\lambda \in \mathbb{R}$ and $0 < q \le \infty$. Then $\|\varphi(t\cdot)\|_{\Phi_{\lambda,\mu}} = t^{\lambda}\|\varphi\|_{\Phi_{\lambda,\mu}}$ for all $\varphi \in \Phi_{\lambda,\mu}(0,\infty)$ and $t > 0$.

The scaling law of $\Phi_{\lambda,q}(0,\infty)$ is summarized in the following lemma:

Lemma 36. *Let $q, r \in (0,\infty)$, $\lambda_0, \lambda_1, \lambda \in (0,\infty)$ and $0 < \theta < 1$. Suppose that $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$. Then for all $\varphi \in L^0(0,\infty)$, we have*

$$\|\varphi\|_{\Phi_{\lambda,q}(0,\infty)} = \left(\| \cdot^{-\lambda_0 r - 1} |\varphi|^r \|_{\Phi_{\theta(\lambda_1-\lambda_0)r-1,\frac{q}{r}}(0,\infty)}\right)^{\frac{1}{r}}.$$

Proof We calculate

$$\begin{aligned}
&\left(\| \cdot^{-\lambda_0 r - 1} |\varphi|^r \|_{\Phi_{\theta(\lambda_1-\lambda_0)r-1,\frac{q}{r}}(0,\infty)}\right)^{\frac{1}{r}} \\
&= \left(\int_0^\infty (\eta^{-\theta(\lambda_1-\lambda_0)r-\lambda_0 r})^{\frac{q}{r}} \varphi(\eta)^q \frac{\mathrm{d}\eta}{\eta}\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \eta^{-\theta(\lambda_1-\lambda_0)q-\lambda_0 q} \varphi(\eta)^q \frac{\mathrm{d}\eta}{\eta}\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \eta^{-\lambda q} \varphi(\eta)^q \frac{\mathrm{d}\eta}{\eta}\right)^{\frac{1}{q}} \\
&= \|\varphi\|_{\Phi_{\lambda,q}(0,\infty)}.
\end{aligned}$$

We now will consider the space $\Phi^{\uparrow}_{\lambda,q}(0,\infty) \equiv \Phi_{\lambda,q}(0,\infty) \cap \mathbb{M}^{\uparrow}(0,\infty)$. We define $\|f\|_{\Phi_{\lambda,q}(0,\infty)} \equiv \|f\|_{\Phi_{\lambda,q}(0,\infty)}$ for $f \in \Phi^{\uparrow}_{\lambda,q}(0,\infty)$. One of the prominent properties in the scale $\{\Phi_{\lambda,q}(0,\infty)\}_{q>0}$ is the monotonicity in the second parameter q.

Lemma 37. *Let $0 < q_0 < q_1 \le \infty$, and let $0 < \lambda < \infty$. Then $\Phi^{\uparrow}_{\lambda,q_0}(0,\infty) \subset \Phi^{\uparrow}_{\lambda,q_1}(0,\infty)$. More quantitatively, $\|\varphi\|_{\Phi_{\lambda,q_1}(0,\infty)} \le (\lambda q_0)^{\frac{1}{q_0}-\frac{1}{q_1}} \|\varphi\|_{\Phi_{\lambda,q_0}(0,\infty)}$ for all $\varphi \in \Phi^{\uparrow}_{\lambda,q_0}(0,\infty)$.*

Proof First let $q_1 = \infty$. Then

$$\|\varphi\|_{\Phi_{\lambda,\infty}(0,\infty)} = \sup_{t>0} t^{-\lambda}\varphi(t) = (\lambda q_0)^{\frac{1}{q_0}} \sup_{t>0} \left(\int_t^\infty \frac{\mathrm{d}\tau}{\tau^{1+\lambda q_0}}\right)^{\frac{1}{q_0}} \varphi(t).$$

Since $\varphi \in \mathbb{M}^{\uparrow}(0,\infty)$, we have

$$\|\varphi\|_{\Phi_{\lambda,\infty}(0,\infty)} \le (\lambda q_0)^{\frac{1}{q_0}} \sup_{t>0} \left(\int_t^\infty \varphi(\tau)^{q_0} \frac{\mathrm{d}\tau}{\tau^{1+\lambda q_0}}\right)^{\frac{1}{q_0}} = (\lambda q_0)^{\frac{1}{q_0}} \|\varphi\|_{\Phi_{\lambda,q_0}(0,\infty)}.$$

If $q_1 < \infty$, then $\|\varphi\|_{\Phi_{\lambda,q_1}(0,\infty)} \le (\|\varphi\|_{\Phi_{\lambda,\infty}(0,\infty)})^{1-\frac{q_0}{q_1}} (\|\varphi\|_{\Phi_{\lambda,q_0}(0,\infty)})^{\frac{q_0}{q_1}}$ by virtue of Hölder's inequality. Thanks to what we have proven in the case $q_1 = \infty$ $\|\varphi\|_{\Phi_{\lambda,q_1}(0,\infty)} \le (\lambda q_0)^{\frac{1}{q_0}-\frac{1}{q_1}} \|\varphi\|_{\Phi_{\lambda,q_0}(0,\infty)}$, as desired.

We move on to Hardy operators. Let $f \in L^1_{\mathrm{loc}}(0,\infty)$. Then define

$$\mathcal{H}f(t) \equiv \frac{1}{t}\int_0^t f(s)\mathrm{d}s, \quad \mathfrak{H}f(t) \equiv \frac{1}{t}\int_t^\infty f(s)\mathrm{d}s \quad (t>0)$$

as long as the integral makes sense. Both operators are called *Hardy operators*.

Example 39. Since $\mathcal{H}\chi_{(0,1)}(t) = \dfrac{\min(1,t)}{t}$ and $\mathfrak{H}\chi_{(0,1)}(t) = \dfrac{\max(1-t,0)}{t}$ for $t>0$. So $\mathcal{H}$ and $\mathfrak{H}$ fail to be $L^1(0,\infty)$-bounded.

Chapters 7, 13 and 18 use Hardy's inequalities.

Theorem 38 (Hardy's inequality)**.** *Let* $1 \le q \le \infty$.

(1) *For* $g \in \Phi_{\mu,q}(0,\infty)$ *and* $\mu > -1$, $\|\mathcal{H}g\|_{\Phi_{\mu,q}(0,\infty)} \le \dfrac{1}{\mu+1}\|g\|_{\Phi_{\mu,q}(0,\infty)}$.

(2) *For* $g \in \Phi_{-\mu,q}(0,\infty)$ *and* $\mu > 1$, $\|\mathfrak{H}g\|_{\Phi_{-\mu,q}(0,\infty)} \le \dfrac{1}{\mu-1}\|g\|_{\Phi_{-\mu,q}(0,\infty)}$.

Proof We omit the proof of the easy case where $q=\infty$. Also, if $q=1$, there will be no need to use Minkowski's inequality. In this case we use Fubini's theorem instead; see Exercise 31.

(1) We write out what we need to prove:

$$\left\{\int_0^\infty \left(t^{-\mu-1}\int_0^t g(\eta)\mathrm{d}\eta\right)^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \le \frac{1}{\mu+1}\left\{\int_0^\infty t^{-q\mu}g(t)^q\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}.$$

We change variables: $\eta = t\tau$ to obtain

$$\left\{\int_0^\infty \left(t^{-\mu-1}\int_0^t g(\eta)\mathrm{d}\eta\right)^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} = \left\{\int_0^\infty \left(t^{-\mu}\int_0^1 g(t\tau)\mathrm{d}\tau\right)^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}.$$

By Minkowski's inequality we have

$$\left\{\int_0^\infty \left(t^{-\mu-1}\int_0^t g(\eta)\mathrm{d}\eta\right)^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \le \int_0^1 \left\{\int_0^\infty (t^{-\mu}g(t\tau))^q\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}\mathrm{d}\tau.$$

If we change variables once again: $\eta = t\tau$, then

$$\left\{\int_0^\infty \left(t^{-\mu}\int_0^t g(\eta)\mathrm{d}\eta\right)^q \frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}} \le \int_0^1 \tau^\mu \left\{\int_0^\infty (t^{-\mu}g(t))^q\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{q}}\mathrm{d}\tau.$$

Since $\mu > -1$, we obtain the desired result once we integrate the right-hand side against τ.

(2) Let $\nu > -1$ satisfy $-\mu = -\nu - 2$. Then we simply change variables in $\|\mathcal{H}g\|_{\Phi_{\nu,q}} \le \dfrac{1}{1+\nu}\|g\|_{\Phi_{\nu,q}}$. We consider $s = t^{-1}$.

It is important that we do not tolerate the case $\mu = -1$ and $q = 1$. In fact, if there exists the estimate $\|\mathcal{H}g\|_{L^1(0,\infty)} \lesssim \|g\|_{L^1(0,\infty)}$, then this would contradict Example 39.

A quasi-normed space is said to be *normable* if there exists a norm equivalent to the quasi-norm. We provide an application of Hardy's inequality.

Corollary 39. *Let $1 < p < \infty$ and $1 \le q \le \infty$. Let $(X, \mathcal{B}, \mu)$ be a measure space. Then for all $f \in L^0(\mu)$, $\|f\|_{L^{p,q}(\mu)} \sim \|f^{**}\|_{\Phi_{p^{-1},q}(0,\infty)}$, so that $L^{p,q}(\mu)$ is metrizable.*

Proof The norm equivalence is a consequence of Hardy's inequality. Since we can check $(f+g)^{**} \le f^{**} + g^{**}$ for any simple functions f and g, we can pass to a limit to have $(f+g)^{**} \le f^{**} + g^{**}$ for any $f, g \in L^0(\mu)$. Thus, $f \mapsto \|f^{**}\|_{\Phi_{p^{-1},q}(0,\infty)}$ satisfies the triangle inequality.

We end some applications of the idea used in Section 1.4.3.

Example 40. Let $1 < p < \infty$.

(1) Let $K_0 f(t) \equiv \displaystyle\int_0^\infty \frac{f(s)}{s+t}\mathrm{d}s$ for $f \in \mathbb{M}^+(0,\infty)$. Then since for any $t > 0$, $K_0 f(t) \equiv \displaystyle\int_0^\infty \frac{f(ts)}{s+1}\mathrm{d}s$, $\|K_0 f\|_{L^p(0,\infty)} \le \|f\|_{L^p(0,\infty)} \displaystyle\int_0^\infty \frac{\mathrm{d}s}{(s+1)s^{\frac{1}{p}}}$. So K_0 can be extended to a linear operator bounded on $L^p(0,\infty)$.

(2) For $x = (x', x_n) \in \mathbb{R}^n_+$, we write $\tilde{x} \equiv (x', -x_n)$. We define the *reflected singular integral operator* by $Kf(x) \equiv \displaystyle\int_{\mathbb{R}^n_+} \frac{f(y)}{|\tilde{x} - y|^n}\mathrm{d}y$ for $f \in \mathbb{M}^+(\mathbb{R}^n_+)$. If we use the Fubini theorem and the boundedness of K_0 above, we see that K is bounded on $L^p(\mathbb{R}^n_+)$.

1.4.4 Inequalities for monotone functions and their applications to Lorentz norms

We present an idea of discretizing the integral here to prove Chiarenza's lemma.

Lemma 40. (Chiarenza's lemma) *Let $q > 0$, $\rho > 0$ and $0 < \theta < 1$, and let $\mu \in C(0,\infty)$ be such that $\lim_{r \downarrow 0} \mu(r) = 0$ and that μ is strictly increasing. Denote by $\nu = \mu^{-1}$ its inverse. Then the following statements are equivalent*:

(1) *The sum* $K \equiv \displaystyle\sum_{j=0}^\infty \theta^j \log\left(\nu\left(\mu(\rho)\theta^{qj}\right)\right)$ *converges.*

(2) $M \equiv \displaystyle\int_0^\rho \frac{\mu(t)^{\frac{1}{q}}}{t}\,dt < \infty.$

Proof We start with a setup: Write $a_j \equiv \theta^j \log \nu(\mu(\rho)\theta^{qj})$ for each $j \in \mathbb{N}_0$, so that $\theta a_j \le a_{j+1}$.

Suppose that (1) holds. After a change of variables, we decompose

$$\int_0^\rho \frac{\mu(t)^{\frac{1}{q}}}{t}\mathrm{d}t = \int_0^{\mu(\rho)} \frac{s^{\frac{1}{q}}}{\nu(s)\mu'(\nu(s))}\mathrm{d}s = \sum_{j=0}^\infty \int_{\mu(\rho)\theta^{q(j+1)}}^{\mu(\rho)\theta^{qj}} \frac{s^{\frac{1}{q}}}{\nu(s)\mu'(\nu(s))}\mathrm{d}s.$$

We estimate

$$\int_0^\rho \frac{\mu(t)^{\frac{1}{q}}}{t}\mathrm{d}t \le \sum_{j=0}^\infty \mu(\rho)\theta^j \int_{\mu(\rho)\theta^{q(j+1)}}^{\mu(\rho)\theta^{qj}} \frac{1}{\nu(s)\mu'(\nu(s))}\mathrm{d}s. \tag{1.2}$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}\int_0^\rho \frac{\mu(t)^{\frac{1}{q}}}{t}\mathrm{d}t &\le \sum_{j=0}^\infty \mu(\rho)\theta^j \int_{\mu(\rho)\theta^{q(j+1)}}^{\mu(\rho)\theta^{qj}} \frac{1}{\nu(s)\mu'(\nu(s))}\mathrm{d}s \\ &= \sum_{j=0}^\infty \mu(\rho)^{\frac{1}{q}}\theta^j \left(\log(\nu(\mu(\rho)\theta^{qj})) - \log(\nu(\mu(\rho)\theta^{q(j+1)}))\right) \\ &= \frac{\mu(\rho)^{\frac{1}{q}}}{\theta}\sum_{j=0}^\infty(\theta a_j - a_{j+1}) = \mu(\rho)^{\frac{1}{q}}K - \mu(\rho)^{\frac{1}{q}}\frac{K}{\theta}\frac{\mu(\rho)^{\frac{1}{q}}}{\theta}a_0, \end{aligned} \tag{1.3}$$

implying that (2) holds.

We now assume that (2) holds. If we switch inequality (1.2) by a reasonable manner, we have

$$\mu(\rho)^{\frac{1}{q}}\sum_{j=0}^\infty(\theta a_j - a_{j+1}) \le \int_0^\rho \frac{\mu(t)^{\frac{1}{q}}}{t}\mathrm{d}t. \tag{1.4}$$

Then, from (1.3) and (1.4), it follows that

$$\sum_{j=0}^\infty(\theta a_j - a_{j+1}) \le M. \tag{1.5}$$

Since $\theta \in (0,1)$, $a_j < 0$ for large j. Thus, we have either $K = -\infty$ or K is finite. If K were not finite, then we would have $\sum\limits_{j=0}^\infty(\theta a_j - a_{j+1}) = \infty$. This is a contradiction to (1.5).

We present an example of discretization.

Example 41. Let $\{a_k\}_{k=1}^\infty$ be a non-negative decreasing sequence. If we write $\mathcal{A}_1 \equiv B(1)$ and $\mathcal{A}_k \equiv B(2^k) \setminus B(2^{k-1})$ for all $k \in \mathbb{N} \cap [2,\infty)$, then $\left\|\sum\limits_{k=1}^\infty a_k\chi_{\mathcal{A}_k}\right\|_{L^{p,r}} \sim \left\{\sum\limits_{k=1}^\infty \left(2^{\frac{n}{p}k}a_k\right)^r\right\}^{\frac{1}{r}}$, since $f^*(2^{kn}|\mathcal{A}_1|) = a_k$.

1.4.5 Exercises

Exercise 26. Let $0 < p < \infty$ and $0 < q \le \infty$.

(1) Argue similarly to Example 28(3) to show that $\chi^*_{B(R)} = \chi_{[0,|B(R)|)}$.

(2) Show that $\chi_{B(R)} \in L^{p,q}(\mathbb{R}^n)$.

(3) Show that $L^\infty_{\rm c}(\mathbb{R}^n) \subset L^{p,q}(\mathbb{R}^n)$.

Exercise 27. Let $0 < p < \infty$.

(1) Use the Lebesgue convergence theorem to show that $\lim_{t\downarrow 0} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{1}{p}} = 0$ and that $\lim_{t\downarrow \infty} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{1}{p}} = 0$ for all $f \in L^p(\mathbb{R}^n)$.

(2) Show by an example that $\lim_{t\downarrow \infty} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{1}{p}} = 0$ and $\lim_{t\downarrow \infty} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{1}{p}} = 0$ fails for all $f \in \mathrm{W}L^p(\mathbb{R}^n)$. Hint: Consider $f(x) = |x|^{-\frac{n}{p}}$, $x \in \mathbb{R}^n$.

Exercise 28. Let $0 < \alpha < n$, and let $1 < q < \frac{n}{\alpha}$. By using (1.11), prove $|\cdot|^{-\alpha} \in \mathcal{M}^{\frac{n}{\alpha}}_q(\mathbb{R}^n)$. Also disprove $|\cdot|^{-\alpha} \in L^{\frac{n}{\alpha}}(\mathbb{R}^n)$ by a direct calculation.

Exercise 29. Let p and q be as in Definition 20. Find the necessary and sufficient condition on the parameter $\gamma \in \mathbb{R}$ ensuring that $f_\gamma \in L^{p,q}(\mathbb{R}^n)$ and $g_\gamma \in L^{p,q}(\mathbb{R}^n)$.

Exercise 30. By using the set E defined by (6.25), prove that $L^{4,\infty}(\mathbb{R})$ is strictly wider than $\mathcal{M}^4_2(\mathbb{R})$. Here, the norm of $L^{4,\infty}(\mathbb{R})$ is given by

$$\|f\|_{L^{4,\infty}} \equiv \sup_{\lambda>0} \lambda \sqrt[4]{\lambda_f(\lambda)}$$

for $f \in L^0(\mathbb{R})$.

Exercise 31.

(1) For $g \in \Phi_{\mu,\infty}(0,\infty)$ and $\mu > -1$, show that $\|\mathcal{H}g\|_{\Phi_{\mu,\infty}(0,\infty)} \le \dfrac{\|g\|_{\Phi_{\mu,\infty}(0,\infty)}}{\mu+1}$ using $\displaystyle\int_0^t s^\mu \mathrm{d}s = \frac{1}{\mu+1}$.

(2) For $g \in \Phi_{-\mu,\infty}(0,\infty)$ and $\mu > 1$, using $\displaystyle\int_t^\infty s^{-\mu}\mathrm{d}s = \frac{1}{\mu-1}$, show that $\|\mathfrak{H}g\|_{\Phi_{-\mu,\infty}(0,\infty)} \le \dfrac{\|g\|_{\Phi_{-\mu,\infty}(0,\infty)}}{\mu-1}$.

(3) For $g \in \Phi_{\mu,1}(0,\infty)$ and $\mu > -1$, show that $\|\mathcal{H}g\|_{\Phi_{\mu,1}(0,\infty)} \le \dfrac{\|g\|_{\Phi_{\mu,1}(0,\infty)}}{\mu+1}$ using Fubini's theorem.

(4) Show that $\|\mathfrak{H}g\|_{\Phi_{-\mu,1}(0,\infty)} \le \dfrac{\|g\|_{\Phi_{-\mu,1}(0,\infty)}}{\mu-1}$ for $g \in \Phi_{-\mu,\infty}(0,\infty)$ and $\mu > 1$ using Fubini's theorem.

Exercise 32. Let $1 < p < \infty$. Work in $\mathbb{R}^2 = \mathbb{R}_t \times \mathbb{R}_s$.

(1) Show that $L^1_s L^p_t(\mathbb{R}^2) = L^{(p,1)}(\mathbb{R}^2_{t,s}) \overset{1}{\hookrightarrow} L^p_t L^1_s(\mathbb{R}^2) = L^{(1,p)}(\mathbb{R}^2_{s,t})$. Show also that $F = f \otimes g$ for some functions $f, g \in \mathbb{M}^+(0,\infty)$ if $F \in L^0(\mathbb{R}^2)$ $0 < \|F\|_{L^p_t L^1_s} = \|F\|_{L^1_s L^p_t} < \infty$.

(2) Establish that $\|\mathcal{H}f\|_{L^p(0,\infty)} \le p'\|f\|_{L^p(0,\infty)}$ for all $f \in L^p(0,\infty)$.

(3) Let $f_\varepsilon(t) = t^{-\frac{1}{p}+\varepsilon}\chi_{(0,1)}(t)$ for $t > 0$. Show that $\lim\limits_{\varepsilon\downarrow 0} \dfrac{\|\mathcal{H}f\|_{L^p(0,\infty)}}{\|f\|_{L^p(0,\infty)}} = p'$.

(4) Show that there is no measurable function f on $(0,\infty)$ such that $0 < \|\mathcal{H}f\|_{L^p(0,\infty)} = p'\|f\|_{L^p(0,\infty)} < \infty$.

Exercise 33. [156, Exercise 1.4.13] Let $0 < p < \infty$, and let $f \in L^0(\mathbb{R}^n)$. Use Lemma 29 to show that $f \in L^{p,1}(\mathbb{R}^n)$ if and only if there exist a decreasing sequence $\{a_j\}_{j=-\infty}^{\infty} \subset [0,\infty)$, $\{\varphi_j\}_{j=-\infty}^{\infty} \subset L^\infty(\mathbb{R}^n)$ and a sequence $\{E_j\}_{j=-\infty}^{\infty}$ of measurable sets safisfying $|\varphi_j| \le a_j\chi_{E_j}$ and $|E_j| \le 2^j$ for each $j \in \mathbb{Z}$ such that $f = \sum\limits_{j=-\infty}^{\infty} \varphi_j$ and $\sum\limits_{j=-\infty}^{\infty} 2^{\frac{j}{p}} a_j < \infty$.

1.5 Young functions and Orlicz spaces

One of the fundamental techniques in the theory of function spaces is considering function spaces slightly different from Lebesgue spaces. This technique is used especially to compensate for the failure of the boundedness of operators. More precisely we consider Young functions and Orlicz spaces which Young functions generate. Actually in this book we need to use Orlicz spaces for the purpose above. The Young functions will generalize the parameter p in the Lebesgue space L^p.

We investigate Young functions in Section 1.5.1. Orlicz spaces will be dealt with in Section 1.5.2. As an application of Orlicz functions, we generalize the average of functions Section 1.5.3. The results in Section 1.5 will be very subtle. However, the Orlicz spaces are nice tools to describe the boundedness properties of operators even when we consider Morrey spaces. Later we will combine Morrey spaces and Orlicz spaces to generate Orlicz–Morrey spaces. Orlicz–Morrey spaces are effectively used for the purpose above.

1.5.1 Young functions

Functions defined over $[0,\infty)$ or $(0,\infty)$ will serve various purposes of measuring the size of other functions. For example, many integral operators are not always $L^p(\mathbb{R}^n)$-bounded. Hence, we are faced with the modification of the definition of $L^p(\mathbb{R}^n)$-spaces. Young functions, which we deal with in Section 1.5.1, will be used for this purpose. By definition Young functions are convex, so that they are almost everywhere differentiable. We collect some inequalities on Young functions and their derivatives. We also define classes Δ_2 and ∇_2 to discuss the boundedness properties of operators acting on Orlicz spaces which Young functions generate.

Definition 22 (Young functions)**.** A function $\Phi : [0,\infty) \to [0,\infty)$ is a *Young function*, if it satisfies the following conditions:

(1) $\Phi(0) = 0$.

(2) Φ is continuous.

(3) Φ is convex. That is, $\Phi((1-\theta)t_1 + \theta t_2) \le (1-\theta)\Phi(t_1) + \theta\Phi(t_2)$ for all $t_1, t_2 \in [0,\infty)$ and $0 < \theta < 1$.

By convention define $\Phi(\infty) \equiv \infty$.

Motivated by the above examples it is convenient to introduce the class $\mathbb{M}^{\uparrow}(0,a)$. Let $\mathbb{M}^{\uparrow}(0,a)$ denote the space of all non-negative and non-decreasing functions on $(0,a)$ for $a \in [0,\infty]$. We do not tolerate the function $\Phi(t) = \infty\chi_{[1,\infty)}(t)$, although this is a convex function satisfying $\Phi(t) = 0$.

Example 42. The following functions are all Young functions.

(1) t^p, $p \ge 1$.

(2) $e^t - 1$

(3) $t\,\log(t+1)$.

There are other examples.

(4) Let $\beta > 0$. If Φ satisfies $\Phi(r) = r$ for $0 < r \le 1$ and $\Phi(r) = r(\log r)^{\beta}$ for $r \gg 1$, then Φ can be arranged to be a Young function.

(5) Let $p > 0$. Define

$$e_p(r) \equiv \begin{cases} \exp(-\exp(r^{-p})) & 0 < r \ll 1, \\ \exp(r^p) & r \gg 1. \end{cases}$$

Then e_p can be arranged to be a Young function.

(6) Define $\tan^{-1} t \equiv \displaystyle\int_0^t \frac{\mathrm{d}s}{1+s^2}$ for $t \in \mathbb{R}$. Then $\Phi(t) = t - \tan^{-1} t$, $t \ge 0$ can be arranged to be an Orlicz function.

(7) The function $\Phi(t) = t\exp(-t^{-1}), t > 0$ has the limit $\Phi(+0) = \lim_{t\downarrow 0}\Phi(t) = 0$. Consequently, if we extend Φ continuously to $[0,\infty)$, then Φ is a Young function since $\Phi''(t) = t^{-4}\Phi(t)$ for all $t > 0$.

Let us start with the following simple estimates for Young functions.

Proposition 41. *For a Young function $\Phi : [0,\infty) \to [0,\infty)$ and $t \geq 0$, $\theta\Phi(t) \geq \Phi(\theta t)$ if $0 < \theta < 1$, or equivalently, $\theta\Phi(t) \leq \Phi(\theta t)$ if $1 < \theta < \infty$.*

Proof Since Φ is convex and $\Phi(0) = 0$, for $0 < \theta < 1$,

$$\Phi(\theta t) = \Phi((1-\theta)0 + \theta t) \leq (1-\theta)\Phi(0) + \theta\Phi(t) = \theta\Phi(t).$$

As a corollary, we can show that the tangent of the line connecting $(0,0)$ and $(a,\Phi(a))$ is increasing in $a \geq 0$.

Lemma 42. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. Then mapping $t \in (0,\infty) \mapsto t^{-1}\Phi(t) \in [0,\infty)$ is increasing.*

Proof This can be derived easily from Proposition 41.

Since Young functions are locally absolutely continuous, they have density: Any Young function can be expressed for some $\varphi \in \mathbb{M}^{\uparrow}(0,\infty)$ as

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s \quad (t \geq 0).$$

We call this equality the canonical representation of Φ. Next, we will treat the information on the density function φ of a Young function.

Lemma 43. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function that is expressed as*

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s \quad (t \geq 0).$$

Then for $t > 0$:

$$\frac{\Phi(t)}{t} \leq \varphi(t) \leq \frac{\Phi(2t)}{t}, \tag{1.1}$$

$$\frac{1}{4}\varphi\left(\frac{t}{2}\right) \leq \int_0^t \frac{\Phi(s)}{s}\mathrm{d}s. \tag{1.2}$$

Proof All the estimates are easy to derive:

$$\frac{\Phi(t)}{t} = \frac{1}{t}\int_0^t \varphi(s)\mathrm{d}s \leq \frac{1}{t}\int_0^t \varphi(t)\mathrm{d}s = \varphi(t)$$

and

$$\varphi(t) = \frac{1}{t}\int_t^{2t} \varphi(t)\mathrm{d}t \leq \frac{1}{t}\int_t^{2t} \varphi(s)\mathrm{d}s \leq \frac{1}{t}\int_0^{2t} \varphi(s)\mathrm{d}s = \frac{\Phi(2t)}{t},$$

which proves (1.1).

To obtain (1.2), we use Fubini's theorem and $\log 2 = 0.69\cdots$ to see

$$\int_0^t \frac{\Phi(s)}{s}\mathrm{d}s = \int_0^t \left(\frac{1}{s}\int_0^s \varphi(u)\mathrm{d}u\right)\mathrm{d}s = \int_0^t \varphi(u)\log\frac{t}{u}\mathrm{d}u \geq \frac{1}{4}\varphi\left(\frac{t}{2}\right).$$

Thus, the proof is therefore complete.

The following definition can be regarded as a passage of the harmonic conjugate of real numbers to functions.

Definition 23 (Conjugate)**.** For a Young function $\Phi : [0,\infty) \to [0,\infty)$, its (*Legendre*) *conjugate function or its Fenchel–Legendre transform is defined by*

$$\Phi^*(t) \equiv \int_0^t \varphi^*(s)\mathrm{d}s \quad (t \geq 0), \tag{1.3}$$

where

$$\varphi^*(v) \equiv \int_0^\infty \chi_{\varphi^{-1}([0,v])}(t)\mathrm{d}t = |\varphi^{-1}([0,v])| \tag{1.4}$$

for every $v > 0$.

We provide a rather general but useful formula on the Fenchel–Legendre transform.

Example 43. Let $1 < p < \infty$, $q_1, q_2 \in \mathbb{R}$. Also let Φ be a Young function satisfying $\Phi(t) = t^p(\log(e+t))^{q_1}(\log(e+t^{-1}))^{q_2}$ for large t and t slightly larger than 0. Then since $\lim_{t\downarrow 0} t\log t = \lim_{t\to\infty} t^{-1}\log t = 0$, $\Phi^*(t) \sim t^{p'}(\log(e+t))^{-\frac{q_1}{p-1}}(\log(e+t^{-1}))^{-\frac{q_2}{p-1}}$ for $t \geq 0$.

Concerning the definition above, we have the following observation:

Lemma 44. *Maintain the notation in Definition 23. Let* $x, v \geq 0$.

(1) *If* $\varphi(x) < v$, *then* $x \leq \varphi^*(v)$.

(2) *If* $\varphi(x) > v$, *then* $x \geq \varphi^*(v)$.

Proof

(1) Note that $[0,x] \subset \varphi^{-1}([0,v])$. Thus,

$$\varphi^*(v) = \int_0^\infty \chi_{\varphi^{-1}([0,v])}(t)\mathrm{d}t \geq \int_0^\infty \chi_{[0,x]}(t)\mathrm{d}t = x.$$

(2) Note that $[0,x] \supset \varphi^{-1}([0,v])$. Thus, we can argue as in (1).

We have the following expression of Φ^*:

Theorem 45. *Let Φ be a Young function that is, $\Phi : [0,\infty) \to [0,\infty)$ is a homeomorphism, and let Φ^* be its conjugate. Then*

$$\Phi^*(t) = \sup\{st - \Phi(s) \,:\, s \in [0,\infty)\} \quad (t \geq 0).$$

Hence, Theorem 45 asserts that Φ^* is the Fenchel–Legendre transform of Φ.

Proof Fix $s, t \geq 0$. We are going to show

$$\Phi(t) + \Phi^*(s) \geq st \tag{1.5}$$

for all $s, t \geq 0$ and that for each $s \geq 0$, there exists $t \geq 0$ for which equality in (1.5) holds.

If $st = 0$, then (1.5) is trivial. Therefore, we may assume that $s, t > 0$. Inserting (1.4) into the canonical representation, we have

$$\begin{aligned}\Phi^*(s) &= \int_0^s \left(\int_0^\infty \chi_{\varphi^{-1}([0,v])}(x) \mathrm{d}x \right) \mathrm{d}v \\ &= \iint_{[0,\infty)^2} \chi_{\{(x,v)\in[0,\infty)^2 \,:\, 0\leq v\leq s,\, \varphi(x)<v\}}(x,v) \mathrm{d}v\mathrm{d}x.\end{aligned}$$

Note that "$0 \leq v \leq s$ and $x \in \varphi^{-1}([0,v])$" is not equivalent to "$0 \leq v \leq s$ and $\varphi(x) < v$".

Meanwhile,

$$\Phi(t) = \int_0^t \varphi(x) \mathrm{d}x = \iint_{[0,\infty)^2} \chi_{\{(x,v)\in[0,\infty)^2 \,:\, 0\leq x\leq t,\, 0\leq v\leq \varphi(x)\}}(x,v) \mathrm{d}v\mathrm{d}x.$$

Observe that

$$[0,t] \times [0,s] \subset \{(x,v) \,:\, 0 \leq v \leq s,\, \varphi(x) \leq v \text{ or } 0 \leq x \leq t,\, 0 \leq v \leq \varphi(x)\}. \tag{1.6}$$

Therefore we conclude

$$\Phi^*(s) + \Phi(t) \geq \int_0^\infty \int_0^\infty \chi_{[0,t]\times[0,s]}(x,v) \mathrm{d}x\mathrm{d}v = st.$$

To show that equality holds for some t, we set $t \equiv \varphi^*(s)$. We obtain a kind of reverse inclusion in (1.6). We aim to show

$$\{(x,v) \in (0,\infty) \times (0,\infty) \,:\, 0 < v < s,\, \varphi(x) < v\} \subset [0,t] \times [0,s]$$

and that

$$\{(x,v) \in (0,\infty) \times (0,\infty) \,:\, 0 < x < t,\, 0 < v < \varphi(x)\} \subset [0,t] \times [0,s].$$

Let $0 < v < s$ satisfy $\varphi(x) < v$. Then $t = \varphi^*(s) \geq \varphi^*(v) \geq x \geq 0$ thanks to Lemma 44. Meanwhile, let $0 \leq x < t$ and $0 \leq v < \varphi(x)$. Since $x < t = \varphi^*(s)$, we have $\varphi(x) \leq s$ thanks to Lemma 44 once again. Therefore $v \leq s$.

Carrying out the same calculation as before, we conclude that $\Phi^*(s) + \Phi(t) = st$.

Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. It turns out that the conjugate function of a convex bijection $\Phi : [0,\infty) \to [0,\infty)$ is given by

$$\Phi^*(t) \equiv \sup\{st - \Phi(s) \,:\, s \geq 0\} \quad (t \geq 0).$$

The following theorem shows that the operation of conjugation is an involution: if we take the conjugate twice then we go back to the original function.

Theorem 46. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. Then $\Phi^{**} = \Phi$.*

Proof Fix $x \in [0,\infty)$. We claim that $\varphi^{**} = \varphi$ except at countable points in $[0,\infty)$. Once this is achieved, since Φ^{**} and Φ have the same canonical representation, it follows that $\Phi^{**} = \Phi$.

Let $v > \varphi(x)$. Then $x \leq \varphi^*(v)$ thanks to Lemma 44. Consequently, for all $\varepsilon > 0$, $\varphi^{**}(x-\varepsilon) \leq v$ again thanks to Lemma 44. Since $v > \varphi(x)$ is arbitrary, $\varphi^{**}(x-\varepsilon) \leq \varphi(x)$. Since $\varepsilon > 0$ is also arbitrary, $\varphi^{**}(x) \leq \varphi(x)$ for all x at which φ^{**} is continuous.

Conversely, let $v < \varphi(x)$. Then $x \geq \varphi^*(v)$ thanks to Lemma 44. Thus, for all $\varepsilon > 0$, $\varphi^{**}(x+\varepsilon) \geq v$ again thanks to Lemma 44. Since $v < \varphi(x)$ is arbitrary, $\varphi^{**}(x+\varepsilon) \geq \varphi(x)$. Since $\varepsilon > 0$ is also arbitrary and φ^{**} is right-continuous, it follows that $\varphi^{**}(x) \geq \varphi(x)$.

If we reexamine the proof, we notice that $\varphi^{**} = \varphi$ on $(0,\infty)$, since both functions are right-continuous.

Lemma 47. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function, and let Φ^* be its conjugate. Then the following*:

(1) *The inequality*

$$\Phi^*\left(\frac{\Phi(t)}{t}\right) \leq \Phi(t) \leq \Phi^*\left(\frac{2\Phi(t)}{t}\right) \tag{1.7}$$

holds for all $t > 0$.

(2) *For all $t, s > 0$ such that $st \leq \Phi(t)$, we have*

$$\Phi^*(s) \leq st. \tag{1.8}$$

(3) *For all $t, s > 0$ such that $st \geq \Phi(t)$, we have*

$$st \leq \Phi^*(2s) \tag{1.9}$$

(4) *Let $s, t, \lambda > 0$. If $\Phi(t) = \Phi^*(s) = \lambda$, then*

$$\lambda \leq ts \leq 2\lambda. \tag{1.10}$$

Proof

(1) Recall that $t \in (0,\infty) \mapsto t^{-1}\Phi(t) \in (0,\infty)$ is increasing in t thanks to Lemma 42. Thus,

$$\Phi^*\left(\frac{\Phi(t)}{t}\right) = \sup_{0<s<t} s\left(\frac{\Phi(t)}{t} - \frac{\Phi(s)}{s}\right) \le t \sup_{0<s<t}\left(\frac{\Phi(t)}{t} - \frac{\Phi(s)}{s}\right) \le \Phi(t).$$

The right inequality is easier to prove. We have only to use Theorem 45:

$$\Phi^*\left(\frac{2\Phi(t)}{t}\right) = \sup_{s>0} s\left(\frac{2\Phi(t)}{t} - \frac{\Phi(s)}{s}\right) \ge \Phi(t).$$

Thus, (1.7) is proven.

(2) Let $st \le \Phi(t)$. Since $t^{-1}\Phi(t)$ is increasing in $t > 0$ again thanks to Lemma 42, we have

$$\begin{aligned}\frac{\Phi^*(s)}{s} &= \sup_{u>0}\left(u - \frac{\Phi(u)}{s}\right)\\ &\le t \sup_{u>0}\left(\frac{u}{t} - \frac{\Phi(u)}{\Phi(t)}\right)\\ &= t \sup_{0<u<t}\left(\frac{u}{t} - \frac{\Phi(u)}{\Phi(t)}\right)\\ &\le t.\end{aligned}$$

Thus, the proof of (1.8) is complete.

(3) Suppose instead that $st \ge \Phi(t)$. Then

$$\frac{\Phi^*(2s)}{s} = \sup_{u>0}\left(2u - \frac{\Phi(u)}{s}\right) \ge 2t - \frac{\Phi(t)}{s} \ge t.$$

Thus, the proof of (1.9) is complete.

(4) It follows from the Young inequality that $st \le \Phi(s) + \Phi(t) = 2\lambda$. Meanwhile, if $\varphi(t) \le s$, then

$$\lambda = \Phi(t) = \int_0^t \varphi(u)\mathrm{d}u \le st.$$

If $\varphi(t) > s$, then $\varphi^*(s) \le t$ from Lemma 44. Thus it follows that $\lambda = \Phi^*(s) = \displaystyle\int_0^s \varphi^*(u)\mathrm{d}u \le st$. Therefore (1.10) is established.

We will be mainly interested in the boundedness of linear or sublinear operators. To obtain the boundedness of operators, we need some conditions on Φ. If Φ is a Young function, it is easy to see that $\Phi(2t) \ge 2\Phi(t)$ for all $t \ge 0$ using Lemma 42. Here, we define two important classes keeping this in mind.

Definition 24 (Δ_2, ∇_2)**.**

(1) A function $\varphi : (0,\infty) \to [0,\infty)$ or $\varphi : [0,\infty) \to [0,\infty)$ is a doubling function, if there exists a constant $C > 0$, called a *doubling constant*, such that $C^{-1}\varphi(t) \le \varphi(s) \le C\varphi(t)$ for all t and s in the domain of φ satisfying $s \le t \le 2s$.

(2) Denote by Δ_2 the set of all convex bijections $\Phi : [0,\infty) \to [0,\infty)$ satisfying the doubling condition; $\Phi(2r) \lesssim \Phi(r)$ for $r > 0$. The implicit constant is again called a *doubling constant*. In this case Φ also satisfies the Δ_2-*condition*.

(3) The set ∇_2 is the set of all convex bijections $\Phi : [0,\infty) \to [0,\infty)$ such that there exists $C > 1$ such that $\Phi(2t) \ge 2C\Phi(t)$ for all $t \ge 0$. In this case one says that Φ satisfies the ∇_2 *condition*.

Example 44. We can list the following examples:

(1) Let $\Phi(t) = t^q$ where $1 < q < \infty$. Then, $\Phi \in \Delta_2 \cap \nabla_2$.

(2) Let $\Phi(t) = t$. Then, $\Phi \in \Delta_2$ but $\Phi \notin \nabla_2$.

(3) Let $\Phi(t) = e^t - t - 1$. Then, $\Phi \in \nabla_2$ but $\Phi \notin \Delta_2$.

Any function $\Phi \in \Delta_2$ grows like a polynomial as the following proposition shows:

Lemma 48. *Let $\Phi \in \Delta_2$. There exists $q \in [2,\infty)$ such that the following properties hold*:

(1) *For all $0 < t \le s < \infty$, $\dfrac{\Phi(s)}{s^{q-1}} \lesssim \dfrac{\Phi(t)}{t^{q-1}}$.*

(2) *For all $r > 0$, $\displaystyle\int_r^\infty \frac{\Phi(t)}{t^{q+1}}\mathrm{d}t \lesssim \frac{\Phi(r)}{r^q}$.*

Proof Let $D \ge 2$ be a doubling constant of Φ. Take $q \equiv 1 + \log_2 D \ge 1$.

(1) For all $0 < t \le s < \infty$, we choose a number $k \in \mathbb{N}_0$ such that $2^{-k-1}s \le t \le 2^{-k}s$. By using convexity and the Δ_2 condition of Φ, we obtain $\dfrac{\Phi(s)}{s^{q-1}} \le \dfrac{\Phi(2^{k+1}t)}{2^{k(q-1)}t^{q-1}} \le 2^{-k(q-1)}D^{k+1} = \dfrac{\Phi(t)}{t^{q-1}}$.

(2) From the above computation, we have

$$\int_r^\infty \frac{\Phi(t)}{t^{q+1}}\mathrm{d}t = \int_r^\infty \frac{\Phi(t)}{t^{q-1}}\frac{\mathrm{d}t}{t^2} \lesssim \frac{\Phi(r)}{r^{q-1}}\int_r^\infty \frac{1}{t^2}\mathrm{d}t = \frac{\Phi(r)}{r^q}.$$

Any function $\Phi \in \nabla_2$ grows faster than $\Phi(t) = t$, as the following proposition shows:

Lemma 49. *Let* $\Phi \in \nabla_2$.

(1) *There exists* $\theta \in (1,\infty)$ *such that for all* $0 < t \le s < \infty$ $\dfrac{\Phi(s)}{s^\theta} \lesssim \dfrac{\Phi(t)}{t^\theta}$.

(2) *There exists* $\theta^\dagger \in (1,\infty)$ *such that for all* $r > 0$, $\displaystyle\int_0^r \frac{\Phi(t)}{t^{\theta^\dagger+1}}\mathrm{d}t \lesssim \frac{\Phi(r)}{r^{\theta^\dagger}}$.

Proof Let $D > 1$ satisfy $\Phi(2t) \ge 2D\Phi(t)$. Take

$$\tau \equiv \begin{cases} \frac{1}{2} + \log_2 D & (D \ge 2), \\ 1 + \frac{1}{2}\log_2 D & (1 < D < 2). \end{cases}$$

Define an auxiliary parameter $\tilde{q}$ by

$$\tilde{q} \equiv \begin{cases} \frac{1}{2} & (D \ge 2), \\ 1 - \frac{1}{2}\log_2 D & (1 < D < 2). \end{cases}$$

(1) We claim that $\theta \equiv \tau + 1 - \tilde{q}$ does the job. For all $0 < t \le s < \infty$, we choose a number $k \in \mathbb{N}_0$ such that $2^{-k-1}s \le t \le 2^{-k}s$. By using the ∇_2 condition, we obtain

$$\Phi(s) \ge \Phi\left(2^k t\right) \ge (2D)^k \Phi(t) = 2^{(1+\log_2 D)k}\Phi(t) \ge \left(\frac{s}{t}\right)^{1+\log_2 D} \frac{\Phi(t)}{2^{1+\log_2 D}},$$

or equivalently,

$$\frac{\Phi(t)}{t^\theta} \le 2^\theta \frac{\Phi(s)}{s^\theta}.$$

(2) We claim that $\theta^\dagger \equiv \tau$ does the job. Since $\tilde{q} \in (0,1)$, we have

$$\int_0^r \frac{\Phi(t)}{t^{\tau+1}}\mathrm{d}t = \int_0^r \frac{\Phi(t)}{t^\theta}\,\frac{\mathrm{d}t}{t^{\tilde{q}}} \lesssim \frac{\Phi(r)}{r^\theta}\int_0^r \frac{1}{t^{\tilde{q}}}\mathrm{d}t \sim \frac{\Phi(r)}{r^\tau}$$

thanks to (1). Thus, the proof is complete.

1.5.2 Orlicz spaces

We define Orlicz spaces here and then investigate some properties similar to Lebesgue spaces; triangle inequality Hölder's inequality, duality and so on. For the time being we will work on a σ-finite measure space $(X, \mathcal{B}, \mu)$. For $f \in L^0(\mu)$, we rewrite the norm $\|f\|_{L^p(\mu)}$ as follows:

$$\|f\|_{L^p(\mu)} \equiv \inf\left(\left\{\lambda \in (0,\infty) \,:\, \int_X \left(\frac{|f(x)|}{\lambda}\right)^p \mathrm{d}\mu(x) \le 1\right\} \cup \{\infty\}\right).$$

After writing the norm in this way, we generalize the $L^p(\mu)$-norm. We define the Orlicz space $L^\Phi(\mu)$ as follows:

Definition 25 (Luxemburg–Nakano norm)**.** Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. Then define the *Luxemburg–Nakano norm* $\|\cdot\|_{L^\Phi(\mu)}$ by

$$\|f\|_{L^\Phi(\mu)} \equiv \inf\left(\left\{\lambda \in (0,\infty) \,:\, \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\mu(x) \le 1\right\} \cup \{\infty\}\right)$$

for $f \in L^0(\mu)$. The *Orlicz space* $L^\Phi(\mu)$ over X is the set of all $f \in L^0(\mu)$ for which $\|f\|_{L^\Phi(\mu)}$ is finite.

Here and below for the sake of simplicity, we assume that Φ is nice.

Example 45.

(1) If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^\Phi(\mu) = L^p(\mu)$ with coincidence of norms.

(2) Let $p > 1, q > 0$ be parameters. The most important example of Orlicz spaces is generated by a function satisfying $\Phi(t) \sim t^p[\log(3+t)]^q$ for all $t \ge 0$. Denote by $\mathrm{L}^p \log^q \mathrm{L}(\mu)$ such a function space. The space $\mathrm{L}\log\mathrm{L}(\mu)$ is occasionally called the Hardy–Littlewood space. If $p = 1$, then write $\mathrm{L}\log^q \mathrm{L}(\mu)$ instead of $\mathrm{L}^p \log^q \mathrm{L}(\mu)$.

(3) If $\Phi(t) = e^t - 1$ for $t \ge 0$, then we write $\exp(\mu)$ instead of $L^\Phi(\mu)$.

(4) Let E be a measurable set, and let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. Then $\|\chi_E\|_{L^\Phi(\mu)} = \dfrac{1}{\Phi^{-1}(\mu(E)^{-1})}$. In fact, assuming that $\mu(E) < \infty$, we see that $\lambda \equiv \|\chi_E\|_{L^\Phi(\mu)}$ solves $\mu(E)\Phi\left(\dfrac{1}{\lambda}\right) = 1$.

As the following theorem shows, $\|\cdot\|_{L^\Phi(\mu)}$ is a complete norm.

Theorem 50. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. Then $L^\Phi(\mu)$ is a Banach space.*

Proof We content ourselves with proving that $\|\cdot\|_{L^\Phi(\mu)}$ is a norm. Let $f, g \in L^\Phi(\mu)$. We are now going to show that

$$\|f+g\|_{L^\Phi(\mu)} \le \|f\|_{L^\Phi(\mu)} + \|g\|_{L^\Phi(\mu)}. \tag{1.11}$$

Let $\varepsilon > 0$ be taken arbitrarily. Then (1.11) can be reduced to showing

$$\|f+g\|_{L^\Phi(\mu)} \le \|f\|_{L^\Phi(\mu)} + \|g\|_{L^\Phi(\mu)} + \varepsilon. \tag{1.12}$$

From the definition of the set defining the norm $\|f\|_{L^\Phi(\mu)}$, we deduce

$$\|f\|_{L^\Phi(\mu)} + \frac{1}{2}\varepsilon \in \left\{\lambda > 0 \,:\, \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\mu(x) \le 1\right\}.$$

Namely, we have

$$\int_X \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi(\mu)}+2^{-1}\varepsilon}\right)\mathrm{d}\mu(x)\le 1.$$

The same can be said for g;

$$\int_X \Phi\left(\frac{|g(x)|}{\|g\|_{L^\Phi(\mu)}+2^{-1}\varepsilon}\right)\mathrm{d}\mu(x)\le 1.$$

Set $\alpha\equiv\|f\|_{L^\Phi(\mu)}+\dfrac{1}{2}\varepsilon$, $\beta\equiv\|g\|_{L^\Phi(\mu)}+\dfrac{1}{2}\varepsilon$ and $\theta\equiv\dfrac{\beta}{\alpha+\beta}$.

Since Φ is convex, we have a pointwise estimate: for all $x\in X$,

$$\begin{aligned}\Phi\left(\frac{|f(x)+g(x)|}{\alpha+\beta}\right)&=\Phi\left((1-\theta)\,\frac{|f(x)|}{\alpha}+\theta\,\frac{|g(x)|}{\beta}\right)\\&\le(1-\theta)\,\Phi\left(\frac{|f(x)|}{\alpha}\right)+\theta\,\Phi\left(\frac{|g(x)|}{\beta}\right).\end{aligned}\tag{1.13}$$

Integrating (1.13) over X, we obtain

$$\int_X \Phi\left(\frac{|f(x)+g(x)|}{\|f\|_{L^\Phi(\mu)}+\|g\|_{L^\Phi(\mu)}+\varepsilon}\right)\mathrm{d}\mu(x)=\int_X \Phi\left(\frac{|f(x)+g(x)|}{\alpha+\beta}\right)\mathrm{d}\mu(x)\le 1,$$

which is equivalent to (1.12).

As is the case with the people who learned Riemann integration theory, occasionally we are not interested in the exact value of integrals. Indeed, in many cases, it is again next to impossible to calculate precisely the value of an integral. Instead, we are mainly interested in the size of functions. Here, we remark that the Orlicz norm has the following equivalent expression:

Theorem 51. *Let $f\in L^0(\mu)$, and let $\Phi:[0,\infty)\to[0,\infty)$ be a Young function. Then*

$$\|f\|_{L^\Phi}\le\inf\left\{\lambda+\lambda\int_X\Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x\,:\,\lambda>0\right\}\le 2\|f\|_{L^\Phi}.$$

Proof By the Fatou lemma, we may assume that $f\in L^\infty(\mu)$ and that $\mu\{f\neq 0\}<\infty$. We may assume that $\|f\|_{L^\Phi}=1$ by normalization. By considering two cases; $\lambda\le 1$ and $\lambda\ge 1$, we obtain $1\le\lambda+\lambda\displaystyle\int_X\Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x$ for all $\lambda>0$. The opposite inequality can be obtained by taking $\lambda\equiv 1$.

We next prove the dual inequality for Orlicz spaces.

Theorem 52. *Let $\Phi:[0,\infty)\to[0,\infty)$ be a Young function, and let Φ^* be its conjugate. Then for all $f\in L^\Phi(\mu)$ and $g\in L^{\Phi^*}(\mu)$, $\|f\cdot g\|_{L^1(\mu)}\le 2\|f\|_{L^\Phi(\mu)}\|g\|_{L^{\Phi^*}(\mu)}$.*

Proof We may assume that $\|f\|_{L^{\Phi}(\mu)} = \|g\|_{L^{\Phi^*}(\mu)} = 1$. In this case, by integrating $|f(x)g(x)| \le \Phi(|f(x)|) + \Phi^*(|g(x)|)$ over X we obtain the desired result.

We end Section 1.5.2 with a duality result on Orlicz spaces.

Theorem 53. *Let $\Phi \in \Delta_2$. Then the dual space of $L^{\Phi}(\mu)$ is isomorphic to $L^{\Phi^*}(\mu)$ in the following sense*:

(1) *For all $f \in L^{\Phi}(\mu)$ and $g \in L^{\Phi^*}(\mu)$, $f \cdot g \in L^1(\mu)$, so that any $g \in L^{\Phi^*}(\mu)$ defines a continuous functional $h \in L^{\Phi}(\mu) \mapsto \int_X h(x)g(x)\mathrm{d}\mu(x) \in \mathbb{C}$.*

(2) *Conversely any continuous functional on $L^{\Phi}(\mu)$ is realized in this way.*

Proof From Theorem 52, (1) follows. We prove (2). Let $L : L^{\Phi}(\mu) \to \mathbb{C}$ be continuous. Since X is σ-finite, we may assume $\mu(X) < \infty$ by a routine truncation procedure.

Let $f \in L^{\Phi}(\mu)$ with $\|f\|_{L^{\Phi}(\mu)} = 1$. We remark that $g \equiv \chi_{\{f \neq 0\}}|f|^{-1}\Phi(|f|)$ satisfies

$$\|g \cdot f\|_{L^1(\mu)} = 1, \quad \int_X \Phi^*(g(x))\mathrm{d}\mu(x) \le \int_X \Phi(|f(x)|)\mathrm{d}\mu(x) = 1$$

thanks to Lemma 47(1). Thus, for any $f \in L^{\Phi}(\mu)$ with $\|f\|_{L^{\Phi}(\mu)} = 1$, there exists a simple function g satisfying $\|g\|_{L^{\Phi^*}(\mu)} \le 1$ such that $\|f \cdot g\|_{L^1(\mu)} = 1$.

Since $\Phi \in \Delta_2$, we can find a number $p > 0$ such that $\Phi(t) \lesssim t^p$ for $t \ge 1$. Thus, since $\mu(X) < \infty$, the mapping $f \in L^p(\mu) \mapsto L(f) \in \mathbb{C}$ is a continuous linear functional, so that there exists $g \in L^{p'}(\mu)$ such that

$$L(f) = \int_X f(x)g(x)\mathrm{d}\mu(x)$$

for all $f \in L^p(\mu)$. Consequently, by a simple truncation procedure once again, we have

$$\|f \cdot g\|_{L^1(\mu)} \le \|L\|_{L^{\Phi}(\mu) \to \mathbb{C}}\|f\|_{L^{\Phi}(\mu)}$$

for all $f \in L^{\Phi}(\mu)$. Once we show that $g \in L^{\Phi^*}(\mu)$ with the estimate $\|g\|_{L^{\Phi^*}(\mu)} \lesssim \|L\|_{L^{\Phi}(\mu) \to \mathbb{C}}$, we see that g realizes L. Again by a simple truncation procedure, we may assume that g is a simple function, so that $g \in L^{\Phi^*}(\mu)$. This means that we can concentrate on the norm estimate. It remains to test this estimate on $f = \chi_{\{g \neq 0\}}|g|^{-1}\Phi^*(|g|)$.

1.5.3 Orlicz-averages

On many occasions it is important to consider the powered average $m_Q^{(p)}(f)$ of functions f over a cube Q for $0 < p < \infty$. We will apply the idea of generalizing p to a Young function $\Phi : [0, \infty) \to [0, \infty)$ to generalize the above notion of the powered average. Here, we are concerned with the case

where $(X, \mathcal{F}, \mu)$ is a probabilty space. More precisely, let Q be a cube with the uniform probability measure $|Q|^{-1}\mathrm{d}x$.

Definition 26 (Orlicz-average)**.** Let $\Phi : [0,\infty) \to (0,\infty)$ a Young function. For a cube Q, define its Φ-*average*, its *Orlicz-average* or its *the mean Luxemberg norm* over Q by

$$\|f\|_{\Phi;Q} \equiv \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\}. \tag{1.14}$$

We do not have to stick to cubes. Instead we can use balls.

For applications, we envisage the following Young functions:

Example 46. Let $Q \in \mathcal{Q}$ and $f \in L^0(Q)$.

(1) If $\Phi(t) = t$, then $\|f\|_{\Phi;Q} = m_Q(|f|)$.

(2) If $\Phi(t) = t^p$, then $\|f\|_{\Phi;Q} = m_Q^{(p)}(|f|)$.

Example 47. Let $Q \in \mathcal{Q}$ and $f \in L^0(Q)$.

(1) If $\Phi(t) = \mathrm{L}\log\mathrm{L}(t) \equiv t\log(e+t)$, then $\|f\|_{\mathrm{L}\log\mathrm{L};Q} \equiv \|f\|_{\Phi;Q}$.

(2) If $\Phi(t) = \mathrm{L}\log\mathrm{L}^\alpha(t) \equiv t(\log(e+t))^\alpha$, then $\|f\|_{\mathrm{L}\log\mathrm{L}^\alpha;Q} \equiv \|f\|_{\Phi;Q}$.

The following is an example of calculating the Φ-average of the indicator functions. We say that a Young function Φ is normalized if $\Phi(1) = 1$.

Example 48. Let $\Phi : [0,\infty) \to [0,\infty)$ be a bijective Young function. Then for all cubes Q and $s \ge 1$, $\|\chi_Q\|_{\Phi;sQ} = \frac{1}{\Phi^{-1}(s^n)}$. Thus, for a normalized Young function Φ, $\|\chi_Q\|_{\Phi;Q} = 1$.

The dilation relation is summarized in the following lemma:

Lemma 54. *Let $Q \in \mathcal{Q}$, and let $\kappa > 0$. Define $R \equiv \{x \in \mathbb{R}^n \,:\, \kappa x \in Q\}$. Then $\|f\|_{\Phi;Q} = \|f(\kappa\cdot)\|_{\Phi;R}$ for all $f \in L^0(Q)$.*

Proof Indeed, by a change of variables,

$$\begin{aligned}
\|f\|_{\Phi;Q} &= \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\} \\
&= \inf\left\{\lambda > 0 \,:\, \frac{1}{|R|}\int_R \Phi\left(\frac{|f(\kappa x)|}{\lambda}\right)\mathrm{d}x \le 1\right\} \\
&= \|f(\kappa\cdot)\|_{\Phi;R}.
\end{aligned}$$

We can generalize the inequality $m_Q(|f|) \le m_Q^{(p)}(|f|)$ for $1 \le p < \infty$, $Q \in \mathcal{Q}$ and $f \in L^0(Q)$.

Lemma 55. *For a cube Q and $f \in L^0(Q)$, $m_Q(|f|) \lesssim \|f\|_{\Phi;Q}$.*

Proof We may suppose that $\|f\|_{\Phi;Q} = 1$. Then

$$\begin{aligned} &m_Q(|f|) \\ &\le \frac{1}{|Q|}\int_Q \chi_{[0,1]}(|f(x)|)|f(x)|\mathrm{d}x + \frac{1}{\Phi(1)|Q|}\int_Q \chi_{(1,\infty]}(|f(x)|)\Phi(|f(x)|)\mathrm{d}x \\ &\le 1 + \frac{1}{\Phi(1)}. \end{aligned}$$

In Lemma 55, we used the fact that $\Phi(1)t \le \Phi(t)$ for $t \ge 1$. Note that $\Phi(1)t \ge \Phi(t)$ for $0 \le t \le 1$. As is guessed from this fact, the value $\Phi(t)$ with $0 \le t \le 1$ does not affect very much. This fact can be summarized in the following proposition:

Proposition 56. *Let $\Phi_1, \Phi_2 : [0,\infty) \to [0,\infty)$ be Young functions. Assume that $\Phi_1(t) \le \Phi_2(t)$ for all $t > t_0$, where $t_0 > 0$ is a fixed number. Then for all cubes Q and all $f \in L^0(Q)$, $\|f\|_{\Phi_1;Q} \lesssim \|f\|_{\Phi_2;Q}$.*

Proof Assume that $\|f\|_{\Phi_2;Q} = 1$. Then

$$\begin{aligned} \frac{1}{|Q|}\int_Q \Phi_1(|f(x)|\chi_{(t_0,\infty]}(|f(x)|))\mathrm{d}x &= \frac{1}{|Q|}\int_Q \Phi_2(|f(x)|\chi_{(t_0,\infty]}(|f(x)|))\mathrm{d}x \\ &\le 1, \end{aligned}$$

which implies $\|f\chi_{(t_0,\infty]}(|f|)\|_{\Phi_1;Q} \le 1$. Meanwhile

$$\frac{1}{|Q|}\int_Q \Phi_1\left(\frac{|f(x)|\chi_{[0,t_0]}(|f(x)|))}{\Phi_1(t_0)}\right)\mathrm{d}x \le \Phi_1\left(\frac{t_0}{\Phi_1(t_0)}\right) \le 1.$$

Thus, $\|f\|_{\Phi_1;Q} \lesssim \|f\|_{\Phi_2;Q}$.

Here, we list a couple of properties of Φ-averages. The first one extends the fact that $|Q| \cdot m_Q^{(p)}(f)$ is increasing in Q for all $p \ge 1$ and $f \in L^0(Q)$.

Lemma 57. *Let $f \in L^0(\mathbb{R}^n)$. Then for all cubes $Q_1 \subset Q_2$,*

$$|Q_1| \cdot \|f\|_{\Phi;Q_1} \lesssim |Q_2| \cdot \|f\|_{\Phi;Q_2}.$$

Proof We may assume that the right-hand side is finite or more strongly equals 1. Then

$$\frac{1}{|Q_2|}\int_{Q_2} \Phi(|Q_2| \cdot |f(x)|)\mathrm{d}x \le 1.$$

Since Φ is a Young function and $Q_1 \subset Q_2$, we have

$$\frac{1}{|Q_1|}\Phi(|Q_1| \cdot |f(x)|) \le \frac{1}{|Q_2|}\Phi(|Q_2| \cdot |f(x)|) \quad (x \in \mathbb{R}^n).$$

It remains to integrate this pointwise estimate over Q_2 using Lemma 42 and discard the contribution of $Q_2 \setminus Q_1$ in the left-hand side.

Let $Q \in \mathcal{Q}$, and let $\mathcal{U}$ be any partition of Q into cubes. Also let $f \in L^1(Q)$. Then by the pigeon-hole principle, we have $m_R(|f|) \geq m_Q(|f|)$ for some $R \in \mathcal{U}$. We will generalize this fact to the Φ-average as follows:

Lemma 58. *Let $f \in L^0(\mathbb{R}^n)$. Then $\|f\|_{\Phi;Q} \leq 2^n \sup_{Q' \in \mathcal{Q}(Q) : \ell(Q') = t'} \|f\|_{\Phi;Q'}$ for any cube $Q \in \mathcal{Q}$ and any positive number $t' \leq \ell(Q)$.*

Proof By the monotone convergence theorem, we may assume $f \in L^\infty(\mathbb{R}^n)$. By the normalization, we may also assume $\|f\|_{\Phi;Q} = 1$. Let $t' \in (0, \ell(Q)]$ be fixed. Let

$$N = \left[1 + \frac{\ell(Q)}{t'}\right] \in [1, \infty).$$

We write $Q = \prod_{j=1}^n [a_j, a_j + \ell(Q)]$, so that $(a_1, a_2, \ldots, a_n)$ is the "bottom" corner of Q. Define

$$\delta(m_j) \equiv \begin{cases} 0 & (m_j < N), \\ Nt' - \ell(Q) & (m_j = N) \end{cases}$$

for $m_j \in \{1, 2, \ldots, N\}$ and

$$Q_m \equiv \prod_{j=1}^n [a_j + (m_j - 1)t' - \delta(m_j), a_j + m_j t' - \delta(m_j)]$$

for $m = (m_1, m_2, \ldots, m_n) \in \{1, 2, \ldots, N\}^n$. Then

$$1 \leq \sum_{m \in \{1,2,\ldots,N\}^n} \chi_{Q_m} \leq 2^n$$

almost everywhere. Consequently,

$$\begin{aligned} 1 &= \frac{1}{|Q|} \int_Q \Phi(|f(x)|) \mathrm{d}x \\ &\leq \sum_{m \in \{1,2,\ldots,N\}^n} \frac{1}{|Q|} \int_{Q_m} \Phi(|f(x)|) \mathrm{d}x \\ &= \sum_{m \in \{1,2,\ldots,N\}^n} \frac{t'^n}{|Q|} \frac{1}{|Q_m|} \int_{Q_m} \Phi(|f(x)|) \mathrm{d}x. \end{aligned}$$

Since

$$\frac{t'N}{\ell(Q)} = \frac{t'}{\ell(Q)} \left[1 + \frac{\ell(Q)}{t'}\right] \leq \frac{t'}{\ell(Q)} \left(1 + \frac{\ell(Q)}{t'}\right) \leq \frac{t'}{\ell(Q)} + 1 \leq 2,$$

it follows that

$$\frac{1}{|Q|} \int_Q \Phi(|f(x)|) \mathrm{d}x \leq 2^n \max_{m \in \{1,2,\ldots,N\}^n} \frac{1}{|Q_m|} \int_{Q_m} \Phi(|f(x)|) \mathrm{d}x.$$

Thus, the right-hand side equals ∞. If the left-hand side is finite, then we argue similarly.

1.5.4 Lebesgue spaces with a variable exponent

Here, we content ourselves with the definitions. Lebesgue spaces with a variable exponent can be defined similarly to Orlicz spaces.

Definition 27 (Variable (exponent) Lebesgue space)**.** Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $p(\cdot) : X \to [1, \infty]$ be a measurable function. Then the *variable exponent Lebesgue space* $L^{p(\cdot)}(\mu)$ *with a variable exponent* is defined by

$$L^{p(\cdot)}(\mu) \equiv \bigcup_{\lambda>0} \{f \in L^0(\mu) \,:\, \rho_p(\lambda^{-1} f) < \infty\},$$

where

$$\rho_p(f) \equiv \| |f|^{p(\cdot)} \chi_{(0,\infty)}(p)\|_{L^1(\mu)} + \|f\chi_{\{\infty\}}(p)\|_{L^\infty(\mu)}.$$

Moreover, for $f \in L^{p(\cdot)}(\mu)$ one defines the *variable Lebesgue norm* by

$$\|f\|_{L^{p(\cdot)}(\mu)} \equiv \inf\left(\left\{\lambda \in (0,\infty) \,:\, \rho_p(\lambda^{-1} f) \le 1\right\} \cup \{\infty\}\right).$$

1.5.5 Exercises

Exercise 34. Show that any homeomorphism from $[0, \infty)$ to $[0, \infty)$ leaves 0 invariant. Hint: Count the number of connected components of the set $[0, \infty) \setminus \{a\}$ for $a \in [0, \infty)$.

Exercise 35.

(1) Let $p > 1$ and $t > 0$. Calculate $\sup_{s\ge 0}(st - s^p)$.

(2) Calculate $\sup_{s\ge 0}(st - e^s + 1)$ for $t > 1$.

Exercise 36. [192, Theorem 1.3] Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function, and let $f \in L^0(\mathbb{R}^n)$.

(1) Show that $\liminf_{r\downarrow 0} \dfrac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \ge |f(x)|$ for almost all $x \in \mathbb{R}^n$.

(2) Assume that $\Phi \in \Delta_2$. Then prove that $\liminf_{r\downarrow 0} \dfrac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} = |f(x)|$ for almost all $x \in \mathbb{R}^n$.

1.6 Smoothness function spaces

The simplest function spaces appearing in analysis seems $\mathrm{BC}(X)$, the set of all complex-valued continuous functions defined on a metric space (X, d). This function space appears as an example or an exercise of complete metric spaces.

The next example we encounter in the course of analysis is the Lebesgue space $L^p(\mathbb{R}^n)$. Unlike the space $\mathrm{BC}(X)$, it had been very difficult to prove its completeness. One of the reasons why the theory of function spaces is prevailing is that the triangle inequality is available in addition to the completeness. Everything is fine as long as we have only to work within the framework of $\mathrm{BC}(X)$ or $L^p(\mathbb{R}^n)$.

However, things are not so nice once we consider many other mathematical problems. For example, these spaces are not sufficient, when we consider the Laplace equation $-\Delta u = f$. For the purpose of tackling this equation, we are led to thinking of Sobolev spaces. Indeed, in this case, it is convenient to introduce

$$\|f\|_{W^{1,2}} = \sqrt{\|f\|_{L^2}{}^2 + \sum_{j=1}^{n} \|\partial_j f\|_{L^2}{}^2} \quad \text{(Sobolev space)}$$

because we can use the integration by parts.

We define Sobolev spaces in Section 1.6.1 as an application of Lebesgue spaces. We define Hölder–Zygmund spaces in Section 1.6.2. Remark that we do not recall the definition of multi-indexes; see [117, Appendix A] or [258, Chapter 2].

1.6.1 Sobolev spaces

The aim of this section is very modest. We intend to define the Sobolev space $W^{m,p}(\mathbb{R}^n)$ and investigate some elementary properties. We do not go into detail, for example, the trace theorem, the extension theorem and so on are not taken up here. We content ourselves with making a brief view of the function spaces.

Since $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is too large, it may be surprising that we can consider the derivatives of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. It may be indirect to define them in the next definition. We work in a domain Ω to consider the function spaces defined there. Here by a domain we mean an open set in $\mathbb{R}^n$. But this indirect definition does work as we shall see.

Definition 28 (Partial derivative)**.** Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0{}^n$. A weak partial α-derivative of f, if it exists, is $g \in L^1_{\mathrm{loc}}(\Omega)$ such that $\displaystyle\int_\Omega f(x)\partial^\alpha \varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega g(x)\varphi(x)\,\mathrm{d}x$ for all $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$.

The following lemma is easy to prove.

Lemma 59. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0{}^n$.*

(1) *The weak partial α-derivative of f, if it exists, is unique.*

(2) *If $f \in C^{|\alpha|}(\mathbb{R}^n)$, then $\partial^\alpha f$ coincides with the usual partial α-derivative.*

Proof The proof is left as an exercise; see Exercise 37.

Example 49. Let $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ and define $\psi(x) \equiv |x|\varphi(x)$ for $x \in \mathbb{R}^n$. Then we have

$$\partial_j \psi(x) = |x|\partial_j \varphi(x) + \frac{x_j}{|x|}\varphi(x) \tag{1.1}$$

in the sense of weak derivatives. To check this, we take $\eta \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ arbitrarily. For the sake of simplicity we can assume $j = n$. Then if we carry out integration by parts, we obtain

$$\begin{aligned}
&\int_{|x_n|>\varepsilon} \psi(x) \cdot \partial_n \eta(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{n-1}} (\psi(x' + \varepsilon e_n)\eta(x' + \varepsilon e_n) - \psi(x' - \varepsilon e_n)\eta(x' - \varepsilon e_n))\,\mathrm{d}x' \\
&\quad - \int_{|x_n|>\varepsilon} \partial_n \psi(x) \cdot \eta(x)\,\mathrm{d}x.
\end{aligned}$$

If we let $\varepsilon \downarrow 0$, then the left-hand side tends to $\displaystyle\int_{\mathbb{R}^n} \psi(x)\partial_n \eta(x)\,\mathrm{d}x$ and the boundary terms of the right-hand side cancel. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(x)\partial_n \eta(x)\,\mathrm{d}x &= -\lim_{\varepsilon \downarrow 0} \int_{|x_n|>\varepsilon} \partial_n \psi(x)\eta(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \left(|x|\partial_j \varphi(x) + \frac{x_j}{|x|}\varphi(x) \right) \eta(x)\,\mathrm{d}x.
\end{aligned}$$

Thus, we obtain (1.1), although $\psi \notin C^1(\mathbb{R}^n)$ in general!

As we have seen before, $C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ is not complete, if we endow it with the $L^p(\mathbb{R}^n)$-norm. If p is finite, then $L^p(\mathbb{R}^n)$ is a completion of $C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ with the $L^p(\mathbb{R}^n)$-norm. The space $L^{\infty}(\mathbb{R}^n)$ is a natural extension of the $L^p(\mathbb{R}^n)$-norm.

The failure of completeness is the case, if we endow $C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ with a norm given by

$$\|f\|_{W^{m,p}} \equiv \sum_{\substack{\alpha \in {\mathbb{N}_0}^n \\ |\alpha| \le m}} \|\partial^{\alpha} f\|_{L^p}.$$

We can say that $W^{m,p}(\mathbb{R}^n)$, whose precise definition we are about to present, overcomes this defect, if $p < \infty$.

Definition 29 (The Sobolev spaces in $L^p(\mathbb{R}^n)$ with m derivatives)**.** Let $m \in \mathbb{N}_0$, and let $1 \le p \le \infty$. Then define *the Sobolev space* $W^{m,p}(\mathbb{R}^n)$ by:

$$\begin{aligned}
&W^{m,p}(\mathbb{R}^n) \\
&\equiv \bigcap_{|\alpha| \le m} \{ f \in L^p(\mathbb{R}^n) : \partial^{\alpha} f \text{ exists in weak sense and belongs to } L^p(\mathbb{R}^n) \}.
\end{aligned}$$

The norm of $f \in W^{m,p}(\mathbb{R}^n)$ is defined by: $\|f\|_{W^{m,p}} \equiv \displaystyle\sum_{\substack{\alpha \in {\mathbb{N}_0}^n \\ |\alpha| \le m}} \|\partial^{\alpha} f\|_{L^p}$.

Let $1 \le p \le \infty$. From the viewpoint of $W^{m,p}(\mathbb{R}^n)$, we see that $f \in L^p(\mathbb{R}^n)$, if and only if $\partial^\alpha f \in L^p(\mathbb{R}^n)$ for all $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le m$.

Theorem 60. *Let $1 \le p \le \infty$ and $m \in \mathbb{N}$. Then the Sobolev space $W^{m,p}(\mathbb{R}^n)$ is a Banach space.*

Proof The proof is left as an exercise; see Exercise 38.

Next, we are going to discuss the mollification and consider the dense subspace of $W^{m,p}(\mathbb{R}^n)$.

Lemma 61. *Let $1 \le p < \infty$. Let $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ be a positive function with integral 1. Then for given $f \in W^{m,p}(\mathbb{R}^n)$ and $t > 0$, we set $f_t \equiv \varphi_t * f$, where $\varphi_t = \dfrac{1}{t^n}\varphi\left(\dfrac{\cdot}{t}\right)$. Then we have $\lim\limits_{t \downarrow 0} f_t = f$ in the topology of $W^{m,p}(\mathbb{R}^n)$.*

Proof Note that $\partial_j f_t = \varphi_t * \partial_j f$ for $j = 1, 2, \dots, n$ because $f \in W^{1,p}(\mathbb{R}^n)$. Therefore, this lemma is true for $|\alpha| = 1$. In the general case we have only to induct on $|\alpha|$.

Corollary 62. *Let $1 \le p < \infty$. Then $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$.*

Proof By Lemma 61, we have only to approximate smooth elements $f \in W^{m,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. Choose a truncation function $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ that equals 1 on $B(1)$. Set $f_m \equiv f\eta\left(\frac{\cdot}{m}\right)$. Then by the Lebesgue convergence theorem f_m tends to f in $W^{m,p}(\mathbb{R}^n)$.

We next consider the relation between the difference quotient and the weak-partial derivative. Classically the differentiation was defined by the limit, that is, it was given by the limit of the difference quotient. However, the weak derivative we take up here seems to have nothing to do with the difference quotient. In this paragraph, we investigate the connection between the difference quotient and the weak-partial derivative. The vector $e_k \equiv (\delta_{kj})_{j=1}^n$ stands for the k-th elementary vector.

Definition 30 (difference quotient)**.** Let $h \in \mathbb{R}\setminus\{0\}$ and $k = 1, 2, \dots, n$. Then for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define its difference quotient $D_k^h f$ by $D_k^h f \equiv \dfrac{f(\cdot + he_k) - f}{h}$.

Theorem 63. *Let $1 < p < \infty$. Then $f \in L^p(\mathbb{R}^n)$ belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $\sup\limits_{j=1,2,\dots,n} \sup\limits_{h \in \mathbb{R}\setminus\{0\}} \|D_j^h f\|_{L^p} \lesssim 1$.*

Proof The proof uses the weak compactness of $L^p(\mathbb{R}^n)$ and is left as an exercise; see Exercise 39.

Now we consider the chain rule. What is totally different from the classical analysis is that if $f \in W^{1,p}(\mathbb{R}^n)$, then we have $|f| \in W^{1,p}(\mathbb{R}^n)$. This property fails for the space $C^1(\mathbb{R}^n)$ because the limit of the difference quotient comes into play. The next theorem is the first step for this purpose.

Theorem 64. *Let $1 < p < \infty$. Assume that $\eta \in \mathrm{BC}^1(\mathbb{C})$ with $\eta(0) = 0$. Also let $f \in W^{1,p}(\mathbb{R}^n)$. Then $\eta \circ f \in W^{1,p}(\mathbb{R}^n)$.*

Here if $f(x) = \pm\infty$, then it will be understood that $\eta \circ f(x) = 0$ and below we shall disregard such a point. We shall not allude to this point later.

Proof The proof is left as an exercise; see Exercise 40.

We next consider the Sobolev embedding theorem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function with the first derivative integrable over $\mathbb{R}$. Then $f(x) = \displaystyle\int_0^x f'(t)\,\mathrm{d}t + f(0)$ and hence f is bounded. In this way if we have some information on the partial derivatives, then we can say more about the function. The Sobolev embedding theorem quantifies such a situation.

We write $Du \equiv (\partial_{x_1} u, \partial_{x_2} u, \dots, \partial_{x_n} u)$ for a function u. For a Banach lattice X, we define $\|Du\|_X \equiv \| \, |Du| \, \|_X$.

Theorem 65 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume $1 \le p < n$. Define q by $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{n}$. Then $\|u\|_{L^q} \lesssim_{p,n} \|Du\|_{L^p}$ for all $u \in C^1_{\mathrm{c}}(\mathbb{R}^n)$.*

Proof The proof is left as an exercise; see Exercise 41.

Function spaces are tools with which to describe the size and the smoothness of functions. Let $p, q, m \in [1, \infty]$ and suppose that m is an integer. For example, $L^q(\mathbb{R}^n)$ measures the size. Since the pointwise multiplication of $L^\infty(\mathbb{R}^n)$-functions is closed in $L^q(\mathbb{R}^n)$, $L^q(\mathbb{R}^n)$ never measures the smoothness of functions. However, $W^{m,p}(\mathbb{R}^n)$ can control somehow the smoothness of functions. What counts about size and smoothness is that the control of smoothness can be transformed into that of size. This aspect of these two quantities can be illustrated by the next theorem.

Theorem 66. *Let $m \in \mathbb{N}$ and $1 \le p < \dfrac{n}{m}$. Then define q by $m - \dfrac{n}{p} = -\dfrac{n}{q}$. Then $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$.*

Proof The proof is left as an exercise; see Exercise 41.

The quantity $m - \dfrac{n}{p}$ in the space $W^{m,p}(\mathbb{R}^n)$ has a special name.

Definition 31 (Differential/Sobolev index)**.** In $W^{m,p}(\mathbb{R}^n)$, the quantity $m - \dfrac{n}{p}$ is referred to as the differential/Sobolev index of functions.

We move on to the extension property. The extension problem is a classical problem in the theory of function spaces with important applications in many fields of mathematical analysis, in particular harmonic analysis and the theory of partial differential equations. Broadly speaking, the problem consists in extending the elements of a space of functions defined on a given subset of $\mathbb{R}^n$ to the whole of $\mathbb{R}^n$, preserving certain differentiability and summability properties.

We consider elementary Lipschitz domains Ω in $\mathbb{R}^n$ of the form

$$\Omega = \Omega_\varphi \equiv \{x = (x', x_n) \in \mathbb{R}^n : x_n < \varphi(x'), x' \in \mathbb{R}^{n-1}\}, \tag{1.2}$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz continuous function.

We consider function spaces on Lipschitz domains above as a model case of bounded domains having a Lipschitz boundary. Since φ is a Lipschitz function, $x \in \Omega_\varphi$ and $y_n > L|y'|$ implies $x + y \in \Omega_\varphi$. The following lemma takes full advantage of the property of Lipschitz domains.

Lemma 67. *Let $1 \le p < \infty$. Let Ω be an elementary domain as above. Then $W^{2,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $W^{2,p}(\Omega)$.*

Proof We start with the setup: Let $\psi \in C_c^\infty(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ be such that it is supported in a cone $\{y_n > L|y'|\}$. Define $\psi_t(x) \equiv t^{-n}\psi(t^{-1}x)$ for $x \in \mathbb{R}^n$ and $t > 0$.

For $f \in W^{2,p}(\Omega)$, we define $f_t \equiv f * \psi_t$. Then $D^k f_t = \psi_t * D^k f$ for $k = 0, 1, 2$ and $f_t \in C^\infty(\Omega)$. Thus, we see that $f_t \to f$ in $W^{2,p}(\mathbb{R}^n)$ as $t \downarrow 0$.

Let $G \equiv \mathbb{R}^n \setminus \overline{\Omega}$. For every $k \in \mathbb{Z}$, set $G_k \equiv \{x \in G : 2^{-k-1} < \rho_n(x) \le 2^{-k}\}$, where $x \in \mathbb{R}^n \mapsto \rho_n(x) \equiv x_n - \varphi(x')$ is the *signed distance* from $x \in \mathbb{R}^n$ to ∂G in the x_n-direction. Let $\tilde{G}_k \equiv G_{k-1} \cup G_k \cup G_{k+1}$. Clearly, $\rho_n(x) \ge 0$ for all $x \in G$. Write $M \equiv \|\varphi\|_{\mathrm{Lip}}$ and $A \equiv 16(M+1)$.

In the sequel, we need the partition of unity subordinate to $\{G_k\}_{k=-\infty}^\infty$.

Lemma 68. *There exists a collection $\{\psi_k\}_{k=-\infty}^\infty \subset C^\infty(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ satisfying the following conditions*:

(1) $\chi_G = \sum_{k=-\infty}^{\infty} \psi_k \le \sum_{k=-\infty}^{\infty} \chi_{\mathrm{supp}(\psi_k)} \le 2\chi_G$;

(2) $|D^\alpha \psi_k| \lesssim_\alpha 2^{k|\alpha|} \chi_{\tilde{\Omega}_k}$ *for all* $k \in \mathbb{Z}, \alpha \in \mathbb{N}_0^n$.

Proof Let $\tau \in C_c^\infty(B(r)) \cap \mathbb{M}^+(\mathbb{R}^n)$ for some small $r > 0$ and $\|\tau\|_{L^1} = 1$. Set $\tilde{\psi}_k \equiv 2^{kn}\tau(2^k \cdot) * \chi_{\rho_n^{-1}(2^{-k-1}, 2^{-k})}$ for each $k \in \mathbb{Z}$. Since $\varphi \in \mathrm{Lip}(\mathbb{R}^n)$, if we choose $r \ll 1$, then we have $\mathrm{supp}(\tilde{\psi}_k) \subset \rho_n^{-1}(0.9 \cdot 2^{-k-1}, 1.1 \cdot 2^{-k})$.

We note $|D^\alpha \tilde{\psi}_k| \lesssim_\alpha 2^{k|\alpha|} \chi_{\tilde{\Omega}_k}$ for all $k \in \mathbb{Z}, \alpha \in \mathbb{N}_0^n$. Thus, if we set $\tilde{\psi} \equiv \sum_{l=-\infty}^{\infty} \tilde{\psi}_l$ and $\psi_k \equiv \dfrac{\tilde{\psi}_k}{\tilde{\psi}}$ for each $k \in \mathbb{Z}$, then we obtain the desired function.

Define

$$d(x) \equiv \sum_{k=-\infty}^{\infty} 2^k \psi_k(x) \quad (x \in \mathbb{R}^n).$$

We are now ready to define Burenkov's extension operator for an elementary Lipschitz domain Ω as in (1.2). Let $l \in \mathbb{N}_0$ and $1 \le p \le \infty$. Write $M \equiv \|\varphi\|_{\mathrm{Lip}}$ and $A \equiv 16(M+1)$. The following lemma explains the role of A:

Lemma 69. *Let $a > 0$ be fixed. Then $(x' - 2^{-k}z', x_n - A2^{-k}z_n) \in \Omega$ for all $k \in \mathbb{Z}$, $x \in G$ and $z \in (-a,a)^{n-1} \times (2a,4a)$.*

Proof This is clear from the definition of the Lipschitz norm.

We are now ready for the construction of Burenkov's extension operator. Choose $\omega \in C_c^\infty((-a,a)^{n-1} \times (2a,4a)) \cap \mathbb{M}^+(\mathbb{R}^n)$ so that $\|\omega\|_{L^1} = 1$. For every $f \in L^1_{\mathrm{loc}}(\Omega)$ and $x \in G$, we set

$$\begin{aligned} f_k(x) &\equiv \int_{(-a,a)^{n-1}\times(2a,4a)} f(x' - 2^{-k}z', x_n - A2^{-k}z_n)\omega(z)\mathrm{d}z \\ &= A^{-1}2^{kn}\int_\Omega \omega(2^k(x'-y'), A^{-1}2^k(x_n - y_n))f(y)\mathrm{d}y \end{aligned}$$

and

$$Tf(x) \equiv \begin{cases} f(x), \text{ if } x \in \Omega, \\ \displaystyle\sum_{k=-\infty}^{\infty} \psi_k(x) f_k(x), \text{ if } x \in G \equiv \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{1.3}$$

Note that $\partial\Omega$ has Lebesgue measure zero, so that Tf is defined almost everywhere on $\mathbb{R}^n$. We note that the operator T in (1.3) is defined by means of a sequence of mollifiers with discrete steps and has a local nature in the sense that the values of the extended function Tf around a point in $\mathbb{R}^n \setminus \Omega$ depend only on the values of f localized around certain "reflected" points inside Ω.

Theorem 70. *Let $l \in \mathbb{N}$. Let $f \in C^l(\Omega)$ satisfy $\partial^\alpha f \in L^\infty(\Omega)$ for any $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le l$, so that its derivative up to any order extends continuously to the boundary. Then $Tf \in C^l(\mathbb{R}^n)$.*

Proof Since $Tf \in C^\infty(\mathbb{R}^n \setminus \partial\Omega)$, it suffices to prove $\lim_{x\in G, x\to x_0} \partial^\alpha Tf(x) = \partial^\alpha f(x_0)$ for all $x_0 \in \partial\Omega$ and $|\alpha| \le l$. We let $x = (x', x_n) \in G$. We set

$$\begin{aligned} &Z(x) \\ &\equiv (\partial^\alpha f(x' - 2^{-k}z', x_n - A2^{-k}z_n) - \partial^\alpha f(x', \varphi(x')))\omega(z), \\ &Z'(x) \\ &\equiv \int_{(-a,a)^{n-1}\times(2a,4a)} (\partial^\beta f(x' - 2^{-k}z', x_n - A2^{-k}z_n) - \partial^\beta f(x', \varphi(x')))\omega(z)\mathrm{d}z \\ &\zeta(s) = \zeta(s; k, x, z, A, \beta) \\ &\equiv \partial^\beta f(x' - 2^{-k}sz', s(x_n - A2^{-k}z_n) + (1-s)\varphi(x')) - \partial^\beta f(x', \varphi(x')). \end{aligned}$$

Since $\sum_{k=-\infty}^{\infty} \partial^\gamma \psi_k(x) = 0$ for any $\gamma \in {\mathbb{N}_0}^n \setminus \{0\}$, we have

$$\begin{aligned} &D^\alpha Tf(x) - \partial^\alpha f(x', \varphi(x')) \\ &= \sum_{k=-\infty}^{\infty} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\psi_k(x) Z'(x) + \sum_{k=-\infty}^{\infty} \psi_k(x) \int_{(-a,a)^{n-1}\times(2a,4a)} Z(z)\mathrm{d}z. \end{aligned}$$

Denote by $\zeta_l = \zeta_l(\cdot; k, x, z, A, \beta)$ the remainder term of the Taylor expansion up to order l at $s = 0$. Since ω satisfies $\omega \in C_c^\infty((-a,a)^{n-1} \times (2a,4a)) \cap \mathbb{M}^+(\mathbb{R}^n)$ and $\sum_{k=-\infty}^{\infty} \partial^\beta \psi_k \equiv 0$ of any multi-index $\beta \in \mathbb{N}_0{}^n \setminus \{0\}$, we obtain

$$\begin{aligned}
&D^\alpha Tf(x) - \partial^\alpha f(x', \varphi(x)) \\
&= \sum_{k=-\infty}^{\infty} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\psi_k(x) \int_{(-a,a)^{n-1}\times(2a,4a)} \zeta_l(1; k, x, z, A, \beta)\omega(z)\mathrm{d}z \\
&\quad + \sum_{k=-\infty}^{\infty} \psi_k(x) \int_{(-a,a)^{n-1}\times(2a,4a)} Z(z)\mathrm{d}z.
\end{aligned}$$

Consequently, we obtain $|D^\alpha Tf(x) - \partial^\alpha f(x', \varphi(x))| \lesssim d(x)$ for all $x \in G$, as required.

Lemma 71. *Let $x \in G$, and let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$|Tf(x)| \lesssim \int_{(-2a,2a)^{n-1}\times(aA,8aA)} |f(x' - (x_n - \varphi(x'))z', x_n - (x_n - \varphi(x'))z_n)|\mathrm{d}z.$$

Proof It suffices to handle each piece f_k, $k \in \mathbb{Z}$. To handle $f_k(x)$, $x \in G$, we may assume $x \in G_k$, so that $2^{-k-1} \le x_n - \varphi(x') \le 2^{-k}$. Observe that

$$\begin{aligned}
&|f_k(x)| \\
&\lesssim 2^{kn} \int_\Omega \chi_{(-a,a)^{n-1}\times(2a,4a)}(2^k(x' - y'), A^{-1}2^k(x_n - y_n))|f(y)|\mathrm{d}y \\
&= 2^{kn} \int_{(-2^{-k}a,2^{-k}a)^{n-1}\times(2^{-k+1}aA,2^{-k+2}aA)} |f(x - w)|\mathrm{d}w \\
&\le 2^{kn} \int_{(-2ax_n+2a\varphi(x'),2ax_n-2a\varphi(x')))^{n-1}\times(A(x_n-\varphi(x')),8A(x_n-\varphi(x')))} |f(x - w)|\mathrm{d}w \\
&\sim \int_{(-2a,2a)^{n-1}\times(aA,8aA)} |f(x' - (x_n - \varphi(x'))z', x_n - (x_n - \varphi(x'))z_n)|\mathrm{d}z.
\end{aligned}$$

We investigate the transformation in Lemma 71.

Lemma 72. *Suppose $0 < a \ll 1$. Let $z \in (-2a, 2a)^{n-1} \times (aA, 8aA)$. If a is small enough, then the mapping*

$$\Phi_z : x \in \mathbb{R}^n \mapsto (x' - (x_n - \varphi(x'))z', x_n - (x_n - \varphi(x'))z_n) \in \mathbb{R}^n,$$

is bi-Lipschitz with the constants independent of z.

Proof Since the Lipschitz norm of the mapping $x \in \mathbb{R}^n \mapsto (-(x_n - \varphi(x'))z', -(x_n - \varphi(x'))z_n) \in \mathbb{R}^n$ is small as long as a is small, the conclusion is clear.

Note that Φ_z is the composition $\Phi_z^2 \circ \Phi_z^1$, where

$$\Phi_z^1(x) \equiv (x', x_n - \varphi(x'))$$
$$\Phi_z^2(x) \equiv (x' - x_n z', x_n + \varphi(x') - x_n z_n).$$

As a consequence, the mapping Φ_z is a homeomorphism (see Exercise 44).

We move on to the interpolation property. We have the interpolation formula:

Theorem 73. *Let φ be a Lipschitz function. Define Ω_φ as above. Then for all $f \in W^{2,p}(\Omega_\varphi)$, $\|Df\|_{L^p(\Omega_\varphi)} \lesssim \|D^2 f\|_{L^p(\Omega_\varphi)} + \|f\|_{L^p(\Omega_\varphi)}$, where the implicit constant depends only on $\|\varphi\|_{\mathrm{Lip}}$.*

We need an inequality for functions defined in a compact interval $[0,1]$. This inequality also explains the meaning of interpolation.

Lemma 74. *Let $f \in C^2[0,1]$. Then $|f'(0)| \lesssim \|f''\|_{L^1([0,1])} + \|f\|_{L^1([0,1])}$.*

Proof If $x \in \left[\frac{3}{4}, 1\right]$ and $y \in \left[\frac{1}{4}, \frac{1}{2}\right]$, then $f(x) - f(y) = (x-y)f'(z)$ for some $z \in \left[\frac{1}{2}, \frac{3}{4}\right]$ by the mean-value theorem. Thus,

$$\begin{aligned} |f'(0)| &\le |f'(0) - f'(z)| + |f'(z)| \\ &\le \|f''\|_{L^1([0,1])} + 4|f(x) - f(y)| \\ &\le \|f''\|_{L^1([0,1])} + 4|f(x)| + |4f(y)|. \end{aligned}$$

If we take the average over $x \in \left[\frac{3}{4}, 1\right]$ and $y \in \left[\frac{1}{4}, \frac{1}{2}\right]$, we obtain the desired result.

Proof of Theorem 73 Write $\Sigma \equiv \{z \in \mathbb{R}^{n-1} : |z| = 1, z_n < -M|z'|\}$. Equip Σ with the (hyper)surface measure σ. We may suppose that $f \in C^\infty(\Omega_\varphi)$ thanks to Lemma 67. Then we have

$$\int_\Sigma |v \cdot Df(x)|^p \mathrm{d}\sigma(v) \sim |Df(x)|^p \quad (x \in \Omega).$$

With $v \in \Sigma$ and $x \in \Omega_\varphi$ fixed, we consider $u(t) \equiv f(x + t\sigma)$. Then thanks to Lemma 74,

$$|v \cdot Df(x)| = |u'(0)| \lesssim \int_0^1 |u(t)|\mathrm{d}t + \int_0^1 |u''(t)|\mathrm{d}t,$$

or equivalently,

$$|v \cdot Df(x)| \lesssim \int_0^1 |f(x + t\sigma)|\mathrm{d}t + \int_0^1 |D^2 f(x + t\sigma)|\mathrm{d}t.$$

Consequently,

$$|Df(x)|^p \lesssim \int_\Sigma \int_0^1 |f(x + t\sigma)|^p \mathrm{d}t\mathrm{d}\sigma(v) + \int_\Sigma \int_0^1 |D^2 f(x + t\sigma)|^p \mathrm{d}t\mathrm{d}\sigma(v).$$

If we integrate this inequality against $x \in \Omega_\varphi$, we obtain

$$\begin{aligned}
&\int_\Omega |Df(x)|^p \mathrm{d}x \\
&\lesssim \int_\Omega \int_\Sigma \int_0^1 |f(x+t\sigma)|^p \mathrm{d}t\mathrm{d}\sigma(v)\mathrm{d}x + \int_\Omega \int_\Sigma \int_0^1 |D^2 f(x+t\sigma)|^p \mathrm{d}t\mathrm{d}\sigma(v)\mathrm{d}x \\
&\lesssim \int_\Omega |f(x)|^p \mathrm{d}x + \int_\Omega |D^2 f(x)|^p \mathrm{d}x.
\end{aligned}$$

Thus, we obtain the desired result.

Corollary 75. *Let $1 \le p < \infty$. For all $f \in W^{2,p}(\Omega_\varphi)$ and $\varepsilon > 0$,*

$$\|Df\|_{L^p(\Omega_\varphi)} \lesssim \varepsilon \|D^2 f\|_{L^p(\Omega_\varphi)} + \varepsilon^{-1} \|f\|_{L^p(\Omega_\varphi)},$$

where the implicit constant depends only on $\|\varphi\|_{\mathrm{Lip}}$.

Proof Let $\varphi_\varepsilon(x') \equiv \varepsilon^{-1}\varphi(\varepsilon x')$, $x' \in \mathbb{R}^{n-1}$. Then we have

$$\|Dg\|_{L^p(\Omega_{\varphi_\varepsilon})} \lesssim \|D^2 g\|_{L^p(\Omega_{\varphi_\varepsilon})} + \|g\|_{L^p(\Omega_{\varphi_\varepsilon})}$$

for all $g \in W^{2,p}(\Omega_{\varphi_\varepsilon})$. For $f \in W^{2,p}(\Omega_\varphi)$, the function $g = f(\varepsilon\cdot)$ belongs to $W^{2,p}(\Omega_{\varphi_\varepsilon})$. Thus,

$$\varepsilon \|Df(\varepsilon\cdot)\|_{L^p(\Omega_{\varphi_\varepsilon})} \lesssim \varepsilon^2 \|D^2 f(\varepsilon\cdot)\|_{L^p(\Omega_{\varphi_\varepsilon})} + \|f(\varepsilon\cdot)\|_{L^p(\Omega_{\varphi_\varepsilon})}.$$

If we change variables, we obtain

$$\|Df\|_{L^p(\Omega_\varphi)} \lesssim \varepsilon \|D^2 f\|_{L^p(\Omega_\varphi)} + \varepsilon^{-1} \|f\|_{L^p(\Omega_\varphi)}.$$

So, we obtain the desired result.

1.6.2 Hölder–Zygmund spaces

Unlike the class $C^r(\mathbb{R}^n)$, $r \in \mathbb{N}_0$, we need to take a couple of steps to define homogeneous Hölder–Zygmund spaces according to the parameter γ which is equipped with the space $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$. We deal with $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ for $0 < \gamma < 2$ and then we pass to the general parameters.

We want to define the homogeneous Hölder–Zygmund space $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ for $0 < \gamma < 1$ using the first order difference as follows.

Definition 32 (Homogeneous Lipschitz space)**.** For $f \in \mathrm{C}(\mathbb{R}^n)$ and $0 < \gamma < 1$, define

$$\|f\|_{\mathrm{Lip}^\gamma} \equiv \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\gamma} = \sup_{x,y\in\mathbb{R}^n, y\neq 0} |y|^{-\gamma} |f(x+y)-f(x)|.$$

The *homogeneous Lipschitz space* $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ is the set of all continuous functions f for which the norm $\|f\|_{\mathrm{Lip}^\gamma}$ is finite.

This definition of $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ is reasonable since the space $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ generalizes the familiar space $\mathrm{Lip}(\mathbb{R}^n)$ made up of all functions f satisfying $\|f\|_{\mathrm{Lip}} \equiv \sup_{x,y\in\mathbb{R}^n, x\neq y} |x-y|^{-1}|f(x)-f(y)| < \infty$. The trouble is that we cannot extend this definition naturally to $\gamma > 1$; the space $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ is nothing but $\mathcal{P}_0(\mathbb{R}^n)$ once we use the same formula using the first order difference.

So we consider another expression of $\mathrm{Lip}^\gamma(\mathbb{R}^n)$. We define $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2$ using the difference operator of order 2.

Definition 33 (Homogeneous Hölder–Zygmund space)**.** For $0 < \gamma < 2$ and a continuous function $f : \mathbb{R}^n \to \mathbb{C}$, define

$$\|f\|_{\dot{\mathcal{C}}^\gamma} \equiv \sup_{x,y\in\mathbb{R}^n, y\neq 0} |y|^{-\gamma}|f(x+y) - 2f(x) + f(x-y)|.$$

The *homogeneous Hölder–Zygmund space* $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ is the set of all continuous functions f for which the norm $\|f\|_{\dot{\mathcal{C}}^\gamma}$ is finite.

Remark that if $f \in C(\mathbb{R}^n)$ satisfies $\|f\|_{\dot{\mathcal{C}}^\gamma} = 0$ in Definition 33, then $f \in \mathcal{P}_1(\mathbb{R}^n)$, so that we regard $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ as a normed space modulo $\mathcal{P}_1(\mathbb{R}^n)$. See Exercise 42.

We will show that $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ and $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ are isomorphic for $0 < \gamma < 1$ in Theorem 82.

We employ Definition 33 to investigate the property of Hölder–Zygmund spaces. Let us now choose $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ so that

$$\chi_{B(1)} \le \psi = \psi(-\cdot) \le \chi_{B(2)}. \tag{1.4}$$

For $j \in \mathbb{Z}$, define

$$\psi^j \equiv 2^{jn}\psi(2^j\cdot), \quad \varphi^j \equiv \psi^j - \psi^{j-1}. \tag{1.5}$$

We show how to use the difference operator of order 2 in the next lemma.

Lemma 76. *Let $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2$. Then $\|\varphi^j * f\|_{L^\infty} \lesssim 2^{-j\gamma}\|f\|_{\dot{\mathcal{C}}^\gamma}$ for all $j \in \mathbb{Z}$.*

Proof We note that $\varphi^j * f(x) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n} \varphi^j(y)(f(x-y)-2f(x)+f(x+y))\mathrm{d}y$ since φ^j has zero integral and φ^j is an even function. It remains to use the triangle inequality and the definition of the norm in Definition 33.

Before we go further, we will develop the argument in Lemma 76. Let β be a multi-index with length 2. By the use of the Taylor expansion we see that $\varphi^j * \varphi^k$ and $\partial^\beta\varphi^j$ are even functions having zero integral and satisfying $|\varphi^j * \varphi^k| \lesssim 2^{\min(j,k)n-2|j-k|}\chi_{B(2^{\min(j,k)+2})}$, $|\partial^\beta\varphi^j| \lesssim 2^{j(2+n)}\chi_{B(2^{j+1})}$ and $|\partial^\alpha\varphi^j| \lesssim 2^{j(n+|\alpha|)}\chi_{B(2^{j+1})}$, which yields $\|\varphi^j * \varphi^k * f\|_{L^\infty} \lesssim 2^{\min(j,k)n-2|j-k|}\|f\|_{\dot{\mathcal{C}}^\gamma}$, $\|\partial^\beta\psi^j * f\|_{L^\infty} \lesssim 2^{j(2-\gamma)}\|f\|_{\dot{\mathcal{C}}^\gamma}$, and $\|\partial^\alpha(\varphi^j * \varphi^k * f)\|_{L^\infty} \lesssim \min(2^{j|\alpha|-k\gamma}, 2^{k|\alpha|-j\gamma})\|f\|_{\dot{\mathcal{C}}^\gamma}$ for all $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$. So we are led to the following estimate:

Lemma 77. *Let $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2$. Then $\|\varphi^k * \varphi^j * f\|_{L^\infty} \lesssim 2^{-2|j-k|-\min(j,k)\gamma}\|f\|_{\dot{\mathcal{C}}^\gamma}$ for all $j, k \in \mathbb{Z}$. In particular, $\sum_{j=-\infty}^{\infty} \varphi^j * \varphi^k * f = \varphi^k * f$ and $\|\partial^\alpha(\varphi^k * f)\|_{L^\infty} \lesssim 2^{k|\alpha|-k\gamma}\|f\|_{\dot{\mathcal{C}}^\gamma}$ for all $k \in \mathbb{Z}$, $\alpha \in {\mathbb{N}_0}^n$ and $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2$ and for all multi-indexes β with $|\beta| = 2$, $\lim_{j\to-\infty} \partial^\beta[\psi^j * f] = 0$ in $L^\infty(\mathbb{R}^n)$.*

Hence $\sum_{j=0}^{\infty} \|\varphi^j * f\|_{L^\infty} \lesssim \sum_{j=0}^{\infty} 2^{-j\gamma}\|f\|_{\dot{\mathcal{C}}^\gamma} \sim \|f\|_{\dot{\mathcal{C}}^\gamma}$ from Lemma 76, which leads us to the following definition:

Definition 34. Let $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2$. The function $H \equiv \sum_{j=0}^{\infty} \varphi^j * f$ is called the *high frequency part* of f. Meanwhile, the function $\tilde{G} \equiv \sum_{j=1}^{\infty} \varphi^{-j} * f$, which is defined formally, is called the *low frequency part* of f.

The trouble is that there is no guarantee that the right-hand side defining $\tilde{G}$ is convergent in some suitable topology. To circumvent this problem, we consider its derivative as follows:

Lemma 78. *Let $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2$. Let $\psi \in C_{\rm c}^\infty(\mathbb{R}^n)$ satisfy* (1.4). *Define φ^j by* (1.5) *for $j \in \mathbb{Z}$. Then $G_\alpha \equiv \sum_{j=-\infty}^{0} \partial^\alpha[\varphi^j * f]$ is convergent in $L^\infty(\mathbb{R}^n)$ whenever $|\alpha| \geq 2$. Hence, G_α is smooth.*

Proof Simply use Lemma 77.

To construct a substitute of the lower part, we need to depend on a geometric property of $\mathbb{R}^n$; $H^1_{\rm DR}(\mathbb{R}^n) = 0$. Applying this fact, we can prove the following:

Lemma 79. *Let $N \in \mathbb{N}$. Suppose that we have $\{f_\alpha\}_{\alpha\in{\mathbb{N}_0}^n, |\alpha|=N} \subset C^\infty(\mathbb{R}^n)$ satisfying the curl-free condition: $\partial_\beta f_\alpha = \partial_{\beta'} f_{\alpha'}$ for all $\alpha, \alpha', \beta, \beta' \in {\mathbb{N}_0}^n$ with $|\alpha| = |\alpha'| = N$ and $\alpha + \beta = \alpha' + \beta'$. Then there exists $f \in C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha f = f_\alpha$.*

Proof We prove this lemma by induction on N. If $N = 1$, then the result is immediate from the Poincaré lemma, or equivalently, $H^1_{\rm DR}(\mathbb{R}^n) = 0$. Let $N \geq 2$. Define $g_{j\gamma} \equiv f_{e_j+\gamma}$ if $j = 1, 2, \ldots, n$ and $|\gamma| = N - 1$. Let $e_j \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the j-th lot. Then $\partial^\delta g_{j\gamma} = \partial^\delta f_{e_j+\gamma} = \partial^{\delta'} f_{e_{j'}+\gamma} = \partial^{\delta'} g_{j'\gamma}$ for all $j, j' = 1, 2, \ldots, n$, $\gamma, \delta, \delta' \in {\mathbb{N}_0}^n$ with $|\gamma| = N - 1$ and $e_j + \delta = e_{j'} + \delta'$. Thus, we are in the position of applying the Poincaré lemma to have $g_\gamma \in C^\infty(\mathbb{R}^n)$ satisfying $g_{j\gamma} = \partial^j g_\gamma$ for all γ with $|\gamma| = N - 1$ and $j = 1, 2, \ldots, n$.

Let $\alpha, \alpha', \gamma, \gamma' \in \mathbb{N}$ satisfy $|\gamma| = |\gamma'| = N-1$ and $|\alpha| = |\alpha| > 0$ and $\alpha+\gamma = \alpha' + \gamma'$. Suppose $\alpha - e_j, \alpha' - e_{j'} \geq 0$. Then $\partial^\alpha g_\gamma = \partial^{\alpha-e_j}\partial^j g_\gamma = \partial^{\alpha-e_j} g_{j\gamma} =$

$\partial^{\alpha-e_j} f_{e_j+\gamma}$ and $\partial^{\alpha'} g_{\gamma'} = \partial^{\alpha'-e_{j'}} \partial^{j'} g_{\gamma'} = \partial^{\alpha'-e_{j'}} g_{j'\gamma'} = \partial^{\alpha'-e_{j'}} f_{e_{j'}+\gamma'}$. Since $(\alpha - e_j) + (e_j + \gamma) = (\alpha' - e_{j'}) + (e_{j'} + \gamma')$, we obtain $\partial^\alpha g_\gamma = \partial^{\alpha'} g_{\gamma'}$, thus, by the induction assumption to have a function $f \in C^\infty(\mathbb{R}^n)$ such that $g_\gamma = \partial^\gamma f$. Consequently, if $|\alpha| = N$, then by choosing j so that $e_j \le \alpha$ we have $f_\alpha = \partial_j g_{\alpha-e_j} = \partial_j \partial^{\alpha-e_j} f = \partial^\alpha f$.

We go back to the analysis of the low frequency part. As before choose $\psi \in C_c^\infty(\mathbb{R}^n)$ so that $\chi_{B(1)} \le \psi = \psi(-\cdot) \le \chi_{B(2)}$. Define $\varphi^j \equiv 2^{jn}\psi(2^j\cdot) - 2^{(j-1)n}\psi(2^{j-1}\cdot)$ for $j \in \mathbb{Z}$. By using Lemma 79, we have the following control of the low frequency part:

Corollary 80. *Let $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfy (1.4). Define φ^j by (1.5) for $j \in \mathbb{Z}$. There exists a function $G \in C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha G = \sum_{j=-\infty}^{0} \partial^\alpha \left[\varphi^j * f\right]$ for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \ge 2$.*

We further investigate the property of G in Corollary 80.

Proposition 81. *Let $0 < \gamma < 2$ and $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$. Define the low frequency part G and the high frequency part H as in Definition 34 and Corollary 80. Then $f - G - H \in \mathcal{P}_1(\mathbb{R}^n)$.*

Proof Let $\alpha \in {\mathbb{N}_0}^n$ satisfy $|\alpha| = 2$. Note that $f - H = \psi * f$, so that $\partial^\alpha(f - H) = \partial^\alpha \psi * f = \sum_{l=-\infty}^{0} \partial^\alpha \psi_l * f = \partial^\alpha G$ thanks to Lemma 76. Thus, $f - H$ differs from G by a linear function.

We show that two different definitions $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ in Definition 32 and $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ in Definition 33 are actually the same for $0 < \gamma < 1$.

Theorem 82. *The two sets $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ and $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ are the same for $0 < \gamma < 1$. Furthermore, their two norms are equivalent.*

It is clear that $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ is a subset of $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$, since the difference of the second order can be decomposed into the difference of the difference of the first order. So, the heart of the matter is to show the opposite inclusion.

Proof Let $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$. We will show that $F \equiv G + H \in \mathrm{Lip}^\gamma(\mathbb{R}^n)$.
We need to show that

$$|F(x+y) - F(x)| \lesssim |y|^\gamma \|f\|_{\dot{\mathcal{C}}^\gamma}. \tag{1.6}$$

Let $j_0 \in \mathbb{Z}$ be chosen so that

$$1 \le 2^{j_0}|y| < 2. \tag{1.7}$$

We define $H_{j_0} \equiv \sum_{j=j_0}^{\infty} \varphi^j * f$, $G_{j_0} \equiv G + H - H_{j_0}$. Once we prove

$$|G_{j_0}(x+y) - G_{j_0}(x)| \lesssim |y|^\gamma \|f\|_{\dot{\mathcal{C}}^\gamma} \tag{1.8}$$

and

$$|H_{j_0}(x+y) - H_{j_0}(x)| \lesssim |y|^\gamma \|f\|_{\dot{\mathcal{C}}^\gamma}, \tag{1.9}$$

then we have (1.6).

To prove (1.8), by Lemma 77 we estimate

$$\begin{aligned} |G_{j_0}(x+y) - G_{j_0}(x)| &\lesssim |y| \cdot \|\nabla G_{j_0}\|_{L^\infty} \\ &= |y| \left\| \sum_{j=-\infty}^{j_0-1} \nabla \left[\varphi^j * f\right] \right\|_{L^\infty} \\ &\le \sum_{j=-\infty}^{j_0-1} |y| \cdot \left\| \nabla \left[\varphi^j * f\right] \right\|_{L^\infty}. \end{aligned}$$

Using Lemma 77 and (1.7), we have

$$\frac{|G_{j_0}(x+y) - G_{j_0}(x)|}{|y|^\gamma} \lesssim \sum_{j=-\infty}^{j_0-1} 2^{j(1-\gamma)} |y|^{1-\gamma} \|f\|_{\dot{\mathcal{C}}^\gamma} \sim \|f\|_{\dot{\mathcal{C}}^\gamma}.$$

To prove (1.9) we use Lemma 76 to give

$$|H_{j_0}(x+y) - H_{j_0}(x)| \lesssim \sum_{j=j_0}^{\infty} \|\varphi^j * f\|_\infty \lesssim \sum_{j=j_0}^{\infty} 2^{-j\gamma} \|f\|_{\dot{\mathcal{C}}^\gamma} \sim |y|^\gamma \|f\|_{\dot{\mathcal{C}}^\gamma},$$

as was to be shown.

From now on, we can identify $\mathrm{Lip}^\gamma(\mathbb{R}^n)$ with $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ for $0 < \gamma < 1$.

So far, we do not consider the meaning of the parameter γ. If $\gamma \in (1,2)$, we can differentiate functions in $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$.

Theorem 83. *Let $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$, and let $1 < \gamma < 2$. Then $f \in C^1(\mathbb{R}^n)$ and $\partial_j f \in \dot{\mathcal{C}}^{\gamma-1}(\mathbb{R}^n)$ for all $j = 1, 2, \dots, n$.*

Proof We use the functions G and H in the proof of Theorem 82. Arguing as before, we can show that $\sum_{l=0}^{\infty} \partial_j[\varphi^l * f]$, $j = 1, 2, \dots, n$, converges uniformly, so that $H \in C^1(\mathbb{R}^n)$. Since f differs from $G + H$ by a linear function, we see that $f \in C^1(\mathbb{R}^n)$. We can show that $\partial_j(G+H) \in \dot{\mathcal{C}}^{\gamma-1}(\mathbb{R}^n)$ by an argument similar to Theorem 82. Hence, $\partial_j f \in \dot{\mathcal{C}}^{\gamma-1}(\mathbb{R}^n)$.

We consider the converse of Theorem 83.

Proposition 84. *Let $f \in C^1(\mathbb{R}^n)$, and let $1 < \gamma < 2$. If $\partial_j f \in \dot{\mathcal{C}}^{\gamma-1}(\mathbb{R}^n)$ for each $j = 1, 2, \dots, n$, then $f \in \dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$.*

Proof Let $x, y \in \mathbb{R}^n$. We may assume that f is real-valued. By the mean-value theorem,

$$f(x+y) - 2f(x) + f(x-y) = y \cdot (\nabla f(x+ty) - \nabla f(x-ty))$$

for some $0 < t < 1$. If we use $\partial_j f \in \dot{\mathcal{C}}^{\gamma-1}(\mathbb{R}^n)$ for each $j = 1, 2, \dots, n$, then we obtain $|f(x+y) - 2f(x) + f(x-y)| \lesssim |y|^\gamma \sum_{j=1}^{n} \|\partial_j f\|_{\mathcal{C}^{\gamma-1}}$.

We can define $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ for any $\gamma > 0$.

Definition 35 ($\mathcal{C}^\gamma(\mathbb{R}^n)$)**.** Let $\gamma > 1$. Then $f \in \mathcal{C}^\gamma(\mathbb{R}^n)$ if and only if $f \in \mathrm{C}^{-1-[-\gamma]}(\mathbb{R}^n)$ and $\partial^\alpha f \in \mathcal{C}^{\gamma+1+[-\gamma]}(\mathbb{R}^n)$ for all $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| = -1-[-\gamma]$.

Note that Theorem 83 and Proposition 84 justify Definition 35.

The spaces handled above are called the homogeneous spaces. We can also define inhomogeneous spaces by adding the lower terms: The *inhomogeneous Hölder–Zygmund space* $\mathcal{C}^\gamma(\mathbb{R}^n)$, $\gamma > 0$, is the set of all continuous functions f with the derivative up to order L, which satisfies $L < \gamma$, such that $\|f\|_{\mathcal{C}^\gamma} = \sum_{|\alpha|<\gamma} \|\partial^\alpha f\|_{L^\infty} + \|f\|_{\mathcal{C}^{[\gamma]}}$ is finite. Likewise it is worth defining inhomogeneous Lipschitz spaces. Let $0 < \alpha < 1$. Write $\mathrm{BC}^\alpha(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \cap \mathrm{Lip}^\alpha(\mathbb{R}^n)$. Equip $\mathrm{BC}^\alpha(\mathbb{R}^n)$ with the norm $\|f\|_{\mathrm{BC}^\alpha(\mathbb{R}^n)} = \|f\|_{L^\infty} + \|f\|_{\mathrm{Lip}^\alpha}$.

Exercises

Exercise 37.

(1) Prove Lemma 59(1) by the use of the Lebesgue differentiation theorem.

(2) Prove Lemma 59(2) by the use of the Stokes theorem.

Exercise 38. Let $1 \le p \le \infty$ and $m \in \mathbb{N}$.

(1) Show that $W^{m,p}(\mathbb{R}^n)$ is a normed space using the fact that $L^p(\mathbb{R}^n)$ is a normed space.

(2) Prove Theorem 60.

Exercise 39. Using the Banach–Alaoglu theorem (see Theorem 89), prove Theorem 63.

Exercise 40. Prove Theorem 64 for $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ and then complete the proof of Theorem 64.

Exercise 41.

(1) Prove Theorem 65 by using Theorems 182 and 193 to follow.

(2) Prove Theorem 66 by inducting on m. Hint: The base case $m = 1$ is Theorem 65.

Exercise 42. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuous function satisfying $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in \mathbb{R}^n$.

(1) Establish that there exist uniquely $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$ such that $f(m_1, m_2, \dots, m_n) = a_0 + a_1 m_1 + \cdots + a_n m_n$ for all $(m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$. Here and below we fix such $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$.

(2) Establish that $f(q_1, q_2, \dots, q_n) = a_0 + a_1 q_1 + \cdots + a_n q_n$ for all $(q_1, q_2, \dots, q_n) \in \mathbb{Q}^n$.

(3) Conclude that $f \in \mathcal{P}_1(\mathbb{R}^n)$ by the continuity argument.

Exercise 43. Let $k, m \in \mathbb{N}$.

(1) Show that $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k} e^{-t^2} = p_k(t) e^{-t^2}$ for some $p_k \in \mathcal{P}_k(\mathbb{R}^n)$.

(2) Show that $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k} \sqrt{t^2+1} = p_k(t)(t^2+1)^{\frac{1-2k}{2}}$ for some $p_k \in \mathcal{P}_k(\mathbb{R}^n)$.

(3) Show that $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k} \dfrac{1}{(1+t^2)^m} = \dfrac{p_k(t)}{(1+t^2)^{m(k+1)}}$ for some $p_k \in \mathcal{P}_{(m-1)k}(\mathbb{R}^n)$.

(4) Show that $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k} e^{-\frac{1}{t}} = \dfrac{p_k(t)}{t^{k+1}} e^{-\frac{1}{t}}$ for some $p_k \in \mathcal{P}_{k-1}(\mathbb{R}^n)$. for $t \neq 0$.

Exercise 44. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz function. That is, $|F(x) - F(y)| \sim |x - y|$ for all $x, y \in \mathbb{R}^n$.

(1) Show that $F(\mathbb{R}^n)$ is closed.

(2) Show that $F(\mathbb{R}^n)$ is open.

(3) Show that F is a homeomorphism.

(4) Suppose we have a bi-Lipschitz mapping Φ defined on $\ell^2(\mathbb{N})$. Can we say that f is a homeomorphism?

1.7 Notes

Section 1.1

General remarks and textbooks in Section 1.1

There are many books on Lebesgue spaces; see [156, 258, 319] for example. We refer to the exhaustive textbooks [29] and [296] for Banach function spaces. We refer to the textbooks [189, 319] for an introduction of integration theory. We refer to the recent textbook [203] for mixed Morrey spaces. The authors hit upon Exercise 9 after the lecture of Kaoru Yoneda.

Section 1.1.1 and 1.1.2

The materials dealt with here are standard and we do not recall to whom the definitions belong.

Section 1.1.3

See [302] for Marcinkiewicz spaces.

In 1961, Benedek and Panzone introduced Lebesgue spaces with mixed norm [27]. See [24, 128, 195, 196, 211] for the studies of mixed Lebesgue spaces such as the boundedness of the Hardy–Littlewood and strong maximal operators, multivariate rearrangements, the theory of variable exponents, and interpolation theory.

Section 1.2

General remarks and textbooks in Section 1.2

Morrey spaces go back to the work of Morrey [323], which handled elliptic differential equations. Actually Morrey pointed out that the function is continuous once the derivative satisfies a weak integrability condition. Peetre and Stampaccia considered Morrey and Campanato spaces [353, 413]. We refer to the textbooks [235, Chapter 13], [258, Chapter 5], [438] and [382, §6.1.5] for Morrey spaces. We refer to [29], [258, Chapter 6] for Banach function spaces.

Section 1.2.1

There are many papers dealing with $|\cdot|^{-\frac{n}{p}}$ in the context of Morrey spaces. Zorko considered this function to explain that smooth functions cannot approximate this function [472, p. 587]. See [304, Proposition 4.1] for Example 6, for example.

The space $\mathbb{M}^p_1(\mathbb{R}^n)$ was defined by Adams, Xiao, Giga and Miyakawa. See [1, (4)], [149, p. 582, §2] as well as [8, p.1631] for $\mathbb{M}^p_1(\mathbb{R}^n)$, Definition 9. Among others, Adams used $\mathbb{M}^p_1(\mathbb{R}^n)$ to characterize the boundedness property of the convolution operator [1, Theorems 1 and 2]. We can find Proposition 15, asserting that $\mathcal{M}^p_q(\mathbb{R}^n)$ is trivial for $0 < p < q < \infty$, in [42, p. 13].

Section 1.2.2

We calculated $\|\chi_{B(x,r)}\|_{\mathcal{M}^p_q}$; see Example 6 [252, Lemma 1], for example. The dyadic rearrangement (Example 10) is due to Olsen [342]. Olsen worked in $\mathbb{R}^3$ and constructed an example in [342, Theorem 10]. Example 10 is a passage to $\mathbb{R}^n$ of [342, Theorem 10].

We considered the power function in Example 7. Example 7(1) can be found in [472], while Example 7(2) and (3) can be found in [42, Examples 1 and 2].

In [390], Sawano, Sugano and Tanaka considered the self-similar increasing/decreasing sequence for Morrey spaces. Examples 9 and 11 come from [390,

Proposition 4.1], based on the proof of [64, Theorem 2.1]. See also [304, Lemma 4.6] and [391, Proposition 2.1] for Examples 9 and 11. The term "wasteful" comes from the proof of the survey paper [391, Proposition 2.1].

Section 1.2.3

Giaquinta pointed out the role of the parameter q; see [148, p. 67] for Theorem 18.

As we saw in this book, we have an approach using a self-similar decreasing sequence for Morrey spaces. See [390, Proposition 4.1] together with detailed calculation [304, p.120] and [393, Lemma 2.1] for Example 9.

As we saw in Theorem 20, $\mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$ whenever $1 < q < \tilde{q} < p$; see the paper [380].

Section 1.2.4

Triebel pointed out that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $0 < q < p < \infty$ does not have $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and the Schwartz space as dense subspaces; see [438, Proposition 2.16] for Proposition 22.

As a modification of Morrey spaces, we can consider the Stummel class considered by Ragusa and Zamboni [365]. See [442] for the recent development. Eridani and Gunawan, pointed out the inclusion between $\mathcal{M}^p_1(\mathbb{R}^n)$ and the Stummel class [115]. See [116] for a further generalization to various directions.

As we saw in Proposition 21, Morrey spaces are not rearrangment invariant, that is, it can happen that $\|f\|_{\mathcal{M}^p_q} > \|g\|_{\mathcal{M}^p_q}$ if $\lambda_f = \lambda_g$. Furusho and Ono proposed to restrict the class of Morrey functions to a special case by using a finite subdivision of cubes; see [137, Definition 1] and [345, 346, 347]. See also [347] for the inclusionship similar to Theorem 19.

Section 1.2.5

The definition of weak Morrey spaces can be found in [405]; see also [2, Section 5]. Applications to PDE can be found in [314, 360]. See [304, Lemma 4.7(ii)] for Example 17. See also [173] for Example 17, a more quantitative approach for the difference between $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ and $\mathcal{M}^p_q(\mathbb{R}^n)$.

In [172], we can find Lemma 23, which compares $\mathrm{W}\mathcal{M}^p_r(\mathbb{R}^n)$ and $\mathcal{M}^p_q(\mathbb{R}^n)$ for $0 < q < r \le p$.

As we saw in Theorem 24, the weak Morrey space $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ is smaller than $\mathcal{M}^p_q(\mathbb{R}^n)$; see [139, p. 2143], [149, Proposition 2.2], [440, p. 4] and [471, p.86] as well as [172]. We refer to [186] for a weighted version.

Section 1.2.6

The terminology of ball Banach function spaces can be tracked back to [178, Definition 2]. See [398, Example 3.3] for an account of why Morrey spaces are not Banach function spaces.

Section 1.3

General remarks and textbooks in Section 1.3

Theory of Herz spaces goes back to the work of Herz [193]. See the textbook [458] for Herz spaces.

Section 1.3.1

Although Wiener did not consider local Morrey spaces when his paper [447] was published, the idea of defining local Morrey spaces can be found in [447]. In fact, Wiener considered the case of $p = 1, 2$ in the context of Fourier analysis. Beurling [31] introduced the spaces $B^p(\mathbb{R}^n)$ and $\dot{B}^p(\mathbb{R}^n)$, whose norm is given by $\|f\|_{B^p} \equiv \sup_{r\geq 1} m^{(p)}_{Q(r)}(f)$ and $\|f\|_{\dot{B}^p} \equiv \sup_{r>0} m^{(p)}_{Q(r)}(f)$ together with the predual $A^p(\mathbb{R}^n)$ so-called a *Beurling algebra*. In this case $B^p(\mathbb{R}^n) = B_{\frac{n}{p}}(L^p)(\mathbb{R}^n)$ and $\dot{B}^p(\mathbb{R}^n) = \dot{B}_{\frac{n}{p}}(L^p)(\mathbb{R}^n)$. Later, to extend Wiener's ideas [447, 448] which describe the behavior of functions at infinity, Feichtinger [126] gave an equivalent norm on $B^p(\mathbb{R}^n)$, which is a special case of norms to describe non-homogeneous Herz spaces $K^{\alpha}_{p,r}(\mathbb{R}^n)$ introduced in [193]. García-Cuerva and Herrero [142] and Alvarez, Guzmán-Partida and Lakey [17] introduced the non-homogeneous central Morrey space $B^{p,\lambda}(\mathbb{R}^n)$ (the non-homogeneous B_σ-Lebesgue space/ the non-homogeneous central Morrey space) and the central Morrey space $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ (the homogeneous B_σ-Lebesgue space/ the homogeneous central Morrey space).

$$\|f\|_{\mathrm{W}B^{p,\lambda}} \equiv \sup_{r\geq 1} r^{-\lambda-\frac{n}{p}} \left(\sup_{t>0} t^p \left|\{|f| > t\} \cap Q(r)\right|\right)^{\frac{1}{p}}$$

and

$$\|f\|_{\mathrm{W}\dot{B}^{p,\lambda}} \equiv \sup_{r>0} r^{-\lambda-\frac{n}{p}} \left(\sup_{t>0} t^p \left|\{|f| > t\} \cap Q(r)\right|\right)^{\frac{1}{p}}.$$

B_σ-Lipschitz and B_σ-Morrey-Campanato spaces were introduced in [248] and [306], respectively. Komori-Furuya, Matsuoka and Nakai studied the boundedness of fractional integral operators on these spaces. Since Morrey and Campanato spaces realize Lebesgue spaces, BMO and Lipschitz spaces as special cases, B_σ-Morrey-Campanato spaces cover $B^{p,\lambda}(\mathbb{R}^n)$, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$, as well as the usual Morrey, Campanato and Lipschitz spaces. The word "CMO" stands for "central mean oscillations" and "CBMO" stands for "central bounded mean oscillations". See Example 102 for [249].

Komori-Furuya, Matsuoka, Nakai and Sawano considered a new framework to grasp central Morrey spaces, Campanato spaces and so on [250, §2]. See [250, 251, 399] for more about B_σ-spaces.

Section 1.3.2

Herz–Morrey spaces are defined by replacing the $L^p(\mathbb{R}^n)$-norm with the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ in some sense. Lu and Xu defined Herz–Morrey spaces [295, Definition 1.1]. Wang and Shu considered the Littlewood–Paley g^μ_λ-function in Herz–Morrey spaces [445]. Herz–Morrey spaces are investigated by Izuki in the setting of variable exponents. For details see [214, 215].

Wu investigated the boundedness property of Hardy operators as well as the Riesz transform in [452, 453, 454]. We refer to [431] for the multilinear case.

Local Morrey spaces and Herz spaces overlap largely [167, Lemma 8.1]; see Theorem 26. Mizuta and Ohno took an approach similar to Theorem 26 [321].

Gao, Tang, Xue and Zhou investigated the boundedness properties of commutators generated by Hardy operators and BMO functions [140, 429].

Herz–Morrey spaces can be defined over locally compact groups. More precisely, Wu and Liu worked in locally compact Vilenkin groups in [451, 455]. Dzhabrailov and Khaligova worked in the anisotropic setting in [110, 111].

Fan and Gao considered the boundedness property of the bilinear fractional integral operator acting on Herz–Morrey spaces [120].

Kuang considered Hilbert/Hausdorff operators acting on Herz–Morrey spaces; see [256, 257].

Section 1.4

General remarks and textbooks in Section 1.4

Edmunds, Kokilashvili and Meskhi dealt with Hardy operators. Especially, [112, Chapter 1] deals intensively with this topic [112].

See [373, Chapter 2] for Hardy's inequality.

See [30, §1.3], [29, Chapter 2 §1], [156, Chapter 1], [147, §1.4], [258, Chapter 8] and [223, Chapter 0, §II] for the distribution of functions and Lorentz spaces. In [147, Chapter 2], we can find some information on the Hardy–Littlewood maximal operator. See [30], [156, §1.4], [417, Chapter V, §3] for a detailed explanation on Lorentz spaces.

Sections 1.4.1 and 1.4.2

Lorentz defined Lorentz spaces in [288, 289]. Luxenberg investigated them as well. The work of Luxenberg on Lorentz spaces is just a part of his paper [296].

Section 1.4.3

We can find inclusion $\Phi^\uparrow_{\lambda,q_0}(0,\infty) \subset \Phi^\uparrow_{\lambda,q_1}(0,\infty)$; see [50, Lemma 1] for Lemma 37. We considered the boundedness of K_0; see [77, Theorem 2.5] for Example 40.

Section 1.4.4

See [364] Lemma 40, Chiarenza's lemma.
We polished [32, §2] to state Example 41.

Section 1.5

General remarks and textbooks in Section 1.5

See the exhaustive textbook and seminar note [29, 299, 366] as well as [147, pp. 22–23], [258, Chapter 4] and [382, §6.1.6] for Orlicz spaces and Young functions. Trudinger pointed out that Orlicz spaces can be used to compensate for the failure of the endpoint case for integral operators. See [439]. We can find some maximal estimates in [147, §5.2] (including [147, Chapter 6]) and [382] and some singular integral estimates in [147, §5.6]. We refer to [258, Chapter 13] for Lebesgue spaces with variable exponents. See also the survey [216].

Section 1.5.1

Young proved an inequality for the Young function and its conjugate [466]; see Theorem 45. We obtained an estimate for ∇_2-functions. See [232, Lemma 1.3.2] for Lemma 49(1) and [387, Lemma 4.3] for Lemma 49(2). We borrowed [297, Example 2.3] for Example 43.

Section 1.5.2

Koshi and Shimogaki obtained an equivalent expression of Orlicz norms [238]; see Theorem 51.

Kita established similar results to Proposition 156 for $\Psi(t) \equiv t$ on $L^{\Phi}(\mathbb{R}^n)$. See [228, 229].

Section 1.5.3

Carro, Pérez, F. Soria and J. Soria defined the Φ-average in [64, §3]; see Definition 26. See [64, (3.1)] for Definition 26 including Example 46.

Section 1.5.4

Orlicz initially considered Lebesgue spaces with a variable exponent [348, §2]. Later on Nakano considered variable Lebesgue norms [336, 337].

In particular, function spaces with variable exponents are necessary in the field of electronic fluid mechanics [374] and the applications to image restoration [69, 187, 276]. Kovacik and Rakosnik [243] gave an application of generalized Lebesgue spaces with variable exponents to Dirichlet boundary value problems for nonlinear partial differential equations with coefficients of a variable growth. Another simple example of the application to differential equations can be found in [121, p. 438, Example], where Fan and Zhao

implicitly showed that the variable Lebesgue spaces can be used to control the non-linear term of differential equations.

See the survey paper [216] for a detailed explanation of the old books [336, 337].

Section 1.6

General remarks and textbooks in Section 1.6

There are many textbooks dealing with Sobolev spaces; see the exhaustive textbooks [12, 13, 309] as well as the textbooks [5], [117, §5.2], [118, Chapter 4], [415, §V.2] and [410, §6.3] for example.

See [117, §5.1], [415, Chapter V, §4] and [416, Chapter VI, §5] for a characterization of Hölder–Zygmund spaces. For the extension, we refer to the textbook [41].

Section 1.6.1

We followed the idea of Burenkov [41]. See [40] for Lemma 68. Theorem 70 is due to Burenkov [41]; see also the textbook [41, Lemma 18]. See [411] for Theorem 66.

Section 1.6.2

See [415] for Definition 33. We followed [384] overall in this section.

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Chapter 2

Fundamental facts in functional analysis

This chapter collects some preliminary facts. The readers may choose to skip Chapter 2. We indicate where we use each theorem in the other parts of this book. We start with normed spaces in Section 2.1, while we concentrate on Hilbert spaces in Section 2.2. Section 2.3 is used for the last chapter of the second book. We will consider integration theory for measurable functions assuming their values in a Banach space.

2.1 Normed spaces and Banach spaces

We do not go into a detailed discussion of normed spaces in Section 2.1. Section 2.1.1 recalls various forms of the Hahn–Banach theorem. Section 2.1.2 is interesting in its own right. We consider a self-improvement of the triangle inequality. Section 2.1.3 is oriented to the operations for Banach spaces.

2.1.1 Hahn–Banach theorem and Banach–Alaoglu theorem

We assume that readers are familiar with the Hahn–Banach theorem and Banach–Alaoglu theorem. We list a couple of variants of the Hahn–Banach theorem.

Theorem 85 (Hahn–Banach theorem/analytic form)**.** *Let V be a complex vector space and $p : V \to \mathbb{C}$ a function satisfying*

$$p(u+v) \leq p(u) + p(v), \quad p(\alpha u) = |\alpha|\, p(u)$$

for all $\alpha \in \mathbb{C}$ and $u, v \in V$. Assume that $l_0 : W_0 \to \mathbb{R}$, which is defined on a subspace W_0, is a linear mapping satisfying $l_0(u) \leq p(u)$ for all $u \in W_0$. Then there exists a linear mapping $L : V \to \mathbb{R}$ such that $L|W_0 = l_0$ and that $|L(u)| \leq p(u)$ for all $u \in V$.

Theorem 86 (Hahn–Banach extension)**.** *Let $\mathcal{Y}$ be a closed subspace of a normed space $\mathcal{X}$. Then a bounded functional $f : \mathcal{Y} \to \mathbb{C}$ can be extended to a bounded functional F defined on $\mathcal{X}$ so that $\| F \|_{\mathcal{X}^*} = \| f \|_{\mathcal{Y}^*}$.*

Theorem 87 (Existence of the norm attainer). *Let $\mathcal{X}$ be a normed space and $x \in \mathcal{X}$. Then x has a norm attainer. Namely, there exists $x^* \in \mathcal{X}^*$ with unit norm such that $x^*(x) = \|x\|_{\mathcal{X}}$.*

Theorem 88 (Hahn–Banach theorem/separation version). *Suppose that A is a closed convex set in a normed space $\mathcal{X}$. If b is a point outside A, then there exists a continuous functional $x^* : \mathcal{X} \to \mathbb{C}$ such that*

$$\sup_{a \in A} \operatorname{Re} x^*(a) < x^*(b).$$

We next recall the Banach–Alaoglu theorem. Recall that the metric space is separable if it is realized as the closure of a countable subset.

Theorem 89 (Banach and Alaoglu). *Let $\mathcal{X}$ be a separable normed space. Assume that $\{x_j{}^*\}_{j=1}^{\infty}$ belongs to the closed unit ball B in $\mathcal{X}^*$. Then there exists a subsequence $\{x_{j_k}{}^*\}_{k \in \mathbb{N}}$ convergent with respect to weak-$*$ topology. That is,*

$$\lim_{k \to \infty} x_{j_k}{}^*(x) = x^*(x)$$

for all $x \in \mathcal{X}$.

We do not recall the proof of Theorems 85, 86, 87, 88 and 89. See [310] for example.

2.1.2 Refinement of the triangle inequality

We will use Section 2.1.2 in Section 3.1.4. We prove an inequality valid for any normed space, which is interesting in its own right and refines the original triangle inequality.

Theorem 90. *We have*

$$\|x+y\|_{\mathcal{X}} \le \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{X}} - \left(2 - \left\| \frac{x}{\|x\|_{\mathcal{X}}} + \frac{y}{\|y\|_{\mathcal{X}}} \right\|_{\mathcal{X}}\right) \min(\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{X}}).$$

for any nonzero elements x and y in a normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$.

Proof Without loss of generality we may assume $\|x\|_{\mathcal{X}} \le \|y\|_{\mathcal{X}}$. Then

$$\begin{aligned}
\|x+y\|_{\mathcal{X}} &= \left\| \frac{\|x\|_{\mathcal{X}}}{\|x\|_{\mathcal{X}}} x + \frac{\|x\|_{\mathcal{X}}}{\|y\|_{\mathcal{X}}} y + \left(1 - \frac{\|x\|_{\mathcal{X}}}{\|y\|_{\mathcal{X}}}\right) y \right\|_{\mathcal{X}} \\
&\le \left\| \frac{\|x\|_{\mathcal{X}}}{\|x\|_{\mathcal{X}}} x + \frac{\|x\|_{\mathcal{X}}}{\|y\|_{\mathcal{X}}} y \right\|_{\mathcal{X}} + \left\| \left(1 - \frac{\|x\|_{\mathcal{X}}}{\|y\|_{\mathcal{X}}}\right) y \right\|_{\mathcal{X}} \\
&= \|x\|_{\mathcal{X}} \left\| \frac{x}{\|x\|_{\mathcal{X}}} + \frac{y}{\|y\|_{\mathcal{X}}} \right\|_{\mathcal{X}} + \|y\|_{\mathcal{X}} - \|x\|_{\mathcal{X}} \\
&= \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{X}} - \left(2 - \left\| \frac{x}{\|x\|_{\mathcal{X}}} + \frac{y}{\|y\|_{\mathcal{X}}} \right\|_{\mathcal{X}}\right) \min(\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{X}}.
\end{aligned}$$

2.1.3 Sum and intersection of Banach spaces

Keeping in mind the definition of Orlicz spaces, we consider the sum and the intersection of quasi-Banach spaces. We will show that duality connects these two notions of the sum and the intersection. We will use Section 2.1.3 mainly in Chapters 18 and 19.

Suppose that we have quasi-Banach spaces $\mathcal{X}_0, \mathcal{X}_1, \mathcal{Y}_0, \mathcal{Y}_1, \mathcal{X}, \mathcal{Y}$ such that

$$\mathcal{X}_0, \mathcal{X}_1 \hookrightarrow \mathcal{X} \quad \mathcal{Y}_0, \mathcal{Y}_1 \hookrightarrow \mathcal{Y},$$

in the sense that $\mathcal{X}_0 \subset \mathcal{X}$, $\mathcal{X}_1 \subset \mathcal{X}$, $\mathcal{Y}_0 \subset \mathcal{Y}$ and $\mathcal{Y}_1 \subset \mathcal{Y}$ and that there exists $M > 0$ such that

$$\|x_0\|_{\mathcal{X}} \le M\|x_0\|_{\mathcal{X}_0}, \quad \|x_1\|_{\mathcal{X}} \le M\|x_1\|_{\mathcal{X}_1},$$

$$\|y_0\|_{\mathcal{Y}} \le M\|y_0\|_{\mathcal{Y}_0}, \quad \|y_1\|_{\mathcal{Y}} \le M\|y_1\|_{\mathcal{Y}_1}$$

for all $x_0 \in \mathcal{X}_0$, $x_1 \in \mathcal{X}_1$, $y_0 \in \mathcal{Y}_0$ and $y_1 \in \mathcal{Y}_1$.

Recall that a linear topology on a linear space $\mathcal{X}$ is a topology of $\mathcal{X}$ in which the multiplication and the addition are continuous. Such a space $\mathcal{X}$ is called a topological vector space. We may assume that $\mathcal{X}$ and $\mathcal{Y}$ are merely topological vector spaces in the sense that $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces equipped with continuous addition and continuous scalar multiplication.

In any of these cases, $\mathcal{X}_0$ and $\mathcal{X}_1$ are called compatible and we can deal with elements in $\mathcal{X}_0$ and $\mathcal{X}_1$ as if they belonged to $\mathcal{X}$. The same applies to $\mathcal{Y}_0$ and $\mathcal{Y}_1$. Thus, we can naturally define $\mathcal{X}_0 + \mathcal{X}_1$ and $\mathcal{Y}_0 + \mathcal{Y}_1$. For example, $\mathcal{X}_0 + \mathcal{X}_1$ is given by

$$\mathcal{X}_0 + \mathcal{X}_1 \equiv \{x \in \mathcal{X} : x = x_0 + x_1,\, x_0 \in \mathcal{X}_0,\, x_1 \in \mathcal{X}_1\}.$$

We automatically make $\mathcal{X}_0 + \mathcal{X}_1$ into a normed space. The norm is given by

$$\|x\|_{\mathcal{X}_0+\mathcal{X}_1} \equiv \inf\{\|x_0\|_{\mathcal{X}_0} + \|x_1\|_{\mathcal{X}_1} : x_0 \in \mathcal{X}_0, x_1 \in \mathcal{X}_1, x = x_0 + x_1\}.$$

Likewise, the norm of $\mathcal{X}_0 \cap \mathcal{X}_1$ is given by $\|x\|_{\mathcal{X}_0\cap\mathcal{X}_1} = \max(\|x\|_{\mathcal{X}_0}, \|x\|_{\mathcal{X}_1})$. It is routine to show that $\mathcal{X}_0 + \mathcal{X}_1$ and $\mathcal{X}_0 \cap \mathcal{X}_1$ are Banach spaces. Suppose that we have a linear operator $T : \mathcal{X}_0 + \mathcal{X}_1 \to \mathcal{Y}_0 + \mathcal{Y}_1$ with the properties that $T(\mathcal{X}_0) \subset \mathcal{Y}_0$, $T(\mathcal{X}_1) \subset \mathcal{Y}_1$ and

$$\|Tx_0\|_{\mathcal{Y}_0} \le C_0\|x_0\|_{\mathcal{X}_0}, \quad \|Tx_1\|_{\mathcal{Y}_1} \le C_1\|x_1\|_{\mathcal{X}_1}$$

for all $x_0 \in \mathcal{X}_0$ and $x_1 \in \mathcal{X}_1$. Here, C_0 and C_1 are independent of x_0 and x_1, respectively.

Example 50. We defined $L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$ as the linear space of all functions which can be expressed as a sum of $L^1(\mathbb{R}^n)$-functions and $L^\infty(\mathbb{R}^n)$-functions in Proposition 21. For $f \in L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$, a simple computation using the definition of the norm, its norm is given by $\|f\|_{L^1+L^\infty} = \inf_{L>0}(L+\|\chi_{\{|f|>L\}}f\|_{L^1})$. See Exercise 48.

Note that for a given $x \in \mathcal{X}_0 + \mathcal{X}_1$ representation $x = x_0 + x_1$ may not be unique. In fact, we can charaterize this condition as follows:

Lemma 91. *Let $\mathcal{X}_0$, $\mathcal{X}_1$ be linear subsets of $\mathcal{X}$. Then for any $x \in \mathcal{X}_0 + \mathcal{X}_1$ the representation $x = x_0 + x_1, x_0 \in \mathcal{X}_0, x_1 \in \mathcal{X}_1$ is unique if and only if $\mathcal{X}_0 \cap \mathcal{X}_1 = \{0\}$.*

Proof Assume that $\mathcal{X}_0 \cap \mathcal{X}_1 \supsetneq \{0\}$, or equivalently, there exists $b \in \mathcal{X}_0 \cap \mathcal{X}_1 \setminus \{0\}$. Then $0 = 0 + 0 = b + (-b)$, which implies that the representation of 0 is not unique.

Suppose that $\mathcal{X}_0 \cap \mathcal{X}_1 = \{0\}$. Assume that $x_0, y_0 \in \mathcal{X}_0$, $x_1, y_1 \in \mathcal{X}_1$ and $z = x_0 + x_1 = y_0 + y_1$. Then $x_0 - y_0 = y_1 - x_1 \in \mathcal{X}_0 \cap \mathcal{X}_1$. Hence $x_0 - y_0 \in \mathcal{X}_0 \cap \mathcal{X}_1 = \{0\}$, which implies $y_0 = x_0$ and $y_1 = x_1$.

Example 51. Let $1 \le p_0, p_1 \le \infty$. Then

$$L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n) = \{f = f_0 + f_1 \,:\, f_0 \in L^{p_0}(\mathbb{R}^n), f_1 \in L^{p_1}(\mathbb{R}^n)\}.$$

Clearly, $L^{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n) \neq \{0\}$. Hence, the representation $f = f_0 + f_1$ is not unique. In this case $\mathcal{X}$ is the set of all $f \in L^0(\mathbb{R}^n)$, $\mathcal{X}_0 = L^{p_0}(\mathbb{R}^n)$, $\mathcal{X}_1 = L^{p_1}(\mathbb{R}^n)$. Remark that $L^0(\mathbb{R}^n)$ can be made into a topological vector space. The topology is given by the functional $f \in L^0(\mathbb{R}^n) \mapsto \int_E \min(1, |f(x)|)\mathrm{d}x \in \mathbb{R}$, where E moves over all measurable sets of finite measure in $\mathbb{R}^n$.

We remark that $\mathcal{X}_0 + \mathcal{X}_1 \hookrightarrow \mathcal{X}$ and we are led to subspaces of $\mathcal{X}_0 + \mathcal{X}_1$ when we consider interpolation spaces of spaces $\mathcal{X}_0, \mathcal{X}_1$. Among other properties, we use the following duality fact:

Theorem 92 (Duality theorem)**.** *Suppose that $(\mathcal{X}_0, \mathcal{X}_1)$ is a compatible couple of Banach spaces. Assume in addition that $\mathcal{X}_0 \cap \mathcal{X}_1$ is dense in $\mathcal{X}_0$ and $\mathcal{X}_1$. Then $(\mathcal{X}_0 \cap \mathcal{X}_1)^* = \mathcal{X}_0{}^* + \mathcal{X}_1{}^*$ and $(\mathcal{X}_0 + \mathcal{X}_1)^* = \mathcal{X}_0{}^* \cap \mathcal{X}_1{}^*$ with coincidence of norms; namely,*

$$\|x^*\|_{\mathcal{X}_0{}^* + \mathcal{X}_1{}^*} = \sup_{x \in \mathcal{X}_0 \cap \mathcal{X}_1 \setminus \{0\}} \frac{|\langle x^*, x\rangle|}{\|x\|_{\mathcal{X}_0 \cap \mathcal{X}_1}} \text{ for all } x^* \in \mathcal{X}_0{}^* + \mathcal{X}_1{}^*,$$

$$\|x^*\|_{\mathcal{X}_0{}^* \cap \mathcal{X}_1{}^*} = \sup_{x \in \mathcal{X}_0 + \mathcal{X}_1 \setminus \{0\}} \frac{|\langle x^*, x\rangle|}{\|x\|_{\mathcal{X}_0 + \mathcal{X}_1}} \text{ for all } x^* \in \mathcal{X}_0{}^* \cap \mathcal{X}_1{}^*.$$

Proof Note that

$$(\mathcal{X}_0 \cap \mathcal{X}_1)^* \hookleftarrow \mathcal{X}_0{}^* + \mathcal{X}_1{}^*,\ (\mathcal{X}_0 + \mathcal{X}_1)^* \approx \mathcal{X}_0{}^* \cap \mathcal{X}_1{}^* \tag{2.1}$$

and that

$$\|x^*\|_{\mathcal{X}_0{}^* + \mathcal{X}_1{}^*} \ge \sup_{x \in \mathcal{X}_0 \cap \mathcal{X}_1} \frac{|\langle x^*, x\rangle|}{\|x\|_{\mathcal{X}_0 \cap \mathcal{X}_1}}, \tag{2.2}$$

$$\|x^*\|_{\mathcal{X}_0{}^* \cap \mathcal{X}_1{}^*} = \sup_{x \in \mathcal{X}_0 + \mathcal{X}_1} \frac{|\langle x^*, x\rangle|}{\|x\|_{\mathcal{X}_0 + \mathcal{X}_1}}. \tag{2.3}$$

Let us prove the reverse inclusion of (2.1). To this end, we take $x^* \in (\mathcal{X}_0 \cap \mathcal{X}_1)^*$. The norm of $\mathcal{X}_0 \oplus \mathcal{X}_1$ is defined by

$$\| (x_0, x_1) \|_{\mathcal{X}_0 \oplus \mathcal{X}_1} = \max(\|x_0\|_{\mathcal{X}_0}, \|x_1\|_{\mathcal{X}_1}) \quad ((x_0, x_1) \in \mathcal{X}_0 \oplus \mathcal{X}_1),$$

which immediately makes $E \equiv \{(x_0, x_1) \in \mathcal{X}_0 \oplus \mathcal{X}_1 \,:\, x_0 = x_1 \in \mathcal{X}_0 \cap \mathcal{X}_1\}$ into a closed subspace of $\mathcal{X}_0 \oplus \mathcal{X}_1$. Furthermore, the dual of $\mathcal{X}_0 \oplus \mathcal{X}_1$ is canonically identified with ${\mathcal{X}_0}^* \oplus {\mathcal{X}_1}^*$, whose norm is given by

$$\| (x_0{}^*, x_1{}^*) \|_{\mathcal{X}_0 \oplus \mathcal{X}_1} = \|x_0{}^*\|_{\mathcal{X}_0} + \|x_1{}^*\|_{\mathcal{X}_1}.$$

Then $l : (x_0, x_1) \in E \mapsto 2^{-1} x^*(x_0 + x_1) \in \mathbb{C}$ is a continuous functional which is dominated by the norm of $\mathcal{X}_0 \oplus \mathcal{X}_1$. Therefore, l extends to a continuous linear functional L on $\mathcal{X}_0 \oplus \mathcal{X}_1$ thanks to the Hahn–Banach theorem, Theorem 86, in such a way that $\|L\|_{(\mathcal{X}_0 \oplus \mathcal{X}_1)^*} \leq \|x^*\|_{(\mathcal{X}_0 \cap \mathcal{X}_1)^*}$ As a result we obtain ${x_0}^*$ and ${x_1}^*$ such that

$$\|x_0{}^*\|_{\mathcal{X}_0{}^*} + \|x_1{}^*\|_{\mathcal{X}_1{}^*} = \|L\|_{(\mathcal{X}_0 \oplus \mathcal{X}_1)^*} \leq \|x^*\|_{(\mathcal{X}_0 \cap \mathcal{X}_1)^*} \tag{2.4}$$

and that $L(x_0, x_1) = \langle x_0{}^*, x_0 \rangle + \langle x_1{}^*, x_1 \rangle$ for all $(x_0, x_1) \in \mathcal{X}_0 \oplus \mathcal{X}_1$. Letting $x_0 = x_1 = x$, we obtain $\langle x^*, x \rangle = L(x, x) = \langle x_0{}^*, x \rangle + \langle x_1{}^*, x \rangle$. Thus, $x^* = x_0{}^* | \mathcal{X}_0 \cap \mathcal{X}_1 + x_1{}^* | \mathcal{X}_0 \cap \mathcal{X}_1$ and we deduce from (2.4) that $\|x^*\|_{\mathcal{X}_0{}^* + \mathcal{X}_1{}^*} \leq \|x_0{}^*\|_{\mathcal{X}_0{}^*} + \|x_1{}^*\|_{\mathcal{X}_1{}^*} \leq \|x^*\|_{(\mathcal{X}_0 \cap \mathcal{X}_1)^*}$. This is the desired converse inequality.

2.1.4 Exercises

Exercise 45. [225] Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T \colon \mathcal{X} \to \mathcal{Y}$ a linear operator. Suppose that there exist constants $M > 0$ and $0 < \varepsilon < 1$ with the following property: For every $y \in \mathcal{Y}$ with unit norm, there exists $x \in \mathcal{X}$ with $\|x\|_{\mathcal{X}} \leq M$ and $\|Tx - y\|_{\mathcal{Y}} < \varepsilon$. Then show that T is onto.

Exercise 46. Let $\mathcal{X}, \mathcal{Y}$ be normed spaces, and let $T : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Define $\|T\|_{\mathcal{X} \to \mathcal{Y}} \equiv \sup\{\|Tx\|_{\mathcal{Y}} \,:\, \|x\|_{\mathcal{X}} = 1\}$. Then show that $\|T\|_{\mathcal{X} \to \mathcal{Y}} = \sup\{\|Tx\|_{\mathcal{Y}} \,:\, \|x\|_{\mathcal{X}} < 1\}$.

Exercise 47 (Clarkson's inequality)**.** [319, Theorem 12.6] Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $1 < p \leq 2$.

(1) Show that $\left|\frac{a+b}{2}\right|^{p'} + \left|\frac{a-b}{2}\right|^{p'} \leq \left(\frac{|a|^p + |b|^p}{2}\right)^{\frac{1}{p-1}}$ for all $a, b \in \mathbb{R}$. (Hint: We may assume $b = 1 \geq a$.)

(2) Show that $\left(\left\|\frac{f+g}{2}\right\|_{L^p(\mu)}\right)^{p'} + \left(\left\|\frac{f-g}{2}\right\|_{L^p(\mu)}\right)^{p'} \leq \left(\frac{(\|f\|_{L^p(\mu)})^p + (\|g\|_{L^p(\mu)})^p}{2}\right)^{p'-1}$ for all $f, g \in L^0(\mu)$. (Hint: Prove first that $\|f\|_{L^{p'}(\mu)}{}^{p'} + \|g\|_{L^{p'}(\mu)}{}^{p'} \leq \|f+g\|_{L^{p'}(\mu)}{}^{p'}$ for $f, g \in \mathbb{M}^+(\mu)$.)

(3) Show that $\left|\frac{a+b}{2}\right|^{p'} + \left|\frac{a-b}{2}\right|^{p'} \leq \frac{|a|^{p'}+|b|^{p'}}{2}$ for all $a, b \in \mathbb{R}$.

(4) Show that $\left(\left\|\frac{f+g}{2}\right\|_{L^{p'}(\mu)}\right)^{p'} + \left(\left\|\frac{f-g}{2}\right\|_{L^{p'}(\mu)}\right)^{p'}$
$\leq \frac{(\|f\|_{L^{p'}(\mu)})^{p'} + (\|g\|_{L^{p'}(\mu)})^{p'}}{2}$ for all $f, g \in L^0(\mu)$.

Exercise 48. In Example 50, we considered the norm of $f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$. Let $L \geq 0$. Then find the minimum of $\|f-h\|_{L^1}$, where $h \in L^0(\mathbb{R}^n)$ moves over all functions satisfying $\|h\|_{L^\infty} = L$.

Exercise 49. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a variable exponent. Let $p_+ \equiv \operatorname{ess.sup}_{x\in\mathbb{R}^n} p(x)$ and $p_- \equiv \operatorname{ess.inf}_{x\in\mathbb{R}^n} p(x)$. Then establish that $L^{p_+}(\mathbb{R}^n) \cap L^{p_-}(\mathbb{R}^n) \subset L^{p(\cdot)}(\mathbb{R}^n) \subset L^{p_+}(\mathbb{R}^n) + L^{p_-}(\mathbb{R}^n)$.

2.2 Hilbert spaces

A Banach space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is said to be a Hilbert space, if it comes with a mapping $\langle \cdot_1, \cdot_2 \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, with the following properties for all $x, y, z \in \mathcal{H}$ and $a \in \mathbb{C}$. The mapping $\langle \cdot_1, \cdot_2 \rangle_{\mathcal{H}}$ is an *inner product*.

(1) $\langle x, x \rangle_{\mathcal{H}} = \|x\|_{\mathcal{H}}{}^2$ for all $x \in \mathcal{H}$.

(2) $\langle x, y \rangle_{\mathcal{H}} = \overline{\langle y, x \rangle_{\mathcal{H}}}$ for all $x, y \in \mathcal{H}$.

(3) $\langle x + y, z \rangle_{\mathcal{H}} = \langle x, z \rangle_{\mathcal{H}} + \langle y, z \rangle_{\mathcal{H}}$ for all $x, y, z \in \mathcal{H}$.

(4) $\langle a \cdot x, y \rangle_{\mathcal{H}} = a \cdot \langle x, y \rangle_{\mathcal{H}}$.

Let $\mathcal{H}$ be a Hilbert space in Section 2.2. Section 2.2.1 handles bounded sequences in Hilbert spaces, while Section 2.2.2 considers the uniform boundedness of operators acting on Hilbert spaces.

2.2.1 Komlos theorem

The Komlos theorem is a tool to create the limit when we have a bounded sequence in a normed space. We do not consider the Komlos theorem itself here. In this book we will use the following simple variant, which we use in Chapter 17.

Proposition 93. *Let $\mathcal{H}$ be a Hilbert space.*

(1) *Let C be a closed convex set in $\mathcal{H}$. Then there exists an element $h_0 \in C$ such that $\|h_0\|_{\mathcal{H}} = \inf\{\|h\|_{\mathcal{H}} : h \in C\}$.*

(2) *Let $f_1, f_2, \ldots$ be a sequence in the closed unit ball of $\mathcal{H}$. Choose $g_k \in \overline{\mathrm{co}}(\{f_{k+1}, f_{k+2}, \ldots\})$ so that $\|g_k\|_{\mathcal{H}} = \min\{\|g\|_{\mathcal{H}} \in \overline{\mathrm{co}}(\{f_{k+1}, f_{k+2}, \ldots\})\}$ for each $k \in \mathbb{N}$. Then $\{g_k\}_{k=1}^{\infty}$ converges to an element in $\mathcal{H}$.*

Proof

(1) This is a standard result in functional analysis. See [409, Theorem 3.2.1], for example.

(2) Write $v_k \equiv \min\{\|g\|_{\mathcal{H}} \in \overline{\mathrm{co}}(\{f_{k+1}, f_{k+2}, \ldots\})\}$. Note that $\|g_k - g_{k'}\|_{\mathcal{H}}{}^2 = 2\|g_k\|_{\mathcal{H}}{}^2 + 2\|g_{k'}\|_{\mathcal{H}}{}^2 - \|g_k + g_{k'}\|_{\mathcal{H}}{}^2 \leq 2{v_k}^2 + 2{v_{k'}}^2 - 4{v_{\min(k,k')}}^2$, since

$$\frac{g_k + g_{k'}}{2} \in \overline{\mathrm{co}}(\{f_{\min(k,k')+1}, f_{\min(k,k')+2}, \ldots\})\}.$$

Note that $\{v_k\}_{k=1}^{\infty}$ is a bounded increasing sequence. Hence, it converges and we obtain the desired result.

2.2.2 Cotlar's lemma

Here, we will collect an orthogonality lemma. For a Hilbert space $\mathcal{H}$ denote by $P(\mathcal{H})$ the set of all projections; a bounded linear operator $P : \mathcal{H} \to \mathcal{H}$ belongs to $P(\mathcal{H})$ if and only if $P = P^*$ and $P^2 = P$. Suppose that we have a sequence of projections $\{P_j\}_{j=1}^{\infty} \in P(\mathcal{H})$. If $P_j P_k = 0$, $j \neq k$, then

$$\|P_1 + P_2 + \cdots + P_N\|_{\mathcal{H}\to\mathcal{H}} \leq 1. \tag{2.1}$$

Of course, in general, (2.1) fails. A typical case is $P_1 = P_2 = \cdots$. In a very simple case, we do have $P_j P_k = 0$, $j \neq k$ in many applications. However, with a slight perturbation, $P_j P_k = 0$, $j \neq k$ no longer holds. What we are going to consider is a substitute of (2.1), when a situation slightly differs from the case $P_j P_k = 0$, $j \neq k$.

Theorem 94. *Suppose that $\{T_j\}_{j\in J} \subset B(\mathcal{H})$ is a finite collection, where $J \subset \mathbb{Z}$ is a finite subset. Assume that a sequence of positive constants $\{\gamma_j\}_{j\in\mathbb{Z}}$ satisfies*

$$A \equiv \max\left(\sup_{j'\in J}\sum_{j\in J}\gamma_{j-j'}, \sup_{j'\in J}\sum_{j\in J}\gamma_{j'-j}\right) < \infty \tag{2.2}$$

and that

$$\| T_{j'}^* T_j \|_{\mathcal{H}\to\mathcal{H}} + \| T_{j'} T_j^* \|_{\mathcal{H}\to\mathcal{H}} \leq {\gamma_{j'-j}}^2 \quad (j, j' \in J). \tag{2.3}$$

Then the operator $T_J \equiv \sum_{j\in J} T_j$ satisfies $\| T_J \|_{\mathcal{H}\to\mathcal{H}} \leq A$.

Proof If we write out in full, then

$$(T_J^* T_J)^m = \sum_{j_1, j_2, \ldots, j_{2m} \in J} T_{j_1}^* T_{j_2} T_{j_3}^* \cdots T_{j_{2m}}. \tag{2.4}$$

We will estimate this sum by majorizing the norms of the individual summands.

First, associating the factors in each summand as

$$T_{j_1}^* T_{j_2} T_{j_3}^* \cdots T_{j_{2m}} = \prod_{l=1}^{m} (T_{j_{2l-1}}^* T_{j_{2l}}),$$

and using inequality in (2.3), we obtain

$$\| T_{j_1}^* T_{j_2} T_{j_3}^* \cdots T_{j_{2m}} \|_{\mathcal{H}\to\mathcal{H}} \le \prod_{l=1}^{m} {\gamma_{j_{2l-1}-j_{2l}}}^2. \tag{2.5}$$

Alternatively, we can associate the factors as

$$T_{j_1}^* T_{j_2} T_{j_3}^* \cdots T_{j_{2m}} = T_{j_1}^* \left(\prod_{l=1}^{m-1} (T_{j_{2l}} T_{j_{2l+1}}^*) \right) T_{j_{2m}}.$$

Since $\|T_{j_1}\|_{\mathcal{H}\to\mathcal{H}} \le \gamma_0 \le A$, and similarly $\|T_{j_{2m}}\|_{\mathcal{H}\to\mathcal{H}} \le A$, thanks to (2.3) we obtain

$$\| T_{j_1}^* T_{j_2} T_{j_3}^* \cdots T_{j_{2m}} \|_{\mathcal{H}\to\mathcal{H}} \le A^2 \prod_{l=1}^{m-1} {\gamma_{j_{2l}-j_{2l+1}}}^2. \tag{2.6}$$

We take the geometric mean of (2.5) and (2.6) and insert this into (2.4). The result is

$$\| (T_J^* T_J)^m \|_{\mathcal{H}\to\mathcal{H}} \le A \sum_{j_1, j_2, \ldots, j_{2m} \in J} \prod_{l=1}^{2m-1} \gamma_{j_l - j_{l+1}}.$$

In the above, we first sum in j_1 and use the fact that $\sum_{j_1 \in J} \gamma_{j_1 - j_2} \le A$. Next, we sum in j_2, using $\sum_{j_2 \in J} \gamma_{j_2 - j_3} \le A$. Continuing in this way for $j_1, j_2, \ldots, j_{2m-1}$ yields $\| (T_J^* T_J)^m \|_{\mathcal{H}\to\mathcal{H}} \le A^{2m-1} \sharp J$. Recall that we assumed J finite. Thus, $\|T_J\|_{\mathcal{H}\to\mathcal{H}} \le A(\sharp J)^{\frac{1}{2m}}$. Finally, we let $m \to \infty$, proving the theorem.

2.2.3 Exercises

Exercise 50. Let $1 < p < \infty$. By using Clarkson's inequality (see Exercise 47), prove an analogue in $L^p([0,1])$ to Proposition 93.

Exercise 51. Suppose that we have $\{T_j\}_{j\in\mathbb{Z}} \subset B(\mathcal{H})$. and a sequence of positive constants $\{\gamma_j\}_{j\in\mathbb{Z}}$ satisfying (2.3) with $J = \mathbb{Z}$ and

$$A \equiv \sum_{j=-\infty}^{\infty} \gamma_j < \infty. \tag{2.7}$$

(1) Show that $\sum_{j=-\infty}^{\infty} T_j T_k^* x$ converges for all $x \in \mathcal{H}$ and $k \in \mathbb{Z}$.

(2) Show that $\sum_{j=-\infty}^{\infty} T_j z$ converges for all $z \in \bigcup_{j\in\mathbb{Z}} \mathrm{Im}(T_j)$.

(3) Construct the limit of $\sum_{j=-\infty}^{\infty} T_j$ in an appropriate sense. Hint: What happens on $\bigcap_{j=-\infty}^{\infty} \ker(T_j)$?

2.3 Bochner integral

The Bochner integral is an advanced topic for beginners. In a word the Bochner integral replaces $\mathbb{R}$ or $\mathbb{C}$ in the range by Banach spaces. In the context of partial differential equations, it appears naturally: Suppose that we are trying to solve a PDE given by

$$u_t(x,t) = \Delta u(x,t) + f(x,t), \quad u_0(x) = u_0(x).$$

Here, $f(\cdot,t) \in L^p(\mathbb{R}^d)$. Then the solution is given by

$$u(x,t) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-s)\Delta}[f(\cdot,s)]\mathrm{d}s.$$

The integral appearing in the right-hand side of this formula is strictly speaking the Bochner integral.

For the sake of simplicity, we assume that $(X, \mathcal{B}, \mu)$ is a complete σ-finite measure space in Section 2.3.

Section 2.3 is interested mainly in Bochner integrals. Section 2.3.1 extends the notion of measurability of functions to Banach spaces-valued functions. Section 2.3.2 considers some convergence theorems in analogy with the ones in the theory of the Lebesgue integral. Section 2.3.3 is devoted to the theory of repeated integrals.

2.3.1 Measurable functions

Let $\mathcal{X}$ be a Banach space. Our present aim, to which we have been referring, is the construction of the integration theory for $\mathcal{X}$-valued functions. The theory of the underlying measure space $(X, \mathcal{B}, \mu)$ is already set up, so it seems appropriate that we start from the definition of the measurability of $\mathcal{X}$-valued functions.

Definition 36. Let $\mathcal{X}$ be a Banach space, and let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose that $\varphi : X \to \mathcal{X}$ is a function.

(1) The function φ is *weakly measurable*, if $b^* \circ \varphi : X \to \mathbb{C}$ is measurable for all $b^* \in \mathcal{X}^*$.

(2) The function φ is *simple*, if there exist $b_1, b_2, \ldots \in \mathcal{X}$ and $A_1, A_2, \ldots \in \mathcal{B}$ such that $\varphi(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x) \cdot b_j$ for μ-almost all $x \in X$.

(3) The function φ is *strongly measurable*, if there exists a sequence of simple functions $\{\varphi_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} \varphi_j(x) = \varphi(x)$ for μ-a.e. $x \in X$.

(4) The function φ is *separably valued*, if there exists a measurable set X_0 such that $\varphi(X_0)$ is separable and that $\mu(X \setminus X_0) = 0$.

The notion of weak measurablity corresponds to the separation of the real-valued functions into the positive part and the negative part. The notion of countably valuedness is used to guarantee that this operation can be completed within countably infinite steps.

The next lemma is useful, when we consider separably valued functions.

Lemma 95. *If $\mathcal{Y}$ is a separable Banach space, then there exists a countable subset $\mathcal{Z}^*$ of the unit ball $(\mathcal{Y}^*)_1$ of $\mathcal{Y}^*$ such that $\|y\|_{\mathcal{Y}} = \sup_{z^* \in \mathcal{Z}^*} |z^*(y)|$ for all $y \in \mathcal{Y}$.*

Proof Pick a countable dense set $Y_0 = \{y_j\}_{j=1}^{\infty}$. For each $j \in \mathbb{N}$ by virtue of the Hahn–Banach theorem, Theorem 87, we can find a norm attainer $z_j^* \in (\mathcal{Y}^*)_1$ of y_j. We have only to set $\mathcal{Z}^* \equiv \{z_j^* : j \in \mathbb{N}\}$.

Below, given an $\mathcal{X}$-valued function $\varphi : X \to \mathcal{X}$, let us write $\|\varphi\|_{\mathcal{X}} \equiv \|\varphi(\cdot)\|_{\mathcal{X}}$, so that $\|\varphi\|_{\mathcal{X}}$ is a non-negative function. We characterize the strong measurablity.

Theorem 96 (Pettis)**.** *Let $\varphi : X \to \mathcal{X}$ be a function. Then φ is strongly measurable, if and only if φ is separably valued and weakly measurable.*

Proof It is straightforward to prove the "only if " part and we leave this for the readers. See Exercise 54. Assume that φ is separably valued and weakly measurable. We may assume by disregarding a set of measure zero in addition that $\varphi(X)$ itself is separable. Let $Y = \overline{\mathrm{Span}(\varphi(X))}$ and $\{y_k\}_{k=-\infty}^{\infty}$ be a countable dense set in Y. Then, by Lemma 95, we obtain $z_1^*, z_2^*, \ldots, z_k^*, \ldots \in (Y^*)_1$ such that $\|y\|_Y = \sup_{j\in\mathbb{N}} |z_k^*(y)|$. From this we conclude that $\|\varphi - z\|_{\mathcal{X}}$ is a measurable function.

Let $j \in \mathbb{N}$. Then define $A_{k,j} \equiv \{\|\varphi - y_k\|_{\mathcal{X}} < j^{-1}\} \subset X$. Set

$$B_{1,j} \equiv A_{1,j}, \quad B_{k,j} \equiv A_{k,j} \setminus (A_{1,j} \cup A_{2,j} \cup \cdots \cup A_{k-1,j}),\ k \geq 2.$$

Then we have only to set $\varphi_j(x) \equiv \sum_{k=1}^{\infty} \chi_{B_{k,j}}(x) \cdot y_k$ for $x \in \mathcal{X}$. From the property of $A_{k,j}, k, j \in \mathbb{N}$, we conclude that $\varphi_j(x)$ tends to $\varphi(x)$ for μ-almost every $x \in \mathcal{X}$. Hence, φ is separably valued.

Having clarified the definition of measurability, we turn to the definition of the Bochner integral.

Now we are going to define integrals for countably simple functions. As we did for usual measurable functions, it is convenient to start from the definition of countably simple functions. But we need be careful : It will not work if we define the countably simple functions as the one taking finitely many values. What differs from our earlier job is that countably simple functions are allowed to take countably many values.

Definition 37 (Integrable countably simple functions)**.** Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $\mathcal{X}$ be a Banach space. A countably simple function φ $X \to \mathcal{X}$ is *integrable*, if it admits a representation $\varphi(x) = \sum_{j=1}^{\infty} \chi_{E_j}(x) b_j$ μ-a.e. $x \in X$ where $\{b_j\}_{j=1}^{\infty} \subset \mathcal{X}$ and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{B}$ satisfy $\sum_{j=1}^{\infty} \|b_j\|_{\mathcal{X}} \cdot \mu(E_j) < \infty$. The above representation is called an integrable representation of φ and one writes $\varphi \simeq \sum_{j=1}^{\infty} \chi_{E_j} b_j$.

Lemma 97. *Suppose that φ is a countably simple integrable function. Then there exists a unique element $\Phi \in \mathcal{X}$, which we will write $\Phi \equiv \int_X \varphi(x) \mathrm{d}\mu(x) \in \mathcal{X}$, such that $b^*(\Phi) = \int_X b^*(\varphi(x)) \mathrm{d}\mu(x)$ for all $b^* \in \mathcal{X}^*$.*

Proof We will prove the existence of Φ. Choose an integrable representation $\varphi \simeq \sum_{j=1}^{\infty} \chi_{E_j} a_j$. Then define an element $\Phi \equiv \sum_{j=1}^{\infty} a_j \, \mu(E_j) \in \mathcal{X}$. Note that the convergence is absolute. Let $b^* \in \mathcal{X}^*$. Then

$$b^*(\Phi) = \sum_{j=1}^{\infty} b^*(a_j) \mu(E_j) = \int_X b^* \circ \varphi(x) \mathrm{d}\mu(x),$$

where we have used the Lebesgue convergence theorem for the second equality.

Uniqueness of Φ follows from the Hahn–Banach theorem, Theorem 87.

We generalize the triangle inequality so that it is adapted to our current setting.

Lemma 98. *Let $\varphi : X \to \mathcal{X}$ be an integrable countably simple function. Then*

$$\left\| \int_X \varphi(x) \mathrm{d}\mu(x) \right\|_{\mathcal{X}} \leq \int_X \|\varphi\|_{\mathcal{X}}(x) \mathrm{d}\mu(x). \tag{2.1}$$

In particular, if φ has an integrable representation $\varphi \simeq \sum_{j=1}^{\infty} a_j \, \chi_{E_j}$, then

$$\left\| \int_X \varphi(x) \mathrm{d}\mu(x) \right\|_{\mathcal{X}} \leq \sum_{j=1}^{\infty} \|a_j\|_{\mathcal{X}} \mu(E_j). \tag{2.2}$$

Proof By the Hahn–Banach theorem, Theorem 87, we can choose y^* from the closed unit ball of $\mathcal{X}^*$ so that

$$\left\|\int_X \varphi(x)\mathrm{d}\mu(x)\right\|_{\mathcal{X}} = y^*\left(\int_X \varphi(x)\mathrm{d}\mu(x)\right) = \int_X y^*\circ\varphi(x)\mathrm{d}\mu(x).$$

By the definition of the dual norm,

$$\left\|\int_X \varphi(x)\mathrm{d}\mu(x)\right\|_{\mathcal{X}} \le \int_X \|\varphi\|_{\mathcal{X}}(x)\mathrm{d}\mu(x),$$

showing (2.1). Since $\|\varphi\|_{\mathcal{X}}(x) \le \sum_{j=1}^{\infty} \|a_j\|_{\mathcal{X}}\chi_{E_j}(x)$ for μ-almost every $x \in X$, (2.2) follows. Therefore, the proof is now complete.

The next lemma shows that the choice of the integrable representation can be made not so wasteful.

Lemma 99. *Suppose that $\varphi : X \to \mathcal{X}$ is a countably simple measurable function and $\varepsilon > 0$. Then φ has an integrable representation $\varphi \simeq \sum_{j=1}^{\infty} a_j\,\chi_{E_j}$ such that*

$$\sum_{j=1}^{\infty} \|a_j\|_X \mu(E_j) < \int_X \|\varphi\|_{\mathcal{X}}(x)\mathrm{d}\mu(x) + \varepsilon. \tag{2.3}$$

Proof Suppose that $\varphi \simeq \sum_{j=1}^{\infty} b_j\chi_{F_j}$ is an integrable representation. Then there exists an increasing sequence of integers $N_1 < N_2 < \cdots$ such that

$$\sum_{j=N_k}^{\infty} \|b_j\|_X \mu(F_j) < \frac{\varepsilon}{2^{k+1}}. \tag{2.4}$$

for each k. Let $N_0 = M_0 = 1$. By partitioning $F_{N_{k-1}}, F_{N_{k-1}+1}, \ldots, F_{N_k}$, we can find a collection of disjoint measurable sets $E_{M_{k-1}}, E_{M_{k-1}+1}, \ldots, E_{M_k}$ and $a_{M_{k-1}}, a_{M_{k-1}+1}, \ldots, a_{M_k} \in \mathcal{X}$

$$\sum_{l=M_{k-1}}^{M_k-1} a_l\,\chi_{E_l} = \sum_{j=N_{k-1}}^{N_k-1} b_j\,\chi_{F_j}. \tag{2.5}$$

Then the disjointness of $\{E_j\}_{j=M_{k-1}}^{M_k-1}$ yields

$$\begin{aligned}\int_X \left\|\sum_{j=N_{k-1}}^{N_k-1} b_j\,\chi_{F_j}(x)\right\|_{\mathcal{X}} \mathrm{d}\mu(x) &= \int_X \left\|\sum_{l=M_{k-1}}^{M_k-1} a_l\,\chi_{E_l}(x)\right\|_{\mathcal{X}} \mathrm{d}\mu(x)\\ &= \sum_{l=M_{k-1}}^{M_k-1} \|a_l\|_{\mathcal{X}}\mu(E_l).\end{aligned}$$

We also have

$$\sum_{l=1}^{\infty} \|a_l\|_{\mathcal{X}} \mu(E_l) = \sum_{k=1}^{\infty} \int_X \left\| \sum_{j=N_{k-1}}^{N_k-1} b_j \, \chi_{F_j}(x) \right\|_{\mathcal{X}} \mathrm{d}\mu(x).$$

By the triangle inequality,

$$\sum_{l=1}^{\infty} \|a_l\|_{\mathcal{X}} \mu(E_l) \leq \int_X \left\| \sum_{j=1}^{N_1-1} b_j \, \chi_{F_j}(x) \right\|_{\mathcal{X}} \mathrm{d}\mu(x) + \sum_{k=1}^{\infty} \sum_{j=N_k}^{\infty} \|b_j\|_X \mu(F_j).$$

We rewrite the right-hand side using f:

$$\sum_{l=1}^{\infty} \|a_l\|_{\mathcal{X}} \mu(E_l) \leq \int_X \left\| f(x) - \sum_{j=N_1}^{\infty} b_j \, \chi_{F_j}(x) \right\|_{\mathcal{X}} \mathrm{d}\mu(x) + \sum_{k=1}^{\infty} \sum_{j=N_k}^{\infty} \|b_j\|_X \mu(F_j).$$

Therefore, we obtain

$$\begin{aligned} \sum_{l=1}^{\infty} \|a_l\|_{\mathcal{X}} \mu(E_l) &\leq \int_X \|f(x)\|_{\mathcal{X}} \mathrm{d}\mu(x) + 2 \sum_{k=1}^{\infty} \sum_{j=N_k}^{\infty} \|b_j\|_X \mu(F_j) \\ &\leq \int_X \|f(x)\|_{\mathcal{X}} \mathrm{d}\mu(x) + \varepsilon. \end{aligned}$$

The number $\varepsilon > 0$ being arbitrary, this is the desired result.

Keeping our observations above, we pass to the general c ase. We are going to define the integrable functions. In the $\mathbb{R}$-valued case we have allowed the integral to take the value $\pm\infty$. However, in the Bochner integral we rule out the possibility for the integral to diverge.

Definition 38 (Bochner integrable functions)**.** A function $\varphi : X \to \mathcal{X}$ is *Bochner integrable*, if there exists a sequence of countably simple functions $\{\varphi_j\}_{j=1}^{\infty}$ such that

$$\lim_{j \to \infty} \varphi_j(x) = \varphi(x) \tag{2.6}$$

for μ-almost every $x \in X$ and that

$$\lim_{j \to \infty} \int_X \|\varphi(x) - \varphi_j(x)\|_{\mathcal{X}} \mathrm{d}\mu(x) = 0. \tag{2.7}$$

If this is the case, define the integral of φ on X by:

$$\int_X \varphi(x) \mathrm{d}\mu(x) = \int_X \varphi(x) \mathrm{d}\mu(x) \equiv \lim_{j \to \infty} \int_X \varphi_j(x) \mathrm{d}\mu(x). \tag{2.8}$$

The symbol $L^1(X; \mathcal{X})$ denotes the set of all Bochner integrable functions.

Lemma 100. *The definition of the integral* $\int_X \varphi(x)\mathrm{d}\mu(x)$ *makes sense. That is, in that the limit* (2.8) *defining* $\int_X \varphi(x)\mathrm{d}\mu(x)$ *does exist and the element* $\int_X \varphi(x)\mathrm{d}\mu(x) \in \mathcal{X}$ *is independent of the choice of* $\{\varphi_j\}_{j=1}^{\infty}$.

Proof We will show that the limit (2.8) exists. To see this, we observe, if $j, k \geq J$

$$\left\| \int_X \varphi_j(x)\mathrm{d}\mu(x) - \int_X \varphi_k(x)\mathrm{d}\mu(x) \right\|_{\mathcal{X}} \leq 2 \sup_{l \geq J} \int_X \|\varphi(x) - \varphi_l(x)\|_{\mathcal{X}}\mathrm{d}\mu(x).$$

Therefore $\left\{ \int_X \varphi_j(x)\mathrm{d}\mu(x) \right\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$ and the limit therefore does exist.

We will show that the integral is independent of the admissible representation of φ. Suppose $\{\psi_j\}_{j=1}^{\infty}$ is another sequence of countably simple integrable functions such that $\lim_{j\to\infty} \psi_j = \varphi$ and that $\lim_{j\to\infty} \int_X \|\varphi(x) - \psi_j(x)\|_{\mathcal{X}}\mathrm{d}\mu(x) = 0$. Then by the triangle inequality we obtain

$$\begin{aligned} &\left\| \int_X \varphi_j(x)\mathrm{d}\mu(x) - \int_X \psi_j(x)\mathrm{d}\mu(x) \right\|_{\mathcal{X}} \\ &\leq \int_X \left(\|\varphi(x) - \varphi_j(x)\|_{\mathcal{X}} + \|\varphi(x) - \psi_j(x)\|_{\mathcal{X}} \right) \mathrm{d}\mu(x), \end{aligned}$$

which tends to 0 as $j \to \infty$. Thus, it follows that both $\left\{ \int_X \varphi_j(x)\mathrm{d}\mu(x) \right\}_{j=1}^{\infty}$ and $\left\{ \int_X \psi_j(x)\mathrm{d}\mu(x) \right\}_{j=1}^{\infty}$ tend to the same limit. Therefore, the integral is independent of the admissible representations of φ.

Lemma 101. *In addition to the same setting as above, we let* $\varphi : X \to \mathcal{X}$ *be a Bochner integrable function. Then*

$$\left\| \int_X \varphi(x)\mathrm{d}\mu(x) \right\|_{\mathcal{X}} \leq \int_X \|\varphi\|_{\mathcal{X}}(x)\mathrm{d}\mu(x).$$

Proof It is just a simple matter to pass to the limit of the countably simple functions. However, it is worth considering the case when $\mathcal{X} = \mathbb{C}$. Writing out the statement in full with $\mathcal{X} = \mathbb{C}$, we can easily imagine what is the heart of the matter in the proof of Lemma 101.

We characterize Bochner integrable functions.

Theorem 102 (Bochner)**.** *The function* $\varphi : X \to \mathcal{X}$ *is Bochner integrable, if and only if* φ *is an integrable countably simple function.*

Proof Suppose that φ is an integrable countably simple function. Then we can take an integrable representation $\varphi \simeq \sum\limits_{j=1}^{\infty} a_j \chi_{E_j}$. Set $\varphi_j \equiv \sum\limits_{k=1}^{j} a_k \chi_{E_k}$ for each $j \in \mathbb{N}$. Then each φ_j is an integrable countably simple function. By the Fatou lemma we have

$$\begin{aligned}\int_X \|\varphi(x) - \varphi_j(x)\|_{\mathcal{X}} \, \mathrm{d}\mu(x) &= \int_X \left\| \sum_{k=j+1}^{\infty} a_k \chi_{E_k}(x) \right\|_{\mathcal{X}} \mathrm{d}\mu(x) \\ &\leq \sum_{k=j+1}^{\infty} \|a_k\|_{\mathcal{X}} \mu(E_k).\end{aligned}$$

The most right-hand side tending to 0 as $k \to \infty$, so does the left-hand side. Therefore, φ is Bochner integrable.

Suppose that φ is Bochner integrable. Then there exists a sequence of countably simple integrable functions $\{\varphi_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \int_X \|\varphi(x) - \varphi_j(x)\|_{\mathcal{X}} \mathrm{d}\mu(x) = 0.$$

A passage to the subsequence, we can assume

$$\int_X \|\varphi(x) - \varphi_j(x)\|_{\mathcal{X}} \mathrm{d}\mu(x) < 2^{-j-1}.$$

In this case, by the triangle inequality, we have

$$\int_X \|\varphi_{j+1}(x) - \varphi_j(x)\|_{\mathcal{X}} \mathrm{d}\mu(x) < \frac{3}{4} \cdot 2^{-j}.$$

If we invoke Lemma 99, then for each $j \in \mathbb{N}$, we can find an integrable representation

$$\varphi_{j+1} - \varphi_j \simeq \sum_{k=1}^{\infty} a_{j+1,k} \chi_{E_{j+1,k}} \text{ with } \sum_{k=1}^{\infty} \|a_{j,k}\|_{\mathcal{X}} \mu(E_{j,k}) < 2^{-j}. \tag{2.9}$$

We also have an integrable representation of $\varphi_1 \colon \varphi_1 \simeq \sum\limits_{k=1}^{\infty} a_{1,k} \chi_{E_{1,k}}$. Using (2.9), we conclude that φ has the following integrable representation

$$\varphi \simeq \sum_{j,k=1}^{\infty} a_{j,k} \chi_{E_{j,k}}.$$

Therefore φ is an integrable countably simple function as well.

2.3.2 Convergence theorems

Having set down the definition of the integrals, we now turn to the convergence theorems such as the dominated convergence theorem in integration theory.

Theorem 103. *Suppose that $\{\varphi_j\}_{j=1}^{\infty}$ is a sequence of strongly integrable functions such that $\{\varphi_j(x)\}_{j=1}^{\infty}$ is a Cauchy sequence for almost every $x \in X$. Let φ be a weakly measurable function such that $\varphi(x) = \lim_{j\to\infty} \varphi(x)$ for μ-almost every $x \in X$. Assume in addition there exists $\psi \in L^1(\mu) \cap \mathbb{M}^+(\mu)$ such that*

$$\|\varphi_j(x)\|_{\mathcal{X}} \le \psi(x)$$

for μ-a.e. $x \in X$. Then, $\varphi \in L^1(X;\mathcal{X})$ and

$$\lim_{j\to\infty} \int_X \varphi_j(x)\mathrm{d}\mu(x) = \int_X \varphi(x)\mathrm{d}\mu(x). \tag{2.10}$$

Proof Since $\bigcup_{j=1}^{\infty} Y_j$ is separable, whenever each Y_j is separable, φ is strongly measurable.

We claim that $\varphi \in L^1(X;\mathcal{X})$ and (2.10) holds. Since $\varphi_j \in L^1(X;\mathcal{X})$, by Theorem 102 we conclude that φ_j is an integrable countably simple function. Furthermore, $\|\varphi(x)-\varphi_j(x)\|_{\mathcal{X}} \le 2\psi(x)$ for μ-almost everywhere $x \in X$. Thus, if we invoke Lebesgue's convergence theorem, we obtain

$$\lim_{j\to\infty} \int_X \|\varphi(x) - \varphi_j(x)\|_{\mathcal{X}}\mathrm{d}\mu(x) = 0.$$

As a result $\varphi \in L^1(X;\mathcal{X})$ and (2.10) holds.

We will show that the space $L^1(X;\mathcal{X})$ is a Banach space.

Theorem 104. *Let $\mathcal{X}$ be a Banach space. The space $L^1(X;\mathcal{X})$ is complete, once we identify the functions with the same integrable representation.*

Proof Let $\{\varphi_j\}_{j=1}^{\infty} \subset L^1(X;\mathcal{X})$ be arbitrary. We have only to prove that, if $\sum_{j=1}^{\infty} \|\varphi_j\|_{L^1(\mathcal{X};X)}$ converges, then so does $\sum_{j=1}^{\infty} \varphi_j$. To do this we pick an integrable representation

$$\varphi_j \simeq \sum_{k=1}^{\infty} a_{j,k}\,\chi_{E_{j,k}} \text{ with } \sum_{k=1}^{\infty} \|a_{j,k}\|_{\mathcal{X}}\mu(E_{j,k}) < \|\varphi_j\|_{L^1(X;\mathcal{X})} + 2^{-j}.$$

Then for $x \in X$, we define $\psi(x) \equiv \sum_{j,k=1}^{\infty} a_{j,k}\,\chi_{E_{j,k}}(x)$, provided the series converges and $\psi(x) = 0$ otherwise. Then ψ has an integrable representation

$\psi \simeq \sum_{j,k=1}^{\infty} a_{j,k}\,\chi_{E_{j,k}}$. Therefore $\psi \in L^1(X;\mathcal{X})$. Since $\psi - \sum_{j=1}^{l} \varphi_j$ has an integrable representation $\psi - \sum_{j=1}^{l} \varphi_j \simeq \sum_{j=l+1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}\,\chi_{E_{j,k}}$, we conclude $\psi = \sum_{j=1}^{\infty} \varphi_j$.

Finally let us characterize the Bochner integral in terms of the dual spaces.

Proposition 105. *Let $\varphi : X \to \mathcal{X}$ be a Bochner integrable function. Then $b \equiv \int_X \varphi(x)\mathrm{d}\mu(x)$ is a unique element in $\mathcal{X}$ satisfying*

$$b^*(b) = \int_X b^*(\varphi(x))\mathrm{d}\mu(x)$$

for all $b^* \in X^*$.

Proof Now that "Bochner integrable function" and "integrable countably simple function" are synonymous, this proposition is Lemma 97 itself.

2.3.3 Fubini's theorem for Bochner integral

In this subsection to discuss Fubini's theorem for Bochner integrals, let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.

Theorem 106. *Suppose that $\varphi \in L^1(X \times Y; \mathcal{X})$. Then we have the following:*

(1)
- *For almost every $x \in X$, the function $\varphi(x, \cdot)$ belongs to $L^1(Y; \mathcal{X})$.*
- *For almost every $y \in Y$, the function $\varphi(\cdot, y)$ belongs to $L^1(X; \mathcal{X})$.*

Below it will be understood tacitly that $\int_Y \varphi(x,y)\mathrm{d}\nu(y) = 0$ and that $\int_X \varphi(x,y)\mathrm{d}\mu(x) = 0$, if the integrand is not integrable.

(2) *We have*

$$\begin{aligned}\iint_{X\times Y} \varphi(x,y)\mathrm{d}\mu \otimes \nu(x,y) &= \int_X \left(\int_Y \varphi(x,y)\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) \\ &= \int_Y \left(\int_X \varphi(x,y)\mathrm{d}\mu(y) \right) \mathrm{d}\nu(y).\end{aligned}$$

Proof

(1) Since $\iint_{X\times Y} \|\varphi(x,y)\|_{\mathcal{X}}\mathrm{d}\mu \otimes \nu(x,y) < \infty$, assertion (1) follows immediately.

(2) Pick an integrable representation of φ: $\varphi \simeq \sum_{j=1}^{\infty} a_j \chi_{G_j}$, where each G_j is a $\mu \otimes \nu$-measurable set. By disregarding a set of measure zero we may

assume $\sum_{j=1}^{\infty} \|a_j\|_{\mathcal{X}} \cdot \int_Y \chi_{G_j}(x,y)\mathrm{d}\nu(y) < \infty$ for all $x \in X$. Then for all $x \in X$, $\int_Y \varphi(x,y)\mathrm{d}\mu(y) = \sum_{j=1}^{\infty} a_j \int_Y \chi_{G_j}(x,y)\mathrm{d}\nu(y)$. Since

$$\sum_{j=1}^{\infty} \int_X \left\| a_j \int_Y \chi_{G_j}(x,y)\mathrm{d}\nu(y) \right\|_{\mathcal{X}} \mathrm{d}\mu(y) = \sum_{j=1}^{\infty} \|a_j\|_{\mathcal{X}} \int_Y \chi_{G_j}(x,y)\mathrm{d}\nu(y) < \infty,$$

we are in the position of using the Lebesgue convergence theorem (Theorem 103) for the Bochner integral to obtain

$$\begin{aligned} \int_X \left(\int_Y \varphi(x,y)\mathrm{d}\mu(y) \right) \mathrm{d}\nu(x) &= \sum_{j=1}^{\infty} a_j \int_{X\times Y} \chi_{G_j}(x,y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \\ &= \iint_{X\times Y} \varphi(x,y)\mathrm{d}\mu \otimes \nu(x,y). \end{aligned}$$

This is the desired result.

Now we pass our theory of Lebesgue integral to the $L^p(X;\mathcal{X})$-space.

Definition 39 ($L^p(X;\mathcal{X})$)**.** Let $1 \le p < \infty$. The space $L^p(X;\mathcal{X})$ is defined as the set of all simple functions $f : X \to \mathcal{X}$ for which the norm

$$\|f\|_{L^p(X;\mathcal{X})} \equiv \left(\int_X \|f\|_{\mathcal{X}}(x)^p \mathrm{d}\mu \right)^{\frac{1}{p}} < \infty.$$

The space $L^\infty(X;\mathcal{X})$ is the set of all simple functions satisfying

$$\|f\|_{L^\infty(X;\mathcal{X})} \equiv \| \, \|f\|_{\mathcal{X}} \, \|_{L^\infty(X)} < \infty.$$

Now we will collect some duality results.

Theorem 107. *Let $\mathcal{X}$ be a separable Banach space. Then the dual space of $L^1(\mathbb{R};\mathcal{X})$ is canonically identified with $L^\infty(\mathbb{R};\mathcal{X}^*)$ in the following manner*:

(1) *If $g \in L^\infty(\mathbb{R};\mathcal{X}^*)$, then there exists a linear functional L_g such that*

$$L_g(\chi_A \cdot b) = \int_A \langle b, g(x)\rangle_{\mathcal{X},\mathcal{X}^*} \mathrm{d}x$$

for all $b \in \mathcal{X}$ and measurable sets A.

(2) *Let $\varphi \in L^1(\mathbb{R};\mathcal{X})^*$. Then there exists $g \in L^\infty(\mathbb{R};\mathcal{X}^*)$ such that $L_g = \varphi$.*

Proof It is not so hard to see that $L^\infty(\mathbb{R};\mathcal{X}^*)$ defines a continuous functional on $L^1(\mathbb{R};\mathcal{X})$. Suppose that $\varphi \in L^1(\mathbb{R} : \mathcal{X})^*$. Then define $\Phi : \mathbb{R} \to \mathcal{X}^*$ by

$$\langle b, \Psi(t)\rangle_{\mathcal{X},\mathcal{X}^*} \equiv \begin{cases} \varphi(\chi_{[0,t]} \cdot b) & t \ge 0, \\ -\varphi(\chi_{[t,0]} \cdot b) & t \le 0 \end{cases} \qquad (b \in \mathcal{X}).$$

For $-\infty < s < t < \infty$, we note that

$$\langle b, \Psi(t)\rangle_{\mathcal{X},\mathcal{X}^*} - \langle b, \Psi(s)\rangle_{\mathcal{X},\mathcal{X}^*} = \varphi(\chi_{[s,t]} \cdot b).$$

Hence, denoting $\mathcal{X}_1$ by the unit ball of $\mathcal{X}$, we obtain

$$\begin{aligned} \|\Psi(t) - \Psi(s)\|_* &= \sup_{b\in\mathcal{X}_1} |\langle b, \Psi(t)\rangle_{\mathcal{X},\mathcal{X}^*} - \langle b, \Psi(s)\rangle_{\mathcal{X},\mathcal{X}^*}| \\ &= \sup_{b\in\mathcal{X}_1} |\varphi(\chi_{[s,t]} \cdot b)| \\ &\le \|\varphi\|_* \cdot |t - s|. \end{aligned} \tag{2.11}$$

Let us define

$$\Phi_j(t) \equiv 2^j(\Psi(t + 2^{-j}) - \Psi(t)) \quad (t \in \mathbb{R}).$$

Then

$$\|\Phi_j(t)\|_{\mathcal{X}^*} \le \|\varphi\|_* \quad (t \in \mathbb{R})$$

by virtue of (2.11). Therefore, by the Banach Alaoglu theorem, Theorem 89, there exists a subsequence $\{\Phi_{j_k}\}_{k\in\mathbb{N}}$ such that the weak-$*$ limit

$$\Phi(t) \equiv \lim_{k\to\infty} \Phi_{j_k}(t)$$

exists for all $t \in \mathbb{Q}$. If $t, s \in \mathbb{Q}$, then

$$\|\Phi(t) - \Phi(s)\|_{\mathcal{X}^*} = \sup_{b\in\mathcal{X}\setminus\{0\}} \frac{|\langle b, \Phi(t) - \Phi(s)\rangle_{\mathcal{X},\mathcal{X}^*}|}{\|b\|_{\mathcal{X}}} \le \|\varphi\|_* |t - s|.$$

Therefore, Φ can be extended to a continuous function from $\mathbb{R}$ to $\mathcal{X}^*$ satisfying

$$\|\Phi(t) - \Phi(s)\|_{\mathcal{X}^*} \le \|\varphi\|_* |t - s|.$$

This implies

$$\begin{aligned} &|\langle b, \Phi(t) - \Phi_{j_k}(t)\rangle_{\mathcal{X},\mathcal{X}^*}| \\ &\le |\langle b, \Phi(t) - \Phi(s) - \Phi_{j_k}(t) + \Phi_{j_k}(s)\rangle_{\mathcal{X},\mathcal{X}^*}| + |\langle b, \Phi(s) - \Phi_{j_k}(s)\rangle_{\mathcal{X},\mathcal{X}^*}| \\ &\le 2\|\varphi\|_* |t - s| + |\langle b, \Phi(s) - \Phi_{j_k}(s)\rangle_{\mathcal{X},\mathcal{X}^*}| \end{aligned}$$

for all $t \in \mathbb{R}$ and $s \in \mathbb{Q}$. Thus, we obtain

$$\limsup_{j\to\infty} \|\Phi(t) - \Phi_{j_k}(t)\|_{\mathcal{X}^*} = \limsup_{j\to\infty} \sup_{b\in\mathcal{X}_1} |\langle b, \Phi(t) - \Phi_{j_k}(t)\rangle_{\mathcal{X},\mathcal{X}^*}| \le 2\|\varphi\|_* |t - s|.$$

Since $s \in \mathbb{Q}$ is arbitrary, this implies $\lim_{k\to\infty} \Phi_{j_k}(t) = \Phi(t)$ for all $t \in \mathbb{R}$. As a result, for each $g \in L^\infty_{\mathrm{c}}(\mathbb{R}^d; \mathcal{X})$,

$$\int_{\mathbb{R}} \langle \Phi(t), g(t)\rangle \mathrm{d}t = \lim_{k\to\infty} \int_{\mathbb{R}} \langle g(t), \Phi_{j_k}(t)\rangle_{\mathcal{X},\mathcal{X}^*} \mathrm{d}t = \varphi(g).$$

Since $L^\infty_{\mathrm{c}}(\mathbb{R}^d; \mathcal{X})$ is dense in $L^1(\mathbb{R}^d; \mathcal{X})$, we see that φ is realized by Φ.

We conclude Section 2.3.3 with another duality result. We recall that a Banach space $\mathcal{X}$ is reflexive if the canonical mapping $Q : \mathcal{X} \to \mathcal{X}^{**}$ is

surjective; see [310, 1.11.6]. The proof is similar to Theorem 107 and we omit the proof.

Theorem 108. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let $1 \le p < \infty$. Suppose that $\mathcal{X}$ is a reflexive Banach space. Then $L^p(X;\mathcal{X})^* = L^{p'}(X;\mathcal{X}^*)$.*

Let $\mathcal{X}$ be a Banach space, and let $\mathrm{Lip}(\mathbb{R};\mathcal{X})$ be the set of all Lipschitz continuous functions $f : \mathbb{R} \to \mathcal{X}$ for which the quantity $\|f\|_{\mathrm{Lip}(\mathbb{R};\mathcal{X})} \equiv \sup_{-\infty<s<t<\infty} |t-s|^{-1}\|f(t)-f(s)\|_{\mathcal{X}}$ is finite. If $\mathcal{X} = \mathbb{C}$, then abbreviate $\mathrm{Lip}(\mathbb{R};\mathcal{X})$ to $\mathrm{Lip}(\mathbb{R})$.

Lemma 109. *Let $\mathcal{X}$ be a Banach space.*

(1) *Let $f \in L^1(\mathbb{R};\mathcal{X}) \cap C(\mathbb{R};\mathcal{X})$ and $g \in \mathrm{Lip}(\mathbb{R};\mathcal{X}^*)$. Then the limit*

$$L_g(f) \equiv \lim_{N\to\infty} \sum_{j=-\infty}^{\infty} \langle g(N^{-1}(j+1)) - g(N^{-1}j), f(N^{-1}j)\rangle$$

exists and satisfies $|L_g(f)| \le \|f\|_{L^1(\mathbb{R};\mathcal{X})}\|g\|_{\mathrm{Lip}(\mathbb{R};\mathcal{X}^*)}$. Thus, any $g \in \mathrm{Lip}(\mathbb{R}^*;\mathcal{X})$ induces a bounded linear functional on $L^1(\mathbb{R};\mathcal{X})$.

(2) *Conversely, any bounded linear functional L on $L^1(\mathbb{R};\mathcal{X})$ is realized as $L = L_g$ for some $g \in \mathrm{Lip}(\mathbb{R};\mathcal{X}^*)$.*

Proof Assertion (1) is straightforward and we omit the proof. For (2), we define

$$\langle g(t), x\rangle \equiv \begin{cases} L(\chi_{[0,t]}x) & t \ge 0, \\ -L(\chi_{[t,0]}x) & t < 0 \end{cases}$$

for $x \in \mathcal{X}$. Then $g \in \mathrm{Lip}(\mathbb{R},\mathcal{X}^*)$. It is not so hard to check $L = L_g$, since functions of the form $\chi_{[a,b]}x$ with $a, b \in \mathbb{R}$ and $x \in X$ span a dense subspace in $L^1(\mathbb{R};\mathcal{X})$.

2.3.4 Exercises

Exercise 52. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $\mathcal{X}$ be a Banach space. Suppose that $\{\varphi_j\}_{j=1}^{\infty} \in L^1(X;\mathcal{X})$ satisfies

$$\sum_{j=1}^{\infty} \int_X \|\varphi_j(x)\|_{\mathcal{X}} \mathrm{d}\mu(x) < \infty.$$

Then show that $\sum_{j=1}^{\infty} \varphi_j(x)$ converges for μ-almost every $x \in X$. Modifying $\sum_{j=1}^{\infty} \varphi_j(x)$ over a set of measure zero, show that $\sum_{j=1}^{\infty} \varphi_j \in L^1(X;\mathcal{X})$ and

$$\int_X \left(\sum_{j=1}^{\infty} \varphi_j\right) \mathrm{d}\mu = \sum_{j=1}^{\infty} \int_X \varphi_j(x)\mathrm{d}\mu(x).$$

Exercise 53. Let $(X, \mathcal{B}, \mu)$ be a measure space. Also let $\mathcal{X}$ be a Banach space.

(1) Let $1 \leq p \leq \infty$. Then show that $L^p(X; \mathcal{X})$ is a Banach space.

(2) Let $\mathcal{E}(\mu)$ be a Banach function space. Denote by $\mathcal{E}(\mu; \mathcal{X})$ the set of all strongly measurable functions $f : X \to \mathcal{X}$ such that $\|f(\cdot)\|_{\mathcal{X}} \in \mathcal{E}(\mu)$. Equip $\mathcal{E}(\mu; \mathcal{X})$ with the norm;

$$\|f\|_{\mathcal{E}(\mu;\mathcal{X})} \equiv \| \, \|f(\cdot)\|_{\mathcal{X}} \, \|_{\mathcal{E}(\mu)} \quad (f \in \mathcal{E}(\mu; \mathcal{X})).$$

Then show that $\mathcal{E}(\mu; X)$ is a Banach space.

Exercise 54. Let $\mathcal{X}$ be a Banach space, and let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $\varphi : X \to \mathcal{X}$ is a strongly measurable function.

(1) Prove that φ is separably valued and weakly measurable, if φ is simple.

(2) Prove that φ is separably valued and weakly measurable.

Exercise 55. Let $\mathcal{X}$ be a Banach space. Show that $C^1([t_0, t_1]; \mathcal{X})$ is dense in $L^1([t_0, t_1]; \mathcal{X})$.

2.4 Notes

Section 2.1

General remarks and textbooks in Section 2.1

See the textbooks [241, Chapter 14], [310] and [409, Chapter 5] for functional analysis, for example.

Section 2.1.1

We can consult [310, 409]. We refer to [310, Theorem 1.9.8] and [310, Proposition 1.9.15] for Theorems 87 and 88, respectively or see [409, Theorem 5.5.5], [409, Theorem 5.5.8] and [409, Theorem 5.5.9] for Hahn–Banach extensions, Theorems 85, 86 and 87, respectively.

See [239] as well.

See [310, 2.6.18 and 2.6.19] for Theorem 89 for the Banach Alaoglu theorem.

Section 2.1.2

Many people strengthened the triangle inequality. We employed one of them in Theorem 90; see [300, Theorem 1].

Section 2.1.3

Section 2.1.3 heavily depends on Lions and Peetre [280]. See [280, Chapter premier] for the definition of sum spaces and intersection spaces. We refer to [280, 3. I] for Theorem 92.

Section 2.2

General remarks and textbooks in Section 2.2

There are many textbooks on Hilbert spaces. Any one of them will suffice for this book. Here we content ourselves with the book by Yoshida [465].

Section 2.2.1

Beigböck, Schachermayer and Veliyev considered a variant of Komlós' theorem. See [26, §2.1] for Proposition 93, while the original Komlós theorem is due to Komlós [244].

Section 2.2.2

Cotlar proved the "so called" Cotlar's lemma, Theorem 94, in [88].

Section 2.3

General remarks and textbooks in Section 2.3

The Bochner integral is named after Salomon Bochner [38]. We refer to [81, Appendix E] and [465] for Bochner integrals. See also the lecture note [223, Chapter 0, §I].

Section 2.3.1

Pettis characterized the strong measurability [357, Theorem 1.1]; see Theorem 96. We followed Pettis [357, Definition 2.1] when we defined the integral of functions assuming their values in a Banach space; see Definition 37.

Section 2.3.2

See the paper by Bochner [38, §3] for Theorem 103. The completeness of the Bochner integrable spaces can be found in [357, Theorem 4.2]; see Theorem 104. Pettis obtained a dense subspace; see [357, Theorem 4.3].

Section 2.3.3

See the paper by Bochner [38, §4] for Theorem 106. Pettis considered the duality with respect to Bochner integral [357, Theorem 3.3]; see Theorem 107. Calderón specified the dual space of $L^1(\mathbb{R};\mathcal{X})$ [52, §32.1]; see Lemma 109.

Chapter 3

Polynomials and harmonic functions

This chapter collects some preliminary facts. The readers may skip Chapter 3. We indicate where we use each theorem in the other parts of this book.

As a preliminary step, in Section 3.1 we collect some fundamental facts on polynomials. Section 3.2 is the main part of Chapter 3, where we consider harmonic functions.

3.1 Preliminary facts on polynomials

Here, we collect some facts on polynomials. Although we use them to investigate Morrey spaces, they are interesting in their own right. We aim to set up some notation and then consider some fundamental properties in Section 3.1.1. We consider the convolution estimates in Section 3.1.2, where the moment conditions come into play. Section 3.1.3 compares the norm of the derivatives and the L^q-norm. In Section 3.1.4 we consider the approximation of functions by polynomials.

3.1.1 The space $\mathcal{P}_k(\mathbb{R}^n)$

Here, we set up some notation for polynomials used in this book. We will equip these spaces with the structure of a Hilbert space. We also investigate the dimension of the space we define here.

Definition 40 ($\mathcal{P}(\mathbb{R}^n)$, $\mathcal{P}_k(\mathbb{R}^n)$)**.** Define $\mathcal{P}_{-1}(\mathbb{R}^n) \equiv \{0\}$. Denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ the set of all polynomials. Denote by $\mathcal{P}_k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, the set of all polynomial functions with degree less than or equal to k.

By the definition $\mathcal{P}(\mathbb{R}^n) \equiv \bigcup_{d=0}^{\infty} \mathcal{P}_d(\mathbb{R}^n)$. Unlike algebra, we regard each element in $\mathcal{P}(\mathbb{R}^n)$ as a function.

We also use the notion of homogeneous polynomials.

Definition 41. Let $k \in \mathbb{N}_0 \cup \{-1\}$. The set $\dot{\mathcal{P}}_k(\mathbb{R}^n)$ is the set of all polynomials P such that $P(tx) = t^k P(x)$ for all $x \in \mathbb{R}^n$ and $t > 0$. The space $\dot{\mathcal{P}}_k(\mathbb{R}^n)$ is

called the *homogeneous polynomial space* of order k, while $\mathcal{P}_k(\mathbb{R}^n)$ is called the *nonhomogeneous polynomial space* of order k.

Here and in the sequel, ${}_n\mathrm{C}_r$ denotes the combination of choosing r objects out of n.

Corollary 110. *For each $k \in \mathbb{N}_0$, one has $\dim_{\mathbb{C}}(\mathcal{P}_k(\mathbb{R}^n)) = {}_{k+n}\mathrm{C}_n$ and $\dim_{\mathbb{C}}(\dot{\mathcal{P}}_k(\mathbb{R}^n)) = {}_{k+n-1}\mathrm{C}_{n-1}$.*

Proof We will deal with the nonhomogeneous spaces due to similarity. We leave the proof for interested readers; see Exercise 56. We calculate

$$\begin{aligned}\dim_{\mathbb{C}}(\mathcal{P}_k(\mathbb{R}^n)) &= \sharp\{(m_0, m_1, \ldots, m_n) \in {\mathbb{N}_0}^{n+1} \,:\, m_0 + m_1 + \cdots + m_n = k\} \\ &= {}_{k+n}\mathrm{C}_n.\end{aligned}$$

We will equip $\mathcal{P}_k(\mathbb{R}^n)$ with the structure of a Hilbert space. Observe that any polynomial P defines a differential operator $P(D) \equiv \sum_{|\alpha|\le d} a_\alpha \partial^\alpha$, where

$$P(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha.$$

Lemma 111. *For $P, Q \in \dot{\mathcal{P}}_k(\mathbb{R}^n)$, $P(D)\overline{Q}$ is a complex number. More precisely, if we let $P(x) \equiv \sum_{|\alpha|=k} a_\alpha x^\alpha$ and $Q(x) \equiv \sum_{|\alpha|=k} b_\alpha x^\alpha$, then $P(D)\overline{Q} = \sum_{|\alpha|=k} \alpha! a_\alpha \overline{b_\alpha}$.*

Proof It suffices to deal with monomials. In this case, the conclusion is clear.

Consequently, for any $P, Q \in \mathcal{P}(\mathbb{R}^n)$ we can define an inner product.

Definition 42. One defines an inner product in $\mathcal{P}$ by $\langle P, Q\rangle_{\mathcal{P}} \equiv P(D)\bar{Q}(0)$.

Here we present an example of taking advantage of the finiteness of $\dim_{\mathbb{C}}(\mathcal{P}_l(\mathbb{R}^n))$.

Example 52. Let $l \in \mathbb{N}_0$, and let $V \subset \mathbb{R}^n$ be an open set. Then there exists $\omega \in C_c^\infty(V)$ satisfying $\int_V \omega(x) x^\alpha \mathrm{d}x = \delta_{0\alpha}$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le l$. In fact, equip $\mathcal{P}_l(\mathbb{R}^n)$ with the inner product

$$\langle P, Q\rangle = \int_{\mathbb{R}^n} P(x)\overline{Q(x)}\Omega(x)\mathrm{d}x \quad (P, Q \in \mathcal{P}_l(\mathbb{R}^n)),$$

where $\Omega \in (\mathbb{M}^+(\mathbb{R}^n) \cap C_{\mathrm{c}}^\infty(\mathbb{R}^n)) \setminus \{0\}$ is a fixed function supported on V. Set $U \equiv \{P \in \mathcal{P}_l(\mathbb{R}^n) \,:\, P(0) = 0\}$ and

$$U^\perp \equiv \left\{ R \in L^1(V) \,:\, \int_{\mathbb{R}^n} R(x)P(x)\Omega(x)\mathrm{d}x = 0, P \in U \right\}.$$

Then any function $R \in U^{\perp} \cap \mathcal{P}_l(\mathbb{R}^n) \setminus \{0\}$ satisfies $\int_V R(x)\Omega(x)\mathrm{d}x \neq 0$; otherwise R is orthogonal to any element in $\mathcal{P}_l(\mathbb{R}^n)$ with respect to this inner product, that is, $R = 0$. Choosing R suitably, we can assume that $\int_V R(x)\Omega(x)\mathrm{d}x = 1$. If we set $\omega \equiv R \cdot \Omega$, then we have the desired function.

3.1.2 Moment inequalities

The moment condition defined in Definition 43 below is an important condition which will lead us to a nontrivial estimate; the triangle inequality is beyond our reach. Sometimes we are faced with the situation where the cancellation of functions must be taken into account. For $a \in \mathbb{R}^n$, we write $\langle a \rangle \equiv \sqrt{1+|a|^2}$.

Definition 43 ($\mathcal{P}_L^{\perp}(\mathbb{R}^n)$)**.** Let $L \in \mathbb{N}_0$. The set $\mathcal{P}_L^{\perp}(\mathbb{R}^n)$ denotes the set of all $f \in L^0(\mathbb{R}^n)$ for which $\langle\cdot\rangle^L f \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} x^{\alpha} f(x)\mathrm{d}x = 0$ for all $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \leq L$. Such a function f satisfies the *moment condition of order* L. In this case, one also writes $f \perp \mathcal{P}_L(\mathbb{R}^n)$. If $f \perp \mathcal{P}_L(\mathbb{R}^n)$ for all $L \in \mathbb{N}$, one writes $f \perp \mathcal{P}(\mathbb{R}^n)$.

Example 53. If $a \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ and $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| = L \in \mathbb{N}_0$, then $\partial^{\alpha} a \in \mathcal{P}_{L-1}(\mathbb{R}^n)$.

Example 54. Let $\rho \in C^{\infty}(\mathbb{R}) \cap \mathbb{M}^+(\mathbb{R})$ be supported in $\left(-\frac{1}{k+1}, \frac{1}{k+1}\right)$ and suppose that $\|\rho\|_{L^1} = 1$. Let $m_{\alpha} \equiv \int_{\mathbb{R}} t^{\alpha}\rho(t)\mathrm{d}t$ for $\alpha \in \mathbb{N}_0$. Set $\rho_k(t) \equiv \frac{1}{k}\rho\left(\frac{t}{k}\right)$ for $t \in \mathbb{R}$. Observe that $\int_{\mathbb{R}} t^{\alpha}\rho_k(t)\mathrm{d}t = k^{\alpha} m_{\alpha}$. If we let $\psi \equiv \sum_{k=1}^{s+1} c_k \rho_k$, then we have $\int_{\mathbb{R}} t^{\alpha}\psi(t)\mathrm{d}t = m_{\alpha} \sum_{k=1}^{s+1} c_k k^{\alpha}$. If we choose the c_k's suitably by using the Vandermonde determinant, we can arrange

$$\int_{\mathbb{R}} t^{\alpha}\psi(t)\mathrm{d}t = \delta_{\alpha\beta} \quad (\beta = 0, 1, \ldots, k), \quad \mathrm{supp}(\psi) \subset (-1, 1).$$

A passage to higher dimension can be done with the tensor product. Thus, it follows that there exists $\psi \in C^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_k^{\perp}(\mathbb{R}^n)$ supported in a cube $(-1, 1)^n$ such that $\int_{\mathbb{R}^n} \psi(x)\mathrm{d}x = 1$ and $2^n\psi(2\cdot) - \psi \in \mathcal{P}_k^{\perp}(\mathbb{R}^n)$.

We start with the inequality without the moment condition. Among the convolutions of functions, we consider those generated by non-negative functions with polynomial decay. When we handle integral operators, we occasionally want to regard $\langle\cdot\rangle^{-n-1}$ and $\chi_{[0,1]^n}$ as the same functions. Although the former spreads more than the latter, the effect away from the origin of the former is not so strong. This idea can be quantified in the following theorem:

Theorem 112. *Let $\lambda > n$. Then for $0 < t \le 1$,*

$$\frac{1}{t^n}\int_{\mathbb{R}^n}\left\langle\frac{y}{t}\right\rangle^{-\lambda}\langle x-y\rangle^{-\lambda}\mathrm{d}y \sim_\lambda \langle x\rangle^{-\lambda} \quad (x\in\mathbb{R}^n) \tag{3.1}$$

with constants independent of t.

Proof We decompose the integral on the left-hand side of (3.1) according to $B(t)$. On $B(t)$, we use Peetre's inequality $\langle x+y\rangle \le \sqrt{2}\langle x\rangle\langle y\rangle$ to obtain

$$\begin{aligned}\frac{1}{t^n}\int_{B(t)}\left\langle\frac{y}{t}\right\rangle^{-\lambda}\langle x-y\rangle^{-\lambda}\mathrm{d}y &\sim \frac{1}{t^n}\int_{B(t)}\langle x-y\rangle^{-\lambda}\mathrm{d}y\\ &\sim \frac{1}{t^n}\int_{B(t)}\langle x\rangle^{-\lambda}\mathrm{d}y\\ &\simeq \langle x\rangle^{-\lambda}.\end{aligned}$$

From this, we obtain estimate (3.1) from below.

Hence it suffices to prove

$$\frac{1}{t^n}\int_{\mathbb{R}^n\setminus B(t)}\left\langle\frac{y}{t}\right\rangle^{-\lambda}\langle x-y\rangle^{-\lambda}\mathrm{d}y \lesssim \langle x\rangle^{-\lambda} \tag{3.2}$$

to complete the proof of (3.1). When $|x|\le 10$, we use $\|f*g\|_{L^1}\le\|f\|_{L^\infty}\|g\|_{L^1}$ for $f\in L^\infty(\mathbb{R}^n)$ and $g\in L^1(\mathbb{R}^n)$ and the result follows.

Hence, it can be assumed that $|x|\ge 10$. Write $M(x;t)\equiv -1+\log_2\dfrac{|x|}{t}$ and

$$\begin{aligned}\mathrm{I} &\equiv \sum_{j=1}^{[M(x;t)]}\frac{1}{t^n}\int_{B(2^jt)}\frac{\langle x-y\rangle^{-\lambda}\mathrm{d}y}{2^{\lambda j}},\\ \mathrm{II} &\equiv \sum_{j=[1+M(x;t)]}^{\infty}\frac{1}{t^n}\int_{B(2^jt)}\frac{\langle x-y\rangle^{-\lambda}\mathrm{d}y}{2^{\lambda j}}.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{t^n}\int_{\mathbb{R}^n\setminus B(t)}\left\langle\frac{y}{t}\right\rangle^{-\lambda}\langle x-y\rangle^{-\lambda}\mathrm{d}y &= \sum_{j=1}^{\infty}\frac{1}{t^n}\int_{B(2^jt)\setminus B(t)}\left\langle\frac{y}{t}\right\rangle^{-\lambda}\langle x-y\rangle^{-\lambda}\mathrm{d}y\\ &\lesssim \sum_{j=1}^{\infty}\frac{1}{t^n}\int_{B(2^jt)}\frac{1}{2^{\lambda j}}\langle x-y\rangle^{-\lambda}\mathrm{d}y = \mathrm{I}+\mathrm{II}.\end{aligned}$$

Note that $|x|\ge\max(10t,2^{j+1}t)$ as long as $1\le j\le M(x;t)=-1+\log_2\dfrac{|x|}{t}$.

Thus, $|x|\ge 10t, y\in B(2^jt)$ and $1\le j\le M(x;t)$, provided $2|x-y|\ge|x|$. Consequently,

$$\mathrm{I}\le\sum_{j=1}^{[M(x;t)]}\frac{1}{t^n}\int_{B(2^jt)}\left\langle\frac{x}{2}\right\rangle^{-\lambda}\frac{1}{2^{\lambda j}}\mathrm{d}y \lesssim \sum_{j=1}^{[M(x;t)]}\langle x\rangle^{-\lambda}\frac{1}{2^{j(\lambda-n)}}\lesssim\frac{1}{\langle x\rangle^\lambda}.$$

Meanwhile,

$$\mathrm{II} \le \sum_{j=[1+M(x;t)]}^{\infty} \frac{1}{t^n} \int_{\mathbb{R}^n} \frac{\mathrm{d}y}{2^{\lambda j}\langle x-y\rangle^{\lambda}} \sim \sum_{j=[1+M(x;t)]}^{\infty} \frac{1}{2^{\lambda j} t^n} \lesssim \frac{t^{\lambda-n}}{\langle x\rangle^{\lambda}}.$$

If we combine I and II, then we obtain the desired inequality.

The following estimates are used many times in this book.

Theorem 113. *Let $N \in \mathbb{N}$ be a constant. Suppose that $\lambda \in \mathbb{R}$ satisfies $\lambda > n+N-1$. Also assume that $a \in C^N(\mathbb{R}^n)$ satisfies the differential inequality:*

$$|\partial^{\alpha} a(x)| \le \langle x\rangle^{-\lambda} \quad (x \in \mathbb{R}^n) \tag{3.3}$$

for $|\alpha| \le N$ and that $\eta \perp \mathcal{P}_{N-1}$ satisfies the differential inequality:

$$|\eta(x)| \le \langle x\rangle^{-\lambda} \quad (x \in \mathbb{R}^n). \tag{3.4}$$

Set $\eta(\cdot;t) \equiv \frac{1}{t^n}\eta\left(\frac{\cdot}{t}\right)$ for $0 < t \le 1$. Then

$$\left| \int_{\mathbb{R}^n} a(y)\eta(x-y;t)\mathrm{d}y \right| \lesssim_{\lambda} t^N \langle x\rangle^{-\lambda} \int_t^2 r^{\lambda-N-n-1}\mathrm{d}r.$$

In particular, when $\lambda > N+n$,

$$\left| \int_{\mathbb{R}^n} a(y)\eta(x-y;t)\mathrm{d}y \right| \lesssim t^N \langle x\rangle^{-\lambda}.$$

Proof When $\frac{1}{8} \le t \le 1$, Theorem 112 covers Theorem 113. Hence, we assume $t \le \frac{1}{8}$. Change the variables of the integral on the left-hand side and then subtract the Taylor polynomial of a with degree $N-1$ from a;

$$\begin{aligned} &\int_{\mathbb{R}^n} a(y)\eta(x-y;t)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} a(x-t\,y)\eta(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \left(a(x-t\,y) - \sum_{\alpha\in\mathbb{N}_0{}^n, |\alpha|\le N-1} \frac{1}{\alpha!}\partial^{\alpha}a(x)(-ty)^{\alpha} \right) \eta(y)\mathrm{d}y. \end{aligned} \tag{3.5}$$

We estimate the integral on the most right-hand side of (3.5) by decomposing the integral of variable y according to $B(t^{-1})$ and obtaining (3.6) and (3.7) below.

Let us begin with working on $B(t^{-1})$. Here by assumption (3.3), the reminder term of the Taylor expansion is estimated by (3.4):

$$\left| a(x-t\,y) - \sum_{\alpha\in\mathbb{N}_0{}^n, |\alpha|\le N-1} \frac{1}{\alpha!}\partial^{\alpha}a(x)(-ty)^{\alpha} \right| \lesssim |t\,y|^N \langle x\rangle^{-\lambda} \tag{3.6}$$

if $|y| \le t^{-1}$. We write $\min(a,b) \equiv a \wedge b$. Then

$$\int_{B(t^{-1})} \frac{|y|^N \mathrm{d}y}{\langle y\rangle^{\lambda}} \le \int_{B(t^{-1})} |y|^N \wedge |y|^{N-\lambda} \mathrm{d}y \simeq \int_0^{t^{-1}} r^{n+N-1}(1 \wedge r^{-\lambda}) \mathrm{d}r.$$

Changing the variables: $r \mapsto r^{-1}$ on the integral on the right-hand side yields

$$\int_0^{t^{-1}} r^{n+N-1}(1 \wedge r^{-\lambda}) \mathrm{d}r = \int_t^{\infty} r^{-n-N-1}(1 \wedge r^{\lambda}) \mathrm{d}r.$$

Consequently,

$$\left| \int_{B(t^{-1})} \left(a(x - t\,y) - \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le N-1} \frac{1}{\alpha!} \partial^{\alpha} a(x) (-ty)^{\alpha} \right) \eta(y) \mathrm{d}y \right| \lesssim \int_t^2 \frac{\mathrm{d}r}{r^{N+n+1-\lambda}}.$$

Assume instead that $|y| \ge t^{-1}$. Then we obtain

$$\begin{aligned} & \left| a(x - t\,y) - \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le N-1} \frac{1}{\alpha!} \partial^{\alpha} a(x) (-ty)^{\alpha} \right| \\ & \quad \le |a(x - t\,y)| + \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le N-1} \left| \frac{1}{\alpha!} \partial^{\alpha} a(x) (-ty)^{\alpha} \right| \\ & \quad \lesssim \langle x - t\,y\rangle^{-\lambda} + |t\,y|^{N-1} \langle x\rangle^{-\lambda} \end{aligned} \tag{3.7}$$

by Peetre's inequality, $\langle x + y\rangle \le \sqrt{2}\langle x\rangle\langle y\rangle$ and (3.3). Hence

$$\begin{aligned} & \left| \int_{\mathbb{R}^n \setminus B(t^{-1})} a(y) \eta(x - y; t) \mathrm{d}y \right| \\ & \quad \lesssim \int_{\mathbb{R}^n \setminus B(t^{-1})} \left(\langle x - t\,y\rangle^{-\lambda} + |t\,y|^{N-1} \langle x\rangle^{-\lambda} \right) \langle y\rangle^{-\lambda} \mathrm{d}y. \end{aligned}$$

We change the variables $y \mapsto z = ty$:

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(t^{-1})} \langle x - t\,y\rangle^{-\lambda} \langle y\rangle^{-\lambda} \mathrm{d}y & \le \int_{\mathbb{R}^n \setminus B(t^{-1})} \langle x - t\,y\rangle^{-\lambda} |y|^{-\lambda} \mathrm{d}y \\ & = t^{\lambda - n} \int_{\mathbb{R}^n \setminus B(t^{-1})} \langle x - z\rangle^{-\lambda} |z|^{-\lambda} \mathrm{d}z \\ & \lesssim t^{\lambda - n} \int_{\mathbb{R}^n \setminus B(1)} \langle x - z\rangle^{-\lambda} \langle z\rangle^{-\lambda} \mathrm{d}z \\ & \le t^{\lambda - n} \int_{\mathbb{R}^n} \langle x - z\rangle^{-\lambda} \langle z\rangle^{-\lambda} \mathrm{d}z. \end{aligned}$$

From Theorem 112, we obtain

$$\int_{\mathbb{R}^n\setminus B(t^{-1})} \langle x-t\,y\rangle^{-\lambda}\langle y\rangle^{-\lambda}\mathrm{d}y \lesssim t^{\lambda-n}\langle x\rangle^{-\lambda}.$$

Recall also that $\lambda > N+n-1$. With this in mind we change the variables: $y \mapsto z = ty$ to obtain

$$\int_{\mathbb{R}^n\setminus B(t^{-1})} |y|^{N-1-\lambda}\mathrm{d}y = t^{\lambda-N-n+1}\int_{\mathbb{R}^n\setminus B(t^{-1})} |z|^{N-1-\lambda}\mathrm{d}z \sim t^{\lambda-N-n+1}.$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(t^{-1})} |t\,y|^{N-1}\langle y\rangle^{-\lambda}\mathrm{d}y &< t^{N-1}\int_{\mathbb{R}^n\setminus B(t^{-1})} |y|^{N-1-\lambda}\mathrm{d}y\\ &\simeq t^{\lambda-n}\\ &\lesssim t^N\int_t^2 r^{\lambda-N-n-1}\mathrm{d}r.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\int_{\mathbb{R}^n\setminus B(t^{-1})}\left(a(x-t\,y)-\sum_{\alpha\in\mathbb{N}_0{}^n,|\alpha|\le N-1}\frac{1}{\alpha!}\partial^\alpha a(x)(-ty)^\alpha\right)\eta(y)\mathrm{d}y\right|\\ &\lesssim t^N\langle x\rangle^{-\lambda}\int_t^2 r^{\lambda-N-n-1}\mathrm{d}r.\end{aligned}$$

Thus, the proof is complete.

The proof of Theorem 113 used the mean-value theorem. We transform Theorem 113 so that we can easily handle it later.

Theorem 114. *Maintain the same assumption on $\lambda, N > 0$ and on $a, \eta \in L^1(\mathbb{R}^n)$ as Theorem 113. For $j \le \nu$ we write $a^j \equiv 2^{jn}a(2^j\cdot), \eta^\nu \equiv 2^{\nu n}\eta(2^\nu\cdot)$. Then $|a^j * \eta^\nu(x)| \lesssim 2^{jn+(j-\nu)N}\langle 2^j x\rangle^{-\lambda}\displaystyle\int_{2^{j-\nu}}^2 r^{\lambda-N-n-1}\mathrm{d}r$ for $x \in \mathbb{R}^n$. In particular, when $\lambda > N+n$, $|a^j * \eta^\nu(x)| \lesssim 2^{jn+(j-\nu)N}\langle 2^j x\rangle^{-\lambda}$.*

Proof Since $a^j * \eta^\nu = (a * \eta^{\nu-j})^j$, simply use Theorem 113 with $t = 2^{j-\nu}$.

Example 55. Let $\{a_{j\nu}\}_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}\times\mathbb{Z}^n)$. Also let $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$. For $j \in \mathbb{Z}$ and $\nu \in \mathbb{Z}^n$, we set $\varphi_{j\nu}(x) \equiv 2^{\frac{jn}{2}}\Delta\psi(2^j x-\nu), x \in \mathbb{R}^n$. Then we claim that $f \equiv \displaystyle\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n} a_{j\nu}\varphi_{j\nu}$ converges in $L^2(\mathbb{R}^n)$ and satisfies

$$\|f\|_{L^2} \lesssim \left(\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n}|a_{j\nu}|^2\right)^{\frac12}. \tag{3.8}$$

In fact, thanks to Theorem 114, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\varphi_{j\nu}(x)\psi_{\tilde{j}\tilde{\nu}}(x)\mathrm{d}x\right|\\&\lesssim 2^{\min(j,\tilde{j})n-\frac{1}{2}(j+\tilde{j})n-2|j-\tilde{j}|}(1+2^{\min(j,\tilde{j})}|2^{-j}\nu-2^{-\tilde{j}}\tilde{\nu}|)^{-2n-2}.\end{aligned}$$

Fix $\tilde{j}$ and $\tilde{\nu}$. Then

$$\begin{aligned}&\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n}2^{\min(j,\tilde{j})n-\frac{1}{2}(j+\tilde{j})n-2|j-\tilde{j}|}(1+2^{\min(j,\tilde{j})}|2^{-j}\nu-2^{-\tilde{j}}\tilde{\nu}|)^{-2n-2}\\&\sim\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}2^{\min(j,\tilde{j})n-\frac{1}{2}(j+\tilde{j})n-2|j-\tilde{j}|}(1+2^{\min(j,\tilde{j})}|2^{-j}x-2^{-\tilde{j}}\tilde{\nu}|)^{-2n-2}\mathrm{d}x\\&\sim\sum_{j\in\mathbb{Z}}2^{\frac{1}{2}(j-\tilde{j})n-2|j-\tilde{j}|}\\&\lesssim 1.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n}\sum_{\tilde{j}\in\mathbb{Z},\tilde{\nu}\in\mathbb{Z}^n}|a_{j\nu}a_{\tilde{j}\tilde{\nu}}|\left|\int_{\mathbb{R}^n}\varphi_{j\nu}(x)\psi_{\tilde{j}\tilde{\nu}}(x)\mathrm{d}x\right|\\&\le 2\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n}\sum_{\tilde{j}\in\mathbb{Z},\tilde{\nu}\in\mathbb{Z}^n}|a_{\tilde{j}\tilde{\nu}}|^2\left|\int_{\mathbb{R}^n}\varphi_{j\nu}(x)\psi_{\tilde{j}\tilde{\nu}}(x)\mathrm{d}x\right|\\&\lesssim\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n}|a_{j\nu}|^2.\end{aligned}$$

As a result, the sum converges in $L^2(\mathbb{R}^n)$ and (3.8) holds.

Duality entails

$$\sqrt{\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n}|\langle f,\varphi_{j\nu}\rangle_{L^2}|^2}\lesssim\|f\|_{L^2}. \tag{3.9}$$

Combining (3.8) and (3.9), we conclude that

$$f\in L^2(\mathbb{R}^n)\mapsto\sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n}\langle f,\varphi_{j\nu}\rangle_{L^2}\varphi_{j\nu}$$

is a bounded linear operator.

3.1.3 Control of derivatives by integrals

If E is not too small, then we can control derivatives of polynomials in $\mathcal{P}_k(\mathbb{R}^n)$ by the L^q-norm defined over E. For an open set Ω and a ball $B(x_0,\rho)$, we write $\Omega(x_0,\rho)\equiv\Omega\cap B(x_0,\rho)$.

Lemma 115. *Let $k \in \mathbb{N}_0 \cup \{-1\}$, $q \geq 1$ and $A \in (0,1)$. Also let $x_0 \in \Omega$ and $0 < \rho < \operatorname{diam}(\Omega)$. If E is a measurable subset in $\Omega(x_0;\rho)$ such that $|E| \geq A|B(\rho)|$, then $|\partial^\beta P(x_0)| \lesssim_{\beta,A} \left(\frac{1}{\rho^{n+|\beta|q}} \int_E |P(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}$ for all $P \in \mathcal{P}_k(\mathbb{R}^n)$ and $\beta \in \mathbb{N}_0{}^n$.*

It should be noted that if $E = B(x_0, \rho)$, then Lemma 115 is trivial from the dilation and translation.

Proof By the translation and the dilation we may assume that $B(x_0, \rho) = B(1)$. The symbol $\mathcal{T}_k$ denotes the set of all polynomials in $\mathcal{P}_k(\mathbb{R}^n)$ of the form

$$P(x) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha$$

with

$$\sum_{|\alpha| \leq k} |a_\alpha|^2 = 1. \tag{3.10}$$

Let $\mathcal{F}$ be the set of all measurable functions f defined on $\mathbb{R}^n$ such that $0 \leq f \leq \chi_{B(1)}$ and that

$$\int_{\mathbb{R}^n} f(x)\mathrm{d}x = \int_{\mathbb{R}^n} \chi_{B(1)}(x) f(x)\mathrm{d}x \geq A.$$

Let us define

$$\gamma(A) \equiv \inf_{P \in \mathcal{T}_k, f \in \mathcal{F}} \int_{B(1)} |P(x)|^q f(x)\mathrm{d}x.$$

Let us prove that the infimum is attained;

$$\gamma(A) = \min_{P \in \mathcal{T}_k, f \in \mathcal{F}} \int_{B(1)} |P(x)|^q f(x)\mathrm{d}x.$$

By the definition of $\gamma(A)$, we can find a sequence of polynomials $\{P_l\}_{l=1}^\infty$ in $\mathcal{T}_k$ and a sequence of measurable function $\{f_l\}_{l=1}^\infty$ in $\mathcal{F}$ such that

$$\gamma(A) \leq \int_{B(1)} |P_l(x)|^q f_l(x)\mathrm{d}x \leq \gamma(A) + \frac{1}{l}.$$

Recall that we are assuming (3.10). Define $N \equiv \sharp\{\alpha \in \mathbb{N}_0{}^n : |\alpha| \leq k\}$. Since the closed unit ball in $\mathbb{R}^N$ is compact for all $N \in \mathbb{N}$ and the unit ball of $L^2(B(1))$ is weakly compact, we can suppose that $P_l \to P \in \mathcal{T}_k$ uniformly over $B(1)$ and that $f_l \to f \in \mathcal{F}$ weakly in $L^2(B(1))$. Thus the pair (P, f) is the one we are looking for.

Since $\chi_E \in \mathcal{F}$, it follows that $\int_E |P(x)|^q \mathrm{d}x \geq \gamma(A)$. Also, since all linear operators in a finite dimensional normed space $\mathcal{P}_k(\mathbb{R}^n)$ are automatically bounded, we have $|\partial^\beta P(0)| \lesssim \left(\sum_{|\alpha| \leq k} |a_\alpha|^2\right)^{\frac{1}{2}} \sim \|P\|_{L^q(B(1))}$ when P

takes the form; $P(x) = \sum_{|\alpha|\le k} a_\alpha x^\alpha$. Thus, we obtain the desired result when $B(x_0,\rho) = B(1)$. In the general case, we can use a scaling argument as we mentioned in the beginning.

3.1.4 Best approximation

We define the best approximation, which we use in Chapters 5 and 14. We will work in an open set Ω in $\mathbb{R}^n$. For $x_0 \in \mathbb{R}^n$ and $\rho > 0$, write $\Omega(x_0,\rho) \equiv \Omega \cap B(x_0,\rho)$. Denote by $\operatorname{diam}(\Omega)$ its diameter.

Lemma 116. *Let $1 < q < \infty$ and $k \in \mathbb{N}_0 \cup \{-1\}$. Also let $f \in L^q_{\mathrm{loc}}(\Omega)$. Then, for all $x_0 \in \overline{\Omega}$ and $\rho \in (0, \operatorname{diam}(\Omega))$, we can find a unique polynomial $P_k(\cdot; x_0, \rho, f) \in \mathcal{P}_k(\mathbb{R}^n)$ such that*

$$\inf_{P\in\mathcal{P}_k(\mathbb{R}^n)} \|f-P\|_{L^q(\Omega(x_0;\rho))} = \|f - P_k(\cdot;x_0,\rho,f)\|_{L^q(\Omega(x_0;\rho))}.$$

The polynomial $P_k(\cdot;x_0,\rho,f)$ is called the *best approximation* of f.

Proof The uniqueness is a consequence of Theorem 90. In fact, if $P_k(\cdot;x_0,\rho,f)$ and $Q_k(\cdot;x_0,\rho,f)$ in $\mathcal{P}_k(\mathbb{R}^n)$ attains the infimum, then we have

$$\begin{aligned}&\|f - P_k(\cdot;x_0,\rho,f) + f - Q_k(\cdot;x_0,\rho,f)\|_{L^q(\Omega(x_0,\rho))}\\ &= \|f - P_k(\cdot;x_0,\rho,f)\|_{L^q(\Omega(x_0,\rho))} + \|f - Q_k(\cdot;x_0,\rho,f)\|_{L^q(\Omega(x_0,\rho))}.\end{aligned}$$

This means that $f-P_k(\cdot;x_0,\rho,f)$ and $f-Q_k(\cdot;x_0,\rho,f)$ are linearly dependent thanks to Theorem 90 and the constant K satisfying $f-P_k(\cdot;x_0,\rho,f) = K(f-Q_k(\cdot;x_0,\rho,f))$ is positive. Since $f-P_k(\cdot;x_0,\rho,f)$ and $f-Q_k(\cdot;x_0,\rho,f)$ have the same $L^q(\Omega(x_0,\rho))$-norm, we see that $K=1$, implying that $P_k(\cdot;x_0,\rho,f)$ is unique.

We prove the existence. Set

$$P(x) \equiv \sum_{|\alpha|\le k} \frac{a_\alpha}{\alpha!}(x-x_0)^\alpha \quad (x\in\mathbb{R}^n).$$

Observe that

$$\left(\sum_{|\alpha|\le k} |a_\alpha|^2\right)^{\frac{1}{2}} \sim \|P\|_{L^q(x_0;\rho)}$$

again thanks to the fact that the norms in a finite dimensional normed space $\mathcal{P}_k(\mathbb{R}^n)$ are equivalent. Thus, when a sequence $\{P_l\}_{l=0}^\infty \in \mathcal{P}_k(\mathbb{R}^n)$ satisfies

$$\inf_{P\in\mathcal{P}_k(\mathbb{R}^n)} \|f-P\|_{L^q(\Omega(x_0;\rho))} = \lim_{l\to\infty}\left(\int_{\Omega(x_0;\rho)} |f(x)-P_l(x)|^q \mathrm{d}x\right)^{\frac{1}{q}},$$

then $\sup_{l\in\mathbb{N}} \sum_{|\alpha|\le k} |a_{l,\alpha}|^2 < \infty$, where

$$P_l(x) = \sum_{|\alpha|\le k} a_{l,\alpha} x^\alpha \quad (x \in \mathbb{R}^n).$$

Consequently, we can resort to the compactness argument.

Example 56. Let $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and let $1 \le q < \infty$ and $k \in \mathbb{N}_0$. Also let $(x_0, \rho) \in \mathbb{R}^{n+1}_+$.

(1) If $q = 1$, we can still show that such a polynomial exists by going through a similar argument. However, we do not have the uniqueness of $P_k(\cdot; x_0, \rho, f)$. In fact, let $k = 0$. Let $f : [0, 2] \to [0, \infty)$ be a measurable function. Then any number between $f^*(1)$, $f^*(1 + 0)$ can be chosen as P.

(2) Let $q = 2$ and $k = 0$, and let f be a square integrable function defined on a cube Q. Then $P_k(\cdot; x_0, \rho, f)(x) = m_Q(u)$ for $x \in Q$.

(3) In general, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $P \in \mathcal{P}_k(\mathbb{R}^n)$, we have $P_k(\cdot; x_0, \rho, f + P) = P_k(\cdot; x_0, \rho, f) + P$.

Before we go further, a helpful remark may be in order.

Remark 3. Let $1 < q < \infty$ and $k \in \mathbb{N}_0 \cup \{-1\}$. Also let $f \in L^q_{\mathrm{loc}}(\Omega)$, $x_0 \in \Omega$ and $\rho \in (0, \mathrm{diam}(\Omega))$. For any polynomial $P \in \mathcal{P}_k(\mathbb{R}^n)$, $\|P_k(\cdot; x_0, \rho, f)\|_{L^q(\Omega(x_0;\rho))} \lesssim \|f\|_{L^q(\Omega(x_0;\rho))}$.

There is another technique of approximating functions. Let $k \in \mathbb{N}_0$ be fixed and let B be a fixed ball. For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ there uniquely exists $P^k_B f \in \mathcal{P}_k(\mathbb{R}^n)$ such that $\int_B f(x)g(x)\mathrm{d}x = \int_B P^k_B f(x)g(x)\mathrm{d}x$ for all $g \in \mathcal{P}_k(\mathbb{R}^n)$ since $\{x^\alpha\}_{|\alpha|\le k}$ is independent in $L^2(B)$. The polynomial $P^k_B f$ is called the *Gram–Schmidt polynomial* of order k for B. Remark that we can also define $P^k_Q f$ for cubes Q in a similar manner.

Proposition 117. *Let $B = B(x_0, r)$ be a ball. Also let $k \in \mathbb{N}$.*

(1) *For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $P^k_B f = r^{-n} P^k_{B(1)} f\left(\dfrac{\cdot - x_0}{r}\right)$.*

(2) *If $f \in \mathcal{P}_k(\mathbb{R}^n)$, then $P^k_B f = f$.*

(3) *There exists $k_B \in \mathcal{P}(\mathbb{R}^{2n})$ such that $P^k_B f(x) = \int_{\mathbb{R}^n} k_B(x, y) f(y)\mathrm{d}y$ and that $|k_B(x, y)| \lesssim |B|^{-1}$ for all $x, y \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof The proof is routine; see Exercise 57.

3.1.5 Exercises

Exercise 56. Prove $\dim_{\mathbb{C}}(\dot{\mathcal{P}}_k(\mathbb{R}^n)) = {}_{k+n-1}\mathrm{C}_{n-1}$ by considering the number of the integer solutions of $x_1 + x_2 + \cdots + x_n = k$.

Exercise 57. Prove Proposition 117 as follows:

(1) Use the change of variables to prove Proposition 117(1).

(2) Use (1) to prove Proposition 117(2).

(3) Use the Riesz representation theorem for $L^2(B)$ to prove that there exists $k_B \in \mathcal{P}(\mathbb{R}^{2n})$ such that $P_B^k f(x) = \displaystyle\int_{\mathbb{R}^n} k_B(x,y) f(y) \mathrm{d}y$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

(4) Use (1) to show that $|k_B(x,y)| \lesssim |B|^{-1}$ for all $x, y \in \mathbb{R}^n$.

3.2 Spherical harmonic functions

When we consider the application of Morrey spaces to partial differential equations, we need the expansion by way of the spherical harmonic functions. This section is a preliminary section for applications to PDE.

Section 3.2.1 deals with harmonic polynomials. Section 3.2.2 considers the norm estimates. We obtain an integration by parts formula in Section 3.2.3.

3.2.1 The spaces $\mathcal{H}_k(\mathbb{R}^n)$ and $\dot{\mathcal{H}}_k(\mathbb{R}^n)$

Here, we are interested in polynomials such that $\Delta P(x) = 0$. Such polynomials are called *harmonic polynomials*. We are interested in the structure of the polynomial linear spaces. Among others, we are interested in the dimension of the linear spaces of harmonic polynomials. For a start, we investigate the image of $\Delta : \mathcal{P}_k(\mathbb{R}^n) \to \mathcal{P}_{k-2}(\mathbb{R}^n)$. We can locate Section 3.2.1 as a setup for the discussion in Section 3.2.2.

Definition 44. Let $k \in \mathbb{N}_0 \cup \{-1\}$. The space $\mathcal{H}_k(\mathbb{R}^n)$ stands for the set of all harmonic polynomials in $\dot{\mathcal{P}}_k(\mathbb{R}^n)$. Also define $\mathcal{H}(\mathbb{R}^n) \equiv \displaystyle\bigcup_{k=0}^{\infty} \mathcal{H}_k(\mathbb{R}^n)$. The space $\dot{\mathcal{H}}_k(\mathbb{R}^n)$ is called the *homogeneous harmonic polynomial space* of order k, while $\mathcal{H}_k(\mathbb{R}^n)$ is called the *nonhomogeneous harmonic polynomial space* of order k.

Algebraically, we can say that $\mathcal{H}_k(\mathbb{R}^n) = \ker(\Delta : \mathcal{P}_k(\mathbb{R}^n) \to \mathcal{P}_{k-2}(\mathbb{R}^n))$, $\dot{\mathcal{H}}_k(\mathbb{R}^n) = \ker(\Delta : \dot{\mathcal{P}}_k(\mathbb{R}^n) \to \dot{\mathcal{P}}_{k-2}(\mathbb{R}^n))$ and $\mathcal{H}(\mathbb{R}^n) = \ker(\Delta : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n))$.

Example 57. We have $\dot{\mathcal{H}}_0(\mathbb{R}^n) = \dot{\mathcal{P}}_0(\mathbb{R}^n)$, $\mathcal{H}_1(\mathbb{R}^n) = \mathcal{P}_1(\mathbb{R}^n)$ and $\dot{\mathcal{H}}_1(\mathbb{R}^n) = \dot{\mathcal{P}}_1(\mathbb{R}^n)$.

We are interested in the structure of these polynomial linear spaces. For a start, we investigate the image of $\Delta : \mathcal{P}_k(\mathbb{R}^n) \to \mathcal{P}_{k-2}(\mathbb{R}^n)$.

Lemma 118. *The Laplacian Δ maps $\mathcal{P}_k(\mathbb{R}^n)$ onto $\mathcal{P}_{k-2}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$.*

Proof Since the polynomials of the form $(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^k$, $a_1, a_2, \ldots, a_n \in \mathbb{R}$ span $\mathcal{P}_k(\mathbb{R}^n)$, we have only to show that such a polynomial is realized as the image of $\Delta : \mathcal{P}_k(\mathbb{R}^n) \to \mathcal{P}_{k-2}(\mathbb{R}^n)$. Since

$$\begin{aligned}&\Delta(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^{k+2}\\ &= (k+2)(k+1)({a_1}^2 + \cdots + {a_n}^2)(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^k,\end{aligned}$$

this is clear.

Since $\mathcal{P}_k(\mathbb{R}^n)$ is a finite dimensional linear space, so are $\mathcal{H}_k(\mathbb{R}^n)$ and $\dot{\mathcal{H}}_k(\mathbb{R}^n)$. In higher dimensions, it is difficult to take a basis. Consequently, we do not go in this direction here. However, we will need to know how large the spaces $\mathcal{H}_k(\mathbb{R}^n)$ and $\dot{\mathcal{H}}_k(\mathbb{R}^n)$ are.

Corollary 119. *Let $k \in \mathbb{N}_0$. Then $\dim_{\mathbb{C}}(\mathcal{H}_k(\mathbb{R}^n)) = {}_{k+n}\mathrm{C}_n - {}_{k+n-2}\mathrm{C}_n = \mathrm{O}(k^{n-1})$ and $\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n)) = {}_{k+n-1}\mathrm{C}_{n-1} - {}_{k+n-3}\mathrm{C}_{n-1} = \mathrm{O}(k^{n-2})$.*

Proof We will deal with the nonhomogeneous spaces; the homogeneous spaces can be handled similarly. See Exercise 59. Using the definition of $\mathcal{H}_k(\mathbb{R}^n)$ and Lemma 118, we calculate

$$\begin{aligned}\dim_{\mathbb{C}}(\mathcal{H}_k(\mathbb{R}^n)) &= \dim\ker(\Delta : \mathcal{P}_k(\mathbb{R}^n) \to \mathcal{P}_{k-1}(\mathbb{R}^n))\\ &= \dim_{\mathbb{C}}(\mathcal{P}_k(\mathbb{R}^n)) - \dim_{\mathbb{C}}(\mathcal{P}_{k-2}(\mathbb{R}^n))\\ &= {}_{k+n}\mathrm{C}_n - {}_{k+n-2}\mathrm{C}_n.\end{aligned}$$

Along this inner product, we decompose $\mathcal{P}_k(\mathbb{R}^n)$.

Proposition 120. *We define a linear mapping $M : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ by $MP(x) \equiv |x|^2P(x)$ for $P \in \mathcal{P}(\mathbb{R}^n)$. Then for all $k \geq 2$, we have $\mathcal{P}_k(\mathbb{R}^n) = \mathcal{H}_k(\mathbb{R}^n) \oplus \mathrm{Im}(M : \mathcal{P}_{k-2}(\mathbb{R}^n) \to \mathcal{P}_k(\mathbb{R}^n))$ and $\dot{\mathcal{P}}_k(\mathbb{R}^n) = \dot{\mathcal{H}}_k(\mathbb{R}^n) \oplus \mathrm{Im}(M : \dot{\mathcal{P}}_{k-2}(\mathbb{R}^n) \to \dot{\mathcal{P}}_k(\mathbb{R}^n))$.*

Proof We consider the nonhomogeneous case; we will deal with the first equality since the second one follows immediately from the first one; see Exercise 59. First, let us check that $\mathcal{H}_k(\mathbb{R}^n)$ and $\mathrm{Im}(M : \mathcal{P}_{k-2}(\mathbb{R}^n) \to \mathcal{P}_k(\mathbb{R}^n))$ are orthogonal. Indeed, for all $P \in \mathcal{P}_k(\mathbb{R}^n)$ and $Q \in \mathcal{P}_{k-2}(\mathbb{R}^n)$,

$$\langle P, MQ\rangle_{\mathcal{P}} = \overline{\langle MQ, P\rangle_{\mathcal{P}}} = \overline{Q(D)[\Delta P]} = \overline{Q(D)0} = 0.$$

Consequently, $\dot{\mathcal{H}}_k(\mathbb{R}^n)$ and $\mathrm{Im}(M : \mathcal{P}_{k-2}(\mathbb{R}^n) \to \mathcal{P}_k(\mathbb{R}^n))$ are orthogonal.

To check that $\dot{\mathcal{H}}_k(\mathbb{R}^n)$ and $\mathrm{Im}(M : \mathcal{P}_{k-2}(\mathbb{R}^n) \to \mathcal{P}_k(\mathbb{R}^n))$ span $\mathcal{P}_k(\mathbb{R}^n)$, we can compare the dimension of these spaces using Corollaries 110 and 119.

3.2.2 Norm estimates for spherical harmonics

A spherical harmonic is a function obtained by restricting harmonic functions to the unit sphere. Here and below we write $S^{n-1} \equiv \{x \in \mathbb{R}^n : |x| = 1\}$. Here, we summarize some useful estimates for spherical harmonics. The estimates will be used to show that some integral operators are $L^p(\mathbb{R}^n)$-bounded. In addition, write $\omega_{n-1} \equiv |S^{n-1}|$.

Definition 45. Equip $\mathcal{H}(\mathbb{R}^n)$ with an inner product given by

$$\langle P, Q\rangle_{\mathcal{H}} = \int_{S^{n-1}} P(x)\overline{Q(x)}\mathrm{d}\sigma(x),$$

where $\mathrm{d}\sigma$ denotes the surface measure. Fix a complete orthonormal system $\{Y_{jk}\}_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}$.

Example 58. In $\mathbb{R}^2$, if we set $P(x,y) \equiv x^2 - y^2$, then

$$\langle P, P\rangle_{\mathcal{H}} = \int_{x^2+y^2=1} (x^2-y^2)^2 \mathrm{d}\sigma(x,y) = \int_0^{2\pi} \cos^2 2\theta \mathrm{d}\theta = \pi.$$

Meanwhile, according to Definition 42,

$$\langle P, P\rangle_{\mathcal{P}} = P(D)P(0) = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)(x^2-y^2) = 4.$$

Example 59. We work in $\mathbb{R}^2$ again. Let $f(x,y) \equiv (x+iy)^m$ for $m \in \mathbb{N}$ Then

$$\langle f, f\rangle_{\mathcal{H}} = \int_{S^1} (x^2+y^2)^{2m} \mathrm{d}\sigma(x,y) = 2\pi,$$

while

$$\langle f, f\rangle_{\mathcal{P}} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m (x-iy)^m = 2^m m!.$$

Concerning this new inner product, we have a sort of the Pythagorean equality.

Proposition 121. *Let $k \in \mathbb{N}_0$. Then*

$$\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |Y_{jk}(x)|^2 = \frac{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}} |x|^{2k}$$

for all $x \in \mathbb{R}^n$.

In the proof, we use the orthogonal group of degree n; $\mathrm{O}(n)$ denotes the set of all orthogonal matrices.

Proof Let $z \in \mathbb{R}^n$ be fixed. We will evaluate both sides at $x = z$. Thanks to homogeneity, we can assume that $z \in S^{n-1}$.

If we set

$$Z_z \equiv \sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} \overline{Y_{jk}(z)} Y_{jk} \in \dot{\mathcal{H}}_k(\mathbb{R}^n), \tag{3.1}$$

which is called *zonal harmonic of degree k with pole at x*, then $Y(z) = \langle Y, Z_z \rangle_{\mathcal{H}}$ for all $Y \in \dot{\mathcal{H}}_k$. By the uniqueness of the Riesz representation theorem for Hilbert spaces, we see that

$$\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} \overline{Y_{jk}(z)} Y_{jk}(x) = \sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} \overline{Y_{jk}(Az)} Y_{jk}(Ax)$$

for all $A \in \mathrm{O}(n)$. Thus, $\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |Y_{jk}(z)|^2$ is a constant. To determine the precise value of this constant, we integrate over S^{n-1} and use the fact that $\{Y_{jk}\}_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}$ is an orthonormal system.

The family $\{Y_{jk}\}_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}$ has the following growth in $L^\infty(S^{n-1})$-norm.

Corollary 122. $|Y_{jk}(x)| \le \sqrt{\dfrac{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}}} = \mathrm{O}(k^{\frac{n}{2}-1})$ *for all* $x \in S^{n-1}$, $k \in \mathbb{N}_0$ *and* $j = 1, 2, \ldots, \dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))$.

Proof Simply use Proposition 121:

$$|Y_{jk}(x)|^2 \le \sum_{l=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |Y_{lk}(x)|^2 = \frac{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}} |x|^{2k} = \frac{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}}.$$

We investigate the relation between the L^2-norm of functions in $P \in \dot{\mathcal{H}}_k(\mathbb{R}^n)$ and that for their gradient.

Lemma 123. *Let $P \in \dot{\mathcal{H}}_k(\mathbb{R}^n)$. Then*

$$\int_{B(1)} |\mathrm{grad}(P)(x)|^2 \mathrm{d}x = k \int_{S^{n-1}} |P(x)|^2 \mathrm{d}\sigma(x).$$

In particular,

$$k(n + 2k - 2) \int_{S^{n-1}} |P(x)|^2 \mathrm{d}\sigma(x) = \int_{S^{n-1}} |\mathrm{grad}(P)(x)|^2 \mathrm{d}\sigma(x) \tag{3.2}$$

for $k \ge 1$.

Proof Since P is harmonic, we obtain

$$|\mathrm{grad}(P)(x)|^2 = |\mathrm{grad}(P)(x)|^2 + P(x)\Delta P(x) = \mathrm{div}(P(x)\mathrm{grad}(P)(x)).$$

By the Stokes theorem, we have

$$\begin{aligned}\int_{B(1)} |\mathrm{grad}(P)(x)|^2 \mathrm{d}x &= \int_{B(1)} \mathrm{div}(P(x)\mathrm{grad}(P)(x))\mathrm{d}x \\ &= \int_{S^{n-1}} (P(x)\mathrm{grad}(P)(x)) \cdot \frac{x}{|x|}\mathrm{d}\sigma(x).\end{aligned}$$

Since $\mathrm{grad}(P)(x) \cdot x = kP(x)$, we obtain the desired result.

When $P \in \dot{\mathcal{H}}_k(\mathbb{R}^n)$, $\mathrm{grad}(P)$ has the following non-trivial bound:

Corollary 124. *Let $P \in \dot{\mathcal{H}}_k(\mathbb{R}^n)$. Then*

$$|\mathrm{grad}(P)(x)| \le \sqrt{\frac{k(n+2k-2)\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}}}\|P\|_{L^2(S^{n-1})}|x|^{k-1} \quad (x \in \mathbb{R}^n).$$

Proof Let $x \in \mathbb{R}^n$ be fixed. Since $P = \sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} \langle P, Y_{jk}\rangle_{\mathcal{H}} Y_{jk}$, we obtain $\mathrm{grad}(P)(x) = \sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} \langle P, Y_{jk}\rangle_{\mathcal{H}} \mathrm{grad}(Y_{jk})(x)$. Notice that there exists a constant $A = A_k > 0$ such that

$$\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |\mathrm{grad}(Y_{jk})(x)|^2 = A|x|^{2k-2}. \tag{3.3}$$

Indeed, $\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |\mathrm{grad}(Y_{jk})(Mx)|^2 = \sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |\mathrm{grad}(Y_{jk})(x)|^2$ for all $M \in \mathrm{O}(n)$. To calculate the precise value of A, we integrate (3.3) over S^{n-1}:

$$A = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |\mathrm{grad}(Y_{jk})(x)|^2 \mathrm{d}x.$$

By (3.2), we obtain

$$\begin{aligned}A &= \frac{k(n+2k-2)}{\omega_{n-1}} \int_{S^{n-1}} \sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |Y_{jk}(x)|^2 \mathrm{d}x \\ &= \frac{k(n+2k-2)}{\omega_{n-1}} \dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n)).\end{aligned}$$

Thus, by Hölder's inequality and the Bessel inequality, we obtain

$$|\mathrm{grad}(P)(x)| \le \|P\|_{L^2(S^{n-1})} \sqrt{\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))} |\mathrm{grad}(Y_{jk})(x)|^2}$$
$$\le \sqrt{\frac{k(n+2k-2)\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}}} \|P\|_{L^2(S^{n-1})} |x|^{k-1}.$$

Consequently, we obtain the desired result.

3.2.3 Laplacian and integration by parts formula

We consider some equalities for homogeneous harmonic functions. Proposition 125 can be located as the cancellation condition which the functions in $\dot{\mathcal{H}}_k(\mathbb{R}^n)$ have.

Proposition 125. *Let $k \in \mathbb{N}_0$. Then $\Delta \frac{P(x)}{|x|^k} = \frac{k(2-k-n)}{|x|^{k+2}} P(x)$ for all $P \in \dot{\mathcal{H}}_k(\mathbb{R}^n)$.*

Proof We remark that $\sum_{j=1}^{n} \partial_j P(x) x_j = kP(x)$ by Euler's formula. (See Exercise 60.) Hence

$$\Delta \frac{P(x)}{|x|^k} = P(x)\Delta \frac{1}{|x|^k} - 2k \sum_{j=1}^{n} \partial_j P(x) \cdot \frac{x_j}{|x|^{k+2}}$$
$$= P(x) \frac{k(k+2)}{|x|^{k+2}} - P(x) \sum_{j=1}^{n} \frac{k}{|x|^{k+2}} - 2k \sum_{j=1}^{n} \partial_j P(x) \cdot \frac{x_j}{|x|^{k+2}}$$
$$= k(2-k-n) \frac{P(x)}{|x|^{k+2}}.$$

Thus, we obtain the desired result.

Example 60. We work in $\mathbb{R}^2$; $n = 2$ (and $k = 4$). We calcluate

$$\Delta \frac{x^4 + y^4 - 6x^2y^2}{(x^2+y^2)^2} = -16 \frac{x^4 + y^4 - 6x^2y^2}{(x^2+y^2)^3}.$$

Recall that a function f defined on $\mathbb{R}^n \setminus \{0\}$ is of homogeneous degree 0, if $f(tx) = f(x)$ for all $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. We will need Proposition 126 below to prove Lemma 214.

Proposition 126. *Let $f, g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a smooth function of homogeneous degree 0. Then $\int_{S^{n-1}} f(x)\Delta g(x) \mathrm{d}\sigma(x) = \int_{S^{n-1}} g(x)\Delta f(x) \mathrm{d}\sigma(x)$.*

Proof Denote by $\dfrac{\partial}{\partial n}$ the normal derivative of the smooth surface. Since $\dfrac{\partial f}{\partial n} = \dfrac{\partial g}{\partial n} = 0$, and hence $\displaystyle\int_{\partial(\Delta(2)\setminus\Delta(1))} \left(f(x)\frac{\partial g}{\partial n}(x) - g(x)\frac{\partial f}{\partial n}(x) \right) \mathrm{d}\sigma(x) = 0$, by the Stokes theorem, we have

$$\int_{B(2)\setminus B(1)} f(x)\Delta g(x)\mathrm{d}x = \int_{B(2)\setminus B(1)} g(x)\Delta f(x)\mathrm{d}x.$$

If we write this by the spherical coordinate, then we obtain the desired result.

3.2.4 Exercises

Exercise 58. Show that $\mathcal{P}_1(\mathbb{R}^n) = \dot{\mathcal{H}}^0(\mathbb{R}^n) \oplus \dot{\mathcal{H}}^1(\mathbb{R}^n)$. Hint: Can we write these spaces explicitly ?

Exercise 59. Let $k, n \in \mathbb{N}$.

(1) Fix $n \in \mathbb{N}$. Prove $\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n)) = {}_{k+n-1}\mathrm{C}_{n-1} - {}_{k+n-3}\mathrm{C}_{n-1} = \mathrm{O}(k^{n-2})$ as $k \to \infty$ by reexaming the proof of Corollary 119.

(2) Prove $\dot{\mathcal{P}}_k(\mathbb{R}^n) = \dot{\mathcal{H}}_k(\mathbb{R}^n) \oplus \mathrm{Im}(M : \dot{\mathcal{P}}_{k-2}(\mathbb{R}^n) \to \dot{\mathcal{P}}_k(\mathbb{R}^n))$ from the decomposition $\mathcal{P}_k(\mathbb{R}^n) = \mathcal{H}_k(\mathbb{R}^n) \oplus \mathrm{Im}(M : \mathcal{P}_{k-2}(\mathbb{R}^n) \to \mathcal{P}_k(\mathbb{R}^n))$. Hint: How do M in Proposition 120 and Δ act on $\mathcal{P}_k(\mathbb{R}^n)$?

Exercise 60 (Euler's formula)**.** Let $\alpha \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a function such that $f(tx_1, tx_2, \ldots, tx_n) = t^\alpha f(x_1, x_2, \ldots, x_n)$ for all $t > 0$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$. Then, by differentiating the above equation in t show that $\displaystyle\sum_{j=1}^{n} x_j f_{x_j}(x) = \alpha f(x)$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$.

3.3 Notes

Section 3.1

General remarks and textbooks in Section 3.1

See [85] for example.

Section 3.1.1

See [85, Theorem 3.8] for the expression of $P(D)\overline{Q}$, Lemma 111. Janson, Taibleson and Weiss used the Gram–Schmidt polynomial. See [220, p.111] for Proposition 117, for example.

Section 3.1.2

See [129, Lemma 3.3].

Section 3.1.3

We can find some quantitative information on the relation between the $L^q(\mathbb{R}^n)$-norm and the $L^\infty(\mathbb{R}^n)$-norm of the derivative; see [59, Lemma 2.I] for Lemma 115.

Section 3.1.4

We followed [59, (3.1)] to find the best approximation; see Lemma 116.

Section 3.2

General remarks and textbooks in Section 3.2

The properties and uses of spherical harmonics can be found in the books by [415, 417] and [410, §3.5].

Section 3.2.1

Corollary 119, which calculates $\dim_{\mathbb{C}}(\mathcal{H}_k(\mathbb{R}^n))$, can be found in [127, (2.55), (2.56)]. Proposition 120, which decomposes $\mathcal{P}_k(\mathbb{R}^n)$ can be found in [127, (2.49)].

Section 3.2.2

Proposition 121 can be found in [127, (2.57)].

Section 3.2.3

The functional equation in Proposition 125 can be found in [127, (2.62)].

The self-adjointness of the spherical Laplacian can be found in [76, Lemma 2.9]; see Proposition 126.

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Chapter 4

Various operators in Lebesgue spaces

One of the primary aims in this book is the boundedness properties of the integral operators. Here, we collect the properties of the linear operators acting on Lebesgue spaces. The boundedness obtained here will be used for analysis of operators acting on Morrey spaces. We take up the following operators.

(1) Hardy–Littlewood maximal operators, including the related maximal operators (Sections 4.1 and 4.2).

(2) Fractional maximal operators (Section 4.3).

(3) Singular integral operators (Section 4.4).

(4) Fractional integral operators (Section 4.5).

The first three operators are used for the purpose of controlling the remaining two operators and these two operators are mainly used to express the solution of the differential equations.

4.1 Maximal operators

Here, we present fundamental results for the Hardy–Littlewood maximal operator.

Section 4.1.1 defines the Hardy–Littlewood maximal operator. We give some examples of calculations and address some important problems concerning its boundedness. In Section 4.1.2 we consider fundamental results of the boundedness of the Hardy–Littlewood maximal operator (Theorems 134 and so on). We develop what we did in Section 4.1.2; Section 4.1.3 will investigate the local counterpart of the results in Section 4.1.2. Section 4.1.4 deals with the Fefferman–Stein vector-valued inequality (Theorem 145). In Section 4.1.5 we apply what we considered in Section 1.5.3 to the definition of the maximal operator. We will obtain a new maximal operator called the Orlicz-maximal operator. As an application of the Orlicz-maximal operator defined in Section 4.1.5, we consider the composition of the maximal operators in Section 4.1.6. We investigate the boundedness property of the Orlicz maximal operator in Section 4.1.7. As it turns out, the Hardy–Littlewood maximal operator can

be regarded as the supremum of the convolution of some classes of functions, and we collect some convolution estimates in Section 4.1.8.

4.1.1 Hardy–Littlewood maximal operator

One of the fundamental tools in the theory of function spaces is the Hardy–Littlewood maximal operator. Here, we define the Hardy–Littlewood maximal operator. We will have a look at the history and some applications here and then do some calculations.

The Hardy–Littlewood maximal operator appears in many mathematical contexts. Here, let us see three examples after giving definitions.

Definition 46 ((classical) Hardy–Littlewood maximal operator)**.** For $f \in L^0(\mathbb{R}^n)$, one defines a function Mf by $M^{\mathcal{B}}f(x) = Mf(x) \equiv \sup_{B\in\mathcal{B}} \frac{\chi_B(x)}{|B|}\int_B |f(y)|\mathrm{d}y$ for $x \in \mathbb{R}^n$. The mapping $M : f \mapsto Mf$ is called the (*ball based*) (*uncentered*) (*Hardy–Littlewood*) *maximal operator* or the maximal function for short.

A sublinear operator $T : L^\infty_{\mathrm{c}}(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ is by definition an operator with the property $|T(f+g)(x)| \le |Tf(x)| + |Tg(x)|$ and $|T(af)(x)| = |a||Tf(x)|$ for all $f, g \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $a \in \mathbb{C}$. In this sense, the Hardy–Littlewood maximal operator is sublinear. Before we show the calculation, we verify that M commutes with dilation; we have the following scaling law, whose proof is straightforward.

Lemma 127. *For any $f \in L^0(\mu)$ and $r > 0$ $M[f(r\cdot)] = Mf(r\cdot)$.*

Proof Simply use the change of variables $x = ry$.

Denote by $\mathcal{Q}$ the set of all cubes whose edges are parallel to coordinate axes.

Remark 4. One may replace $\mathcal{B}$ with $\mathcal{Q}$ to obtain $M^{\mathcal{Q}}f(= Mf)$, which one occasionally does; one can define $M^{\mathcal{Q}}f = Mf \equiv \sup_{Q\in\mathcal{Q}} m_Q(|f|)\chi_Q$. However for the time being one will study the Hardy–Littlewood maximal operator defined by $\mathcal{B}$ unless otherwise stated.

Let $f \in L^0(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. According to Definition 46 of Mf, we note that $\{Mf > \lambda\}$ is an open set. Hence, $Mf \in L^0(\mathbb{R}^n)$. Let us explain why the Hardy–Littlewood maximal operator is important.

Example 61. The first example is closely related to PDE. Let I_α be the *fractional integral operator* or the *Riesz potential* given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y \quad (x \in \mathbb{R}^n) \tag{4.1}$$

for $f \in \mathbb{M}^+(\mathbb{R}^n)$. Let $1 \le p < \dfrac{n}{\alpha}$ and define q so that $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$. Due to its importance in this book, we will show $I_\alpha f(x) \lesssim Mf(x)^{\frac{p}{q}} \|f\|_{L^p}{}^{1-\frac{p}{q}}$ for $x \in \mathbb{R}^n$ in Lemma 181.

Here, we work on $\mathbb{R}$ for the time being. We find $M\chi_{(-R,R)}(t)$ for $t \in \mathbb{R}$, $R > 0$ for the uncentered maximal operator M.

Proposition 128. *We have* $M\chi_{(-R,R)}(t) = \min\left(1, 2R(R+|t|)^{-1}\right)$, $t \in \mathbb{R}$, *for all* $R > 0$. *In particular,* $M\chi_{[-1,1]}(t) = \min\left(1, 2(1+|t|)^{-1}\right)$ *for* $t \in \mathbb{R}$.

Proof Since we are considering the average of the function of $\chi_{(-R,R)}$, we have $M\chi_{(-R,R)}(t) \le 1$ for all $t \in \mathbb{R}$. Let $|t| < R$. Then

$$M\chi_{(-R,R)}(t) \ge \lim_{r \downarrow 0} \frac{1}{2r} \int_{t-r}^{t+r} \chi_{(-R,R)}(y) \mathrm{d}y = \lim_{r \downarrow 0} \frac{1}{2r} \int_{t-r}^{t+r} \mathrm{d}y = 1.$$

For $t \ge R$, we use $\max\limits_{s \in (-R,R)} \dfrac{|(s,t) \cap (-R,R)|}{|(s,t)|} = \dfrac{2R}{R+t}$. For $t \le -R$, we use symmetry.

Thus, $M\chi_{[-1,1]}$ is not an $L^1(\mathbb{R})$-function: $\displaystyle\int_{-\infty}^{\infty} M\chi_{[-1,1]}(t) \mathrm{d}t = \infty$. Mimicking this argument, we can establish that $Mf \notin L^1(\mathbb{R}^n)$ when $f \in L^0(\mathbb{R}^n) \setminus \{0\}$.

We generalize Proposition 128 as follows:

Example 62. Let $f \in \mathbb{M}^+(\mathbb{R})$ be such that $f(t) = 0$ if $t \le 0$ and that $f(t) \ge f(s)$ if $t > s > 0$. Then $Mf(t) = \dfrac{1}{t} \displaystyle\int_0^t f(s) \mathrm{d}s$ for all $t > 0$.

Although we could find $M\chi_{[-1,1]}(t)$, $t \in \mathbb{R}$ in Proposition 128, it is in general hard to find $Mf(x)$, $x \in \mathbb{R}^n$ for $f \in L^0(\mathbb{R}^n)$.

Next, we pass Proposition 128 to higher dimensions to have the following useful expression.

Proposition 129. *For* $R \in \mathcal{Q}$ *and* $x \in \mathbb{R}^n$, $M^{\mathcal{Q}}\chi_R(x) \sim_n \dfrac{|R|}{|R| + |x - c(R)|^n}$, *where the implicit constant does not depend only on* n. *In particular, for all* $x \in \mathbb{R}^n$, $M^{\mathcal{B}}\chi_{B(r)}(x) \sim_n \left(\dfrac{r}{|x|+r}\right)^n$.

Proof The proof is based on a geometric observation. We content ourselves with its outline. We will distinguish two cases.

(1) Let $x \in 3R$. In this case, we can show that

$$\frac{1}{3^n} \le M^{\mathcal{Q}}\chi_R \le 1, \quad 1 \le 1 + \frac{|x - c(R)|}{\ell(R)} \le 1 + 3n.$$

(2) Let $x \in 3^{l+1}R \setminus 3^l R$ for some $l \in \mathbb{N}$. In this case, we can show that

$$\frac{1}{3^{(l+1)n}} \le M^{\mathcal{Q}}\chi_R(x) \le \frac{4^n}{3^{ln}}, \quad \frac{3^l}{2} \le 1 + \frac{|x - c(R)|}{\ell(R)} \le 1 + n \cdot 3^{l+1}.$$

Since M is not bounded on $L^1(\mathbb{R}^n)$, we need something more to describe the boundedness of M near $L^1(\mathbb{R}^n)$. Orlicz spaces are used for this purpose; see Chapter 13.

The following crude estimate is useful on many occasions.

Example 63. Let $Q \in \mathcal{Q}$, and let E be a measurable set containing $x \in \mathbb{R}^n$ and contained by Q. Observe that $M^{\mathcal{Q}}\chi_E \ge \dfrac{|E|}{|Q|}$ on Q. Indeed, for $x \in Q$,

$$M^{\mathcal{Q}}\chi_E(x) = \sup\left\{ \frac{|E \cap S|}{|S|} \ : \ S \ \text{ satisfies } \ x \in S \in \mathcal{Q} \right\} \ge \frac{|E|}{|Q|}.$$

We can readily replace cubes by balls.

The following technique is used to localize the problem: $M^{\mathcal{Q}}[\chi_{\mathbb{R}^n \setminus 5Q} f](y)$, $y \in Q$ has a standard estimate while $M^{\mathcal{Q}}[\chi_{5Q} f](y)$, $y \in Q$ must be handled in a more careful manner. Let $Q \in \mathcal{Q}$. The symbol $\mathcal{Q}^\sharp(Q)$ stands for the set of all cubes containing Q.

Lemma 130 (Nonlocal/global estimate of the maximal operator)**.** *For all $f \in L^0(\mathbb{R}^n)$ and cubes Q, we have*

$$M[\chi_{\mathbb{R}^n \setminus 5Q} f](x) \sim \sup_{R \in \mathcal{Q}^\sharp(Q)} m_R(\chi_{\mathbb{R}^n \setminus 5Q}|f|) \lesssim \sum_{k=1}^{\infty} m_{2^{k+1}Q}(|f|) \quad (x \in Q).$$

Remark that cubes can be replaced by balls and that $5Q$ can be readily replaced by κQ for any $\kappa > 1$ once the implicit constant can be allowed to depend on κ. The right inequality is clear but we frequently use it. However, it will turn out that we need to pay attention to the difference between the summation and the supremum.

Proof The right inequality is clear, so we prove the left equivalence. One inequality is clear from the definition of M. We write out $M[\chi_{\mathbb{R}^n \setminus 5Q} f](x)$ in full:

$$M[\chi_{\mathbb{R}^n \setminus 5Q} f](x) = \sup_{R \in \mathcal{Q}} \frac{\chi_R(x)}{|R|} \int_{R \setminus 5Q} |f(y)| \mathrm{d}y.$$

In order that $\chi_R(x) \displaystyle\int_{R \setminus 5Q} |f(y)| \mathrm{d}y$ be not zero, we need to have $x \in R$ and $R \setminus 5Q \neq \emptyset$. Thus, R meets both Q and $\mathbb{R}^n \setminus 5Q$. If $R \in \mathcal{Q}$ is a cube that meets both Q and $\mathbb{R}^n \setminus 5Q$, then $\ell(R) \ge 2\ell(Q)$ and $2R \supset Q$. Thus, the desired result follows.

The next theorem is an example of controls of linear operators by means of the Hardy–Littlewood maximal operator. A radial and decreasing function means a function Φ such that there exists $\varphi \in \mathbb{M}^{\downarrow}(0,\infty)$ such that $\Phi(x) = \varphi(|x|)$ for all $x \in \mathbb{R}^n$.

Theorem 131. *Let $\Phi : \mathbb{R}^n \to [0,\infty)$ be an integrable, continuous and radial decreasing function. Then $|\Phi * f(x)| \le \|\Phi\|_{L^1} Mf(x)$ holds for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.*

Proof We may assume that $f \in \mathbb{M}^+(\mathbb{R}^n)$ by the triangle inequality for integrals. Let $\varphi : [0,\infty) \to [0,\infty)$ be a function taking the form $\varphi(r) = \sum_{j=1}^{k} a_j \chi_{[0,r_j]}(r)$ for $r \ge 0$. Then

$$|\Phi * f(x)| \le \sum_{j=1}^{k} a_j |B(x,r_j)| \frac{1}{|B(x,r_j)|} \int_{B(x,r_j)} |f(y)| \mathrm{d}y \le \|\Phi\|_{L^1} Mf(x).$$

A passage to the limit, makes this inequality valid for any integrable, continuous and radial decreasing function Φ.

Example 64. We list some examples of functions to which Theorem 131 is applicable.

(1) $\Phi = \chi_{B(1)}$.

(2) Gaussian: $\Phi(x) = \dfrac{1}{\sqrt{(4\pi)^n}} \exp\left(-\dfrac{|x|^2}{4}\right)$ for $x \in \mathbb{R}^n$.

(3) Let $j \in \mathbb{N}_0$ and $m > n$. Define $\eta_{j,m}(x) \equiv 2^{jn}(1 + 2^j|x|)^{-m}$ for $x \in \mathbb{R}^n$. The function $\eta_{j,m}$ is called the *η-function*.

4.1.2 Hardy–Littlewood maximal inequality

In view of Proposition 128, we see that Mf never belongs to $L^1(\mathbb{R}^n)$ even when $f \in L^1(\mathbb{R}^n)$. What can be said for $L^1(\mathbb{R}^n)$ functions? We will answer this question by proving the weak-$(1,1)$ estimate. As an application of the weak-$(1,1)$ estimate and the trivial $L^\infty(\mathbb{R}^n)$-estimate, we will obtain various important inequalities.

First of all, we state a geometrical lemma. It is important that the covering lemma can be located as an assertion on geometry but that it comes into play in analysis. Consequently, the covering lemma below can be viewed as an interface of geometry and analysis.

Theorem 132 (Finite Vitali's $5r$-covering lemma)**.** *Suppose that we have a finite family $\{B_\lambda\}_{\lambda\in\Lambda}$ of J balls. If we relabel $1,2,\dots,J$, then, for the newly labeled family $B_1, B_2, \dots, B_J$, there exists $1 \le K \le J$ such that the following condition is fulfilled*;

(1) $B_1, B_2, \dots, B_K$ *are disjoint,*

(2) *there exists a mapping* $\iota : \{1, 2, \dots, J\} \to \{1, 2, \dots, K\}$ *such that* $B_j \subset 3\, B_{\iota(j)}$.

We can readily replace balls by cubes.

Proof We induct on J. When $J = 1$, $K = 1$ trivially does the job and Theorem 132 is true in this case.

Let $J_0 \in \mathbb{N}$ be fixed. Suppose that we have finite family $\{B^*_\lambda\}_{\lambda\in\Lambda}$ of J balls with $J \le J_0$ and assume that we can relabel $1, 2, \dots, J$ to have $B^*_1, B^*_2, \dots, B^*_J$ so that;

(1) $B^*_1, B^*_2, \dots, B^*_K$ are disjoint,

(2) there exists a mapping $\iota : \{1, 2, \dots, J\} \to \{1, 2, \dots, K\}$ such that $B^*_j \subset 3\, B^*_{\iota(j)}$.

Now suppose that we have $J_0 + 1$ balls. We relabel $B_1, B_2, \dots, B_{J_0+1}$ so that B_1 has the largest radius. Again, if necessary, we change the label of $B_2, B_3, \dots, B_{J_0+1}$. If there are more than one ball with the maximum radius, then choose one of them among $B_2, B_3, \dots, B_{J_0+1}$. If B_1 intersects B_j for some $j = 2, 3, \dots, J_0 + 1$, then $B_j \subset 3\, B_1$ since $r(B_j) \le r(B_1)$.

Among $\{B_j\}_{j=1}^{J_0+1}$, we can find at most J_0 balls such that $B_j \subset 3\, B_1$ does not hold, since B_1 is automatically excluded. Then such a ball B_j does not intersect B_1. Thus, among $\{B_j\}_{j=1}^{J_0+1}$, let $\{B_j\}_{j=2}^{J+1}$ be a family of balls such that $B_j \cap B_1 = \emptyset$ for all $j \in [2, J + 1]$. Note that $B_j \subset 3B_1$ for $j > J$ in view of the definition of J.

If we apply the induction assumption to $\{B_j\}_{j=2}^{J+1}$, a relabeling yields $B_2, \dots, B_K$ and $J(j) = 2, 3, \dots, K$ such that $B_j \subset 3\, B_{J(j)}$ unless $2 \le j \le J$. Then define a function $\iota : \mathbb{N} \to \{1, 2, \dots, n\}$ by

$$\iota(j) \equiv \begin{cases} 1, & (j = 1), \\ J(j), & (2 \le j \le J + 1), \\ 1, & (j > J + 1). \end{cases}$$

Thus, the balls $B_1, B_2, \dots, B_K$ satisfy the condition of the theorem and the proof is complete.

Usually, we mean by the "5r-covering lemma" the following theorem (due to Wiener), whose proof we omit due to similarity:

Theorem 133 (Infinite Vitali's 5r-covering lemma)**.** *Suppose that we have a family* $\{B_\lambda\}_{\lambda\in\Lambda}$ *of balls of general cardinality. If the radii of the balls in the family is bounded from above by a positive constant, then we can find* $\Lambda_0 \subset \Lambda$ *such that*

(1) $\{B_\lambda\}_{\lambda\in\Lambda_0}$ *is disjoint,*

(2) *there exists a mapping* $\iota : \Lambda_0 \to \Lambda$ *such that* $B_\lambda \subset 5\, B_{\iota(\lambda)}$.

Now we investigate Mf for $L^1(\mathbb{R}^n)$-functions f.

We prove the following important estimate called the *weak*-$(1,1)$ *boundedness* or the $L^1(\mathbb{R}^n)$-$L^{1,\infty}(\mathbb{R}^n)$ boundedness of M.

Theorem 134 (Hardy–Littlewood maximal inequality)**.** *For all* $f \in L^1(\mathbb{R}^n)$ *and* $\lambda > 0$, $\lambda|\{Mf > \lambda\}| \le 3^n \|f\|_{L^1}$.

We will not discuss whether we can replace 3^n by a smaller number. The best constant C that can be used instead of 3^n is called the weak-$(1,1)$ constant.

Proof By the inner regularity (1.1) of the Lebesgue measure, it suffices to show

$$\lambda|K| \le 3^n \|f\|_{L^1} \tag{4.2}$$

for any compact set K contained in $\{Mf > \lambda\}$.

Let $x \in K$. By the definition of $M = M^{\mathcal{B}}$, there exists a ball B_x centered at x such that

$$m_{B_x}(|f|) > \lambda. \tag{4.3}$$

By the compactness of K, there exists a finite collection $x_1, x_2, \dots, x_J$ of points such that $K \subset \bigcup_{j=1}^{J} B_{x_j}$. To simplify the notation, let $B_j \equiv B_{x_j}$. By Theorem 132, if we relabel the B_j's, for some $L \le J$, we have the following:

(1) The family $\{B_j\}_{j=1}^{L}$ of balls is disjoint. That is,

$$\sum_{j=1}^{L} \chi_{B_j} \le 1, \tag{4.4}$$

(2) We have inclusion:

$$K \subset \bigcup_{j=1}^{J} B_j \subset \bigcup_{j=1}^{L} 3\, B_j. \tag{4.5}$$

Hence, if we use (4.5), (4.3) and (4.4) in that order, then

$$|K| \le \sum_{j=1}^{L} |3B_j| = 3^n \sum_{j=1}^{L} |B_j| \le \sum_{j=1}^{L} \frac{3^n}{\lambda} \int_{B_j} |f(x)| \mathrm{d}x \le \frac{3^n}{\lambda} \|f\|_{L^1}.$$

Thus, (4.2) and hence Theorem 134 are proven.

If we define the *weak* $L^p(\mathbb{R}^n)$*-space* $\mathrm{W}L^p(\mathbb{R}^n) = L^{p,\infty}(\mathbb{R}^n)$, $p \in [1,\infty)$ as the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{L^{p,\infty}} \equiv \sup_{\lambda>0} \lambda |\{|f| > \lambda\}^{\frac{1}{p}}$ is finite, then it follows that M is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.

Corollary 135. *Suppose that f is a locally integrable function. Then*

$$|\{y \in \mathbb{R}^n \,:\, Mf(y) > 2\lambda\}| \le \frac{3^n}{\lambda} \int_{\{y\in\mathbb{R}^n\,:\,|f(y)|>\lambda\}} |f(x)|\mathrm{d}x. \tag{4.6}$$

Proof We calculate

$$\begin{aligned} |\{y \in \mathbb{R}^n \,:\, Mf(y) > 2\lambda\}| &\le |\{y \in \mathbb{R}^n \,:\, M[\chi_{(\lambda,\infty]}(|f|)f](y) > \lambda\}| \\ &\le \frac{3^n}{\lambda} \int_{\{y\in\mathbb{R}^n\,:\,|f(y)|>\lambda\}} |f(x)|\mathrm{d}x. \end{aligned}$$

To investigate the Hardy–Littlewood maximal operator further, it is sometimes useful to consider its dyadic version. The next lemma is almost trivial and we omit its proof.

Lemma 136 (The nesting structure of dyadic cubes)**.** *When two dyadic cubes intersect, one is contained in the other.*

Recall that the symbol $\mathcal{D}(\mathbb{R}^n)$ denotes the set of all dyadic cubes:

$$\mathcal{D} = \mathcal{D}(\mathbb{R}^n) \equiv \{Q_{jk} \,:\, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

Likewise, for a right-open cube $Q = \prod_{j=1}^{n} [a_j, b_j)$, the set $\mathcal{D}_1(Q)$ of the children of Q is the set of all right-open cubes obtained by bisecting Q. Define inductively $\mathcal{D}_k(Q)$ by $\mathcal{D}_k(Q) \equiv \bigcup_{R\in\mathcal{D}_{k-1}(Q)} \mathcal{D}_1(R)$. Most likely, we let $Q = [-N, N)^n$ or $Q = [0, N)^n$.

Theorem 137 (Sunrise lemma)**.** *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for all $\lambda > 0$,*

$$|\{y \in \mathbb{R}^n \,:\, Mf(y) > \lambda\}| \ge \frac{1}{2^n\lambda} \int_{\{y\in\mathbb{R}^n\,:\,|f(y)|>\lambda\}} |f(x)|\mathrm{d}x.$$

Proof We may assume that $f \in L^1(\mathbb{R}^n)$ by using the monotone convergence theorem. Then we will obtain a disjoint family $\{Q_j\}_{j\in J}$ of dyadic cubes such that $\lambda < m_{Q_j}(|f|)$. By maximality, we have $\lambda < m_{Q_j}(|f|) \le 2^n\lambda$.

From the Lebesgue differentiation theorem, Theorem 12,

$$\{y \in \mathbb{R}^n \,:\, |f(y)| > \lambda\} \subset \{y \in \mathbb{R}^n \,:\, M^{\mathcal{D}} f(y) > \lambda\} = \sum_{j\in J} Q_j$$

a.e.. Thus, we have

$$\begin{aligned} \frac{1}{\lambda} \int_{\{y\in\mathbb{R}^n\,:\,|f(y)|>\lambda\}} |f(x)|\mathrm{d}x &\le \frac{1}{\lambda} \int_{\sum_{j\in J} Q_j} |f(x)|\mathrm{d}x \\ &\le \sum_{j\in J} 2^n |Q_j| \\ &= 2^n |\{y \in \mathbb{R}^n \,:\, M^{\mathcal{D}} f(y) > \lambda\}| \\ &\le 2^n |\{y \in \mathbb{R}^n \,:\, Mf(y) > \lambda\}|. \end{aligned}$$

Thus, Theorem 137 is proven.

We are now oriented to the localized estimate. If we consider the boundedness property of the operators acting on Morrey spaces or Lebesgue spaces, it is sometimes important to use the information for their behavior around cubes or balls.

Example 65 (Sunrise lemma)**.** For all $f \in L^0(\mathbb{R}^n)$, $\lambda > 0$ and cubes Q,

$$|\{z \in Q \,:\, M[\chi_Q f](z) > \lambda\}| \geq \frac{1}{2^n \lambda} \int_{\{z \in Q \,:\, |f(z)| > \lambda\}} |f(y)| \mathrm{d}y$$

We can modify the argument above to have an estimate for the maximal function of the measure.

Example 66. Let μ be a non-negative finite Radon measure and define (temporarily) the Hardy–Littlewood maximal operator of measures by

$$M\mu(x) \equiv \sup \left\{ \frac{\mu(Q)}{|Q|} \,:\, Q \in \mathcal{Q},\, c(Q) = x \right\} \quad (x \in \mathbb{R}^n). \tag{4.7}$$

Then modifying the argument above, we can show

$$\lambda |\{x \in \mathbb{R}^n \,:\, M\mu(x) > \lambda\}| \lesssim \mu(\mathbb{R}^n) \tag{4.8}$$

for $\lambda > 0$. Let δ_0 be the measure massed at the origin. It is also noteworthy that $M\delta_0(x) \sim |x|^{-n}$ for $x \in \mathbb{R}^n$. The proof of (4.8) is left as an exercise; see Exercise 63.

Although in Proposition 128 we explained that the function Mf is never integrable, we still substitute this inequality for the weak variant.

Theorem 138. *Let $0 < p < 1$, $E \subset \mathbb{R}^n$ be a measurable set with finite measure, and let $f \in L^1(\mathbb{R}^n)$. Then $\|Mf\|_{L^p(E)} \lesssim |E|^{\frac{1}{p}-1} \|f\|_{L^1}$.*

The idea that lies behind it is the use of the finiteness of the measure and the weak-$(1,1)$ boundedness of operators. This estimate is called the *Kolmogorov inequality*. Note that this type of estimate is valid for any weak-$(1,1)$ bounded linear operator.

Proof This is a consequence of

$$\int_E Mf(x)^p \mathrm{d}x = \int_0^\infty p\lambda^{p-1} |\{x \in E \,:\, Mf(x) > \lambda\}| \mathrm{d}\lambda,$$

which follows from Theorem 5, and

$$|\{x \in E \,:\, Mf(x) > \lambda\}| \lesssim \min\{|E|, \lambda^{-1} \|f\|_{L^1}\} \quad (\lambda > 0),$$

which in turn follows from Theorem 134.

We extend Theorem 134. We consider the uncentered maximal operator M generated by balls. For $w \in \mathbb{M}^+(\mathbb{R}^n)$ and measurable set E, we define the weighted measure $w(E)$ by $w(E) \equiv \int_E w(x)\mathrm{d}x$. We will prove a sort of weighted estimate, which will often be of importance in this book.

Proposition 139 (Weak dual inequality of Stein-type)**.** *Let $w \in \mathbb{M}^+(\mathbb{R}^n)$. Then $w\{Mf > \lambda\} \le \frac{3^n}{\lambda}\int_{\mathbb{R}^n} |f(x)|Mw(x)\mathrm{d}x$ for all $f \in L^0(\mathbb{R}^n)$. More precisely, $w\{Mf > \lambda\} \le \frac{3^n}{\lambda}\int_{\{Mf>\lambda\}} |f(x)|Mw(x)\mathrm{d}x$.*

We also call the above inequality the weighted weak $(1,1)$-inequality. In Proposition 139, by letting $w \equiv 1$, we can recover Theorem 133.

Proof By the monotone convergence theorem, it can be assumed that w is bounded. For $\lambda > 0$, set $E_\lambda \equiv \{Mf > \lambda\}$. In view of the inner regularity of the measure $w(x)\mathrm{d}x$, it suffices to show

$$w(K) \le \frac{3^n}{\lambda}\int_{\mathbb{R}^n} |f(x)|Mw(x)\mathrm{d}x$$

for any compact set K contained in E_λ.

If we combine the compactness of K and Theorem 132, we can define a finite collection $B_1, B_2, \ldots, B_M$ of balls such that:

(1) $B_1, B_2, \ldots, B_M$ are disjoint:

$$\sum_{j=1}^{M} \chi_{B_j} \le 1. \tag{4.9}$$

(2) The inequality

$$m_{B_j}(|f|) > \lambda \tag{4.10}$$

holds for $j = 1, 2, \ldots, M$.

(3) K is covered by the triple of the balls $B_1, B_2, \ldots, B_M$, namely,

$$K \subset \bigcup_{j=1}^{M} 3\,B_j. \tag{4.11}$$

By the definition of the Hardy–Littlewood maximal operator M, we have

$$\frac{w(3B_j)}{|B_j|} \le 3^n \inf_{y\in B_j} Mw(y). \tag{4.12}$$

By (4.11), we have $w(K) \le \sum_{j=1}^{M} w(3\,B_j)$. If we use (4.10), (4.12), and (4.9), in that order, then

$$\begin{aligned} w(K) &\le \frac{1}{\lambda} \sum_{j=1}^{M} \int_{B_j} |f(x)| \mathrm{d}x \cdot \frac{w(3\,B_j)}{|B_j|} \\ &\le \frac{3^n}{\lambda} \sum_{j=1}^{M} \int_{B_j} |f(x)| M w(x) \mathrm{d}x \\ &\le \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(x)| M w(x) \mathrm{d}x. \end{aligned}$$

Thus, the proof is complete.

If we combine Proposition 139 with the trivial equality $\|M\|_{L^\infty \to L^\infty} = 1$, we obtain the weighted $L^p(\mathbb{R}^n)$-estimate.

Corollary 140 (Strong dual inequality of Stein-type)**.** *For all* $1 < p < \infty$, *and* $f, w \in L^0(\mathbb{R}^n)$, $\displaystyle\int_{\mathbb{R}^n} Mf(x)^p |w(x)| \mathrm{d}x \le 3^n 2^p p' \int_{\mathbb{R}^n} |f(x)|^p M w(x) \mathrm{d}x$.

Proof By replacing w with $|w|$, we may suppose in addition that $w \ge 0$ almost everywhere. First of all, whenever $\| f \|_{L^\infty} \le \lambda$, $Mf \le \lambda$. Together with the subadditivity $M[f+g] \le Mf + Mg$, we deduce that $\{Mf > 2\lambda\} \subset \{M[\chi_{(\lambda,\infty]}(|f|)f] > \lambda\}$. By Proposition 139,

$$w\{Mf > 2\lambda\} \le w\{M[\chi_{(\lambda,\infty]}(|f|)f] > \lambda\} \le \frac{1}{\lambda} \int_{\{|f|>\lambda\}} |f(x)| \cdot M w(x) \mathrm{d}x.$$

We use (1.4) for the function Mf and the measure $w(x)\mathrm{d}x$. If we change variables: $\sigma = 2\lambda$, then

$$\begin{aligned} \int_{\mathbb{R}^n} Mf(x)^p w(x) \mathrm{d}x &= p \int_{\mathbb{R}^n} \left(\int_0^{Mf(x)} \sigma^{p-1} \mathrm{d}\sigma \right) w(x) \mathrm{d}x \\ &= 2^p p \int_{\mathbb{R}^n} \left(\int_0^{2^{-1} Mf(x)} \lambda^{p-1} \mathrm{d}\lambda \right) w(x) \mathrm{d}x. \end{aligned} \tag{4.13}$$

By virtue of (4.13) and Proposition 139,

$$\begin{aligned} &\int_{\mathbb{R}^n} Mf(x)^p w(x) \mathrm{d}x \\ &= 2^p p \int_0^\infty \left(\int_{\mathbb{R}^n} \chi_{(2\lambda,\infty]}(Mf(x)) w(x) \mathrm{d}x \right) \lambda^{p-1} \mathrm{d}\lambda \\ &\le 3^n 2^p p \int_0^\infty \left(\int_{\mathbb{R}^n} \chi_{(\lambda,\infty)}(|f(x)|) |f(x)| M w(x) \mathrm{d}x \right) \lambda^{p-2} \mathrm{d}\lambda. \end{aligned}$$

By Fubini's theorem (Theorem 4), we calculate the 1-dimensional integral precisely to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} Mf(x)^p w(x)\mathrm{d}x \\
&\leq 3^n 2^p p \int_0^\infty \left(\int_{\mathbb{R}^n} \chi_{(\lambda,\infty)}(|f(x)|)|f(x)|Mw(x)\mathrm{d}x \right) \lambda^{p-2}\mathrm{d}\lambda \\
&= 3^n 2^p p \int_{\mathbb{R}^n} \left(\int_0^\infty \chi_{(\lambda,\infty)}(|f(x)|)\lambda^{p-2}\mathrm{d}\lambda \right) |f(x)|Mw(x)\mathrm{d}x \\
&= 3^n 2^p p \int_{\mathbb{R}^n} \left(\int_0^{|f(x)|} \lambda^{p-2}\mathrm{d}\lambda \right) |f(x)|Mw(x)\mathrm{d}x \\
&= 3^n 2^p p' \int_{\mathbb{R}^n} |f(x)|^p Mw(x)\mathrm{d}x.
\end{aligned}$$

Therefore, the proof is complete.

In particular, we obtain the $L^p(\mathbb{R}^n)$-inequality of the Hardy–Littlewood maximal operator M as a corollary by letting $w \equiv 1$:

Theorem 141 ($L^p(\mathbb{R}^n)$-inequality of M)**.** *Let $1 < p < \infty$. Then we have $\|Mf\|_{L^p} \leq 3^n p' \|f\|_{L^p}$ for all $f \in L^0(\mathbb{R}^n)$.*

Proof By the monotone convergence theorem, we may assume that $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Reexamine the proof of Corollary 140 above. We start with

$$|\{Mf > \lambda\}| \leq \frac{3^n}{\lambda} \int_{\{Mf>\lambda\}} |f(x)|\mathrm{d}x.$$

Going through the same argument using the Layer-Cake formula, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} Mf(x)^p \mathrm{d}x &\leq 3^n p' \int_{\mathbb{R}^n} Mf(x)^{p-1}|f(x)|\mathrm{d}x \\
&\leq 3^n p' \left(\int_{\mathbb{R}^n} Mf(x)^p \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, we can arrange this inequality to obtain the desired conclusion.

About the constant obtained in Theorem 141, we have the following clarifying remark:

Remark 5. Theorem 141 is very important and we use it many many times in this book. Usually, it is not necessary to learn the constant $3^n p'$ by heart. However, it is not so difficult to keep track of the constants in the proof.

Example 67. Let $p \in (1,\infty)$ and $r \in [1,\infty)$. Let $T : L^\infty_{\mathrm{c}}(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ be a sublinear operator such that T can be extended to a bounded operator both

$T : L^1(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ and $T : L^p(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$. In analogy to Corollary 140, we can prove that T is bounded from $L^{p_\theta}(\mu) \to L^{q_\theta}(\mu)$, where p_θ and q_θ are given by

$$\frac{1}{p_\theta} = 1 - \theta + \frac{\theta}{p}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{r}$$

for some $0 < \theta < 1$.

The boundedness of the Hardy–Littlewood maximal operator M is important because it is less likely to control many other operators in terms of the given function itself. Since the Hardy–Littlewood maximal operator enlarges the functions in a suitable way, we still have a hope to control many integral operators using M.

We conclude Section 4.1.2 with the proof of the Lebesgue differentiation theorem, Theorem 12. To this end, we have only to show that the Lebesgue measure does not charge the set E of all points for which (1.7) fails. We decompose E; we define

$$E_k \equiv \left\{ x \in \mathbb{R}^n \,:\, \limsup_{r \downarrow 0} m_{B(x,r)}(|f - f(x)|) > \frac{1}{k} \right\} \tag{4.14}$$

for $k \in \mathbb{N}$. Once we prove E_k has Lebesgue measure 0, then E will have Lebesgue measure 0. Condition (1.7) is local. Hence, we can consider $\chi_{B(r)} f$ instead of f. We can assume that $f \in L^1(\mathbb{R}^n)$.

Furthermore, we can argue by using the ε-δ argument to obtain (1.7) for $g \in L^1(\mathbb{R}^n) \cap \mathrm{BC}(\mathbb{R}^n)$.

Let $\varepsilon > 0$ be arbitrary. Let us conclude the proof of the theorem by showing that $|E_k| \le 2(3^n + 1)\, k\, \varepsilon$.

Choose $g \in L^1(\mathbb{R}^n) \cap \mathrm{BC}(\mathbb{R}^n)$ by density so that $\|f - g\|_{L^1} \le \varepsilon$. Then have

$$E_k = \left\{ x \in \mathbb{R}^n \,:\, \limsup_{r \downarrow 0} m_{B(x,r)}(|f - g + g(x) - f(x)|) > \frac{1}{k} \right\}.$$

We estimate E_k from above to have

$$\begin{aligned} E_k &\subset \left\{ x \in \mathbb{R}^n \,:\, M[f-g](x) + |f(x) - g(x)| > \frac{1}{k} \right\} \qquad (4.15) \\ &\subset \left\{ x \in \mathbb{R}^n \,:\, M[f-g](x) > \frac{1}{2k} \right\} \cup \left\{ x \in \mathbb{R}^n \,:\, |f(x) - g(x)| > \frac{1}{2k} \right\}. \end{aligned}$$

If we combine (4.15) with the subadditivity, Chebychev's inequality and the weak $(1,1)$-maximal inequality, $|E_k| \le 2(3^n + 1)\, k\, \|f - g\|_{L^1} \le 2(3^n + 1)\, k\, \varepsilon$. The number $\varepsilon > 0$ being arbitrary, we obtain (1.7).

Before we go further, we give a fundamental relation between Mf and f.

Lemma 142. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $Mf(x) \ge |f(x)|$ for almost all $x \in \mathbb{R}^n$.*

We end this section with an analogy to Lebesgue differentiation theorem for the polynomials.

Proof By the Lebesgue differentiation theorem,

$$Mf(x) \geq \lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \mathrm{d}y = |f(x)|$$

for almost all $x \in \mathbb{R}^n$.

We present another application of the idea employed in the proof above.

Lemma 143. *Let $k \in \mathbb{N}_0$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $\lim_{r \downarrow 0} P^k_{Q(x,r)} f(x) = f(x)$ for almost all $x \in \mathbb{R}^n$.*

Proof Similar to the Lemma 142; see Exercise 64.

4.1.3 Local estimates for the Hardy–Littlewood maximal operator

So far, we have obtained some global estimates. However, when handling some linear operators, we will face the situation where some local estimates are necessary. As an application of the dual inequality of Stein-type, we will obtain some related estimates useful in the situation above.

Example 68. For all $f \in L^0(\mathbb{R}^n)$ and balls $B(x,r)$, we have

$$\begin{aligned} \int_{B(x,r)} Mf(y)^p \mathrm{d}y &= \int_{\mathbb{R}^n} Mf(y)^p \chi_{B(x,r)}(y) \mathrm{d}y \\ &\leq 3^n p' \int_{\mathbb{R}^n} |f(y)|^p M\chi_{B(x,r)}(y) \mathrm{d}y, \end{aligned}$$

which can be regarded as a local norm estimate at all scales.

It is noteworthy that the most right-hand side in Example 68 can be rewritten using Fubini's theorem. We transform Example 68 to the form which we use, which is called a local estimate of the Hardy–Littlewood maximal operator.

Theorem 144. *Let $1 < p < \infty$. Then for all $f \in L^0(\mathbb{R}^n)$ and all balls $B(x,r)$,*

$$\|Mf\|_{L^p(B(x,r))} \lesssim r^{\frac{n}{p}} \left\{ \int_r^\infty \left(\int_{B(x,t)} |f(y)|^p \mathrm{d}y \right) \frac{\mathrm{d}t}{t^{n+1}} \right\}^{\frac{1}{p}}.$$

Proof Simply combine Example 68 and Fubini's theorem.

Example 69. Let $1 < p < \infty$. Consider the inequality

$$\|Mf\|_{L^p(B(r))} \leq c(r) \|f\|_{L^p(r^{-n} M\chi_{B(r)})} \quad (f \in L^0(\mathbb{R}^n)),$$

and let $c^*(r)$ be the minimal value of $c(r)$ in the inequality above. Then we claim $c^*(r) = c^*(1)r^{\frac{n}{p}}$, which redues matters to the case where $r = 1$. Indeed, we calculate

$$\begin{aligned} c^*(r)^p &= \sup_f \frac{\displaystyle\int_{B(r)} Mf(x)^p \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^n} \frac{|f(x)|^p}{(|x|+r)^n}\mathrm{d}x} = \sup_f \frac{\displaystyle r^{\frac{n}{p}}\int_{B(1)} Mf(ry)^p \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^n} \frac{|f(ry)|^p}{(|y|+1)^n}\mathrm{d}y} \\ &= \sup_f \frac{\displaystyle r^{\frac{n}{p}}\int_{B(1)} M(f(r\cdot))(y)^p \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^n} \frac{|f(ry)|^p}{(|y|+1)^n}\mathrm{d}y} = r^{\frac{n}{p}} \sup_g \frac{\displaystyle\int_{B(1)} Mg(x)^p \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^n} \frac{|g(x)|^p \mathrm{d}x}{(|x|+1)^n}} \\ &= c^*(1)^p r^n. \end{aligned}$$

4.1.4 Fefferman–Stein vector-valued maximal inequality

The Fefferman–Stein vector-valued maximal inequality, which we prove in Section 4.1.4, asserts that

$$\left\| \left(\sum_{j=1}^{\infty} Mf_j{}^q \right)^{\frac{1}{q}} \right\|_{L^p} \lesssim_{p,q} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p}$$

for all sequences $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$. Note that this inequality differs from

$$\left\| M\left[\left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right] \right\|_{L^p} \lesssim_{p,q} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p},$$

which is a direct consequence of Theorem 141.

A basic idea of analysis is the decomposition of a function into a countable sum of elementary pieces of functions. The sum must be countable; otherwise the sum may fail to be measurable. Because the sum is made up of functions which are not complicated, we pay attention to each summand. Therefore, we handle each term separately. The Hardy–Littlewood maximal operator controls in some sense functions whose structures are not clear. Therefore, the Fefferman–Stein vector-valued maximal inequality, whose left-hand side contains a countable sum of the Hardy–Littlewood maximal functions of functions, is useful.

The Fefferman–Stein vector-valued maximal inequality is elementary and it is used mainly in the study of Triebel–Lizorkin spaces. We cannot cover Triebel–Lizorkin spaces since it is beyond the scope of this book. See [146, 176, 156, 157, 415, 416, 382] for more about this direction. However, recently more and more applications of this inequality were found. Theorem 145 is extremely important and is used hundreds of times in this book.

Theorem 145 (Fefferman–Stein vector-valued maximal inequality)**.** *Let* $1 < p < \infty$ *and* $1 < q \leq \infty$*. Then for all sequences* $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$*,*

$$\left\| \left(\sum_{j=1}^{\infty} Mf_j{}^q \right)^{\frac{1}{q}} \right\|_{L^p} \lesssim_{p,q} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p}.$$

To formulate the vector-valued inequalities in general, we adopt the following notation; the index set J is usually taken as $J = \mathbb{N}, \mathbb{N}_0, \mathbb{Z}$. However here we present the definition generally.

Definition 47 (Vector-valued norm)**.** Let $0 < p, q \leq \infty$, and let J be a countable set.

(1) For a system $\{f_j\}_{j\in J} \subset L^0(\mathbb{R}^n)$ of functions, define

$$\|\{f_j\}_{j\in J}\|_{\ell^q(L^p)} \equiv \left(\sum_{j\in J} \|f_j\|_{L^p}{}^q \right)^{\frac{1}{q}}.$$

The space $\ell^q(L^p, \mathbb{R}^n)$ is the set of all collections $\{f_j\}_{j\in J}$ for which the quantity $\|\{f_j\}_{j\in J}\|_{\ell^q(L^p)}$ is finite.

(2) For a sequence $\{f_j\}_{j\in J}$ of functions we define a function $\|\{f_j\}_{j\in J}\|_{\ell^q} : \mathbb{R}^n \to [0, \infty]$ by

$$\|\{f_j\}_{j\in J}\|_{\ell^q} \equiv \left(\sum_{j\in J} |f_j|^q \right)^{\frac{1}{q}}$$

(3) For a system $\{f_j\}_{j\in J} \subset L^0(\mathbb{R}^n)$ of functions, define

$$\|\{f_j\}_{j\in J}\|_{L^p(\ell^q)} \equiv \| \|\{f_j\}_{j\in J}\|_{\ell^q} \|_{L^p}.$$

The space $L^p(\ell^q, \mathbb{R}^n)$ is the set of all collections $\{f_j\}_{j\in J}$ for which the quantity $\|\{f_j\}_{j\in J}\|_{\ell^q(L^p)}$ is finite.

(4) For a measure space $(X, \mathcal{B}, \mu)$, define $L^p(\ell^q, \mu)$ and $\ell^q(L^p, \mu)$ similarly.

These two norms are called *vector-valued norms*. A natural modification is made in the above when $q = \infty$.

Example 70. We extend Hölder's inequality. Let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Let $(X, \mathcal{B}, \mu)$ be a measure space. By original Hölder's inequality $\|\{a_j b_j\}_{j=1}^{\infty}\|_{\ell^1} \leq \|\{a_j\}_{j=1}^{\infty}\|_{\ell^q} \|\{b_j\}_{j=1}^{\infty}\|_{\ell^{q'}}$ for sequences $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ and by the one $\|F \cdot G\|_{L^1(\mu)} \leq \|F\|_{L^p(\mu)} \|G\|_{L^{p'}(\mu)}$ for $F, G \in L^0(\mu)$, $\|\{F_j G_j\}_{j=1}^{\infty}\|_{L^1(\ell^1,\mu)} \leq \|\{F_j\}_{j=1}^{\infty}\|_{L^p(\ell^q,\mu)} \|\{G_j\}_{j=1}^{\infty}\|_{L^{p'}(\ell^{q'},\mu)}$.

We plan to prove Theorem 145 after proving a lemma.

Lemma 146. *Let $1 \le p, q \le \infty$, and let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $L^p(\mathbb{R}^n)$-functions such that $f_j = 0$ a.e. if j is large enough. Then we can take $\{g_j\}_{j=1}^{\infty} \in L^{p'}(\ell^{q'})$ with norm 1 such that $\|\{f_j\}_{j=1}^{\infty}\|_{L^p(\ell^q)} = \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} f_j(x) g_j(x) \mathrm{d}x$. If $\{f_j\}_{j=1}^{\infty} \subset \mathbb{M}^+(\mathbb{R}^n)$, then we can arrange that $\{g_j\}_{j=1}^{\infty} \subset \mathbb{M}^+(\mathbb{R}^n)$.*

Proof Suppose $1 < q < \infty$ for simplicity. There is nothing to prove if $f_j = 0$ a.e. for all $j \in \mathbb{N}_0$; assume otherwise. In this case, we recall the construction of the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$: Set $g_j \equiv \frac{1}{A} \overline{\mathrm{sgn}(f_j)} |f_j|^{q-1} \|\{f_j\}_{j=1}^{\infty}\|_{\ell^q}{}^{p-q}$. Here A is a normalization constant: $A \equiv \|f_j\|_{L^p(\ell^q)}^{p-1}$. Since $(\|\{g_j\}_{j=1}^{\infty}\|_{\ell^{q'}})^{p'} = (\|\{f_j\}_{j=1}^{\infty}\|_{\ell^q})^p$, we have $\|\{g_j\}_{j=1}^{\infty}\|_{L^{p'}(\ell^{q'})} = 1$. Furthermore, since $\sum_{j=1}^{\infty} f_j g_j = \frac{1}{A} (\|\{f_j\}_{j=1}^{\infty}\|_{\ell^q})^p$, we have $\|\{f_j\}_{j=1}^{\infty}\|_{L^p(\ell^q)} = \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} f_j(x) g_j(x) \mathrm{d}x$.

We prove Theorem 145. First we rephrase Theorem 145. We will show

$$\int_{\mathbb{R}^n} \|\{Mf_j(x)\}_{j=1}^{\infty}\|_{\ell^q}{}^p \, \mathrm{d}x \lesssim_{p,q} \int_{\mathbb{R}^n} \|\{f_j(x)\}_{j=1}^{\infty}\|_{\ell^q}{}^p \mathrm{d}x.$$

By the monotone convergence theorem, we can assume that $f_j \equiv 0$ for large j.

Proof A simple case 1: $q = p, \infty$.

If $p = q$, Theorem 145 is clear by the $L^p(\mathbb{R}^n)$-boundedness of M and the monotone convergence theorem. If $q = \infty$, we invoke a trivial pointwise estimate: $\sup_{j \in \mathbb{N}} Mf_j(x) \le M \left[\sup_{j \in \mathbb{N}} |f_j| \right](x)$ for $x \in \mathbb{R}^n$.

Case 2: $p > q$. Keeping in mind that the left-hand side is at least finite, we resort to duality. Set $r \equiv \frac{p}{q}$. Then there exists $g \in L^{r'}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ with norm 1 such that $\int_{\mathbb{R}^n} \|\{f_j(x)\}_{j=1}^{\infty}\|_{\ell^q}{}^p \mathrm{d}x = \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} Mf_j(x)^q g(x) \mathrm{d}x$ by virtue of the duality $L^r(\mathbb{R}^n)$-$L^{r'}(\mathbb{R}^n)$. By Corollary 140 and Hölder's inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \|\{Mf_j(x)\}_{j=1}^{\infty}\|_{\ell^q}{}^p \mathrm{d}x &\lesssim \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |f_j(x)|^q Mg(x) \mathrm{d}x \\ &\lesssim \|Mg\|_{L^{r'}}{}^p \int_{\mathbb{R}^n} \|\{f_j(x)\}_{j=1}^{\infty}\|_{\ell^q}{}^p \mathrm{d}x. \end{aligned}$$

Since we know that M is $L^{r'}(\mathbb{R}^n)$-bounded, $\|Mg\|_{L^{r'}} \lesssim 1$. Hence

$$\int_{\mathbb{R}^n} \|\{Mf_j(x)\}_{j=1}^{\infty}\|_{\ell^q}{}^p \mathrm{d}x \lesssim \int_{\mathbb{R}^n} \|\{f_j(x)\}_{j=1}^{\infty}\|_{\ell^q}{}^p \mathrm{d}x.$$

Therefore, the proof is complete.

Case 3: $p < q < \infty$**.** Let $\theta \equiv \sqrt{p}$, $s \equiv \dfrac{p}{\theta}$ and $t \equiv \dfrac{q}{\theta}$. Then

$$\int_{\mathbb{R}^n} \|\{Mf_j(x)\}_{j=1}^\infty\|_{\ell^q}{}^p \mathrm{d}x = \|\{Mf_j{}^\theta\}_{j=1}^\infty\|_{L^s(\ell^t)}^s.$$

By Lemma 146, there exists $\{g_j\}_{j=1}^\infty \subset L^{s'}(\ell^{t'}, \mathbb{R}^n)$ such that

$$\|\{Mf_j{}^\theta\}_{j=1}^\infty\|_{L^s(\ell^t)} = \sum_{j=1}^\infty \int_{\mathbb{R}^n} Mf_j(x)^\theta g_j(x)\mathrm{d}x, \quad \|\{g_j\}_{j=1}^\infty\|_{L^{s'}(\ell^{t'})} = 1.$$

By Corollary 140, we can move M to the function g_j. Using Hölder's inequality, we have

$$\begin{aligned}\sum_{j=1}^\infty \int_{\mathbb{R}^n} Mf_j(x)^\theta g_j(x)\mathrm{d}x &\lesssim_{p,q} \sum_{j=1}^\infty \int_{\mathbb{R}^n} |f_j(x)|^\theta Mg_j(x)\mathrm{d}x \\ &\le \|\{|f_j|^\theta\}_{j=1}^\infty\|_{L^s(\ell^t)}{}^{\frac{s}{t}} \|\{Mg_j\}_{j=1}^\infty\|_{L^{s'}(\ell^{t'})}.\end{aligned}$$

The relation for s, t is reversed if we pass to the conjugate index; $s' > t'$. Thus, by Step 2, $\|\{Mg_j\}_{j=1}^\infty\|_{L^{s'}(\ell^{t'})} \lesssim 1$. Consequently, we have

$$\int_{\mathbb{R}^n} \|\{Mf_j(x)\}_{j=1}^\infty\|_{\ell^q}{}^p \mathrm{d}x \lesssim \int_{\mathbb{R}^n} \|\{f_j(x)\}_{j=1}^\infty\|_{\ell^q}{}^p \mathrm{d}x.$$

Therefore, the proof is complete.

We will show that the Fefferman–Stein vector-valued inequality carries over to various function spaces in this book.

Example 71. Let $\Phi \in \nabla_2 \cap \Delta_2$ and $1 < u < \infty$. Then using Lemmas 48 and 49, we can find $\theta, \theta^\dagger \in (1, \infty)$ such that $\displaystyle\int_r^\infty \frac{\Phi(s)}{s^{\theta+1}}\mathrm{d}s \lesssim \frac{\Phi(r)}{r^\theta}$ for all $r > 0$ and that $\displaystyle\int_0^r \frac{\Phi(t)}{t^{\theta^\dagger+1}}\mathrm{d}t \lesssim \frac{\Phi(r)}{r^{\theta^\dagger}}$ for all $r > 0$.

Let $\{f_j\}_{j=1}^\infty \in L^\Phi(\ell^u, \mathbb{R}^n)$. Then by the Fefferman–Stein vector valued inequality for Lebesgue spaces, we have

$$\begin{aligned}V_\lambda &\equiv \left|\left\{x \in \mathbb{R}^n : \|\{Mf_j\}_{j=1}^\infty\|_{\ell^u}(x) > \lambda\right\}\right| \qquad (4.16)\\ &\lesssim \lambda^{-\theta} \int_{\mathbb{R}^n} \|\{f_j\}_{j=1}^\infty\|_{\ell^u}(x)^\theta \chi_{(0,\lambda)}(\|\{f_j\}_{j=1}^\infty\|_{\ell^u}(x))\mathrm{d}x \\ &\quad + \lambda^{-\theta^\dagger} \int_{\mathbb{R}^n} \|\{f_j\}_{j=1}^\infty\|_{\ell^u}(x)^{\theta^\dagger} \chi_{[\lambda,\infty]}(\|\{f_j\}_{j=1}^\infty\|_{\ell^u}(x))\mathrm{d}x.\end{aligned}$$

Since $\Phi \in \Delta_2$, by the Layer-Cake formula

$$\int_{\mathbb{R}^n} \Phi(\|\{Mf_j\}_{j=1}^\infty\|_{\ell^u}(x))\mathrm{d}x \lesssim \int_{\mathbb{R}^n} \frac{\Phi(\lambda)}{\lambda} V_\lambda \mathrm{d}\lambda, \qquad (4.17)$$

if we insert (4.16) into (4.17), then we obtain the vector-valued modular inequality

$$\int_{\mathbb{R}^n} \Phi(\|\{Mf_j\}_{j=1}^\infty\|_{\ell^u}(x))\mathrm{d}x \lesssim \int_{\mathbb{R}^n} \Phi(\|\{f_j\}_{j=1}^\infty\|_{\ell^u}(x))\mathrm{d}x \tag{4.18}$$

holds. In particular, $\|\{Mf_j\}_{j=1}^\infty\|_{L^\Phi(\ell^u)} \lesssim \|\{f_j\}_{j=1}^\infty\|_{L^\Phi(\ell^u)}$ holds for all $\{f_j\}_{j=1}^\infty \in L^\Phi(\ell^u, \mathbb{R}^n)$.

4.1.5 Orlicz-maximal operators

As we will see, the Hardy–Littleowood maximal operators are used to control various linear operators. Sometimes we need to consider the two-fold iteration of this maximal operator. Since M is not $L^1(\mathbb{R}^n)$-bounded, it does not make sense to consider such an iteration in $L^1(\mathbb{R}^n)$. However, we have an equivalent expression of the composition of M in terms of the Orlicz-maximal operator. We first define the Orlicz-maximal operator. Recall that we defined $(\Phi; Q)$-average $\|f\|_{\Phi;Q}$ of $f \in L^0(Q)$ in Definition 26. Motivated by Definition 26, we define the Φ-maximal operator.

Definition 48. Let $\Phi : [0,\infty) \to (0,\infty)$ be a Young function. The (*Hardy–Littlewood*) Φ-*maximal operator* is given by $M_\Phi^{\mathcal{Q}} f \equiv \sup_{Q\in\mathcal{Q}} \chi_Q \|f\|_{\Phi;Q}$ for $f \in L^0(\mathbb{R}^n)$. Likewise define $M_\Phi^{\mathcal{B}}$. The symbol M_Φ denotes either $M_\Phi^{\mathcal{Q}}$ or $M_\Phi^{\mathcal{B}}$.

We do not have to stick to cubes. Instead we can use balls. However, here we use cubes because we need to bisect them. Since $L^\Phi(\mu)$ is a normed space for any measure space $(X, \mathcal{F}, \mu)$, M_Φ is a sublinear operator. Sometimes we are faced with the powered maximal operator after we use Hölder's inequality. Let $0 < \eta < \infty$, and let $f \in L^0(\mathbb{R}^n)$. We define the *powered Hardy–Littlewood maximal operator* $M^{(\eta)} = M^{(\eta),\mathcal{B}}$ by

$$M^{(\eta)} f(x) \equiv \sup_{R>0} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|^\eta \mathrm{d}y \right)^{\frac{1}{\eta}} = \sup_{R>0} m_{B(x,R)}^{(\eta)}(f)$$

for $x \in \mathbb{R}^n$. Likewise $M^{(\eta)} = M^{(\eta),\mathcal{Q}}$ is defined. We have the following example:

Example 72. Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function.

(1) Note that M and $M^{(u)}$ for $1 < u < \infty$ fall under the scope of this class of operators; simply take $\Phi(t) = t^u$ for $1 \le u < \infty$.

(2) Let $\alpha \in \mathbb{R}$ and let $\Phi : [0,\infty) \to [0,\infty)$ be a convex function satisfying $\Phi(t) = t(\log(e+t))^\alpha$. In this case we write $M_{\mathrm{L}(\log \mathrm{L})^\alpha} \equiv M_\Phi$. If $\Phi(t) = t\log(e+t)$, then we write $M_{\mathrm{L}\log\mathrm{L}} \equiv M_{\mathrm{L}(\log \mathrm{L})^\alpha}$.

(3) The function $\Phi(t) = e^t - 1$ is convex. In this case we write $M_{\exp \mathrm{L}} \equiv M_\Phi$.

Let us start with comparison with M and the Hardy–Littlewood Φ-maximal operator.

Corollary 147. *For any Young function $\Phi : [0,\infty) \to [0,\infty)$ and $f \in L^0(\mathbb{R}^n)$ $Mf \lesssim M_\Phi f$.*

Proof Simply compare the definition of Mf and $M_\Phi f$ using Lemma 55.

As before, the maximal operator M_Φ is bigger than the identity operator.

Corollary 148. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that Φ is a Young function. Then $|f(x)| \lesssim M_\Phi f(x)$ for almost all $x \in \mathbb{R}^n$.*

Proof This is clear since $|f(x)| \le Mf(x) \lesssim M_\Phi f(x)$ for any Lebesgue point x of f; see Lemma 142 and Corollary 147.

We investigate the boundedness property of the Orlicz maximal operators. We first establish the following weak-type inequality:

Proposition 149. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that $\Phi : [0,\infty) \to [0,\infty)$ is a Young function. Then for all $\lambda > 0$,*

$$|\{y \in \mathbb{R}^n \,:\, M_\Phi f(y) > \lambda\}| \le \frac{3^n}{\lambda} \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x.$$

Proof By the inner regularity (1.1) of the Lebesgue measure, it suffices to show

$$|K| \le 3^n \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \tag{4.19}$$

for any compact set K contained in $\{M_\Phi f > \lambda\}$.

Let $x \in K$ be fixed. By the definition of M, there exists $Q_x \in \mathcal{Q}$ containing x such that

$$\|f\|_{\Phi;Q_x} > \lambda. \tag{4.20}$$

By expanding Q_x slightly, we may assume $x \in \mathrm{Int}(Q_x)$.

By the compactness of K, there exists a finite collection $x_1, x_2, \dots, x_J$ of points such that $K \subset \bigcup_{j=1}^{J} Q_{x_j}$. To simplify the notation, write $Q_j \equiv Q_{x_j}$. By Theorem 132, if we relabel the Q_j's, for some $L \le J$, the collection $\{Q_j\}_{j=1}^{L}$ is disjoint, that is,

$$\sum_{j=1}^{L} \chi_{Q_j} \le 1, \tag{4.21}$$

and

$$K \subset \bigcup_{j=1}^{J} Q_j \subset \bigcup_{j=1}^{L} 3\,Q_j. \tag{4.22}$$

Hence, if we use (4.22), (4.20) and (4.21) in that order, then

$$|K| \le \sum_{j=1}^{L} |3Q_j| \le 3^n \sum_{j=1}^{L} \int_{Q_j} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \le \frac{3^n}{\lambda} \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x.$$

Thus, (4.19) and hence Proposition 149 are proven.

We generalize (4.6) and Theorem 137 as follows:

Proposition 150. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that $\Phi : [0,\infty) \to [0,\infty)$ is a bijective Young function. Then for all $\lambda > 0$,*

$$|\{y \in \mathbb{R}^n \,:\, M_\Phi f(y) > (1 + \Phi^{-1}(1))\lambda\}| \le \frac{3^n}{\lambda} \int_{\{y \in \mathbb{R}^n \,:\, |f(y)| > \lambda\}} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x.$$

Proof Since $\lambda > 0$ is fixed, we can go through a similar argument to (4.6) using Example 48.

In Proposition 150 we also have the opposite inequality.

Proposition 151. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that $\Phi : [0,\infty) \to [0,\infty)$ is a Young function. Then there exists $D \gg 1$ such that, for all $\lambda > 0$,*

$$|\{y \in \mathbb{R}^n \,:\, M^{\mathcal{Q}}_\Phi f(y) > \lambda\}| \ge \int_{\{y \in \mathbb{R}^n \,:\, |f(y)| > D\lambda\}} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x. \tag{4.23}$$

For the proof, we employ the dyadic maximal operator. Let $f \in L^0(\mathbb{R}^n)$. We define *dyadic maximal operator* $M^{\mathcal{D}}$ by

$$M^{\mathcal{D}} f(x) \equiv \sup_{Q \in \mathcal{D}(\mathbb{R}^n)} \chi_Q(x) m_Q(|f|) \quad (x \in \mathbb{R}^n).$$

As an extension of $M^{\mathcal{D}}$ consider the dyadic Φ-maximal operator: for all $f \in L^0(\mathbb{R}^n)$,

$$M^{\mathcal{D}}_\Phi f \equiv \sup_{Q \in \mathcal{D}(\mathbb{R}^n)} \|f\|_{\Phi;Q} \chi_Q. \tag{4.24}$$

Since $\mathcal{Q}(\mathbb{R}^n) \supset \mathcal{D}(\mathbb{R}^n)$, $M^{\mathcal{D}}_\Phi f \le M^{\mathcal{Q}}_\Phi f$.

Proof We have an analogy to Corollary 148: For almost all $x \in \mathbb{R}^n$ $|f(x)| (\le M^{\mathcal{D}} f(x)) \lesssim M^{\mathcal{D}}_\Phi f(x)$. It suffices to show that

$$|\{y \in \mathbb{R}^n \,:\, M^{\mathcal{D}}_\Phi f(y) > \lambda\}| \ge \int_{\{y \in \mathbb{R}^n \,:\, M^{\mathcal{D}}_\Phi f(y) > \lambda\}} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x.$$

We consider the maximal dyadic cubes Q contained in

$$\Omega_\lambda \equiv \{y \in \mathbb{R}^n \,:\, M^{\mathcal{D}}_\Phi f(y) > \lambda\}$$

satisfying $\|f\|_{\Phi;Q} > \lambda$. Let $\mathcal{S}$ be a collection of all such cubes. Then

$$\begin{aligned} |\{y \in \mathbb{R}^n \,:\, M^{\mathcal{D}}_\Phi f(y) > \lambda\}| &= \sum_{Q \in \mathcal{S}} |Q| \\ &\ge \sum_{Q \in \mathcal{S}} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \\ &= \int_{\{y \in \mathbb{R}^n \,:\, M^{\mathcal{D}}_\Phi f(y) > \lambda\}} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x, \end{aligned}$$

as required.

So far, we have seen how the operator M_Φ arises. Here, we are interested in showing that M_Φ is bounded on $L^p(\mathbb{R}^n)$ under some suitable conditions.We propose the following condition:

Definition 49. Let $1 \leq p < \infty$. A Young function $\Phi : [0,\infty) \to [0,\infty)$ belongs to the class B_p if $\displaystyle\int_1^\infty \frac{\Phi(t)}{t^{p+1}}\mathrm{d}t < \infty$. In this case write $\Phi \in B_p$.

Example 73. Let $1 < p < \infty$.

(1) Since $\Phi(t) \geq \Phi(1)t$ for any Young function $\Phi : [0,\infty) \to [0,\infty)$ and $t \geq 0$, we see that $B_1 = \emptyset$.

(2) Let $\Phi_1(t) = t^q$ for $1 \leq q < \infty$. Then $\Phi_1 \in B_p$ if and only if $q < p$.

(3) Let $1 \leq q < \infty$ and $r \in \mathbb{R}$. Let Φ_2 be a Young function such that $\Phi_2(t) \sim t^q \log(3+t)$ for $t \gg 1$. Then $\Phi_2 \in B_p$ if and only if $q < p$.

The class B_p is important because this class characterizes the boundedness of M_Φ on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Theorem 152. *Let $1 < p < \infty$, and let $\Phi : [0,\infty) \to [0,\infty)$ be a bijective Young function. Then M_Φ is bounded on $L^p(\mathbb{R}^n)$ if and only if $\Phi \in B_p$.*

Proof Suppose $\Phi \in B_p$. Let $C_0 \equiv 1 + \Phi^{-1}(1)$. Then by Theorem 5 and Proposition 150,

$$\begin{aligned}\int_{\mathbb{R}^n} M_\Phi f(y)^p \mathrm{d}y &\sim \int_0^\infty \lambda^{p-1} |\{y \in \mathbb{R}^n \,:\, M_\Phi f(y) > C_0\lambda\}| \mathrm{d}\lambda \\ &\lesssim \int_0^\infty \lambda^{p-2} \left(\int_{\{y\in\mathbb{R}^n \,:\, |f(y)|>\lambda\}} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \right) \mathrm{d}\lambda \\ &= \int_1^\infty \frac{\Phi(t)}{t^{p+1}}\mathrm{d}t \cdot \int_{\mathbb{R}^n} |f(y)|^p \mathrm{d}y.\end{aligned}$$

Conversely suppose that M_Φ is bounded on $L^p(\mathbb{R}^n)$. Let D be a constant as in Proposition 151. Then

$$\begin{aligned}\infty &> \int_{\mathbb{R}^n} M_\Phi \chi_{[0,1]^n}(y)^p \mathrm{d}y \\ &= p\int_0^\infty \lambda^{p-1} |\{y \in \mathbb{R}^n \,:\, M_\Phi \chi_{[0,1]^n}(y) > 2\lambda\}| \mathrm{d}\lambda \\ &\geq p\int_0^\infty \lambda^{p-1} \int_{\{y\in\mathbb{R}^n \,:\, \chi_{[0,1]^n}(y) > D\lambda\}} \Phi\left(\frac{\chi_{[0,1]^n}(x)}{\lambda}\right) \mathrm{d}x \\ &\sim \int_D^\infty \frac{\Phi(s)}{s^{p+1}} \mathrm{d}s,\end{aligned}$$

implying $\Phi \in B_p$.

4.1.6 Composition of the maximal operators

As is mentioned before, the Φ-maximal operators can be used for the purpose of the composition of the maximal operators. To see this, we start with localizing the sunrise lemma. To localize our estimate, it is useful to use the dyadic cubes for a fixed compact or right-open cube Q.

Definition 50 ($\mathcal{D}(Q)$)**.** Let $\mathcal{D}(Q)$ be the collection of all dyadic subcubes of Q. That is, all those cubes obtained by dividing Q into 2^n congruent cubes of half its length, dividing each of those into 2^n congruent cubes, and so on. For the cube Q and $k = 0, 1, \dots$ $\mathcal{D}_k(Q)$ is the set of all cubes obtained by bisecting Q k times. Write $\mathcal{D}(Q) \equiv \bigcup_{k=0}^{\infty} \mathcal{D}_k(Q)$. In particular, by convention, Q itself belongs to $\mathcal{D}_0(Q)$ and hence $\mathcal{D}(Q)$.

Example 74. Let $Q \equiv [-2^N, 2^N)^n$ for $N \in \mathbb{Z}$. Then $\mathcal{D}_1(Q) \subset \mathcal{D}(\mathbb{R}^n)$.

The following lemma is the localized version of the so-called "Wiener-Stein equivalence".

Lemma 153 (Sunrise lemma for M_Φ)**.** *Suppose that $\Phi : [0,\infty) \to [0,\infty)$ is a normalized Young function and that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$. Then, for $Q \in \mathcal{Q}$ and any number $t > \|f\|_{\Phi,Q}$,*

$$\begin{aligned} \int_{\{x\in Q:\, f(x)>t\}} \Phi\left(\frac{f(y)}{2^n t}\right) \mathrm{d}y &\leq |\{x \in Q :\, M_\Phi[f\chi_Q](x) > t\}| \\ &\lesssim \int_{\{x\in Q:\, 2f(x)>t\}} \Phi\left(\frac{2f(y)}{t}\right) \mathrm{d}y. \quad (4.25) \end{aligned}$$

In particular, if $\Phi \in \Delta_2$ then

$$\begin{aligned} \int_{\{x\in Q:\, f(x)>t\}} \Phi\left(\frac{f(y)}{t}\right) \mathrm{d}y &\leq |\{x \in Q :\, M_\Phi[f\chi_Q](x) > t\}| \\ &\lesssim \int_{\{x\in Q:\, 2f(x)>t\}} \Phi\left(\frac{f(y)}{t}\right) \mathrm{d}y. \end{aligned}$$

Proof For $x \in \mathbb{R}^n$ write $f_1(x) \equiv f(x)$ if $2f(x) > t$ and $f_1(x) = 0$ otherwise and let $f_2 \equiv f - f_1$. As we did in Proposition 150, using Example 48, we have

$$M_\Phi[f\chi_Q](x) \leq M_\Phi[f_1\chi_Q](x) + \frac{t}{2}.$$

This implies

$$\{x \in Q :\, M_\Phi[f\chi_Q](x) > t\} \subset \left\{x \in Q :\, M_\Phi[f_1\chi_Q](x) > \frac{t}{2}\right\}.$$

It follows from the infinite $5r$-covering lemma that there exists the set of disjoint cubes $\{Q_j\}$ such that

$$Q_j \subset Q, \quad \frac{t}{2} < \|f_1\|_{\Phi,Q_j}, \quad \left\{x \in Q :\, M_\Phi[f_1\chi_Q](x) > \frac{t}{2}\right\} \subset \bigcup_j 3Q_j.$$

We now see that $\frac{1}{|Q_j|}\int_{Q_j}\Phi\left(\frac{2f_1(x)}{t}\right)\mathrm{d}x > 1$; if it were not, then by the definition of the mean Luxemberg norm we would have $\frac{t}{2} \geq \|f_1\|_{\Phi,\,Q_j}$, which is a contradiction. This implies that $|Q_j| < \int_{Q_j}\Phi\left(\frac{2f_1(x)}{t}\right)\mathrm{d}x$ and that the right inequality of (4.25).

Since, $t > \|f\|_{\Phi,\,Q}$, there exists a disjoint collection of maximal (with inclusion) dyadic cubes $\{Q_j\} \subset \mathcal{D}(Q) \setminus \{Q\}$ such that

$$t < \|f\|_{\Phi,\,Q_j}, \quad \{x \in Q : M_\Phi[f\chi_Q](x) > t\} \supset \bigcup_j Q_j.$$

Let $\tilde{Q}_j \in \mathcal{D}(Q)$ be the dyadic parent of Q_j, the unique dyadic cube containing Q_j with side-length twice that of Q_j. Thus, from Example 47, we have $\|f\|_{\Phi,\,Q_j} \leq 2^n\|f\|_{\Phi,\,\tilde{Q}_j}$ and by the maximality of Q_j $\|f\|_{\Phi,\,Q_j} \leq 2^n\|f\|_{\Phi,\,\tilde{Q}_j} \leq 2^n t$. It follows from the definition of the mean Luxemberg norm and the fact that $\Phi \in \mathbb{M}^{\uparrow}[0,\infty)$ satisfies $1 \geq m_{Q_j}\left(\Phi\left(\frac{f}{\|f\|_{\Phi,\,Q_j}}\right)\right) \geq m_{Q_j}\left(\Phi\left(\frac{f}{2^n t}\right)\right)$. This yields that $|Q_j| \geq \int_{Q_j}\Phi\left(\frac{f(x)}{2^n t}\right)\mathrm{d}x$. Thus, we obtain the left inequality of (4.25) by observing further that $\bigcup_j Q_j \supset \{x \in Q : f(x) > t\}$, which holds by the Lebesgue differential theorem.

We use the following criteria to check that M_Ψ is bounded.

Lemma 154. *Let $\Psi : [0,\infty) \to [0,\infty)$ be a normalized Young function. Suppose that normalized Young functions Φ, Φ_1, $\Phi_2 : [0,\infty) \to [0,\infty)$ fulfill, for some positive constants C_1 and C_2,*

$$\Phi_1(C_1 t) - 1 \leq \int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\mathrm{d}s \leq \Phi_2(C_2 t) \quad (t > 1).$$

Then, for any compact $Q \in \mathcal{Q}$ and any $f \in \mathbb{M}^+(\mathbb{R}^n)$,

$$\|f\|_{\Phi_1,\,Q} \lesssim \|M_\Psi[f\chi_Q]\|_{\Phi;Q} \lesssim \|f\|_{\Phi_2,\,Q}.$$

Hence, $M_{\Phi_1} f \lesssim M_\Phi \circ M_\Psi f \lesssim M_{\Phi_2} f$. In particular, the boundedness of M_Ψ is independent of the values of $\Phi(t)$ and $\Psi(t)$, $0 < t \leq 1$.

Proof First, we verify $\|M_\Psi[f\chi_Q]\|_{\Phi;Q} \lesssim \|f\|_{\Phi_2,\,Q}$. Let $t > 0$. By virtue of Lemma 153, $|\{x \in Q : M_\Psi[f\chi_Q](x) > t\}| \lesssim \int_{Q\cap\{2f>t\}}\Psi\left(\frac{2f(x)}{t}\right)\mathrm{d}x$. It

follows from this inequality that

$$\begin{aligned}
&\int_Q \Phi\left(M_\Psi[f\chi_Q](x)\right)\mathrm{d}x \\
&= \int_0^\infty |\{x\in Q:\, M_\Psi[f\chi_Q](x)>t\}|\Phi'(t)\mathrm{d}t \\
&\le |Q| + C\int_1^\infty \left(\int_{Q\cap\{2f>t\}} \Psi\left(\frac{2f(x)}{t}\right)\mathrm{d}x\right)\Phi'(t)\mathrm{d}t \\
&= |Q| + C\int_{Q\cap\{2f>1\}} \left(\int_1^{2f(x)} \Psi\left(\frac{2f(x)}{t}\right)\Phi'(t)\mathrm{d}t\right)\mathrm{d}x \\
&\le |Q| + C\int_Q \Phi_2(2C_2 f(x))\mathrm{d}x.
\end{aligned}$$

For any $\lambda>0$, replacing $f(x)$ by $\frac{f(x)}{\lambda}$, we have

$$m_Q\left(\Phi\left(\frac{M_\Psi[f\chi_Q]}{\lambda}\right)\right) \le 1 + Cm_Q\left(\Phi_2\left(\frac{2C_2 f}{\lambda}\right)\right).$$

This yields, by the definition of the mean Luxemburg norm, for some $C>1$

$$\frac{1}{|Q|}\int_Q \Phi\left(\frac{M_\Psi[f\chi_Q](x)}{C\|f\|_{\Phi_2,\,Q}}\right)\mathrm{d}x \le 1,$$

which proves the desired inequality.

Next, we verify the remaining inequality. Without loss of generality, we may assume $\|M_\Psi[f\chi_Q]\|_{\Phi;Q}=1$. This means that

$$\frac{1}{|Q|}\int_Q \Phi\left(M_\Psi[f\chi_Q](x)\right)\mathrm{d}x \le 1.$$

We now claim that then $\|f\|_{\Psi,\,Q}\le 1$. If it were not, then we must have that the above integral mean is bigger than one, which contradicts our normalization above, by virtue of the fact that for almost every $x\in Q$ $\|f\|_{\Psi,\,Q}\le M_\Psi[f\chi_Q](x)$.

We claim

$$\|f\|_{\Phi_1,\,Q} \lesssim 1. \tag{4.26}$$

In fact, since $\|f\|_{\Psi,\,Q}\le 1$,

$$\begin{aligned}
|Q| &\ge \int_Q \Phi\left(M_\Psi[f\chi_Q](x)\right)\mathrm{d}x \\
&= \int_0^\infty |\{x\in Q:\, M_\Psi[f\chi_Q](x)>t\}|\Phi'(t)\mathrm{d}t.
\end{aligned}$$

Thanks to Lemma 153,

$$\begin{aligned} |Q| &\geq \int_1^\infty \left(\int_{Q\cap\{f>t\}} \Psi\left(\frac{f(x)}{2^n t}\right) \mathrm{d}x \right) \Phi'(t)\mathrm{d}t \\ &= \int_{Q\cap\{f>1\}} \left(\int_1^{f(x)} \Psi\left(\frac{f(x)}{2^n t}\right) \Phi'(t)\mathrm{d}t \right) \mathrm{d}x. \end{aligned}$$

By our assumption,

$$\begin{aligned} |Q| &\geq \int_{Q\cap\{f>2^n\}} \left(\int_1^{f(x)/2^n} \Psi\left(\frac{f(x)}{2^n t}\right) \Phi'(t)\mathrm{d}t \right) \mathrm{d}x \\ &\geq \int_{Q\cap\{f>2^n\}} \Phi_1(2^{-n}C_1 f(x))\mathrm{d}x - |Q\cap\{f>2^n\}| \\ &\geq \int_Q \Phi_1(2^{-n}C_1 f(x))\mathrm{d}x - (\Phi_1(C_1)+1)|Q|. \end{aligned}$$

Let $C \equiv \dfrac{\Phi_1(C_1)+2}{2^{-n}C_1}$. Then the above estimate reads $\dfrac{1}{|Q|}\displaystyle\int_Q \Phi_1\left(\frac{f(x)}{C}\right)\mathrm{d}x \leq 1$ and, hence, (4.26) follows. The proof is now complete.

Example 75. Let $Q \in \mathcal{Q}$, and let $f \in L^0(Q)$. Then $m_Q(M[\chi_{3Q}f]) \lesssim \|f\|_{\mathrm{L}\log\mathrm{L},3Q}$ from Lemma 75 with $\Phi(t) = \Psi(t) = t$, $t \geq 0$.

As a special case of $\Phi_1(t) = \Phi_2(t) = t\log(e+t)$ and $\Phi(t) = \Psi(t) = t$, $t \geq 0$, we have the equivalent expression of the composition of the maximal operators.

Corollary 155. *One has $M \circ Mf \simeq M_{\mathrm{L}\log\mathrm{L}} f$ for any $f \in L^0(\mathbb{R}^n)$.*

Proof Simply combine Example 65 and Lemma 75 with $\Phi(t) = \Psi(t) = t$ and $\Phi_1(t) = \Phi_2(t) = t\log(e+t)$, $t \geq 0$.

Example 76. For any $f \in L^0(\mathbb{R}^n)$ and $\lambda > 0$,

$$|\{x \in \mathbb{R}^n \,:\, M\circ Mf(x) > \lambda\}| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left(3 + \frac{|f(x)|}{\lambda}\right) \mathrm{d}x$$

from Proposition 150 and Corollary 155.

4.1.7 Local boundednss of the Φ-maximal operators

As an application of the results in the previous section, we further investigate the property of M_Ψ; we will obtain its local boundedness. We recall that by a normalized Young function, we mean a Young function Φ satisfying $\Phi(1) = 1$.

Proposition 156. *Let $\Phi, \Psi : [0,\infty) \to [0,\infty)$ be normalized Young functions. Then the following are equivalent.*

(1) *The maximal operator M_Ψ is locally bounded in the norm determined by Φ. That is,*

$$\|M_\Psi f\|_{\Phi;Q} \lesssim \|f\|_{\Phi;Q} \tag{4.27}$$

for all measurable functions f and cubes Q.

(2) *There exists a constant $C > 0$ such that*

$$\int_1^t \Psi\left(\frac{t}{s}\right) \Phi'(s)\mathrm{d}s \le \Phi(Ct) \quad (t \ge 1). \tag{4.28}$$

(3) *There exists a constant $C > 0$ such that*

$$\int_t^\infty \Phi\left(\frac{t}{s}\right) \Psi'(s)\mathrm{d}s \le \Phi(Ct) \quad (t \ge 1). \tag{4.29}$$

Proof We have already proven that (4.28) implies (4.27); see Lemma 154. First, we verify (4.27) implies (4.28). Suppose that (4.27), namely, assume that there exists a constant $C_0 \ge 1$ such that $\displaystyle\int_Q \Phi\left(\frac{M_\Psi[f\chi_Q](x)}{C_0\|f\|_{\Phi;Q}}\right)\mathrm{d}x \le |Q|$. We now claim that then $\|f\|_{\Psi,Q} \lesssim C_0\|f\|_{\Phi;Q}$. If it were not, then we would have that the above integral mean is bigger than one, which contradicts our normalization above, by virtue of the fact that $\|f\|_{\Psi,Q} \le M_\Psi[f\chi_Q](x)$ for almost every $x \in Q$. Thus we have by Lemma 154

$$\begin{aligned}
|Q| &\ge \int_Q \Phi\left(\frac{M_\Psi[f\chi_Q](x)}{C_0\|f\|_{\Phi;Q}}\right)\mathrm{d}x \\
&= \int_0^\infty |\{x \in Q : M_\Psi[f\chi_Q](x) > C_0\|f\|_{\Phi;Q}\cdot s\}|\,\Phi'(s)\mathrm{d}s \\
&\ge \int_1^\infty \int_{Q\cap\{|f|>C_0\|f\|_{\Phi;Q}\cdot s\}} \Psi\left(\frac{|f(x)|}{2^nC_0\|f\|_{\Phi;Q}\cdot s}\right)\mathrm{d}x\,\Phi'(s)\mathrm{d}s \\
&\ge \int_{Q\cap\{|f|>2^nC_0\|f\|_{\Phi;Q}\}} \int_1^{\frac{|f(x)|}{2^nC_0\|f\|_{\Phi;Q}}} \Psi\left(\frac{|f(x)|}{2^nC_0\|f\|_{\Phi;Q}\cdot s}\right)\Phi'(s)\mathrm{d}s\mathrm{d}x.
\end{aligned}$$

If we test the above inequality on $f \equiv \chi_R$ with some appropriate $R \in \mathcal{Q}(Q)$ and let $t \equiv (2^nC_0\|f\|_{\Phi;Q})^{-1}$, then $\displaystyle\int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\mathrm{d}s \le \frac{|Q|}{|R|}$. Observing that $\Phi(2^nC_0t) = \Phi\left(\dfrac{1}{\|f\|_{\Phi;Q}}\right) = \dfrac{|Q|}{|R|}$, we obtain (4.28).

Next, we verify (4.29) implies (4.28). Carrying out integration by parts, we have

$$\begin{aligned}\int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\mathrm{d}s &= \left[\Psi\left(\frac{t}{s}\right)\Phi(s)\right]_1^t + t\int_1^t \Psi'\left(\frac{t}{s}\right)\Phi(s)\,\frac{\mathrm{d}s}{s^2}\\ &= \Phi(t)-\Psi(t)+\int_t^\infty \Phi\left(\frac{t}{s}\right)\Psi'(s)\mathrm{d}s\\ &\le \Phi(t)+\Phi(Ct)\le\Phi(Ct).\end{aligned}$$

Here, we have used changed variables $\frac{t}{s}\mapsto s$. The converse, (4.28) implies (4.29), is similar, once we notice that

$$\begin{aligned}\Psi(t) &\le \frac{1}{\Phi(2)-\Phi(1)}\int_1^2 \Psi\left(\frac{2t}{s}\right)\Phi'(s)\mathrm{d}s\\ &\le \frac{1}{\Phi(2)-\Phi(1)}\int_1^{2t} \Psi\left(\frac{2t}{s}\right)\Phi'(s)\mathrm{d}s\\ &\le \frac{1}{\Phi(2)-\Phi(1)}\Phi(2Ct)\\ &\le \Phi(Ct), t>1.\end{aligned}$$

Thus, the proof is complete.

Earlier we investigated the condition on Φ for which M_Φ is bounded on $L^p(\mathbb{R}^n)$. Here, we present the opposite type of boundedness: We investigate when the maximal operator $M^{(p)}$ is locally bounded in the norm determined by Φ.

Theorem 157. *Let $\Phi:[0,\infty)\to[0,\infty)$ be a normalized Young function, and let $1\le p<\infty$. Suppose that $s^{1-p}\Phi'(s)$, $s>1$, is nondecreasing. Then the following are equivalent*:

(a) *The maximal operator $M^{(p)}$ is locally bounded in the norm determined by Φ.*

(b) *There exists a constant $C>0$ such that $t^p\displaystyle\int_1^t \frac{\Phi'(s)}{s^p}\mathrm{d}s\le\Phi(Ct)$ for all $t>1$.*

(c) *There exists some constant $K>1$ such that $2K^p\Phi(t)\le\Phi(Kt)$ for all $t>1$.*

Proof We have already proven the equivalence between (a) and (b) in Proposition 156; simply let $\Psi(t)\equiv t^p$. We verify that (b) implies (c). By assumption, $s^{1-p}\Phi'(s)$ is nondecreasing, we have for $\mu>1$

$$\Phi(C\mu t)\ge(\mu t)^p\int_t^{\mu t}\frac{\Phi'(s)}{s^p}\mathrm{d}s\ge\mu^p t\Phi'(t)\int_t^{\mu t}\frac{1}{s}\mathrm{d}s\ge\mu^p\Phi(t)\log\mu,$$

where we have used the fact that $\frac{\Phi(t)}{t} \leq \Phi'(t)$ which holds from the convexity. It follows by letting μ be big enough so that $C^{-p}\log\mu > 2$ and then, by setting $K = C\mu$ so that

$$2K^p\Phi(t) \leq \Phi(Kt).$$

This is the desired inequality.

We verify the converse: We verify (c) implies (b). If we let $N \equiv [1+\log_K t]$, then

$$\begin{aligned}\int_1^t \frac{\Phi'(s)}{s^p}\mathrm{d}s &\leq \sum_{j=0}^{N}\int_{K^{-j-1}t}^{K^{-j}t}\frac{\Phi'(s)}{s^p}\mathrm{d}s \leq \sum_{j=0}^{N}\frac{\Phi(K^{-j}t)}{(K^{-j-1}t)^p}\\ &= \frac{K^p}{t^p}\sum_{j=0}^{N}K^{pj}\Phi(K^{-j}t) \leq \frac{K^p}{t^p}\Phi(t)\sum_{j=0}^{N}2^{-j} \leq \frac{1}{t^p}\Phi(2K^pt).\end{aligned}$$

This proves the desired inequality.

As a special case we can recover the classical result on the ∇_2-condition.

Corollary 158. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function.*

(1) *Let $\Phi \in \nabla_2$. Then $\|M[f\chi_Q]\|_{\Phi;Q} \approx \|f\|_{\Phi;Q}$ for all $f \in L^0(\mathbb{R}^n)$.*

(2) *Conversely, if the above estimate is true, the $\Phi \in \nabla_2$.*

Proof

(1) We can readily check that Φ satisfies (b) with $p = 1$ in Theorem 157 using Lemma 49.

(2) Simply observe that (c) with $p = 1$ in Theorem 157 is equivalent to $\Phi \in \nabla_2$.

4.1.8 Estimates for convolutions

Here we collect the estimates for convolutions. Recall that the convolution $f * g$ is given by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y$$

as long as the integral makes sense.

We present an auxiliary estimate using the Hardy–Littlewood maximal operator.

Proposition 159. *Suppose that $\rho \in \mathbb{M}^{\downarrow}[0,\infty)$ and that $\tau \in \mathbb{M}^{+}(\mathbb{R}^n)$ is given by*

$$\tau(x) = \rho(|x|) \text{ for all } x \in \mathbb{R}^n. \tag{4.30}$$

Then for all $t > 0$ $|\tau_t * f(x)| \leq \|\tau\|_{L^1}Mf(x)$. for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

Proof This simply paraphrases Theorem 131.

We will provide a method to transform $|x-y|^{-\beta}$ to another expression here.

The following lemma is interesting in its own right.

Lemma 160. *Let $\beta > 0$. Then for all $x, y \in \mathbb{R}^n$,*

$$\int_0^\infty \chi_{B(x,l)}(y)\,\frac{\mathrm{d}l}{l^{\beta+1}} = \frac{|x-y|^{-\beta}}{\beta}.$$

Proof The proof is simple. Indeed, we have

$$\int_0^\infty \chi_{B(x,l)}(y)\,\frac{\mathrm{d}l}{l^{\beta+1}} = \int_0^\infty \chi_{(|x-y|,\infty)}(l)\,\frac{\mathrm{d}l}{l^{\beta+1}} = \int_{|x-y|}^\infty \frac{\mathrm{d}l}{l^{\beta+1}} = \frac{|x-y|^{-\beta}}{\beta}$$

by using Fubini's theorem, which is the desired result.

Here we present applications of Lemma 160.

Lemma 161. *Let $\beta \geq 0$. Let $\varphi \in \mathbb{M}^+(\mathbb{R}^n)$. Then for all $r \geq 0$,*

$$\int_{\mathbb{R}^n\setminus B(r)} \frac{\varphi(x)}{|x|^\beta}\mathrm{d}x = \beta \int_r^\infty \left(\int_{B(t)\setminus B(r)} \varphi(x)\mathrm{d}x \right) \frac{\mathrm{d}\ell}{\ell^{\beta+1}}.$$

In particular, for all $z \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \frac{\varphi(x)}{|x-z|^\beta}\mathrm{d}x = \beta \int_0^\infty \left(\int_{B(z,t)} \varphi(x)\mathrm{d}x \right) \frac{\mathrm{d}\ell}{\ell^{\beta+1}}.$$

Proof By Lemma 160 and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B(r)} \frac{\varphi(x)}{|x|^\beta}\mathrm{d}x &= \beta \int_{\mathbb{R}^n\setminus B(r)} \varphi(x) \left(\int_0^\infty \chi_{B(l)}(x)\,\frac{\mathrm{d}l}{l^{\beta+1}} \right) \mathrm{d}x \\
&= \beta \int_0^\infty \left(\int_{\mathbb{R}^n} \varphi(x)\chi_{B(l)\setminus B(r)}(x)\,\frac{\mathrm{d}x}{l^{\beta+1}} \right) \mathrm{d}l \\
&= \beta \int_r^\infty \left(\int_{B(l)\setminus B(r)} \varphi(x)\,\frac{\mathrm{d}x}{l^{\beta+1}} \right) \mathrm{d}l.
\end{aligned}$$

The second formula is a translation of the first formula with $r = 0$ by z.

We transform Lemma 161 to handle a different type of function.

Corollary 162. *Let $\beta > 0$ and $r > 0$. Let $\varphi \in \mathbb{M}^+(\mathbb{R}^n)$. Then*

$$\begin{aligned}
\beta 2^{-\beta} \int_r^\infty \left(\int_{B(t)} \varphi(x)\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{\beta+1}} &\leq \int_{\mathbb{R}^n} \frac{\varphi(x)\mathrm{d}x}{(|x|+r)^\beta} \\
&\lesssim_\beta \int_r^\infty \left(\int_{B(t)} \varphi(x)\mathrm{d}x \right) \frac{\mathrm{d}t}{t^{\beta+1}}.
\end{aligned}$$

Proof We note that

$$\int_{\mathbb{R}^n} \frac{\varphi(x)}{(|x|+r)^\beta}\mathrm{d}x \le r^{-\beta}\int_{B(r)} \varphi(x)\mathrm{d}x + \int_{\mathbb{R}^n\setminus B(r)} \frac{\varphi(x)}{|x|^\beta}\mathrm{d}x$$

and that

$$\int_{\mathbb{R}^n} \frac{\varphi(x)}{(|x|+r)^\beta}\mathrm{d}x \ge 2^{-\beta} r^{-\beta}\int_{B(r)} \varphi(x)\mathrm{d}x.$$

If we decompose the integral dyadically in the second term in the right-hand side and use Lemma 161 and

$$\beta\int_r^\infty \left(\int_{B(r)} \varphi(x)\mathrm{d}x\right)\frac{\mathrm{d}\ell}{\ell^{\beta+1}} = r^{-\beta}\int_{B(r)} \varphi(x)\mathrm{d}x,$$

then we obtain the desired result.

We collect some other convolution estimates of convolution type.

Proposition 163. *Let $\gamma \in \mathbb{R}$. Then for all $x \in \mathbb{R}^n$ and $r, \rho > 0$,*

$$\begin{aligned}
&\int_{B(r)\setminus B(x,\rho)} |x-y|^\gamma \mathrm{d}y \\
&\lesssim \rho^\gamma \min(r^n, \rho^n)\chi_{(0,|x|+r)}(\rho)\begin{cases} 1 & (\gamma < -n), \\ \log\left(3 + \dfrac{|x|+r}{\rho}\right) & (\gamma = -n), \\ \left(\dfrac{|x|+r}{\rho}\right)^{\gamma+n} & (\gamma > -n). \end{cases}
\end{aligned}$$

Proof We may suppose $\rho < |x| + r$; otherwise $B(x,\rho) \supset B(r)$. Let $m = \left[\log_2 \frac{|x|+r}{\rho}\right]_+$. For $k > m$, we note $B(r) \cap \left(B(x, 2^{k+1}\rho)\setminus B(x, 2^k\rho)\right) = \emptyset$, since $2^k\rho \ge |x| + r$. Hence $B(r) \subset B(x, 2^k\rho)$. Thus,

$$\begin{aligned}
\int_{B(r)\setminus B(x,\rho)} |x-y|^\gamma \mathrm{d}y &= \sum_{k=0}^m \int_{B(r)\cap(B(x,2^{k+1}\rho)\setminus B(x,2^k\rho))} |x-y|^\gamma \mathrm{d}y \\
&\lesssim \sum_{k=0}^m (2^k\rho)^\gamma \int_{B(r)\cap B(x,2^{k+1}\rho)} \mathrm{d}y \\
&\lesssim \sum_{k=0}^m (2^k\rho)^\gamma \min(r^n, 2^{(k+1)n}\rho^n) \\
&\lesssim \rho^\gamma \min(r^n,\rho^n)\sum_{k=0}^m 2^{k(\gamma+n)},
\end{aligned}$$

and the desired result follows since for any $t \in \mathbb{R}$,

$$\sum_{k=0}^m 2^{kt} \le \begin{cases} \dfrac{1}{1-2^t} & (t<0), \\ m+1 & (t=0), \\ \dfrac{2^{(m+1)t}}{2^t-1} & (t>0). \end{cases}$$

We also collect another type of estimate.

Proposition 164. *Let $\beta \geq 0$, $x \in \mathbb{R}^n$, $\rho > 0$ and $r > 0$.*

(1) *If $\gamma < -\beta - n$, then*

$$\int_{\mathbb{R}^n \setminus B(x,\rho)} |x-y|^\gamma \, (|y|+r)^\beta \, \mathrm{d}y \lesssim (|x|+\rho+r)^\beta \rho^{\gamma+n}.$$

(2) *If $\gamma < -\beta$, then*

$$\sup_{y \in \mathbb{R}^n \setminus B(x,\rho)} |x-y|^\gamma \, (|y|+r)^\beta \leq 2^\beta (|x|+\rho+r)^\beta \rho^\gamma.$$

Proof

(1) We decompose

$$\begin{aligned} \mathrm{I} &\equiv \int_{\mathbb{R}^n \setminus B(x,\rho)} |x-y|^\gamma (|y|+r)^\beta \mathrm{d}y \\ &= \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}\rho) \setminus B(x,2^k\rho)} |x-y|^\gamma (|y|+r)^\beta \mathrm{d}y \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{I} &\leq \rho^\gamma \sum_{k=0}^{\infty} 2^{k\gamma} \int_{B(x,2^{k+1}\rho) \setminus B(x,2^k\rho)} (|y|+r)^\beta \mathrm{d}y \\ &\lesssim \rho^\gamma \sum_{k=0}^{\infty} 2^{k\gamma} \left(2^{k+1}\rho\right)^n \left(|x| + 2^{k+1}\rho + r\right)^\beta \\ &= 2^{n+\beta} \rho^{\gamma+n} \left(|x|+\rho+r\right)^\beta \sum_{k=0}^{\infty} 2^{k(\gamma+\beta+n)} \\ &\sim \rho^{\gamma+n} \left(|x|+\rho+r\right)^\beta. \end{aligned}$$

(2) Mimic the proof above.

4.1.9 Exercises

Exercise 61. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Also let $a > 0$ and $g : \mathbb{R} \to [0,\infty)$ is a function such that $g|[a,\infty)$ is decreasing and $g|(-\infty,a]$ is increasing. Then show that $\|f \cdot g\|_{L^1} \leq \|g\|_{L^1} Mf(a)$ by mimicking the proof of Proposition 159. Here M denotes the uncentered Hardy–Littlewood maximal operator.

Exercise 62. Let $0 < a < b < 1$, and let Q be a cube. Then show that $M^{\mathcal{Q}} \chi_{bQ \setminus aQ} \geq (b^n - a^n) \chi_Q$ by considering $m_Q(\chi_{bQ \setminus aQ})$.

Exercise 63. Let μ be a finite Radon measure on $\mathbb{R}^n$ and $\lambda > 0$.

(1) Let K be a compact set. Show that there exists a finite collection $\{\tilde{Q}_j\}_{j=1}^{\tilde{J}}$ of cubes such that $K \subset \bigcup_{j=1}^{\tilde{J}} \tilde{Q}_j$ and that $\mu(\tilde{Q}_j) \geq \lambda |\tilde{Q}_j|$ for all $j = 1, 2, \dots, \tilde{J}$.

(2) Let K be a compact set. Use the finite $5r$-covering lemma (Theorem 132) to show that there exists a disjoint finite collection $\{Q_j\}_{j=1}^{J}$ of cubes such that $K \subset \bigcup_{j=1}^{J} 3Q_j$ and that $\mu(Q_j) \geq \lambda |Q_j|$ for all $j = 1, 2, \dots, J$.

(3) Prove (4.8).

Exercise 64. Let $k \in \mathbb{N}_0$ and denote by $P^k_{Q(x,r)} f$ the *Gram–Schmidt polynomial* of order k for a cube $Q(x, r)$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

(1) Prove that $|P^k_{Q(x,r)} f(x)| \lesssim Mf(x)$ using Proposition 117.

(2) Prove Lemma 143 by mimicking the proof of Lemma 142.

4.2 Sharp maximal operators

On many occasions, we are faced with the necessity of estimating the norm of functions. Especially we need to control the image of functions in terms of the norms. It is often difficult to do this because the definition of operators is complicated. However, when we can use the mean-value theorem, the sharp maximal operator, which is defined by the difference of functions, is a useful tool. In some sense, we decompose measurable functions into the sum of less oscillating functions. Here, we investigate the sharp maximal operator to be able to deal with and take full advantage of the oscillating properties of functions and operators.

We define the sharp maximal operator and then investigate some fundamental inequality in Section 4.2.1. In Section 4.2.2 we will expand the idea employed in Section 4.2.1 so that it is applicable to any measurable functions. Section 4.2.3 is a further development of Sections 4.2.1 and 4.2.2 and considers a fundamental pointwise estimate of measurable functions.

4.2.1 Sharp-maximal inequalities

Here, we present an important estimate from the viewpoint given above. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We define the *sharp maximal operator* $M^\sharp f$ by

$$M^\sharp f \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q}{|Q|} \int_Q |f(y) - m_Q(f)| \mathrm{d}y = \sup_{Q \in \mathcal{Q}} m_Q(|f - m_Q(f)|) \chi_Q.$$

A trivial but interesting observation is that

$$\int_{Q_j^l} |f(x) - m_{Q_j^l}(f)| \mathrm{d}x \le \int_{Q_j^l} M^\sharp f(x) \mathrm{d}x. \tag{4.1}$$

We prove an estimate which is useful when we control the singularity of functions:

Theorem 165 (Sharp-maximal inequality)**.** *Let $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then if*

$$|\{|f| > \lambda\}| < \infty \tag{4.2}$$

for all $\lambda > 0$, $\|f \cdot g\|_{L^1} \le 6 \|M^\sharp f \cdot M^{\mathcal{D}} g\|_{L^1}$.

Proof Let $k \in \mathbb{Z}$ be fixed. We can assume that $f \in L^\infty(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ and that $g \in L^\infty_{\mathrm{c}}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ by the truncation argument. We partition $\{M^{\mathcal{D}} g > 2^k\}$ into a collection of maximal dyadic cubes such that $m_{Q_j^k}(g) > 2^k$. We write $\{M^{\mathcal{D}} g > 2^k\} = \prod_{j \in J^k} Q_j^k$, where $\{Q_k^j\}_{j \in J^k}$ is a disjoint collection of dyadic cubes. We will form a Calderón–Zygmund decomposition with respect to $\mathcal{D}$. Define

$$G^k \equiv \chi_{\{M^{\mathcal{D}} g \le 2^k\}} g + \sum_{j \in J^k} m_{Q_k^j}(g) \chi_{Q_j^k}, \quad B^k \equiv g - G^k.$$

We will move $k \in \mathbb{Z}$ now. For each $j' \in J^{k+1}$, there exists $j \in J^k$ such that $Q_{j'}^{k+1} \subset Q_j^k$. Therefore, we have $\int_{Q_j^k} (B^{k+1}(x) - B^k(x)) \mathrm{d}x = 0$ for all $j \in J^k$. In view of the Lebesgue differentiation theorem, $|B^k(x) - B^{k+1}(x)| = |G^k(x) - G^{k+1}(x)| \le 3 \cdot 2^{n+k}$ for almost all $x \in \mathbb{R}^n$. Since $g \in L^\infty(\mathbb{R}^n)$, $B^k \equiv 0$ if $k \gg 1$. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x) g(x) \mathrm{d}x &= \int_{\mathbb{R}^n} f(x) G^k(x) \mathrm{d}x + \int_{\mathbb{R}^n} f(x) B^k(x) \mathrm{d}x \\
&= \int_{\mathbb{R}^n} f(x) G^k(x) \mathrm{d}x + \sum_{l=k}^{\infty} \int_{\mathbb{R}^n} f(x) (B^l(x) - B^{l+1}(x)) \mathrm{d}x \\
&=: \mathrm{I}_k + \mathrm{II}_k.
\end{aligned}$$

We will show that $\lim_{k \to -\infty} \mathrm{I}_k = 0$. Let $\varepsilon > 0$ be arbitrary. We split

$$\mathrm{I}_k = \int_{\{f > \varepsilon\}} f(x) G^k(x) \mathrm{d}x + \int_{\{f \le \varepsilon\}} f(x) G^k(x) \mathrm{d}x.$$

Choose $k \in \mathbb{Z} \cap (-\infty, \log_2 \varepsilon)$. Note again that $|G^k(x)| \le 2^{k+n}$ almost everywhere by the maximality of the cubes Q_j^k and the Lebesgue differentiation theorem. Since

$$\mathrm{I}_k \le 2^{k+n} \|f\|_{L^\infty} |\{f > \varepsilon\}| + \varepsilon \|g\|_{L^1},$$

it follows from (4.2) that $\lim_{k\to-\infty} \mathrm{I}_k = 0$. As for II_k, we have

$$|\mathrm{II}_k| \le \sum_{l=k}^{\infty} \sum_{j\in J^l} \int_{Q_j^l} |f(x) - m_{Q_j^l}(f)| \cdot |B^l(x) - B^{l+1}(x)| \mathrm{d}x$$
$$\le 3 \sum_{l=k}^{\infty} 2^{n+l} \sum_{j\in J^l} \int_{Q_j^l} |f(x) - m_{Q_j^l}(f)| \mathrm{d}x.$$

We deduce $|\mathrm{II}_k| \le 3 \sum_{l=-\infty}^{\infty} 2^{n+l} \int_{\{M^{\mathcal{D}} g > 2^l\}} M^\sharp f(x) \mathrm{d}x \le 6 \|M^\sharp f \cdot M^{\mathcal{D}} g\|_{L^1}$ from (4.1). The proof is therefore complete.

We transform Theorem 165 into the form which we actually use.

Theorem 166 (Sharp maximal inequality)**.** *Let $1 < p < \infty$. Then $\|f\|_{L^p} \lesssim_p \|M^\sharp f\|_{L^p}$ for $f \in L^0(\mathbb{R}^n)$ with $\min(Mf, 1) \in L^p(\mathbb{R}^n)$.*

We really have to assume that $\min(1, Mf) \in L^p(\mathbb{R}^n)$ as the example of the function $f \equiv 1$ shows.

Proof By truncation, we may suppose that $f \in L^\infty(\mathbb{R}^n)$. In this case, $Mf \in L^p(\mathbb{R}^n)$. Hence by Corollary 148, we have $f \in L^p(\mathbb{R}^n)$, so that (4.2) is satisfied. Let $w \in L^{p'}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ be a function with norm 1 satisfying

$$\|f\|_{L^p} \le 2 \int_{\mathbb{R}^n} f(x) w(x) \mathrm{d}x.$$

By Theorem 165, we have $\|f\|_{L^p} \le 12 \|M^\sharp f \cdot M^{\mathcal{D}} w\|_{L^1}$. It remains to use Hölder's inequality and the boundedness of $M^{\mathcal{D}}$.

We consider a pointwise estimate of the oscillation of functions.

We use the following notation: $\mathcal{D}(Q_0)$ is the set of all dyadic cubes with respect to a cube Q_0. The *mean oscillation* of $f \in L^0(Q)$ of level $\lambda \in (0,1)$ is given by $\omega_\lambda(f; Q) \equiv \inf_{c\in\mathbb{C}} ((f-c)\chi_Q)^*(\lambda|Q|)$, where $*$ denotes the decreasing rearrangement for functions.

Definition 51 ($M^{\sharp,\mathcal{D}}_{\lambda;Q_0}$)**.** Let $0 < \lambda < 1$ and $Q_0 \in \mathcal{Q}$ be fixed, and define the *local distributional dyadic sharp maximal operator* or the *distributional dyadic sharp maximal operator* in Q_0 by $M^{\sharp,\mathcal{D}}_{\lambda;Q_0} f \equiv \sup_{Q\in\mathcal{D}(Q_0)} \omega_\lambda(f; Q)\chi_Q$ for $f \in L^0(\mathbb{R}^n)$.

4.2.2 Distributional maximal function and median

When we consider the Hardy–Littlewood maximal operator, we need to assume that f is locally integrable; otherwise $Mf \equiv \infty$. This is somewhat inconvenient. For example, $Mf, f \in L^1(\mathbb{R}^n) \setminus \{0\}$ is merely in the space

$\mathrm{W}L^1(\mathbb{R}^n) = L^{1,\infty}(\mathbb{R}^n)$, so that it is not always locally integrable. Hence, we are interested in the maximal operator which yields something for any measurable function. Recall that the decreasing rearrangement of $f \in L^0(\mathbb{R}^n)$ is given as follows:

$$f^*(t) \equiv \inf\{\lambda > 0 : |\{|f| > \lambda\}| < t\} \quad (t > 0).$$

See Definition 19. We will consider a maximal operator which can be useful for any measurable function.

For $Q \in \mathcal{Q}$, the median of $f \in L^0(Q)$, which is close to $(\chi_Q f)^* \left(2^{-1}|Q|\right)$, will be an important role in the sequel.

Definition 52 (Median)**.** Let $Q \in \mathcal{Q}$, and $f \in L^0(Q)$ be a real-valued function. Define $\mathrm{MED}(f;Q) \equiv \{a \in \mathbb{R} : a \text{ satisfies (4.3)}\}$ where condition (4.3) is given by

$$|\{x \in Q : f(x) > a\}|, \quad |\{x \in Q : f(x) < a\}| \leq \frac{1}{2}|Q|. \tag{4.3}$$

Denote by $\mathrm{Med}(f;Q)$ any element in $\mathrm{MED}(f;Q)$. The quantity $\mathrm{Med}(f;Q)$ is called the *median of* f over Q. One can regard $\mathrm{Med}(f;Q)$, as if it were a mapping $Q \mapsto \mathrm{Med}(f;Q)$.

Note that $\mathrm{MED}(f;Q)$ is an interval. As the following example shows, $\mathrm{MED}(f;Q)$ contains more than 1 point.

Example 77. $\mathrm{MED}(\chi_{[0,1]\times[0,2]^{n-1}} + 2\chi_{[1,2]\times[0,2]^{n-1}}; [0,2]^n) = [1,2]$.

We collect some elementary properties of the median.

Proposition 167. *Let $f, g \in L^0(Q)$ be real-valued functions defined on a cube Q. Assume that f, g are finite almost everywhere on Q.*

(1) $\mathrm{MED}(f;Q) \neq \emptyset$.

(2) *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Extend Φ to a continuous mapping from $\overline{\mathbb{R}}$ to itself. Then*

$$\mathrm{MED}(\Phi \circ f;Q) = \{\Phi(t) : t \in \mathrm{MED}(f;Q)\}.$$

In particular, $\mathrm{MED}(f + c;Q) = \{c + \alpha : \alpha \in \mathrm{MED}(f;Q)\}$.

(3) *If $f \leq g$, then* $\sup \mathrm{MED}(f;Q) \leq \sup \mathrm{MED}(g;Q)$.

(4) *Let $0 < a < 1$. If $\lambda \in \mathrm{MED}(f;Q)$, then $|\lambda| \leq f^*(2^{-1}a|Q|)$.*

Proof

(1) Since $\lim_{\lambda\to\infty} |\{x \in Q : f(x) > \lambda\}| = 0$, the number

$$\lambda_0 \equiv \inf\left(\left\{\lambda > 0 : |\{x \in Q : f(x) > \lambda\}| < \frac{|Q|}{2}\right\} \cup \{\infty\}\right)$$

is finite. If $\lambda < \lambda_0$, then $2|\{x \in Q : f(x) \geq \lambda\}| \geq 2|\{x \in Q : f(x) > \lambda\}| \geq |Q|$. Letting $\lambda \uparrow \lambda_0$, we have $2|\{x \in Q : f(x) \geq \lambda_0\}| \geq |Q|$. Hence, by passing to the complement again, we obtain $2|\{x \in Q : f(x) < \lambda_0\}| \leq |Q|$. If $\lambda > \lambda_0$, then $2|\{x \in Q : f(x) > \lambda\}| \leq |Q|$. Letting $\lambda \downarrow \lambda_0$, we obtain $2|\{x \in Q : f(x) > \lambda_0\}| \leq |Q|$. Thus, $\lambda_0 \in \mathrm{MED}(f; Q)$.

(2) This is clear from the definition.

(3) Assume otherwise; $\sup \mathrm{MED}(f; Q) > \sup \mathrm{MED}(g; Q)$. We choose $\beta \in (\sup \mathrm{MED}(g; Q), \sup \mathrm{MED}(f; Q))$. Then

$$|\{x \in Q : g(x) > \beta\}| \leq |\{x \in Q : g(x) > \mathrm{Med}(g; Q)\}| \leq \frac{1}{2}|Q|.$$

Meanwhile,

$$\begin{aligned} |\{x \in Q : g(x) < \beta\}| > \frac{1}{2}|Q| &\geq |\{x \in Q : f(x) < \beta\}| \\ &\geq |\{x \in Q : g(x) < \beta\}|, \end{aligned}$$

which is a contradiction.

(4) Assume otherwise, assume $\lambda > f^*(2^{-1}a|Q|)$ or $\lambda < -f^*(2^{-1}a|Q|)$. Then if $\lambda > f^*(2^{-1}a|Q|)$, then

$$|\{x \in Q : f(x) \leq f^*(2^{-1}a|Q|)\}| \leq |\{x \in Q : f(x) < \lambda\}| \leq \frac{1}{2}|Q|$$

since $\lambda \in \mathrm{MED}(f; Q)$. Meanwhile, thanks to Lemma 29,

$$\begin{aligned} |\{x \in Q : f(x) > f^*(2^{-1}a|Q|)\}| &\leq |\{x \in Q : |f(x)| > f^*(2^{-1}a|Q|)\}| \\ &\leq \lambda_f(f^*(2^{-1}a|Q|)) \\ &\leq \frac{a}{2}|Q|, \end{aligned}$$

which is a contradiction since $|Q| > \frac{1}{2}|Q| + \frac{a}{2}|Q|$.

If $\lambda < -f^*(2^{-1}a|Q|)$, then we consider $-f$ instead of f. Thanks to the above argument, we obtain a contradiction.

Roughly speaking, $(\chi_Q f)^*(\tau|Q|)$ is an average of the function $f \in L^0(Q)$ which is not always locally integrable. We have the following version of the Lebesgue differentiation theorem:

Lemma 168 (Fujii's lemma)**.** *Let $0 < \tau < 1$. Let $f \in L^0(\mathbb{R}^n)$ be real-valued. Then $\lim_{Q \to x} (f\chi_Q)^*(\tau|Q|) = |f(x)|$ for almost every $x \in \mathbb{R}^n$. That is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $||f(x)| - (f\chi_Q)^*(\tau|Q|)| < \varepsilon$ whenever a cube Q satisfies $|Q| < \delta$ and $x \in Q$.*

Note that there is no need to assume that f is integrable.

Proof Since the proof is similar to that of Corollary 169 to follow, we omit the proof.

The following equality is a version of the Lebesgue differentiation theorem for the medium.

Corollary 169 (Fujii's lemma II)**.** *Let $f \in L^0(\mathbb{R}^n)$ be a real-valued function. Then there exists a set E of measure 0 with the following property: For all $x \in \mathbb{R}^n \setminus E$ and $\varepsilon > 0$, there exists $r > 0$ such that for all cubes Q satisfying $x \in Q \subset Q(x, r)$ and $\lambda \in \mathrm{MED}(f; Q)$, $|f(x) - \lambda| < \varepsilon$.*

Proof For every $j \in \mathbb{N}$, we define

$$s_j(x) = \sum_{k=-\infty}^{\infty} 2^{-j}(k-1)\chi_{E_{j,k}}(x),$$

where

$$E_{j,k} \equiv \left\{x \in \mathbb{R}^n \,:\, 2^{-j}(k-1) \le f(x) < 2^{-j}k\right\}.$$

Then

$$0 \le f(x) - s_j(x) \le 2^{-j}, \quad \mathrm{Inf}(\mathrm{MED}(s_j; Q)) \le \mathrm{Med}(f; Q)$$

for every $Q \in \mathcal{Q}(\mathbb{R}^n)$. Let

$$F \equiv \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{x \in E_{j,k} \,:\, \lim_{|Q|\to 0, x\in Q} \frac{|E_{j,k} \cap Q|}{|Q|} = 1\right\}.$$

Then F differs from $\mathbb{R}^n$ by a set E of measure zero.

If $x \in F = \mathbb{R}^n \setminus E$, then by the definition of F, for every $j \in \mathbb{N}$, we have $x \in E_{j,k}$ and $3|E_{j,k} \cap Q| > 2|Q|$ for some $k \in \mathbb{N}$ and for some sufficiently small Q. Fix such a cube Q, then $\mathrm{MED}(s_j; Q) = \{2^{-j}(k-1)\}$. Since $2|\{y \in Q : f(y) \ge \mathrm{Med}(f; Q)\}| \ge |Q|$, there exists $y \in E_{j,k}$ such that $\mathrm{Med}(f; Q) \ge f(y)$. Since $2^{-j}(k-1) = s_j(x) = s_j(y)$, we have

$$\begin{aligned} |\mathrm{Med}(f; Q) - f(x)| &\le |\mathrm{Med}(f; Q) - s_j(y)| + |f(x) - s_j(y)| \\ &\le \mathrm{Med}(f; Q) - 2^{-j}(k-1) + f(x) - s_j(x) \\ &\le 2^{-j+1}. \end{aligned}$$

Since j is arbitrary, we conclude the proof.

4.2.3 Generalized dyadic grid and the Lerner–Hytönen decomposition

As an application of the distributional maximal operator, we obtain the Lerner–Hytönen decomposition. Roughly speaking, given any $f \in L^0(\mathbb{R}^n)$, we want its decomposition into better functions. The Calderón–Zygmund decomposition, dealt with in Section 4.5.2, is insufficient because it requires that f be locally integrable.

The decomposition is not linear. So we suppose that the decomposition may heavily depend on f. To have such a decomposition we need the following observation below.

Theorem 170. *Let Q^0 be a (right-open) cube. For $h \in L^\infty(Q^0) \cap \mathbb{M}^+(\mathbb{R}^n)$, we write $\gamma_0 \equiv m_{Q^0}(h)$. Fix $\tau \geq 2^{n+1}$. Define*

$$\mathcal{Z}_j \equiv \left\{Q \,:\, Q \in \mathcal{D}(Q^0) :\, m_Q(h) > \gamma_0 \tau^j\right\}$$

for $j \in \mathbb{N}$ and $\mathcal{Z}_0 = \{Q_k^0\}_{k\in K_0} \equiv \{Q^0\}$. Let $D_j \equiv \bigcup \mathcal{Z}_j = \bigcup\limits_{Q\in\mathcal{Z}_j} Q$ for $j \in \mathbb{N}_0$. Considering the maximal cubes $\{Q_k^j\}_{k\in K_j} \subset \mathcal{D}(Q^0)$ in $\mathcal{Z}_j$ with respect to inclusion, we can write $D_j = \bigcup\limits_{k\in K_j} Q_k^j$ for $j \in \mathbb{N}_0$, where the family $\{Q_k^j\}_{k\in K_j} \subset \mathcal{D}(Q^0)$ is disjoint for each $j \in \mathbb{N}_0$. Let $E_k^j \equiv Q_k^j \setminus D_{j+1}$ for $j \in \mathbb{N}_0$ and $k \in K_j$.

(1) *We have $\gamma_0\tau^j < m_{Q_k^j}(h) \leq 2^n\gamma_0\tau^j$ for $k \in K_j$ with $j = 1, 2, \ldots$ and $\gamma_0 = m_{Q_k^0}(h)$ for $k \in K_0$.*

(2) *The family $\mathcal{E} \equiv \{E_k^j\}_{j\in\mathbb{N}_0, k\in K_j}$ is disjoint, decomposes Q^0 and satisfies $|Q_k^j| \leq \dfrac{\tau|E_k^j|}{\tau - 2^n} \leq \dfrac{1}{2}|E_k^j|$ for all $j \in \mathbb{N}_0$ and $k \in K_j$.*

Proof

(1) If $j = 0$, then the assertion is clear from the maximality. Let us suppose $j \geq 1$. The left inequality is a consequence of the fact that we chose Q_k^j from $\mathcal{Z}_j$. If we consider the dyadic parent R of Q_k^j, then $m_R(h) \leq \gamma_0\tau^j$. Otherwise, instead of Q_k^j R would have been chosen as an element of $\mathcal{Z}_j$. Since $m_{Q_{j,k}}(h) \leq 2^n m_R(h)$, we obtain the right inequality.

(2) A geometric observation shows that two dyadic cubes never intersect unless one is not included in the other. We freeze $j \in \mathbb{N}$ and $k \in K_j$. From the observation above, we have

$$D_{j+1} \cap Q_k^j = \bigcup \left\{Q_{k^*}^{j+1} \,:\, k^* \in K_{j+1}, \quad Q_{k^*}^{j+1} \subset Q_k^j\right\}$$

because $Q_{k^*}^{j+1} \supsetneq Q_k^j$ never happens thanks to the maximality of Q_k^j and the fact that $\tau \geq 2^{n+1}$. Note that $|Q_{k^*}^{j+1}|\gamma_0\tau^{j+1} \leq \displaystyle\int_{Q_{k^*}^{j+1}} h(x)\mathrm{d}x$, since $Q_{k^*}^{j+1} \in \mathcal{Z}_{j+1}$. If we add this estimate over all $k^* \in K_{j+1}$ such that $Q_{k^*}^{j+1} \subset Q_k^j$, we obtain $|D_{j+1} \cap Q_k^j|\gamma_0\tau^{j+1} \leq \displaystyle\int_{Q_k^j} h(x)\mathrm{d}x \leq |Q_k^j|2^n\gamma_0\tau^j$. Consequently, if we consider the complement of D_{j+1}, then we obtain the desired result.

Although the set of dyadic cubes has a good geometric property, there are many dyadic cubes that dyadic cubes cannot cover: there does not exist a dyadic cube R such that $Q(1)$ cannot be covered by R. However, if we consider 2^n translated families, we can have this property.

Definition 53 ($\mathfrak{D}_a$)**.** Let $a = (a_1, a_2, \dots, a_n) \in \{0, 1/3\}^n$. The *generalized dyadic grid* $\mathfrak{D}_a$ is defined by $\mathfrak{D}_a \equiv \{Q : Q - (-1)^{\log_2 \ell(Q)} \ell(Q) a \in \mathcal{D}\}$. If $n = 1$, then we write $\mathfrak{D}_+ \equiv \mathfrak{D}_{(1)}$.

Example 78. Let $n = 1$. Then

$$\dots, \left[\frac{1}{3}, \frac{4}{3}\right), \left[\frac{4}{3}, \frac{7}{3}\right), \left[\frac{7}{3}, \frac{10}{3}\right), \left[\frac{10}{3}, \frac{13}{3}\right), \dots \in \mathfrak{D}_+$$

and

$$\dots, \left[-\frac{1}{6}, \frac{1}{3}\right), \left[\frac{1}{3}, \frac{5}{6}\right), \left[\frac{5}{6}, \frac{4}{3}\right), \left[\frac{4}{3}, \frac{11}{6}\right), \left[\frac{11}{6}, \frac{7}{3}\right), \dots \in \mathfrak{D}_+.$$

Furthermore, $[-1, 1] \subset \left(-\frac{8}{3}, \frac{4}{3}\right] \in \mathcal{D}_+$.

The family $\mathfrak{D}_a, a \in \{0, 1/3\}^n$ is useful because we can control any cubes in the following sense:

Lemma 171. *Let Q be any cube. Then there exist $a = (a_1, a_2, \dots, a_n) \in \{0, 1/3\}^n$ and $R \in \mathfrak{D}_a \cap \mathcal{Q}^\sharp(Q)$ such that $|R| \le 6^n |Q|$.*

Proof We work in $\mathbb{R}$; $n = 1$. By translation, we may assume $Q = (a, b)$, where $0 < a < b$. Choose $k_0 \in \mathbb{Z}$ such that $2^{-k_0 - 1} \le 3(b - a) < 2^{-k_0}$. If $(a, b) \cap 2^{-k_0}\mathbb{Z} = \emptyset$, then (a, b) is contained in an interval of the form $2^{-k_0}[m_0, m_0 + 1)$. Thus, we may assume $(a, b) \cap 2^{-k_0}\mathbb{Z} \neq \emptyset$. Let $m_0 2^{-k_0} \in (a, b)$. Then $m 2^{-k_0} \notin (a, b)$; otherwise $b - a \ge 2^{-k_0}$ and this is a contradiction. Then

$$\begin{aligned}(a, b) &\subset (m_0 2^{-k_0} - 2^{-k_0}/3, m_0 2^{-k_0} + 2^{-k_0}/3) \\ &\subset (m_0 2^{-k_0} - 2^{1-k_0}/3, m_0 2^{-k_0} + 2^{-k_0}/3).\end{aligned}$$

Thus, if we choose $R = (m_0 2^{-k_0} - 2^{1-k_0}/3, m_0 2^{-k_0} + 2^{-k_0}/3) \in \mathfrak{D}_+$, then we have $\ell(R) = 2^{-k_0} \le 6(b - a) = 6\ell(Q)$.

Motivated by the generalized dyadic grid $\mathfrak{D}_a$, we introduce the notion of generalized dyadic grids.

Definition 54 (Generalized dyadic grid)**.** A *generalized dyadic grid* $\mathfrak{D}$ is the set of all cubes of the form

$$x + [0, 2^l)^n \quad (x \in \mathbb{R}^n, l \in \mathbb{Z})$$

with the property: If $Q, R \in \mathfrak{D}$, then $Q \cap R \in \{Q, R, \emptyset\}$.

In addition to generalized dyadic grids, we need the notion of sparse families. Roughly speaking, the notion of sparse families extends the one of disjoint families. A sparse family is an "almost disjoint" one. We need some flexibility in the disjointness because in many cases it is difficult to obtain the disjointness.

Definition 55 (Sparse family)**.**

(1) Let $\tau > 0$. A set of cubes $\mathfrak{A}$ is τ-*sparse*, if there exists a disjoint collection $\{K(Q)\}_{Q\in\mathfrak{A}}$ such that $K(Q)$ contained in Q and that satisfies $|K(Q)| \geq \tau|Q|$ for each $Q \in \mathfrak{A}$. If there is no need to specify the precise value of τ, $\mathfrak{A}$ is called simply *sparse*. Each $K(Q)$ is called the *nutshell* of Q.

(2) Let $\mathbf{a} \in \{0, 1/3\}^n$ be fixed, and let $\tau > 0$. Suppose that we have a family $\mathcal{S}_j = \{Q_k^j\}_{k\in K_j}$ of disjoint cubes in $\mathfrak{D}_{\mathbf{a}}$ for each $j \in \mathbb{N}_0$. Assume in addition that $\bigcup_{k\in K_{j+1}} Q_k^{j+1} \subset \bigcup_{k\in K_j} Q_k^j$ and that $\sum_{k\in K_{j+1}} |Q_k^{j+1} \cap R| \leq \tau|R|$ for all $R \in \mathcal{S}_j$, $j \in \mathbb{N}_0$. Then $\bigcup_{j=0}^{\infty} \mathcal{S}_j$ is a τ-sparse family with a level structure.

If there is no need to specify τ, we omit τ.

Example 79.

(1) Any family $\mathfrak{Q}$ of disjoint cubes is 1-sparse. In fact for any cube Q we can choose its nutshell $K(Q) = Q$. In particular, if Q is a cube, then $\{Q\}$ is a sparse family.

(2) In Theorem 170, the family $\{Q^0\} \cup \{Q_k^j\}_{j\in\mathbb{N},k\in K_j}$ is a sparse family of cubes with a level structure. Furthermore, we have a decomposition in the sense of the almost everywhere pointwise estimate:

$$\chi_{Q^0} Mf \leq \gamma_0\tau\chi_{E^0} + \sum_{j=1}^{\infty}\sum_{k\in K_j} \gamma_0\tau^{k+1}\chi_{E_k^j},$$

where E^0 is the nutshell of Q^0 and each E_k^j is the nutshell of Q_k^j.

(3) The family $\{3Q\}_{Q\in\mathcal{D}_0}$ is a 3^{-n}-sparse family with $K(Q) = Q$. This is a misleading but important example.

(4) Let $0 < \eta \leq 1$. Suppose that we have famlies $\mathfrak{S}_j, j \in \mathbb{N}_0$ of disjoint dyadic cubes satisfying $\sum_{Q\in\mathfrak{S}_{j+1}} \chi_Q \leq \sum_{Q\in\mathfrak{S}_j} \chi_Q \leq 1$ for all $j \in \mathbb{N}_0$ and $\sum_{Q\in\mathfrak{S}_{j+1}} |Q \cap R| \leq \eta|R|$ for all $R \in \mathfrak{S}_j$ and $j \in \mathbb{N}_0$. Then $\bigcup_{j=0}^{\infty} \mathfrak{S}_j$ is an η-sparse family with a level structure.

We generalize Theorem 170 as follows:

Theorem 172. *Let Q^0 be a (right-open) cube, and let $0 \le \alpha < n$. Write $a_0 \equiv |Q^0|^{\frac{\alpha}{n}} m_{3Q^0}(f)$ and $a \equiv 18^{n+1}$. For each $j \in \mathbb{N}_0$ and $f \in L^\infty_{\rm c}(Q^0) \cap \mathbb{M}^+(\mathbb{R}^n)$, define*

$$D_j \equiv \bigcup \left\{ Q \in \mathcal{D}(Q^0) : |Q|^{\frac{\alpha}{n}} m_{3Q}(f) \ge a_0 a^j \right\} (\subset Q^0).$$

Considering the maximal cubes $Q^j_k, k \in K_j$ with respect to inclusion, we can write $D_j = \bigcup_{k=1}^{\infty} Q^j_k$, where the cubes $Q^j_k, k \in K_j$ are pairwise disjoint. Then $\mathcal{S} \equiv \{Q^j_k\}_{j \in \mathbb{N}_0, k \in K_j}$ is sparse.

Note that $D_0 = Q^0$ in the above.

Proof Let $j \in \mathbb{N}$ and $k \in K_j$. By the maximality of each Q^j_k, we see that

$$a_0 a^j \le |Q^j_k|^{\frac{\alpha}{n}} m_{3Q^j_k}(f) \le 2^n a_0 a^j.$$

Let $E^j_k \equiv Q^j_k \setminus D_{j+1}$. Then the family $\{E^j_k\}_{j \in \mathbb{N}_0, k \in K_j}$ is pairwise disjoint and decomposes Q^0. Consequently, we need only verify that $2|E^j_k| \ge |Q^j_k|$. Notice that, if $k' \in K_{j+1}$ satisfies $Q^{j+1}_{k'} \subset Q^j_k$, then

$$a_0 a^{k+1} \le |Q^{j+1}_{k'}|^{\frac{\alpha}{n}} m_{3Q^{j+1}_{k'}}(f) \le |Q^j_k|^{\frac{\alpha}{n}} m_{3Q^{j+1}_{k'}}(f) \le |Q^j_k|^{\frac{\alpha}{n}} M[f\chi_{3Q^j_k}](x)$$

for all $x \in Q^{j+1}_{k'}$. This entails

$$Q^j_k \cap D_{j+1} = Q^j_k \cap \left(\bigcup_{\tilde{k} \in K_{j+1}} Q^{j+1}_{\tilde{k}} \right) \subset \left\{ x \in \mathbb{R}^n : M[f\chi_{3Q^j_k}](x) \ge \frac{a_0 a^{k+1}}{|Q^j_k|^{\frac{\alpha}{n}}} \right\}.$$

The weak-$(1,1)$ boundedness of M (see Theorem 134) yields

$$\begin{aligned} |Q^j_k \cap D_{j+1}| &\le 3^n \cdot \frac{|Q^j_k|^{\frac{\alpha}{n}}}{a_0 a^{k+1}} \cdot \int_{3Q^j_k} f(x) \mathrm{d}x \\ &\le 3^n \cdot 2^{n-\alpha} a_0 a^k \cdot \frac{|Q^j_k|^{\frac{\alpha}{n}}}{a_0 a^{k+1}} \cdot \frac{3^n |Q^j_k|}{|Q^j_k|^{\frac{\alpha}{n}}} \\ &= \frac{18^n}{a} |Q^j_k| \\ &\le \frac{1}{2} |Q^j_k|, \end{aligned}$$

which implies $2|E^j_k| \ge |Q^j_k|$. Thus $\mathcal{S} \equiv \{Q^j_k\}_{j \in \mathbb{N}_0, k \in K_j}$ is sparse.

Likewise, we can prove the following:

Theorem 173. *Let Q^0 be a (right-open) cube, and let $0 \le \alpha < n$. Write $a_0 \equiv |Q^0|^{\frac{\alpha}{n}} m_{Q^0}(f)$ and $a \equiv 18^{n+1}$. For each $j \in \mathbb{N}_0$ and $f \in L^\infty_{\rm c}(Q^0) \cap \mathbb{M}^+(\mathbb{R}^n)$. define*

$$D_j \equiv \bigcup \left\{ Q \in \mathcal{D}(Q^0) : |Q|^{\frac{\alpha}{n}} m_Q(f) \ge a_0 a^j \right\} (\subset Q^0).$$

Considering the maximal cubes $Q_k^j, k \in K_j$ with respect to inclusion, we can write $D_j = \bigcup_{k \in K_j} Q_k^j$, where the cubes $Q_k^j, k \in K_j$ are pairwise disjoint. Then $\mathcal{S} \equiv \{Q_k^j\}_{j \in \mathbb{N}_0, k \in K_j}$ is sparse.

Note again that $D_0 = Q^0$ in the above.

The next inequality is a very strong theorem to control the singularity of functions. The following theorem is called the Lerner–Hytönen decomposition of the function, which is important. For a cube R we define

$$\omega_{2^{-n-2}}(f; R) \equiv (\chi_R(f - \mathrm{Med}(f; R)))^*(2^{-n-2}|R|).$$

Although there is ambiguity of the definition of $\mathrm{Med}(f; R)$, we fix one of them and define $\omega_{2^{-n-2}}(f; R)$. For a right-open cube $Q^0 \equiv \prod_{j=1}^n [a_j, b_j)$, recall that we defined the dyadic cubes $\mathcal{D}(Q^0)$ with respect to Q^0 by bisecting Q^0 in a finite number of steps.

Theorem 174 (Lerner–Hytönen decomposition)**.** *Let $f : Q^0 \to \mathbb{R}$ be a measurable function defined on a right-open cube Q^0. Then there exists a sparse family $\{Q_k^j\}_{j \in \mathbb{N}_0,\, k \in K_j} \subset \mathcal{D}(Q^0)$ with a level structure such that $\{Q_k^0\}_{k \in K_0} = \{Q^0\}$ and that*

$$\chi_{Q^0}|f - \mathrm{Med}(f; Q^0)| \le \sum_{j=0}^{\infty} \sum_{k \in K_j} \omega_{2^{-n-2}}(f; Q_k^j) \chi_{Q_k^j}.$$

Here, the inequality is understood as the one for almost every point.

For the proof we use the following notation: Recall that given a cube Q, $\mathcal{D}_1(Q)$ denotes the set of all dyadic children. We rely on a somewhat distorted stopping time argument. For any fixed measurable function f on a fixed cube Q^0 and $Q \in \mathcal{D}(Q^0)$, let $\mathfrak{S}_0(Q) \equiv \{Q\}$, and define $\mathfrak{S}_1(Q) \subset \mathcal{D}(Q^0)$ to be the collection of all maximal dyadic subcubes R of Q such that

$$\max_{Q' \in \mathcal{D}_1(R)} |\mathrm{Med}(f, Q') - \mathrm{Med}(f; Q^0)| > \omega_{2^{-n-2}}(f; Q). \tag{4.4}$$

Suppose that we have defined $\mathfrak{S}_0(Q)$, $\mathfrak{S}_1(Q)$, $\mathfrak{S}_2(Q), \dots, \mathfrak{S}_{k-1}(Q)$ for each $Q \in \mathcal{D}(Q^0)$. Then define $\mathfrak{S}_k(Q) \equiv \bigcup_{R \in \mathfrak{S}_{k-1}(Q)} \mathfrak{S}_1(R)$ for each $Q \in \mathcal{D}(Q^0)$.

The proof of Theorem 174 hinges on the repeated use of the following lemma:

Lemma 175. *Let $f \in L^0(\mathbb{R}^n)$ be a real-valued function defined on a cube Q^0.*

(1) *Almost everywhere*

$$\begin{aligned}&\chi_{Q^0}|f - \mathrm{Med}(f; Q^0)| \\ &\le \omega_{2^{-n-2}}(f; Q^0)\chi_{Q^0} + \sum_{Q \in \mathfrak{S}_1(Q^0)} \chi_Q |f - \mathrm{Med}(f; Q)|.\end{aligned}$$

(2) *Write* $\Omega(Q^0) \equiv \bigcup_{Q\in\mathfrak{S}_1(Q^0)} Q \subset Q^0$. *Then* $2|\Omega(Q^0)| \le |Q_0|$.

Once we prove Lemma 175, for each $J \in \mathbb{N}$, we have

$$\chi_{Q^0}|f - \mathrm{Med}(f;Q^0)| \le \sum_{j=0}^{J} \sum_{Q\in\mathfrak{S}_j(Q)} \omega(f;Q)\chi_Q + \sum_{Q\in\mathfrak{S}_J(Q)} \chi_Q|f - \mathrm{Med}(f;Q)|$$

and $\sum_{Q\in\mathfrak{S}_J(Q)} |Q| \le 2^{-J}|Q^0|$. Hence, letting $J \to \infty$, we have

$$\chi_{Q^0}|f - \mathrm{Med}(f;Q^0)| \le \sum_{j=0}^{\infty} \sum_{Q\in\mathfrak{S}_j(Q)} \omega(f;Q)\chi_Q$$

almost everywhere. Thus, the proof of Theorem 174 will be complete once we prove Lemma 175.

Proof of Lemma 175 We may assume that $\mathrm{Med}(f;Q^0) = 0$ by considering $f - \mathrm{Med}(f;Q^0)$ instead of f itself. In this case $\omega_{2^{-n-2}}(f;Q^0) = (\chi_{Q^0}f)^*(2^{-n-2}|Q^0|)$.

(1) We distinguish two cases. We first assume $x \in \Omega(Q^0)$ and then assume $x \in Q^0 \setminus \Omega(Q^0)$.

Fix $x \in \Omega(Q^0)$. For the disjoint subcubes $\mathfrak{S}_1(Q^0)$ of Q^0, we write

$$\mathrm{A}(x) \equiv \sum_{Q\in\mathfrak{S}_1(Q^0)} \mathrm{Med}(f;Q)\chi_Q(x),$$
$$\mathrm{B}(x) \equiv \sum_{Q\in\mathfrak{S}_1(Q^0)} \chi_Q(x)(f(x) - \mathrm{Med}(f;Q)),$$

so that $\chi_{\Omega(Q^0)}(x)f(x) = \mathrm{A}(x) + \mathrm{B}(x)$. Note that the quantity $\mathrm{B}(x)$ matches the second term in the right-hand side of the conclusion; we have only to handle $\mathrm{A}(x)$. From the maximality it follows that any $Q \in \mathfrak{S}_1(Q^0)$ satisfies the opposite estimate in place of $Q' \in \mathcal{D}(Q)$. That is, $|\mathrm{Med}(f;Q)| \le \omega_{2^{-n-2}}(f;Q^0)$. Thus, since $\mathfrak{S}_1(Q^0)$ is disjoint, $|\mathrm{A}(x)| \le \omega(f;Q^0)\chi_{\Omega(Q^0)}(x)$.

Let $x \in Q^0 \setminus \Omega(Q^0)$ instead. Then we have $|\mathrm{Med}(f;R)| \le \omega_{2^{-n-2}}(f;Q^0)$ for any cube $R \in \mathcal{D}(Q^0)$ containing x. It should be noted that, for any $x \in Q^0 \setminus \Omega(Q^0)$, such a cube R can be chosen as small as we wish. Thus, $|f(x)| \le \omega_{2^{-n-2}}(f;Q^0)$ for almost all $x \in Q^0 \setminus \Omega(Q^0)$ by the Fujii lemma; see Corollary 169.

(2) Let $Q \in \mathfrak{S}_1(Q^0)$ be arbitrary, so that some $Q' \in \mathcal{D}_1(Q)$ satisfies

$$|\mathrm{Med}(f;Q')| > \omega_{2^{-n-2}}(f;Q^0). \tag{4.5}$$

Let $\frac{1}{4} < \nu < \frac{1}{2}$. Since Q' is a dyadic child of Q, $|Q| = 2^n|Q'|$. Hence

$$|\mathrm{Med}(f;Q')| \le (\chi_{Q'}f)^*(\nu|Q'|) \le (\chi_Q f)^*(2^{-n}\nu|Q|). \tag{4.6}$$

Thus we obtain $\omega_{2^{-n-2}}(f;Q^0) < (\chi_Q f)^*(2^{-n}\nu|Q|)$ combining (4.5) and (4.6). In view of Lemma 30, we have

$$|Q \cap \{|f| > \omega_{2^{-n-2}}(f;Q^0)\}| \ge |Q \cap \{|f| \ge (\chi_Q f)^*(2^{-n}\nu|Q|)\}| \ge \frac{\nu|Q|}{2^n}.$$

If we add this estimate over $Q \in \mathfrak{S}_1(Q^0)$ and use Lemma 29, then we obtain

$$\begin{aligned}\frac{\nu}{2^n}|\Omega(Q^0)| &= \frac{\nu}{2^n}\sum_{Q\in\mathfrak{S}_1(Q^0)}|Q| \\ &\le |Q^0 \cap \{|f| > \omega_{2^{-n-2}}(f;Q^0)\}| \\ &= |Q^0 \cap \{|f| > (\chi_{Q_0}f)^*(2^{-n-2}|Q^0|)\}| \\ &\le 2^{-n-2}|Q^0|.\end{aligned}$$

It remains to let $\nu \uparrow \frac{1}{2}$.

We conclude Section 4.2.3 with the remark on the quantity $\omega_{2^{-n-2}}(f;Q)$.

Remark 6. Let f be a measurable function defined on a cube Q. Then for any $c \in \mathbb{R}$, $\omega_{2^{-n-1}}(f;Q) \le 2(\chi_Q(f-c))^*(2^{-n-2}|Q|)$. In fact,

$$\begin{aligned}\chi_Q(x)|f(x) - \mathrm{Med}(f;Q)| &\le \chi_Q(x)|f(x)-c| + |\mathrm{Med}(f;Q) - c| \\ &\le \chi_Q(x)|f(x)-c| + (\chi_Q(f-c))^*(2^{-n-2}|Q|).\end{aligned}$$

If one considers the distribution function, then one obtains the desired result.

4.2.4 Exercises

Exercise 65. We let

$$\begin{aligned}\mathfrak{D}_{2k} &\equiv \{[j\cdot 4^k - 2\cdot 4^k/3, j\cdot 4^k + 4^k/3)\}_{j\in\mathbb{Z}} \\ \mathfrak{D}_{2k+1} &\equiv \{[2j\cdot 4^k - 2\cdot 4^k/3, 2j\cdot 4^k + 2\cdot 4^k/3)\}_{j\in\mathbb{Z}}\end{aligned}$$

for $k \in \mathbb{Z}$. Let $0 < q \le p < \infty$.

(1) [273] Show that $\mathfrak{D} \equiv \{\mathfrak{D}_k\}_{k=-\infty}^{\infty}$ is a generalized dyadic grid.

(2) [273] Show that for any compact set K there exist $k \in \mathbb{Z}$ and $Q \in \mathfrak{D}_k$ such that $K \subset Q$.

(3) Show that $\|f\|_{\mathcal{M}^p_q} \sim \sup_{k\in\mathbb{Z}}\sup_{Q\in\mathfrak{D}_k}|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}$ for $f \in L^0(\mathbb{R}^n)$.

Exercise 66. Let $f \in L^0(Q_0)$ with $Q_0 \in \mathcal{Q}$, and let $\lambda \in (0,1)$.

(1) Show that $\omega_\lambda(f;Q_0) = 0$ if and only if f is constant.

(2) Show that $M^{\sharp,\mathcal{D}}_{\lambda;Q_0}f = 0$ if and only if f is constant.

4.3 Fractional maximal operators

The fractional maximal operator, which we define below in Section 4.3.1, substitutes for fractional integral operators. These operators behave similarly. However, in this book, we encounter the difference between these two things. We will obtain the local estimate and the sparse estimates in Sections 4.3.2 and 4.3.3, respectively.

4.3.1 Fractional maximal operators

The fractional maximal operator M_α of order $\alpha \in [0,n)$ is defined by

$$M_\alpha f(x) \equiv \sup_{Q \in \mathcal{Q}} \chi_Q(x) \ell(Q)^{\alpha-n} \int_Q |f(y)| \mathrm{d}y$$

for $f \in L^0(\mathbb{R}^n)$, so that $M = M_0$ is the Hardy–Littlewood maximal operator. In connection with the Hardy–Littlewood maximal operator, we will consider this generalization in Section 4.3.1.

Example 80. It seems that Morrey spaces are useful when we consider the boundedness property of the fractional maximal operator M_α, $0 < \alpha < n$. In fact, M_α maps $\mathcal{M}^{\frac{n}{\alpha}}_1(\mathbb{R}^n)$ boundedly to $L^\infty(\mathbb{R}^n)$.

As is the case with the Hardy–Littlewood maximal operator, we have the endpoint weak estimate.

Theorem 176. *Let $0 < \alpha < n$. Then $\|M_\alpha f\|_{WL^{\frac{n}{n-\alpha}}} \lesssim \|f\|_{L^1}$ for all $f \in L^1(\mathbb{R}^n)$.*

Proof Simply modify the case of $\alpha = 0$; see Exercise 67.

The following theorem, called the Hardy–Littlewood–Sobolev theorem, is fundamental.

Theorem 177. *Let $1 < p < q < \infty$ and $0 < \alpha < n$ satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $\|M_\alpha f\|_{L^q} \lesssim \|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n)$.*

Before the proof we remark that arithmetic shows

$$\left(q - \frac{n}{n-\alpha}\right) \times \left(\frac{(n-\alpha)p}{nu}\right)' = q.$$

Proof By the monotone convergence theorem we may assume that $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$; see Exercise 68. For $\lambda > 0$, we let $\Omega_\lambda \equiv \{M_\alpha f > \lambda\}$. We observe that

$$(\|M_\alpha f\|_{L^q})^q = q \int_0^\infty \lambda^{q-1} |\Omega_\lambda| \mathrm{d}\lambda \lesssim \int_0^\infty \lambda^{q-1-\frac{n}{n-\alpha}} (\|f \chi_{\Omega_\lambda}\|_{L^1})^{\frac{n}{n-\alpha}} \mathrm{d}\lambda$$

by the use of Theorem 176. We choose $u \in (0,1)$ so that $n(1-u) = p\alpha$. Then by Hölder's inequality, $\|f\chi_{\Omega_\lambda}\|_{L^1} \le \left(\|f\chi_{\Omega_\lambda}\|_{L^{\frac{nu}{n-\alpha}}}\right)^u (\|f\|_{L^p})^{1-u}$. Thus,

$$\begin{aligned}(\|M_\alpha f\|_{L^q})^q &\lesssim \int_0^\infty \lambda^{q-1-\frac{n}{n-\alpha}} \left(\|f\chi_{\Omega_\lambda}\|_{L^{\frac{nu}{n-\alpha}}}\right)^{\frac{nu}{n-\alpha}} (\|f\|_{L^p})^{\frac{n(1-u)}{n-\alpha}} \mathrm{d}\lambda \\ &\simeq (\|f\|_{L^p})^{\frac{n(1-u)}{n-\alpha}} \int_{\mathbb{R}^n} |f(x)|^{\frac{nu}{n-\alpha}} M_\alpha f(x)^{q-\frac{n}{n-\alpha}} \mathrm{d}x.\end{aligned}$$

By Hölder's inequality, we obtain

$$\int_{\mathbb{R}^n} |f(x)|^{\frac{nu}{n-\alpha}} M_\alpha f(x)^{q-\frac{n}{n-\alpha}} \mathrm{d}x \le (\|f\|_{L^p})^{\frac{nu}{n-\alpha}} (\|M_\alpha f\|_{L^q})^{q-\frac{n}{n-\alpha}}.$$

As a result, $(\|M_\alpha f\|_{L^q})^q \lesssim (\|f\|_{L^p})^{\frac{n}{n-\alpha}} (\|M_\alpha f\|_{L^q})^{q-\frac{n}{n-\alpha}}$. If we arrange this inequality, we conclude $\|M_\alpha f\|_{L^q} \lesssim \|f\|_{L^p}$.

4.3.2 Local estimates for the maximal operators and the fractional maximal operators

Here, we collect the local estimates for M_α for $0 \le \alpha < n$. We tolerate the case $\alpha = 0$ here.

Theorem 178. *Let $0 \le \alpha < n$, $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $\gamma \in \mathbb{R}$.*

(1) *Suppose $1 < p_1 \le p_2 < \infty$. Let $W(x) \equiv (1+|x|)^{p_1\gamma}$, $x \in \mathbb{R}^n$. Then the inequality*

$$\|M_\alpha f\|_{L^{p_2}(Q(1))} \lesssim \|f\|_{L^{p_1}(W)} = \|f \cdot W^{1/p_1}\|_{L^{p_1}}, \tag{4.1}$$

where the implicit constant is independent of f holds for all $f \in L^0(\mathbb{R}^n)$ if and only if

$$n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha \le \frac{n}{p_1} + \gamma. \tag{4.2}$$

(2) *Let $1 < p_1 < \infty$ and $0 < p_2 < \infty$ satisfy (4.2). For $r > 0$, we let $W_r(x) \equiv (r+|x|)^{p_1\gamma} \sim r^{p_1\gamma} M\chi_{B(r)}(x)^{-\frac{p_1\gamma}{n}}$ for $x \in \mathbb{R}^n$. Then*

$$\|M_\alpha f\|_{L^{p_2}(Q(r))} \lesssim r^{\alpha - \frac{n}{p_1} + \frac{n}{p_2} - \gamma} \|f\|_{L^{p_1}(W_r)} \tag{4.3}$$

for all $r > 0$ and for all $f \in L^0(\mathbb{R}^n)$. In particular,

$$\|M_\alpha f\|_{L^{p_2}(Q(r))} \lesssim r^{\frac{n}{p_2}} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^{p_1}}{(|x|+r)^{n-\alpha p_1}} \mathrm{d}x\right)^{\frac{1}{p_1}},$$

or equivalently

$$\|M_\alpha f\|_{L^{p_2}(Q(r))} \lesssim r^{\frac{n}{p_2}} \left\{\int_r^\infty \left(\int_{B(t)} |f(x)|^{p_1} \mathrm{d}x\right) \frac{\mathrm{d}t}{t^{n-\alpha p_1+1}}\right\}^{\frac{1}{p_1}} \tag{4.4}$$

for all $r > 0$ and for all $f \in L^0(\mathbb{R}^n)$.

Proof Inequality (4.3) is a conseqneuce of (4.1) and the dilation argument. Consequently, we discuss the equivalence between (4.1) and (4.2).

Assume (4.2). Choose $q_1 \in (1, p_1]$ and $q_2 \in [p_2, \infty)$ so that $\alpha = \dfrac{n}{q_1} - \dfrac{n}{q_2}$. Such a choice of q_1 and q_2 is possible because $\dfrac{n}{p_1} - \dfrac{n}{p_2} \le \alpha$ Then by the Hardy–Littlewood–Sobolev theorem and Hölder's inequality,

$$\begin{aligned}\|M_\alpha[\chi_{Q(3)}f]\|_{L^{p_2}(Q(1))} &\lesssim \|M_\alpha[\chi_{Q(3)}f]\|_{L^{q_2}(Q(1))}\\ &\lesssim \|f\|_{L^{q_1}(Q(3))}\\ &\lesssim \|f\|_{L^{p_1}(Q(3))}\\ &\lesssim \|f\|_{L^{p_1}(W)},\end{aligned}$$

and similar to Lemma 130

$$\begin{aligned}\|M_\alpha[\chi_{\mathbb{R}^n\setminus Q(3)}f]\|_{L^{p_2}(Q(1))} &\lesssim \sup_{r\ge 3}\frac{1}{r^{n-\alpha}}\int_{Q(r)\setminus Q(3)}|f(y)|\mathrm{d}y\\ &\lesssim \sup_{r\ge 1} r^\alpha m^{p_1}_{Q(r)}(f)\\ &\lesssim \|f\|_{L^{p_1}(W)},\end{aligned}$$

Thus, (4.1) holds.

Conversely, assume that (4.1) holds. Then

$$\frac{1}{r^{n-\alpha}}\int_{Q(r)\setminus Q(3)}|f(y)|\mathrm{d}y \lesssim \|f\|_{L^{p_1}(W)}$$

for all $f \in L^0(\mathbb{R}^n)$, or equivalently,

$$\frac{1}{r^{n-\alpha}}\int_{Q(r)\setminus Q(3)}|g(y)|(1+|y|)^{-\gamma}\mathrm{d}y \lesssim \|g\|_{L^{p_1}}$$

for all $g \in L^0(\mathbb{R}^n)$ and $r \ge 3$. Consequently, we obtain

$$\sup_{r>0}\frac{1}{r^{n-\alpha}}\left(\int_{Q(r)\setminus Q(3)}(1+|y|)^{-p_1'\gamma}\mathrm{d}y\right)^{\frac{1}{p_1'}} < \infty,$$

which forces $-n+\alpha-\gamma+\dfrac{n}{p_1'} \le 0$. If $p_1 \ne p_2$, use the truncated power function $|\cdot|^\beta\chi_{B(1)}$ to have $\alpha \ge \frac{n}{p_1} - \frac{n}{p_2}$.

Example 81. Let $1 < p < \infty$. Fix $r > 0$ and $x \in \mathbb{R}^n$. We consider the following "partial maximal function" $\overline{M}_r f(x) \equiv \sup_{t\ge r} m_{B(x,t)}(|f|)$ together with its powered version: $\overline{M}_r^{(p)} f(x) \equiv \sup_{t\ge r} m^{(p)}_{B(x,t)}(f)$. Then one can prove that

$$\|Mf\|_{L^p(B(x,r))} \simeq \|M(f\chi_{B(x,2r)})\|_{L^p(B(x,r))} + r^{\frac{n}{p}}\overline{M}_{2r}f(x).$$

Hence $r^{\frac{n}{p}}\overline{M}_r f(x) \lesssim \|Mf\|_{L^p(B(x,r))} \lesssim r^{\frac{n}{p}}\overline{M}^{(p)}_{2r} f(x)$. Note that

$$\overline{M}^{(p)}_{2r} f(x) \lesssim r^{\frac{n}{p}} \left(\int_r^\infty \left(\int_{B(x,t)} |f(y)|^p \mathrm{d}y \right) \frac{\mathrm{d}t}{t^{n+1}} \right)^{\frac{1}{p}}$$

thanks to (4.4).

4.3.3 Sparse estimate for fractional maximal operators

Here, we obtain a sparse estimate for fractional maximal operators. As shown later, the sparse estimate will be of great use.

Theorem 179. *Let $0 \le \alpha < n$. Suppose that we have $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ and a (right-open) cube Q^0. For a sparse family $\mathcal{S} \subset \mathcal{D}(Q^0)$ with the nutshell $\{E_Q\}_{Q\in\mathcal{S}}$, write*

$$L^{\mathcal{S}}_\alpha f \equiv \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} m_{3Q}(f)\chi_{E_Q} \text{ and } c^{\mathcal{D}^\sharp(Q^0)}_{\infty,\alpha}(f) \equiv \sup_{Q\in\mathcal{D}^\sharp(Q^0)} |Q|^{\frac{\alpha}{n}} m_{3Q}(f).$$

Then, there exists a sparse family $\mathcal{S} = \mathcal{S}_f \subset \mathcal{D}(Q^0)$ that depends on f such that $M_\alpha f(x) \lesssim_{\alpha,n} L^{\mathcal{S}}_\alpha f(x) + c^{\mathcal{D}^\sharp(Q^0)}_{\infty,\alpha}(f)$ for all $x \in Q^0$.

It matters that the sparse family depends on f.

Proof We follow the construction in Theorem 172. Recall that $\mathcal{D}(Q^0)$ is the set of dyadic cubes of Q^0. We may suppose that f is not zero almost everywhere; otherwise simply take $\mathcal{S} = \{Q^0\}$. We write $\widetilde{M}_\alpha f(x) \equiv \sup_{Q\in\mathcal{D}(Q^0)} \chi_Q(x)|Q|^{\frac{\alpha}{n}} m_{3Q}(f)$ for each $x \in \mathbb{R}^n$, so that for all $x \in Q^0$, $M_\alpha f(x) \le C\widetilde{M}_\alpha f(x) + c^{\mathcal{D}^\sharp(Q^0)}_{\infty,\alpha}(f)$. We set $\mathcal{S} \equiv \{Q^j_k\}_{j\in\mathbb{N}_0, k\in K_j}$. Since $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n) \setminus \{0\}$, we have $x \in D_j \setminus D_{j+1}$ for some $j \in \mathbb{Z}$. Since $x \in D_j$, $x \in Q^j_k$ for some $k \in K_j$. Thus, $x \in E^j_k \subset \mathbb{R}^n \setminus D_{j+1}$. In this case, $\widetilde{M}_\alpha f(x) \le a_0 a^{j+1} \le a|Q^j_k|^{\frac{\alpha}{n}} m_{3Q^j_k}(f)$. Since the family $\{E^j_k\}_{j\in\mathbb{N}_0,k\in K_j}$ is pairwise disjoint and decomposes Q^0, we conclude that $\widetilde{M}_\alpha f(x) \lesssim L^{\mathcal{S}}_\alpha f(x)$ for all $x \in Q^0$. This completes the proof.

We have the following variant of Theorem 179:

Remark 7. Let $\mathfrak{D}_{\mathbf{a}}$ be the dyadic grid with $\mathbf{a} \in \{0, 1/3\}^n$. Denote by $\mathfrak{D}^\sharp_{\mathbf{a}}(Q^0)$ the set of all cubes in $\mathfrak{D}_{\mathbf{a}}$ containing a cube Q^0. Then define

$$M^{\mathfrak{D}^\sharp_{\mathbf{a}}(Q^0)}_\alpha f \equiv \sup_{Q\in\mathfrak{D}^\sharp_{\mathbf{a}}(Q^0)} \ell(Q)^\alpha m_Q(|f|)\chi_Q$$

for $f \in L^0(\mathbb{R}^n)$. For a sparse family $\mathcal{S} \subset \mathfrak{D}^\sharp_{\mathbf{a}}(Q^0)$ with the nutshell $\{E_Q\}_{Q\in\mathcal{S}}$, write

$$L^{\mathcal{S}}_\alpha f \equiv \sum_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} m_Q(f)\chi_{E_Q} \text{ and } c^{\mathfrak{D}^\sharp_{\mathbf{a}}(Q^0)}_{\infty,\alpha}(f) \equiv \sup_{Q\in\mathfrak{D}^\sharp_{\mathbf{a}}(Q^0)} |Q|^{\frac{\alpha}{n}} m_Q(f).$$

Then, for any fixed cube $Q^0 \in \mathfrak{Q}$, there exists a sparse family $\mathcal{S} = \mathcal{S}_f \subset \mathfrak{D}(Q^0)$ such that, for all $x \in Q^0$, $M_\alpha^{\mathfrak{D}_\mathfrak{a}} f(x) \lesssim_{\alpha,n} L_\alpha^{\mathcal{S}} f(x) + c_{\infty,\alpha}^{\mathfrak{D}_\mathfrak{a}^\sharp(Q^0)}(f)$.

4.3.4 Exercises

Exercise 67. Prove Theorem 176 by the use of $(a+b)^{\frac{n-\alpha}{n}} \le a^{\frac{n-\alpha}{n}} + b^{\frac{n-\alpha}{n}}$ for $a, b \ge 0$ and $0 \le \alpha < n$.

Exercise 68. Let $\{f_j\}_{j=1}^\infty \subset \mathbb{M}^+(\mathbb{R}^n)$ be an increasing sequence, and let $0 \le \alpha < n$. Then show that $M_\alpha \left[\lim_{j\to\infty} f_j\right] = \lim_{j\to\infty} M_\alpha[f_j]$.

4.4 Fractional integral operators

One of the crucial properties of Morrey spaces is that Morrey spaces can describe the boundedness property of fractional integral operators more precisely than Lebesgue spaces. The fractional integral operators come about when we consider the inverse of Laplacian: the fractional integral operator of order 2 is a fundamental solution of the Laplace equation.

The aim of Section 4.4 is to introduce fractional integral operators and then to discuss their $L^p(\mathbb{R}^n)$-boundedness property.

We define and investigate fractional maximal operators in Section 4.4.1. We expand what we obtained in Section 4.4.1 to the local setting and to the sparse setting in Sections 4.4.2 and 4.4.3. Sections 4.4.4 and 4.4.5 will explain why fractional integral operators are useful. Section 4.4.5, which handles the Bessel operator, does not fall under the scope of other sections in Section 4.4. However, since it is related to fractional integral operators, we deal with the Bessel operator in Section 4.4.5.

4.4.1 Fractional integral operators on Lebesgue spaces

Another important class of singular integral operators is the fractional integral operator I_α (of order α) defined by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y \quad (x \in \mathbb{R}^n). \tag{4.1}$$

Here, $0 < \alpha < n$. Section 4.4.1 examines the close connection between M_α and I_α.

Example 82. Let $0 < \alpha < n$ and $0 < \beta < n$. If $f(x) \equiv |x|^{-\beta} \chi_{B(1)}(x)$ for $x \in \mathbb{R}^n$, then a simple calculation shows $I_\alpha f(x) \sim |x|^{-\alpha-\beta}$ as $x \to 0$.

Example 83. Let $0 < \alpha < n$. Observe that $I_\alpha \chi_{B(1)}(x) > 0$ for all $x \in \mathbb{R}^n$ and $I_\alpha \chi_{B(1)} \in L^\infty(\mathbb{R}^n)$. Furthermore, $I_\alpha \chi_{B(1)}(x) \sim |x|^{\alpha-n}$ for $|x| \geq 2$. Consequently, $I_\alpha \chi_{B(1)}(x) \sim (|x|+1)^{\alpha-n}$ for $x \in \mathbb{R}^n$. Thus a dilation and translation argument yields $I_\alpha \chi_{B(x_0,r)}(x) \sim r^\alpha (|x-x_0|+r)^{\alpha-n}$. A similar conclusion $I_\alpha \chi_Q(x) \sim \ell(Q)^n (|x-c(Q)|+\ell(Q))^{\alpha-n}$ holds for cubes.

Example 84. Let $0 < \alpha < n$ and $f \in \mathbb{M}^+(\mathbb{R}^n)$. Using Lemma 161, we write

$$I_\alpha f(x) = (n-\alpha) \int_0^\infty \left(\frac{1}{l^{n+1-\alpha}} \int_{B(x,l)} f(y) \mathrm{d}y \right) \mathrm{d}l.$$

This formula will be used frequently to analyze $I_\alpha f$.

One of the fundamental questions about $I_\alpha f(x)$ is whether the integral defining $I_\alpha f(x)$ converges absolutely.

Here, we will present the necessary and sufficient condition for the integral above to converge for almost all $x \in \mathbb{R}^n$.

Lemma 180. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The integral defining $I_\alpha f(x)$ converges absolutely for almost all $x \in \mathbb{R}^n$ if and only if*

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{1+|y|^{n-\alpha}} \mathrm{d}y < \infty. \tag{4.2}$$

Proof Suppose first that (4.2) holds. Let us check that the integral converges for almost all $y \in B(x,1)$, where $x \in \mathbb{R}^n$ is chosen arbitrarily. It is not so hard to see that

$$\int_{B(x,1)} I_\alpha [\chi_{B(x,1)} |f|](z) \mathrm{d}z < \infty,$$

which implies $I_\alpha[\chi_{B(x,1)}|f|](y) < \infty$ for almost all $y \in B(x,1)$ thanks to Theorem 182(1) to follow. Therefore, it remains to show that $I_\alpha[\chi_{\mathbb{R}^n \setminus B(x,1)}|f|](y) < \infty$ for almost all $y \in B(x,1)$.

Note that (4.2) implies

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{1+|x-y|^{n-\alpha}} \mathrm{d}y < \infty. \tag{4.3}$$

Thus

$$\begin{aligned} I_\alpha[\chi_{\mathbb{R}^n \setminus B(x,1)}|f|](y) &\leq \int_{\mathbb{R}^n \setminus B(x,1)} \frac{|f(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y \\ &\leq \int_{\mathbb{R}^n \setminus B(x,1)} \frac{|f(y)|}{1+|x-y|^{n-\alpha}} \mathrm{d}y \\ &< \infty. \end{aligned}$$

Thus, $I_\alpha[|f|](y) < \infty$ for almost all $y \in B(x,1)$.

Assume instead that $I_\alpha[|f|](x) < \infty$ for almost all $x \in \mathbb{R}^n$. Then choose a point $x_0 \in \mathbb{R}^n$ such that $I_\alpha[|f|](x_0) < \infty$. Then, we have

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B(x_0, 2|x_0|)} \frac{|f(y)|\mathrm{d}y}{1+|y|^{n-\alpha}} &\le 2^{n-\alpha} \int_{\mathbb{R}^n \setminus B(x_0, 2|x_0|)} \frac{|f(y)|\mathrm{d}y}{|x_0 - y|^{n-\alpha}} \\ &\le 2^{n-\alpha} I_\alpha[|f|](x_0) < \infty. \end{aligned} \tag{4.4}$$

Since f is assumed locally integrable, we obtain

$$\int_{B(x_0, 2|x_0|)} \frac{|f(y)|}{1+|y|^{n-\alpha}} \mathrm{d}y \le \int_{B(x_0, 2|x_0|)} |f(y)| \mathrm{d}y < \infty. \tag{4.5}$$

Combining (4.4) and (4.5) gives (4.2).

The part (2) of the next lemma is known as the Hedberg lemma or the Hedberg inequality [190].

Lemma 181. *Let $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \dfrac{n}{\alpha}$. Define $s > 0$ by $\dfrac{1}{s} = \dfrac{1}{p} - \dfrac{\alpha}{n}$.*

(1) *The integral defining $I_\alpha f(x)$ converges absolutely for almost everywhere $x \in \mathbb{R}^n$.*

(2) *The estimate $|I_\alpha f(x)| \lesssim M f(x)^{\frac{p}{s}} (\|f\|_{L^p})^{1-\frac{p}{s}}$ holds for a.e. $x \in \mathbb{R}^n$.*

Proof It suffices to prove (2); (1) follows from the $L^p(\mathbb{R}^n)$-boundedness of M if $p > 1$ and from the weak $L^p(\mathbb{R}^n)$-boundedness of M if $p = 1$. We can also assume that $f \in \mathbb{M}^+(\mathbb{R}^n)$ since the integral kernel of I_α is positive. We have only to insert

$$\frac{1}{l^n} \int_{B(x,l)} f(y)\mathrm{d}y \lesssim M f(x), \ \frac{1}{l^n} \int_{B(x,l)} f(y)\mathrm{d}y \lesssim \left(\frac{1}{l^n} \int_{B(x,l)} f(y)^p \mathrm{d}y \right)^{\frac{1}{p}}.$$

into Example 84 to obtain the pointwise estimate. More precisely

$$\begin{aligned} I_\alpha f(x) &\le (n-\alpha) \int_0^\infty \left(\frac{1}{\ell^{n-\alpha+1}} \int_{B(x,\ell)} f(y)\mathrm{d}y \right) \mathrm{d}\ell \\ &\lesssim \int_0^\infty \min\left(M f(x), \left(\frac{1}{l^n} \int_{B(x,l)} f(y)^p \mathrm{d}y \right)^{\frac{1}{p}} \right) \frac{d\ell}{\ell^{1-\alpha}} \\ &\lesssim \int_0^\infty \min\left(M f(x), \frac{1}{l^{\frac{n}{p}}} \|f\|_{L^p} \right) \frac{d\ell}{\ell^{1-\alpha}} \\ &\lesssim M f(x)^{\frac{p}{s}} (\|f\|_{L^p})^{1-\frac{p}{s}}. \end{aligned}$$

Thus, the proof is complete.

Thanks to the $L^p(\mathbb{R}^n)$-boundedness of M with $1 < p < \infty$ and the weak $L^1(\mathbb{R}^n)$-boundedness of M, we obtain:

Theorem 182 (Hardy–Littlewood-Sobolev)**.** *Let $0 < \alpha < n$.*

(1) *For all $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, $\lambda|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}|^{\frac{n-\alpha}{n}} \lesssim \|f\|_{L^1}$.*

(2) *Assume that the parameters p and q satisfy $1 < p < \dfrac{n}{\alpha}$ and $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$. Then $\|I_\alpha f\|_{L^q} \lesssim \|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n)$.*

Proof Simply combine Theorems 134 and 141 and Lemma 181.

We end this section with the relation between the fractional integral operator I_α and the fractional maximal operators $M_{\alpha+\varepsilon}$ and $M_{\alpha-\varepsilon}$, which is a good contrast to the relation between the fractional integral operator I_α and the fractional maximal operator M_α.

Lemma 183. *Let $0 < \alpha < n$, and let $\varepsilon \in (0, \min(\alpha, n-\alpha))$. Then we have*

$$I_\alpha f(x) \lesssim \sqrt{M_{\alpha-\varepsilon} f(x) M_{\alpha+\varepsilon} f(x)} \quad (x \in \mathbb{R}^n)$$

for all $f \in \mathbb{M}^+(\mathbb{R}^n)$.

Proof Simply insert the estimate

$$\frac{1}{\ell^{n-\alpha}} \int_{B(x,\ell)} f(y)\mathrm{d}y \le \min(\ell^{-\varepsilon} M_{\alpha-\varepsilon} f(x), \ell^{\varepsilon} M_{\alpha+\varepsilon} f(x)) \quad (\ell > 0)$$

into Example 84.

We can generalize Lemma 183 as follows: Let $0 \le \alpha_1 < \alpha < \alpha_2 < n$. Then, for all $f \in \mathbb{M}^+(\mathbb{R}^n)$,

$$I_\alpha f(x) \lesssim M_{\alpha_1} f(x)^{\frac{\alpha_2-\alpha}{\alpha_2-\alpha_1}} M_{\alpha_2} f(x)^{\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}} \quad (x \in \mathbb{R}^n).$$

4.4.2 Local estimates for the fractional integral operators

We collect local estimates for fractional integral operators.

Theorem 184. *Let $0 < \alpha < n$, $1 \le p_1 \le p_2 \le \infty$ and $\gamma \in \mathbb{R}$. We define $W(x) \equiv (1+|x|)^{p_1\gamma}$, $x \in \mathbb{R}^n$.*

(1) *If $1 < p_1 \le p_2 < \infty$, then the inequality*

$$\|I_\alpha f\|_{L^{p_2}(Q(1))} \lesssim \|f\|_{L^{p_1}(W)} \equiv \|f \cdot W^{\frac{1}{p_1}}\|_{L^{p_1}}, \tag{4.6}$$

where the implicit constant is independent of f holds for all $f \in \mathbb{M}^+(\mathbb{R}^n)$, if and only if $n\left(\dfrac{1}{p_1} - \dfrac{1}{p_2}\right) \le \alpha < \dfrac{n}{p_1} + \gamma$.

(2) *If $p_1 = 1 \le p_2 < \infty$, then inequality (4.6) holds for all $f \in \mathbb{M}^+(\mathbb{R}^n)$ if and only if $n\left(1 - \dfrac{1}{p_2}\right) < \alpha \le n + \gamma$.*

(3) *If* $1 < p_1 \le \infty$, *then inequality* (4.6) *with* $p_2 = \infty$ *holds for all* $f \in \mathbb{M}^+(\mathbb{R}^n)$ *if and only if* $\dfrac{n}{p_1} < \alpha < \dfrac{n}{p_1} + \gamma$.

(4) *If* $p_1 = 1$ *and* $p_2 = \infty$, *then* (4.6) *fails for any* $0 < \alpha < n$ *and* $\gamma \in \mathbb{R}$.

Remark that (4) reveals the difficulty of fractional integral operators and that at the same time (4) is the source of the interest of analyzing fractional integral operators.

Proof Define p_3 by $\dfrac{1}{p_3} = \dfrac{1}{p_1} - \dfrac{\alpha}{n}$. Let $f \in \mathbb{M}^+(\mathbb{R}^n)$.

(1) Assume $n\left(\dfrac{1}{p_1} - \dfrac{1}{p_2}\right) \le \alpha < \dfrac{n}{p_1} + \gamma$, so that $p_2 \le p_3$. Consequently, by Hölder's inequality and the Hardy–Littlewood–Sobolev theorem, we obtain $\|I_\alpha[\chi_{Q(3)}f]\|_{L^{p_2}(Q(1))} \lesssim \|I_\alpha[\chi_{Q(3)}f]\|_{L^{p_3}} \lesssim \|f\|_{L^{p_3}} \le \|f\|_{L^{p_1}(W)}$. Meanwhile,

$$\begin{aligned} I_\alpha[\chi_{\mathbb{R}^n\setminus Q(3)}f](x) &\sim \int_{\mathbb{R}^n} \frac{f(y)\mathrm{d}y}{(1+|y|)^{n-\alpha}} \\ &\lesssim \|f\|_{L^{p_1}(W)} \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}y}{(1+|y|)^{(n-\alpha+\gamma)p_1'}}\right)^{\frac{1}{p_1'}} \\ &\sim \|f\|_{L^{p_1}(W)} \end{aligned}$$

for any $x \in Q(1)$, since $\alpha < \dfrac{n}{p_1} + \gamma$.

Assume (4.6) instead. Then since $|x-y| \lesssim |y| + 1$ for any $x \in Q(1)$, $\displaystyle\int_{\mathbb{R}^n} \frac{f(y)\mathrm{d}y}{(1+|y|)^{n-\alpha}} \lesssim \|f\|_{L^{p_1}(W)}$, or equivalently, $\displaystyle\int_{\mathbb{R}^n} \frac{g(y)\mathrm{d}y}{(1+|y|)^{n-\alpha+\gamma}} \lesssim \|g\|_{L^{p_1}}$ for all $g \in \mathbb{M}^+(\mathbb{R}^n)$. Thus, duality entails

$$\int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{p_1'(n-\alpha+\gamma)}}\mathrm{d}y < \infty. \tag{4.7}$$

This implies $\alpha < \dfrac{n}{p_1} + \gamma$. Finally, from Theorem 178, we deduce $\dfrac{n}{p_1} - \dfrac{n}{p_2} \le \alpha$. Thus, the proof is complete.

(2) Assume $n\left(1 - \dfrac{1}{p_2}\right) < \alpha \le n + \gamma$. Then going through a similar argument to (1) and using the Hardy–Littlewood–Sobolev theorem, we obtain

$$\|I_\alpha[\chi_{Q(3)}f]\|_{L^{p_2}(Q(1))} \lesssim \|I_\alpha[\chi_{Q(3)}f]\|_{\mathrm{w}L^{p_3}(Q(1))} \lesssim \|f\|_{L^{p_1}(W)}.$$

We estimate $I_\alpha[\chi_{\mathbb{R}^n\setminus Q(3)}f]$ similarly to (1). We thus obtain (4.6) keeping in mind that the case where $\alpha = n + \gamma$ is included in (4.7).

Conversely, assume that (4.6) holds with $p_2 = \dfrac{n}{n-\alpha}$. Then for any $0 < r \ll 1$, $I_\alpha[\chi_{B(r)}](x) \sim r^n|x|^{-n+\alpha}$ on $|x| > 2r$ thanks to Example 83. As a result, $\|I_\alpha[\chi_{B(r)}]\|_{L^{p_2}(Q(1))} \gtrsim r^n \log(r^{-1})$, while $\|\chi_{B(r)}\|_{L^1} \sim r^n$. This contradicts (4.6). Thus, $p_2 \neq \frac{n}{n-\alpha}$. If we go through the same argument as before, we conclude $n\left(1 - \dfrac{1}{p_2}\right) < \alpha \le n + \gamma$. See Exercise 70.

(3) If $\dfrac{n}{p_1} < \alpha$, then by Hölder's inequality we can show that $f \in L^{p_1}(\mathbb{R}^n) \mapsto \chi_{Q(1)} I_\alpha[\chi_{Q(1)} f] \in L^\infty(\mathbb{R}^n)$ is a bounded linear operator. Consequently, we can go through the same argument as before. If $\dfrac{n}{p_1} = \alpha$, then

$$f(x) \equiv \chi_{B(e^{-1})}(|x|)|x|^{\frac{n}{p_1}} \log \frac{1}{|x|} \quad (x \in \mathbb{R}^n)$$

is an $L^{p_1}(\mathbb{R}^n)$-function which is mapped to an unbounded function by I_α. Other parts are the same as the model case (1).

(4) Should $\|I_\alpha f\|_{L^\infty(Q(1))} \lesssim \|f\|_{L^1(W)}$ hold for all $f \in L^1(W)$, then we would have $|I_\alpha[gW^{-1}](x)| \lesssim \|g\|_{L^1}$ for all $g \in L^1(\mathbb{R}^n)$. Hence, by duality for any $x \in \mathbb{R}^n$,

$$\frac{1}{|x - \cdot|^{n-\alpha}(1 + |\cdot|)^\gamma} \in L^\infty(\mathbb{R}^n).$$

This is clearly false for any $x \in \mathbb{R}^n$, since $n - \alpha > 0$.

We extend Theorem 184 to the case of $p_1 = 1$ and $0 < p_2 < \infty$.

Corollary 185. *Let $0 < p < \infty$ and $n\left(1 - \frac{1}{p}\right)_+ < \alpha < n$. Then for all $\delta > 0$, $r > 0$ and for all $f \in \mathbb{M}^+(\mathbb{R}^n)$*

$$\|I_\alpha f\|_{L^p(B(r))} \lesssim r^{\frac{n}{p}} \int_{\mathbb{R}^n} \frac{f(x)}{(|x| + r)^{n-\alpha}} \mathrm{d}x.$$

Proof By Hölder's inequality, we may assume $p > 1$. Then we are in the position of using Theorem 184 and the dilation argument.

We consider what happens for $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ in Corollary 185.

Corollary 186. *Let $1 < p_1 \le \infty$, $0 < p_2 \le \infty$ and*

$$n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ < \alpha < n,$$

or let $1 < p_1 < p_2 < \infty$ and

$$\alpha = \frac{n}{p_1} - \frac{n}{p_2}.$$

Then for all $\delta > 0$

$$\|I_\alpha f\|_{L^{p_2}(B(r))} \lesssim r^{\frac{n}{p_2}-\delta} \left\{ \int_{\mathbb{R}^n} \frac{f(x)^{p_1}}{(|x|+r)^{n-(\alpha+\delta)p_1}} \mathrm{d}x \right\}^{\frac{1}{p_1}} \tag{4.8}$$

for all $r > 0$ *and for all* $f \in \mathbb{M}^+(\mathbb{R}^n)$. *In particular, if* $\delta = \dfrac{n}{p_1} - \alpha$, *then*

$$\|I_\alpha f\|_{L^{p_2}(B(r))} \lesssim r^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L^{p_1}} \tag{4.9}$$

for all $f \in L^{p_1}(\mathbb{R}^n)$ *and* $r > 0$ *and if* $0 < \delta < \dfrac{n}{p_1} - \alpha$, *then*

$$\|I_\alpha f\|_{L^{p_2}(B(r))} \lesssim r^{\frac{n}{p_2}-\delta} \left\{ \int_r^\infty \left(\int_{B(t)} f(x)^{p_1} \mathrm{d}x \right) \frac{\mathrm{d}t}{t^{n-(\alpha+\delta)p_1+1}} \right\}^{\frac{1}{p_1}} \tag{4.10}$$

for all $r > 0$ *and for all* $f \in \mathbb{M}^+(\mathbb{R}^n)$.

Proof By Hölder's inequality, we may assume $p_2 > p_1$ to prove (4.8). Then we can use the scaling argument. See Exercise 71 for more.

Let $p_1 > 1$ and $\delta = 0$ in (4.8). We consider $f(x) \equiv |x|^{-\alpha}(\log|x|)^{-1}\chi_{B(2)}(x)$ for $x \in \mathbb{R}^n$. Then the right-hand side of (4.8) is finite whilst $(I_\alpha f)(x) = \infty$ for all $x \in \mathbb{R}^n$. Hence (4.8) cannot hold with $\delta = 0$.

Remark 8. If $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha < n$, then

$$\|M_\alpha f\|_{L^{p_2}(B(r))} \lesssim r^{\frac{n}{p_2}} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^{p_1}}{(|x|+r)^{n-\alpha p_1}} \mathrm{d}x \right)^{\frac{1}{p_1}},$$

for all $r > 0$ and $f \in L^0(\mathbb{R}^n)$. Consequently if I_α is replaced by M_α then inequality (4.10) also holds for $\delta = 0$.

In the same spirit we can prove the following estimate:

Lemma 187. *Let* $0 < \alpha < n$. *Let* $1 < q < t < \infty$ *satisfy*

$$\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}.$$

Let $B \equiv B(x,r)$ *be a ball. Then for all* $f \in \mathbb{M}^+(\mathbb{R}^n)$,

$$\|I_\alpha f\|_{L^t(B)} \lesssim \|f\|_{L^q(2B)} + r^{\frac{n}{t}} \int_r^\infty s^{\alpha-n} \|f\|_{L^1(B(x,s))} \frac{\mathrm{d}s}{s}.$$

Proof We calculate

$$\begin{aligned} \|I_\alpha f\|_{L^t(B)} &\lesssim \|I_\alpha[\chi_{2B} f]\|_{L^t(B)} + \|I_\alpha[f - \chi_{2B} f]\|_{L^t(B)} \\ &\lesssim \|f\|_{L^q(2B)} + \|I_\alpha[f - \chi_{2B} f]\|_{L^t(B)} \\ &\lesssim \|f\|_{L^q(B)} + r^{\frac{n}{t}} \int_r^\infty s^{\alpha-n} \|f\|_{L^1(B(x,s))} \frac{\mathrm{d}s}{s}. \end{aligned}$$

4.4.3 Sparse estimate of the fractional integral operators

We are now interested in the sparse estimate of I_α.

Let $0 < \alpha < n$. We start with an auxiliary operator. Suppose that we have a cube Q^0 and a family $\mathcal{S}$. Define the *discrete fractional integral operator* $I_\alpha^{\mathcal{S}} f$ generated by $\mathcal{S}$ and the *global operator* $C_{\alpha,\infty}(f)$ by

$$I_\alpha^{\mathcal{S}} f \equiv \sum_{Q\in\mathcal{S}} \ell(Q)^\alpha m_{3Q}(f)\chi_Q \text{ and } C_{\alpha,\infty}(f) \equiv \sum_{k=0}^{\infty} \ell(2^k Q^0)^\alpha m_{2^k Q^0}(f).$$

If $\mathcal{S} = \mathcal{D}(\mathbb{R}^n)$, then $I_\alpha^{\mathcal{S}} f$ is simply called the *discrete fractional integral operator*. In particular, for $\mathcal{S} = \mathcal{D}(Q^0)$, we have

$$I_\alpha^{\mathcal{D}(Q^0)} f \equiv \sum_{Q\in\mathcal{D}(Q^0)} \ell(Q)^\alpha m_{3Q}(f)\chi_Q.$$

We begin with a preparatory but useful estimate.

Proposition 188. *Let $0 < \alpha < n$. For $f \in \mathbb{M}^+(\mathbb{R}^n)$, $I_\alpha f \sim I_\alpha^{\mathcal{D}} f$.*

Proof It suffices to compare the integral kernel after we fix x and y in $\mathbb{R}^n$: Let $K(x,y) \equiv \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \ell(Q)^{\alpha-n}\chi_Q(x)\chi_{3Q}(y)$. Let ν_0 be the largest integer such that $x \in Q$ and $y \in 3Q$ for some $Q \in \mathcal{D}_\nu(\mathbb{R}^n)$. Observe that $2^{-\nu_0} \sim |x-y|$. In fact, if we let $Q' \in \mathcal{D}_{\nu+1}(\mathbb{R}^n)$ be the unique cube such that $x \in Q'$, then $y \notin 3Q'$. Thus, $|x-y| \gtrsim \ell(Q') \sim 2^{-\nu_0}$. Since $x, y \in 3Q$, we have $|x-y| \lesssim \ell(Q) = 2^{-\nu_0}$. Thus, $K(x,y) \sim \sum_{j=-\infty}^{\nu_0} 2^{-j(n-\alpha)} \sim |x-y|^{-n+\alpha}$. Consequently, the integral kernels of these operators are equivalent.

We can stop tripling Q in Proposition 188 by the use of the maximal operator. We define

$$\mathcal{D}_j^{\vec{e}}(\mathbb{R}^n) \equiv \{2^j([0,1)^n + m + (-1)^j 3^{-1}\vec{e}) : m \in \mathbb{Z}^n\},$$

for $\vec{e} \in \{0,1,-1\}^n$ and $j \in \mathbb{Z}$ and

$$\mathcal{D}^{\vec{e}}(\mathbb{R}^n) = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j^{\vec{e}}(\mathbb{R}^n)$$

for $\vec{e} \in \{0,1,-1\}^n$.

Corollary 189. *For $f \in \mathbb{M}^+(\mathbb{R}^n)$, $I_\alpha f \lesssim \sup_{\vec{e}\in\{0,1,-1\}^n} \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \ell(Q)^\alpha m_Q(f)\chi_Q$ holds.*

Proof An important observation is that for any cube $Q \in \mathcal{Q}$ there exists $P \in \bigcup_{\vec{e}\in\{0,1,-1\}^n} \mathcal{D}^{\vec{e}}$ such that $P \subset Q$ and $|Q| \le 3^n|P|$. If we combine this observation with Proposition 188, we obtain the desired result.

The following is the sparse decomposition of I_α.

Lemma 190. *Let $0 < \alpha < n$ and $f \in L^\infty_{\rm c}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$. Then, for any cube $Q^0 \in \mathcal{Q}$, there exists a sparse family $\mathcal{S} \subset \mathcal{D}(Q^0)$ such that, for all $x \in Q^0$, $I_\alpha f(x) \lesssim I^{\mathcal{S}}_\alpha f(x) + C_{\alpha,\infty}(f)$.*

Proof For all $x \in Q^0$ we have $I_\alpha f(x) \lesssim I^{\mathcal{D}(Q^0)}_\alpha f(x) + C_{\alpha,\infty}(f)$. We use the construction in Theorem 172 with $\alpha > 0$ there. Write $\mathcal{D}^j_k \equiv \{Q \in \mathcal{D}(Q^j_k) : m_Q(f) \le a_0 a^{j+1}\}$. Fix $x \in Q^j_k$, $j \in \mathbb{N}_0$ and $k \in K_j$. Since $\alpha > 0$, we have $\sum_{Q \in \mathcal{D}^j_k} \chi_Q(x) \ell(Q)^\alpha \lesssim \ell(Q^j_k)^\alpha$. Thus, we have

$$\begin{aligned}\sum_{Q \in \mathcal{D}^j_k} \chi_Q(x) \ell(Q)^\alpha m_{3Q}(f) &\le a_0 a^{j+1} \sum_{Q \in \mathcal{D}^j_k} \chi_Q(x) \ell(Q)^\alpha \\ &\lesssim a_0 a^{j+1} \ell(Q^j_k)^\alpha \\ &\lesssim \ell(Q^j_k)^\alpha m_{3Q^j_k}(f)\end{aligned}$$

for all $Q^j_k \in \mathcal{S}$. Consequently, $I^{\mathcal{D}(Q^0)}_\alpha f(x) \lesssim \sum_{Q \in \mathcal{S}} \chi_Q(x) \ell(Q)^\alpha m_{3Q}(f)$ for all $x \in Q^0$. This completes the proof.

Following the procedure of Theorem 170 with h replaced by f, we can find a sparse family $\mathcal{S}$ such that the following estimate holds:

Theorem 191. *For $f \in \mathbb{M}^+(\mathbb{R}^n)$ and $0 < \alpha < n$, there exists a sparse family $\mathcal{S} = \mathcal{S}_f$ such that $I_\alpha f(x) \sim \sum_{Q \in \mathcal{S}} \ell(Q)^\alpha m_{3Q}(f) \cdot \chi_Q(x)$ for $x \in \mathbb{R}^n$.*

Proof Since the proof is similar to Lemma 190, we omit the proof.

4.4.4 Fundamental solution to the elliptic differential operators

Let us see how I_α with $\alpha = 2$ comes about in the context of other fields of mathematics. This operator can be regarded as a majorant operator of the fundamental solution to the elliptic differential equations.

Theorem 192. *Let $n \ge 3$ and let $A = \{a_{ij}\}^n_{i,j=1}$ be a positive definite matrix with inverse $A^{-1} = \{a^{ij}\}^n_{i,j=1}$. Let L be a constant differential operator given by $L \equiv -\sum_{i,j=1}^n a_{ij} \partial_i \partial_j$. Define*

$$\Gamma(x) \equiv \frac{1}{(n-2)\omega_{n-1}\sqrt{\det A}} \left(\sum_{i,j=1}^n a^{ij} x_i x_j \right)^{1-\frac{n}{2}} \qquad (x \in \mathbb{R}^n).$$

Then $\varphi = L[\Gamma * \varphi] = \Gamma * [L\varphi]$ for all $\varphi \in C^\infty_{\rm c}(\mathbb{R}^n)$.

Proof A direct calculation shows that $L[\Gamma * \varphi](x) = \Gamma * [L\varphi](x)$ in the classical case. Since the conclusion is translation invariant, we may suppose that $x = 0$. We write out this quantity in full:

$$\Gamma * [L\varphi](0) = \int_{\mathbb{R}^n} \frac{1}{(n-2)\omega_{n-1}\sqrt{\det A}} \left(\sum_{i,j=1}^{n} a^{ij} y_i y_j \right)^{1-\frac{n}{2}} L\varphi(y)\mathrm{d}y.$$

Denote by $\sqrt{A} = \{b_{ij}\}_{i,j=1}^n$ the unique positive definite matrix such that $(\sqrt{A})^2 = A$. Then a change of variables yields:

$$\Gamma * [L\varphi](0) = \int_{\mathbb{R}^n} \frac{1}{(n-2)\omega_{n-1}} |y|^{1-\frac{n}{2}} L\varphi(\sqrt{A}y)\mathrm{d}y.$$

Write $\varphi_A(y) \equiv \varphi(\sqrt{A}y)$. Note that

$$\Delta\varphi_A(y) = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} [b_{kj}b_{lj}[\partial_l\partial_k\varphi](\sqrt{A}y)] = -L\varphi(\sqrt{A}y).$$

Thus, we obtain

$$\Gamma * [L\varphi](0) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |y|^{2-n}(-\Delta\varphi_A)(y)\mathrm{d}y.$$

Let $\tau \in C^\infty((-\infty,\infty))$ be a radial function such that $\chi_{(1,\infty)} \le \tau \le \chi_{(0,\infty)}$. To apply the Stokes theorem, we delete the singularity:

$$\Gamma * [L\varphi](0) = \frac{1}{(n-2)\omega_{n-1}} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \tau\left(\frac{|y|}{\varepsilon}\right) |y|^{2-n}(-\Delta\varphi_A)(y)\mathrm{d}y.$$

By the Stokes theorem, we have

$$\Gamma * [L\varphi](0) = -\frac{1}{(n-2)\omega_{n-1}} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \Delta\left[\tau\left(\frac{|y|}{\varepsilon}\right) |y|^{2-n}\right] \varphi_A(y)\mathrm{d}y.$$

Note that

$$\left|\Delta\left[\tau\left(\frac{|y|}{\varepsilon}\right) \frac{1}{(n-2)\omega_{n-1}} |y|^{2-n}\right]\right| \lesssim \varepsilon^{-n}\chi_{B(\varepsilon)}(y).$$

Thus, thanks to the continuity of φ at 0,

$$\begin{aligned}\Gamma * [L\varphi](0) &= -\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \Delta\left[\tau\left(\frac{|y|}{\varepsilon}\right) \frac{1}{(n-2)\omega_{n-1}} |y|^{2-n}\right] \varphi_A(0)\mathrm{d}y \\ &= -\varphi_A(0) \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \Delta\left[\tau\left(\frac{|y|}{\varepsilon}\right) \frac{1}{(n-2)\omega_{n-1}} |y|^{2-n}\right] \mathrm{d}y.\end{aligned}$$

By the use of the polar coordinate, we have

$$\begin{aligned}\Gamma * [L\varphi](0) &= -\frac{\varphi_A(0)}{n-2}\int_0^\varepsilon r^{n-1}\left(\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{\mathrm{d}r}\right)\left[\tau\left(\frac{r}{\varepsilon}\right)\frac{r^{2-n}}{(n-2)\omega_{n-1}}\right]\mathrm{d}r\\ &= -\frac{\varphi_A(0)}{n-2}\int_0^\varepsilon \frac{d}{\mathrm{d}r}\left(r^{n-1}\frac{d}{\mathrm{d}r}r^{2-n}\tau\left(\frac{r}{\varepsilon}\right)\right)\mathrm{d}r\\ &= -\frac{\varphi_A(0)}{n-2}\int_0^\varepsilon \frac{d}{\mathrm{d}r}\left(r^{n-1}(2-n)r^{1-n}\tau\left(\frac{r}{\varepsilon}\right)+\frac{1}{\varepsilon}r^{2-n}\tau'\left(\frac{r}{\varepsilon}\right)\right)\mathrm{d}r\\ &= \varphi_A(0)=\varphi(0).\end{aligned}$$

Thus, we obtain the desired result.

As an application of Theorem 192, we obtain an estimate of functions in terms of their derivative. The fractional integral operator I_1 of order 1 is also important as the following theorem shows:

Theorem 193. *Let $n \geq 2$. Then $|f| \lesssim I_1[|\nabla f|]$ for all $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$.*

Proof Fix $x \in \mathbb{R}^n$. We suppose $n \geq 3$; the case of $n = 2$ can be handled similarly except in that we need to handle the kernel $\log|x-y|$. We omit the proof for the case of $n = 2$. Thanks to Theorem 192, we have $f(x) \simeq_n I_2[\Delta f](x)$. Since $n \geq 3$, we can perform the integration by parts to have

$$f(x) \simeq_n \sum_{k=1}^n \int_{\mathbb{R}^n} \frac{x_k - y_k}{|x-y|^n}\partial_k f(y)\mathrm{d}y,$$

so that by the triangle inequality for integrals, we have the desired result.

4.4.5 The Bessel potential operator $(1-\Delta)^{-\frac{s}{2}}, s > 0$

The operator dealt with in Section 4.4.5 lies outside the main stream of Section 4.4. However due to the similarity and the strong connection with fractional integral operartors, we consider the Bessel potential operator $(1-\Delta)^{-\frac{s}{2}}, s > 0$. We will show the multiplicative law $(1-\Delta)^{-s_1}(1-\Delta)^{-s_2} = (1-\Delta)^{-s_1-s_2}$ and the inverse formula $(1-\Delta)(1-\Delta)^{-1}f = (1-\Delta)^{-1}(1-\Delta)f = f$ for suitable functions f.

We start with the definition of the Bessel kernel.

Definition 56. Let $s > 0$. Then we define the *Bessel kernel* of order s by

$$G_s(x) \equiv \lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\exp(-|\varepsilon\xi|^2)e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi \quad (x \in \mathbb{R}^n).$$

Roughly speaking, $G_s(x)$ is the integral $\displaystyle\int_{\mathbb{R}^n}\frac{e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi$. However, if $0 < s \leq n$, then $\displaystyle\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi = \infty$, so that we need to take an indirect way to define G_s. We need to justify:

Lemma 194. *Let $s > 0$. Then the limit defining $G_s(x)$ exists for each $x \in \mathbb{R}^n$.*

Proof We use integration by parts to the expression. To this end we transform

$$\int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\xi|^2)e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi = \frac{1}{|x|^{2n}}\int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\xi|^2)(-\Delta_\xi)^n e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi.$$

Since $\sup_{t>0} t^m \exp(-t) < \infty$ for each $m \in \mathbb{N}$,

$$\begin{aligned}&\int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\xi|^2)(-\Delta_\xi)^n e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi\\ &= \int_{\mathbb{R}^n} \exp(-|\varepsilon\xi|^2)e^{ix\cdot\xi}(-\Delta_\xi)^n\left[(1+|\xi|^2)^{-\frac{s}{2}}\right]\mathrm{d}\xi + \mathrm{o}(1)\end{aligned}$$

as $\varepsilon \downarrow 0$. Thanks to the Lebesgue convergence theroem, we have

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} \exp(-|\varepsilon\xi|^2)e^{ix\cdot\xi}(-\Delta_\xi)^n\left[(1+|\xi|^2)^{-\frac{s}{2}}\right]\mathrm{d}\xi\\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}(-\Delta_\xi)^n\left[(1+|\xi|^2)^{-\frac{s}{2}}\right]\mathrm{d}\xi.\end{aligned}$$

As a result, we have an expression:

$$G_s(x) = \frac{1}{(2\pi)^n|x|^{2n}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}(-\Delta_\xi)^n\left[(1+|\xi|^2)^{-\frac{s}{2}}\right]\mathrm{d}\xi \quad (x \in \mathbb{R}^n).$$

Likewise, for any $l \gg 1$, we have

$$G_s(x) = \frac{1}{(2\pi)^n|x|^{2l}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}(-\Delta_\xi)^l\left[(1+|\xi|^2)^{-\frac{s}{2}}\right]\mathrm{d}\xi \quad (x \in \mathbb{R}^n).$$

From this expression, we learn:

Corollary 195. *Let $s > 0$. Then $G_s \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and satisfies $G_s(x) = \mathrm{O}(|x|^{-l})$ as $x \to \infty$ for any $l \in \mathbb{N}$.*

Hence, we are interested in the behavior of this function near the origin. We obtain an asymptotic expression of G_s near the origin.

Proposition 196. *Let $s > 0$. Then $G_s(x) = \mathrm{O}(|x|^{-n+s})$ as $x \to 0$.*

Proof Let $\eta(x) \equiv \exp(-|x|^2)$, $x \in \mathbb{R}^n$. Fix $x \in B(1)$. We write

$$g_{s,0}(x) \equiv \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\eta(\xi)e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi \quad (x \in \mathbb{R}^n)$$

and for $l \in \mathbb{N}$

$$g_{s,l}(x) \equiv \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{(\eta(2^{-l}\xi) - \eta(2^{1-l}\xi))e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}}\mathrm{d}\xi \quad (x \in \mathbb{R}^n).$$

We decompose $G_s(x) = \sum_{l=0}^{\infty} g_{s,l}(x)$. It is easy to see that $g_{s,0}$ is a bounded function, so that we have only to take care of $g_{s,l}$. If we perform integration by parts, $|g_{s,l}(x)| \lesssim (2^l|x|)^{-N}2^{l(n-s)}$ for any $N \in \mathbb{N}$ and $x \in \mathbb{R}^n$. If we add this estimate over $l \in \mathbb{N}$, we obtain

$$\sum_{l=1}^{\infty} |g_{s,l}(x)| \lesssim \sum_{l=1}^{\infty} \min(2^{l(n-s)}, 2^{-s}|x|^{-n}) = \mathrm{O}(|x|^{-n+s})$$

as $x \to 0$.

As a result, although G_s is defined in a tricky manner, this is integrable.

Corollary 197. *Let $s > 0$. Then $G_s \in L^1(\mathbb{R}^n)$.*

If we use Corollary 197, then we see that the operator $f \in L^1(\mathbb{R}^n) \mapsto G_s * f \in L^1(\mathbb{R}^n)$ is a bounded linear operator.

As we will prove, we have the following multiplicative law:

Theorem 198. *Let $s_1, s_2 > 0$. Then $G_{s_1+s_2} = G_{s_1} * G_{s_2}$.*

Proof We remark that in the course of the proof of Corollary 197, we have showed that

$$\sup_{\varepsilon>0} \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\xi|^2)e^{ix\cdot\xi}}{(1+|\xi|^2)^{\frac{s}{2}}} \mathrm{d}\xi \right| \lesssim \min(|x|^{-n+s}, |x|^{-n-1}).$$

We calculate

$$\begin{aligned}
&(2\pi)^{2n} G_{s_1} * G_{s_2}(x) \\
&= (2\pi)^{2n} \int_{\mathbb{R}^n} G_{s_1}(y) G_{s_2(x-y)} \mathrm{d}y \\
&= \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\xi|^2)e^{iy\cdot\xi}}{(1+|\xi|^2)^{\frac{s_1}{2}}} \mathrm{d}\xi \right) \left(\int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\zeta|^2)e^{i(x-y)\cdot\zeta}}{(1+|\zeta|^2)^{\frac{s_2}{2}}} \, d\zeta \right) \mathrm{d}y
\end{aligned}$$

in the topology of $L^1(\mathbb{R}^n)$. We observe

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\xi|^2)e^{iy\cdot\xi}}{(1+|\xi|^2)^{\frac{s_1}{2}}} \mathrm{d}\xi \right) \left(\int_{\mathbb{R}^n} \frac{\exp(-|\varepsilon\zeta|^2)e^{i(x-y)\cdot\zeta}}{(1+|\zeta|^2)^{\frac{s_2}{2}}} \, d\zeta \right) \mathrm{d}y \\
&= \lim_{\delta\downarrow 0} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{e^{iy\cdot\xi-|\varepsilon\xi|^2} \mathrm{d}\xi}{(1+|\xi|^2)^{\frac{s_1}{2}}} \right) \left(\int_{\mathbb{R}^n} \frac{e^{i(x-y)\cdot\zeta-|\varepsilon\zeta|^2} \, d\zeta}{(1+|\zeta|^2)^{\frac{s_2}{2}}} \right) \frac{dy}{\exp(\delta|y|^2)} \\
&= \lim_{\delta\downarrow 0} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\exp(-\varepsilon^2(|\xi|^2+|\zeta|^2))}{(1+|\xi|^2)^{\frac{s_1}{2}}(1+|\zeta|^2)^{\frac{s_2}{2}}} \left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\zeta+iy\cdot\xi} \frac{dy}{\exp(\delta|y|^2)} \right) \mathrm{d}\xi \, d\zeta
\end{aligned}$$

thanks to Corollary 195, Proposition 196 and Fubini's theorem. Using the heat kernel, we calculate

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{\exp(-|\varepsilon\xi|^2)e^{iy\cdot\xi}}{(1+|\xi|^2)^{\frac{s_1}{2}}}\mathrm{d}\xi\right)\left(\int_{\mathbb{R}^n}\frac{\exp(-|\varepsilon\zeta|^2)e^{i(x-y)\cdot\zeta}}{(1+|\zeta|^2)^{\frac{s_2}{2}}}\,d\zeta\right)\mathrm{d}y\\
&=\lim_{\delta\downarrow 0}\left(\frac{\pi}{\delta}\right)^{\frac{n}{2}}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\exp(-\varepsilon^2(|\xi|^2+|\zeta|^2))e^{ix\cdot\zeta}}{(1+|\xi|^2)^{\frac{s_1}{2}}(1+|\zeta|^2)^{\frac{s_2}{2}}}\exp\left(-\frac{1}{4\delta}|\xi-\zeta|^2\right)\mathrm{d}\xi\,d\zeta\\
&=(2\pi)^n\int_{\mathbb{R}^n}\frac{\exp(-|\varepsilon\xi|^2)^2e^{ix\cdot\zeta}}{(1+|\xi|^2)^{\frac{s_1+s_2}{2}}}\mathrm{d}\xi.
\end{aligned}$$

Thus, $G_{s_1}*G_{s_2}=G_{s_1+s_2}$ follows.

Among $G_s, s>0$, G_2 is important because this is used to express the solution u of $(1-\Delta)u=f$.

Theorem 199. *For all $f\in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, $(1-\Delta)[G_2*f]=G_2*[(1-\Delta)f]=f$.*

Proof Let $x\in\mathbb{R}^n$ be fixed. We calculate

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}(1-\Delta)f(y)\frac{\exp(-\varepsilon|\xi|^2)e^{i(x-y)\cdot\xi}}{(1+|\xi|^2)}\mathrm{d}\xi\mathrm{d}y\\
&=\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}f(y)\exp(-\varepsilon|\xi|^2)e^{i(x-y)\cdot\xi}\mathrm{d}\xi\mathrm{d}y\\
&=\frac{\sqrt{\varepsilon^{-n}\pi^n}}{(2\pi)^n}\int_{\mathbb{R}^n}f(y)\exp\left(-\frac{1}{4\varepsilon}|x-y|^2\right)\mathrm{d}\xi\mathrm{d}y\to f(x)
\end{aligned}$$

as $\varepsilon\downarrow 0$, showing that $G_2*[(1-\Delta)f]=f$. Likewise, We can show that

$$\begin{aligned}
&(1-\Delta)[G_2*f](x)\\
&=\lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}(1-\Delta)f(y)\frac{\exp(-\varepsilon|\xi|^2)e^{i(x-y)\cdot\xi}}{(1+|\xi|^2)}\mathrm{d}\xi\mathrm{d}y\\
&=f(x).
\end{aligned}$$

4.4.6 Morrey's lemma

We sharpen Theorem 193. Here we investigate an inequality which eventually led to the definition of Morrey spaces.

Lemma 200. *Let $B(x,r)$ be a ball. Then for all $u\in C^1(\overline{B(x,r)})$,*

$$\int_{B(x,r)}|u(x)-u(y)|\mathrm{d}y\le\frac{r^n}{n}\int_{B(x,r)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\mathrm{d}y.$$

Proof Fix any point $w\in S^{n-1}$ and $s\in(0,r]$. Then

$$|u(x+sw)-u(x)|=\left|\int_0^s w\cdot\nabla u(x+tw)\mathrm{d}t\right|\le\int_0^s|\nabla u(x+tw)|\,\mathrm{d}t.$$

Hence for $s \in (0, r]$,

$$\begin{aligned}\|u(x+s\cdot)-u(x)\|_{L^1(S^{n-1})} &\le \int_{S^{n-1}} \left(\int_0^s t^{-n+1} |\nabla u(x+tw)|\, t^{n-1} \mathrm{d}t \right) \mathrm{d}\sigma(w) \\ &= \int_{B(s)} |y|^{-n+1} |\nabla u(x+y)|\, \mathrm{d}y \\ &= \int_{B(x,s)} |x-y|^{-n+1} |\nabla u(y)|\, \mathrm{d}y \\ &\le \int_{B(x,r)} |x-y|^{-n+1} |\nabla u(y)|\, \mathrm{d}y.\end{aligned}$$

Multiply by s^{n-1} and integrate from 0 to r against s to have

$$\int_{B(x,r)} |u(x)-u(y)| \mathrm{d}y \le \frac{r^n}{n} \int_{B(x,r)} |x-y|^{-n+1} |\nabla u(y)|\, \mathrm{d}y.$$

4.4.7 Exercises

Exercise 69. Let $f(\theta) \equiv \dfrac{\theta^p}{p} A + \dfrac{\theta^{-p'}}{p'} B$ with $A, B > 0$. Find the minimum of $f : [0, \infty) \to [0, \infty)$.

Exercise 70. Let $f(x) \equiv (1+|x|)^\theta, x \in \mathbb{R}^n$ with $\theta \in \mathbb{R}$. Let $p_0 = 1 < p_1 < \infty$.

(1) Find the necessary and sufficient condition on θ for $f \in L^\infty(\mathbb{R}^n)$.

(2) Find an analogue to (4.7) to show $\alpha \le n + \gamma$ if (4.6) holds.

Exercise 71. Prove Corollary 186. Hint: Use Theorem 184(1).

Exercise 72. [424, p. 252] Let Q be a dyadic cube, and let $0 < \alpha < n$. Then for any $f \in \mathbb{M}^+(\mathbb{R}^n)$, show that

$$\sum_{R \in \mathcal{D}(Q)} \ell(R)^\alpha \int_R f(x) \mathrm{d}x \lesssim \ell(Q)^\alpha \int_Q f(x) \mathrm{d}x$$

using the decomposition

$$\sum_{R \in \mathcal{D}(Q)} \ell(R)^\alpha \int_R f(x) \mathrm{d}x = \sum_{l=0}^{\infty} \sum_{R \in \mathcal{D}_l(Q)} \ell(R)^\alpha \int_R f(x) \mathrm{d}x$$

4.5 Singular integral operators

As well as the Hardy–Littlewood maximal operator, singular integral operators are important in harmonic analysis. One of our goals of this book is the

consideration of the boundedness of integral operators. Section 4.5, presents the definition of singular integral operators which are already general enough to applications. Then we consider the $L^p(\mathbb{R}^n)$-boundedness of singular integral operators. This boundedness will be a key to the boundedness of singular integral operators acting on Morrey spaces.

We take up examples of singular integral operators in Section 4.5.1. The Hilbert transform and the Riesz transform will be typical examples of singular integral operators. Section 4.5.2 generalizes what we considered in Section 4.5.1. Here, we define Calderón–Zygmund operators and then we consider their fundamental properties. We do not embark on the boundedness properties other than the $L^2(\mathbb{R}^n)$-boundedness here. Section 4.5.3 is a preparatory but important step to deal with singular operators. Here, we obtain a decomposition method called the Calderón–Zgymund decomposition. Based on the preliminary observation made in Sections 4.5.1 – 4.5.3, we investigate the boundedness properties in Section 4.5.4. As the name suggests, singular integral operators have the singularity. More precisely, as it turns out, the integral kernels have the singularity at the diagonal $\{(x,x) : x \in \mathbb{R}^n\}$. Consequently, it is natural to avoid such a singularity by truncating the integration domains. Section 4.5.5 is devoted to such an attempt. Additional examples of singular integral operators will be provided in Section 4.5.6. Finally, we investigate the sparse estimates and the local estimates in Sections 4.5.7 and 4.5.8, respectively.

4.5.1 Riesz transform

Before we give the precise definition of singular integral operators, we will consider one concrete example of the Riesz transform. We will prove the following theorem:

Theorem 201. *Let $j = 1, 2, \dots, n$. Then the linear operator, called the Riesz transform,*

$$R_j f(x) \equiv \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) \mathrm{d}y \quad (x \in \mathbb{R}^n),$$

which is initially defined for $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, extends to a bounded linear operator on $L^2(\mathbb{R}^n)$.

We will prove Theorem 201 as the special case of Example 92. It is not so hard to show that the limit defining $R_j f$ exists for $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$; see Exercise 74.

Example 85. The most prominent case is the case where $n = 1$. In this case,

$$Hf(t) \equiv \lim_{\varepsilon \downarrow 0} \left(\int_{-\infty}^{-\varepsilon} \frac{f(t-s)}{s} \mathrm{d}s + \int_{\varepsilon}^{\infty} \frac{f(t-s)}{s} \mathrm{d}s \right) \quad (t \in \mathbb{R}).$$

This transform is called the *Hilbert transform*. In particular, for $f = \chi_{[-1,1]}$, we have $Hf(t) = \log\left|\frac{t+1}{t-1}\right|$ for almost all $t \in \mathbb{R}$.

The Riesz transform arises naturally.

Example 86. Let $n \geq 2$. Let $f \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$. Let I_1 be the fractional integral operator of order 1; $I_1 f(x) = \displaystyle\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} f(y)\mathrm{d}y, x \in \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$

$$\partial_j I_1 f(x) = -\lim_{\varepsilon \downarrow 0}(n-1)\int_{\mathbb{R}^n \setminus B(x,r)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\mathrm{d}y.$$

In fact, since $I_1 f(x) = \displaystyle\int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-1}}\mathrm{d}y$, $\partial_j I_1 f(x) = \displaystyle\int_{\mathbb{R}^n} \frac{\partial_j f(x-y)}{|y|^{n-1}}\mathrm{d}y$. Let $\eta \in C^{\infty}(\mathbb{R}) \cap \mathbb{M}^{\uparrow}(\mathbb{R})$ satisfy $\chi_{(1,\infty)} \leq \eta \leq \chi_{(1/2,\infty)}$. Then

$$\partial_j I_1 f(x) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \frac{\eta(\varepsilon|y|)}{|y|^{n-1}} \partial_j f(x-y)\mathrm{d}y.$$

By integration by parts, we have

$$\partial_j I_1 f(x) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \partial_j \left(\frac{\eta(\varepsilon|y|)}{|y|^{n-1}}\right) f(x-y)\mathrm{d}y.$$

By the Leibniz rule,

$$\partial_j \left(\frac{\eta(\varepsilon|y|)}{|y|^{n-1}}\right) = (1-n)\frac{\eta(\varepsilon|y|)y_j}{|y|^{n+1}} + \frac{\varepsilon y_j \eta'(\varepsilon|y|)}{|y|^{n+1}}.$$

Since $y \mapsto y_j$ is an odd function,

$$\int_{\mathbb{R}^n} \frac{\varepsilon y_j \eta'(\varepsilon|y|)}{|y|^{n+1}} f(x-y)\mathrm{d}y = \int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{\varepsilon y_j \eta'(\varepsilon|y|)}{|y|^{n+1}} (f(x-y) - f(x))\mathrm{d}y.$$

By the triangle inequality, we have,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \frac{\varepsilon y_j \eta'(\varepsilon|y|)}{|y|^{n+1}} f(x-y)\mathrm{d}y\right| \\
&\leq \int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{\varepsilon|y_j|\eta'(\varepsilon|y|)}{|y|^{n+1}} |f(x-y) - f(x)|\mathrm{d}y \\
&\leq \|\eta'\|_{L^{\infty}(0,\infty)} \sup_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} |f(x-y) - f(x)| \int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{\varepsilon|y_j|}{|y|^{n+1}}\mathrm{d}y \\
&= \|\eta'\|_{L^{\infty}(0,\infty)} \sup_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} |f(x-y) - f(x)| \int_{\frac{1}{2}<|y|<1} \frac{\varepsilon|y_j|}{|y|^{n+1}}\mathrm{d}y.
\end{aligned}$$

As a result,

$$\partial_j I_1 f(x) = (1-n)\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \frac{y_j \eta(\varepsilon|y|)}{|y|^{n+1}} f(x-y)\mathrm{d}y$$

by the continuity of f. We decompose

$$\begin{aligned}&\int_{\mathbb{R}^n} \frac{y_j \eta(\varepsilon|y|)}{|y|^{n+1}} f(x-y)\mathrm{d}y \\ &= \int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{y_j \eta(\varepsilon|y|)}{|y|^{n+1}} f(x-y)\mathrm{d}y + \int_{\mathbb{R}^n\setminus B(\varepsilon)} \frac{y_j}{|y|^{n+1}} f(x-y)\mathrm{d}y.\end{aligned}$$

Since $y \mapsto y_j$ is an odd function, we have

$$\int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{y_j \eta(\varepsilon|y|)}{|y|^{n+1}} f(x-y)\mathrm{d}y = \int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{y_j \eta(\varepsilon|y|)}{|y|^{n+1}} (f(x-y)-f(x))\mathrm{d}y.$$

As a result,

$$\begin{aligned}&\left|\int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{y_j \eta(\varepsilon|y|)}{|y|^{n+1}} f(x-y)\mathrm{d}y\right| \\ &\le \int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{|y_j| \eta(\varepsilon|y|)}{|y|^{n+1}} |f(x-y)-f(x)|\mathrm{d}y \\ &\le \sup_{y\in B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} |f(x-y)-f(x)| \int_{B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} \frac{|y_j|}{|y|^{n+1}}\mathrm{d}y \\ &= \sup_{y\in B(\varepsilon)\setminus B\left(\frac{\varepsilon}{2}\right)} |f(x-y)-f(x)| \int_{B(1)\setminus B\left(\frac{1}{2}\right)} \frac{|y_j|}{|y|^{n+1}}\mathrm{d}y.\end{aligned}$$

Consequently, $\partial_j I_1 f(x) = -(n-1)\lim\limits_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\mathrm{d}y.$

4.5.2 Calderón–Zygmund operators

Now we start a general theory keeping in mind the example of the Riesz transform. The generalized Calderón–Zygmund operator and the theory that lies behind are key tools in this book. The definition, presented below, is highly generalized in view of the singular integral operators we have been dealing with.

Definition 57 (Calderon–Zygmund operator)**.** An $L^2(\mathbb{R}^n)$-bounded linear operator T is a (*generalized*) *Calderón–Zygmund operator* (*with the kernel* K), if it satisfies the following conditions:

(1) There exists

$$K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, x = y\})$$

such that, for all $f \in L^2_{\mathrm{c}}(\mathbb{R}^n)$, we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\mathrm{d}y \text{ for almost all } x \notin \operatorname{supp}(f). \tag{4.1}$$

(2) There exist constants D_1 and D_2 such that K satisfies the following estimates.

(i) Size condition:

$$|K(x,y)| \le D_1|x-y|^{-n}, \tag{4.2}$$

if $x \neq y$.

(ii) Hörmander's condition:

$$|K(x,z)-K(y,z)| + |K(z,x)-K(z,y)| \le D_2 \frac{|x-y|}{|x-z|^{n+1}}, \tag{4.3}$$

if $0 < 2|x-y| < |z-x|$.

If $K(x,y) = K_0(x-y)$ for some function K_0, then T is called a *singular integral operator of convolution type*.

Example 87. Let $\{a_{j\nu}\}_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}\times\mathbb{Z}^n)$. Also let $\psi \in C_{\rm c}^\infty(\mathbb{R}^n)$ be supported on $Q(3)$. For $j \in \mathbb{Z}$ and $\nu \in \mathbb{Z}^n$, we set $\varphi_{j\nu}(x) \equiv 2^{\frac{jn}{2}}\Delta\psi(2^j x - \nu), x \in \mathbb{R}^n$. Since

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \ell(Q)^{-a-n}\chi_{3Q}(x)\chi_{3Q}(y) \sim |x-y|^{-n-a}$$

for all $a > 0$, according to Example 55, the operator

$$f \in L^2(\mathbb{R}^n) \mapsto \sum_{j\in\mathbb{Z},\nu\in\mathbb{Z}^n} \langle f, \varphi_{j\nu}\rangle_{L^2}\varphi_{j\nu}$$

is a Calderón–Zygmund operator, which is not convolution type.

Let $S^{n-1} \equiv \{x \in \mathbb{R}^n : |x| = 1\}$ as before and equip S^{n-1} with the surface measure σ.

Definition 58. A function $K \in L^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ is called a *homogeneous smooth Calderón–Zygmund kernel* if K is a function of homogeneous of degree zero satisfying $\displaystyle\int_{S^{n-1}} K(x)\mathrm{d}\sigma(x) = 0$. For such a function K, the corresponding singular integral operator T of convolution type is defined by $Tf(x) = \displaystyle\int_{\mathbb{R}^n} K(x-y)f(y)\mathrm{d}y$ as long as the definition makes sense.

Example 88. Let $j = 1, 2, \ldots, n$. The *j-th Riesz transform*, given by

$$R_j f(x) \equiv \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\mathrm{d}y \quad (x \in \mathbb{R}^n), \tag{4.4}$$

is an example of singular integral operators of convolution type with the homogeneous smooth Calderón–Zygmund kernel $K(x) = \frac{x_j}{|x|}$. We write

$$R_j f(x) \equiv \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\mathrm{d}y \quad (x \in \mathbb{R}^n).$$

When there is a singularity in the integral kernel, we use p.v. in this way to approximate the integral.

Example 89. Surprisingly enough, the identity operator id_{L^2} is a generalized singular integral operator with the kernel $K = 0$.

Example 90. Let $n \geq 3$ and $f \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$. Also let I_2 be the fractional integral operator of order 2; $I_2 f(x) \equiv \displaystyle\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} \mathrm{d}y \quad (x \in \mathbb{R}^n)$. A similar technique to Example 86 shows that

$$\partial_j \partial_k I_2 f(x) = n(n-2) \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{(x_j - y_j)(x_k - y_k) f(y)}{|x-y|^{n+2}} \mathrm{d}y \tag{4.5}$$

if $1 \leq j < k \leq n$ and

$$\partial_j{}^2 I_2 f(x) = (n-2) \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{(n(x_j - y_j)^2 - |x-y|^2) f(y)}{|x-y|^{n+2}} \mathrm{d}y \tag{4.6}$$

if $1 \leq j \leq n$. See Exercise 73. Since $\partial_j \partial_k I_2$ is $L^2(\mathbb{R}^n)$-bounded thanks to Example 92 to follow, we see that $\partial_j \partial_k I_2$ is a singular integral operator of convolution type.

In what follows we abbreviate this type of operator to CZ-operator. The next proposition explains why we want to pose symmetric condition (4.3).

Lemma 202. *For any CZ-operator T and $f \in L^2_{\mathrm{c}}(\mathbb{R}^n)$, its dual T^* on the complex Hilbert space $L^2(\mathbb{R}^n)$ satisfies*

$$T^* f(x) = \int_{\mathbb{R}^n} \overline{K(y,x)} f(y) \mathrm{d}y \text{ for a.e. } x \notin \mathrm{supp}(f). \tag{4.7}$$

Proof We take a function $g \in L^2_{\mathrm{c}}(\mathbb{R}^n)$ whose support does not meet that of f. Then by (4.1)

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) \overline{T^* g(x)} \mathrm{d}x &= \int_{\mathbb{R}^n} Tf(x) \overline{g(x)} \mathrm{d}x \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} K(x,y) f(y) \overline{g(x)} \mathrm{d}x \mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(y) \overline{\left(\int_{\mathbb{R}^n} \overline{K(x,y)} g(x) \mathrm{d}x \right)} \mathrm{d}y. \end{aligned}$$

Thus, since g is an arbitrary $L^2_{\mathrm{c}}(\mathbb{R}^n)$-function with its support contained in $\mathbb{R}^n \setminus \mathrm{supp}(f)$, we have

$$T^* f(x) = \int_{\mathbb{R}^n} \overline{K(y,x)} f(y) \mathrm{d}y \text{ for almost everywhere } x \notin \mathrm{supp}(f).$$

This is the desired result.

Note that the condition on the kernel K is symmetric in the variables x and y.

Corollary 203. *For a CZ-operator T, its adjoint T^* is a CZ-operator again.*

We are interested in what is truly required for the boundedness of singular integral operators T. However, it may happen that T is the zero mapping or the identity. These cases should be avoided. Consequently, we are interested in operators which are appropriately non-degenerate. Denote by $\mathrm{SO}(n)$ the set of all orthonormal matrices with determinant 1.

Definition 59. One says that T is a *genuine Calderon–Zygmund operator* if it is a Calderon–Zygmund operator with the kernel K and for $n \geq 2$ there exist $a > 0$ and a rotation $\mathcal{R} \in \mathrm{SO}(n)$ such that $K(x,y) \gtrsim |x-y|^{-n}$ for all $x \in \mathbb{R}^n$ and for all $y \in C_x \equiv x + \mathcal{R}(C)$, where

$$C \equiv \{y = (y', y_n) \equiv (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : y_n > a\,|y'|\}\,.$$

If $n = 1$ then we assume that $K(x,y) \gtrsim |x-y|^{-1}$ for all $x \in \mathbb{R}$ and for all $y > x$ or for all $x \in \mathbb{R}$ and for all $y < x$.

A function F defined on $\mathbb{R}^n \setminus \{0\}$ is said to be homogeneous of degree k if $F(tx) = t^k F(x)$ for $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$.

Example 91.

(1) The Hilbert transform H, in which case $K(x,y) \equiv \frac{1}{x-y}$, is a genuine Calderon–Zygmund operator because $K(x,y) \geq |x-y|^{-1}$ for all $x \in \mathbb{R}$ and for all $y < x$.

(2) Suppose that we have $\Omega \in C(S^{n-1}) \backslash \{0\}$, which is homogeneous of degree zero and satisfies $\int_{S^{n-1}} \Omega(\eta) \mathrm{d}\eta = 0$. We let T be a Calderon–Zygmund operator whose kernel K takes the form $K(x,y) = \dfrac{1}{|x-y|^n} \Omega\left(\dfrac{x-y}{|x-y|}\right)$. The properties of Ω imply that there exist $\eta_0 \in S^{n-1}$ and $\delta > 0$ such that $\Omega(\eta) \gtrsim 1$ for all $\eta \in S^{n-1} \cap B\,(\eta_0, \delta)$. So, T is a genuine singular integral operator.

We will show that there are Calderón–Zygmund operators amply. Here, we present an application of Theorem 114 to create an example to which the results in Section 4.5.1 are available.

Example 92. Let $K \in C^1(S^{n-1})$, and let $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n \setminus \{0\})$ be such that

$$\int_{S^{n-1}} K(x) \mathrm{d}\sigma(x) = 0, \quad \sum_{l=-\infty}^{\infty} \varphi(2^l x) = \chi_{\mathbb{R}^n \setminus \{0\}}(x) \quad (x \in \mathbb{R}^n).$$

In order that we apply Cotlar's inequality, we set

$$T_k(x) \equiv \int_{\mathbb{R}^n} \varphi(2^k(x-y)) K\left(\frac{x-y}{|x-y|}\right) f(y) \mathrm{d}y \quad (x \in \mathbb{R}^n) \tag{4.8}$$

for $f \in L^2(\mathbb{R})$. By means of the integral kernel K_{k_1,k_2}, we write

$$T_{k_1} \circ T_{k_2} f(x) = \int_{\mathbb{R}^n} K_{k_1,k_2}(x,y) f(y) \mathrm{d}y$$

for $x \in \mathbb{R}^n$ and for and $k_1, k_2 \in \mathbb{Z}$. Then, thanks to Theorem 114 with $N = 1$ and $\lambda = n + 1$,

$$|K_{k_1,k_2}(x,y)| \lesssim \frac{2^{n \max(k_1,k_2) - |k_1 - k_2|}}{(1 + 2^{\max(k_1,k_2)}|x-y|)^{n+1}} \sum_{|\alpha| \le 1} \|\partial^\alpha K\|_{L^\infty(S^{n-1})} \quad (x, y \in \mathbb{R}^n)$$

for all $k_1, k_2 \in \mathbb{Z}$. Consequenlty, by the use of Young's inequality we have $\|T_{k_1} \circ T_{k_2} f\|_{L^2} \lesssim 2^{-|k_1 - k_2|} \left(\sum_{|\alpha| \le 1} \|\partial^\alpha K\|_{L^\infty(S^{n-1})} \right) \|f\|_{L^2}$. Thus, we are in the position of using Cotlar's inequality (Theorem 94) to obtain

$$\left\| \sum_{k=-k_1}^{k_2} T_k f \right\|_{L^2} \lesssim \left(\sum_{|\alpha| \le 1} \|\partial^\alpha K\|_{L^\infty(S^{n-1})} \right) \|f\|_{L^2}$$

for all $k_1, k_2 \in \mathbb{Z}$. Meanwhile for $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, we know that the limit $Tf(x) = \lim_{k_1,k_2 \to \infty} \sum_{k=-k_1}^{k_2} T_k f(x)$ exists for all $x \in \mathbb{R}^n$. By the Fatou lemma, we have $\|Tf\|_{L^2} \lesssim \|f\|_{L^2}$. Since $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$ with the bound $\|T\|_{L^2 \to L^2} \lesssim \sum_{|\alpha| \le 1} \|\partial^\alpha K\|_{L^\infty(S^{n-1})}$, in particular proving Theorem 201.

Recall that $\dot{\mathcal{H}}_k(\mathbb{R}^n)$ denotes the homogeneous harmonic polynomial space of order $k \in \mathbb{N}_0$. Fix an orthonormal basis $\{Y_{jk}\}_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}$ in $\dot{\mathcal{H}}_k(\mathbb{R}^n)$.

Example 93. Let $k \in \mathbb{N}_0$ and $j = 1, 2, \ldots, \dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))$. Fix $x \in \mathbb{R}^n$. Then

$$|Y_{jk}(x)| \le \sqrt{\frac{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}}} |x|^k \lesssim (k+1)^{\frac{n}{2}-1} |x|^k$$

thanks to Corollary 122 and

$$|\mathrm{grad}(Y_{jk})(x)| \le \sqrt{\frac{k(n+2k-2)\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}{\omega_{n-1}}} |x|^{k-1} \lesssim (k+1)^{\frac{n}{2}} |x|^{k-1}$$

thanks to Corollary 124. Consequently, if we consider the integral operator T_{jk} given by

$$T_{jk} f(x) \equiv \int_{\mathbb{R}^n} \frac{Y_{jk}(x-y)}{|x-y|^{n+k}} f(y) \mathrm{d}y \quad (x \in \mathbb{R}^n)$$

for $k \ge 1$ and $j = 1, 2, \ldots, \dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))$, then $\|T_{jk} f\|_{L^2} \lesssim k^{\frac{n}{2}} \|f\|_{L^2}$ for all $f \in L^2(\mathbb{R}^n)$ with the implicit constants independent of such j and k.

4.5.3 Calderón–Zygmund decomposition

In principle we are oriented to the boundedness property of singular integral operators. However, due to the strong singularity of the integral kernel, it is difficult to handle these operators. Hence, we need some weapons to be able to deal with singular integral operators. As we have mentioned, we are fascinated with the method of decomposition of functions into countable elementary pieces. That is, we will look into each function in much depth. However no matter how deeply we investigate the functions, we need to stop at a certain stage. Consequently, we have to judge whether we stop or continue. To make such a decision, we need a tool, a collection of dyadic cubes.

With the definition of the dyadic cubes in mind, we define the dyadic average operators.

Definition 60 (Dyadic average operator)**.** Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. The *dyadic average operator E_j of generation j*, or shortly, the *dyadic averaging function* $x \mapsto E_j f(x)$ stands for the average of f over a unique cube $Q \in \mathcal{D}_j(\mathbb{R}^n)$ containing x. That is, $E_j f(x) \equiv E_j[f](x) \equiv \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \chi_Q(x) m_Q(f)$ for $x \in \mathbb{R}^n$.

Example 94. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$E_0 f = \sum_{Q \in \mathcal{D}_0(\mathbb{R}^n)} m_Q(f)\chi_Q = \sum_{m \in \mathbb{Z}^n} \chi_{m+[0,1)^n}(f)\chi_{m+[0,1)^n}.$$

If $f \in L^1(\mathbb{R}^n)$, we have $|E_j f(x)| \le 2^{jn}\|f\|_{L^1}$ for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Hence, the next lemma holds trivially.

Lemma 204. *We have* $\lim_{j \to -\infty} E_j f(x) = 0$ *for all* $f \in L^1(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$.

The next theorem is the heart of the matter. To formulate this theorem, we define the *quadrant* as a set of the form

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \,:\, (a_1 x_1, a_2 x_2, \dots, a_n x_n) \in (0,\infty)^n\}$$

for some $(a_1, a_2, \dots, a_n) \in \{-1, 1\}^n$.

Theorem 205 (Construction of Calderón–Zygmund cubes)**.** *Let $\lambda > 0$, and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume that $E_\lambda \equiv \{M^{\mathcal{D}} f > \lambda\}$ does not contain any quadrant and is not empty. Then we have a disjoint collection $\{Q_j\}_{j \in J} \subset \mathcal{D}(\mathbb{R}^n)$ such that $E_\lambda = \bigcup_{j \in J} Q_j$ and that $\lambda < m_{Q_j}(|f|) \le 2^n \lambda$ for all $j \in J$.*

By the maximal inequality, if $f \in \bigcup_{1 \le p < \infty} L^p(\mathbb{R}^n) \setminus \{0\}$, then the assumption on quadrants in Theorem 205 is satisfied.

Proof By the definition of the dyadic maximal operator, in particular, by the property of "sup", we have $E_\lambda = \bigcup_{j \in \mathbb{Z}} \{E_j[|f|] > \lambda\}$. From assumption on

quadrants, the set of j satisfying $x \in \{E_j[|f|] > \lambda\}$ is bounded from below for each $x \in E_\lambda$. Hence, there exists j^* such that $E_j[\,|f|\,](x) \le \lambda$ for all $j < j^*$. Hence, we can partition E_λ:

$$E_\lambda = \bigcup_{j\in\mathbb{Z}} \bigcap_{k\in\mathbb{Z}\cap(-\infty,j)} \{x \in \mathbb{R}^n \,:\, E_j[\,|f|\,](x) > \lambda,\ E_k[\,|f|\,](x) \le \lambda\}.$$

From the property of dyadic cubes we can partition

$$\bigcap_{k\in\mathbb{Z}\cap(-\infty,j)} \{x \in \mathbb{R}^n \,:\, E_j[\,|f|\,](x) > \lambda,\ E_k[\,|f|\,](x) \le \lambda\} \tag{4.9}$$

into the disjoint sum of dyadic cubes $Q_m^j \in \mathcal{D}_j(\mathbb{R}^n)$ with $m \in M_j$. A rearrangement of Q_m^j's yields a partition of E_λ: $E_\lambda = \sum_{j\in J} Q_j$ and $\{Q_j\}_{j\in J} = \{Q_{jm}\}_{m\in M_j,\, j\in\mathbb{Z}}$, where M_j is at most countable.

Let $x \in Q_j =: Q_{j_0 m_0}$. From the choice of $Q_{j_0 m_0}$, we have $m_{Q_j}(|f|) = E_{j_0}[\,|f|\,](x) > \lambda$.

Let $Q_j^* \in \mathcal{D}_{j_0-1}(\mathbb{R}^n)$ be the dyadic parent of the dyadic cube Q_j. That is, the unique cube containing Q_j with volume as 2^n times Q_j. From (4.9), the choice of $Q_{j_0 m_0}$, we have $m_{Q_j^*}(\,|f|\,) = E_{j_0-1}[\,|f|\,](x) \le \lambda$. From this formula, we conclude $m_{Q_j}(\,|f|\,) \le 2^n m_{Q_j^*}(\,|f|\,) = 2^n E_{j_0-1}[\,|f|\,](x) \le 2^n\lambda$.

We consider the Calderón–Zygmund decomposition keeping the setup above in mind. The Calderón–Zygmund decomposition is one of the fundamental tools to analyze locally integrable functions. Another important decomposition is the Lerner–Hytönen decomposition considered in Theorem 174.

Theorem 206 (Calderón–Zygmund decomposition)**.** *Let $f \in L^1(\mathbb{R}^n)$, and let $\lambda > 0$. Then we can find a set $\{Q_j\}_{j\in J}$ of dyadic cubes and a family $\{g\} \cup \{b_j\}_{j\in J}$ of countable collections of $L^1(\mathbb{R}^n)$-functions such that*:

(1) *f admits the following decomposition*: $f = g + \sum_{j\in J} b_j$.

(2) (*The $L^1(\mathbb{R}^n)$-condition*) *The $L^1(\mathbb{R}^n)$-norm of g is less than that of f*:

$$\|g\|_{L^1} \le \|f\|_{L^1}. \tag{4.10}$$

(3) (*The $L^\infty(\mathbb{R}^n)$-condition*) *The function g is $L^\infty(\mathbb{R}^n)$-bounded:*

$$\|g\|_{L^\infty} \le 2^n\lambda. \tag{4.11}$$

(4) (*The support condition*) *The function b_j is supported on the closure of Q_j; namely* $\operatorname{supp}(b_j) \subset \overline{Q_j}$.

(5) (*The moment condition*) $b_j \perp \mathcal{P}_0(\mathbb{R}^n)$.

(6) (*Partition of the set* $\{M^{\mathcal{D}}f > \lambda\}$) *The family* $\{Q_j\}_{j\in J}$ *is disjoint and* $\{\, M^{\mathcal{D}}f > \lambda \,\} = \sum_{j\in J} Q_j$.

This decomposition is called the Calderón–Zygmund decomposition (*at height* λ).

Proof If f is zero, then we simply set $J = \emptyset$ and $f = g$. So we assume that $f \neq 0$ and hence $E_\lambda \equiv \{\, M^{\mathcal{D}}f > \lambda \,\}$ is empty. Then by the Lebesgue differentiation theorem, Theorem 12, we see that $g = f$ and $b = 0$ suffice. See also Lemma 142 below. Assume otherwise. Choose a disjoint conllection $\{Q_j\}_{j\in J}$ of dyadic cubes such that

$$\lambda \le m_{Q_j}(|f|) \le 2^n\lambda, \quad E_\lambda = \sum_{j\in J} Q_j \tag{4.12}$$

according to Theorem 205. Define $b_j \equiv \chi_{Q_j}(f - m_{Q_j}(f))$. Based on b_j, we define g by $g \equiv f - \sum_{j\in J} b_j$, which yields a decomposition of f. Here note that the sum makes sense in the topology of $L^1(\mathbb{R}^n)$ and in the sense of almost everywhere convergence. Since the supports of b_j's are nonoverlapping, from the definition of b_j, the moment condition (4.10) follows. It remains to show (4.11).

Let $x \in Q_j$. Since $\{Q_j\}_{j\in J}$ is disjoint, we have $g(x) = m_{Q_j}(f)$. Hence $|g(x)| \le 2^n\lambda$ by (4.12).

Conversely, let $x \notin \sum_{j\in J} Q_j$. Inequality (4.11) follows since for almost all $x \in \mathbb{R}^n$, $|g(x)| = |f(x)| \le M^{\mathcal{D}}f(x) \le \lambda$ by the Lebesgue differentiation theorem.

A helpful remark may be in order.

Remark 9. Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. If one reexamines the above proof, then one notices $b = \sum_{j\in J} b_j$ in the topology of $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Theorems 205 and 206 are widely known as the Calderón–Zygmund decomposition or the Calderón–Zygmund theory and these theorems have been applied to harmonic analysis.

Example 95. Let $\lambda > 0$, and let $\mathfrak{D}$ be a generalized dyadic grid or a family $\mathcal{D}(Q)$ for some right-open cube. We consider $E_\lambda \equiv \{M^{\mathfrak{D}}f > \lambda\}$ for $f \in L^0(\mathbb{R}^n)$ or $L^0(Q)$, where $M^{\mathfrak{D}}$ denotes the maximal operator generated by $\mathfrak{D}$. Assume that E_λ is not empty. Choose the family $\mathcal{N}$ of maximal cubes $Q \in \mathfrak{D}$ satisfying $3Q \subset E_\lambda$. Note that $\mathcal{N}$ is disjoint. Since E_λ is an open set, if $x \in E_\lambda$, then we can find a dyadic cube Q such that $x \in Q \subset 3Q \subset E_\lambda$. If necessary we can assume $Q \in \mathcal{N}$. Thus, $E_\lambda = \sum_{Q\in\mathcal{N}} Q$. We define $b_Q \equiv (f - m_Q(f))\chi_Q$

for $Q \in \mathcal{N}$. Set $b \equiv \sum_{Q\in\mathcal{N}} b_Q$ and $g \equiv f - b$. Since $10Q \setminus E_\lambda \neq \emptyset$, we have $m_Q(|f|) \lesssim \lambda$ if we go through a similar argument. Thus, $|g| \lesssim \lambda$. If we let $\mathfrak{D} = \mathcal{D}(Q)$ for a compact cube Q, we will obtain a family with disjoint interior.

4.5.4 Weak-$(1,1)$ boundedness and strong-(p,p) boundedness

With our weapon obtained in Section 4.5.3 we are going to prove two theorems for CZ operators, the weak-$(1,1)$ boundedness and the strong-(p,p) boundedness, for $1 < p < \infty$. Recall that given $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. There is a (possibly empty) family of disjoint (dyadic) cubes $\{Q_j\}_j$ such that $\lambda \leq m_{Q_j}(|f|) \leq 2^n\lambda$. Using these cubes we have a decomposition $f = g + \sum_{j\in J} b_j$ which enjoys the following properties: $b_j \in L^1(Q_j) \cap \mathcal{P}_0^\perp(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$. From the definition of g we deduce $\|g\|_{L^1} \leq \|f\|_{L^1}$.

Although T fails to be $L^1(\mathbb{R}^n)$, more precisely, T cannot be extended continuously to an $L^1(\mathbb{R}^n)$-bounded operator, we have a substitute called the weak-$L^1(\mathbb{R}^n)$ estimate.

Theorem 207. *Suppose that T is a singular integral operator. Let D_1 and D_2 be constants in* (4.2) *and* (4.3)*, respectively. Then T is weak-$(1,1)$ bounded. That is, $\lambda\,|\{\,|Tf| > \lambda\}| \lesssim (\|T\|_{L^2\to L^2} + D_1 + D_2)\|f\|_{L^1}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\lambda > 0$.*

Proof If necessary, we may suppose that $\|T\|_{L^2\to L^2} + D_1 + D_2 \leq 1$ by considering $\dfrac{T}{\|T\|_{L^2\to L^2} + D_1 + D_2}$. Keeping to the above notation, we have

$$|\{|Tf| \geq \lambda\}| \leq \left|\left\{|Tg| \geq \frac{\lambda}{2}\right\}\right| + \left|\left\{\left|T\left(\sum_{j\in J} b_j\right)\right| \geq \frac{\lambda}{2}.\right\}\right|.$$

The first term is easy to estimate. In fact, using the $L^2(\mathbb{R}^n)$-boundedness as usual, we have

$$\left|\left\{|Tg| \geq \frac{\lambda}{2}\right\}\right| \lesssim \frac{1}{\lambda^2}\int_{\mathbb{R}^n} |Tg(x)|^2 \mathrm{d}x \lesssim \frac{1}{\lambda^2}\int_{\mathbb{R}^n} |g(x)|^2 \mathrm{d}x \lesssim \frac{1}{\lambda}\|g\|_{L^1} \lesssim \frac{1}{\lambda}\|f\|_{L^1}.$$

Hence

$$\left|\left\{|Tg| \geq \frac{\lambda}{2}\right\}\right| \lesssim \frac{1}{\lambda}\|f\|_{L^1}. \tag{4.13}$$

To estimate the second term we need the following observation:

Claim 208. *For any $j \in J$, $\displaystyle\int_{\mathbb{R}^n\setminus 2\sqrt{n}\,Q_j} |Tb_j(x)|\mathrm{d}x \lesssim \|b_j\|_{L^1}$.*

Admitting Claim 208, let us finish the proof of the theorem. Write $Z \equiv \bigcup_{j\in J} 2\sqrt{n}Q_j$. Firstly we proceed as follows:

$$\left|\left\{\left|T\left(\sum_{j\in J} b_j\right)\right| \geq \frac{\lambda}{2}\right\}\right| \leq \left|(\mathbb{R}^n \setminus Z) \cap \left\{\left|T\left(\sum_{j\in J} b_j\right)\right| \geq \frac{\lambda}{2}\right\}\right| + |Z|$$
$$\lesssim \frac{1}{\lambda}\left\|T\left(\sum_{j\in J} b_j\right)\right\|_{L^1(\mathbb{R}^n\setminus Z)} + \sum_{k\in J}|Q_k|.$$

Recall that $\sum_{k\in J}|Q_k| = \left|\bigcup_{k\in J} Q_k\right| = |\{M^{\mathcal{D}} f > \lambda\}| \lesssim \frac{1}{\lambda}\|f\|_{L^1}$. By the triangle inequality we obtain

$$\left\|T\left(\sum_{j\in J} b_j\right)\right\|_{L^1(\mathbb{R}^n\setminus Z)} \leq \sum_{j\in J}\|Tb_j\|_{L^1(\mathbb{R}^n\setminus Z)} \leq \sum_{j\in J}\int_{\mathbb{R}^n\setminus 2\sqrt{n}Q_j}|Tb_j(x)|\mathrm{d}x.$$

We can estimate $\sum_{j\in J}\|b_j\|_{L^1(Q_j)}$ because the b_j have support in Q_j and the Q_j are disjoint. If we invoke Claim 208, then we obtain

$$\left\|T\left(\sum_{j\in J} b_j\right)\right\|_{L^1(\mathbb{R}^n\setminus Z)} \lesssim \sum_{j\in J}\|b_j\|_{L^1(Q_k)} \lesssim \int_{\mathbb{R}^n}(|f(x)| + |g(x)|)\mathrm{d}x \lesssim \|f\|_{L^1}.$$

Thus, the proof is over modulo Claim 208.

Let us prove Claim 208. For this purpose we can write out $Tb_j(x)$ in full:

$$Tb_j(x) = \int_{\mathbb{R}^n} K(x,y)b_j(y)\mathrm{d}y \text{ for all } \text{ for a.e. } x \in \mathbb{R}^n \setminus 2\sqrt{n}\,Q_j.$$

We can deduce from the moment condition and the Hörmander condition

$$|Tb_j(x)| = \left|\int_{\mathbb{R}^n}(K(x,y) - K(x,c(Q_j)))b_j(y)\mathrm{d}y\right| \lesssim \int_{Q_j}\frac{|y-c(Q_j)|}{|x-y|^{n+1}}|b_j(y)|\mathrm{d}y.$$

By integrating over $\mathbb{R}^n \setminus 2\sqrt{n}\,Q_j$ we have, with the aid of Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus 2\sqrt{n}\,Q_j}|Tb_j(x)|\mathrm{d}x &\lesssim \int_{\mathbb{R}^n\setminus 2\sqrt{n}\,Q_j}\left(\int_{Q_j}\frac{|y-c(Q_j)|}{|x-y|^{n+1}}|b_j(y)|\mathrm{d}y\right)\mathrm{d}x\\ &= \int_{Q_j}\left(\int_{\mathbb{R}^n\setminus 2\sqrt{n}\,Q_j}\frac{|y-c(Q_j)|}{|x-y|^{n+1}}\mathrm{d}x\right)|b_j(y)|\mathrm{d}y\\ &\lesssim \int_{Q_j}\left(\int_{|x-y|>|x-c(Q_j)|}\frac{|y-c(Q_j)|}{|x-y|^{n+1}}\mathrm{d}x\right)|b_j(y)|\mathrm{d}y\\ &\lesssim \int_{Q_j}|b_j(y)|\mathrm{d}y.\end{aligned}$$

Thus, we have finished the proof of the claim and hence the theorem.

Although it is impossible to have a pointwise control of singular integral operators in terms of given functions, we can control their integral in terms of the original functions in $L^p(\mathbb{R}^n)$-norm.

Theorem 209. *Let $1 < p < \infty$ and T be a singular integral operator, which is initially defined on $L^2(\mathbb{R}^n)$. Let D_1 and D_2 be constants in* (4.2) *and* (4.3)*, respectively. Then T can be extended to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ so that $\|Tf\|_{L^p} \lesssim_p (\|T\|_{L^2 \to L^2} + D_1 + D_2)\|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.*

We still write T defined on $L^p(\mathbb{R}^n)$.

Proof We may suppose that $\|T\|_{L^2 \to L^2} + D_1 + D_2 \leq 1$ as before. We distinguish two cases keeping in mind that the case $p = 2$ is contained in the assumption.

Case 1 : $1 < p < 2$ In this case we will use interpolation of $L^2(\mathbb{R}^n)$-boundedness and weak-$(1,1)$ boundedness, the same technique that we used for the maximal operator.

Case 2 : $p > 2$ In this case we use the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p'}(\mathbb{R}^n)$, for all $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,

$$\begin{aligned}
\|Tf\|_{L^p} &= \sup\left\{ \left| \int_{\mathbb{R}^n} Tf(x)\overline{g(x)}\mathrm{d}x \right| \,:\, g \in C_c^\infty(\mathbb{R}^n), \|g\|_{L^{p'}} \leq 1 \right\} \\
&= \sup\left\{ \left| \int_{\mathbb{R}^n} f(x)\overline{T^*g(x)}\mathrm{d}x \right| \,:\, g \in C_c^\infty(\mathbb{R}^n), \|g\|_{L^{p'}} \leq 1 \right\} \\
&\leq \|f\|_{L^p}\|T^*g\|_{L^{p'}}.
\end{aligned}$$

Recall that T^* is again a singular integral operator and that $1 < p' < 2$. Thus for all $g \in L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $\|T^*g\|_{L^{p'}} \lesssim \|g\|_{L^{p'}}$.

Combining these estimates allows us to conclude that T can be extended to a bounded operator on $L^p(\mathbb{R}^n)$.

4.5.5 Truncation and pointwise convergence

Having proven the boundedness property, we now turn to the maximal operator of the truncated singular integral to check that the definition of singular integral operators can be realized as the limit of truncated singular integral operators: We now turn to the problem of pointwise convergence. To do this, it is very effective that we use the maximal operator, as this is the case when we investigated the Lebesgue point of measurable functions. The maximal operator associated with our present situation is the following one.

Definition 61 (Truncated maximal singular integral operator)**.** Let T be a singular integral operator with the kernel K and define the *truncated maximal*

singular integral operator $\tilde{T}$ by

$$\tilde{T}f(x) \equiv \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,y) f(y) \mathrm{d}y \right| \quad (x \in \mathbb{R}^n)$$

for $f \in L^0(\mathbb{R}^n)$ as long as the definition makes sense.

Importantly, at this moment, we suppose that T itself is defined for any function $f \in \bigcup_{1 \le p < \infty} L^p(\mathbb{R}^n)$, in which case we are confortable with defining $\tilde{T}f$ for such functions.

Example 96. If $f \in L^p(\mathbb{R}^n)$ for some $1 \le p < \infty$, then the definition of $\tilde{T}f(x)$ makes sense for any $x \in \mathbb{R}^n$ thanks to the size condition (4.2).

The Cotlar inequality below will control the truncated maximal singular integral operators.

Theorem 210 (Cotlar)**.** *Suppose that T is a singular integral operator and $0 < \nu \le 1$. Let D_1 and D_2 be constants in* (4.2) *and* (4.3)*, respectively. Then*

$$\tilde{T}f(x) \lesssim_\nu (\|T\|_{L^2 \to L^2} + D_1 + D_2)(M^{(\nu)}[Tf](x) + Mf(x)) \quad (x \in \mathbb{R}^n) \tag{4.14}$$

for all $f \in \bigcup_{1 \le p < \infty} L^p(\mathbb{R}^n)$.

Proof We may suppose that $\|T\|_{L^2 \to L^2} + D_1 + D_2 \le 1$ as before. Fix $\varepsilon > 0$ and $x \in \mathbb{R}^n$. We will estimate $\int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,y) f(y) \mathrm{d}y$ independently of ε.

Let $z \in B\left(x, \frac{1}{2}\varepsilon\right)$. We will decompose f according to $B(x, 2\varepsilon)$: Let $f_0 \equiv \chi_{B(x,\varepsilon)} \cdot f$ and $f_1 \equiv f - f_0$. As for the estimate of f_0, we will employ the size condition $|K(x,y)| \lesssim |x-y|^{-n}$. It follows from the definition of f_0 that

$$\left| \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,y) f(y) \mathrm{d}y \right| = \left| \int_{B(x,2\varepsilon) \setminus B(x,\varepsilon)} K(x,y) f_0(y) \mathrm{d}y \right| \lesssim Mf(x).$$

Next we will consider the estimate of f_1. We decompose the estimate further:

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,y) f_1(y) \mathrm{d}y \\ &= \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} (K(x,y) - K(z,x)) f_1(y) \mathrm{d}y + \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(z,y) f_1(y) \mathrm{d}y. \end{aligned}$$

By the Hörmander condition (4.3), the first term is readily estimated by the Hardy–Littlewood maximal function. Thus, we have

$$\left| \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,y) f_1(y) \mathrm{d}y \right| \lesssim Mf(x) + |Tf_1(z)|. \tag{4.15}$$

Thus, from (4.5.5) and (4.15) we have obtained

$$\left|\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} K(x,y)f(y)\mathrm{d}y\right| \lesssim Mf(x)+|Tf_1(z)| \lesssim Mf(x)+|Tf_0(z)|+|Tf(z)|$$

as long as x and z satisfy $|x-y|\le \frac{\varepsilon}{2}$. Taking the ν-power and then averaging this estimate over $B(x,\varepsilon)$, we have

$$\left|\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} K(x,y)f(y)\mathrm{d}y\right|^{\nu} \lesssim Mf(x)^{\nu}+m^{(\nu)}_{B(x,\varepsilon)}(Tf_0)^{\nu}+m^{(\nu)}_{B(x,\varepsilon)}(Tf)^{\nu}. \tag{4.16}$$

Since $0<\nu<1$, we are in the position of applying Kolmogorov's inequality (Theorem 138), to have

$$m^{(\nu)}_{B(x,\varepsilon)}(Tf_0) \lesssim Mf(x). \tag{4.17}$$

Combining (4.16) and (4.17) gives (4.14).

We now prove the a.e. convergence of singular integral operators, keeping the estimate of the maximal singular integral operators in mind. As a preparatory step, we will show that many functions behave well.

Theorem 211. *Let T be a singular integral operator with the kernel K. Assume that the limit*

$$\lim_{\varepsilon\downarrow 0}\int_{B(x,1)\setminus B(x,\varepsilon)} K(x,y)\mathrm{d}y \tag{4.18}$$

exists for almost all $x\in\mathbb{R}^n$. Then for all $f\in C^{\infty}_{\mathrm{c}}(\mathbb{R}^n)$,

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} K(x,y)f(y)\mathrm{d}y \tag{4.19}$$

exists for such $x\in\mathbb{R}^n$. In particular, if $K(x,x-y)=-K(x,y-x)$, then (4.18) *and hence* (4.19) *hold.*

Proof By the kernel condition the matter is reduced to the convergence of

$$\lim_{\varepsilon\downarrow 0}\int_{B(x,1)\setminus B(x,\varepsilon)} K(x,y)f(y)\mathrm{d}y. \tag{4.20}$$

However, by assumption (4.18), (4.20) is reduced to showing

$$\lim_{\varepsilon\downarrow 0}\int_{B(x,1)\setminus B(x,\varepsilon)} K(x,y)(f(x)-f(y))\mathrm{d}y. \tag{4.21}$$

By the mean-value theorem we have $|f(x)-f(y)|\lesssim |x-y|$. Combining this with the size condition $|K(x,y)|\lesssim |x-y|^{-n}$, we see that the integrand of (4.21) is integrable. Therefore, the limit in question does exist.

Combining Cotlar's inequality with Theorem 211, we can prove that the truncated singular integral operator converges if and only if K enjoys a sort of cancellation condition.

Theorem 212. *Suppose that T is a singular integral kernel and that K is an associated kernel. Then the limit*

$$T_0 f(x) \equiv \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,y) f(y) \mathrm{d}y$$

exists for almost all $x \in \mathbb{R}^n$, if and only if the limit

$$\lim_{\varepsilon \downarrow 0} \int_{B(x,1) \setminus B(x,\varepsilon)} K(x,y) \mathrm{d}y$$

exists for almost every $x \in \mathbb{R}^n$. If this is the case, then for all $1 < p < \infty$ there exists a constant c_p such that

$$\|T_0 f\|_{L^p} \leq c_p \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n)$$

and there exists a constant $c_1 > 0$ such that

$$|\{|T_0 f| > \lambda\}| \leq \frac{c_1}{\lambda} \|f\|_{L^1}, \text{ for all } f \in L^1(\mathbb{R}^n) \text{ and } \lambda > 0.$$

In particular, $K(x, x-y) = -K(x, y-x)$, then these estimates hold.

We remark that singular integral operators satisfy the condition of the above theorem. Hence, we have the weak-$(1,1)$ estimate as well. Below let us say that a generalized singular integral operator is standard if the cancellation condition (4.18) is fulfilled. In this case we say that the kernel is standard as well. As an example of standard singular integral operators, we can list the Riesz transform.

4.5.6 Examples of singular integral operators

We have seen that the Riesz transform and hence the Hilbert transform fall under the scope of Section 4.5. We will present further examples. When we consider the integration by parts involving functions with a certain singularity, singular integral operators arise naturally. Proposition 213 naturally extends Example 90.

Proposition 213. *Let $n \geq 3$ and let $A = \{a_{ij}\}_{i,j=1}^n$ be a positive definite matrix with inverse $A^{-1} = \{a^{ij}\}_{i,j=1}^n$. Let $\tau \in C^\infty(\mathbb{R})$ be such that $\chi_{(0,\infty)} \leq \tau \leq \chi_{(1,\infty)}$. Define*

$$\Gamma(x) \equiv \frac{1}{(n-2)\omega_{n-1}\sqrt{\det A}} \left(\sum_{i,j=1}^n a^{ij} x_i x_j \right)^{1-\frac{n}{2}}$$

for $x \in \mathbb{R}^n$. For each $i, j = 1, 2, \dots, n$, write

$$c_{ij} \equiv \int_{\mathbb{R}^n} \{\partial_{ij}[\tau(|x|)]\Gamma(x) + \partial_j[\tau(|x|)]\partial_i\Gamma(x) + \partial_i[\tau(|x|)]\partial_j\Gamma(x)\}\,\mathrm{d}x.$$

We define

$$\text{p.v.} \int_{\mathbb{R}^n} \partial_{ij}\Gamma(x)\mathrm{d}x \equiv \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n \setminus B(\varepsilon)} \partial_{ij}\Gamma(x)\mathrm{d}x.$$

Then for all $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \Gamma(x)\partial_{ij}\varphi(x)\mathrm{d}x = \text{p.v.} \int_{\mathbb{R}^n} \partial_{ij}\Gamma(x)\mathrm{d}x + c_{ij}\varphi(0).$$

Proof Let $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ with $\varphi(0) = 1$ by normalization. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Gamma(x)\partial_{ij}\varphi(x)\mathrm{d}x - \text{p.v.} \int_{\mathbb{R}^n} \partial_{ij}\Gamma(x)\cdot\varphi(x)\mathrm{d}x \\
&= \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \tau\left(\frac{|x|}{\varepsilon}\right)\Gamma(x)\partial_{ij}\varphi(x)\mathrm{d}x - \text{p.v.} \int_{\mathbb{R}^n} \partial_{ij}\Gamma(x)\mathrm{d}x \\
&= \lim_{\varepsilon\downarrow 0} \left(\int_{\mathbb{R}^n} \partial_{ij}\left[\tau\left(\frac{|x|}{\varepsilon}\right)\right]\Gamma(x)\varphi(x)\mathrm{d}x + \int_{\mathbb{R}^n} \partial_j\left[\tau\left(\frac{|x|}{\varepsilon}\right)\right]\partial_i\Gamma(x)\varphi(x)\mathrm{d}x \right) \\
&\quad + \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \partial_i\left[\tau\left(\frac{|x|}{\varepsilon}\right)\right]\partial_j\Gamma(x)\varphi(x)\mathrm{d}x.
\end{aligned}$$

As before, in view of the size of the support, using the change of variables, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Gamma(x)\partial_{ij}\varphi(x)\mathrm{d}x - \text{p.v.} \int_{\mathbb{R}^n} \partial_{ij}\Gamma(x)\cdot\varphi(x)\mathrm{d}x \\
&= \lim_{\varepsilon\downarrow 0} \left(\int_{\mathbb{R}^n} \partial_{ij}\left[\tau\left(\frac{|x|}{\varepsilon}\right)\right]\Gamma(x)\mathrm{d}x + \int_{\mathbb{R}^n} \partial_j\left[\tau\left(\frac{|x|}{\varepsilon}\right)\right]\partial_i\Gamma(x)\mathrm{d}x \right) \\
&\quad + \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \partial_i\left[\tau\left(\frac{|x|}{\varepsilon}\right)\right]\partial_j\Gamma(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \{\partial_{ij}[\tau(|x|)]\Gamma(x) + \partial_j[\tau(|x|)]\partial_i\Gamma(x) + \partial_i[\tau(|x|)]\partial_j\Gamma(x)\}\,\mathrm{d}x.
\end{aligned}$$

This is the desired result.

We integrate by parts to obtain the decay of the inner product. We fix a complete orthonormal system $\{Y_{jk}\}_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}$ as we did in Definition 45.

Lemma 214. *Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be a homogeneous smooth Calderón–Zygmund kernel.*

Also let $L \equiv |x|^2\Delta$ be a differential operator. We set

$$a_{jk} \equiv \int_{S^{n-1}} K(z)Y_{jk}(z)\mathrm{d}\sigma(z) \quad (k \in \mathbb{N}_0, 1 \le j \le \dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))).$$

Then $|a_{jk}| \lesssim_n \sup_{|k|\le n} \|L^j[|\cdot|^n K]\|_{L^\infty} k^{-2n}$.

Proof By Proposition 126 we have

$$\int_{S^{n-1}} L[|x|^n K(x)] \cdot \frac{Y_{jk}(x)}{|x|^k} \mathrm{d}\sigma(x) = \int_{S^{n-1}} |x|^n K(x) L\left[\frac{Y_{jk}(x)}{|x|^k}\right] \mathrm{d}\sigma(x).$$

Thanks to Proposition 125, we have

$$\begin{aligned}&\int_{S^{n-1}} L[|x|^n K(x)] \cdot \frac{Y_{jk}(x)}{|x|^k} \mathrm{d}\sigma(x)\\ &= -k(k+n-2) \int_{S^{n-1}} |x|^n K(x) |x|^2 \frac{Y_{jk}(x)}{|x|^k} \mathrm{d}\sigma(x).\end{aligned}$$

From the definition of S^{n-1}, we have

$$\begin{aligned}&\int_{S^{n-1}} L[|x|^n K(x)] \cdot \frac{Y_{jk}(x)}{|x|^k} \mathrm{d}\sigma(x)\\ &= -k(k+n-2) \int_{S^{n-1}} |x|^n K(x) \frac{Y_{jk}(x)}{|x|^k} \mathrm{d}\sigma(x).\end{aligned}$$

If we go through the same argument n times, we obtain

$$a_{jk} = (-k)^{-n}(k+n-2)^{-n} \int_{S^{n-1}} L^j[|x|^n K(x)] Y_{km}(x) \mathrm{d}\sigma(x).$$

It remains to use $k+n-2 \geq k$.

With these preliminary observations in mind, we present an example of singular integral operators. Roughly speaking, we will see that we can handle the operator of the form $Tf(x) = k(x) T_0 f(x)$, where T_0 is a singular integral operator and $k \in L^\infty(\mathbb{R}^n)$. We consider the kernel $(x,z) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto K(x,z) \in \mathbb{C}$ below.

Theorem 215. *Let $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the following conditions*:

(1) *$K(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for all $x \in \mathbb{R}^n$;*

(2) *$K(x,tz) = t^{-n} K(x,z)$ for every $t > 0$ and $(x,z) \in \mathbb{R}^n \times S^{n-1}$;*

(3) *$\displaystyle\int_{S^{n-1}} K(x,z) \mathrm{d}\sigma(z) = 0$;*

(4) *$\displaystyle\max_{|\alpha| \leq 2n} \left\| \frac{\partial^\alpha K}{\partial z^\alpha} \right\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} < \infty.$*

Define

$$Tf(x) \equiv \int_{\mathbb{R}^n} K(x, x-y) f(y) \mathrm{d}y \quad (x \in \mathbb{R}^n)$$

for $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$. Then T extends to an $L^p(\mathbb{R}^n)$-bounded linear operator.

Proof We expand the kernel K using $\{\{Y_{jk}\}_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}\}_{k=0}^{\infty}$. Due to (3), we have

$$K(x,z)=\sum_{k=1}^{\infty}\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}\int_{S^{n-1}}K(x,y)Y_{jk}(y)\mathrm{d}\sigma(y)\cdot\frac{Y_{jk}(z)}{|z|^{k+n}}.$$

Along this decomposition, we decompose T to write

$$Tf(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{\dim_{\mathbb{C}}(\dot{\mathcal{H}}_k(\mathbb{R}^n))}T_{jk}f(x). \tag{4.22}$$

We claim that the series converges absolutely thanks to the strong decay in each summand. Due to Example 93,

$$\left\|\int_{\mathbb{R}^n}\frac{Y_{jk}(z)}{|z|^{k+n}}f(\cdot-z)\mathrm{d}z\right\|_{L^p}\lesssim(1+k)^{\frac{n}{2}}\|f\|_{L^p}.$$

We use (1), (2) and (4). Since we have

$$\left\|\int_{S^{n-1}}K(\cdot,y)Y_{jk}(y)\mathrm{d}\sigma(y)\right\|_{L^\infty}\lesssim k^{-2n}$$

due to Lemma 214, we conclude from Corollary 124 that (4.22) takes place in the topology of the operator norm $\|\cdot\|_{L^p\to L^p}$. Thus, T is $L^p(\mathbb{R}^n)$-bounded.

We can also discuss the pointwise convergence for this class of operators.

Theorem 216. *Let $K\in L^0(\mathbb{R}^n\times\mathbb{R}^n)$ satisfy the following conditions*:

(1) $K(x,\cdot)\in C^\infty(\mathbb{R}^n\setminus\{0\})$ *for all* $x\in\mathbb{R}^n$;

(2) $K(x,tz)=t^{-n}K(x,z)$ *for every* $t>0$ *and* $(x,z)\in\mathbb{R}^n\times S^{n-1}$;

(3) $\displaystyle\int_{S^{n-1}}K(x,z)\mathrm{d}\sigma(z)=0$ *for all* $x\in\mathbb{R}^n$;

(4) $\displaystyle\max_{|\alpha|\le 2n}\left\|\frac{\partial^\alpha K}{\partial z^\alpha}\right\|_{L^\infty(\mathbb{R}^n\times S^{n-1})}<\infty.$

Let $f\in L^p(\mathbb{R}^n)$ with $1<p<\infty$. For any $\varepsilon>0$, we set

$$T_\varepsilon f(x)\equiv\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}K(x,x-y)f(y)\mathrm{d}y\quad(x\in\mathbb{R}^n).$$

Then there exists a function $Tf\in L^p(\mathbb{R}^n)$ such that $\lim_{\varepsilon\downarrow 0}\|T_\varepsilon f-Tf\|_{L^p}=0$.

Proof This is a general fact on singular integral operators together with the decomposition (4.22). Use Theorem 211 as well.

4.5.7 Sparse estimate of singular integral operators

Having set down some elementary properties, we now embark on the sparse estimate. Hence, we are now converting our results above to some forms useful to applications. We will need to estimate the quantities related to Tf, where $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. First, we handle the local oscillation $\omega_{2^{-n-2}}(Tf;Q)$.

Lemma 217. *Let T be a singular integral operator, and let Q be a cube. Then*

$$\omega(\mathrm{Re}(Tf);Q)+\omega(\mathrm{Im}(Tf);Q)\lesssim\sum_{l=0}^{\infty}2^{-l}m_{2^lQ}(|f|)$$

for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

Proof We consider the real part solely. We set $c\equiv\mathrm{Re}(T[f\cdot\chi_{\mathbb{R}^n\setminus 3Q}](c(Q)))$. Then there exist $D_1, D_2 > 0$ such that

$$\begin{aligned}|\mathrm{Re}(Tf(x))-c|&\le D_1|T[f\chi_{3Q}](x)|+D_1\int_{\mathbb{R}^n\setminus Q}\frac{\ell(Q)|f(y)|}{|y-c(Q)|^{n+1}}\mathrm{d}y\\&\le D_2|T[f\chi_{3Q}](x)|+D_2\sum_{l=0}^{\infty}2^{-l}m_{2^lQ}(|f|).\end{aligned}$$

Since T is weak-$(1,1)$ bounded, there exists a constant $D_3 > 0$ such that

$$|\{x\in Q\,:\,|\mathrm{Re}(T[f\chi_{3Q}](x))|>\lambda\}|\le\frac{D_3}{\lambda}\int_{3Q}|f(y)|\mathrm{d}y$$

for all $\lambda > 0$. Thus,

$$\left|\left\{x\in Q\,:\,|\mathrm{Re}(Tf(x))-c|>9^{n+2}D_2D_3m_{4Q}(|f|)+D_2\sum_{l=0}^{\infty}\frac{m_{2^lQ}(|f|)}{2^l}\right\}\right|\le\frac{|Q|}{2^{n+2}}.$$

Consequently, we have the desired result.

As an application of the Lerner–Hytönen decomposition (see Theorem 174) and Lemma 217, we obtain the following control of singular integral operators. The following estimate, which can be viewed as a direct corollary of Lemma 217, converges singular integral operators to non-negative integral operators in some sense. Consequently, this estimate is a bridge that connects singular integral operators with non-negative integral operators.

Lemma 218. *Let Q^0 be a cube, and let $f \in L^1(Q^0)$. Then*

$$\chi_{Q^0}|Tf-\mathrm{Med}(Tf;Q^0)|\lesssim\sum_{l=0}^{\infty}\sum_{L\in\mathcal{L}}2^{-l}\chi_Lm_{2^lL}(|f|).$$

Here, $\mathcal{L}\subset\mathcal{D}(Q^0)$ is a sparse family of dyadic cubes that depends on f.

Proof If we use the Lerner–Hytönen decomposition, we can find a sparse family $\mathcal{L} \equiv \{Q_k^j\}_{j\in\mathbb{N}_0,\, k\in K_j} \subset \mathcal{D}(Q^0)$ with a level structure such that

$$\chi_{Q^0}|Tf - \mathrm{Med}(Tf; Q^0)| \le \sum_{j=0}^{\infty} \sum_{k\in K_j} \omega_{2^{-n-2}}(Tf; Q_k^j)\chi_{Q_k^j}.$$

It remains to resort to Lemma 217.

4.5.8 Local estimates for singular integral operators

As we did for other operators we obtain the local estimate of singular integral operators.

Theorem 219. *Let $1 < q < \infty$. Then*

$$\|Tf\|_{L^q(B(x,r))} \lesssim r^{\frac{n}{q}} \int_r^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L^q(B(x,t))} \mathrm{d}t \tag{4.23}$$

for all $x \in \mathbb{R}^n$ and $r > 0$ and for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

The idea of the proof is similar. The only difference from the case of the maximal operator lies in the point where we handle the integral kernel with the size condition.

Proof We split $f = f_1 + f_2$, where $f_1 \equiv \chi_{B(x,2r)} f$ and $f_2 \equiv f - f_1$. For f_1, apply the $L^q(\mathbb{R}^n)$-boundedness of T and, for f_2, apply the size condition together with the triangle inequality.

We transform Theorem 219 into the form which we use.

Corollary 220. *Fix $x \in \mathbb{R}^n$. Let $1 < q < \infty$ and $0 < \delta < \frac{n}{q}$. Then*

$$\|Tf\|_{L^q(B(x,r))} \lesssim r^{\frac{n}{q}-\delta} \left(\int_r^{\infty} \left(\int_{B(x,t)} |f(z)|^q dz \right) \frac{\mathrm{d}t}{t^{n-\delta q+1}} \right)^{\frac{1}{q}}$$

for all $r > 0$ and for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

Proof It suffices to apply Hölder's inequality to the integral in the right-hand side of (4.23).

4.5.9 Exercises

Exercise 73.

(1) Prove (4.5) and (4.6) using the Stokes theorem. Alternatively, we may argue as follows: Let $\rho \in C^\infty(\mathbb{R}^n)$ satisfy $\chi_{(1,\infty)} \le \rho \le \chi_{(2,\infty)}$. For $\varepsilon > 0$, we set $I_{2,\varepsilon} f(x) \equiv \displaystyle\int_{\mathbb{R}^n} \frac{\rho(\varepsilon|x-y|) f(y)}{|x-y|^{n-2}} \mathrm{d}y \quad (x \in \mathbb{R}^n)$. Prove first that $I_{2,\varepsilon} f$ converges in some suitable topology.

(2) Show that $\partial_j \partial_k I_2$ is a Calderón–Zygmund operator if $1 \le j, k \le 2$.

Exercise 74. Let $x \in \mathbb{R}^n$. Show that the limit defining $R_j f(x)$ exists for $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ using $\int_{B(1)\setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\mathrm{d}y = \int_{B(1)\setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} (f(y) - f(x))\mathrm{d}y$ and the mean-value theorem.

4.6 Notes

Section 4.1

General remarks and textbooks in Section 4.1

See the fundamental textbooks and lecture notes [104, Chapter 2], [146, Chapter II, §2], [156, §2.1], [223, Chapter 1] [293, §1.1], [319, §1.10], [417, Chapter 2, §3], [416, Chapter I, §3 and Chapter II, §1], [382, §1.4] and [410, §2.3] for the maximal operator considered in this book.

For the Fefferman–Stein vector-valued inequality, we refer to [146, Chapter V, §4], where a different approach can be found.

See [118, §1.5] for various covering lemmas such as Vitali's covering lemma and Besicovitch's covering lemma.

We viewed the Hedberg inequality in the context of Hedberg's inequality; see Lemma 181 for more about Example 61.

Section 4.1.1

The history of the Hardy–Littlewood maximal operator dates back to 1930 [182]. The original aim was to apply it to the functions on the unit disk $\Delta(1)$ on the complex plane. Consequently, originally the Hardy–Littlewood maximal operator was used for the analysis over the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Here, to study the function spaces on $\mathbb{R}$, we work in $\mathbb{R}$. A natural passage to higher dimensions is due to Wiener [449]. His purpose was to apply it to the ergodic theory.

Section 4.1.2

In retrospect, the proof of Theorem 141 essentially relies upon the interpolation theorem due to Marcinkiewicz [302] whose proof Zygmund gave [473]. Wiener proved the sunrise lemma in [449]; see Theorem 137.

Section 4.1.3

We showed how to calculate the maximal operator on a cube Q for functions supported outside $2Q$. Lemma 130 is well known. See [387, Lemma 4.2] for example. Fefferman and Stein considered Example 68.

See [43] for the relation between the Kantorovitch operator and the Hardy–Littlewood maximal operator on $[0,1]$.

Section 4.1.4

See [124] for the Fefferman–Stein vector-valued inequality, Theorem 145. Our proof depends on the method of [378, Theorem 1.3].

Section 4.1.5

Carro, Pérez, F. Soria and J. Soria used the maximal operator $M_{\mathrm{L}\log\mathrm{L}}$ to investigate the dual inequality of Stein-type; see [64, Theorem 3.1].

Section 4.1.6

Gogatishvili, Mustafayev and Agcayazi considered the composition of the maximal operator and fractional maximal operators. See [152, Theorem 4.1] with $\alpha = 0$ for Corollary 155.

Section 4.1.7

Sawano, Sugano and Tanaka characterized the condition on the boundedness for the maximal operator M_Ψ to be locally bounded in the norm determined by Φ including the powered maximal operator $M^{(\eta)}$ in [392, Proposition 5.7], [392, Claim 2.18] and [392, Corollary 2.19]; see Proposition 156, Theorem 157 and Corollary 158, respectively.

Section 4.1.8

Recall that we considered the integral expression of the integral kernel $|\cdot|^{\alpha-n}$. See the survey paper [391, Lemma 3.2] for Lemma 160. We recorded a general estimate on convolution by Burenkov and Guliyev; see [46, Lemmas 5,6 and 7] for Propositions 163, 164(1) and 164(2), respectively.

Section 4.2

General remarks and textbooks in Section 4.2

See the fundamental textbooks [29, Chapter 5 §8], [104, Chapter 6], [146, Chapter II, §3], [157, §3.4], [293], [416, Chapter IV, §2] and [410, §6.2].

Section 4.2.1

Lerner proved the sharp maximal inequality, Theorem 165, [267, Theorem 2.1]. The sharp maximal inequality, Theorem 166, is due to Fefferman and Stein; see [125, Theorem 5] as well as [125, (4.1)].

Section 4.2.2

Fujii refined the Lebesgue differentiation theorem as in Lemma 168 [136].

Section 4.2.3

Lerner considered the local mean oscillation and the local sharp maximal function in [269, Section 4] see also [270, Section 2.3]. Theorem 174 has a history. Lerner prove Theorem 174 with an extra term in [270]. Later on Hytönen proved Theorem 174 as it stands. We followed [270, Theorem 4.5]. We followed observations on $\mathfrak{D}_a$ made by Lerner [270, Proposition 2.1]. See [269, Lemma 3.1] for Lemma 171.

We constructed a sparse family in Theorem 170; see the work by Tanaka [424].

Section 4.3

General remarks and textbooks in Section 4.3

We refer to the fundamental textbooks [156, Exercise 1.2.6] and [293, §3.2]. See also [104, p. 89].

The fractional maximal operator can control the operator of the form $V^{\alpha}(-\Delta+V)^{-\beta}$ with $0 \le \alpha \le \beta \le 1$ [421, Theorem 1].

Section 4.3.1

Adams and Xiao calculated $I_{\alpha}[|\cdot|^{-\beta}]$ in [8, Example 5.5]; see Example 82. Theoren 182 is known as the Hardy–Littlewood–Sobolev inequality [184, 185, 411]. We employed the method of Hedberg [190]. Lemma 181, known as the Hedberg inequality, is nowadays a standard method of proving the Hardy–Littlewood–Sobolev inequality. It is nowadays common to use the maximal operator and fractional maximal operators to control fractional integral operators. See [446, p.146] for Lemma 183 and [7, Proposition 3.1.2] for its variants.

Next, we consider fractional integral operators with rough kernel.

Definition 62 (Fractional integral operators with rough kernel)**.** Let $s > 1$, and let $\Omega \in L^0(\mathbb{R}^n)$ be a function homogeneous of degree zero such that $\Omega|S^{n-1} \in L^s(S^{n-1})$. Let $0 < \alpha < n$.

(1) One defines the *fractional maximal operator* $M_{\Omega,\alpha}$ *with rough kernel* Ω by

$$M_{\Omega,\alpha}f(x) \equiv \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(r)} |\Omega(y)f(x-y)|\mathrm{d}y \quad (x \in \mathbb{R}^n).$$

(2) One defines the *fractional integral operator* $I_{\Omega,\alpha}$ *with rough kernel* Ω by

$$I_{\Omega,\alpha}f(x) \equiv \int_{\mathbb{R}^n} \frac{\Omega(y)f(x-y)}{|x-y|^{n-\alpha}}\mathrm{d}y \quad (x \in \mathbb{R}^n).$$

These operators go back to [101, p. 29]. See also the textbook [293, §3.6]. See [293, Chapter 2] for a counterpart to singular integrals together with a detailed motivation. See [311, Theorem 1.1] for the Morrey–BMO boundedness and [**?**, 375] for the Morrey boundedness.

Section 4.3.2

See [164, Theorem 4.1] for Theorem 178 with $\alpha = 0$ and $p_2 = 1$.

Section 4.3.3

See [91, Theorem 5.1] for the sparse estimate of the fractional maximal operator M_α. Theorem 179, a sparse estimate for fractional maximal operators, is [334, Lemma 2.3].

Section 4.4

See [90, Proposition 2.3] for the sparse estimate of fractional integral operators I_α.

General remarks and textbooks in Section 4.4

See [4], [7, §1.2.2 and Chapter 3], [231], [293], [319] and [416, p. 354] for fractional integral operators. See [117, Chapter 5, Theorem 4], [150, Theorem 7.19] for Morrey's lemma.

There are many applications of fractional integral operators. For example, we can find applications of fractional integral operators to the restriction problem in [416, p. 352].

See [7, §1.2.3], [319, Chapter 7] and [415, Chapter III, §2] for the Bessel kernel.

Section 4.4.1

Example 84 can be found in [320, 394] for example. See [16] for the Sobolev embedding in terms of Morrey spaces over metric measure spaces. We followed [319] for Lemma 180.

Section 4.4.4

We have some explicit formulas of the solution of the variable coefficient elliptic differential equations; see [60] for Theorem 192.

We know that functions are controlled by the Riesz potential I_1 of their gradient; see [439, Lemma 1] as well as [415, Section V.2.3] for Theorem 193.

Section 4.4.2

We followed the works by Burenkov, Gogatishvili, Guliyev and Mustafayev for the local estimate of fractional integral operators; see [46, Lemma 8], [44, Lemma 3.3] and [44, Lemma 3.6] for Theorem 184.

Section 4.4.3

The idea of decomposing fractional integral operators, as we did in Proposition 188, dates back to 1995; see the paper [354, Section 3] by Pérez. See [199] for a non-linear version of the discrete fractional integral operator appearing in Proposition 188. The sparse estimate used here is [334, Lemma 3.1]; see Theorem 191.

Section 4.4.5

The Bessel potential is a potential named after Friedrich Wilhelm Bessel, a German astronomer, mathematician, physicist and geodesist.

Section 4.4.6

See the original paper [323] by Morrey for Morrey's lemma. Cianchi and Pick used Morrey's lemma to show that embedding relation between Sobolev (type) spaces and generalized Campanato spaces [79].

Section 4.5

General remarks and textbooks in Section 4.5

See the fundametal papers [28, 55] for the theory of singular integral operators. A general overview can be found in [53].

See [104, Theorem 2.11], [293, §1.2], [416, Chapter I, §4] (for general doubling measures), [416, Chapter IV, §3] (using the dyadic cubes) [382] for the Calderón–Zygmund decomposition.

For further results on singular integral operators considered in this book, we refer to the textbooks [104, Chapters 3,4 and 5], [146, Chapter II, §5], [147, Chapter 5], [223, Chapter 4], [293, §2.1], [415, Chpaters II and III], [417, Chapter VI], [416, Chapter 1, §5], [382, §1.5] and [410, §6.4].

Although we did not consider the boundedness of singular integral operators from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$, this is due to Spanne, Peetre and Stein [352, 405, 414].

Section 4.5.1

The weak-$(1,1)$ boundedness of the Riesz transform is due to Kolmogorov [237]. Its strong-(p,p) boundedness is due to M. Riesz [367, 368]. See also [51, 287].

Section 4.5.2

We refer to [83] for Definition 57. We borrowed the term "genuine singular integral operator" from [48]; see also [47, p. 34].

Section 4.5.3

We refer to [54, Lemmas 1 and 2] for Theorem 205.

Section 4.5.4

See [54, Lemma 2] for the weak estimate of singular integral operators, Theorem 207.

Section 4.5.5

See [54, Chapter 4] for Theorem 211.

Section 4.5.6

Theorems 215 and 216 (see [76, Theorem 2.5]) including Lemma 214 (see [76, (2.6)]) are some of the important results in the theory of elliptic differential operators; see [463] for an application of Theorem 215 together with the technique of the proof.

Section 4.5.7

Lerner obtained a control of $\omega(\mathrm{Re}(Tf);Q)$ and $\omega(\mathrm{Im}(Tf);Q)$ as in Lemma 217; see [271, Proposition 2.3].

Section 4.5.8

We followed Guliyev's works [162, 163] for the local estimates; see Theorem 219. Corollary 220, the local estimate for singular integral operators, is [47, Corollary 1].

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Chapter 5

BMO spaces and Morrey–Campanato spaces

Let us look at Morrey's original observation in 1938 [323]. For $0 < \theta < 1$, recall that the *Lipschitz space of order* θ denotes the set of all continuous functions F for which $\|F\|_{\mathrm{Lip}^\theta} \equiv \sup_{x,y\in\mathbb{R}^n, x\neq y} |x-y|^{-\theta}|F(x)-F(y)|$ is finite. Roughly, he showed that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ has a weak derivative in $\mathcal{M}^p_1(\mathbb{R}^n)$ with $p > n$, then $f \in \mathrm{Lip}^{1-\frac{n}{p}}(\mathbb{R}^n)$.

However, it is not easy to guarantee that a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ has a weak derivative, let alone that it belongs to $\mathcal{M}^p_1(\mathbb{R}^n)$ for some $p \in [1,\infty)$. Often, it is taken for granted that the functions are locally integrable. Consequently, there is a gap between what is assumed and what needs to be proven.

It is desirable to obtain some smoothness of locally integrable functions from inequalities derived from integration. Actually, this chapter is interested in how to replace the condition for a derivative belonging to $\mathcal{M}^p_1(\mathbb{R}^n)$. Morrey–Campanato spaces can be used for this purpose. As the limiting case, the bounded mean oscillation (BMO) space, arises naturally.

Section 5.1 investigates the BMO space since it is a fundamental function space. Section 5.2 handles operators made up of BMO functions. We define Morrey–Campanato spaces in Section 5.3. Section 5.3, which includes an auxiliary observation to Morrey–Campanato spaces, shows that Morrey–Campanato spaces are isomorphic to known function spaces.

5.1 The space BMO($\mathbb{R}^n$) and commutators

Generally speaking, the commutator is an operator of the form $[A,B] = AB - BA$, where A and B are linear operators. Here, we are mainly concerned with the case where A is a singular integral operator defined and discussed in Section 4.5 and B is a pointwise multiplier generated by BMO functions; for $b \in \mathrm{BMO}(\mathbb{R}^n)$, consider the operator $f \mapsto b \cdot f$. In Section 5.1.1 we will define $\mathrm{BMO}(\mathbb{R}^n)$ to be the set of all $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-functions a for which

$$\|a\|_* \equiv \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_Q |a(x) - m_Q(a)|\mathrm{d}x$$

is finite. This mapping makes sense despite ambiguity caused by the additive constant in $\mathrm{BMO}(\mathbb{R}^n)$ as long as we consider the commutator. This type of operator will be used when we consider the elliptic differential operators. We aim in Section 5.1 to define the commutator described above more precisely. One of the prominent properties of $\mathrm{BMO}(\mathbb{R}^n)$ is the distributional inequality which is called the John–Nirenberg inequality and is investigated in Section 5.1.2. The John–Nirenberg inequality will be a key tool when we consider operators involoved with BMO.

5.1.1 The space BMO

As we saw in Example 85, some important operators dealt with in this book fail to be $L^\infty(\mathbb{R}^n)$-bounded. The space $\mathrm{BMO}(\mathbb{R}^n)$ will compensate for this problem.

Definition 63 ($\mathrm{BMO}(\mathbb{R}^n)$)**.** Define $\|f\|_* \equiv \sup_{Q\in\mathcal{Q}} m_Q(|f - m_Q(f)|)$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. One says that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ has *bounded mean oscillation* (abbreviated to $f \in \mathrm{BMO}(\mathbb{R}^n)$), if $\|f\|_* < \infty$. One also calls $\mathrm{BMO}(\mathbb{R}^n)$ the $\mathrm{BMO}(\mathbb{R}^n)$ *space*.

Concerning terminology, clarifying remarks may be in order.

Remark 10.

(1) Strictly speaking, the $\mathrm{BMO}(\mathbb{R}^n)$-norm is not a norm appearing in function analysis: in fact, the "$\mathrm{BMO}(\mathbb{R}^n)$-norm" annihilates nonzero constants. Conversely, if a function f has 0 "$\mathrm{BMO}(\mathbb{R}^n)$-norm", then f is a constant almost everywhere. Consequently, consider the quotient linear space, and redefine the "$\mathrm{BMO}(\mathbb{R}^n)$-norm" there. Then $\mathrm{BMO}(\mathbb{R}^n)$ as a quotient linear space is a Banach space. However, the $\mathrm{BMO}(\mathbb{R}^n)$ space is not regarded as a quotient space in this book; so there are nonzero functions whose "$\mathrm{BMO}(\mathbb{R}^n)$-norm" is 0.

(2) Nevertheless, it is noteworthy that one can make the set $\mathrm{BMO}(\mathbb{R}^n)$ into a Banach space $(\mathrm{BMO}(\mathbb{R}^n), \|\cdot\|_{\mathrm{BMO}^+})$ by the norm given by $\|f\|_{\mathrm{BMO}^+} \equiv \|f\|_{L^1(Q(1))} + \|f\|_{\mathrm{BMO}}$. The price to pay here is to distort symmetry.

Example 97.

(1) In the sense of Remark 10, $L^\infty(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n)$, since $m_Q(|f|) \le \|f\|_{L^\infty}$ for all $f \in L^\infty(\mathbb{R}^n)$.

(2) Let $f(x) \equiv \log|x|$ for $x \in \mathbb{R}^n$. Then $f \in \mathrm{BMO}(\mathbb{R}^n)$. In fact, we need to check $m_Q(|f - m_Q(f)|) \lesssim 1$ for all cubes Q. By the scaling $f \mapsto f(a\cdot)$, we may assume that $|Q| = 1$. If $|Q| = 1$ and $|c(Q)| \le 2n$, then $m_Q(|f|) \lesssim 1$, so that it is clear that $m_Q(|f - m_Q(f)|) \lesssim 1$. Meanwhile if $|Q| = 1$ and $|c(Q)| > 2n$, then we can use $|\log|y| - \log|z|| \lesssim 1$ for all $y, z \in Q$.

Note that $f \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $M^\sharp f \in L^\infty(\mathbb{R}^n)$.

The next proposition reveals to us how large each $\mathrm{BMO}(\mathbb{R}^n)$ function is. Here and in what follows we tacitly mean by $f \in \mathrm{BMO}(\mathbb{R}^n)$ that f belongs to the normed space $\mathrm{BMO}(\mathbb{R}^n)$ or that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $\|f\|_* < \infty$. The confusion can rarely occur.

Proposition 221. *For all* $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\displaystyle\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(1)}(f)|}{1+|x|^{n+1}}\mathrm{d}x \lesssim \|f\|_*$.

Proof Let $Q \in \mathcal{Q}$. Then

$$|m_Q(f) - m_{2Q}(f)| = m_Q(|f - m_{2Q}(f)|) \le 2^n m_{2Q}(|f - m_{2Q}(f)|) \le 2^n \|f\|_*.$$

As a result,

$$|m_Q(f) - m_{2^j Q}(f)| \le 2^n j \|f\|_* \tag{5.1}$$

for $j \in \mathbb{N}$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(1)}(f)|}{1+|x|^{n+1}}\mathrm{d}x \\
&\le \int_{Q(1)} \frac{|f(x) - m_{Q(1)}(f)|}{1+|x|^{n+1}}\mathrm{d}x + \sum_{j=1}^{\infty} \int_{Q(2^j)} \frac{|f(x) - m_{Q(1)}(f)|}{1+|x|^{n+1}}\mathrm{d}x \\
&\le \sum_{j=1}^{\infty} \frac{1}{2^{(j-1)(d+1)}} \int_{Q(2^j)} |f(x) - m_{Q(1)}(f)|\mathrm{d}x \\
&\lesssim \sum_{j=1}^{\infty} \frac{j}{2^j} \cdot \left(\|f\|_* + \frac{1}{|Q(2^j)|} \int_{Q(2^j)} |f(x) - m_{Q(2^j)}(f)|\mathrm{d}x \right).
\end{aligned}$$

By the definition of the $\mathrm{BMO}(\mathbb{R}^n)$-norm, we have

$$\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(1)}(f)|}{1+|x|^{n+1}}\mathrm{d}x \lesssim \sum_{j=1}^{\infty} \frac{j}{2^j} \cdot \|f\|_* \lesssim \|f\|_*.$$

This is what we want.

In the next section we deal with more precise estimates. Despite the fact that $\|1\|_* = 0$, we are still eager to regard $\mathrm{BMO}(\mathbb{R}^n)$ as a Banach space without distorting the symmetry. To do this, we first observe the following, which is an obstacle when we regard $\mathrm{BMO}(\mathbb{R}^n)$ as a normed space.

Lemma 222. *Suppose that* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *satisfies* $\|f\|_* = 0$. *Then* f *is a.e. constant. The converse is also true.*

Proof This is a direct consequence of Proposition 221.

It should be noted that $\mathbb{C}$ is embedded into $L^\infty(\mathbb{R}^n)$ naturally. Consequently, if we (re)define

$$\mathrm{BMO}(\mathbb{R}^n) \equiv \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \|f\|_* < \infty\}/\mathbb{C},$$

then $\mathrm{BMO}(\mathbb{R}^n)$ is a normed space. We write $f \bmod \mathbb{C}$, if we want to disregard the difference of additive constants of f.

We can use Proposition 221 to show that $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space.

Theorem 223. *The space* $\mathrm{BMO}(\mathbb{R}^n)$ *is a Banach space.*

Proof Let $f_j \in \mathrm{BMO}(\mathbb{R}^n)$ for $j = 1, 2, \dots$ and assume $\sum_{j=1}^{\infty} \|f_j\|_* < \infty$. We assume $m_{Q(1)}(f_j) = 0$ for each j, so that $f_j \in L^1((M\chi_{Q(1)})^{\frac{n+1}{n}})$ and $\|f_j\|_{L^1((M\chi_{Q(1)})^{\frac{n+1}{n}})} \lesssim \|f_j\|_*$ thanks to Proposition 221. Thus $F \equiv \sum_{j=1}^{\infty} f_j$ converges in $L^1((M\chi_{Q(1)})^{\frac{n+1}{n}})$. We claim that $F \in \mathrm{BMO}(\mathbb{R}^n)$ and that $\sum_{j=1}^{\infty} f_j = F$ in $\mathrm{BMO}(\mathbb{R}^n)$.

Given $R \in \mathcal{Q}$, we have

$$m_R(|F - m_R(F)|) \le \sum_{j=1}^{\infty} m_R(|f_j - m_R(f_j)|) \le \sum_{j=1}^{\infty} \|f_j\|_*$$

by Lebesgue's convergence theorem. Taking the supremum over $R \in \mathcal{Q}$, we see that $F \in \mathrm{BMO}(\mathbb{R}^n)$.

In the same way we can prove

$$m_R\left(\left|F - \sum_{j=1}^{J} f_j - m_R\left(F - \sum_{j=1}^{J} f_j\right)\right|\right) \le \sum_{j=J+1}^{\infty} \|f_j\|_*.$$

Thus, we have proven that $\sum_{j=1}^{\infty} f_j = F$ in $\mathrm{BMO}(\mathbb{R}^n)$.

As a result, $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space.

Although we have remarked that we have to identify functions modulo additive constants, we are still able to embed $\mathrm{BMO}(\mathbb{R}^n)$ to the set of all measurable functions. To do this, we choose a special representative with integral over $Q(1)$ zero. Nevertheless, we usually regard $\mathrm{BMO}(\mathbb{R}^n)$ as a function space modulo $\mathbb{C}$.

5.1.2 John–Nirenberg inequality

We go into the detailed discussion of the property of $\mathrm{BMO}(\mathbb{R}^n)$. For the time being, we are oriented to showing that $\mathrm{BMO}(\mathbb{R}^n)$ is close to $L^\infty(\mathbb{R}^n)$. The John–Nirenberg inequality below is a fundamental tool when we handle $\mathrm{BMO}(\mathbb{R}^n)$-functions.

Theorem 224 (John–Nirenberg inequality)**.** *There exists* $c > 0$ *such that*

$$|\{x \in Q : |b(x) - m_Q(b)| > \lambda\}| \lesssim_n |Q| \exp\left(-\frac{c\,\lambda}{\|b\|_{\mathrm{BMO}}}\right)$$

for all $\lambda > 0$, cubes Q, and non-constant $\mathrm{BMO}(\mathbb{R}^n)$*-functions b, or equivalently,*

$$|\{x \in Q : |b(x) - m_Q(b)| > \lambda \|b\|_{\mathrm{BMO}}\}| \lesssim_n |Q| \exp(-c\,\lambda)$$

for all $\lambda > 0$, cubes Q, and $\mathrm{BMO}(\mathbb{R}^n)$*-functions b.*

Proof Here and below, we suppose that Q is a right-open cube. If λ is small, then the assertion is trivial; the right-hand side is more than the constant multiple of $|Q|$ and the left-hand side never exceeds $|Q|$. With this in mind, let us prove

$$\left|\left\{x \in Q : |b(x) - m_Q(b)| > k\,2^{n+2}\,\|b\|_{\mathrm{BMO}}\right\}\right| \le 2^{1-k}|Q| \tag{5.2}$$

by induction on k. Once this is achieved, we let $\lambda = k\,2^{n+2}\,\|b\|_{\mathrm{BMO}}$ and then pass to the continuous variables, we obtain the inequality.

When $k = 1$, then (5.2) is trivial. Assume that

$$\left|\left\{x \in R : |b(x) - m_R(b)| > k\,2^{n+1}\,\|b\|_{\mathrm{BMO}}\right\}\right| \le 2^{1-k}|R| \tag{5.3}$$

for all cubes R. We aim to prove the inequality

$$\left|\left\{x \in Q : |b(x) - m_Q(b)| > (k+1)\,2^{n+1}\,\|b\|_{\mathrm{BMO}}\right\}\right| \le 2^{-k}|Q| \tag{5.4}$$

for all cubes Q. To this end, we can assume that

$$|\{x \in Q : |b(x) - m_Q(b)| > (k+1)\,2^{n+1}\,\|b\|_{\mathrm{BMO}}\}| > 0;$$

otherwise there is nothing to prove.

By the Lebesgue differentiation theorem (see Theorem 253 below), there exists a set Z of measure zero and $S \in \mathcal{D}(Q) \setminus \{Q\}$ containing x such that $m_S(|b - m_Q(b)|) > 2\|b\|_{\mathrm{BMO}}$ as long as $x \in Q \setminus Z$ satisfies $|b(x) - m_Q(b)| > (k+1)\,2^{n+1}\,\|b\|_{\mathrm{BMO}}$. Choose a maximal cube $S(x)$ among such cubes. By the maximality, we have $m_{S(x)}(|b - m_Q(b)|) \le 2^{n+1}\|b\|_{\mathrm{BMO}}$. Hence, $|m_{S(x)}(b) - m_Q(b)| \le m_{S(x)}(|b - m_Q(b)|) \le 2^{n+1}\|b\|_{\mathrm{BMO}}$. If $S(x)$ and $S(y)$ are disjoint, maximality forces $S(x)$ and $S(y)$ to be identical. Thus we have a countable collection $D_{b,k+1}(Q) \equiv \{Q_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{D}(Q)$ of disjoint cubes such that

$$2^{-n-1}|m_{Q_\lambda}(b) - m_Q(b)| \le \|b\|_{\mathrm{BMO}} \le 2^{-1} m_{Q_\lambda}(|b - m_Q(b)|), \tag{5.5}$$

$$\{x \in Q : |b(x) - m_Q(b)| > (k+1)\,2^{n+1}\,\|b\|_{\mathrm{BMO}}\} \subset \bigcup_{\lambda\in\Lambda} Q_\lambda \cup Z. \tag{5.6}$$

Since $\{Q_\lambda\}_{\lambda\in\Lambda}$ is a family of non-overlapping dyadic cubes with respect to Q, and inequality (5.5) holds, we have

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |Q_\lambda| &\le \frac{1}{2\|b\|_{\mathrm{BMO}}} \sum_{\lambda\in\Lambda} \int_{Q_\lambda} |b(x) - m_Q(b)|\mathrm{d}x \\
&\le \frac{1}{2\|b\|_{\mathrm{BMO}}} \int_{Q} |b(x) - m_Q(b)|\mathrm{d}x \\
&\le \frac{|Q|}{2}.
\end{aligned} \tag{5.7}$$

Hence,

$$\begin{aligned}
&|\{x \in Q \,:\, |b(x) - m_Q(b)| > (k+1)\, 2^{n+1} \|b\|_{\mathrm{BMO}}\}| \\
&\quad = \sum_{\lambda \in \Lambda} |\{x \in Q_\lambda \,:\, |b(x) - m_Q(b)| > (k+1)\, 2^{n+1} \|b\|_{\mathrm{BMO}}\}| \\
&\qquad \text{(from (5.6) and the fact that } \{Q_\lambda\}_{\lambda \in \Lambda} \text{ is non-overlapping)} \\
&\quad \leq \sum_{\lambda \in \Lambda} |\{x \in Q_\lambda \,:\, |b(x) - m_{Q_\lambda}(b)| > k\, 2^{n+1} \|b\|_{\mathrm{BMO}}\}| \quad \text{(from (5.5))} \\
&\quad \leq \sum_{\lambda \in \Lambda} 2^{1-k} |Q_\lambda| \quad \text{(from the induction assumption (5.3) on } k) \\
&\quad \leq 2^{-k} |Q| \quad \text{(from (5.7)).}
\end{aligned}$$

Thus (5.4) was proven.

The John–Nirenberg inequality yields a corollary. This shows that any $\mathrm{BMO}(\mathbb{R}^n)$ function is $L^p(\mathbb{R}^n)$-integrable for all $1 \leq p < \infty$, and as a result we can say that $\mathrm{BMO}(\mathbb{R}^n)$ is close to $L^\infty(\mathbb{R}^n)$.

Corollary 225. *Let $1 \leq p < \infty$. Then, $m_Q^{(p)}(b - m_Q(b)) \lesssim_p \|b\|_{\mathrm{BMO}}$ for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all cubes Q. More precisely, there exists a constant D independent of p, Q and b such that $m_Q^{(p)}(b - m_Q(b)) \leq Dp \|b\|_{\mathrm{BMO}}$.*

Proof Denote by Γ the Gamma function. We use the Layer-Cake formula:

$$\begin{aligned}
\int_Q |b(x) - m_Q(b)|^p \mathrm{d}x &= p \int_0^\infty \lambda^{p-1} |\{x \in Q \,:\, |b(x) - m_Q(b)| > \lambda\}| \mathrm{d}\lambda \\
&\lesssim p \int_0^\infty \lambda^{p-1} \exp\left(-\frac{c_1\, \lambda}{\|b\|_{\mathrm{BMO}}}\right) \mathrm{d}\lambda \\
&\simeq p\, \Gamma(p) \cdot \|b\|_{\mathrm{BMO}}{}^p \\
&= \Gamma(p+1) \cdot \|b\|_{\mathrm{BMO}}{}^p.
\end{aligned}$$

Since we know $\sqrt[p]{\Gamma(p+1)} \sim p$ for $1 \leq p < \infty$, we obtain the desired result.

We have the following integrability estimate:

Corollary 226. *There exists a constant $\theta > 0$ depending on n such that the following inequality:*

$$\frac{1}{|Q|} \int_Q \exp\left(\theta \frac{|b(x) - m_Q(b)|}{\|b\|_{\mathrm{BMO}}}\right) \mathrm{d}x \lesssim 1$$

holds for all non-constant $\mathrm{BMO}(\mathbb{R}^n)$-functions b and all cubes Q.

Proof By the Cauchy–Hadamard test, $\lim_{j\to\infty} \frac{j}{\sqrt[j]{j!}} = \lim_{j\to\infty} \frac{j!(j+1)^{j+1}}{j^j(j+1)!} = e < 3$. Use a constant D in Corollary 225. Set $\theta \equiv \frac{1}{3D}$. Then,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \exp\left(\theta\frac{|b(x)-m_Q(b)|}{\|b\|_{\mathrm{BMO}}}\right)\mathrm{d}x &= \sum_{j=0}^{\infty}\frac{1}{j!|Q|}\int_Q\left(\theta\frac{|b(x)-m_Q(b)|^j}{\|b\|_{\mathrm{BMO}}^j}\right)\mathrm{d}x\\ &\le \sum_{j=0}^{\infty}\frac{(D\theta)^j\, j^j}{j!}\\ &\lesssim 1.\end{aligned}$$

Thus, the proof is complete.

5.1.3 Exercises

Exercise 75. Let Γ be the Gamma function given by

$$\Gamma(p) \equiv \int_0^\infty t^{p-1}e^{-t}\mathrm{d}t \quad (p>0).$$

(1) Show that $\Gamma(p) = (p-1)!$ for all $p \in \mathbb{N}$.

(2) Show that $\sqrt[p]{\Gamma(p+1)} \sim p$ for $p \ge 1$.

Exercise 76. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $Q \in \mathcal{Q}$. Set $D_{b,0}(Q) \equiv \{Q\}$. For a cube $T \in \mathcal{Q}$ and $k \in \mathbb{N}_0$, choose $D_{b,k+1}(T)$ satisfying (5.5) and (5.6) with Q replaced by T. Set $\mathcal{S}_0 \equiv \{Q\}$ and $\mathcal{S}_{j+1} \equiv \bigcup_{T\in\mathcal{S}_j} D_{b,j+1}(T)$ for $j \in \mathbb{N}_0$. Then reexamine the proof of Theorem 224 to show that $\{\mathcal{S}_j\}_{j=0}^\infty$ is 1/2-sparse.

Exercise 77. [344, Lemma 2.1] Let $1 \le p < \infty$ and $Q \in \mathcal{Q}$. Then show that

$$\left[\int_Q\left\{\int_Q |u(x)-u(y)|\mathrm{d}y\right\}^p\right]^{\frac{1}{p}} \le 2\left[\int_Q\left|\int_Q (u(x)-u(y))\mathrm{d}y\right|^p\right]^{\frac{1}{p}}$$

for all $u \in L^p(Q)$. Hint: Use the decomposition $u(x)-u(y) = u(x)-m_Q(u)+m_Q(u)-u(y)$ to the left-hand side.

Exercise 78. [331, Lemma 2.2] Let $F : \mathbb{C} \to \mathbb{C}$ be a Lipschitz mapping.

(1) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Show that $m_Q(|F\circ f(x) - m_Q(F\circ f)|) \le \|F\|_{\mathrm{Lip}} m_Q(|f(x)-m_Q(f)|)$.

(2) If $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then show that $F\circ f \in \mathrm{BMO}(\mathbb{R}^n)$ together with the estimate $\|F\circ f\|_* \le \|F\|_{\mathrm{Lip}}\|f\|_*$.

5.2 Commutators

Commutators will play a key role when we consider elliptic differential equations. Especially, commutators generated by BMO and singular integral operators will arise as the solution operators of the elliptic differential equations. We will consider commutators generated by BMO and singular integral operators and commutators generated by BMO and fractional integral operators in Sections 5.2.1 and 5.2.2, respectively.

5.2.1 Commutators generated by BMO and singular integral operators

Section 5.2.1 takes up an operator of the form

$$[a,T]f(x) \equiv \int_{\mathbb{R}^n} (a(x)-a(y))K(x,y)f(y)\mathrm{d}y, \tag{5.1}$$

where T is a singular integral operator whose kernel is K and $a \in \mathrm{BMO}(\mathbb{R}^n)$. Such an operator $[a,T]$ is called a *commutator*. We will notice that $[a,T]$ behaves similarly to T. Let $1 < p,q,r < \infty$ and assume $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then if $a \in L^q$, then $[a,T]$ maps $L^p(\mathbb{R}^n)$ boundedly into $L^r(\mathbb{R}^n)$. But the cancellation should give something else. Assume in addition that $T_0 f(x) \equiv \int_{\mathbb{R}^n} K(x,y)f(y)\mathrm{d}y$ defines an $L^p(\mathbb{R}^n)$-bounded operator. We aim in Section 5.2.1 to apply the sharp maximal inequality to prove the boundedness of this operator. Because of the singularity of the kernel of $[a,T]$, we have to begin with the task of making clear what (5.1) means.

Keeping in mind that we are going to use the sharp maximal operator $M^\sharp$, let us fix a cube Q. Although $\mathrm{BMO}(\mathbb{R}^n)$ differs from $L^\infty(\mathbb{R}^n)$, it is still close to $L^\infty(\mathbb{R}^n)$. Once we choose a cube Q, the restriction of any element in $\mathrm{BMO}(\mathbb{R}^n)$ to Q is in $L^u(Q)$, for all $u < \infty$ because of the John Nirenberg inequality. In view of this observation we are tempted to rewrite (5.1) as follows:

$$[a,T]f(x) = (a(x)-m_Q(a))T_0 f(x) - T_0[(a-m_Q(a))f](x). \tag{5.2}$$

The first term of (5.2) seems nice, because K is assumed $L^p(\mathbb{R}^n)$-bounded. However, we are still at a loss because $(a-m_Q(a))f$ does not belong to $L^p(\mathbb{R}^n)$. Thus we have to work more. We decompose the above formula once more;

$$\begin{aligned}[a,T]f(x) = (a(x)-m_Q(a))T_0 f(x) - T[(a-m_Q(a))\chi_{2Q}\cdot f](x)\\ - \int_{\mathbb{R}^n\setminus 2Q} K(x,y)(a(y)-m_Q(a))f(y)\mathrm{d}y.\end{aligned}$$

Now that $x \in Q$ and $(a-m_Q(a))\chi_{2Q}\cdot f \in L^q(\mathbb{R}^n)$ for some $1 < q < p$ by virtue of the John–Nirenberg inequality, it looks nice. The integral of the third term does not contain the singularity.

Speaking more precisely, we have the following information:

Lemma 227. *Let T be a singular integral operator. Then*

$$\int_{\mathbb{R}^n\setminus 2Q} |K(x,y)(a(y)-m_Q(y))f(y)|\mathrm{d}y \lesssim \ell(Q)^{-\frac{n}{p}}\|a\|_*\cdot\|f\|_{L^p}.$$

Proof By the size condition

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2Q} |K(x,y)(a(y)-m_Q(y))f(y)|\mathrm{d}y\\
&\lesssim \sum_{j=1}^{\infty}\frac{1}{(2^j\ell(Q))^n}\int_{2^{j+1}Q\setminus 2^jQ}|a(y)-m_Q(y)|\cdot|f(y)|\mathrm{d}y\\
&\lesssim \sum_{j=1}^{\infty}\frac{\|f\|_{L^p}}{(2^j\ell(Q))^n}\left(\int_{2^{j+1}Q\setminus 2^jQ}|a(y)-m_Q(y)|^{p'}\mathrm{d}y\right)^{\frac{1}{p'}}\\
&\lesssim \sum_{j=1}^{\infty}\frac{j}{(2^j\ell(Q))^{\frac{n}{p}}}\|a\|_*\|f\|_{L^p}\\
&\lesssim \ell(Q)^{-\frac{n}{p}}\|a\|_*\cdot\|f\|_{L^p}.
\end{aligned}$$

Here for the second inequality we used (5.1).

Therefore, this is an appropriate candidate of the definition of $[a,T]f$. To justify the observation above, we will prove the following lemma.

Lemma 228. *Let $f\in\bigcup_{1\le p<\infty}L^p(\mathbb{R}^n)$, T be a singular integral operator and $a\in\mathrm{BMO}(\mathbb{R}^n)$. Given a cube Q, we set*

$$\begin{aligned}
S_Qf(x)\equiv{}&(a(x)-m_Q(a))T_0f(x)-T[(a-m_Q(a))\chi_{2Q}\cdot f](x)\\
&-\int_{\mathbb{R}^n\setminus 2Q}K(x,y)(a(y)-m_Q(a))f(y)\mathrm{d}y.
\end{aligned}$$

Then if Q and R are cubes with $Q\subset R$, then $S_Qf(x)=S_Rf(x)$ for almost every $x\in Q(\subset R)$.

Proof Note that

$$\begin{aligned}
&S_Qf(x)-S_Rf(x)\\
&=T[(a-m_R(a))\chi_{2R\setminus 2Q}\cdot f](x)-\int_{2R\setminus 2Q}K(x,y)(a(y)-m_R(a))f(y)\mathrm{d}y.
\end{aligned}$$

By the kernel condition this is equal to 0. The lemma is therefore proven.

Definition 64 (Commutator)**.** Given $a\in\mathrm{BMO}(\mathbb{R}^n)$ and a singular integral operator T with associated kernel K, define the *commutator* $[a,T]$ by

$$\begin{aligned}
[a,T]f(x)\equiv{}&\lim_{j\to\infty}(a(x)-m_{Q(j)}(a))T_0f(x)-T[(a-m_{Q(j)}(a))\chi_{2Q(j)}\cdot f](x)\\
&-\int_{\mathbb{R}^n\setminus 2Q(j)}K(x,y)(a(y)-m_{Q(j)}(a))f(y)\mathrm{d}y.
\end{aligned}$$

Note that the above definition makes sense because the limit exists for a.e. $x \in \mathbb{R}^n$. In view of Lemma 228, the following proposition is clear.

Proposition 229. *Let $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. If R is a cube and T is a singular integral operator with the kernel K, then*

$$[a,T]f(x) = (a(x) - m_R(a))T_0 f(x) - T[(a - m_R(a))\chi_{2R} \cdot f](x) - \int_{\mathbb{R}^n \setminus 2R} K(x,y)(a(y) - m_R(a))f(y)\mathrm{d}y$$

for a.e. $x \in R$.

Proof The proof is left as an exercise for the interested readers. See Exercise 79.

We now prove the boundedness of commutators, where we present an application of the sharp maximal inequality.

Lemma 230. *Let T be a singular integral operator, and let $r > 0$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Then*

$$M^\sharp([a,T]f)(x) \lesssim \|a\|_* \left(M^{(r)} \circ Tf(x) + M^{(r)} f(x) \right) \quad (x \in \mathbb{R}^n). \tag{5.3}$$

Proof Now let us estimate $M^\sharp([a,T]f)(x)$. To do this, we use the decomposition of $[a,T]f$ for a given cube Q above. Firstly, we write

$$\begin{aligned} F_1(x) &\equiv (a(x) - m_Q(a))T_0 f(x), \\ F_2(x) &\equiv T[(a - m_Q(a))\chi_{2Q} \cdot f](x), \\ F_3(x) &\equiv \int_{\mathbb{R}^n \setminus 2Q} K(x,y)(a(y) - m_R(a))f(y)\mathrm{d}y \end{aligned}$$

for the sake of simplicity. Let $1 < r < p$ be an auxiliary parameter fixed throughout.

We treat F_1 by using $m_Q(|F_1 - m_Q(F_1)|) \le 2m_Q(|F_1|)$. Let us recall that we have been writing

$$m_Q^{(t)}(G) \equiv \left(\frac{1}{|Q|} \int_Q |G(x)|^t \mathrm{d}x \right)^{\frac{1}{t}}, \ M^{(t)}F(x) \equiv M[|F|^t](x)^{\frac{1}{t}}$$

for $t > 0$ and $F, G \in L^0(\mathbb{R}^n)$. Under this notation we have

$$m_Q(|F_1 - m_Q(F_1)|) \lesssim m_Q^{(r')}(|a - m_Q(a)|) \cdot m_Q^{(r)}(|Tf|) \lesssim \|a\|_* M^{(r)}[Tf](x).$$

The most right-hand side matches the first term of (5.3).

As for F_2, we use the $L^{\sqrt{r}}(\mathbb{R}^n)$-boundedness of T as well.

$$\begin{aligned} m_Q(|F_2 - m_Q(F_2)|) &\le 2m_Q(|F_2|) \\ &\le 2m_Q^{(r)}(|F_2|) \\ &\le \frac{2}{|Q|} \|T[(a - m_Q(a))\chi_{2Q} \cdot f]\|_{L^{\sqrt{r}}}. \end{aligned}$$

We now employ the John–Nirenberg inequality:

$$m_Q(|F_2 - m_Q(F_2)|) \lesssim m_{2Q}^{(\sqrt{r})}(|(a - m_Q(a))f|) \lesssim \|a\|_* M^{(r)} f(x).$$

The most right-hand side matches the second term of (5.3).

Finally for the estimate of F_3, we take full advantage of the oscillation property of $M^\sharp$. Note that if $x, y \in Q$ then

$$|F_3(x) - F_3(y)| \lesssim \int_{\mathbb{R}^n \setminus 2Q} |K(x,z) - K(x,y)| \cdot |a(z) - m_R(a)| \cdot |f(z)| \mathrm{d}z.$$

Let $\ell \le 2\ell(Q)$. Then, by the Hörmander condition of the kernel K,

$$\frac{1}{|z - c(Q)|^{n+1}} = (n+1) \int_0^\infty \frac{\chi_{B(c(Q),\ell)}(z) \mathrm{d}\ell}{\ell^{n+2}},$$

and the fact that $(\mathbb{R}^n \setminus 2Q) \cap B(x, \frac{1}{2}\ell) = \emptyset$, we have

$$\begin{aligned} |F_3(x) - F_3(y)| &\lesssim \ell(Q) \int_{\mathbb{R}^n \setminus 2Q} \frac{|a(z) - m_R(a)| \cdot |f(z)|}{|z - c(Q)|^{n+1}} \mathrm{d}z \\ &\lesssim \ell(Q) \int_{2\ell(Q)}^\infty \left(\frac{1}{\ell^{n+2}} \int_{B(c(Q),\ell)} |a(z) - m_R(a)| \cdot |f(z)| \mathrm{d}z \right) \mathrm{d}\ell. \end{aligned}$$

If we invoke the John–Nirenberg inequality, then we obtain

$$\begin{aligned} &|F_3(x) - F_3(y)| \\ &\lesssim \ell(Q) \int_{2\ell(Q)}^\infty \|a - m_R(a)\|_{L^{r'}(B(c(Q),\ell))} \left(\int_{B(c(Q),\ell)} |f(z)|^r \mathrm{d}z \right)^{\frac{1}{r}} \frac{\mathrm{d}\ell}{\ell^{n+2}} \\ &\lesssim \ell(Q) \, \|a\|_* \cdot M^{(r)} f(x) \int_{2\ell(Q)}^\infty \log\left(2 + \frac{\ell}{\ell(Q)}\right) \frac{d\ell}{\ell^2} \\ &\lesssim \|a\|_* M^{(r)} f(x), \end{aligned}$$

which leads us to (5.3).

We will prove that commutators of the above type are $L^p(\mathbb{R}^n)$-bounded for all $1 < p < \infty$.

Theorem 231. *Let $1 < p < \infty$ and $a \in \mathrm{BMO}(\mathbb{R}^n)$. Then $[a, T]$ extends to an $L^p(\mathbb{R}^n)$-bounded operator so that it satisfies the norm estimate $\| [a,T] \|_{L^p \to L^p} \lesssim_T \|a\|_*$.*

Proof By the truncation, we may assume that $a \in L^\infty(\mathbb{R}^n)$. We will show that $\|[a,T]f\|_{L^p} \lesssim_T \|a\|_* \|f\|_{L^p}$ for $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

Now that $\min(\, |[a,T]f|, 1\,) \in L^p(\mathbb{R}^n)$, we are in the position of applying the sharp maximal inequality (see Theorem 166) to obtain

$$\| [a,T]f \|_{L^p} \lesssim \|M^\sharp([a,T]f)\|_{L^p}. \tag{5.4}$$

Let $r = \sqrt{p}$. Since M and T are bounded on $L^q(\mathbb{R}^n)$ for all $1 < q < \infty$, it follows that

$$\|M^\sharp([a,T]f)\|_{L^p} \lesssim \|a\|_* \left(\|M^{(r)} \circ Tf\|_{L^p} + \|M^{(r)} f\|_{L^p}\right) \lesssim \|a\|_* \|f\|_{L^p}. \quad (5.5)$$

Combining (5.4) and (5.5) gives the desired estimate.

We consider the convergence problem of the integral defining commutators. Start with *truncated commutators*. Let $\varepsilon > 0$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a singular integral operator. Then define

$$[a,T]_\varepsilon f(x) \equiv \int_{\mathbb{R}^n \setminus B(\varepsilon)} (a(x) - a(y)) K(x,y) f(y) \mathrm{d}y$$

as long as the right-hand side makes sense. If $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $a \in \mathrm{BMO}(\mathbb{R}^n)$, then $[a,T]f(x) = \lim_{\varepsilon \downarrow 0} [a,T]_\varepsilon f(x)$ exists for almost all $x \in \mathbb{R}^n$, since $f, a \cdot f \in L^2(\mathbb{R}^n)$ and

$$[a,T]f(x) = a(x) \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(\varepsilon)} K(x,y) f(y) \mathrm{d}y - \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(\varepsilon)} a(y) K(x,y) f(y) \mathrm{d}y.$$

For a given $f \in L^p(\mathbb{R}^n)$, we seek to consider the almost everywhere convergence as $\varepsilon \downarrow 0$. We need to consider the maximal operator of truncated commutators.

Lemma 232. *Let $a \in \mathrm{BMO}(\mathbb{R}^n)$. Also let $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$. Then the operator*

$$Wf(x) \equiv \sup_{\varepsilon > 0} \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} |a(x) - a(y)| \varphi\left(\frac{x-y}{\varepsilon}\right) |f(y)| \, \mathrm{d}y \quad (x \in \mathbb{R}^n),$$

is a bounded operator on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$.

Proof Let $f \in L^p(\mathbb{R}^n)$. Set $r \equiv \sqrt{p}$. Write

$$W_\varepsilon f(x) \equiv \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} |a(x) - a(y)| \varphi\left(\frac{x-y}{\varepsilon}\right) f(y) \mathrm{d}y \quad (x \in \mathbb{R}^n).$$

Consider the operator given by $W_E \equiv \sup_{\varepsilon \in E} |W_\varepsilon f|$ for each subset $E \subset (0,\infty)$, so that $Wf(x) = W_{(0,\infty)}[|f|](x)$. By a simple approximation argument, we have $Wf(x) = W_{(0,\infty)\cap\mathbb{Q}}[|f|](x)$. Furthermore, since $W_{E_j}[|f|](x) \uparrow Wf(x)$ for any sequence of increasing finite subsets $\{E_j\}_{j=1}^\infty$ such that $\bigcup_{j=1}^\infty E_j = (0,\infty)\cap\mathbb{Q}$, we have only to prove $\|W_E f\|_{L^p} \lesssim \|f\|_{L^p}$ for any finite subset $E \subset (0,\infty)$ with the

implicit constant does not depend on E. Furthermore, by another truncation argument, we may assume that $a \in L^\infty(\mathbb{R}^n)$. We will prove $\|Wf\|_{L^p} \lesssim \|f\|_{L^p}$ for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. We will resort to the sharp maximal operator estimate. We define

$$M_E^\sharp \vec{F}(x) = \sup_{Q \in \mathcal{Q}} \chi_Q(x) m_Q(\|\vec{F} - m_Q(\vec{F})\|_{\ell^\infty(E)})$$

for any vector-valued function $F = \{F_\varepsilon\}_{\varepsilon \in E} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)^E$.

We will choose $F_\varepsilon = W_\varepsilon f$ for each $\varepsilon \in E$. In fact, we fix $x \in \mathbb{R}^n$ and a cube Q and then, letting $r = \sqrt{p}$, we will prove

$$\begin{aligned} &m_Q\left(\sup_{\varepsilon \in E} |W_\varepsilon f - W_\varepsilon[(a - m_Q(a))\chi_{\mathbb{R}^n \setminus 2Q} f](c(Q))|\right) \\ &\lesssim \|a\|_* M^{(r)} f(x) + \|a\|_* M^{(r)} T_* f(x). \end{aligned}$$

Let $z \in Q$ and $\varepsilon \in E$ be arbitrary. By the triangle inequality,

$$\begin{aligned} &\left||a(z) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right) - |m_Q(a) - a(y)|\varphi\left(\frac{c(Q)-y}{\varepsilon}\right)\chi_{\mathbb{R}^n \setminus 2Q}(y)\right| \\ &\le \left||a(z) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right) - |m_Q(a) - a(y)|\varphi\left(\frac{c(Q)-y}{\varepsilon}\right)\right|\chi_{\mathbb{R}^n \setminus 2Q}(y) \\ &\quad + |a(z) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right)\chi_{2Q}(y). \end{aligned}$$

By the triangle inequality once again,

$$\begin{aligned} &\left||a(z) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right) - |m_Q(a) - a(y)|\varphi\left(\frac{c(Q)-y}{\varepsilon}\right)\chi_{\mathbb{R}^n \setminus 2Q}(y)\right| \\ &\le \left||m_Q(a) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right) - |m_Q(a) - a(y)|\varphi\left(\frac{c(Q)-y}{\varepsilon}\right)\right|\chi_{\mathbb{R}^n \setminus 2Q}(y) \\ &\quad + \left||a(z) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right) - |m_Q(a) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right)\right|\chi_{\mathbb{R}^n \setminus 2Q}(y) \\ &\quad + |a(z) - a(y)| \cdot \frac{1}{\varepsilon^n}\varphi\left(\frac{z-y}{\varepsilon}\right)\chi_{2Q}(y). \end{aligned}$$

By the triangle inequality and by the mean value theorem,

$$\begin{aligned} &\frac{1}{\varepsilon^n}\left||a(z) - a(y)|\varphi\left(\frac{z-y}{\varepsilon}\right) - |m_Q(a) - a(y)|\varphi\left(\frac{c(Q)-y}{\varepsilon}\right)\chi_{\mathbb{R}^n \setminus 2Q}(y)\right| \\ &\lesssim \frac{|a(y) - m_Q(a)|}{(|y - c(Q)| + \ell(Q))^{n+1}}\ell(Q) + |a(z) - m_Q(a)| \cdot \frac{1}{\varepsilon^n}\varphi\left(\frac{z-y}{\varepsilon}\right) \\ &\quad + |m_Q(a) - a(y)| \cdot \frac{1}{\varepsilon^n}\varphi\left(\frac{z-y}{\varepsilon}\right)\chi_{2Q}(y). \end{aligned}$$

Note that M is bounded on $L^r(\mathbb{R}^n)$. We observe that

$$\begin{aligned}
&m_Q\left(\int_{\mathbb{R}^n} |m_Q(a)-a(y)|\cdot\frac{1}{\varepsilon^n}\varphi\left(\frac{\cdot-y}{\varepsilon}\right)\chi_{2Q}|f(y)|\mathrm{d}y\right)\\
&\le m_Q\left(M\left[|m_Q(a)-a|\cdot\chi_{2Q}f\right]\right)\\
&\le m_Q^{(\sqrt{r})}\left(M\left[|m_Q(a)-a|\cdot\chi_{2Q}f\right]\right)\\
&\lesssim m_{2Q}^{(\sqrt{r})}\left(|m_Q(a)-a|\cdot f\right).
\end{aligned}$$

Consequently, by the John–Nirenberg inequality,

$$\begin{aligned}
&|W_\varepsilon f(z)-W_\varepsilon[(a-m_Q(a))\chi_{\mathbb{R}^n\setminus 2Q}f](c(Q))|\\
&\le \|a\|_* M^{(r)}f(x)+|a(z)-m_Q(a)|Mf(z).
\end{aligned}$$

Thus, $M_E^\sharp[\{W_\varepsilon f\}_{\varepsilon\in E}](x)\lesssim \|a\|_* M^{(r)}f(x)+\|a\|_* M^{(r)}\circ Mf(x)$ for all $x\in\mathbb{R}^n$. If we use the vector-valued sharp maximal inequality, then we have the desired result.

Although the indicator function of balls is not smooth, we can take advantage of the positivity of the integral kernel.

Corollary 233. *Let $a\in \mathrm{BMO}(\mathbb{R}^n)$, and let $1<p<\infty$. Define*

$$M_a f(x)\equiv \sup_{Q\in\mathcal{Q}}\frac{\chi_Q(x)}{|Q|}\int_Q |a(x)-a(y)|\cdot|f(y)|\mathrm{d}y \quad (x\in\mathbb{R}^n)$$

for $f\in L^p(\mathbb{R}^n)$. Then M_a is bounded on $L^p(\mathbb{R}^n)$.

Proof Let φ be a smooth bump function at zero, that is $\varphi\in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ satisfies $\chi_{B(1)}\le\varphi\le\chi_{B(2)}$. Then

$$M_a f(x)\lesssim \sup_{\varepsilon>0}\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}|a(x)-a(y)|\varphi\left(\frac{x-y}{\varepsilon}\right)|f(y)|\mathrm{d}y.$$

Thus, we are in the position of using Lemma 232.

Let $\varphi\in C^\infty(\mathbb{R}^n)$. We will consider a smooth variant of the truncation in view of Corollary 233.

We write

$$K_{\varphi,\varepsilon}(x,y)\equiv\varphi\left(\frac{x-y}{\varepsilon}\right)K(x,y)\quad(x,y\in\mathbb{R}^n).$$

Lemma 234. *Let $a\in\mathrm{BMO}(\mathbb{R}^n)$. Let $\varphi\in C^\infty(\mathbb{R}^n)$ satisfy $0\notin\mathrm{supp}(\varphi)$, $\|\varphi\|_{L^\infty}+\|\nabla\varphi\|_{L^\infty}\lesssim 1$ and $\chi_{\mathbb{R}^n\setminus B(1)}\le\varphi\le 1$. Also let T be a singular integral operator with the kernel K. Then the operator*

$$[a,T]_*^\varphi f(x)\equiv\sup_{\varepsilon>0}\left|\int_{\mathbb{R}^n}(a(x)-a(y))K_{\varphi,\varepsilon}(x,y)f(y)\mathrm{d}y\right|\quad(x\in\mathbb{R}^n),$$

is a bounded linear operator on $L^p(\mathbb{R}^n)$ for any $1<p<\infty$.

Proof It suffices to prove $\|[a,T]^{\varphi}_{*}f\|_{L^p} \lesssim \|f\|_{L^p}$ for all $f \in L^{\infty}_{\mathrm{c}}(\mathbb{R}^n)$. It can be assumed that $a \in L^{\infty}(\mathbb{R}^n)$ by the truncation. As before, we replace $\sup_{\varepsilon>0}$ by $\sup_{\varepsilon\in E}$ for a finite set $E \subset (0,\infty)$. Let $r = \sqrt{p}$ as before.

Let $f \in L^p(\mathbb{R}^n)$. We write

$$[a,T]^{\varphi,\varepsilon}f(x) \equiv \int_{\mathbb{R}^n} (a(x)-a(y))K_{\varphi,\varepsilon}(x,y)f(y)\mathrm{d}y$$

and

$$T^{\varphi,\varepsilon}f(x) \equiv \int_{\mathbb{R}^n} K_{\varphi,\varepsilon}(x,y)f(y)\mathrm{d}y, \quad T^{\varphi,*}f(x) = \sup_{\varepsilon>0}|T^{\varphi,\varepsilon}f(x)|.$$

Consider the sharp maximal operator estimate. In fact, we will prove

$$\begin{aligned} &m_Q(|[a,T]^{\varphi,\varepsilon}f - T^{\varphi,\varepsilon}[(a-m_Q(a))\chi_{\mathbb{R}^n\setminus 2Q}f](c(Q))|) \\ &\lesssim \|a\|_* M^{(r)}f(x) + \|a\|_* M^{(r)} \circ T^{\varphi,*}f(x) \end{aligned}$$

for any cube Q containing $x \in \mathbb{R}^n$.

Let $z \in \mathbb{R}^n$. Then

$$[a,T]^{\varphi,\varepsilon}f(z) - T^{\varphi,\varepsilon}[(a-m_Q(a))\chi_{\mathbb{R}^n\setminus 2Q}f](c(Q)) = \int_{\mathbb{R}^n} F_{\varepsilon}(z,y)\mathrm{d}y,$$

where

$$\begin{aligned} F_{\varepsilon}(z,y) \equiv &-(a(y)-m_Q(a))\left(K_{\varphi,\varepsilon}(z,y) - K_{\varphi,\varepsilon}(c(Q),y)\right)\chi_{\mathbb{R}^n\setminus 2Q}(y)f(y) \\ &+ (a(z)-m_Q(a))K_{\varphi,\varepsilon}(z,y)f(y) \\ &+ (m_Q(a)-a(y))K_{\varphi,\varepsilon}(z,y)\chi_{2Q}(y)f(y). \end{aligned}$$

For the first term, we will use the John–Nirenberg inequality and the fact that φ is smooth to give

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} (a(y)-m_Q(a))\left(K_{\varphi,\varepsilon}(z,y) - K_{\varphi,\varepsilon}(c(Q),y)\right)\chi_{\mathbb{R}^n\setminus 2Q}(y)f(y)\mathrm{d}y\right| \\ &\lesssim \|a\|_* M^{(r)}f(z). \end{aligned}$$

For the second term, we will use the John–Nirenberg inequality again to give

$$\frac{1}{|Q|}\int_Q \left|\int_{\mathbb{R}^n} (a(z)-m_Q(a))K_{\varphi,\varepsilon}(z,y)f(y)\mathrm{d}y\right|\mathrm{d}z \lesssim \|a\|_* M^{(r)}[T^{\varphi,*}f](z)$$

For the third term, we will use the maximal singular integral operator to give

$$\left|\int_{\mathbb{R}^n} (m_Q(a)-a(y))K_{\varphi,\varepsilon}(z,y)\chi_{2Q}(y)f(y)\mathrm{d}y\right| \lesssim T^{\varphi,*}[(a-m_Q(a))\chi_{2Q}f](z)$$

If we integrate this estimate over $z \in Q$, we obtain

$$\left| \int_{\mathbb{R}^n} (m_Q(a) - a(y)) K_{\varphi,\varepsilon}(z,y) \chi_{2Q}(y) f(y) \mathrm{d}y \right| \lesssim \|a\|_* M^{(r)} \circ T^{\varphi,*} f(x).$$

Thus,

$$M^\sharp [[a,T]_*^\varphi f](x) \lesssim \|a\|_* M^{(r)} f(x) + \|a\|_* M \circ T^{\varphi,*} f(x).$$

Since $a \in L^\infty(\mathbb{R}^n)$ and $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, we are in the position of using the sharp maximal inequality to have the desired result.

We are oriented to the almost everywhere convergence of truncated commutators. To this end, we will prove that the maximal operators of truncated commutators are bounded.

Theorem 235. *Let $a \in \mathrm{BMO}(\mathbb{R}^n)$. Also let T be a singular integral operator with the kernel K. Then the operator*

$$[a,T]_* f(x) \equiv \sup_{\varepsilon > 0} \left| \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} (a(x) - a(y)) K(x,y) f(y) \mathrm{d}y \right| \quad (x \in \mathbb{R}^n)$$

is a bounded linear operator on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$.

Proof Let $\varphi \in C^\infty(\mathbb{R}^n)$ satisfy $0 \notin \mathrm{supp}(\varphi)$, $\|\varphi\|_{L^\infty} + \|\nabla \varphi\|_{L^\infty} \lesssim 1$ and $\chi_{\mathbb{R}^n \setminus B(1)} \le \varphi \le 1$ as in Lemma 234. We simply use

$$[a,T]_* f(x) \lesssim [a,T]_*^\varphi f(x) + M_a f(x) \quad (x \in \mathbb{R}^n)$$

and Lemma 234.

We apply Theorem 235 to some special cases. Although we do not take up elliptic differential equations in this book, Theorem 236 can be used for elliptic differential equations.

Theorem 236. *Let $a \in \mathrm{BMO}(\mathbb{R}^n)$. Assume that $K \in L^\infty(\mathbb{R}^n \times S^{n-1})$ satisfies the following:*

(1) *$K(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for all $x \in \mathbb{R}^n$;*

(2) *$K(x,tz) = t^{-n} K(x,z)$ for every $t > 0$ and $(x,z) \in \mathbb{R}^n \times S^{n-1}$;*

(3) *$\displaystyle\int_{S^{n-1}} K(x,z) \mathrm{d}\sigma(z) = 0$ for all $x \in \mathbb{R}^n$;*

(4) *$\displaystyle\max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha K}{\partial z^\alpha} \right\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} < \infty.$*

Let $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$. For any $\varepsilon > 0$, we set

$$[a,T]_\varepsilon f(x) \equiv \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x, x-y)(a(x) - a(y)) f(y) \mathrm{d}y.$$

Then there exist $[a,T]f \in L^p(\mathbb{R}^n)$ such that $\lim_{\varepsilon \downarrow 0} \|[a,T]_\varepsilon f - [a,T]f\|_{L^p} = 0$.

Proof The integral kernel K satisfies the cancellation condition. Hence, we can use the expansion in Theorem 215.

5.2.2 Commutators generated by BMO and fractional integral operators

In addition to the above commutator and the kernel K of T satisfying the size condition, we consider the commutator generated by BMO and the fractional integral operator I_α given by

$$[a, I_\alpha]f(x) \equiv \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{(a(x) - a(y))}{|x-y|^{n-\alpha}} f(y) \mathrm{d}y \quad (x \in \mathbb{R}^n). \tag{5.6}$$

Needless to say what $[a, T]$ is to $[a, I_\alpha]$ is what T is to I_α. Consequently, $[a, I_\alpha]$ can be located as the mixture of $[a, T]$ and I_α. As we will see the commutator $[a, I_\alpha]$ behaves like I_α.

Lemma 237. *For all $0 < \alpha < n$, $1 < r < \infty$ and $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, we have*

$$M^\sharp([a, I_\alpha]f)(x) \lesssim \|a\|_*(M^{(r)}_\alpha f(x) + M^{(r)} I_\alpha f(x)) \quad (x \in \mathbb{R}^n).$$

Proof Let Q be a fixed cube, and let $x \in Q$. We will show that

$$m_Q(|[a, I_\alpha]f - m_Q(I_\alpha[(a - m_Q(a))\chi_{\mathbb{R}^n \setminus 3Q} f])|) \lesssim \|a\|_*(M^{(r)}_\alpha f(x) + M^{(r)} I_\alpha f(x))$$

for $x \in \mathbb{R}^n$. To this end, by replacing a with $a - m_Q(a)$, we may assume $m_Q(a) = 0$. We decompose

$$[a, I_\alpha]f = a \cdot I_\alpha f - I_\alpha[a \cdot \chi_{3Q} f] - I_\alpha[a \cdot \chi_{\mathbb{R}^n \setminus 3Q} f].$$

By the John–Nirenberg inequality we have

$$m_Q(|a \cdot I_\alpha f|) \le m^{(r')}_Q(a) m^{(r)}_Q(I_\alpha f) \lesssim \|a\|_* M^{(r)} \circ I_\alpha f(x).$$

We also have

$$m_Q(I_\alpha[a \cdot \chi_{3Q} f]) \lesssim \ell(Q)^\alpha m_{3Q}(|a \cdot f|) \lesssim \ell(Q)^\alpha m^{(r)}_{3Q}(f)\|a\|_* \lesssim \|a\|_* M^{(r)}_\alpha f(x)$$

again by the John–Nirenberg inequality. Finally, by the mean–value theorem and the John–Nirenberg inequaity, for all $x, y \in Q$, we have

$$|I_\alpha[(a - m_Q(a))\chi_{\mathbb{R}^n \setminus 3Q} f](x) - I_\alpha[(a - m_Q(a))\chi_{\mathbb{R}^n \setminus 3Q} f](y)| \lesssim \|a\|_* M^{(r)}_\alpha f(x).$$

Thus, the proof is complete.

With this pointwise estimate, we can easily obtain the boundedness of $[a, I_\alpha]$.

Theorem 238. *Let $1 < p < q < \infty$ and $0 < \alpha < n$ satisfy $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$. Then $[a, I_\alpha]$, defined initially on $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, extends to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof The proof is akin to that of Theorem 231. See Exercise 80.

5.2.3 Exercises

Exercise 79. Prove Proposition 229 using (4.1).

Exercise 80. Let $1 < r < p$. Using Lemma 237, the sharp maximal inequality and the boundedess of $M_\alpha^{(r)}$ and I_α, prove Theorem 238.

5.3 Morrey–Campanato spaces

In 1938, C. Morrey investigated the elliptic differential operators. His technique became the theory of normed spaces. However, around the 1960's, S. Campanato proposed a different approach. Quite often, the Morrey norm has an equivalent expression to this Campanato technique. He proposed to consider the difference when we consider Campanato spaces. Later his technique grew up to be another theory of normed spaces. Named after Campanato, the normed spaces are named (Morrey–)Campanato spaces. We call them Campanato spaces for short here and below. We will define (Morrey–)Campanato spaces in Section 5.3.1. In Section 5.3.1, we will also establish that Morrey spaces and Campanato spaces are equivalent. Campanato spaces are connected with Hölder–Zygmund spaces, which we establish in Section 5.3.2.

5.3.1 Morrey–Campanato spaces

In the definition of the Morrey norm we did not tolerate the case where $p < 0$. To consider such cases, we will need to consider function spaces modulo $\mathcal{P}_k(\mathbb{R}^n)$. We formulate our idea.

Definition 65 ($\mathcal{L}_k^{\lambda,q}(\mathbb{R}^n)$)**.** Let $\lambda \in \mathbb{R}$ and $1 \le q \le \infty$, and fix $k \in \mathbb{N}_0 \cup \{-1\}$. Then define the *Morrey–Campanato space* $\mathcal{L}_k^{\lambda,q}(\mathbb{R}^n)$ as the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{L}_k^{\lambda,q}} \equiv \sup_{(x,\rho)\in\mathbb{R}^{n+1}_+} \left\{ \inf_{P\in\mathcal{P}_k(\mathbb{R}^n)} \rho^\lambda |B(\rho)|^{-\frac{1}{q}} \|f-P\|_{L^q(B(x,\rho))} \right\} < \infty.$$

Before we investigate $\mathcal{L}_k^{\lambda,q}(\mathbb{R}^n)$, a helpful remark may be in order.

Remark 11. Let $\lambda \in \mathbb{R}$, $1 \le q \le \infty$, and $k \in \mathbb{N}_0 \cup \{-1\}$. The function space $\mathcal{L}_k^{\lambda,q}(\mathbb{R}^n)$ is a function space modulo $\mathcal{P}_k(\mathbb{R}^n)$. Hence, it is natural to write $[f]$ to denote the element in $\mathcal{L}_k^{\lambda,q}(\mathbb{R}^n)$. However, to simplify the notation, one often writes f instead of $[f]$.

Here we consider some concrete cases.

Example 98. Let $\lambda < 0$ and $1 \le q < \infty$. Then $\mathcal{L}^{q,\lambda}_{-1}(\mathbb{R}^n) = \{0\}$ by the Lebesgue differentiation theorem.

Example 99. Let $1 \le q < \infty$. Then $\mathcal{L}^{0,q}_0(\mathbb{R}^n)$ is nothing but $\mathrm{BMO}(\mathbb{R}^n)$ thanks to the John–Nirenberg inequality.

Example 100. Let $\lambda \in \mathbb{R}$ and $1 \le q < \infty$. Then the Morrey–Campanato space $\mathcal{L}^{\lambda,q}_0(\mathbb{R}^n)$ is the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for which the quantity

$$\|f\|_{\mathcal{L}^{\lambda,q}_0} \equiv \sup_{(x,\rho)\in\mathbb{R}^{n+1}_+} \left\{ \inf_{c\in\mathbb{C}} \rho^\lambda |B(\rho)|^{-\frac{1}{q}} \|f-c\|_{L^q(B(x,\rho))} \right\} < \infty.$$

In general thanks to Lemma 115, the infimum is attained: We have

$$\|f\|_{\mathcal{L}^{\lambda,q}_k} \equiv \sup_{(x,\rho)\in\mathbb{R}^{n+1}_+} \left\{ \min_{P\in\mathcal{P}_k(\mathbb{R}^n)} \rho^\lambda |B(\rho)|^{-\frac{1}{q}} \|f-P\|_{L^q(B(x,\rho))} \right\}$$

in the above. We will use the following notation for this minimum.

Definition 66. Let $\lambda \in \mathbb{R}$, $1 \le q < \infty$ and $k \in \mathbb{N}_0 \cup \{-1\}$. Also let $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$.

(1) Define $P_k(\cdot; x_0, \rho, f) \equiv \mathrm{argmin}_{P\in\mathcal{P}_k(\mathbb{R}^n)} \|f-P\|_{L^q(B(x_0;\rho))}$, the *best approximation of u up to order k in the ball $B(x_0,\rho)$*. Regard this polynomial as a function of the first variable. Other variables play the role of the parameters.

(2) Define $a_\alpha(x_0;\rho,f) \equiv \partial^\alpha P_k(x_0;x_0,\rho,f)$ for any $\alpha \in \mathbb{N}_0$ with $|\alpha| \le k$.

Thanks to Remark 3, we have the following norm equivalence $\mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$: $\|f\|_{\mathcal{L}^{\lambda,q}_k} \sim \sup_{(x,\rho)\in\mathbb{R}^{n+1}_+} \left\{ \rho^\lambda |B(\rho)|^{-\frac{1}{q}} \|f - P^k_{B(x,\rho)} f\|_{L^q(B(x,\rho))} \right\}$.

We investigate the property of $a_\alpha(\cdot;\rho,f)$ keeping in mind that $\mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$ is monotone in q.

Lemma 239. *Let $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}_0 \cup \{-1\}$. Also let $f \in \mathcal{L}^{\lambda,1}_k(\mathbb{R}^n)$. Then $|a_\alpha(x_0;2\rho,f) - a_\alpha(x_0;\rho,f)| \lesssim \rho^{-|\alpha|-\lambda} \|f\|_{\mathcal{L}^{\lambda,1}_k}$ for all $x_0 \in \mathbb{R}$ and $\rho > 0$.*

Proof Thanks to Lemma 115, we have

$$\begin{aligned}
&|a_\alpha(x_0;2\rho,f) - a_\alpha(x_0;\rho,f)| \\
&= |\partial^\alpha P_k(x_0;x_0,2\rho,f) - \partial^\alpha P_k(x_0;x_0,\rho,f)| \\
&\lesssim \rho^{-n-|\alpha|} \|P_k(\cdot;x_0,2\rho,f) - P_k(\cdot;x_0,\rho,f)\|_{L^1(B(x_0,\rho))}.
\end{aligned}$$

Meanwhile, by the definition of f, we have

$$\begin{aligned}
&\|P_k(\cdot;x_0,2\rho,f) - P_k(\cdot;x_0,\rho,f)\|_{L^1(B(x_0,\rho))} \\
&\le \|P_k(\cdot;x_0,2\rho,f) - u\|_{L^1(B(x_0,\rho))} + \|f - P_k(\cdot;x_0,\rho,f)\|_{L^1(B(x_0,\rho))} \\
&\lesssim \rho^{n-\lambda} \|f\|_{\mathcal{L}^{\lambda,1}_k}.
\end{aligned}$$

Combining these two estimates gives the desired result.

In particular, if $1 \le q < \infty$, then $|a_\alpha(x_0;2\rho,f) - a_\alpha(x_0;\rho,f)| \lesssim \rho^{-|\alpha|-\lambda}\|f\|_{\mathcal{L}^{\lambda,1}_k}$ for all $f \in \mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$.

We have the following analogy to Lemma 239:

Lemma 240. *Let $\lambda \in \mathbb{R}$, $1 \le q < \infty$ and $k \in \mathbb{N}_0 \cup \{-1\}$. Also let $f \in \mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$. Then $|a_\alpha(x_0;\rho,f) - a_\alpha(y_0;\rho,f)| \lesssim \rho^{-|\alpha|-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_k}$ for all $x_0, y_0 \in \mathbb{R}$ and $\rho > 0$ such that $|x_0 - y_0| < \rho$.*

Proof Using $B(x_0,\rho) \cup B(y_0,\rho) \subset B(x_0,2\rho)$, we argue as in Lemma 239.

Importantly, the number k does not affect the definition of $\mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$ as long as k is large enough.

Theorem 241. *Let $\lambda \in \mathbb{R}$, and let $1 \le q \le \infty$. Then for any $l \in \mathbb{Z} \cap (\max(-1,[-\lambda]),\infty)$, $\mathcal{L}^{\lambda,q}_{\max(-1,[-\lambda])}(\mathbb{R}^n)$ and $\mathcal{L}^{\lambda,q}_l(\mathbb{R}^n)$ are isomorphic.*

Proof It follows from the definition of the norm that the natural mapping $[f] \in \mathcal{L}^{\lambda,q}_{\max(-1,[-\lambda])}(\mathbb{R}^n) \mapsto [f] \in \mathcal{L}^{\lambda,q}_l(\mathbb{R}^n)$ is continuous.

To see that we can construct the inverse mapping, it suffices to prove that there exists $P = P^{l-1}_{B(x_0,\rho)}f \in \mathcal{P}_l(\mathbb{R}^n)$ such that

$$\rho^\lambda |B(\rho)|^{-\frac{1}{q}}\|f - P^{l-1}_{B(x_0,\rho)}\|_{L^q(B(x,\rho))} \lesssim \|f\|_{\mathcal{L}^{\lambda,q}_l}.$$

for any $[f] \in \mathcal{L}^{\lambda,q}_l(\mathbb{R}^n)$ and for any ball $B = B(x_0,\rho)$.

We start with constructing the constant a_α to this end. We choose the best approximation $P_l(x;x_0,\rho,f)$ of f up to order l in the ball $B(x_0,\rho)$. As long as $B(x_0,\rho) \supset B(y_0,\rho')$ and $\rho' \le \rho \le 2\rho'$, $|a_\alpha(x_0;\rho,f) - a_\alpha(y_0;\rho',f)| \lesssim \rho^{-|\alpha|-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_l}$ due to Lemmas 239 and 240. As a result, $a_\alpha \equiv \lim_{\rho\to\infty} a_\alpha(x_0;\rho,f)$ exists and is independent of x_0. Furthermore $|a_\alpha(x_0;\rho,f) - a_\alpha| \lesssim \rho^{-|\alpha|-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_l}$.

Let $B \equiv B(x_0,\rho)$ be a ball once again. For $x \in \mathbb{R}^n$, write $Q^l_B(x) \equiv P_l(x;x_0,\rho,f) - \sum_{|\alpha|=l}(a_\alpha(x_0;\rho,f) - a_\alpha)(x-x_0)^\alpha$ and $P^l_B(x) \equiv Q^l_B(x) - P(x)$, so that $f(x) - Q^l_B(x) = f(x) - P^l_B(x) + \sum_{|\alpha|=l}(a_\alpha(x_0;\rho,f) - a_\alpha)(x-x_0)^\alpha$. Thus, $m^{(q)}_B(f - Q^l_B) \lesssim m^{(q)}_B(f - P^l_B) + \sum_{|\alpha|=l}|a_\alpha(B) - a_\alpha|\rho^l \lesssim \rho^{-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_l}$. This means that we can take $P = Q^l_B$ in the above.

Theorem 241 with $\lambda > 0$ shows that Morrey–Campanato spaces and Morrey spaces coincide in the following sense:

Theorem 242. *Let $1 \le q \le p < \infty$.*

(1) *In the sense of continuous embedding, $\mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{\frac{n}{p},q}_k(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0 \cup \{-1\}$.*

(2) *Let* $f \in \mathcal{L}_0^{\frac{n}{p},q}(\mathbb{R}^n)$.

(*i*) *Let* $x_0 \in \mathbb{R}^n$. *Then the quantity* $u_\infty \equiv \lim_{R\to\infty} a_0(x_0;R,u) \in \mathbb{C}$ *exists, where* $a_0(x_0;R,u)$ *is defined in Definition* 66 *with* $\alpha = 0$ *and* $\rho = R$. *Furthemore* u_∞ *is independent of* x_0.

(*ii*) *The function* $u - u_\infty$ *is independent of the representative of* u. *That is, if* v *differs from* u *by an additive constant and* $v_\infty \equiv \lim_{R\to\infty} a_0(x_0,\rho;v)$, *then* $v - v_\infty = u - u_\infty$.

(*iii*) *We have* $\|u - u_\infty\|_{\mathcal{M}_q^p} \lesssim \|f\|_{\mathcal{L}_0^{\frac{n}{p},q}}$.

Proof

(1) This is trivial from the definition of the norms.

(2) This is nothing but Theorem 241 with $\lambda > 0$.

5.3.2 Morrey–Campanato spaces and Hölder–Zygmund spaces

The homogeneous Hölder–Zygmund space $\dot{\mathcal{C}}^\gamma(\mathbb{R}^n)$ (see in Section 1.6.2) and the Morrey–Campanato space $\mathcal{L}_k^{\lambda,q}(\mathbb{R}^n)$ defined in Section 5.3.1 are closely connected. We investigate the relation between Morrey–Campanato spaces and Hölder–Zygmund spaces.

Lemma 243. *Let* $\psi \in C^\infty(\mathbb{R}^n)$ *be a function in Example* 54. *Write* $\psi_\rho \equiv \rho^{-n}\psi(\rho^{-1}\cdot)$ *for* $\rho > 0$. *Let* $\lambda \in \mathbb{R}$, $1 \le q \le \infty$ *and* $k \in \mathbb{N}_0 \cup \{-1\}$, *and let* $f \in \mathcal{L}_k^{\lambda,q}(\mathbb{R}^n)$.

(1) *Let* $\rho, \rho' > 0$ *satisfy* $\rho' \le \rho \le 2\rho'$, *and let* $\lambda \in \mathbb{R}$. *Then we have* $\|\partial^\alpha(f * \psi_\rho) - \partial^\alpha(f * \psi_{\rho'})\|_{L^\infty} \lesssim \rho^{-\lambda-|\alpha|}\|f\|_{\mathcal{L}_k^{\lambda,q}}$.

(2) *Let* $\rho > 0$ *and* $\lambda < 0$. *Then whenever* $\alpha \in \mathbb{N}_0$ *satisfies* $|\alpha| < -\lambda$, $f \in C^{[-\lambda]}(\mathbb{R}^n)$ *and* $\|\partial^\alpha(f * \psi_\rho) - \partial^\alpha f\|_{L^\infty} \lesssim \rho^{-\lambda-|\alpha|}\|f\|_{\mathcal{L}_k^{\lambda,q}}$.

Proof

(1) Since

$$\begin{aligned}
&\|\partial^\alpha(f * \psi_\rho) - \partial^\alpha(f * \psi_{\rho'})\|_{L^\infty} \\
&= \|\partial^\alpha[(f - P_k(\cdot;x_0,\rho,f))] * \psi_\rho - \partial^\alpha[(f - P_k(\cdot;x_0,\rho,f))] * \psi_{2^{-n}}\|_{L^\infty} \\
&\le \|(f - P_k(\cdot;x_0,\rho,f)) * \partial^\alpha\psi_\rho\|_{L^\infty} + \|(f - P_k(\cdot;x_0,\rho,f)) * \partial^\alpha\psi_{\rho'}\|_{L^\infty},
\end{aligned}$$

we have $\|\partial^\alpha(f * \psi_\rho) - \partial^\alpha(f * \psi_{\rho'})\|_{L^\infty} \lesssim \rho^{-\lambda-|\alpha|}\|f\|_{\mathcal{L}_k^{\lambda,q}}$.

(2) Since $f - f * \psi_\rho = \sum_{L=1}^{\infty}(f * \psi_{2^{-L}\rho} - f * \psi_{2^{-L+1}\rho})$ converges in $C^{[-\lambda]}(\mathbb{R}^n)$, $f \in C^{[-\lambda]}(\mathbb{R}^n)$ and $f - f * \psi_\rho$ satisfy the desired estimate from (1).

We will show that any function $f \in \mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$ is represented by a continuous function and that the value of q is not important in the definition of $\mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$ if $\lambda < 0$.

Theorem 244. *Let $1 \le q \le \infty$, $\lambda < 0$ and $k \in \mathbb{Z} \cap [-1, -\lambda)$. Then any $f \in \mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$ is almost everywhere equal to a continuous function $g \in \mathrm{C}(\mathbb{R}^n)$ such that $\|g\|_{\mathcal{L}^{\lambda,\infty}_k} \lesssim \|f\|_{\mathcal{L}^{\lambda,q}_k}$.*

Proof Let $B \equiv B(x_0, r)$ be a ball. Also let $P^k_B f$ be the Gram–Schmidt polynomial of f of order k over the ball B. Take ψ to be a function defined in Example 54. Thanks to Lemma 243, f is almost everywhere equal to a continuous function $g \equiv \lim_{l\to\infty} f * \psi_{2^{-l}r} \in \mathrm{C}(\mathbb{R}^n)$. Meanwhile since we have $\|f * \psi_r - P^k_B f\|_{L^\infty} = \|(f - P^k_B f) * \psi_r\|_{L^\infty} \lesssim r^{-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_k}$ and $g - P^k_B f = f * \psi_r - P^k_B f + \sum_{l=1}^{\infty} f * \psi_{2^{-l}r} - f * \psi_{2^{-l+1}r}$. we obtain $\|g\|_{\mathcal{L}^{\lambda,\infty}_k} \lesssim \|f\|_{\mathcal{L}^{\lambda,q}_k}$.

We investigate the differential feature of Morrey–Campanato spaces keeping in mind that q does not affect the definition of the function spaces.

Theorem 245. *Let $\lambda < 0$, and let $K \in \mathbb{Z} \cap [0, -\lambda)$. Then $f \in \mathcal{L}^{\lambda,\infty}_{[-\lambda]}(\mathbb{R}^n)$ if and only if $f \in \mathrm{C}^K(\mathbb{R}^n)$ and $\partial^\alpha f \in \mathcal{L}^{\lambda+K,\infty}_{[-\lambda-K]}(\mathbb{R}^n)$ for all $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le K$.*

Proof We may suppose $K = 1$ and hence $\lambda < -1$; since Theorem 244 shows that $f \in \mathrm{C}(\mathbb{R}^n)$, the case of $K = 0$ is covered by Theorem 244. We may induct on K for the case $K \ge 2$. Let $f \in \mathcal{L}^{\lambda,q}_{[-\lambda]}(\mathbb{R}^n)$. From Lemma 243, $f \in \mathrm{C}^1(\mathbb{R}^n)$. Since $\partial_j f - \partial_j P_k(\cdot; x_0, \rho, f) = \partial_j f - \partial_j(f * \psi_\rho) + (f - P_k(\cdot; x_0, \rho, f)) * \partial_j \psi_\rho$, we see that $\|\partial_j f - \partial_j P_k(\cdot; x_0, \rho, f)\|_{L^\infty} \lesssim \rho^{-\lambda-1}\|f\|_{\mathcal{L}^{\lambda,q}_k}$. Thus $\partial^\alpha f \in \mathcal{L}^{\lambda+1,\infty}_{[-\lambda-1]}(\mathbb{R}^n)$ for all $\alpha \in {\mathbb{N}_0}^n$ with $|\alpha| \le 1$ if $f \in \mathcal{L}^{\lambda,\infty}_{[-\lambda]}(\mathbb{R}^n)$.

Suppose that $f \in C^1(\mathbb{R}^n)$ and that each component of the gradient of f belongs to $\mathcal{L}^{\lambda+1,\infty}_{[-\lambda-1]}(\mathbb{R}^n)$. Let $B \equiv B(x_0, \rho)$ be a ball. Then by the fundamental theorem of calculus $f(x) - f(x_0) = \sum_{j=1}^n (x_j - x_{j,0}) \int_0^1 \partial_j f(x_0 + t(x - x_0))\mathrm{d}t$. Setting $P(x) \equiv \sum_{j=1}^n (x_j - x_{j,0}) \int_0^1 P_{[-\lambda-1]}(\partial_j f(x_0 + t(\cdot - x_0)); x_0, \rho)(x)\mathrm{d}t$, we have

$$
\begin{aligned}
&\|f - f(x_0) - P\|_{L^\infty(B)} \\
&\le r \sum_{j=1}^n \int_0^1 \|\partial_j f(x_0 + t(\cdot - x_0)) - P_{[-\lambda]}(\partial_j f(x_0 + t(\cdot - x_0)); x_0, \rho)\|_{L^\infty(B)}\mathrm{d}t \\
&\le n r^{-\lambda} \sum_{j=1}^n \|\partial_j f\|_{\mathcal{L}^{\lambda+1,\infty}_{[-\lambda-1]}}.
\end{aligned}
$$

Thus, $f \in \mathcal{L}^{\lambda,q}_{[-\lambda]}(\mathbb{R}^n)$.

As we mentioned, Morrey–Campanato spaces and Hölder–Zygmund spaces are closely connected. More precisely, we have the following norm equivalence.

Theorem 246. *Let $\lambda > 0$ and $1 \le q \le \infty$. Then $\dot{\mathcal{C}}^\lambda(\mathbb{R}^n) \sim \mathcal{L}^{-\lambda,q}_{[\lambda]}(\mathbb{R}^n)$.*

Proof We may assume $0 < \lambda < 2$ in view of Theorem 245 and the structure of the class $\{\dot{\mathcal{C}}^\lambda(\mathbb{R}^n)\}_{\lambda>0}$.

Let $f \in \mathcal{L}^{\lambda,q}_1(\mathbb{R}^n)$. We know that f is continuous thanks to Lemmas 239 and 240. Let us obtain the norm estimate. Let $x, y \in \mathbb{R}^n$ be fixed. Suppose $x \neq y$. Choose r so that $r < |x-y|$. Then we have

$$\begin{aligned}
&P^1_{B(x+y,r)}f(x+y) - 2P^1_{B(x,r)}f(x) + P^1_{B(x-y,r)}f(x-y)\\
&= P^1_{B(x+y,r)}f(x+y) - P^1_{B(x,|x-y|)}f(x+y) - 2(P^1_{B(x,r)}f(x) - P^1_{B(x,|x-y|)}f(x))\\
&\quad + P^1_{B(x-y,r)}f(x-y) - P^1_{B(x,|x-y|)}f(x-y).
\end{aligned}$$

Since $\lambda < 0$, thanks to Lemmas 239 and 240, we have

$$\begin{aligned}
|P^1_{B(x+y,r)}f(x+y) - P^1_{B(x,|x-y|)}f(x+y)| &\lesssim |x-y|^{-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_k},\\
|P^1_{B(x,r)}f(x) - P^1_{B(x,|x-y|)}f(x)| &\lesssim |x-y|^{-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_k},\\
|P^1_{B(x-y,r)}f(x-y) - P^1_{B(x,|x-y|)}f(x-y)| &\lesssim |x-y|^{-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_k}.
\end{aligned}$$

So, $|P^1_{B(x+y,r)}f(x+y) - 2P^1_{B(x,r)}f(x) + P^1_{B(x-y,r)}f(x-y)| \lesssim |x-y|^{-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_k}$. Letting $r \downarrow 0$, we obtain $|f(x+y) - 2f(x) + f(x-y)| \lesssim |x-y|^{-\lambda}\|f\|_{\mathcal{L}^{\lambda,q}_k}$ thanks to Lemma 143. Thus, $f \in \dot{\mathcal{C}}^{-\lambda}(\mathbb{R}^n)$.

Conversely, let $f \in \dot{\mathcal{C}}^{-\lambda}(\mathbb{R}^n)$. Then choose ψ^j and φ^j for $j \in \mathbb{Z}$ as in (1.4) and (1.5), respectively. Fix a cube $Q = Q(z,r)$ for the purpose of estimating $\inf_{P\in\mathcal{P}_k(\mathbb{R}^n)} \rho^\lambda |B(r)|^{-\frac{1}{q}} \|f - P\|_{L^q(B(z,r))}$. If we replace r by a slightly larger number, we may assume that $\log_2 r \in \mathbb{Z}$. Set $r \equiv 2^{-j}$. Write $f^j \equiv \psi^j * f$. Then thanks to Lemma 76, we have $\|f^{j+1} - f^j\|_{L^\infty} \lesssim 2^{j\lambda}\|f\|_{\dot{\mathcal{C}}^{-\lambda}}$. Since f is continuous, there is a decomposition $f - f^j = \sum_{k=j}^{\infty}(f^{k+1} - f^k)$ in $L^\infty(\mathbb{R}^n)$. So $\|f - f^j\|_{L^\infty} \lesssim 2^{j\lambda}\|f\|_{\dot{\mathcal{C}}^{-\lambda}}$. Meanwhile, if P denotes the Taylor polynomial of f^j up to order 1 around z, then we have

$$\left\|f^j - P\right\|_{L^\infty(B(z,2^{-j}))} \lesssim \sup_{|\alpha|=2} 2^{-2j}\|\partial^\alpha f^j\|_{L^\infty(B(z,2^{-j}))} \lesssim 2^{\lambda j}\|f\|_{\dot{\mathcal{C}}^{-\lambda}}$$

thanks to Lemma 115. We see that $f \in \mathcal{L}^{\lambda,q}_1(\mathbb{R}^n)$ putting these observations together.

5.3.3 Exercises

Exercise 81. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Show that $|m_Q(f) - m_{kQ}(f)| \lesssim \log(k+2)\|f\|_*$ for any $k \ge 1$ and any cube Q.

Exercise 82. Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$. If $f \in \mathcal{M}^p_1(\mathbb{R}^n)$, then show that (4.2) is satisfied.

Exercise 83. [279, Theorem 1] Let $\lambda \le 0$ and $k \in \mathbb{N}_0$, and let $f \in \mathcal{L}^{\lambda,1}_k(\mathbb{R}^n) \setminus \{0\}$. For all cubes Q, write $P^k_Q f$ for the Gram–Schmidt polynomial of order k for Q. Show that

$$|\{x \in Q : |Q|^\lambda |f(x) - P^k_Q f(x)| > \lambda\}| \lesssim \exp\left(-\frac{\lambda}{\|f\|_{\mathcal{L}^{\lambda,1}_k}}\right)|Q|$$

using Proposition 117.

5.4 Notes

Section 5.1

General remarks and textbooks in Section 5.1

See the textbooks and lecture notes [29, Chapter 5], [104, Chapter 6], [146, Chapter II, §3], [147], [157, Chapter 3], [223, Chapter 3], [231], [293], [416, Chapter IV], [382, §3.3] and [432] for the theory of BMO.

Section 5.1.1

Fefferman and Stein defined the space BMO in [125]. Theorem 223 is essentially due to Fefferman and Stein; [125, (1.2)]. See [404] for the definition of $\mathrm{BMO}^+(\mathbb{R}^n)$.

See [363] for the closure of $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$.

Section 5.1.2

See the original paper [221] by John and Nirenberg, where we can find a variant for functions defined on cubes.

Section 5.2

General remarks and textbooks in Section 5.2

See the textbooks [157, §3.5] and [293].

Section 5.2.1

The boundedness of commutators $[b, T]$ appearing in this section is due to Coifman, Meyer and Weiss [84]. Chanillo considered the boundedness property

of $[b, I_\alpha]$ in Chanillo [66]. See [84, Theorem 1] for Theorem 231. See [401, Theorem 2] for Theorem 235 including Lemmas 232 and 234.

Chianrenza, Frasca and Longo pointed out that the commutators in Theorem 236 arise naturally; see [76, Theorem 2.5] for Theorem 236. We refer to [464] for the application of Theorem 236. See [161] for its generalization.

We can generalize the above commutators, given by (5.6), as follows: Let $\vec{b} = (b_1, b_2, \dots, b_m)$. Define the multicommutator $T_{\vec{b}}$

$$T_{\vec{b}} f(x) \equiv \int_{\mathbb{R}^n} K(x,y) f(y) \prod_{j=1}^{m} (b_j(x) - b_j(y)) \mathrm{d}y,$$

where K is the Calderón–Zygmund kernel satisfying (4.1), (4.2) and (4.3). Here, f is an appropriate function that needs to be specified after a calculation.

This operator dates back to the paper by Coifman, Rochberg and Weiss in 1976 [84].

Section 5.2.2

See [36] for example.

Section 5.3

General remarks and textbooks in Section 5.3

The earliest overview of Morrey–Campanato spaces seems to be [353]. The next one seems [423]. We refer to [258, Chapter 5] for Morrey–Campanato spaces. We also refer to the survey [359, §4]. See [108] for the characterization of Morrey spaces and Morrey–Campanato spaces in terms of the heat kernel.

Section 5.3.1

A series of results in this section are basically based on the works by Campanato [57, 58, 59, 60]. As we saw in Theorem 241, Janson, Taibleson and Weiss pointed out that the order of the polynomials does not affect so much the definition of Morrey–Campanato spaces [220, Theorem 1]. Morrey–Campanato spaces are known to be equivalent to Morrey spaces in many cases. See [427, 460, 461] for weighted characterizations. Several characterizations of weighted Morrey–Campanato spaces are obtained by Tang in [427].

The Morrey–Campanato space $\mathcal{L}^{\lambda,q}_k(\mathbb{R}^n)$ and the Lipschitz space $\mathrm{Lip}^\theta(\mathbb{R}^n)$ are equivalent. This equivalence goes back to the works by Campanato and Meyer; see [58, Theorema, p. 183] and [312, Theorem, p. 718], where both authors showed that these norms are equivalent to the $\mathrm{Lip}^\theta(\mathbb{R}^n)$ norm. See [353, p. 72] for an account of these facts. See also [329, Theorem 3.1].

Campanato and Meyers independently showed that Morrey–Campanato spaces and Morrey spaces coincide in many cases as we saw in Theorem 242 see [57], [59, Theorem 6.II] and [313]. Duong, Xiao and Yan expanded this idea by the use of the heat kernel in [109].

Section 5.3.2

Jonsson, Sjögen and Wallin proved Theorem 244 [222]. Janson, Taibleson and Weiss pointed out that Morrey–Campanato spaces are isomorphic to Hölder–Zygmund spaces; see [220, Lemma (3.4)], [220, Theorem 2] and [220, Theorem 3] for Lemma 243, Theorems 244 and 245, respectively. Grevholm proved Theorem 246 in [159, Theorem 3.1]. Greenwald also gave a different proof of Theorem 246 [160]. See the booklet by DeVore and Sharpley [100] and the lecture note [220] by Janson, Taibleson and Weiss for Theorem 246. We now aim to consider function spaces related to Morrey–Campanato spaces. We content ourselves with definitions. We consider CMO and CBMO spaces. Let $d \in \mathbb{N}_0 \cup \{-1\}$, $1 \le p \le \infty$ and $-\frac{n}{p} \le \lambda < d$, and define

$$\|f\|_{\mathrm{CMO}^{p,\lambda,d}} \equiv \sup_{r \ge 1} \frac{1}{r^\lambda} \left(\frac{1}{r^n} \int_{Q(r)} |f(x) - P_Q^d f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}$$

$$\|f\|_{\mathrm{CBMO}^{p,\lambda,d}} \equiv \sup_{r > 0} \frac{1}{r^\lambda} \left(\frac{1}{r^n} \int_{Q(r)} |f(x) - P_Q^d f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

Example 101. Chen and Lau [70] and García-Cuerva [141] introduced the central mean oscillation space $\mathrm{CMO}^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathrm{CMO}^p} \equiv \sup_{r \ge 1} \left(\frac{1}{|Q(r)|} \int_{Q(r)} |f(x) - f_{Q(r)}|^p \mathrm{d}x \right)^{\frac{1}{p}},$$

So, $\mathrm{CMO}^p(\mathbb{R}^n) = \mathrm{CMO}^{p,0,0}(\mathbb{R}^n)$.

Example 102. Lu and Yang [291, 292] introduced the central bounded mean oscillation space $\mathrm{CBMO}^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathrm{CBMO}^p} \equiv \sup_{r > 0} \left(\frac{1}{|Q(r)|} \int_{Q(r)} |f(x) - f_{Q(r)}|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

So, $\mathrm{CBMO}^p(\mathbb{R}^n) = \mathrm{CBMO}^{p,0,0}(\mathbb{R}^n)$. Let $\mathcal{C}$ be the space of all constant functions and $\mathcal{X}/\mathcal{C}$ mean the quotient space of $\mathcal{X}$ by $\mathcal{C}$. Then $\mathrm{CMO}^p(\mathbb{R}^n)$ and $\mathrm{CBMO}^p(\mathbb{R}^n)$ are expressed by $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$, respectively, with $E = L^p(\mathbb{R}^n)/\mathcal{C}$, $\|f\|_{E(Q(r))} = \|f - f_{Q(r)}\|_{L^p(Q(r))}$ and $\sigma = \frac{n}{p}$.

Fu obtained λ-central BMO estimates for commutators of N-dimensional Hardy operators [131, Theorem 1.1].

Chapter 6

General metric measure spaces

Morrey spaces can describe the local properties of functions over $\mathbb{R}^n$. This applies to a general metric measure space. We remark that a metric measure space means a couple (X, d, μ) of a metric space (X, d) and a Borel measure μ. We do not recall the notion of metric spaces; see [241, §4] for metric measure spaces. Recall that a Borel measure μ is defined for the σ-field generated by open sets.

This chapter considers two typical cases. The first is where the Lebesgue measure is replaced by a general Radon measure on $\mathbb{R}^n$. Recall that a Radon measure in a metric space (X, d) is a Borel measure such that $\mu(K) < \infty$ for any compact set K. Here, as examples the fractal sets generated by systems of iteration functions are envisioned. Although fractal theory is not discussed due to its vastness, the ternary Cantor set is recalled.

The second is where the underlying space differs from Euclidean space. Hence, any (X, d) will do as long as (X, d) is a metric space. Graphs are typical examples. An (*unoriented*) *graph* is a pair $G = (V, E)$ comprising a set V of vertices together with a set E of edges, which are 2-element subsets of V. The (graph) distance between x and y is the minimal integer k such that there exist $k - 1$ elements in between x and y. Since graphs are not the primary purpose of this book, this chapter does not go into detail. But let us mention that we use graphs in Examples 109 and 110. One of our motivations is an extension to the general metric measure setting. Metric measure spaces have already been discussed in this book: A previous chapter considered the weighted measure $w(x)\mathrm{d}x$. This chapter is interested in cases where the norm bound is independent of the weights. Such an estimate can be obtained if the weighted case is considered. This aspect is examined later in this book. Section 6.1 establishes the theory of Lebesgue spaces in Euclidean space with general Radon measures, while Section 6.2 focuses on the theory of Lebesgue spaces in metric measure spaces.

6.1 Maximal operators on Euclidean spaces with general Radon measures

We suppose that we have a Radon measure μ on $\mathbb{R}^n$. As we have seen, the Hardy–Littlewood maximal operator M on Euclidean space $\mathbb{R}^n$ plays an

important role in analysis. When we work on a metric measure space, we need to generalize the definition of M. A natural candidate of this definition is as follows:

$$M'f(x) \equiv \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \mathrm{d}\mu(y) \quad (x \in \mathbb{R}^n)$$

and

$$Mf(x) \equiv \sup_{z\in\mathbb{R}^n, r>0} \frac{\chi_{B(z,r)}(x)}{\mu(B(z,r))} \int_{B(z,r)} |f(y)| \mathrm{d}\mu(y) \quad (x \in \mathbb{R}^n).$$

Since we have no information on $\mu(B(x,r))$ and $\mu(B(z,r))$, these two operators must be treated differently. However, if we have enough information on the relation between $\mu(B(x,r))$ and $\mu(B(x,2r))$ for $x \in \mathbb{R}^n$ and $r > 0$, then we can identify M and M' as follows: Assume that there exists a constant $C > 0$ such that the *doubling condition* $\mu(B(x,r)) \le \mu(B(x,2r)) \le C\mu(B(x,r))$ holds for all $x \in \mathbb{R}^n$ and $r > 0$. Then $M'f(x) \le Mf(x) \le CM'f(x)$, $x \in \mathbb{R}^n$ for any $f \in L^0(\mu)$. In fact, since

$$M'f(x) = \sup_{z=x, r>0} \frac{1}{\mu(B(z,r))} \int_{B(z,r)} |f(y)| \mathrm{d}\mu(y),$$

we have $M'f(x) \le Mf(x)$. Meanwhile, since $B(z,r) \subset B(x,2r)$ for all $z \in B(x,r)$,

$$\begin{aligned} Mf(x) &= \sup_{z\in\mathbb{R}^n, r>0} \frac{\chi_{B(z,r)}(x)}{\mu(B(z,r))} \int_{B(z,r)} |f(y)| \mathrm{d}\mu(y) \\ &\le \sup_{z\in\mathbb{R}^n, r>0} \frac{\chi_{B(z,r)}(x)}{\mu(B(z,r))} \int_{B(x,2r)} |f(y)| \mathrm{d}\mu(y). \end{aligned}$$

Thus, thanks to the doubling condition, we have

$$Mf(x) \le C \sup_{r>0} \frac{1}{\mu(B(x,2r))} \int_{B(x,2r)} |f(y)| \mathrm{d}\mu(y).$$

In addition to the identification of M and M', we can prove the boundedness of these operators. For all $\lambda > 0$ and $f \in L^1(\mu)$, we will show that

$$\mu\{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\} \le \frac{C'}{\lambda} \int_{\mathbb{R}^n} |f(x)| \mathrm{d}\mu(x).$$

See Theorem 267. Consequently, if the measure is doubling, then we can develop the theory of the maximal operator. A metric measure space is called a space of homogeneous type if its measure is doubling.

However, there are many important measures which are not doubling. Here are examples of non-doubling measures.

Example 103. Let $\nu \equiv (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{R}^n$ be a fixed vector. Denote by $\langle \cdot, \cdot \rangle$ the inner product. If we let $\mu(E) \equiv \int_E e^{2\langle x, \nu \rangle} \mathrm{d}x$ for a Lebesgue measurable set E, then we obtain a non-doubling measure μ. This measure seems to be artificial but actually this measure naturally arises. In fact, if we consider $\Delta_\nu \equiv \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial {x_j}^2} + 2\nu_j \frac{\partial}{\partial x_j} \right)$, then

$$\int_{\mathbb{R}^n} f(x) \Delta_\nu g(x) \mathrm{d}\mu(x) = -\langle \nabla f, \nabla g \rangle_{L^2(\mu)} \tag{6.1}$$

for all $f, g \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$. See Exercise 84.

Thus, we are interested in the theory of metric spaces equipped with measures which are not always non-doubling. To handle general Radon measures, we need a covering lemma, which will be obtained in Section 6.1.1. Using this covering lemma, we investigate the boundedness property of these maximal operators in Section 6.1.2. As an application of the boundedness property, we establish a differentiation theorem in Section 6.1.3. Section 6.1.4 further develops what we did in Sections 6.1.1 and 6.1.2. Instead of using balls or cubes, we concentrate on the dyadic cubes. At first glance this seems to be a mere generalization of what we have been doing. However, Chapter 7 will convincingly explain why such a generalization is useful. We take up an example of metric measure spaces in Section 6.1.5.

6.1.1 Covering lemmas on Euclidean spaces

Unlike the general setting, in the Euclidean case, we can take advantage of the geometric structure of the space. We need a substitute of Besicovitch's covering lemma (Theorem 249). We start with the following sharp covering lemma.

Theorem 247. *For all $k > 1$ there exists an integer $N = N_k$, depending only on the dimension and k, that satisfies the following:*

Let $\{B(x_\lambda, r_\lambda)\}_{\lambda \in L}$ be a family of balls in Euclidean space endowed with a standard distance. Suppose that $\sup_{\lambda \in L} r_\lambda < \infty$.

Then we can take disjoint subfamilies

$$\{B(x_\rho, r_\rho)\}_{\rho \in L^1}, \{B(x_\rho, r_\rho)\}_{\rho \in L^2}, \dots, \{B(x_\rho, r_\rho)\}_{\rho \in L^N}$$

such that $\bigcup_{\lambda \in L} B(x_\lambda, r_\lambda) \subset \bigcup_{j=1,2,\dots,N} \bigcup_{\rho \in L^j} B(x_\rho, k r_\rho)$.

We aim to prove Theorem 247 via an auxiliary step. If $k \geq 5$, we set $N_k \equiv 1$ and we can use Theorem 133, the $5r$-covering lemma to prove Theorem 247. Hence, for the proof of Theorem 247, we may assume that $k \leq 5$. This may be viewed also as a substitute of Vitali's covering lemma. To prove Theorem 247, we prove it by posing another assumption on centers of balls.

Lemma 248. *Let $k > 1$, and let $\{B(x_\lambda, r_\lambda)\}_{\lambda \in L}$ be a family of balls and assume that $\sup_{\lambda \in L} r_\lambda \leq \sqrt{k} \inf_{\lambda \in L} r_\lambda < \infty$. Then we can take disjoint subfamilies as in Theorem 247.*

Proof As we mentioned, we may suppose that $k \leq 5$; otherwise we can use the classical $5r$-covering lemma (Theorem 133).

First of all, a scaling allows us to assume $\sup_{\lambda \in L} r_\lambda = 1$. (We are working on Euclidean space. Hence, we are able to multiply the scalar.)

In this part we divide the family of balls. More precisely we proceed as follows: Let $\mathcal{Q}_0$ be a family of dyadic cubes of side length 1. Here, we are now considering cubes of the form $\mathrm{Q} = \prod_{j=1}^{n} [m_j, m_j+1)$, where the m_j's are integers. We abbreviate the dyadic cubes in $\mathcal{Q}_0$ to "cubes" for short. Let $\mathrm{Q}^0 \equiv [0,1)^n$ be a unit cube. We divide the cubes into subfamilies: If $\overrightarrow{m} = (m_1, m_2, \ldots, m_n)$ is an element of ${\mathbb{N}_0}^n$, we set

$$\mathcal{Q}_{\overrightarrow{m}} \equiv \{\mathrm{Q} \in \mathcal{Q}_0 \,:\, \mathrm{Q} - (\overrightarrow{p} + \overrightarrow{m}) = \mathrm{Q}^0 \text{ for some } \overrightarrow{p} \in (4\mathbb{Z})^n\},$$

where $4\mathbb{Z} \equiv \{4l \,:\, l \in \mathbb{Z}\}$. Note that the cubes in $\mathcal{Q}_{\overrightarrow{m}}$ satisfy the following property: Suppose that Q and Q' both belong to $\mathcal{Q}_{\overrightarrow{m}}$ and that Q and Q' are different, then the distance between the two cubes is larger than 3. Hence, if the center of the ball B is in Q and the center of the ball B' is in Q', then B and B' are disjoint.

Next we define

$$L^{\overrightarrow{m}} \equiv \bigcup_{Q \in \mathcal{Q}_{\overrightarrow{m}}} \{\lambda \in L \,:\, x_\lambda \in Q\}.$$

Taking into account the preceding paragraphs we may assume that all centers of the balls are in Q^0. In fact once this is proven, by the last paragraph we can take the balls satisfying the property of this lemma from $\mathcal{Q}_{\overrightarrow{m}}$ for any $\overrightarrow{m} \in {\mathbb{N}_0}^n$. For any $\overrightarrow{m} \in {\mathbb{N}_0}^n$, we obtain families $\mathcal{B}^{(1)}_{\overrightarrow{m}}, \ldots, \mathcal{B}^{(N_k)}_{\overrightarrow{m}}$. Translation shows the number N_k is not dependent on $\overrightarrow{m}$. Hence, our desired family is $\{\mathcal{B}^{(j)}_{\overrightarrow{m}}\}_{\overrightarrow{m} \in {\mathbb{N}_0}^n, j \leq N_k}$.

Hence, in what follows let us assume that all centers of the balls are in Q^0 and that $\sup_{\lambda \in L} r_\lambda = 1$ by normalization.

First take a ball $B(x_{\lambda_1}, r_{\lambda_1})$ arbitrarily from the family $\{B(x_\lambda, r_\lambda)\}_{\lambda \in L}$. The assumption $k^{-\frac{1}{2}} \leq \inf_{\lambda \in L} r_\lambda$ ensures that the radius of each ball is between $k^{-\frac{1}{2}}$ and 1. Thus, the ball $B(x_{\lambda_1}, kr_{\lambda_1})$ contains all balls $B(x_\lambda, r_\lambda)$ such that $d(x_\lambda, x_{\lambda_1})$ is less than $\sqrt{k} - 1$.

Next take a ball $B(x_{\lambda_2}, r_{\lambda_2})$ such that $d(x_{\lambda_2}, x_{\lambda_1}) \geq \sqrt{k} - 1$. We may choose it arbitrarily as long as this condition is satisfied. As in the proceeding paragraph, the ball $B(x_{\lambda_2}, kr_{\lambda_2})$ contains all balls $B(x_\lambda, r_\lambda)$ such that $d(x_\lambda, x_{\lambda_2})$ is less than $\sqrt{k} - 1$.

In this way we repeatedly take a ball $B(x_{\lambda_{j'}}, r_{\lambda_{j'}})$ such that $d(x_{\lambda_{j'}}, x_{\lambda_j}) \geq \sqrt{k}-1$ for all $j = 1, 2, \ldots, j'-1$. This procedure will be stopped at the J_0th step where we obtain $\bigcup_{\lambda\in L} B(x_\lambda, r_\lambda) \subset \bigcup_{j'=1}^{J_0} B(x_{\lambda_{j'}}, kr_{\lambda_{j'}})$.

In fact this procedure stops in finite times: Precisely speaking, we claim that J_0 appearing in the last part is bounded by the constant that depends only on $k > 1$ and n. Since all radii of the balls are at most 1, all balls are contained in $[-1, 2]^n$. By the construction of $\{x_{\lambda_j}\}$, we have $d(x_{\lambda_j}, x_{\lambda_{j'}}) \geq \sqrt{k}-1$ for all $j < j' \leq J_0$. Thus, we have J_0 disjoint balls whose radii are more than $\frac{\sqrt{k}-1}{2}$. Precisely speaking, $\left\{ B\left(x_j, \frac{\sqrt{k}-1}{2}\right)\right\}_{j=1,2,\ldots,J_0}$ is disjoint. Note that $B\left(x_j, \frac{\sqrt{k}-1}{2}\right)$ is contained in $[-1, 2]^n$ for all $j = 1, 2, \ldots, J_0$. Hence, we have $J_0 \left(\frac{\sqrt{k}-1}{2}\right)^n |B(1)| \leq 3^n$. Thus, J_0 is bounded by the quantity which depends only on $k > 1$ and n. We denote this bound by N; $J_0 \leq N$. If J_0 is less than N, we formally define $L^j \equiv \emptyset$ for $j > J_0$. Thus, we obtain the desired families.

Next we prove Theorem 247. That is, we want to eliminate the assumption

$$\sup_{\lambda\in L} r_\lambda \leq \sqrt{k} \inf_{\lambda\in L} r_\lambda < \infty.$$

We use an abbreviation; $k\,B$ stands for $B(x, k\,r)$ when $B = B(x, r)$ and $k > 0$.

Proof of Theorem 247 Again we may assume that $\sup_{\lambda\in L} r_\lambda = 1$. First we take the subfamilies $\mathcal{B}_{j,p}$ inductively as follows (j runs through all positive integers and p through $[1, N]$, where N is a number obtained in the Lemma 248): First we define $\mathcal{X}_1$ by

$$\mathcal{X}_1 \equiv \left\{ B(x_\lambda, r_\lambda) \,:\, \lambda \in L, \quad r_\lambda > k^{-\frac{1}{2}} \right\}.$$

Let $\mathcal{B}_{1,p}$ be families obtained from $\mathcal{X}_1$, using the Lemma 248. Suppose we have obtained the families of the balls $\mathcal{B}_{l,p}$ with $l = 1, 2, \ldots, j, p = 1, 2, \ldots, N$ and that X_l with $l = 1, 2, \ldots, j$ are defined as the subsets of $\{B(x_\lambda, r_\lambda)\}_{\lambda\in L}$. Then we define

$$X_{j+1} \tag{6.2}$$
$$\equiv \left\{ B = B(x_\lambda, r_\lambda) \,:\, k^{-\frac{j+1}{2}} < r_\lambda \leq k^{-\frac{j}{2}}, B \subset \bigcup_{1\leq p\leq N, 1\leq l\leq j} \bigcup_{B'\in\mathcal{B}_{l,p}} kB' \right\}.$$

We apply the Lemma 248 to this subset to obtain $\mathcal{B}_{j+1,p}$ with $p = 1, 2, \ldots, N$, which enjoy the following properties: $\{\mathcal{B}_{j+1,p}\}_{1\leq p\leq N}$ are disjoint subfamilies

and

$$\bigcup_{B\in X_{j+1}} B \subset \bigcup_{p=1,2,\dots,N} \bigcup_{B\in\mathcal{B}_{j+1,p}} k\,B.$$

By the definition of X_j we have;

(1) whenever $B(x_\lambda, r_\lambda) \in \mathcal{X}_j$, $r_\lambda \le \dfrac{1}{\sqrt{k}^{j-1}}$,

(2) $B(x_\lambda, r_\lambda)$ is not contained in $\displaystyle\bigcup_{l=1}^{j-1}\bigcup_{p=1}^{N}\bigcup_{B\in\mathcal{B}_{l,p}} kB$.

Next we claim that there is an integer N', which depends only on k, that satisfies the following: If $|j-l| > N'$, then $B' \in \bigcup_p \mathcal{B}_{j,p}$, and if $B \in \bigcup_p \mathcal{B}_{l,p}$, then $B \cap B' = \emptyset$.

In fact suppose that $B \cap B' \ni x$ and $l > j$. Then by the property noted above, there exists $y \in B \setminus kB'$. Let c be the center of B'. If E is a subset of $\mathbb{R}^n$, $\operatorname{diam}(E)$ denotes the diameter of E. Under this notation and setting we have

$$d(x,y) \le \operatorname{diam}(B), \quad d(c,y) \ge \frac{k}{2}\operatorname{diam}(B'), \quad \text{and} \quad d(c,x) \le \frac{1}{2}\operatorname{diam}(B').$$

Thus $\dfrac{(k-1)}{2}\operatorname{diam}(B') < \operatorname{diam}(B)$. By the construction of X_j, we have

$$\operatorname{diam}(B) \le \frac{2}{\sqrt{k}^{l-1}}, \quad \operatorname{diam}(B') \ge \frac{2}{\sqrt{k}^{j}},$$

hence we have $\dfrac{2}{\sqrt{k}^{l-1}} \ge \dfrac{(k-1)}{\sqrt{k}^{j}}$. Since $k > 1$, this is possible only if the difference of j and l is small. That is, $\log \frac{2\sqrt{k}}{k-1} > (l-j)\log\sqrt{k}$, implying that $N = \left[1 + k^{-\frac{1}{2}} \log \frac{2\sqrt{k}}{k-1}\right]$ is enough.

Put $\mathcal{G}_{j,p} \equiv \displaystyle\bigcup_{i\,:\,i\equiv j \pmod{N'}} \mathcal{B}_{i,p}$. Then $\{\mathcal{G}_{j,p}\}_{j\le N', p\le N}$ does satisfy all demands of the theorem thanks to the property of N'.

We will need a covering technique called Besicovitch's covering lemma.

Theorem 249 (Besicovitch)**.** *Let $\mathcal{G} \equiv \{B_\lambda\}_{\lambda\in\Lambda}$ be a family of balls. Suppose the diameters of balls $\{B_\lambda\}_{\lambda\in\Lambda}$ are bounded. $R \equiv \sup_{\lambda\in\Lambda} r(B_\lambda) < \infty$. Then there exists an integer N depending only on the dimension that enjoys the following property:*

We can pick N disjoint subfamilies $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_N$ from $\mathcal{G}$ so that, any element in $A \equiv \{c(B_\lambda)\}_{\lambda\in\Lambda}$, the set of centers of balls $\{B_\lambda\}_{\lambda\in\Lambda}$, belongs to a ball in some $\mathcal{G}_j$.

Proof We may assume that A is bounded. We use the $5r$-covering lemma (see Theorem 132) for the family

$$\Lambda_0^* \equiv \left\{ \lambda \,:\, \lambda \in \Lambda,\, r(B_\lambda) > \frac{9R}{10} \right\}.$$

Then we obtain a countable family of balls $\{B_\lambda\}_{\lambda\in\Lambda_0}$ satisfying two conditions below:

(1) $\displaystyle\bigcup_{\lambda\in\Lambda_0^*} \frac{1}{5}B_\lambda \subset \bigcup_{\lambda\in\Lambda_0} B_\lambda,$

(2) $\left\{\dfrac{1}{5}B_\lambda\right\}_{\lambda\in\Lambda_0}$ is disjoint.

Since we assume A is bounded, $\sharp\Lambda_0$ is finite. Enumerate $\{B_\lambda\}_{\lambda\in\Lambda_0}$: Then we obtain a subfamily $B_1, B_2, \ldots, B_{N(1)}$ of $\{B_\lambda\}_{\lambda\in\Lambda}$ such that $\displaystyle\bigcup_{\lambda\in\Lambda_0^*} \frac{1}{5}B_\lambda \subset \bigcup_{j=1}^{N(1)} B_\lambda$ and that $\left\{\dfrac{1}{5}B_j\right\}_{j=1}^{N(1)}$ is disjoint. This is the first generation of the balls. Let $A_1 \equiv A \setminus \displaystyle\bigcup_{j=1}^{N(1)} B_\lambda$ and $\Lambda_1^{**} \equiv \{\lambda \in \Lambda \,:\, c(B_\lambda) \in A_1\}$.

Observe that $\displaystyle\sup_{\lambda\in\Lambda_1^{**}} r(B_\lambda) \le \frac{9R}{10}$. Therefore, we are in the position of applying the above observation to $\{B_\lambda\}_{\Lambda_1^{**}}$. Consequently, we can find $B_{N(1)+1}, B_{N(1)+1}, \ldots, B_{N(2)}$ from $\{B_\lambda\}_{\Lambda_1^{**}}$ with the following properties.

(1) If $\lambda \in \Lambda$ satisfy $r(B_\lambda) \ge \dfrac{81R}{100}$, then there exists $\rho \in \Lambda_0 \cup \Lambda_1$ so that $c(B_\lambda) \in B_\rho$,

(2) The family $\left\{\dfrac{1}{5}B_{N(1)+1}, \dfrac{1}{5}B_{N(1)+1}, \ldots, \dfrac{1}{5}B_{N(2)}\right\}$ is disjoint.

Note that $\dfrac{1}{5}B_j$ and $\dfrac{1}{5}B_k$ are disjoint, if $j \in \Lambda_0$ and $k \in \Lambda_1$. Indeed, $c(B_k)$, the center of B_k, does not belong to B_j and $\dfrac{81}{100}r(B_j) \le r(B_k) \le r(B_j)$. This is the second generation of the balls.

Repeating this procedure we can find a sequence of balls $\{B_j\}_{j\in J}$ from $\{B_\lambda\}_{\lambda\in\Lambda}$ so that it satisfies the following properties.

(1) $J = \{1, 2, \ldots, j_0\}$ or $J = \mathbb{N}$,

(2) $A \subset \displaystyle\bigcup_{j\in J} B_j,$

(3) if $J = \mathbb{N}$, then $r(B_j) \downarrow 0$ as $j \to \infty$,

(4) $\left\{\frac{1}{5}B_j\right\}_{j\in J}$ is disjoint.

Note that (2) will guarantee that any element in A, belongs to a ball in some $\mathcal{G}_j$. The generation of the ball B is defined to be the number j satisfying $B = B_\rho$ with $\rho \in \Lambda_j$.

A trivial but important observation is that different balls B_j and B_k intersect, then either $j > k$ or $k < j$ holds. Let $k \in \mathbb{N}$ be fixed, keeping the fact above in mind. Then we set

$$I_k \equiv \{j \in \mathbb{N} : j < k,\, B_j \cap B_k \neq \emptyset\}, \quad J_k \equiv \{j \in I_j : r_j \leq 3r_k\}.$$

We will prove that there exists $M_n > 0$ such that $\sharp I_k \leq M_n$ by proving $\sharp(I_k \setminus J_k) \lesssim 1$ and $\sharp J_k \lesssim 1$. Let $j \in J_k$. Then $r_k \leq r_j \leq 3r_k$. Since $\left\{\frac{1}{5}B_j\right\}_{j\in J}$ is disjoint, it follows that $J_k \lesssim_n 1$.

Let $j_1, j_2 \in I_k \setminus J_k$. If the generation of B_{j_1} and that of B_{j_2} are different, then the balls B_{j_1} and B_{j_2} meet in a point and $c(B_j)$ lies sufficiently close to the boundary of B_{j_1} and B_{j_2}. A geometric observation therefore shows the angle $\angle c(B_{j_1})c(B_j)c(B_{j_2}) \gtrsim 1$.

For each $j \in I_k \setminus J_k$, consider a point at which B_j and the segment $\overline{c(B_j)c(B_k)}$ meet. Then the observation of the above paragraph says that such intersection points are torn apart. Even from this observation, we conclude that the number of generations appearing in $I_k \setminus J_k$ is bounded by a number depending on n.

Of course, for each generation, the number of balls that can intersect B_k is also bounded by a number depending on n. Therefore, $\sharp(I_k \setminus J_k) \lesssim_n 1$.

In view of this observation we conclude that $\sharp I_k$ is bounded by a constant independent of k, say M_n.

We define a mapping $\sigma : \mathbb{N} \to \{1, 2, \ldots, M_n + 1\}$ as follows: First we set $\sigma(j) \equiv j$ for $j \leq M_n + 1$. If $j \geq M_n + 2$, then we define inductively

$$\begin{aligned} &\sigma(j) \\ &\equiv \min\{l = 1, 2, \ldots, M_n + 1 : B_j \cap B_m = \emptyset \text{ for all } m \in \sigma^{-1}(l) \cap [1, j-1]\}. \end{aligned} \tag{6.3}$$

We remark that by the pigeon hole principle the set appearing in (6.3) is not empty, which guarantees that $\sigma(j)$ is well defined. From the property of M_n, we learn that $\{B_j\}_{j\in\sigma^{-1}(l)}$ is disjoint for each $l = 1, 2, \ldots, M_n + 1$. Thus we have $\{\{B_j\}_{j\in\sigma^{-1}(l)}\}_{l=1}^{M_n+1}$ is the desired family.

6.1.2 Maximal operators on Euclidean spaces with general Radon measures

We consider the maximal operator given by

$$M^{\mathcal{B}}_{\kappa,\mathrm{uc}} f(x) = M_{\kappa,\mathrm{uc}} f(x) \equiv \sup_{B\in\mathcal{B}} \frac{\chi_B(x)}{\mu(\kappa B)} \int_B |f(y)| \mathrm{d}\mu(y) \quad (x \in \mathbb{R}^n).$$

The parameter $\kappa > 0$ is called the *modification rate*.

We cannot hope for the boundedness of $M_{1,\mathrm{uc}}$ as the following example shows:

Example 104. Let $\mu(x) = e^{|x|^2}\mathrm{d}x$. Here, we will show that $M_{1,\mathrm{uc}}$ can be unbounded by disproving

$$\mu\{x \in \mathbb{R}^n \,:\, M_{1,\mathrm{uc}} f(x) > \lambda\} \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \mathrm{d}\mu(x) \quad (f \in L^1(\mu), \lambda > 0).$$

For simplicity we assume $n = 2$. Assume to the contrary that this weak-$(1,1)$ maximal inequality holds. If we take $f \equiv \dfrac{\chi_{B(r)}}{\mu(B(r))}$, $r > 0$ then

$$\mu\{x \in \mathbb{R}^2 \,:\, M_{1,\mathrm{uc}} f(x) > \lambda\} \le \frac{C}{\lambda}.$$

Letting $r \downarrow 0$, we obtain

$$\mu\{x \in \mathbb{R}^2 \,:\, M_{1,\mathrm{uc}} \delta(x) > \lambda\} \le \frac{C}{\lambda},$$

where

$$M_{1,\mathrm{uc}} \delta(x) \equiv \frac{1}{\mu(B(x/2, |x|/2))}.$$

Let $\lambda \equiv \dfrac{1}{\mu(B((r,0),r))}$ for $r > 0$. Then we claim

$$\{x \in \mathbb{R}^2 \,:\, M_{1,\mathrm{uc}} \delta(x) > \lambda\} \supset B(2r).$$

To check this, we let $x \in B(2r)$ and prove $M_{1,\mathrm{uc}}\delta(x) > \lambda$. To this end by rotation we may assume the second component of x is 0 and the first component of x is positive. Hence, we have $x \in B((r,0),r)$. Then

$$M_{1,\mathrm{uc}} \delta(x) = \frac{1}{\mu(B(x/2, |x|/2))} > \frac{1}{\mu(B((r,0),r))} = \lambda.$$

This proves the claim.

From this claim, for all $r > 0$, we have $\mu(B(2r)) \le C\mu(B((r,0),r))$, namely,

$$\int_{s^2+t^2\le 4r^2} e^{s^2+t^2} \mathrm{d}s\mathrm{d}t \le C \int_{(s-r)^2+t^2\le r^2} e^{s^2+t^2} \mathrm{d}s\mathrm{d}t.$$

If we use the polar coordinate, then we obtain

$$2\pi\int_0^{2r} ue^{u^2}\mathrm{d}u \le C\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2r\cos\theta} ue^{u^2}\mathrm{d}u\mathrm{d}\theta = C\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(e^{4r^2\cos^2\theta}-1)\mathrm{d}\theta.$$

Thus,

$$\pi(1-e^{-4r^2}) \le C\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(e^{-4r^2\sin^2\theta}-e^{-4r^2})\mathrm{d}\theta.$$

If we let $r\to\infty$, then we obtain $3.14 < \pi \le 0$, which is a contradiction.

We modify the Hardy–Littlewood maximal operator. We now consider

$$M_{k,\mathrm{uc}}f(x) \equiv \sup_{x\in B(z,r),r>0}\frac{1}{\mu(B(z,kr))}\int_{B(z,r)}|f(y)|d\mu(y),$$

which is called the uncentered maximal operator on $\mathbb{R}^n$. Here $f\in L^0(\mu)$. The additional parameter k is necessary because the maximal operator $M_{1,\mathrm{uc}}$ can be unbounded as Example 104 shows.

Theorem 250. *Let μ be a Radon measure on $\mathbb{R}^n$. We have the dual inequality*;

$$\int_{\{M_{b,\mathrm{uc}}f>\lambda\}}|g(x)|d\mu(x) \lesssim_{a,b} \frac{1}{\lambda}\int_{\mathbb{R}^n} M_{a,\mathrm{uc}}g(x)|f(x)|d\mu(x)$$

for $b>a>1$ and $f,g\in L^0(\mu)$.

Using Theorem 247, we prove Theorem 250.

Proof Fix $R>0$. We truncate the maximal function again. Put again

$$M^R_{b,\mathrm{uc}}f(x) \equiv \sup_{x\in B(z,r),R>r>0}\frac{1}{\mu(B(z,br))}\int_{B(z,r)}|f(y)|d\mu(y).$$

Fix $\lambda>0$. We abbreviate $E_b \equiv \{x\in\mathbb{R}^{\mathrm{n}} : M^R_{b,\mathrm{uc}}f(x)>\lambda\}$. For all $x\in E_b$ by its definition there exists $r_x\in(0,R)$ and $y_x\in\mathbb{R}^n$ such that

$$x\in B(y_x,r_x) \tag{6.4}$$

and that

$$\frac{1}{\mu(B(y_x,br_x))}\int_{B(y_x,r_x)}|f(z)|d\mu(z)>\lambda. \tag{6.5}$$

Note that $\sup_{x\in E_b} r_x$ is at most R. Hence, we can apply Theorem 247. Applying the theorem with $k=b/a>1$, we obtain a countable subset $A\subset E_b$ such that $\{B(y_x,r_x)\}_{x\in A}$ satisfies

$$\bigcup_{x\in E_b}B(y_x,r_x)\subset\bigcup_j B\left(x_j,\frac{b}{a}r_j\right) \tag{6.6}$$

and

$$\sum_j \chi_{B(x_j,r_j)} \lesssim_{a,b} 1. \tag{6.7}$$

Using (6.4)–(6.7), we have

$$\begin{aligned}
\int_{E_b} g(x)d\mu(x) &\leq \int_{\bigcup_j B(x_j,\frac{b}{a}r_j)} g(x)d\mu(x) \\
&\leq \sum_j \frac{1}{\mu(B(x_j,br_j))} \int_{B(x_j,\frac{b}{a}r_j)} g(x)d\mu(x) \times \mu(B(x_j,br_j)) \\
&\leq \sum_j \left(\inf_{x\in B(x_j,r_j)} M^R_{a,\mathrm{uc}} g(x) \right) \frac{1}{\lambda} \int_{B(x_j,r_j)} |f(y)|d\mu(y) \\
&\leq \frac{C_{a,b}}{\lambda} \int_{\mathbb{R}^n} |f(y)| M^R_{a,\mathrm{uc}} g(y) d\mu(y).
\end{aligned}$$

Tending R to ∞, we obtain the desired result.

As an application, we can prove the vector-valued inequality.

Theorem 251. *If $1 < p, k < \infty$ and $1 < q \leq \infty$, then*

$$\left\| \left(\sum_{j=1}^{\infty} (M_{k,\mathrm{uc}} f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mu)} \lesssim_{p,q,k} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mu)}$$

for all $\{f_j\}_{j=1}^{\infty} \subset L^0(\mu)$.

We leave the proof of Theorem 251; see Exercise 88.

Example 105. In considering the weighted norm inequalities it could not be better if it held that

$$\int_{\{x\in\mathbb{R}^n : M_{\kappa,\mathrm{uc}} f(x) > \lambda\}} |g(x)| d\mu(x) \leq C_\kappa \int_{\mathbb{R}^n} |f(x)| M_{\kappa,\mathrm{uc}} g(x) \mathrm{d}\mu(x), \tag{6.8}$$

but this fails in general. Here, we will disprove (6.8) with $\kappa = 3$. We consider the case $n = 2$. We define a measure μ by posing a weight function f given below:

$$f(x) \equiv \chi_{\mathbb{R}^n \setminus B(1)}(x) + \sum_{m=1}^{\infty} \frac{1}{m!} \chi_{B(2^{-m+1}) \setminus B(2^{-m})}(x).$$

Let $\mu \equiv f(x)\mathrm{d}x$.

We disprove (6.8) by reduction to the absurdity. Suppose we have inequality (6.8) with $\kappa = 3$. First of all fix an integer α. We are going to let α tend to infinity later.

Set $R_m \equiv \mu(B((2^{-m},0),3\cdot 2^{-m}))^{-1}$. Then

$$B(2^{1-m}) \subset \{(x,y)\in\mathbb{R}^2 \,:\, M_3\delta_0(x,y) > R_m\},$$

where δ_0 denotes a dirac measure massed on the origin. In fact, we let $(x,y)\in B(2^{1-m})$. By the rotation invariance of the sets $B(2\cdot 2^{-m})$ and $\{(x,y)\in\mathbb{R}^2 \,:\, M_3\delta_0(x,y)>\lambda\}$, we may assume that $0\le x < 2^{-k+1}$ and $y=0$. Since $0,(x,0)\in B((x/2,0),(1+s)x/2)$ for all $s>0$, we have $M_3\delta_0(x,0) > \mu(B((x/2,0),(1+s)x/2))^{-1}$. If $s>0$ is sufficiently small, we have $M_3\delta_0(x,0) > \mu(B((x/2,0),(1+s)x/2))^{-1} > R_k$.

It follows from the claim that we have

$$\int_{B(2^{1-m})} |g(x)|\mathrm{d}\mu(x) \le C\mu(B((2^{-m},0),3\cdot 2^{-m}))M_3 g(0).$$

Let $\varphi\in\mathbb{M}^+(\mu)$ satisfy $\|\varphi\|_{L^1(\mu)}=1$ supported on a small ball whose center is the origin. For $r\ll 1/2$ we take a function g_r of the form $g_r = \sum_{j=1}^{\alpha}\frac{1}{r^2}\varphi(r^2\cdot - x_j^r)$, where α is chosen arbitrarily and x_j^r satisfies

$$\lim_{r\downarrow 0} x_j^r = \left(2^{-m+1}\cos\frac{2\pi j}{\alpha}, 2^{-m+1}\sin\frac{2\pi j}{\alpha}\right)$$

and that

$$\begin{aligned}&\mathrm{supp}(g_r)\\ &\subset B(2^{-m+1})\bigcap\left(\bigcup_{j=1}^{\alpha} B\left(\left(2^{-m+1}\cos\frac{2\pi j}{\alpha}, 2^{-m+1}\sin\frac{2\pi j}{\alpha}\right), r\right)\right).\end{aligned}$$

Tending $r\downarrow 0$, we have $\alpha \lesssim \mu(B((2^{-m},0),3\cdot 2^{-m}))M_3\mu_{m,\alpha}(0)$, where denoting δ_x as the dirac measure supported on x, we set

$$\mu_{m,p} \equiv \sum_{j=1}^{\alpha}\delta_{\left(2^{-m+1}\cos\frac{2\pi j}{\alpha}, 2^{-m+1}\sin\frac{2\pi j}{\alpha}\right)}.$$

By the definition of $M_3\mu_{m,\alpha}(0)$ we have

$$\begin{aligned}&M_3\mu_{m,\alpha}(0)\\ &= \sup_{\substack{(y,r)\in\mathbb{R}^{n+1}_+\\ 0\in B(y,r)}} \frac{\sharp\left\{1\le j\le\alpha \,:\, \left(2^{-k+1}\cos\left(\frac{2\pi j}{\alpha}\right), 2^{-k+1}\sin\left(\frac{2\pi j}{\alpha}\right)\right)\in B(y,r)\right\}}{\mu(B(y,3r))}.\end{aligned}$$

For a finite set $J=\{j_1,j_2,\ldots,j_m\}$ with $1\le j_1<j_2<\ldots<j_m\le\alpha$ we set

$$\mathrm{P}_j \equiv \left(2^{-k+1}\cos\left(\frac{2\pi j}{\alpha}\right), 2^{-k+1}\sin\left(\frac{2\pi j}{\alpha}\right)\right)\quad (j\in J)$$

$$S_J \equiv \inf\left\{\mu(B(y,3r)) \,:\, 0, \mathrm{P}_{j_1}, \ldots, \mathrm{P}_{j_m} \in B(y,r)\right\}.$$

Then $M_3\mu_{k,\alpha}(0)$ can be written as $M_3\mu_{k,p}(0) = \max_{J\subset\{1,2,\ldots,\alpha\}} \dfrac{\sharp J}{S_J}$.

Fixing α, if $\sharp J \geq 2$, we have by geometric observation that $\mu(S_J) \geq \dfrac{C_\alpha}{(m-1)!}$. Notice also that $\mu(B((2^{-k},0),3\cdot 2^{-k})) = \mathrm{O}\left(\dfrac{1}{m!}\right)$. Thus, we have $\lim_{k\to\infty} \dfrac{\mu(B((2^{-k},0),3\cdot 2^{-k}))}{S_J} = 0$. Furthermore, $S_{\{j\}} \sim \mu(B((2^{-k},0),3\cdot 2^{-k}))$, where $\sim$ is independent of α and k. Thus, keeping α fixed, we have

$$\alpha \lesssim \limsup_{k\to\infty} M_3\mu_{k,p}(0) \lesssim 1$$

where the implicit constant is independent on α. Since α is arbitrary, we obtained the desired contradiction and (6.8) is disproven.

6.1.3 Differentiation theorem

Having proven that the maximal operator $M_{k,\mathrm{uc}}$ is bounded when $k > 1$, we can now prove the differentiation theorem. A cube Q is *doubling* if $\mu(2Q) \leq 2^{n+1}\mu(Q)$. Or more generally, for $k > 1$, a cube Q is a *k-doubling cube* if $\mu(kQ) \leq k^{n+1}\mu(Q)$. Doubling balls are defined in a similar fashion. Due to the geometric structure of $\mathbb{R}^n$, we can prove that there are sufficiently many doubling cubes no matter how irregular the measure μ is.

Lemma 252. *Let μ be a Radon measure. Then there exists a μ-null set N such that for all $x \in \mathbb{R}^n \setminus N$, we can find a sequence $\{Q(x,r_j)\}_{j=1}^\infty$ of doubling cubes or a sequence $\{B(x,r_j)\}_{j=1}^\infty$ of doubling balls such that $r_j \downarrow 0$.*

Proof We consider balls; for cubes, a similar technique can be used. Let us set

$$N \equiv \left\{x \in \mathbb{R}^n \,:\, \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{r^n} = 0\right\}. \tag{6.9}$$

We claim that μ does not charge N. In fact, for all $x \in N$ and $\varepsilon > 0$, there exists $r_x \in (0,1)$ such that $\mu(B(x,r_x)) \leq \varepsilon {r_x}^n$. Let $R > 0$ be arbitrary. By Lemma 268 with $\delta = 1$ (Vitali's covering lemma), for any compact set K contained in $N \cap B(R)$, we can find $x_1, x_2, \ldots, x_L \in K$ such that

$$\sum_{l=1}^L \chi_{B(x_l, r_{x_l}/3)} \leq 1, \quad K \subset \bigcup_{l=1}^L B(x_l, r_{x_l}).$$

In fact, we can apply Lemma 268 to $\{B(x,r_x/3)\}_{x\in K}$. Then

$$\mu(K) \leq \sum_{l=1}^L \mu(B(x_l, r_{x_l})) \leq \varepsilon \sum_{l=1}^L {r_{x_l}}^n \lesssim \varepsilon \sum_{l=1}^L \left|B\left(x_l, \frac{r_{x_l}}{3}\right)\right| \lesssim \varepsilon |B(x, 1+R)|.$$

Since K is an arbitrary compact set, we have $\mu(N \cap B(R)) \lesssim \varepsilon |B(x, 1+R)|$. Letting $\varepsilon \downarrow 0$, we have $\mu(N \cap B(R)) = 0$. Finally, since $R > 0$ is arbitrary, $\mu(N) = 0$.

Let us show that N is the set we are looking for. For all $x \in \mathbb{R}^n \setminus N$, we can find a non-negative decreasing sequence $\{r_j\}_{j=1}^{\infty}$ such that

$$\sqrt{2} \liminf_{\rho \downarrow 0} \frac{\mu(B(x,\rho))}{\rho^n} > \frac{\mu(B(x,2r_j))}{(2r_j)^n}$$

and that

$$\sqrt{2} \frac{\mu(B(x,r))}{r^n} \geq \liminf_{\rho \downarrow 0} \frac{\mu(B(x,\rho))}{\rho^n} > 0$$

for all $r \leq 2\sqrt{n} r_1$. Thus, it follows that

$$\mu(B(x,2r_j)) \leq \sqrt{2} \left(\liminf_{\rho \downarrow 0} \frac{\mu(B(x,\rho))}{\rho^n} \right) (2r_j)^n \leq 2^{n+1} \mu(B(x,r_j)).$$

Thus, we obtain the desired result.

A direct consequence of Lemma 252 is the Lebesgue differentiation theorem for general measures, whose proof we have been postponed.

Theorem 253. *Let μ be a Radon measure on $\mathbb{R}^n$. Let $f \in L^1_{\mathrm{loc}}(\mu)$. Then*

$$\sup \left\{ \left| f(x) - \frac{1}{\mu(Q)} \int_Q f(y) \mathrm{d}\mu(y) \right| \,:\, \mu(2Q) \leq 2^{n+1} \mu(Q),\, Q \ni x, |Q| \leq r \right\}$$

goes to 0 as $r \downarrow 0$ for μ-a.e. $x \in \mathbb{R}^n$.

In particular, when f is a Lebesgue measurable function on $\mathbb{R}^n$,

$$\lim_{j \to \infty} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \frac{\chi_Q(x)}{|Q|} \int_Q f(z) \mathrm{d}z = f(x) \tag{6.10}$$

for almost all $x \in \mathbb{R}^n$.

Proof The conclusion follows from by the use of $M_{2,\mathrm{c}}$.

A truncation of f allows us to assume that $f \in L^1(\mu)$. Let $\varepsilon, \varepsilon' > 0$ be arbitrary. Choose a function $g \in L^1(\mu) \cap \mathrm{C}(\mathbb{R}^n)$ so that $\|f - g\|_{L^1(\mu)} < \varepsilon'$. Set

$$\begin{aligned} E_\varepsilon &\equiv E_\varepsilon(f) \\ &\equiv \left\{ x \in \mathbb{R}^n \,:\, \limsup_{r \downarrow 0} \left[\sup \left\{ \left| f(x) - \frac{1}{\mu(Q)} \int_Q f(y) \mathrm{d}\mu(y) \right| \,:\, ** \right\} \right] > 2\varepsilon \right\}, \end{aligned}$$

where $**$ is an abbreviation of the condition $\mu(2Q) \leq 2^{n+1}\mu(Q)$, $Q \ni x, |Q| \leq r$. Notice that $E_\varepsilon(f) = E_\varepsilon(f-g)$. In view of inclusion

$$\begin{aligned} &E_\varepsilon \\ &\subset \left\{ x \in \mathbb{R}^n \,:\, \limsup_{r \downarrow 0} \left[\sup \left\{ \left| \frac{1}{\mu(Q)} \int_Q f(y) - g(y) \mathrm{d}\mu(y) \right| \,:\, ** \right\} \right] > \varepsilon \right\} \\ &\quad \cup \{ x \in \mathbb{R}^n \,:\, |f(x) - g(x)| > \varepsilon \} \\ &\subset \left\{ x \in \mathbb{R}^n \,:\, M_{2,\mathrm{c}}[f-g](x) > 2^{-n-1} n^{-n} \varepsilon \right\} \cup \{ x \in \mathbb{R}^n \,:\, |f(x) - g(x)| > \varepsilon \}, \end{aligned}$$

Theorem 267 and the Chebychev inequality, we have

$$\mu(E_\varepsilon) \le \frac{2^{n+1}n^n+1}{\varepsilon}\|f-g\|_{L^1(\mu)} \le (2^{n+1}n^n+1)\frac{\varepsilon'}{\varepsilon}.$$

Since $\varepsilon' > 0$ is arbitrary, it follows that $\mu(E_\varepsilon) = 0$. Since $\varepsilon > 0$ is arbitrary, we conclude the proof of Theorem 253.

6.1.4 Universal estimates

One of the strong thrusts of studying general metric measure spaces is to obtain some estimates independent of the underlying measures; see Chapter 7. Sometimes such estimates are obtained with the constants independent of the dimension. If the constant is independent of the dimension, we can make the dimension as high as we wish. This fact is useful from the viewpoint of applications in stochastic analysis and so on. Meanwhile, we intend to study the general metric measure spaces which will be an application to analysis in Euclidean spaces. The application will be obtained later in this book.

We start Section 6.1.4 by presenting a result on the dyadic maximal operator. We define

$$M^{\mathcal{D}(\mu)}f(x) \equiv \sup_{Q\in\mathcal{D}(\mu)} \frac{\chi_Q(x)}{\mu(Q)}\int_Q |f(z)|\mathrm{d}\mu(z) \quad (x\in\mathbb{R}^n),$$

where $\mathcal{D}(\mu)$ stands for the set of all cubes Q in $\mathcal{D}(\mathbb{R}^n)$ with $\mu(Q) > 0$. By using the fact that the dyadic cubes are nested, we can prove the following:

Theorem 254 (Universal weak $L^1(\mu)$-estimate for $M^{\mathcal{D}(\mu)}$)**.** *Let μ be a Radon measure on $\mathbb{R}^n$, $\lambda > 0$ and $f \in L^1(\mu)$. Then*

$$\mu\{x\in\mathbb{R}^n \,:\, M^{\mathcal{D}(\mu)}f(x) > \lambda\} \le \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n\,:\,M^{\mathcal{D}(\mu)}f(x)>\lambda\}} |f(y)|\mathrm{d}\mu(y).$$

Proof Mimic the proof of Theorem 134.

A direct consequence of Theorem 254 is the $L^p(\mu)$-boundedness.

Theorem 255 (Universal $L^p(\mu)$-estimate for $M^{\mathcal{D}(\mu)}$)**.** *Let μ be a Radon measure on $\mathbb{R}^n$, and let $1 < p < \infty$. Then $\|M^{\mathcal{D}(\mu)}f\|_{L^p(\mu)} \le (p')^{\frac{1}{p}}\|f\|_{L^p(\mu)}$ for all $f \in L^p(\mu)$.*

Proof If we integrate the inequality in Theorem 254 against $\lambda^{p-1}\mathrm{d}\lambda$ over $(0,\infty)$, we first obtain

$$\int_{\mathbb{R}^n} M^{\mathcal{D}(\mu)}f(x)^p\mathrm{d}\mu(x) \le p'\int_{\mathbb{R}^n} M^{\mathcal{D}(\mu)}f(x)^{p-1}|f(x)|\mathrm{d}\mu(x). \tag{6.11}$$

We may assume that $f \in L^\infty(\mu)$ and that $\mu\{f \ne 0\} < \infty$. In this case, $M^{\mathcal{D}(\mu)}f \in L^p(\mu)$. See Exercise 87. By Hölder's inequality and (6.11), we obtain

$$\int_{\mathbb{R}^n} M^{\mathcal{D}(\mu)}f(x)^p\mathrm{d}\mu(x) \le p'\int_{\mathbb{R}^n} |f(x)|^p\mathrm{d}\mu(x). \tag{6.12}$$

Thus, we obtain the desired result.

Let $f \in L^0(\mu)$. We define

$$M_{\log}^{\mathcal{D}(\mu)} f(x) \equiv \sup_{Q \in \mathcal{D}(\mu)} \chi_Q(x) \exp\left(\frac{1}{\mu(Q)} \int_Q \log|f(y)| \mathrm{d}\mu(y)\right) \quad (x \in \mathbb{R}^n). \tag{6.13}$$

Theorem 256 (Universal $L^p(\mu)$-estimate for $M_{\log}^{\mathcal{D}(\mu)}$)**.** *Let $0 < p < \infty$. Then $\|M_{\log}^{\mathcal{D}(\mu)} f\|_{L^p(\mu)} \le e^{\frac{1}{p}} \|f\|_{L^p(\mu)}$ for all $f \in L^0(\mu)$.*

Proof Let $0 < r < p$. Then

$$\|M_{\log}^{\mathcal{D}(\mu)} f\|_{L^p(\mu)} \le \|[M^{\mathcal{D}(\mu)}(|f|^r)]^{\frac{1}{r}}\|_{L^p(\mu)} \le \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \|f\|_{L^p(\mu)}.$$

If we send $r \downarrow 0$, we obtain the desired result.

Define $\mathcal{Q}(\mu)$ to be the set of all cubes with positive μ-measures. We let $\mathcal{D}(\mu; Q) \equiv \mathcal{D}(Q) \cap \mathcal{Q}(\mu)$. We consider the local universal estimates. For $0 \le \alpha < n$, denote $M_\alpha^{\mathcal{D}(\mu;Q)}$ the fractional maximal operator generated by dyadic cubes in a cube Q. That is, for $f \in L^0(Q)$ we define

$$M_\alpha^{\mathcal{D}(\mu;Q)} f(x) \equiv \sup_{R \in \mathcal{D}(\mu;Q)} \chi_R(x) \frac{1}{\mu(R)^{1-\frac{\alpha}{n}}} \int_R |f(z)| \mathrm{d}\mu(z) \quad (x \in \mathbb{R}^n).$$

We also define

$$M_{\log}^{\mathfrak{D}(\mu;Q)} f(x) \equiv \sup_{R \in \mathcal{D}(\mu;Q)} \chi_R(x) \exp\left(\frac{1}{\mu(R)} \int_R \log|f(z)| \mathrm{d}\mu(z)\right) \quad (x \in \mathbb{R}^n).$$

If μ is the Lebesgue measure, then we write $M_\alpha^{\mathcal{D}(Q)}$ and $M_{\log}^{\mathfrak{D}(Q)}$ instead of $M_\alpha^{\mathcal{D}(\mu;Q)}$ and $M_{\log}^{\mathfrak{D}(\mu;Q)}$, respectively.

We have the following universal estimates:

Theorem 257 (Universal estimates over cubes)**.** *Let Q be a cube and $0 \le \alpha < n$.*

(1) *Let $f \in L^1(\mu)$, $\lambda > 0$, and let $\Omega_\lambda \equiv \{x \in Q : M_\alpha^{\mathcal{D}(\mu;Q)} f(x) > \lambda\}$. Then*

$$\lambda \mu(\Omega_\lambda)^{\frac{n-\alpha}{n}} \le \|f \chi_{\Omega_\lambda}\|_{L^1(\mu)}. \tag{6.14}$$

(2) *Let $f \in L^p(\mu)$ with $1 < p \le \frac{n}{\alpha}$. Set $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then*

$$\|M_\alpha^{\mathcal{D}(\mu;Q)} f\|_{L^q(\mu)} \le \left(\frac{q}{q - \frac{n}{n-\alpha}}\right)^{\frac{n-\alpha}{n}} \|f\|_{L^p(\mu)}.$$

In particular, if $\alpha = 0$, then

$$\|M^{\mathcal{D}(\mu;Q)} f\|_{L^p(\mu)} \le p' \|f\|_{L^p(\mu)}. \tag{6.15}$$

(3) *Let $f \in L^1(\mu)$. Then $\|M_{\log}^{\mathcal{D}(\mu;Q)} f\|_{L^1(\mu)} \le e\|f\|_{L^1(\mu)}$.*

Proof Throughout the proof, we may assume that $f \in L^\infty(\mu)$ by Fatou's lemma.

(1) We partition Ω_λ by choosing maximal cubes contained in Ω_λ: $\Omega_\lambda = \bigcup_{\theta\in\Theta} Q_\theta$, where each $Q_\theta \in \mathcal{D}(\mu;Q)$ satisfies $\dfrac{1}{\mu(Q_\theta)^{1-\frac{\alpha}{n}}} \displaystyle\int_{Q_\theta} |f(z)|\mathrm{d}\mu(z) > \lambda$. Thus,

$$\lambda\mu(\Omega_\lambda)^{\frac{n-\alpha}{n}} \le \lambda \sum_{\theta\in\Theta} \mu(Q_\theta)^{\frac{n-\alpha}{n}} \le \sum_{\theta\in\Theta} \|f\chi_{Q_\theta}\|_{L^1(\mu)} \le \|f\chi_{\Omega_\lambda}\|_{L^1(\mu)}.$$

(2) We observe that

$$\begin{aligned}(\|M_\alpha^{\mathcal{D}(\mu;Q)} f\|_{L^q(\mu)})^q &= q\int_0^\infty \lambda^{q-1}\mu(\Omega_\lambda)\mathrm{d}\lambda \\ &\le q\int_0^\infty \lambda^{q-1-\frac{n}{n-\alpha}} (\|f\chi_{\Omega_\lambda}\|_{L^1(\mu)})^{\frac{n}{n-\alpha}}\mathrm{d}\lambda.\end{aligned}$$

We choose $u \ge 0$ so that $n(1-u) = p\alpha$. Then by Hölder's inequality

$$\|f\chi_{\Omega_\lambda}\|_{L^1(\mu)} \le \left(\|f\chi_{\Omega_\lambda}\|_{L^{\frac{nu}{n-\alpha}}(\mu)}\right)^u (\|f\|_{L^p(\mu)})^{1-u}.$$

Thus,

$$\begin{aligned}&(\|M_\alpha^{\mathcal{D}(\mu;Q)} f\|_{L^q(\mu)})^q \\ &\le q\int_0^\infty \lambda^{q-1-\frac{n}{n-\alpha}} \left(\|f\chi_{\Omega_\lambda}\|_{L^{\frac{nu}{n-\alpha}}(\mu)}\right)^{\frac{nu}{n-\alpha}} (\|f\|_{L^p(\mu)})^{\frac{n(1-u)}{n-\alpha}}\mathrm{d}\lambda \\ &= \frac{q}{q-\frac{n}{n-\alpha}} (\|f\|_{L^p(\mu)})^{\frac{n(1-u)}{n-\alpha}} \int_Q |f(x)|^{\frac{nu}{n-\alpha}} M_\alpha^{\mathcal{D}(\mu;Q)} f(x)^{q-\frac{n}{n-\alpha}}\mathrm{d}\mu(x).\end{aligned}$$

Observe that

$$\left(q-\frac{n}{n-\alpha}\right) \times \left(\frac{(n-\alpha)p}{nu}\right)' = \frac{q(p-1)n}{(p-1)n} = q.$$

By Hölder's inequality, we obtain

$$\begin{aligned}&\int_Q |f(x)|^{\frac{nu}{n-\alpha}} M_\alpha^{\mathcal{D}(\mu;Q)} f(x)^{q-\frac{n}{n-\alpha}}\mathrm{d}\mu(x) \\ &\le (\|f\|_{L^p(\mu)})^{\frac{nu}{n-\alpha}} (\|M_\alpha^{\mathcal{D}(\mu;Q)} f\|_{L^q})^{q-\frac{n}{n-\alpha}}.\end{aligned}$$

As a result,

$$(\|M_\alpha^{\mathcal{D}(\mu;Q)} f\|_{L^q(\mu)})^q \le \frac{q}{q-\frac{n}{n-\alpha}} (\|f\|_{L^p(\mu)})^{\frac{n}{n-\alpha}} (\|M_\alpha^{\mathcal{D}(\mu;Q)} f\|_{L^q})^{q-\frac{n}{n-\alpha}}.$$

If we arrange this inequality, we obtain the desired result.

(3) Let $u > 0$. Note that

$$M_{\log}^{\mathcal{D}(Q)} f(x) \le M^{\mathcal{D}(\mu;Q),(u)} f(x) \equiv \left(M_0^{\mathcal{D}(\mu;Q)} [|f|^u](x) \right)^{\frac{1}{u}} \quad (x \in \mathbb{R}^n).$$

Thus, by (6.15)

$$\begin{aligned} \|M_{\log}^{\mathcal{D}(Q)} f\|_{L^1(\mu)} &\le \|M^{\mathcal{D}(\mu;Q),(u)} f\|_{L^1(\mu)} \\ &= (\|M^{\mathcal{D}(\mu;Q)}[|f|^u]\|_{L^{\frac{1}{u}}(\mu)})^{\frac{1}{u}} \\ &\le \left(\frac{1}{1-u} \right)^{\frac{1}{u}} \|f\|_{L^1(\mu)}. \end{aligned}$$

If we let $u \downarrow 0$, then we obtain the desired result.

Denote by $M_\alpha^{\mathcal{D}(\mu)}$ the fractional maximal operator for $0 \le \alpha < n$. That is, for a measurable function f on Q we define

$$M_\alpha^{\mathcal{D}(\mu)} f(x) \equiv \sup_{R \in \mathcal{D}(\mu)} \chi_R(x) \frac{1}{\mu(R)^{1-\frac{\alpha}{n}}} \int_R |f(z)| \mathrm{d}\mu(z).$$

We also define

$$M_{\log}^{\mathcal{D}(\mu)} f(x) \equiv \sup_{R \in \mathcal{D}(\mu)} \chi_R(x) \exp\left(\frac{1}{\mu(R)} \int_R \log |f(z)| \mathrm{d}\mu(z) \right).$$

We have the following universal estimates:

Theorem 258 (Universal estimates over $\mathbb{R}^n$)**.** *Let $0 \le \alpha < n$.*

(1) *Let $f \in L^1(\mu)$, $\lambda > 0$, and let $\Omega_\lambda \equiv \{x \in Q : M_\alpha^{\mathcal{D}(\mu)} f(x) > \lambda\}$. Then*

$$\lambda \mu(\Omega_\lambda)^{\frac{n-\alpha}{n}} \le \|f \chi_{\Omega_\lambda}\|_{L^1(\mu)}.$$

(2) *Let $f \in L^p(\mu)$ with $1 < p \le \dfrac{n}{\alpha}$. Set $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$. Then*

$$\|M_\alpha^{\mathcal{D}(\mu)} f\|_{L^q(\mu)} \le \left(\frac{q}{q - \frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}} \|f\|_{L^p(\mu)}.$$

In particular, $\|M^{\mathcal{D}(\mu)} f\|_{L^p(\mu)} \le p' \|f\|_{L^p(\mu)}$.

(3) *Let $f \in L^1(\mu)$. Then*

$$\|M_{\log}^{\mathcal{D}(\mu)} f\|_{L^1(\mu)} \le e \|f\|_{L^1(\mu)}. \tag{6.16}$$

Proof Let $Q_j \equiv [-2^{-j}, 2^j)$ for $j \in \mathbb{N}$ and use Theorem 258. Since the constants are independent of j, we can let $j \to \infty$ to obtain the desired results.

Instead of $\mathcal{D}(\mathbb{R}^n)$, it is sometimes convenient to use different families. We do not go into detail; we content ourselves with the case $n = 1$.

Lemma 259 (Three lattice theorem)**.** *Write $\mathcal{Z}_l(\mathbb{R}) = \{3Q : Q \in \mathcal{D}_l(\mathbb{R})\}$ for each $l \in \mathbb{Z}$. Then each $\mathcal{Z}_l(\mathbb{R})$, $l \in \mathbb{Z}$, is partitioned into 3 disjoint subfamilies $\{\widetilde{\mathcal{E}}_l^k(\mathbb{R})\}_{k=1}^3$ so that the following properties hold, where $\mathcal{E}_l^k = \{Q \in \mathcal{D}_l : 3Q \in \widetilde{\mathcal{E}}_l^k(\mathbb{R})\}$.*

(1) *Let $k = 1, 2, 3$. For any cubes $Q_1, Q_2 \in \bigcup_{l=-\infty}^{\infty} \widetilde{\mathcal{E}}_l^k(\mathbb{R})$ which meet at a point, one is contained in the other.*

(2) *For all $Q \in \mathcal{D}_l(\mathbb{R})$, $k = 1, 2, 3$ and $l \in \mathbb{Z}$, there exists $Q' \in \mathcal{E}_l^k(\mathbb{R})$ such that $Q \subset 3Q' \subset 5Q$.*

(3) *For any $k = 1, 2, 3$ and $l \in \mathbb{Z}$, $\sum_{Q \in \widetilde{\mathcal{E}}_l^k} \chi_Q = 1$.*

Proof Let

We define $\mathcal{E}_l^k(\mathbb{R})$ with $k = 1, 2, 3$ and $l = 0, -1, -2, -3$ as follows:

$$\begin{aligned}
\mathcal{E}_0^1(\mathbb{R}) &\equiv \{\dots, [-3, -2), [0, 1), [3, 4), [6, 7), \dots\},\\
\mathcal{E}_{-1}^1(\mathbb{R}) &\equiv \{\dots, [-2, 0), [4, 6), [10, 12), [16, 18), \dots\},\\
\mathcal{E}_{-2}^1(\mathbb{R}) &\equiv \{\dots, [0, 4), [12, 16), [24, 28), [36, 40), \dots\},\\
\mathcal{E}_{-3}^1(\mathbb{R}) &\equiv \{\dots, [-8, 0), [16, 24), [40, 48), [64, 72), \dots\},\\
\mathcal{E}_0^2(\mathbb{R}) &\equiv \{\dots, [-2, -1), [1, 2), [4, 5), [7, 8), \dots\},\\
\mathcal{E}_{-1}^2(\mathbb{R}) &\equiv \{\dots, [0, 2), [6, 8), [12, 14), [18, 20), \dots\},\\
\mathcal{E}_{-2}^2(\mathbb{R}) &\equiv \{\dots, [4, 8), [16, 20), [28, 32), [40, 44), \dots\},\\
\mathcal{E}_{-3}^2(\mathbb{R}) &\equiv \{\dots, [0, 8), [24, 32), [48, 56), [72, 80), \dots\},\\
\mathcal{E}_0^1(\mathbb{R}) &\equiv \{\dots, [-1, 0), [2, 3), [5, 6), [8, 9), \dots\},\\
\mathcal{E}_{-1}^1(\mathbb{R}) &\equiv \{\dots, [2, 4), [8, 10), [14, 16), [20, 22), \dots\},\\
\mathcal{E}_{-2}^1(\mathbb{R}) &\equiv \{\dots, [8, 12), [20, 24), [32, 36), [44, 48), \dots\}\\
\mathcal{E}_{-3}^1(\mathbb{R}) &\equiv \{\dots, [8, 16), [32, 40), [56, 64), [80, 88), \dots\}.
\end{aligned}$$

From the definition of $\mathcal{E}_{2l}^k(\mathbb{R})$ and $\mathcal{E}_{2l-1}^k(\mathbb{R})$ with $k = 1, 2, 3$ and $l \in \mathbb{Z}$, we can guess the defintion of each $\mathcal{E}_l^k(\mathbb{R})$. We leave the interested readers to check that the family $\{\mathcal{E}_l^k(\mathbb{R})\}_{k=1,2,3,l\in\mathbb{Z}}$ satisfies (1)–(3).

6.1.5 Examples of metric measure spaces

Here we take up some examples. The first one is the p-adic field.

Example 106. Let $p \in \mathbb{N}$ be a fixed prime number. Equip $\mathbb{Q}$ with a distance function as follows: the distance between $x, y \in \mathbb{Q}$ is $d_p(x,y) = p^{-m}$, where the integers q, r, m satisfy $x - y = p^m q/r$ and the integers q and r are not multiples of p; see [155]. Denote by $\mathbb{Q}_p$ the completion with respect to d_p of $\mathbb{Q}$. Since $\mathbb{Q}_p$ carries the structure of a locally compact Abelian group, $\mathbb{Q}_p$ has the invariant Haar measure μ.

We next consider the Gauss measure space as an example of metric measure spaces. Denote by $\mathcal{B}(\mathbb{R}^n)$ the Borel σ-field over $\mathbb{R}^n$. The *Gauss measure space* is a triplet $(\mathbb{R}^n, |\cdot|, \gamma)$, where

$$\gamma(E) \equiv \pi^{-\frac{n}{2}} \int_E e^{-|x|^2} \mathrm{d}x \quad (E \in \mathcal{B}(\mathbb{R}^n)). \tag{6.17}$$

The measure γ is called the Gauss measure. We distort the underlying measure $\mathrm{d}x$; the ambient space $\mathbb{R}^n$ together with the distance remains intact. Here, we content ourselves with introducing the notion of *maximal admissible balls* and the *mother of a maximal admissible ball* due to Mauceri and Meda.

Definition 67. Let $a > 0$ and define $m(x) \equiv \min(1, |x|^{-1})$.

(1) Define $\mathcal{B}_a \equiv \{B(x,r) : (x,r) \in \mathbb{R}^{n+1}_+, 0 < r \le am(x)\}$.

(2) Write $[0,x] \equiv \{tx : t \in [0,1]\}$ for $x \in \mathbb{R}^n$.

(3) Any ball $B \in \mathcal{B}_a$ is *maximal* if $r(B) = am(c(B))$.

(4) For each maximal ball $B \in \mathcal{B}_a$ which does not contain the origin, $M(B)$ denotes the *maximal ball* in $\mathcal{B}_a$ with center at a point on the segment $[0, c(B)]$ such that the boundary of $M(B)$ contains $c(B)$. Call $M(B)$ the *mother* of B. In other words, the relation between $B = B(x,r)$ and its mother $M(B) = B(z,R)$ satisfy $R = am(z)$, $|z| + R = |x|$ and $|x|z = |z|x \neq 0$. For notational convenience, we set $M^0(B) \equiv B$.

(5) If $M(B)$ does not contain the origin, then we may consider the *mother of* $M(B)$, which we denote by $M^2(B)$. Therefore, given any maximal ball B in $\mathcal{B}_a$, we find a *chain of maximal balls*, B, $M(B)$, $M^2(B)$, $\ldots$, $M^k(B)$, with the property that $M^j(B)$ is the mother of $M^{j-1}(B)$ for $j \in \{1, 2, \ldots, k\}$, and $M^k(B)$ contains the origin.

Example 107. Let $a > 0$. Let $\{a_j\}_{j=0}^{\infty}$ be a sequence defined in Lemma 262. If the ball $B = B(z,r)$ satisfies that $|z| = a_j$ and $r = \frac{a}{a_j}$ for some $j \in \mathbb{N}$, then B is a maximal ball in $\mathcal{B}_a$.

The class $\mathcal{B}_a$ has the following control:

Proposition 260. *Let $B = B(c_B, r) \in \mathcal{B}_a$, $a > 0$, and let $x \in B$. Then $-a^2 - 2a \le |c(B)|^2 - |x|^2 \le 2a$.*

Proof The proof is left as an exercise. See Exercise 85.

We establish some subtle geometric relations between the maximal admissible balls and their mothers.

Lemma 261. *Let $a > 0$. If the maximal ball $B \in \mathcal{B}_a$ does not contain 0, then $B \subset (a+2)M(B)$ and $M(B) \subset (2a+2)B$. Consequently, $\gamma(M(B)) \sim \gamma(B)$ with implicit constants depending only on a and n.*

Proof By the definition of $M(B)$, we have $r(M(B)) = am(c(M(B)))$, $r(B) = am(c(B))$ and $|c(M(B))| + r(M(B)) = |c(B)|$. Using the fact that $c(B)$ is on the boundary of $M(B)$, together with the continuity of the function m, we see that $(a+1)^{-1}m(c(M(B))) \le m(c(B)) \le (a+1)m(c(M(B)))$. Hence, $(a+1)^{-1}r(M(B)) \le r(B) \le (a+1)r(M(B))$. This implies that for any $z \in B$, $|z - c(M(B))| \le |z - c(B)| + |c(B) - c(M(B))| < r(B) + r(M(B)) \le (a+2)r(M(B))$. That is, $B \subset (a+2)M(B)$. Meanwhile, for any $z \in M(B)$, we have

$$|z - c(B)| \le |z - c(M(B))| + |c(M(B)) - c(B)| < 2r(M(B)) \le 2(a+1)r(B).$$

which in turn implies that $M(B) \subset (2a+2)B$. Finally, we have $\gamma(B) \le \gamma((a+1)M(B)) \lesssim \gamma(M(B))$ and $\gamma(M(B)) \le \gamma((2a+2)B) \lesssim \gamma(B)$, which completes the proof of Lemma 261.

We next handle a special recurrence formula to deal with the Gauss measure space.

Lemma 262. *Fix a constant $a > 0$. Let $\varphi(s) \equiv s + \dfrac{a}{s}$ for $s > 0$. Let $\{a_j\}_{j=0}^{\infty}$ be the sequence recursively defined by*

$$a_0 \equiv 1 \qquad \text{and} \qquad a_{j+1} \equiv a_j + \frac{a}{a_j} = \varphi(a_j) \quad \text{for } j \in \mathbb{N}. \tag{6.18}$$

(1) *For all $j \in \mathbb{N}$, $\sqrt{2aj} \le a_j \le \sqrt{(a^2+2a)j}$.*

(2) *For all $i, j \in \mathbb{N}$, $2a(j-i) \le a_j^2 - a_i^2 \le (j-i)(a^2+2a)$.*

Proof

(1) Indeed, this follows from

$$\sqrt{s^2+2a} < s + \frac{a}{s} < \sqrt{s^2+a^2+2a}, \qquad s \in [1, \infty),$$

and the fact that the sequences $\{\sqrt{2aj}\,\}_{j=1}^{\infty}$ and $\{\sqrt{(a^2+2a)j}\,\}_{j=1}^{\infty}$ can be recursively defined by

$$\begin{cases} s_1 = \sqrt{2a} \\ s_{j+1} = \sqrt{s_j^2 + 2a} \end{cases} \qquad \text{and} \qquad \begin{cases} s_1 = \sqrt{a^2+2a} \\ s_{j+1} = \sqrt{s_j^2 + a^2 + 2a}, \end{cases}$$

respectively.

(2) For any $j \in \mathbb{N}_0$, since ${a_{j+1}}^2 - {a_j}^2 = 2a + \dfrac{a^2}{{a_j}^2}$ from (6.18) we have

$$2a \le {a_{j+1}}^2 - {a_j}^2 \le 2a + a^2. \tag{6.19}$$

Consequently, for all $j \ge i \ge 0$, by ${a_j}^2 - {a_i}^2 = \sum_{k=1}^{j-i} ({a_{k+i}}^2 - {a_{k+i-1}}^2)$ and (6.19), we see that (2) holds.

Lemma 263. *Let $a > 0$ and B be a maximal ball in $\mathcal{B}_a$. Suppose that the sequence $\{a_j\}_{j=0}^{\infty}$ is defined as in* (6.18)*. Then, the following hold for each $j \in \mathbb{N}$:*

(1) *if $|c(B)| \le 1 + a$, then $\gamma(B) \sim 1$;*

(2) *if $a_j \le |c(B)| < a_{j+1}$ for some $j \in \mathbb{N}$, then either $|c(M(B))| \le 1 + a$ or $a_{j-1} \le |c(M(B))| < a_j$;*

(3) *if $a_j \le |c(B)| < a_{j+1}$, then $\gamma(B) \sim e^{-a_j^2} j^{-\frac{n}{2}}$;*

(4) *if $|c(B)| > 1 + a$ and $\{M^k(B) : 0 \le k \le k_0\}$ is a chain of maximal balls in $\mathcal{B}_a$, with the property that $M^k(B)$ is the mother of $M^{k-1}(B)$ for $k \in \{1, 2, \dots, k_0\}$, where k_0 is the smallest integer such that $|c_{M^{k_0}(B)}| \le 1 + a$, then $e^{-(a^2+2a)k}\left(1 - \frac{k}{k_0+1}\right)^{\frac{n}{2}} \lesssim \frac{\gamma(B)}{\gamma(M^k(B))} \lesssim e^{-2ak}$ for all $k \in \{1, 2, \dots, k_0\}$.*

Here, the implicit positive constants in (1)–(4) *depend only on a and n.*

Proof

(1) The proof is left as an exercise; see Exercise 86.

(2) For the purpose of showing $|c(M(B))| < a_j$ if $|c(M(B))| > 1 + a$, we show that, when $|c(M(B))| > 1 + a$, we have $a_{j-1} \le |c(M(B))| < a_j$. This follows from reductio ad absurdum. Indeed, if $|c(M(B))| \ge 1 + a$ and $|c(M(B))| \ge a_j$, then by observing that $\varphi(s) = s + \frac{a}{s}$ increases on $[\sqrt{a}, \infty)$, we obtain

$$|c(B)| = |c(M(B))| + |c(M(B))|^{-1} a \ge a_j + \frac{a}{a_j} = a_{j+1},$$

which contradicts assumption $|c(B)| < a_{j+1}$. Therefore, we have $|c(M(B))| < a_j$. Likewise, we have $a_{j-1} \le |c(M(B))|$; the details are left to the reader. This proves (2).

(3) Lemma 262 yields $\sqrt{2aj} \le a_j \le |c(B)| < a_{j+1} \le \sqrt{(a^2 + 2a)(j+1)}$, which, combined with $|c(B)| \ge a_j > 1$, implies $r(B) = am(c(B)) = |c(M(B))|^{-1} a \sim j^{-\frac{1}{2}}$. From $a_j \le |c(B)| < a_{j+1}$ and (6.19), it follows that $e^{-a^2-2a} < e^{a_j^2 - a_{j+1}^2} < e^{a_j^2 - |c(B)|^2} \le 1$. Hence, $e^{a_j^2 - |c(B)|^2} \sim 1$ or, equivalently, $e^{-|c(B)|^2} \sim e^{-a_j^2}$. This and the aforementioned $r(B) \sim j^{-\frac{1}{2}}$, yield that $\gamma(B) \sim e^{-|c(B)|^2} r(B)^n \sim e^{-a_j^2} j^{-\frac{n}{2}}$. Hence, (3) holds.

(4) Without loss of generality, we may assume that $k_0 \geq 2$, since $\gamma(B) = \gamma(M^0(B))$ and $\gamma(M(B)) \sim \gamma(B)$ by Lemma 261. Notice that

$$|c(B)| > |c(M(B))| > \cdots > |c_{M^{k_0-1}(B)}| > 1 + a =: a_1. \tag{6.20}$$

Thus, there exists some $j \in \mathbb{N}$ such that $a_j \leq |c(B)| < a_{j+1}$. Applying (6.20) and (2) repeatedly, we conclude that, for all $k \in \{1, 2, \ldots, k_0\}$,

$$a_{j-k} \leq |c_{M^k(B)}| < a_{j+1-k}. \tag{6.21}$$

In particular, when $k = k_0 - 1$, by (6.20) and (6.21), we have $a_1 < |c_{M^{k_0-1}(B)}| < a_{j-k_0+2}$. From this and the fact that $\{a_j\}_{j=0}^{\infty}$ is strictly increasing, it follows that $j \geq k_0$. For $1 \leq k \leq k_0 - 1$, by (3) and (6.21), we obtain

$$\frac{\gamma(B)}{\gamma(M^k(B))} \sim \frac{e^{-a_j^2} j^{-\frac{n}{2}}}{e^{-a_{j-k}^2}(j-k)^{-\frac{n}{2}}} \sim e^{-(a_j^2 - a_{j-k}^2)} \left(1 - \frac{k}{j}\right)^{\frac{n}{2}}.$$

Moreover, from Lemma 262(2), we infer that

$$e^{-k(a^2+2a)} \left(1 - \frac{k}{k_0}\right)^{\frac{n}{2}} \lesssim e^{-(a_j^2 - a_{j-k}^2)} \left(1 - \frac{k}{j}\right)^{\frac{n}{2}} \lesssim e^{-2ak}.$$

Thus, when $1 \leq k \leq k_0 - 1$,

$$e^{-k(a^2+2a)} \left(1 - \frac{k}{k_0+1}\right)^{\frac{n}{2}} \lesssim \frac{\gamma(B)}{\gamma(M^k(B))} \lesssim e^{-2ak}.$$

By Lemma 261, we have $\gamma(M^{k_0}(B)) \sim \gamma(M^{k_0-1}(B))$. Thus

$$\frac{\gamma(B)}{\gamma(M^{k_0}(B))} \sim \frac{\gamma(B)}{\gamma(M^{k_0-1}(B))},$$

which implies that

$$e^{-k_0(a^2+2a)} \left(1 - \frac{k_0}{k_0+1}\right)^{\frac{n}{2}} \lesssim \frac{\gamma(B)}{\gamma(M^{k_0}(B))} \lesssim e^{-2ak_0}.$$

This concludes the proof of (4).

For later applications, we establish the following covering lemma. The key point is that the number of balls that cover the annulus $B(|a_{j+1}|) \setminus B(|a_j|)$ with $j \in \mathbb{N}_0$ can be estimated from above (see Lemma 264(D)are below), which plays a key role in applications. For simplicity, we only consider the balls in $\mathcal{B}_1$ in Lemma 264 below; similar results hold for the balls in $\mathcal{B}_a$ with general $a > 0$.

Lemma 264. *Let $a \equiv 1$ and $\{a_j\}_{j=0}^{\infty}$ be defined as in* (6.18)*, namely, $a_0 \equiv 1$, and $a_{j+1} \equiv a_j + \frac{1}{a_j}$ when $j \in \mathbb{N}$. Then, there exists a family $\{B_{j,i}\}_{j \in \mathbb{N}_0,\, i \in I_j}$ of maximal balls in $\mathcal{B}_1$ with the following property*:

(A) *Each $B_{j,i}$ is centered at the boundary of $B(a_j)$ and with radius $\frac{1}{a_j}$.*

(B) $\mathbb{R}^n = \bigcup_{j=0}^{\infty} \bigcup_{i\in I_j} 4B_{j,i}$.

(C) *The family $\{\frac{1}{2}B_{j,i}\}_{j\in\mathbb{N}_0,\, i\in I_j}$ is disjoint.*

(D) *For any $j \in \mathbb{N}_0$, I_j is an index set such that $\sharp I_j \lesssim_n (j+1)^n$ with the implicit positive constant depending only on n.*

Proof Set $A_0 \equiv B(1)$. For $j \in \mathbb{N}$, set $A_j \equiv B(a_j) \setminus B(a_{j-1})$. Then, $\mathbb{R}^n = \bigcup_{j=0}^{\infty} A_j$. For each $j \in \mathbb{N}_0$, let $\mathcal{A}_j = \{B_{j,i}\}_{i\in I_j}$, where I_j is an index set, be the collection of maximal balls with the following property:

(a) For any ball $B \in \mathcal{A}_j$, B is centered at the boundary of $B(a_j)$ and with radius $\frac{1}{a_j}$.

(b) Any two balls in $\mathcal{A}_j$ are disjoint.

(c) For any ball $B \notin \mathcal{A}_j$ but centered at the boundary of $B(a_j)$ and with radius $\frac{1}{a_j}$, there exists some ball $B' \in \mathcal{A}_j$ which intersects B.

The existence of such $\mathcal{A}_j$ is guaranteed by Zorn's lemma. With this setup, we establish that this collection satisfies the desired properties.

(A) Clearly, this is a consequence of (a).

(B) We distinguish two cases: $x \in A_0$ and $x \in A_j$ for some $j \in \mathbb{N}$.

For any $x \in A_0$ and $B \in \mathcal{A}_0$, since $|c(B)| = 1$ and $r(B) = 1$, we have $|x - c(B)| \le |x| + |c(B)| < 2 \le 4r(B)$, which implies that

$$A_0 \subset \bigcup_{i\in I_0} 4B_{0,i} \subset \bigcup_{j=0}^{\infty} \bigcup_{i\in I_j} 4B_{j,i}.$$

We now assume that $x \in A_j$ for some $j \in \mathbb{N}$; to establish (B) it suffices to show that there exists some ball $B \in \mathcal{A}_j$ such that $x \in 4B$. Indeed, if, for all balls $B \in \mathcal{A}_j$, we have $x \notin 4B$, then $|c(B)| = a_j \ge 1$, $r(B) = \frac{1}{a_j}$ and $|x - c(B)| \ge 4r(B) = \frac{4}{a_j}$. Consider the ball centered at $a_j \frac{x}{|x|}$ and with radius $\frac{1}{a_j}$, denoted below by B^*, namely, $B^* \equiv B(a_j \frac{x}{|x|}, \frac{1}{a_j})$. Note that $||x| - a_j| < a_j - a_{j-1} = \frac{1}{a_{j-1}}$ when $x \in A_j$, and that $\frac{2}{a_j} \ge \frac{1}{a_{j-1}}$. Since

$$|c(B) - z| \ge |c(B) - x| - \left|x - a_j \frac{x}{|x|}\right| - \left|a_j \frac{x}{|x|} - z\right|,$$

for any $z \in B^*$ and $B \in \mathcal{A}_j$, we have

$$|c(B) - z| \ge \frac{4}{a_j} - ||x| - a_j| - \frac{1}{a_j} \ge \frac{3}{a_j} - \frac{1}{a_{j-1}} \ge \frac{1}{a_j}.$$

Thus, B^* does not intersect any ball in $\mathcal{A}_j$, which contradicts (c). Thus, for all $j \in \mathbb{N}$, $A_j \subset \bigcup_{i\in I_j} 4B_{j,i} \subset \bigcup_{k=0}^{\infty} \bigcup_{i\in I_k} 4B_{k,i}$. This shows the property (B).

(C) By virtue of (a) and (b), together with the definition of A_j, we see that any ball B from $\mathcal{A}_j$ only intersects the balls in $\mathcal{A}_{j-1}$ or $\mathcal{A}_{j+1}$. If $B \in \mathcal{A}_j$ and $B' \in \mathcal{A}_{j-1} \bigcup \mathcal{A}_{j+1}$, by noticing that the width of the annulus $\{A_j\}_{j=1}^{\infty}$ is decreasing, we see that $\frac{1}{2}B \cap \frac{1}{2}B' = \emptyset$. Hence, we obtain the desired result.

(D) From (b) and a volume argument, it follows easily that $\sharp\mathcal{A}_j$ is bounded by a positive constant multiple of $|a_j|^{2n}$. That is, $\sharp I_j \lesssim j^n \sim (j+1)^n$ for $j \in \mathbb{N}$ by virtue of Lemma 262. Similarly, we see that $\sharp I_0 \lesssim 1$.

The Gauss measure γ is locally doubling in the following sense:

Lemma 265. *Let $C_{p,q,n} \equiv \frac{32}{7}\log(8^{1+\frac{npq}{p-q}})$. If a ball $B(x,r)$ satisfies that $\gamma(B(x,8r)) \le 8^{1+\frac{npq}{p-q}}\gamma(B(x,r))$, then either $|x| < 2r$ or $B(x,r) \in \mathcal{B}_{C_{p,q,n}}$.*

Proof It suffices to establish that $B(x,r) \in \mathcal{B}_{C_{p,q,n}}$ when $|x| \ge 2r$. If $x \in B(2)$, then $4m(x) \ge 2$, which together with the fact $|x| \ge 2r$ gives that $r \le 1 \le 2m(x)$. Thus, $B(x,r) \in \mathcal{B}_2 \subset \mathcal{B}_{C_{p,q,n}}$.

Now assume that $|x| \ge 2$. In this case, $m(x) = |x|^{-1}$. Notice that

$$\min_{y\in B(x,r)} |y|^2 = (|x|-r)^2, \qquad \max_{y\in B\left(x-\frac{3r}{2|x|}x,\ \frac{1}{4}r\right)} |y|^2 = \max\left\{\left(|x|-\frac{5}{4}r\right)^2\right\}$$

and that $B(x-\frac{3r}{2|x|}x,\ \frac{1}{4}r) \subset B(x,8r)$. Consequently, we have

$$\begin{aligned} 8^{1+\frac{npq}{p-q}}\pi^{-\frac{n}{2}}e^{-(|x|-r)^2}|B(x,r)| &\ge 8^{1+\frac{npq}{p-q}}\gamma(B(x,r)) \ge \gamma(B(x,8r)) \\ &\ge \gamma\left(B\left(x-\frac{3r}{2|x|}x,\ \frac{1}{4}r\right)\right) \ge \pi^{-\frac{n}{2}}e^{-(|x|-\frac{5}{4}r)^2}|B(x,r)|, \end{aligned}$$

which together with $r \le 2^{-1}|x|$ yields that

$$e^{\frac{7}{32}|x|r} \le e^{\frac{1}{4}r\left(2|x|-\frac{9}{4}r\right)} = e^{(|x|-r)^2-\left(|x|-\frac{5}{4}r\right)^2} \le 8^{1+\frac{npq}{p-q}}.$$

Consequently, $|x|r < \frac{32}{7}\log(8^{1+\frac{npq}{p-q}}) = C_{p,q,n}$. Hence, $r < C_{p,q,n}m(x)$. Thus $B(x,r) \in \mathcal{B}_{C_{p,q,n}}$, which completes the proof of Lemma 265.

We can generalize the above setting. Let (X,d,μ) be a metric measure space. A function $m : X \to (0,\infty)$ is called *admissible* if for any given $\tau \in (0,\infty)$, there exists a constant $\Theta_\tau \in [1,\infty)$ such that for all $x, y \in X$ satisfying $d(x,y) \le \tau m(x)$, it holds that

$$\Theta_\tau{}^{-1}m(y) \le m(x) \le \Theta_\tau\, m(y). \tag{6.22}$$

For any $a > 0$, denote by $\mathcal{B}_a$ the *set of all balls* $B \subset X$ *such that* $r(B) \leq am(c(B))$. Balls in $\mathcal{B}_a$ are referred to as *admissible balls* with scale a. If $r(B) = am(c(B))$, then B is called *maximal*.

Definition 68. The triple (X, d, μ) is called a *locally doubling measure metric space* if, for every $a > 0$, there exist constants $D_a \in (1, \infty)$ and $R_a \in (1, \infty)$ such that for all $B \in \mathcal{B}_a$,

$$\mu(2B) \leq D_a \mu(B) \tag{6.23}$$

and

$$\mu(2B) \geq R_a \mu(B). \tag{6.24}$$

Conditions (6.23) and (6.24) are respectively referred to as the *locally doubling condition* and the *locally reverse doubling condition*.

Example 108. A typical but non-trivial example of the locally doubling measure metric spaces is the *Gauss measure space* $(\mathbb{R}^n, |\cdot|, \gamma)$, where

$$\gamma(E) \equiv \pi^{-\frac{n}{2}} \int_E e^{-|x|^2} \mathrm{d}x.$$

For all $x \in \mathbb{R}^n$, let $m(x) \equiv \min\{1, |x|^{-1}\}$. As above, for any $a > 0$, we say that $B \in \mathcal{B}_a$ if and only if $r(B) \leq am(c(B))$. If $B \in \mathcal{B}_a$ and $x \in B$, then

$$(a+1)^{-1} m(x) \leq m(c(B)) \leq (a+1) m(x), \tag{6.25}$$

which implies that m is an admissible function in the sense of (6.22). For all $B \in \mathcal{B}_a$, we have $2B \in \mathcal{B}_{2a}$. Proposition 260 further implies that

$$\begin{aligned} \gamma(2B) &= \int_{2B} \pi^{-\frac{n}{2}} e^{-|x|^2} \mathrm{d}x \\ &\leq \pi^{-\frac{n}{2}} e^{-|c(B)|^2+4a} |2B| \\ &\leq 2^n e^{a^2+6a} \int_B \pi^{-\frac{n}{2}} e^{-|x|^2} \mathrm{d}x \\ &\leq 2^n e^{a^2+6a} \gamma(B), \end{aligned}$$

whenever $x \in B \in \mathcal{B}_a$. This shows that γ satisfies (6.23). Moreover, Proposition 266 below also shows that γ satisfies (6.24). Therefore, $(\mathbb{R}^n, |\cdot|, \gamma)$ is a locally doubling measure metric space.

Proposition 266. *In* $(\mathbb{R}^n, |\cdot|, \gamma)$*, for any* $a > 0$*, there exists* $C_a > 0$ *such that* $\gamma(2B) \geq (1 + C_a)\gamma(B)$ *for all balls* $B \in \mathcal{B}_a$*.*

Proof Suppose that the ball B is centered at $c(B)$ and with radius $r(B)$. We prove the conclusion by considering two cases: $|c(B)| \leq 2r(B)$ and $|c(B)| > 2r(B)$.

First, we consider the case $|c(B)| \le 2r(B)$. Since $B \in \mathcal{B}_a$, we have $r(B) \le am(c(B)) \le a$. For all $z \in 2B \setminus B$, by $|c(B)| \le 2r(B)$, we conclude that $|z| \le |z - c(B)| + |c(B)| < 4r(B) \le 4a$, which in turn implies that

$$\frac{\gamma(2B)}{\gamma(B)} \ge 1 + \frac{e^{-16a^2}|2B \setminus B|}{|B|} = 1 + e^{-16a^2}(2^n - 1). \tag{6.26}$$

Thus, we obtain the desired result in this case.

We now turn to the opposite case $|c(B)| > 2r(B)$. Notice that in this case $2B$ does not contain the origin. Consider the ball with the center $w \equiv \big(|c(B)| - \frac{3}{2}r(B)\big)\frac{c(B)}{|c(B)|}$ and radius $\frac{r(B)}{2}$. A geometric observation implies that $B\big(w, \frac{r(B)}{2}\big) \subset 2B \setminus B$. Moreover, for any $z \in B\big(w, \frac{r(B)}{2}\big)$, we have

$$|z| \le |z - w| + |w| < \frac{r(B)}{2} + |c(B)| - \frac{3}{2}r(B) = |c(B)| - r(B),$$

which implies that

$$\gamma\left(B\left(w, \frac{r(B)}{2}\right)\right) \ge \pi^{-\frac{n}{2}} e^{-(|c(B)|-r(B))^2}\left|B\left(w, \frac{r(B)}{2}\right)\right|.$$

Meanwhile, for all $z \in B$, we have $|z| \ge |c(B)| - |c(B) - z| > |c(B)| - r(B)$ and $\gamma(B) \le \pi^{-\frac{n}{2}} e^{-(|c(B)|-r(B))^2}|B|$. Thus

$$\frac{\gamma(2B)}{\gamma(B)} - 1 = \frac{\gamma(2B \setminus B)}{\gamma(B)} \ge \frac{\gamma(B(w, \frac{r(B)}{2}))}{\gamma(B)} \ge \frac{|B(w, \frac{r(B)}{2})|}{|B|} = \frac{1}{2^n}. \tag{6.27}$$

Thus, we obtain the desired result in this case.

We then see from (6.26) and (6.27) that $\gamma(2B) \ge (1 + 2^{-n})\gamma(B)$ for all $B \in \mathcal{B}_a$. This finishes the proof of Proposition 266.

6.1.6 Exercises

Exercise 84. Prove (6.1) by integration by parts.

Exercise 85. Let $a > 0$. Use $|x - c(B)| \le a \min(1, |x|^{-1})$ for all $x \in B$ such that B is a ball in $\mathcal{B}_a$ to show $-a^2 - 2a \le |c(B)|^2 - |x|^2 \le 2a$.

Exercise 86. Let $a > 0$ and define $m(x) \equiv \min(1, |x|^{-1})$. Write $b \equiv \frac{a}{a+1}$.

(1) Prove that γ given by (6.17) is a probability measure.

(2) Let B be a maximal ball with $c(B) \in B(1 + a)$.

   (a) Show that $r(B) \ge b$.

   (b) Show that $\gamma(B) \ge \omega_n \pi^{-\frac{n}{2}} e^{-2a^2-4a-1} b^n$.

   (c) Show that $\gamma(B) \sim |B| \exp(-|c(B)|^2)$ with the implicit constant dependent on a.

Exercise 87.

(1) Prove (6.11) using Theorem 254.

(2) Prove (6.12). Hint: We may suppose $\operatorname{supp}(f) \subset Q \subset \mathcal{D}(\mu)$. Then find $\chi_{\mathbb{R}^n \setminus Q} M^{\mathcal{D}(\mu)} f$.

Exercise 88. Prove Theorem 251 using Theorems 247 and 250.

Exercise 89. Let $f \in L^0(\mu)$, where μ is a Radon measure on $\mathbb{R}^n$. Then show that $|f(x)| \le k^n M_{k,uc} f(x)$ for almost every $x \in \mathbb{R}^n$ by considering k-doubling cubes.

6.2 Maximal operators on metric measure spaces with general Radon measures

Keeping in mind what we did in Section 6.1, we work on a metric measure space (X, d, μ). A triple (X, d, μ) is a *metric measure space* if (X, d) is a metric space and μ is a Borel measure, where Borel sets in (X, d) are defined as the σ-algebra generated by open sets. Then μ is *doubling*, if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all $x \in X$ and $r > 0$. When we work on a metric measure space, we need to generalize the definition of M. A natural candidate of this definition is as follows:

$$M' f(x) \equiv \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \mathrm{d}\mu(y) \quad (x \in X)$$

and

$$Mf(x) \equiv \sup_{z \in X, r>0} \frac{\chi_{B(z,r)}(x)}{\mu(B(z,r))} \int_{B(z,r)} |f(y)| \mathrm{d}\mu(y) \quad (x \in X).$$

The former is the centered maximal operator, while the latter is the uncentered maximal operator. Section 6.2.1 considers the boundedness of the maximal operator. What differs from Section 6.1.2 is the lack of the strong covering lemma. We have to use the $5r$-covering lemma instead. Sections 6.2.1 and 6.2.2 parallel Section 6.1.2. We collect some estimates on the modified maximal operators. In Section 6.2.3 with an example we explain that the modification considered in Section 6.2 is actually necessary.

6.2.1 Weak-$(1,1)$ estimate and strong-(p,p) estimate

Section 6.2.1, considers the modified maximal operator in the separable metric space (X, d, μ) such that all balls are non-degenerate, in the sense that

$\mu(B(x,r)) > 0$ for all $x \in X$ and $r > 0$. Define the *centered modified maximal operator*

$$M_{k,\mathrm{c}} f(x) = M_{k,\mathrm{c},\mu} f(x) \equiv \sup_{r>0} \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} |f(y)| d\mu(y)$$

and the *uncentered modified maximal operator*

$$M_{k,\mathrm{uc}} f(x) = M_{k,\mathrm{uc},\mu} f(x) \equiv \sup_{y\in X, r>0} \frac{\chi_{B(y,r)}(x)}{\mu(B(y,kr))} \int_{B(y,r)} |f(z)| d\mu(z).$$

We will see how large k must be in order that we have the boundedness of $M_{k,\mathrm{c}}$. Section 6.2.1 investigates the property of the centered Hardy–Littlewood maximal operator. We will omit the analysis of the uncentered Hardy–Littlewood maximal operator. In general metric measure spaces, the following theorem is fundamental.

Theorem 267. *Let (X,d) be a separable metric space. Let μ be a Radon measure such that $\mu(B(x,r)) < \infty$. Then $M_{2,\mathrm{c}}$ is weak-$(1,1)$ bounded and the weak-$(1,1)$ constant is less than or equal to* 1:

$$\mu\{x \in X : M_{2,\mathrm{c}} f(x) > \lambda\} \le \frac{1}{\lambda} \int_X |f(x)| d\mu(x)$$

for all $f \in L^0(\mu)$.

Since $M_{2,\mathrm{c}} \ge M_{3,\mathrm{uc}}$, we have a similar assertion for $M_{3,\mathrm{uc}}$. Sometimes one abbreviates $\{x \in X : M_{2,\mathrm{c}} f(x) > \lambda\}$ to $\{M_{2,\mathrm{c}} f > \lambda\}$. Since the proof of Theorem 267 uses a delicate covering lemma, we postpone its proof.

Recall that the Radon property guarantees that any measurable set can be approximated from inside by compact sets.

Remark 12. Let $k > 0$. By the Radon property, we can replace $B(x,r)$ by $\overline{B}(x,r)$ in the definition of $M_{k,\mathrm{c}}$. Thus, for $f \in L^0(\mu)$ we have $M_{k+\delta} f(x) \uparrow M_{k,\mathrm{c}} f(x)$ as $\delta \downarrow 0$.

The next covering lemma refines Vitali's covering lemma, which leads us to obtain the weak-$(1,1)$ boundedness of $M_{2,\mathrm{c}}$.

To prove Theorem 267, we need to refine the "so called" $5r$-covering lemma.

Lemma 268. *Suppose that we have a finite family $\{B(x_j,r_j)\}_{j\in\{1,2,\dots,N\}}$ of balls. Then for $\delta > 0$, we can take a subfamily $\{B(x_j,r_j)\}_{j\in A}$ such that*

(1) *$\{B(x_j,r_j)\}_{j\in A}$ is disjoint,*

(2) $\bigcup_{j=1}^{N} B(x_j, \delta r_j) \subset \bigcup_{j\in A} B(x_j, (2+\delta) r_j)$.

This extends Vitali's covering lemma: The lemma is precisely Vitali's covering lemma if $\delta = 1$.

Proof We select $j_1 \in \{1, 2, \dots, N\}$ so that $r_{j_1} = \max\{r_1, r_2, \dots, r_N\}$. If

$$\bigcup_{j\in\{1,2,\dots,N\}} B(x_j, \delta r_j) \subset B(x_{j_1}, (2+\delta) r_{j_1}),$$

we have nothing else to do. Let us assume otherwise in the sequel. We define

$$\Lambda_1 \equiv \{j \in \{1, 2, \dots, N\} \,:\, B(x_j, \delta r_j) \subset B(x_{j_1}, (2+\delta) r_{j_1})\}.$$

We inductively define the subsets of $\{1, 2, \dots, N\}$ and $j_1, j_2, \dots, j_p \in \{1, 2, \dots, N\}$ as follows:
Let $q \geq 2$. Suppose that $j_1, j_2, \dots, j_{q-1}$ and the subsets $\Lambda_1, \Lambda_2, \dots, \Lambda_{q-1} \subset \{1, 2, \dots, N\}$ are defined. Then we define

$$\Lambda_q \equiv \left\{ j \in \{1, 2, \dots, N\} \,:\, B(x_j, \delta r_j) \cap \bigcup_{k=j_1}^{j_{q-1}} B(x_j, (2+\delta) r_j) \neq B(x_j, \delta r_j) \right\}.$$

Next, we take $j_q \in \Lambda_q$ so that $r_{j_q} = \max_{j\in\Lambda_q} r_j$ and This procedure will terminate because we are dealing with the finite number of the balls. Suppose we have stopped in the p-th step after we selected j_p and Λ_p. We will verify that $A \equiv \{j_1, j_2, \dots, j_p\}$ satisfies all requirements of the lemma.

To verify this we fix $j \in \{1, 2, \dots, N\}$. We have three possibilities.

(*a*) $j \in \{j_1, j_2, \dots, j_p\}$.

(*b*) $r_j = r_{j_p}$ and $j \notin \{j_1, j_2, \dots, j_p\}$.

(*c*) $r_{j_k} \geq r_j > r_{j_{k+1}}$ for some $k \in \{1, 2, \dots, p-1\}$ and $j \notin \{j_1, j_2, \dots, j_p\}$.

We want to show that $B(x_j, \delta r_j) \subset \bigcup_{j\in A} B(x_j, (2+\delta) r_j)$; once this is achieved, we have (2). If (*a*) happens, this inclusion is clear from the definition of $j_1, j_2, \dots, j_p$. Assume (*c*) in the sequel. Another possibility (*b*) can be dealt with in a similar manner. A geometric observation together with (*c*) shows that $j \notin \Lambda_k$ and hence $B(x_j, \delta r_j) \subset \bigcup_{k\in\{j_1,j_2,\dots,j_k\}} B(x_k, (2+\delta) r_k)$ by the definition of Λ_k and $r_{j_{k+1}}$. Thus, our claim is justified.

It remains to show that the balls $\{B(x_j, r_j)\}_{j\in A}$ are disjoint. Indeed suppose $k < k'$, so that we have

$$r_{j_k} \geq r_{j_{k'}}. \tag{6.1}$$

By the definition of Λ_j, we have $B(x_{j_{k'}}, \delta r_{j_{k'}})$ is not contained in $B(x_{j_k}, (2+\delta) r_{j_k})$, since $j_k < j_{k'}$. Thus $x_{j_{k'}} \notin B(x_{j_k}, 2r_{j_k})$. This implies

$$d(x_{j_k}, x_{j_{k'}}) \geq 2r_{j_k}. \tag{6.2}$$

Combining (6.1) and (6.2), we conclude that $\{B(x_j, r_j)\}_{j\in A}$ is disjoint. So, (1) is verified.

We now prove Theorem 267.

Proof Fix $\lambda > 0$. We define $E_k \equiv \{x \in X : M_{k,\mathrm{c}} f(x) > \lambda\}$ for $k > 2$. By Remark 12, it follows that

$$\bigcup_{k>2} \{x \in X : M_{k,\mathrm{c}} f(x) > \lambda\} = \{x \in X : M_{2,\mathrm{c}} f(x) > \lambda\}. \tag{6.3}$$

Let $\delta > 0$ and $k \equiv 2 + \delta$. For all $x \in E_k$, by its definition there exists $r_x > 0$ such that $\displaystyle\int_{B(x,r_x)} |f(y)| d\mu(y) > \lambda \mu(B(x, kr_x))$. Since μ is a Radon measure, E_k is an open set. Since X is separable, with the aid of the Linderöf covering theorem, asserting that any covering of X has its countable subcovering as long as X is separable, we can take $x_j \in E_k, j = 1, 2, \ldots$ such that $E_k \subset \bigcup_{j=1}^{\infty} B(x_j, \delta r_{x_j})$.

We fix $N \in \mathbb{N}$. By Lemma 268 there exists a subset $A \subset \{1, 2, \ldots, N\}$ such that

$$\bigcup_{j=1}^{N} B(x_j, \delta r_{x_j}) \subset \bigcup_{l\in A} B(x_l, (2+\delta) r_{x_l}) \tag{6.4}$$

and that

$$\sum_{l\in A} \chi_{B(x_l, r_{x_l})} \le 1. \tag{6.5}$$

By the definition of E_k, we also have

$$\mu(B(x_l, (2+\delta) r_{x_l})) \le \frac{1}{\lambda} \int_{B(x_l, r_{x_l})} |f(x)| d\mu(x) \tag{6.6}$$

for $l \in A$.

Combining (6.4)–(6.6) gives

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{N} B(x_j, \delta r_{x_j})\right) &\le \mu\left(\bigcup_{l\in A} B(x_l, (2+\delta) r_{x_l})\right) \\ &\le \sum_{l\in A} \mu(B(x_l, (2+\delta) r_{x_l})) \\ &\le \sum_{l\in A} \frac{1}{\lambda} \int_{B(x_l, r_{x_l})} |f(x)| d\mu(x) \\ &\le \frac{1}{\lambda} \|f\|_{L^1(\mu)}. \end{aligned} \tag{6.7}$$

Letting N tend to infinity in (6.7), we derive $\lambda \mu(E_k) \le \|f\|_{L^1(\mu)}$. Tending $k \downarrow 2$, we conclude the proof of Theorem 267.

If we interpolate between Theorem 267 and an estimate $\|M_{2,\mathrm{c}}f\|_{L^\infty(\mu)} \le \|f\|_{L^\infty(\mu)}$, we obtain the following:

Theorem 269. *Let $p > 1$ and (X, d, μ) be a separable metric measure space. Then $\|M_{2,\mathrm{c}}f\|_{L^p(\mu)} \le 2\,(p')^{\frac{1}{p}}\,\|f\|_{L^p(\mu)}$ for all $f \in L^0(\mu)$.*

Proof The technique of the proof is known as the interpolation. The proof is similar to Theorem 140. See Exercise 91.

6.2.2 Vector-valued boundedness of the Hardy–Littlewood maximal operators

As an application of Lemma 268, we will prove the following vector-valued inequality for the Hardy–Littlewood maximal operator. It turns out that $k \ge 22$ suffices as the following theorem shows:

Theorem 270. *If $1 < p < \infty$ and $1 < q \le \infty$, then*

$$\left\| \|\{M_{22,\mathrm{c}}f_j\}_{j=1}^\infty\|_{\ell^q} \right\|_{L^p(\mu)} \lesssim_{p,q} \left\| \|\{f_j\}_{j=1}^\infty\|_{\ell^q} \right\|_{L^p(\mu)}. \tag{6.8}$$

When $q = \infty$, (6.8) reads;

$$\left\| \sup_{j\in\mathbb{N}} M_{22,\mathrm{c}}f_j \right\|_{L^p(\mu)} \lesssim_p \left\| \sup_{j\in\mathbb{N}} |f_j| \right\|_{L^p(\mu)}. \tag{6.9}$$

We leave the proof of Theorem 270 to interested readers, since it is similar to the classical case; see Exercise 93. If $p > q$, then we can say a little more; see Theorem 274.

First of all, to prove Theorem 270, we obtain a weighted inequality of Stein-type by using again Lemma 268.

Proposition 271 (Stein's weak dual inequality)**.** *For all $f, g \in L^0(\mu)$ and $\lambda > 0$,*

$$\int_{\{M_{7,\mathrm{c}}f > \lambda\}} |g(x)|d\mu(x) \le \frac{1}{\lambda}\int_X |f(x)|M_{2,\mathrm{c}}g(x)d\mu(x).$$

Proof Fix $k > 7$ keeping in mind Remark 12. Let $E = \{x \in X : M_{k,\mathrm{c}}f > \lambda\}$. By the definition of E, for all $x \in E$, there exists $r_x > 0$ such that

$$\frac{1}{\mu(B(x, kr_x))}\int_{B(x,r_x)} |f(x)|d\mu(x) > \lambda.$$

We will take $a \equiv (k-7)/2$. Since E is an open set, again by the Linderöf covering theorem, there exist $x_j, j = 1, 2, \dots$ such that $E \subset \bigcup_{j=1}^\infty B(x_j, ar_{x_j})$.

Fix $N \in \mathbb{N}$ arbitrarily. We claim that

$$\int_{\bigcup_{j=1,2,\dots,N} B(x_j, ar_{x_j})} |g(x)|d\mu(x) \le \frac{1}{\lambda}\int_X |f(x)|M_{2,\mathrm{c}}g(x)d\mu(x). \tag{6.10}$$

By Lemma 268 there exists a subfamily of balls $\{B(x_j, ar_{x_j})\}_{j\in J}$ with $J \subset \{1, 2, \dots, N\}$ that satisfies the following properties.

$$\sum_{j\in J} \chi_{B(x_j, r_{x_j})} \le 1, \tag{6.11}$$

$$\bigcup_{j=1}^{N} B(x_j, ar_{x_j}) \subset \bigcup_{j\in J} B(x_j, (2+a)r_{x_j}). \tag{6.12}$$

A geometric observation shows, for all $x \in B(x_j, r_{x_j})$,

$$B(x_j, kr_{x_j}) \subset B(x, 3(2+a)r_{x_j}).$$

In fact, for all $y \in B(x, (2+a)r_{x_j})$,

$$d(x_j, y) \le d(x, x_j) + d(y, x) \le r_{x_j} + 3(2+a)r_{x_j} = (7+2a)r_{x_j} = kr_{x_j}.$$

Consequently, for all $x \in B(x_j, r_{x_j})$,

$$\begin{aligned} &\frac{1}{\mu(B(x_j, kr_{x_j}))} \int_{B(x_j,(2+a)r_{x_j})} |g(y)|d\mu(y) \\ &\quad\le \frac{1}{\mu(B(x, (6+2a)r_{x_j}))} \int_{B(x,(3+a)r_{x_j})} |g(y)|d\mu(y) \\ &\quad\le M_{2,\mathrm{c}}g(x). \end{aligned} \tag{6.13}$$

Using (6.11)–(6.13), we obtain

$$\begin{aligned} &\int_{\bigcup_{j=1,2,\dots,N} B(x_j, ar_{x_j})} |g(x)|d\mu(x) \\ &\quad\le \int_{\bigcup_{j\in J} B(x_j,(2+a)r_{x_j})} |g(x)|d\mu(x) \le \sum_{j\in J} \int_{B(x_j,(2+a)r_{x_j})} |g(x)|d\mu(x) \\ &\quad\le \frac{1}{\lambda} \sum_{j\in J} \int_{B(x_j,r_{x_j})} |f(x)|d\mu(x) \cdot \frac{1}{\mu(B(x_j, kr_{x_j}))} \int_{B(x_j,(2+a)r_{x_j})} |g(x)|d\mu(x) \\ &\quad\le \frac{1}{\lambda} \sum_{j\in J} \int_{B(x_j,r_{x_j})} |f(x)|M_{2,\mathrm{c}}g(x)d\mu(x) \le \frac{1}{\lambda} \int_X |f(x)|M_{2,\mathrm{c}}g(x)d\mu(x). \end{aligned}$$

Thus, (6.10) is proven. By the definition of E we have

$$\int_{\{M_{k,\mathrm{c}}f>\lambda\}} |g(x)|d\mu(x) \le \frac{1}{\lambda} \int_X |f(x)|M_{2,\mathrm{c}}g(x)d\mu(x).$$

Letting $k \downarrow 7$, we finally obtain Proposition 271. Thus, we have finished.

By interpolation, we obtain the strong inequality.

Corollary 272 (Stein's strong dual inequality)**.** *If $p > 1$ and $f \in L^0(\mu)$, then* $\int_X (M_{7,\mathrm{c}} f(x))^p |g(x)| d\mu(x) \le 2^p p' \int_X |f(x)|^p M_{2,\mathrm{c}} g(x) d\mu(x)$.

Proof For a positive function w, $\|\cdot\|_{L^\infty(w\mathrm{d}\mu)}$ denotes the $L^\infty(\mu)$-norm of functions with respect to the weighted measure $w\mathrm{d}\mu$. Since

$$\|M_{7,\mathrm{c}} f\|_{L^\infty(|g|\mathrm{d}\mu)} \le \|f\|_{L^\infty(M_{2,\mathrm{c}} g\mathrm{d}\mu)} \tag{6.14}$$

is clear (see Exercise 90(2)), this is again just a matter of interpolation of (6.14) and Proposition 271.

We use the following analogy, which is used below to obtain Theorem 270. The proof is only the adaptation of the parameter k of Proposition 271 and Corollary 272 respectively; we omit its proof.

Proposition 273. *Let $f, g \in L^0(\mu)$.*

(a) *The estimate* $\int_{\{M_{22,\mathrm{c}} f > \lambda\}} |g(x)| d\mu(x) \le \frac{1}{\lambda} \int_X |f(x)| M_{7,\mathrm{c}} g(x) d\mu(x)$ *holds for all $\lambda > 0$.*

(b) *If $p > 1$, then* $\int_X (M_{22,\mathrm{c}} f(x))^p |g(x)| d\mu(x) \lesssim_p \int_X |f(x)|^p M_{7,\mathrm{c}} g(x) d\mu(x)$.

At last we are in the position of proving Theorem 270. If $p \ge q > 1$, a little more can be said. We have the following:

Theorem 274. *If $p \ge q > 1$, then for all sequences $\{f_j\}_{j=1}^\infty \subset L^0(\mu)$,*

$$\left\| \|\{M_{7,\mathrm{c}} f_j\}_{j=1}^\infty\|_{\ell^q} \right\|_{L^p(\mu)} \lesssim_{p,q} \left\| \|\{f_j\}_{j=1}^\infty\|_{\ell^q} \right\|_{L^p(\mu)}. \tag{6.15}$$

Proof The proof is a reexamination of the proof of Theorem 145. See Exercise 92.

6.2.3 Examples of metric measure spaces which require modification

So far, k needs to be greater than or equal to 2 in order that $M_{k,\mathrm{c}}$ is bounded. We will show by an example that the condition $k \ge 2$ is optimal for the centered modified maximal operator.

Example 109. Let $l \in \mathbb{N}$ and $j \in \{1, 2, \dots, l\}$. Consider the graph X with vertices x_l and x_{lj} and edges joining x_l with x_{l+1} and x_l with each x_{lj}. Let μ be a counting measure on X; $\mu(E) = \sharp E$. Consider (X, d, μ) as a metric measure space. We will disprove that $M_{k,\mathrm{c}}$ is weak-$(1,1)$ bounded for any $0 < k < 2$.

Fix $l \in \mathbb{N} \cap [2, \infty)$ and $j \in \{1, 2, \dots, l\}$. Then $B(x_{lj}, kr) = \{x_l, x_{lj}\}$ for $r \in (1, 2 \cdot k^{-1})$. Note that δ_a, the Dirac delta massed at any $a \in X$, is a

function defined in X. We have $M_{k,\mathrm{c}}\delta_{x_l}(x_{lj}) = \dfrac{1}{2}$ for any $j = 1, 2, \ldots, l$. We also have $M_{k,\mathrm{c}}\delta_{x_l}(x_{l\pm1}) = \dfrac{1}{2}$ and $M_{k,\mathrm{c}}\delta_{x_l}(x_l) = 1$. For any other point $z \in X$, $M_{k,\mathrm{c}}\delta_{x_l}(z) = 0$. Consequently, $\{x \in X : 2M_{k,\mathrm{c}}\delta_{x_l}(x) \geq 1\}$ is made up of $l+3$ points. Since the μ-measure of this set is $l+3$ and δ_{x_l} has the $L^1(\mu)$-norm 1, we see that $M_{k,\mathrm{c}}$ fails to be weak-$(1,1)$ bounded. Likewise, we see that $M_{k,\mathrm{c}}$ fails to be strong-(p,p) bounded for any $1 < p < \infty$.

We will show by an example that the condition $k \geq 3$ is optimal for the uncentered modified maximal operator.

Example 110. Let positive integers l and j move over the set $\{(l,j) \in \mathbb{N}^2 : j \leq l\}$. Consider the graph X with vertices x_l, x_{lj} and y_{lj} and edges joining x_l with x_{l+1}, y_{lj} with x_{lj} and x_l with x_{lj}. Let μ_0 be a counting measure on X as before. Define

$$w(x) \equiv \sum_{l=1}^{\infty} \chi_{\{x_l, y_{l1}, y_{l2}, \ldots, y_{ll}\}} + \sum_{l=1}^{\infty} \frac{1}{l} \chi_{\{x_{l1}, x_{l2}, \ldots, x_{ll}\}}.$$

Set $\mu \equiv w\mu_0$, the weighted measure. Consider (X, d, μ) as a metric measure space. We will disprove that $M_{k,\mathrm{uc}}$ is weak-$(1,1)$ bounded for any $0 < k < 3$. If $r \in (1, 3 \cdot k^{-1})$, $l \in \mathbb{N} \cap [3, \infty)$ and $j = 1, 2, \ldots, l$, then

$$B(x_{lj}, kr) = \{x_l, x_{l-1}, x_{l+1}, y_{lj}\} \cup \{x_{lm}\}_{m=1}^{l}.$$

Thus, $M\delta_{x_l}(y_{lm}) \geq \dfrac{1}{5}$ for any $m = 1, 2, \ldots, l$. Thus, as before $M_{k,\mathrm{uc}}$ fails to be weak-$(1,1)$ bounded and strong-$(1,1)$ bounded for any $1 < p < \infty$.

We will present another example of compact metric measure spaces. First, we define a set X on which we work.

Definition 69 (Definition of X)**.** Define a set X as follows:

(1) One writes $\Delta(z, r) \equiv \{w \in \mathbb{C} : |w - z| < r\}$.

(2) Let $A_k \equiv \{z \in \mathbb{C} : |z| = 3^{-k}\}$ for $k \in \mathbb{N}_0$.

(2) Define $X_0 \equiv \{0\} \cup \bigcup_{k=0}^{\infty} A_k \subset \mathbb{C}$.

(3) Let $X \equiv {X_0}^{\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ be the cross product.

(4) Let $\mathbf{O} \equiv (0, 0, \ldots) \in X$.

We draw graphs of A_0 and X_0.
Note that A_0 is an annulus and X_0 is the union and X is a countable product of X_0.

Here and below, until the end of this section, we adopt the following rules:

(1) The letter z without subindex denotes the point in $\mathbb{C}$.

(2) Points in X are written in bold letters such as $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

(3) Symbols such as x_j, y_j, z_j and so on are complex numbers and they denote the j-th component of elements in X.

The point is that we give a "singular" metric on X. The precise definition is as follows:

(1) The integer
$$N_0 \tag{6.16}$$
is chosen so that $\log_3 N_0$ is a big integer.

(2) Denote by $[\cdot]$ a Gauss symbol and define $N(\delta) \equiv \max(1, [\log_{N_0} \delta^{-1}])$ for $\delta > 0$.

(3) Let $\mathbf{x} = \{x_j\}_{j=1}^{\infty}$ and $\mathbf{y} = \{y_j\}_{j=1}^{\infty}$ be points in X, and define the distance $d(\mathbf{x}, \mathbf{y})$ of $\mathbf{x}$ and $\mathbf{y}$ by
$$d(\mathbf{x}, \mathbf{y}) \equiv \inf\{\delta > 0 : |x_j - y_j| \leq \delta \text{ for all } j \leq N(\delta)\}.$$

(4) One defines a sphere S_k by
$$S_k \equiv (A_k)^{N(3^{-k})} \times X_0 \times X_0 \times \cdots \tag{6.17}$$
for each $k \in \mathbb{N}$.

At this moment, about the definition of the natural number N_0, we keep in mind that $N \in \mathbb{M}^{\downarrow}(0, \infty)$.

Note that
$$d(\mathbf{x}, \mathbf{y}) = \inf\{\delta > 0 : |x_1 - y_1| \leq \delta, |x_2 - y_2| \leq \delta, \ldots, |x_{N(\delta)} - y_{N(\delta)}| \leq \delta\}$$
and that
$$S_k = \{\mathbf{z} = (z_1, z_2, \ldots) \in {X_0}^{\mathbb{N}} : |z_1| = |z_2| = \cdots = |z_{N(3^{-k})}| = 3^{-k}\}.$$
Thus, $\operatorname{diam}(X) \equiv \sup_{x,y \in X} d(x, y) = 2$. Let us check that d is a metric function and that $\chi_{S_k} \in L^0(\mu)$.

Lemma 275. *The function d is a metric function. That is,*
$$0 \leq d(\mathbf{x}, \mathbf{y}) < \infty \quad (\mathbf{x}, \mathbf{y} \in X), \tag{6.18}$$
$$d(\mathbf{x}, \mathbf{y}) = 0 \Longrightarrow \mathbf{x} = \mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in X), \tag{6.19}$$
$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \quad (\mathbf{x}, \mathbf{y} \in X), \tag{6.20}$$
$$d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \quad (\mathbf{x}, \mathbf{y}, \mathbf{z} \in X). \tag{6.21}$$

Furthermore, the d-topology is exactly the product topology of X.

Proof Since $2 \in \{\delta > 0 : |x_j - y_j| \le \delta \text{ for all } j \le N(\delta)\}$, (6.18) is clear. If $\mathbf{x} \neq \mathbf{y}$, then $|x_{j_0} - y_{j_0}| > \delta$ for some $\delta > 0$ and $j_0 \in \mathbb{N}$. Therefore, if we choose $\delta^* > 0$ so that $N(\delta^*) > j_0$, then we have $|x_{j_0} - y_{j_0}| > \min(\delta, \delta^*)$ and $j_0 < N(\min(\delta, \delta^*))$. This implies

$$\min(\delta, \delta^*) \notin \{r > 0 : |x_1 - y_1| \le r, |x_2 - y_2| \le r, \dots, |x_{N(r)} - y_{N(r)}| \le r\}.$$

Hence, $d(\mathbf{x}, \mathbf{y}) \ge \min(\delta, \delta^*)$, which shows (6.19). Equality (6.20) follows immediately from the definition of d. Next, we check (6.21). To this end, we take $\varepsilon > 0$. Then by the definition of $d(\mathbf{x}, \mathbf{y})$ and $d(\mathbf{y}, \mathbf{z})$, we can find $\delta_1 \in (d(\mathbf{x}, \mathbf{y}), d(\mathbf{x}, \mathbf{y}) + \varepsilon)$ and $\delta_2 \in (d(\mathbf{y}, \mathbf{z}), d(\mathbf{y}, \mathbf{z}) + \varepsilon)$ so that $|x_j - y_j| \le \delta_1$ for all $j \le N(\delta_1)$ and that $|y_j - z_j| \le \delta_2$ for all $j \le N(\delta_2)$. Note that $N(\delta_1 + \delta_2) \le \min(N(\delta_1), N(\delta_2))$, we have $|x_j - y_j| \le \delta_1$, $|y_j - z_j| \le \delta_2$ for all $j \le N(\delta_1 + \delta_2)$. Hence $|x_j - z_j| \le \delta_1 + \delta_2$ for all $j \le N(\delta_1 + \delta_2)$. Consequently, $d(\mathbf{x}, \mathbf{z}) \le \delta_1 + \delta_2 \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain (6.21).

Finally, we investigate the relation between two topologies. Since any d-open set is open with respect to the product topology, the product topology is not weaker than the d-topology. However, we can express Δ as a union of d-open balls as long as X is given by

$$X = \Delta(z_1, r_1) \times \Delta(z_2, r_2) \times \cdots \times \Delta(z_{\tilde{N}}, r_{\tilde{N}}) \times X_0 \times X_0 \times \cdots$$

with some $\tilde{N} \in \mathbb{N}$, $(z_1, z_2, \dots, z_{\tilde{N}}) \in {X_0}^{\tilde{N}}$ and $(r_1, r_2, \dots, r_{\tilde{N}}) \in (0, \infty)^{\tilde{N}}$. Therefore, two topologies coincide.

Lemma 276. *Let $r > 0$ and $\mathbf{x} = (x_1, x_2, \dots) \in X$. Then*

$$B(\mathbf{x}, r) = \Delta(x_1, r) \times \Delta(x_2, r) \times \cdots \times \Delta(x_{N(r)}, r) \times X_0 \times X_0 \times \cdots .$$

In particular, $\mu(B(\mathbf{x}, r)) = \prod_{j=1}^{N(r)} \mu_0(\Delta(x_j, r))$.

Proof From the definition of the open ball, we have

$$\begin{aligned} B(\mathbf{x}, r) &= \{\mathbf{y} = \{y_j\}_{j=1}^{\infty} \in X : d(\{y_j\}_{j=1}^{\infty}, \{x_j\}_{j=1}^{\infty}) < r\} \\ &= \bigcup_{\delta \in (0,r)} \bigcap_{j=1}^{N(\delta)} \{\mathbf{y} = \{y_j\}_{j=1}^{\infty} \in X : |y_j - x_j| \le \delta\}. \end{aligned}$$

Since $N : (0, \infty) \to \mathbb{R}$ is left-continuous and assumes its value in $\mathbb{Z}$, if δ is slightly less than r, $N(\delta) = N(r)$. Together with the monotonicity of the most-right hand side of the above equality, we conclude

$$\begin{aligned} &B(\mathbf{x}, r) \\ &= \{\mathbf{y} = \{y_j\}_{j=1}^{\infty} \in X : |y_1 - x_1| < r, |y_2 - x_2| < r, \dots, |y_{N(r)} - x_{N(r)}| < r\}. \end{aligned}$$

Consequently, Lemma 276 was proven.

The measure is given by way of the product: Let $\mathcal{H}^1$ denote the 1-dimensional Hausdorff measure.

(1) One defines a function $w_0 : X_0 \to [0, \infty)$ by $w_0 \equiv \gamma \sum_{k=0}^{\infty} \frac{1}{(k!)^k} \chi_{A_k}$, where γ is chosen so that $\int_{X_0} w_0(z)\, d\mathcal{H}^1(z) = 1$.

(2) Define a measure on X_0 by $\mu_0 \equiv w_0\, d\mathcal{H}^1$.

(3) Define a measure μ on X by $\mu \equiv \mu_0 \times \mu_0 \times \cdots = w_0\, d\mathcal{H}^1 \times w_0\, d\mathcal{H}^1 \times \cdots$.

As for the measures μ and μ_0, we prove the following estimate:

Lemma 277. *For $k \in \mathbb{N}_0$, $\mu_0(A_k)$ and $\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right)$ are comparable in the following sense*:

$$\frac{2\pi\gamma}{(3 \cdot k!)^k} = \mu_0(A_k) \le \mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \le \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \frac{1}{\gamma \cdot (k+1)!}\right).$$

Proof From the definition of μ_0, we see

$$\mu_0(A_k) = \int_{A_k} w_0(z)\, d\mathcal{H}^1(z) = \frac{\gamma}{(k!)^k} \int_{|z|=3^{-k}} d\mathcal{H}^1(z) = \frac{2\pi\gamma}{(3 \cdot k!)^k}.$$

Hence

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) = \sum_{j=k}^{\infty} \frac{2\pi\gamma}{(3 \cdot j!)^j} = \frac{2\pi\gamma}{(3 \cdot k!)^k} \sum_{j=k}^{\infty} \frac{(3 \cdot k!)^k}{(3 \cdot j!)^j}. \tag{6.22}$$

This observation yields the lower bound for $\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right)$. It remains to obtain the upper bound for $\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right)$.

First of all, let us assume that $k = 0$. Note that $\mu_0(A_0) = 2\pi\gamma$. Hence, when $k = 0$, from (6.22), we deduce

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) = \mu_0\left(\bigcup_{j=0}^{\infty} A_j\right) = 1 \le 2\pi\gamma\left(1 + \frac{1}{\gamma}\right).$$

Note that $\mu_0(A_1) = \frac{2\pi\gamma}{3}$. Hence, when $k = 1$, again from (6.22), we deduce

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \le \mu_0\left(\bigcup_{j=0}^{\infty} A_j\right) = 1 \le \frac{2\pi\gamma}{3}\left(1 + \frac{1}{2\gamma}\right).$$

Let $k \in \mathbb{N} \cap [2, \infty)$ below. Using (6.22), we calculate that

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \leq \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \sum_{j=k+1}^{\infty} \frac{(3 \cdot k!)^k}{(3 \cdot j!)^k (3 \cdot j!)}\right).$$

If $j \geq k+1$, then $k! \times j \leq j!$. Thus, we have

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \leq \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \sum_{j=k+1}^{\infty} \frac{1}{3 \cdot j^k (k+1)!}\right).$$

Note that $\gamma \in (0,1)$ and that

$$\sum_{j=k+1}^{\infty} \frac{1}{j^k} \leq \sum_{j=2}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} - 1 < 1$$

for all $k \in \mathbb{N} \cap [2, \infty)$. Thus, we obtain

$$\mu_0\left(\bigcup_{j=k}^{\infty} A_j\right) \leq \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \frac{1}{(k+1)!}\right) \leq \frac{2\pi\gamma}{(3 \cdot k!)^k}\left(1 + \frac{1}{\gamma \cdot (k+1)!}\right).$$

Thus, Lemma 277 was proven.

The next lemma is an easy consequence of a simple geometic observation and the inequality $\sin^{-1} r \geq r$ for $r \in \left(0, \frac{\pi}{2}\right)$. Let $\mathcal{H}^1(C)$ be the arc length of the arc C.

Lemma 278. *Let $R > 3$ and $(x_0, y_0) \in \mathbb{R}^2$. In $\mathbb{R}^2$, suppose that two circles $x^2 + y^2 = 1$ and $(x - x_0)^2 + (y - y_0)^2 = R^2$ intersect. Then the arclength L of the arc, given by $\{(x, y) : x^2 + y^2 = 9, (x - x_0)^2 + (y - y_0)^2 \leq R^2\}$, is greater than or equal to*

$$6 \cos^{-1} \frac{2}{3}.$$

Proof A rotation allows us to assume $y_0 = 0 < x_0$. Two circles $x^2 + y^2 = 1$ and $(x - x_0)^2 + y^2 = R^2$ intersect if and only if $R - 1 < x_0 < 1 + R$. Set

$$\begin{aligned}\Theta(x_0, R) &\equiv |\{\theta \in [-\pi, \pi] : (3\cos\theta - x_0)^2 + (3\sin\theta)^2 \leq R^2\}| \\ &= 2\cos^{-1}\left(\frac{{x_0}^2 + 9 - R^2}{6x_0}\right).\end{aligned}$$

Then, L is given by

$$L = 3\Theta(x_0, R). \tag{6.23}$$

Since $x_0 < 1 + R$ and $R > 3$, we have

$$\Theta(x_0, R) \geq 2\cos^{-1}\left(\frac{10 + 2R}{6(1 + R)}\right) \geq 2\cos^{-1}\left(\frac{2}{3}\right). \tag{6.24}$$

Estimates (6.23) and (6.24) yield the desired lower bound of L.

The weak $(1,1)$-constant in our general setting is sharp.

Proposition 279. *There exist a separable space (X, d) and a measure μ such that $M_{k,\mathrm{c}}$ is bounded if and only if $k \geq 2$. Furthermore, the weak-$(1,1)$ constant of $M_{2,\mathrm{c}}$ is 1 on this space.*

Proof We may assume $\frac{3}{2} < k < 2$, since $M_{k,\mathrm{c}}$ is decreasing as k increases. Suppose that $M_{k,\mathrm{c}}$ is bounded. We want to derive a contradiction. We begin with constructing an approximation of Dirac delta. Let $g_r \equiv \dfrac{1_{B(\mathbf{0},r)}}{\mu(B(\mathbf{0},r))}$. Then $M_{k,\mathrm{c}} g_r$ tends pointwise to $M_{k,\mathrm{c}}\delta_{\mathbf{0}}$ as $r \downarrow 0$, where $\delta_{\mathbf{0}}$ is a point mass at $\mathbf{0}$. Let $\mathbf{a}$ be a point defined as follows:
Put $K_n \equiv 2 + [\log_{10} 3^n]$. Define $\mathbf{a} = \mathbf{a_n} = \{3^{-n}\chi_{[1,K_n]}(j)\}_{j\in\mathbb{N}} \in D$.

Take $\lambda \equiv \dfrac{1}{\mu(B(\mathbf{a}, k\cdot 3^{-n}))}$. We have

$$\{x \in X \,:\, M_{k,\mathrm{c}}\delta_{\mathbf{0}}(x) > \lambda\} \supset B(\mathbf{0}, 3^{-n}).$$

Thus, we have $\mu(B(\mathbf{0}, 3^{-n})) \lesssim \lambda^{-1}$. That is, $\mu(B(\mathbf{0}, 3^{-n})) \lesssim \mu(B(\mathbf{a}, k\cdot 3^{-n}))$. By the definition of ν, there are infinitely many integers n such that

$$\left\{\frac{\nu(\{x+\sqrt{-1}y \in D \,:\, x^2+y^2 \leq 9^{-n}\})}{\nu(\{x+\sqrt{-1}y \in D \,:\, (x-3^{-n})^2+y^2 \leq k^2\cdot 9^{-n}\})}\right\}^{(2+[\log_{10} 3^n])} \lesssim 1.$$

Take the limit of this quantity as n, running through such an integer, to infinity. We obtain a contradiction, since the constant is strictly less than 1. Likewise we can calculate the weak-$(1,1)$ constant $M_{2,\mathrm{c}}$.

6.2.4 Exercises

Exercise 90. Let (X, d, μ) be a metric measure space.

(1) Show that for any ball $B(x,r)$ and for any $y \in B(x,r)$,

$$\frac{1}{\mu(B(x,2r))}\int_{B(x,r)} |f(y)|\mathrm{d}\mu(y) \leq M_{2,\mathrm{c}} f(x).$$

(2) Show that for any ball $B(x,r)$ and for any $y \in B(x,r)$,

$$\frac{1}{\mu(B(x,2r))}\int_{B(x,r)} |f(y)|\mathrm{d}\mu(y) \leq \|f\|_{L^\infty(\mu)}.$$

Exercise 91. Let $1 < p < \infty$ and $f \in L^0(\mu)$.

(1) Write out the definition in full using Theorem 5 to have

$$\|M_{2,\mathrm{c}} f\|_{L^p(\mu)}{}^p = \int_0^\infty p\lambda^{p-1}\mu\{M_{2,\mathrm{c}} f > \lambda\}\mathrm{d}\lambda. \tag{6.25}$$

(2) Reexamine the proof of Theorem 267 to give

$$\mu\{M_{2,\mathrm{c}}f > \lambda\} \le \frac{1}{\lambda}\int_X \chi_{(\lambda,\infty]}(Mf(y))|f(y)|\mathrm{d}\mu(y). \tag{6.26}$$

(3) Combine (6.25) and (6.26) and reexamine the proof of Theorem 141 to have $\|M_{2,\mathrm{c}}f\|_{L^p(\mu)} \le p'\|f\|_{L^p(\mu)}$.

Exercise 92. [378] Let $p \ge q > 1$. Suppose that we have $\{f_j\}_{j=1}^\infty \subset L^1(\mu) \cap L^\infty(\mu)$ such that $f_j = 0$ for any $j \gg 1$.

(1) Let $p = q$. Use Theorem 267 to show Theorem 274.

(2) Let $p > q$. Abbreviate $\frac{p}{q}$ to r and let r' be a conjugate exponent of r.

$(2-A)$ Show that $\left(\left\|\|\{M_{7,\mathrm{c}}f_j\}_{j=1}^\infty\|_{\ell^q}\right\|_{L^p(\mu)}\right)^q = \left\|\sum_{j=1}^\infty (M_{7,\mathrm{c}}f_j)^q\right\|_{L^r(\mu)}$.

$(2-B)$ Dualize the right-hand side to find $g \in \mathbb{M}^+(\mu)$ such that $\|g\|_{L^{r'}(\mu)} = 1$ such that

$$\left(\left\|\|\{M_{7,\mathrm{c}}f_j\}_{j=1}^\infty\|_{\ell^q}\right\|_{L^p(\mu)}\right)^q = \int_X \left(\sum_{j=1}^\infty M_{7,\mathrm{c}}f_j(x)^q\right) g(x)d\mu(x).$$

With this in mind, we fix $g \in L^{r'}(\mu) \cap \mathbb{M}^+(\mu)$.

$(2-C)$ Using Corollary 272, prove that

$$\left(\left\|\|\{M_{7,\mathrm{c}}f_j\}_{j=1}^\infty\|_{\ell^q}\right\|_{L^p(\mu)}\right)^q \lesssim_{p,q} \int_X \sum_{j=1}^\infty |f_j(x)|^q M_{2,\mathrm{c}}g(x)d\mu(x).$$

$(2-D)$ Use Hölder's inequality twice to conclude the proof of Theorem 274.

Exercise 93. Let $1 < p < q \le \infty$.

(1) If $q = \infty$, then use $\sup_{j\in\mathbb{N}} M_{22,\mathrm{c}}f_j \le M_{22,\mathrm{c}}\left(\sup_{j\in\mathbb{N}}|f_j|\right)$ to conclude the proof of Theorem 270. Hence, (6.15) is clear, if $q = \infty$.

(2) It remains to show when $p < q < \infty$. Let $q > p$ in what follows and take another $r < q$ so close to q that $s = \dfrac{pr}{q} > 1$.

(3) Use the duality twice to obtain

$$\int_X \left(\sum_{j=1}^\infty (M_{22,\mathrm{c}}f_j(x))^q\right)^{\frac{1}{r}} g(x)d\mu(x) \tag{6.27}$$

$$= \sup_{\sum_k h_k{}^{r'}=1, h_k\in\mathbb{M}^+(\mu)} \int_X \left(\sum_{j=1}^\infty \left(M_{22,\mathrm{c}}f_j(x)^{\frac{q}{r}}h_j(x)\right) g(x)\right) d\mu(x).$$

Below we will write sup instead of $\sup_{\sum_k h_k{}^{r'}=1, h_k\in\mathbb{M}^+(\mu)}$ and we abbreviate

$$\mathrm{I} \equiv \sup \int_X \left(\sum_{j=1}^{\infty} \left(M_{22,\mathrm{c}} f_j(x)^{\frac{q}{r}} h_j(x) \right) g(x) \right) d\mu(x).$$

(4) Use Fubini's theorem to show

$$\mathrm{I} = \sup \sum_{j=1}^{\infty} \int_X \left(M_{22,\mathrm{c}} f_j(x)^{\frac{q}{r}} h_j(x) \right) g(x) d\mu(x)$$

(5) Use Proposition 273 and the fact that $q > r > 1$ to have

$$\begin{aligned} \mathrm{I} &\lesssim_{p,q} \sup \sum_{j=1}^{\infty} \int_X |f_j(x)|^{\frac{q}{r}} M_{7,\mathrm{c}}[h_j g](x) d\mu(x) \\ &= \sup \int_X \left(\sum_{j=1}^{\infty} |f_j(x)|^{\frac{q}{r}} M_{7,\mathrm{c}}[h_j g](x) \right) d\mu(x). \end{aligned} \tag{6.28}$$

(6) Use Hölder's inequality to obtain

$$\begin{aligned} &\sup \int_X \left(\sum_{j=1}^{\infty} |f_j(x)|^{\frac{q}{r}} M_{7,\mathrm{c}}[h_j g](x) \right) d\mu(x) \\ &\leq \sup \int_X \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{1}{r}} \left(\sum_{j=1}^{\infty} (M_{7,\mathrm{c}}[h_j g](x))^{r'} \right)^{\frac{1}{r'}} d\mu(x) \end{aligned}$$

and

$$\begin{aligned} &\sup \int_X \left(\sum_{j=1}^{\infty} |f_j(x)|^{\frac{q}{r}} M_{7,\mathrm{c}}[h_j g](x) \right) d\mu(x) \\ &\leq \sup \int_X \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{1}{r}} \left(\sum_{j=1}^{\infty} (M_{7,\mathrm{c}}[h_j g](x))^{r'} \right)^{\frac{1}{r'}} d\mu(x) \\ &\quad \times \sup \left\{ \int_X \left(\sum_{j=1}^{\infty} (M_{7,\mathrm{c}}[h_j g](x))^{r'} \right)^{\frac{s'}{r'}} d\mu(x) \right\}^{\frac{1}{s'}} \end{aligned}$$

(7) Note that $q > p$ implies $s' > r'$. Use Theorem 270 with parameters $s' > r'$ to obtain

$$\left\{\int_X \left(\sum_{j=1}^{\infty}(M_{7,\mathrm{c}}[h_j g](x))^{r'}\right)^{\frac{s'}{r'}} d\mu(x)\right\}^{\frac{1}{s'}}$$

$$\lesssim_{p,q} \left\{\int_X \left(\sum_{j=1}^{\infty}(h_j(x)g(x))^{r'}\right)^{\frac{s'}{r'}} d\mu(x)\right\}^{\frac{1}{s'}}.$$

(8) Show that

$$\sup\left\{\int_X \left(\sum_{j=1}^{\infty}(M_{7,\mathrm{c}}[h_j g](x))^{r'}\right)^{\frac{s'}{r'}} d\mu(x)\right\}^{\frac{1}{s'}} \lesssim_{p,q} 1. \tag{6.29}$$

(9) Use (6.29) to complete the proof.

Exercise 94. [122, Lemma 2.8] The metric space (X, d) has the *segment property* if for any $x, y \in X$ there exists a continuous curve $\gamma : [0, 1] \to X$ connecting x and y such that $d(x, y) = d(x, \gamma(t)) + d(\gamma(t), y)$ for all $t \in [0, 1]$. Assume that (X, d, μ) is a doubling metric space satisfying the segment property. Then show that the function $r \in (0, \infty) \mapsto \mu(B(x, r)) \in (0, \infty)$ is continuous for each $x \in X$.

Exercise 95. In Theorem 267, prove that we cannot improve the conclusion in the following sense: Let $0 < \rho < 1$. Then we cannot have

$$\mu(\{x \in X : M_{3,\mathrm{uc}} f(x) > \lambda\}) \le \frac{\rho}{\lambda}\int_X |f(x)|d\mu(x) \tag{6.30}$$

for any $f \in L^0(\mu)$.

Exercise 96. In Theorem 267, prove that we cannot improve the conclusion in the following sense: Let $0 < \rho < 1$. Then there does not exist a constant C_ρ such that

$$\mu(\{x \in X : M_{3-\rho,\mathrm{uc}} f(x) > \lambda\}) \le \frac{C_\rho}{\lambda}\int_X |f(x)|d\mu(x) \tag{6.31}$$

for any $f \in L^0(\mu)$.

6.3 Notes

Section 6.1

General remarks and textbooks in Section 6.1

See standard textbooks [112, Chapter 3], [231], [293] and [459].

See [147, §1.3] for various covering lemmas adapted to quasi-metrics.

Fu, Yang and Yuan considered the commutators in [130, Theorem 1.15]. Futamura, Mizuta and Shimomura consided the Sobolev embedding in the setting of general metric measure spaces and variable exponents [138, Theorem 4.1]. García-Cuerva and Martell obtained the vector-valued estimates for singular integral operators [143, Section 2] and the existence of the principal values [144, Theorem 1(f)]. Gorka considered Morrey–Campanato spaces in the setting of metric measure spaces [154].

Universal estimates for dyadic maximal operators can be found in [223, p. 11, Theorem]; see Theorem 254.

Section 6.1.1

The boundedness of the maximal operator $M_{\kappa,\mathrm{uc}}$ associated with general Radon measure goes back to [339, Section 3]. See [338] as well. Later, Sawano and Terasawa pointed out that the modification rate κ can be decreased [378, 433]. Theorem 247 is [378, Theorem 1.5].

See [68] for the commutators generated by $\mathrm{RBMO}(\mu)$ and I_α, where Chen and Sawyer considered a modification constant that differs from the one considered by Tolsa in [435].

Section 6.1.2

Tolsa pointed out that the modification rate κ in $M_{\kappa,\mathrm{uc}}$ must satisfy $\kappa > 1$ for the boundedness of $M_{\kappa,\mathrm{uc}}$; see [435, p. 127]. This idea was reinvestigated by Sawano [378].

Section 6.1.3

As we saw in Theorem 253, Tolsa proved that the analogue of the Lebesgue differentiation theorem is available [435, Remark 2.3]. The covering lemma and the vector-valued inequality can be found in [378]; see [378, Theorem 1.5] and [378, Theorem 1.7] for Theorems 247 and 251, respectively.

Section 6.1.4

We considered the universal $L^p(\mu)$-estimate for $M_{\log}^{\mathcal{D}(\mu)}$ defined in (6.13). See [209, p. 786] for Theorem 256. Theorem 257(3) is recorded as [209, Lemma

2.13]. Lerner and Nazarov proposed to distort dyadic cubes as in Lemma 259; see [92, Lemma 4.8].

Section 6.1.5

Mauceri and Meda considered the admissible balls in [308]. Proposition 260 is due to Mauceri and Meda [308, Proposition 2.1]. Lemma 262 can be found in [308, p. 298].

Fabes, Gutiérrez and Scotto dealt with singular integral operators associated with the Gauss measure [119], where we can find a relation between the Ornstein–Uhlenbeck process and singular integral operators they handle. See also [174, 175, 183, 443] for more. See [145, Lemma 2.4] and [308, Lemma 5.1] for more results related to Lemma 264. The corresponding version in the locally doubling measure metric spaces was proven in [284, Lemma 2.7].

Lemma 265, which guarantess that a ball is doubling under a certain condition, can be found in [281, Lemma 3.9]. Example 108 together with estimate (6.25) is [278, Lemma 2.1].

Carbonaro, Mauceri and Meda considered the Hardy space H^1 and BMO associated with the Gauss measure in [62, 63].

We can use the Gauss measure to show the boundedness of the Riesz transform; see [358].

Section 6.2

General remarks and textbooks in Section 6.2

The textbooks [112, 147, 458] cover the topic exhaustively. A Radon measure μ satisfying $\mu(B(x,r)) \lesssim r^d$ is called the *Frostman measure*. This plays an important role in various aspects in analysis. See [9, 338, 339, 340, 434, 435, 436]. This type of assumption sometimes appears in the analysis of operators handled in this book. See [112, Chapter 6] for fractional integral operators on metric measure spaces with general Radon measures, which includes the analysis of the maximal operator. See [112, Chapter 8, §3] for singular integral operators. See [233] for fractional integral operators on non-homogeneous spaces. See [307] for analysis with general Radon measures.

Section 6.2.1

Terasawa proved Theorem 267 when the function $(x,r) \mapsto \mu(B(x,r))$, $x \in X$ and $r > 0$ is continuous in r [433, Theorem 2.4]. Sawano proved Theorem 267 when X is separable [378, Theorem 2.1] based on Terasawa's result. See [378, Corollary 2.1] for Theorem 269.

See [234, 246] for more about the maximal operators on non-homogeneous spaces.

Section 6.2.2

In most cases, the vector-valued extension of the maximal operator associated with general Radon measures is done. For the modified maximal operator, see [378, Theorem 1.3] and [378, Theorem 2.1] for Theorems 270 and 274, respectively. For the dyadic maximal operator associated with general Radon measures, we refer to the textbook [89, Theorem A.18].

Section 6.2.3

Stempak constructed Examples 109 and 110; see [419, Example 2.1] and [419, Example 2.2], respectively. See also [378, 418] for more examples showing that the modification is necessary.

Chapter 7

Weighted Lebesgue spaces

Chapter 7 examines weighted norm inequalities. Weighted function spaces arise naturally in many contexts of mathematics. Roughly speaking, weighted measure spaces, which generate weighted function spaces, are useful although they break some symmetries. We note that a group can be used to determine the symmetry of a space. In fact, a group action allows the symmetry of the spaces to be measured. For example, $\mathrm{O}(n)$, the group of all orthonormal matrices, acts naturally on $\mathbb{R}^n$, and $\mathbb{R}^n$ acts on itself as a translation. Hence, we can say that $\mathbb{R}^n$ is symmetric enough.

However, when handling something in $\mathbb{R}^n$, often the whole space $\mathbb{R}^n$ is neglected. Occasionally, some open sets in $\mathbb{R}^n$ are used instead of $\mathbb{R}^n$ itself. For example, when considering some differential equations, functions are defined in open sets. Chapter 7 concentrates on weights that break the symmetry. Another example of breaking the symmetry is listing the probability space. Since the whole mass 1 cannot be distributed evenly, a density function of measures or some discrete measures must be considered. The Gaussian distribution is a natural one appearing in many branches of sciences. This is clearly not an invariant measure under the translations, although it is one under $\mathrm{O}(n)$. Consequently, some symmetries must be surrendered.

One way to break the symmetry is to consider weighted measures. For the time being by a weight we mean a measurable function in a measure space which has a positive value almost everywhere.

As usual, here Euclidean space is considered. However, in different types of spaces, the measure changes. For example, the Lebesgue measure $\mathrm{d}x$ will be changed into the *weighted measure* $w(x)\mathrm{d}x$, where w is a weight.

As shown in an earlier chapter, the Hardy–Littlewood maximal operator can be used to control other important operators. Chapter 7 is interested in the weights for which the Hardy–Littlewood maximal operator is bounded by investigating the weighted estimates for Lebesgue spaces, including some sharp estimates on constants. Weighted Morrey spaces are discussed in Chapter 15.

Weighted Lebesgue spaces arise naturally when considering the change of variables. Since the Hardy–Littlewood maximal operator is a fundamental operator, we are interested in the inequality

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\mathrm{d}x \lesssim \int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x.$$

We characterize weights w for which the above inequality holds. Lebesgue spaces have a suitable characterization for this inequality. It matters that we have to measure in some sense the property of weights to see that the weights are close to the *constant weight* 1.

The Lebesgue measure of the measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$, as before. For a measurable set E and $f \in L^1(E)$, recall that $m_E(f)$ denotes the average of f over E with respect to the Lebesgue measure:

$$m_E(f) = \frac{1}{|E|}\int_E f(x)\mathrm{d}x.$$

See (1.6). Here and below a *weight* is a locally integrable function with a positive value almost everywhere on $\mathbb{R}^n$. For a weight w and a measurable set E, the *weight* of E is by

$$w(E) \equiv \int_E w(x)\mathrm{d}x.$$

The weighted Lebesgue space $L^p(w)$ with $0 < p < \infty$ represents the set of all $f \in L^0(\mathbb{R}^n)$ for which the norm

$$\|f\|_{L^p(w)} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

is finite. Among others it is quite important to consider power weights. Let $\alpha \in \mathbb{R}$. We set $\|f\|_{L^p(|x|^\alpha)} \equiv \|f\|_{L^p(w)}$, where $w(x) = |x|^\alpha$, $x \in \mathbb{R}^n$. The *power weighted Lebesgue space* $L^p(|x|^\alpha)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{L^p(|x|^\alpha)} < \infty$. If $n = 1$, then write $L^p(|t|^\alpha)$ instead of $L^p(|x|^\alpha)$. It is also important to consider the weighted Lebesgue space on the half line. The *power weighted Lebesgue space* $L^p(t^\alpha)$ or $L^p(r^\alpha)$ over $(0,\infty)$ is the set of all $f \in L^0(0,\infty)$ for which

$$\|f\|_{L^p(t^\alpha)} = \|f\|_{L^p(r^\alpha)} \equiv \left(\int_0^\infty |f(t)|^p t^\alpha \mathrm{d}t\right)^{\frac{1}{p}} < \infty.$$

Although we do not go into the details, the Jacobi hypergroup is the weighted Lebesgue spaces. Let $\alpha \geq \beta \geq -1$ with $(\alpha,\beta) \neq (-1,-1)$. Then we set $\mu(t) \equiv (\sinh t)^{1+\alpha}(\cosh t)^{1+\beta}\mathrm{d}t$ for $t > 0$. The space $L^2(\mathbb{R}_+,\Delta) = L^2(\mathbb{R}_+,\mu)$ is the weighted L^2-space called the *Jacobi hypergroup*.

Section 7.1 focuses on the *one-weight* (*maximal*) *inequality*

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\mathrm{d}x \lesssim \int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x.$$

Generally speaking when the weights on the both sides are the same, the inequality is called a *one-weight inequality* or a *one-weight norm inequality*. A more general inequality of the form $\|Tf\|_{L^q(v)} \lesssim \|f\|_{L^p(w)}$ is called a *two-weight inequality* or a *two-weight norm inequality*. Section 7.2 deals with the two-weight norm inequality of Hardy operators, (fractional) maximal operators and singular integral operators. We will obtain characterizations different from those in Section 7.1.

7.1 One-weighted norm inequality

Although the following example is not used in the context of weighted norm inequalities, this example is useful to see that the weighted norms arise naturally.

Example 111. Let $E(\mathbb{R}^n)$ be a ball Banach function space, so that there exists a constant c_k such that $\|f\|_{L^1(B(k))} \leq c_k \|f\|_E$ for all $f \in E(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Setting $w \equiv \sum_{k=1}^{\infty} (2^k c_k)^{-1} \chi_{B(k)}$, we obtain $E(\mathbb{R}^n) \hookrightarrow L^1(w)$. We have a similar embedding for Banach function spaces over a measure space $(X, \mathcal{B}, \mu)$.

For $1 \leq p \leq \infty$, we will consider the class A_p. Here, we distinguish three cases to define and investigate A_p. See Sections 7.1.1, 7.1.2 and 7.1.3 for the cases of $p = 1$, $1 < p < \infty$ and $p = \infty$, respectively. Section 7.1.4 is of different nature compared with Sections 7.1.1, 7.1.2 and 7.1.3; Sections 7.1.1, 7.1.2 and 7.1.3 are oriented to the Hardy–Littlewood maximal operator and singular integral operators, while Section 7.1.4 is related to fractional integral operators and fractional maximal operators.

In Sections 7.1.5 and 7.1.6 we consider applications of A_p defined and investigated in Sections 7.1.1, 7.1.2 and 7.1.3. Section 7.1.5 will provide a tool to create some inequalities based on the boundedness property of the weighted Lebesgue spaces. Section 7.1.6 will solve the A_2-conjecture by employing some key tools such as the Lerner–Hytönen decomposition obtained in Section 4.2.3.

7.1.1 The class A_1

We start with the weighted estimate for the case of $p = 1$. We will introduce the class A_1 for this purpose. We are thus interested in the weighted weak-$(1,1)$ maximal inequality

$$w\{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\} \leq \frac{C}{\lambda} \|f\|_{L^1(w)}. \tag{7.1}$$

Here, $\lambda > 0$ and $f \in L^0(\mathbb{R}^n)$. This is the weighted counterpart of the weak-$(1,1)$ inequality of the Hardy–Littlewood maximal operator. Before we investigate the necessary and sufficient condition, we consider an example. Among the weights, we are particularly interested in the power weights. Here, by a power weight we mean a weight w of the form $w(x) = |x|^\alpha$, $x \in \mathbb{R}^n$. We are now interested in the case of $w(x) \equiv |x|^\alpha$ with $\alpha \in \mathbb{R}$.

Example 112. Let $w(x) \equiv |x|^\alpha$ with $\alpha \in \mathbb{R}$. If w satisfies the weighted weak-$(1,1)$ maximal inequality, then $\alpha > -n$. In fact, we can simply test the weighted weak-$(1,1)$ maximal inequality on $f = \chi_{B((2,2,\dots,2),1)}$ keeping in mind that $B((2,2,\dots,2),1)$ is away from the singularity.

Example 113. Let $\alpha > -n$, and let Q be a cube. Then

$$\int_Q |x|^\alpha \mathrm{d}x \sim \max(\ell(Q), |c(Q)|)^\alpha |Q|.$$

This can be checked by distinguishing two cases: $c(Q)$ is far from the origin, that is, $|c(Q)| > 4n\ell(Q)$, and otherwise.

We are now oriented to a general case of weights. We answer the question of characterizing w for which (7.1) holds.

Definition 70. For a weight w, define its A_1-*characteristic* by

$$[w]_{A_1} \equiv \left\| \frac{Mw}{w} \right\|_{L^\infty} = \inf \left[\bigcap_{Q \in \mathcal{Q}} \{\alpha > 0 \,:\, m_Q(w) \le \alpha \operatorname{essinf}_{x \in Q} w(x)\} \right].$$

The quantity is also called the A_1-*constant*. A weight w is an A_1-*weight* if $[w]_{A_1}$ is finite. The class A_1 collects all A_1-weights.

Roughly speaking, the class A_1 is close to the constant weight 1. We consider a concrete but important example.

Example 114. Let $w(x) \equiv |x|^{-n+\delta}$, $x \in \mathbb{R}^n$ with $\delta \in (0, n]$. Then $[w]_{A_1} \sim \delta^{-1}$. In fact, similar to Example 113, we have the following two-sided estimate:

$$Mw(x) \sim \max\left(w(x), \frac{1}{|B(3|x|)|} \int_{B(3|x|)} |y|^{-n+\delta} \mathrm{d}y \right) \sim \frac{1}{\delta} w(x),$$

as is easily seen from the observation that $0 \in B(x, 3r)$ or $0 \notin B(x, 3r)$ for any ball $B(x, r)$.

Although $\mathrm{W}L^1(w)$ is not a Banach space and M is not a linear operator, we still define

$$\|M\|_{L^1(w) \to \mathrm{W}L^1(w)} \equiv \sup_{f \in L^1(w) \setminus \{0\}} \frac{\|Mf\|_{\mathrm{W}L^1(w)}}{\|f\|_{L^1(w)}}.$$

The weak-$(1,1)$ constant of the maximal operator is comparable to the A_1-constant of w.

Theorem 280. *For any weight w, the maximal operator M is bounded from $L^1(w)$ to $\mathrm{W}L^1(w)$ if and only if $w \in A_1$. Furthermore, if this is the case, $\|M\|_{L^1(w) \to \mathrm{W}L^1(w)} \sim [w]_{A_1}$.*

Proof The "if" part is easy; simply use Corollary 140 and $Mw \le [w]_{A_1} w$. For the "only if" part, consider the inequality $\chi_Q m_Q(|f|) \le Mf$ and $\lambda = m_Q(|f|)$ for given a cube. Then $w(Q) \le w\{x \in \mathbb{R}^n \,:\, Mf(x) \ge m_Q(|f|)\} \le C m_Q(|f|)^{-1} \|f\|_{L^1(w)}$, or equivalently, $w(Q) m_Q(|g|w^{-1}) \le C\|g\|_{L^1}$. Then we

are in the position of using the duality $L^1(\mathbb{R}^n)$-$L^\infty(\mathbb{R}^n)$. The above observations include the proof of $\|M\|_{L^1(w)\to \mathrm{W}L^1(w)} \sim [w]_{A_1}$. Thus, the proof is complete.

We now present some examples of A_1-weights. Roughly speaking there are two ways to create A_1-weights. Theorem 281 below, which uses the maximal operator of measures, can be used to create some concrete examples, while Theorem 284 can be used for the theoretical aspects.

Theorem 281. *Assume that μ is a Radon measure on $\mathbb{R}^n$ such that $M\mu(x)$ is finite for a.e. $x \in \mathbb{R}^n$. Then for $0 < \delta < 1$, $w \equiv (M\mu)^\delta$ is an A_1-weight, where $M\mu$ is given by* (4.7)*. Furthermore, $A_1(w) \lesssim_\delta 1$.*

It matters that the implicit constant above does not depend on μ.

Proof We have to establish $m_Q(w) \lesssim w(x)$ whenever Q is a cube containing x. We split μ according to $10Q$. Write $\mu|A \equiv \mu(A \cap \cdot)$, the restriction of μ to a measurable subset A. Split μ by $\mu = \mu_1 + \mu_2$ with $\mu_1 \equiv \mu|10Q$ and $\mu_2 \equiv \mu|(\mathbb{R}^n \setminus 10Q)$. We will prove

$$m_Q({M\mu_1}^\delta) \lesssim w(x),\ m_Q({M\mu_2}^\delta) \lesssim w(x)$$

for all $x \in Q$. To deal with μ_1 we use Example 66, the Kolmogorov inequality, to obtain

$$m_Q({M\mu_1}^\delta) \lesssim m_{10Q}({M\mu_1}^\delta) \lesssim \left(\frac{\mu(10Q)}{|10Q|}\right)^\delta \lesssim w(x).$$

To handle μ_2, we will employ a pointwise estimate, which we encounter frequently. Let $y \in Q$. First we write out $M\mu_2(y)$ in full:

$$M\mu_2(y) = \sup\left\{\frac{\mu(R \setminus 10Q)}{|R|} \ :\ R \in \mathcal{Q}, y \in R\right\}.$$

A geometric observation readily shows $\ell(R) \geq 3\ell(Q)$ whenever R is a cube intersecting both Q and $\mathbb{R}^n \setminus 10Q$. Therefore such an R engulfs Q, if we triple R. In view of this fact, we deduce

$$M\mu_2(y) \leq 3^n \sup\left\{\frac{\mu(S)}{|S|} \ :\ S \in \mathcal{Q},\, Q \subset S\right\} \leq 3^n\, M\mu(x).$$

This pointwise estimate readily gives $m_Q({M\mu_2}^\delta) \lesssim w(x)$. Hence, the theorem is now proven.

The following proposition gives us a concrete example of A_1-weights.

Proposition 282. *Let $0 \leq a < n$. Then $w(x) \equiv |x|^{-a}, x \in \mathbb{R}^n$ belongs to A_1.*

Proof This simply paraphrases Example 114. Alternatively, we argue as follows: Let $a \in (0, n)$. In this case it suffices to take $\mu = \delta_0$, the point measure

massed at the origin and $\delta = \frac{a}{n}$, which is a special case of Theorem 281. Then w equals $M\mu$ modulo multiplicative constants.

We completely characterize the power weights in A_1.

Theorem 283. *Let $\alpha \in \mathbb{R}$. Then $|\cdot|^\alpha \in A_1$ if and only if $-n < \alpha \leq 0$.*

Proof Let $\alpha > 0$. Then $M[|\cdot|^\alpha] \equiv \infty$. Thus, this simply paraphrases Example 114 and Proposition 282 again.

We now invoke the *Rubio de Francia iteration algorithm* to create more examples of A_1-weights: As usual let $L^0(\mathbb{R}^n)$ denote the set of all measurable functions f. Of course, we identify two functions if they are different by a set of measure zero. Denote by M^k the k-fold composition of the Hardy–Littlewood maximal operator M.

Theorem 284. *Let $(E(\mathbb{R}^n), \|\cdot\|_E)$ be a Banach space of $L^0(\mathbb{R}^n)$ such that $Mf \in E(\mathbb{R}^n)$ together with the estimate*

$$\|Mf\|_E \leq D\|f\|_E \tag{7.2}$$

holds whenever $f \in E(\mathbb{R}^n)$, where D is independent of f. Assume in addition that any function in $E(\mathbb{R}^n)$ is finite almost everywhere. Let $g \in E(\mathbb{R}^n)$ with norm 1. Define $G \equiv |g| + \sum_{k=1}^{\infty} \frac{M^k g}{(2D)^k}$. Then G belongs to $E(\mathbb{R}^n)$ and satisfies the following properties:

$$\|G\|_E \leq 2\|g\|_E = 2 \tag{7.3}$$

and

$$[G]_{A_1} \leq 2D. \tag{7.4}$$

Proof Since the series definining G converges absolutely in the topology of $E(\mathbb{R}^n)$, $G \in E(\mathbb{R}^n)$. This also implies (7.3). Hence, G is finite almost everywhere by our assumption. Note that G is everywhere positive because $Mg > 0$. Finally (7.4) is easy to check using (7.2) and the sublinearity of f, $G \in A_1$.

We end Section 7.1.1 with the characterization of Morrey spaces in terms of A_1-weights. The following proposition can be used to express the Morrey norm in terms of the weighted Lebesgue spaces:

Proposition 285. *Let $1 < q \leq p < \infty$, and let $0 < \theta < \frac{q}{p}$. Then for all $f \in L^0(\mathbb{R}^n)$, $\|f\|_{\mathcal{M}^p_q} \sim \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(M^{(\frac{1}{1-\theta})}\chi_Q)}$.*

Proof Since $M\chi_Q \geq \chi_Q$, $\|f\|_{\mathcal{M}^p_q} \leq \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(M^{(\frac{1}{1-\theta})}\chi_Q)}$. To show the opposite inequality, we decompose

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(M^{(\frac{1}{1-\theta})}\chi_Q)} \lesssim \sum_{j=0}^{\infty} |Q|^{\frac{1}{p}-\frac{1}{q}} 2^{-j\frac{n(1-\theta)}{q}} \|f\|_{L^q(2^jQ)} \lesssim \|f\|_{\mathcal{M}^p_q}.$$

Note that the series is convergent since $\theta < \dfrac{q}{p}$.

Example 115. Suppose that we have an operator T defined on $L^q(w)$ for any $w \in A_1$. Then T can be restricted to $\mathcal{M}^p_q(\mathbb{R}^n)$ due to Proposition 285. In Theorem 306, we will see that singular integral operators fall under this scope.

7.1.2 The class A_p

Having set down the necessary and sufficient condition for the weighted weak-$(1,1)$ maximal inequality to hold, we are now oriented to the weak-(p,p) maximal inequality

$$\lambda[w\{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\}]^{\frac{1}{p}} \lesssim \|f\|_{L^p(w)} \quad (f \in L^0(\mathbb{R}^n),\, \lambda > 0)$$

and the strong-(p,p) maximal inequality

$$\|Mf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \quad (f \in L^0(\mathbb{R}^n)).$$

Here, $1 < p < \infty$. We will introduce the class A_p. It should be noted that the weighted weak-$(1,1)$ maximal inequality is equivalent to

$$\lambda[w\{x \in \mathbb{R}^n \,:\, Mf(x) \geq \lambda\}] \leq C\|f\|_{L^1(w)} \quad (f \in L^1(w),\, \lambda > 0).$$

Needless to say, by the Chebyshev inequaltiy, the strong-(p,p) maximal inequality implies the weak-(p,p) maximal inequality. Furthermore, as it turns out, for a weight w these two inequalities occur at the same time. However, to trace the behavior of the constants C, we will consider these two inequalities separately. Before we obtain the necessary and sufficient condition for the weighted weak/strong-$(1,1)$ maximal inequality to hold, we give an example.

Example 116. Let $1 < p < \infty$ and $\alpha \in \mathbb{R}^n$. As is easily seen by mimicking Example 112, if w satisfies the weighted weak-(p,p) maximal inequality, then $\alpha > -n$. Furthermore, we need to have $\alpha < n(p-1)$. In fact, if $\alpha \geq n(p-1)$, then $f(x) \equiv \dfrac{\chi_{B(1)}(2x)}{|x|^n \log|x|}$, $x \in \mathbb{R}^n$ belongs to $L^p(|x|^\alpha)$. However, $Mf \equiv \infty$, since $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

For the time being, we seek in Section 7.1.2 to show that the following class of weights is the solution to the problem of characterizing the weak/strong-(p,p) maximal inequalities.

Definition 71. Let $1 < p < \infty$. A locally integrable weight w is an A_p-*weight* or belongs to A_p-*class*, if $0 < w < \infty$ almost everywhere, and $[w]_{A_p} \equiv \sup_{Q \in \mathcal{Q}} m_Q(w) m_Q^{(\frac{1}{p-1})}(w^{-1}) < \infty$. The quantity $[w]_{A_p}$ is referred to as the A_p-*constant* or the A_p-*characteristic*. The class A_p collects all A_p-weights.

Remark 13. It is easy to check $A_1 \subset A_p \subset A_q \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by Hölder's inequality whenever $1 \leq p \leq q < \infty$.

Before we give some examples of the A_p-weights, we justify the definition, which shows that the weighted weak-$(1,1)$ maximal inequality is sufficient for the weight to belong to the class A_p. As we will see, it is not so hard to see that $w \in A_p$ if the weak-(p,p) maximal inequality holds. In fact we have the following:

Proposition 286. *Let $1 < p < \infty$ and $N > 0$. If a weight w satisfies the weak-(p,p) maximal inequality*

$$\lambda[w\{x \in \mathbb{R}^n : Mf(x) > \lambda\}]^{\frac{1}{p}} \leq N\|f\|_{L^p(w)} \quad (f \in L^p(w),\, \lambda > 0),$$

then $N \geq ([w]_{A_p})^{\frac{1}{p}}$.

Proof Let $f \in L^0(\mathbb{R}^n)$. If we fix a cube Q, then $Mf(x) \geq m_Q(|f|)$ for all $x \in Q$. Consequently,

$$m_Q(|f|)w(Q)^{\frac{1}{p}} \leq N\|f\|_{L^p(w)} \quad (f \in L^p(w)),$$

or equivalently,

$$m_Q(w^{-\frac{1}{p}}|g|)w(Q)^{\frac{1}{p}} \leq N\|g\|_{L^p} \quad (g \in L^p(\mathbb{R}^n)).$$

By the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$, we have $m_Q(w^{-\frac{1}{p-1}})^{\frac{p-1}{p}}w(Q)^{\frac{1}{p}} \leq N$. Thus, $w \in A_p$ together with the estimate $N^p \geq [w]_{A_p}$ since Q is arbitrary.

We polish Example 116 having somewhat clarified how the class A_p is natural.

Example 117. Let $w(x) = |x|^{-n+\delta}$, $x \in \mathbb{R}^n$ with $\delta \in (0,n)$. Then $[w]_{A_p} \sim \delta^{-1}$. We calculate $m_Q(w)m_Q^{(\frac{1}{p-1})}(w^{-1})$ according to the position of Q. If $0 \notin 2Q$, then $m_Q(w)m_Q^{(\frac{1}{p-1})}(w^{-1}) \sim m_Q(w(c(Q)))m_Q^{(\frac{1}{p-1})}(w(c(Q))^{-1}) = 1$. If $0 \in 2Q$, then $m_Q(w)m_Q^{(\frac{1}{p-1})}(w^{-1}) \sim \delta^{-1}\ell(Q)^{-\delta}\ell(Q)^{\delta} = \delta^{-1}$.

Since $L^p(\mathbb{R}^n)$ and $L^{p'}(\mathbb{R}^n)$ are dual to each other, it is natural to ask ourselves whether the classes A_p and $A_{p'}$ are strongly related. Actually, the following equality, whose easy proof we omit (see Exercise 97), follows from the definition.

Lemma 287. *If $w \in A_p$ with $1 < p < \infty$, then we have $w^{-\frac{1}{p-1}} \in A_{p'}$ with $[w^{-\frac{1}{p-1}}]_{A_{p'}} = ([w]_{A_p})^{p'-1}$.*

Motivated by Lemma 287, we occasionally call $w^{-\frac{1}{p-1}}$ the *dual weight* of $w \in A_p$, $1 < p < \infty$.

We apply Lemma 287 to the power weights. Example 118 concludes our calculation on the power weights.

Example 118. Let $1 < p < \infty$, and let $w(x) \equiv |x|^{-n+\delta}$, $x \in \mathbb{R}^n$, with $0 < \delta < n$. Then $v(x) \equiv w(x)^{-\frac{p'}{p}} = |x|^{(p'-1)(n-\delta)}$, $x \in \mathbb{R}^n$, is an $A_{p'}$-weight with $[v]_{A_{p'}} \sim \delta^{-(p'-1)}$. As a result, $W(x) \equiv v(x) = |x|^{(p-1)(n-\delta)}$, $x \in \mathbb{R}^n$, is an A_p-weight with $[W]_{A_p} \sim \delta^{-p'+1} = \delta^{-\frac{1}{p-1}}$.

We completely characterize the power weights in A_p for $1 < p < \infty$. Pay attention to the endpoint. The condition with $p = 1$ obtained in Theorem 288 differs from the one in Theorem 283.

Theorem 288. *Let $\alpha \in \mathbb{R}$, and let $1 < p < \infty$. Then $|\cdot|^\alpha \in A_p$ if and only if $-n < \alpha < n(p-1)$.*

Proof The necessity follows from Example 116. The sufficiency follows from Examples 117 and 118 together with the fact that $1 \in A_p$.

Here, we show that the A_p-condition is sufficient for the weak-(p,p) maximal inequality to hold. We need the weak-type estimate later:

Theorem 289. *Let $w \in A_p$ with $p \in [1, \infty)$. Then*

$$\sup_{\lambda>0} \lambda^p w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \lesssim ([w]_{A_p} \|f\|_{L^p(w)})^p$$

for all $f \in L^0(\mathbb{R}^n)$. Furthermore, if N is a constant satisfying

$$\sup_{\lambda>0} \lambda^p w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq (N\|f\|_{L^p(w)})^p$$

for all $f \in L^0(\mathbb{R}^n)$, then $N \gtrsim [w]_{A_p}$.

Proof If $p = 1$, this is already proven earlier in Theorem 280. If $p > 1$, carefully examine the proof of the case of $p = 1$; see Exercise 99.

We check that the power here is optimal including the case of $p = 1$.

Example 119. Let $1 < p < \infty$. Let $w(x) \equiv |x|^{(p-1)(n-\delta)}, x \in \mathbb{R}^n$ with $0 < \delta < 1$. Then for $f \equiv \chi_{B(1)}|\cdot|^{-n+\delta}$, we have $\|f\|_{L^p(w)} \sim \delta^{-\frac{1}{p}}$. Meanwhile, if we choose $\kappa > 0$ appropriately, then

$$w\{x \in \mathbb{R}^n : Mf(x) > \kappa\delta^{-1}\} = w(B(1)) \sim 1.$$

Thus, if the estimate $\lambda w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq D([w]_{A_p})^A \|f\|_{L^p(w)}$ holds, then $\delta^{-1} \leq D\delta^{-A(p-1)-\frac{1}{p}}$ for all $\delta > 0$, indicating that $-1 \geq -A(p-1) - \dfrac{1}{p}$, or equivalently, $Ap \geq 1$. Suppose we have

$$\lambda w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq D([w]_{A_1})^A \|f\|_{L^1(w)}$$

for any $\lambda > 0$. If we interpolate between this inequality and

$$\|Mf\|_{L^\infty(w)} \leq \|f\|_{L^\infty(w)} \leq D[w]_{A_1} \|f\|_{L^\infty(w)},$$

then

$$\lambda[w\{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\}]^{\frac{1}{p}} \le D'([w]_{A_1})^{\frac{A}{p}} \|f\|_{L^p(w)}.$$

Going through a similar argument using Example 114, we obtain that the power in the weighted weak-$(1,1)$ maximal inequality is sharp.

We now consider the weighted strong-(p,p) maximal inequality. As is announced before, the power of $[w]_{A_p}$ differs from the one in the weighted weak-(p,p) maximal inequality.

Theorem 290. *Let $w \in A_p$ with $1 < p < \infty$. Then for all $f \in L^p(w)$,*

$$\|Mf\|_{L^p(w)} \lesssim_{p,n} ([w]_{A_p})^{\frac{1}{p-1}} \|f\|_{L^p(w)}.$$

The proof hinges on the universal estimate. Let $W : \mathbb{R}^n \to (0,\infty)$ be a weight. Denote by $M_{\mathrm{c},W}$ the centered weighted maximal operator given by

$$M_{\mathrm{c},W}^{\mathcal{Q}} f(x) = M_{\mathrm{c},W} f(x) \equiv \sup_{Q \in \mathcal{Q},\, c(Q)=x} \frac{1}{W(5Q)} \int_Q |f(y)|\, W(y)\mathrm{d}y. \tag{7.5}$$

By virtue of Theorem 133, it is easy to see that $M_{\mathrm{c},W}$ is $L^p(W)$-bounded.

Let $\sigma \equiv w^{-\frac{1}{p-1}}$ be the dual weight.

Proof Let $f \in L^0(\mathbb{R}^n)$. Let Q be a cube containing a point $x \in \mathbb{R}^n$. By the definition of the A_p-weights we have $\dfrac{w(5Q)}{|5Q|}\left(\dfrac{\sigma(5Q)}{|5Q|}\right)^{p-1} \le [w]_{A_p}$. Hence, (without using Hölder's inequality) by the definition of the maximal operator we deduce

$$\begin{aligned} m_Q(|f|) &\lesssim [w]_{A_p}^{\frac{1}{p-1}} \left(\frac{|Q|}{w(5Q)}\left(\frac{1}{\sigma(5Q)}\int_Q |f(y)|\mathrm{d}y\right)^{p-1}\right)^{\frac{1}{p-1}} \\ &= [w]_{A_p}^{\frac{1}{p-1}} \left(\frac{|Q|}{w(5Q)}\left(\frac{1}{\sigma(5Q)}\int_Q |f(y)|\sigma(y)^{-1}\sigma(y)\mathrm{d}y\right)^{p-1}\right)^{\frac{1}{p-1}}. \end{aligned}$$

(It is noteworthy that we do not use $w \in A_p$ elsewhere.) Using the definition of the weighted maximal operators, we obtain

$$\begin{aligned} m_Q(|f|) &\le [w]_{A_p}^{\frac{1}{p-1}} \left(\frac{|Q|}{w(5Q)}\left(\inf_{z\in Q} M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}](z)\right)^{p-1}\right)^{\frac{1}{p-1}} \\ &= [w]_{A_p}^{\frac{1}{p-1}} \left(\frac{1}{w(5Q)}\int_Q \left(\inf_{Z\in Q} M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}](Z)\right)^{p-1}\mathrm{d}z\right)^{\frac{1}{p-1}} \\ &\le [w]_{A_p}^{\frac{1}{p-1}} \left(\frac{1}{w(5Q)}\int_Q M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}](z)^{p-1}\mathrm{d}z\right)^{\frac{1}{p-1}} \\ &= [w]_{A_p}^{\frac{1}{p-1}} \left(\frac{1}{w(5Q)}\int_Q M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}](z)^{p-1}w(z)^{-1}w(z)\mathrm{d}z\right)^{\frac{1}{p-1}} \\ &\le \left([w]_{A_p} M_{\mathrm{c},w}\left[M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}]^{p-1}w^{-1}\right](x)\right)^{\frac{1}{p-1}} \end{aligned}$$

for all $x \in Q$. Or equivalently, since Q is chosen arbitrarily,

$$Mf(x) \lesssim_{p,n} \left([w]_{A_p} M_{\mathrm{c},w}\left[M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}]^{p-1}w^{-1}\right](x)\right)^{\frac{1}{p-1}} \quad (x\in\mathbb{R}^n).$$

Inserting this pointwise estimate into $\|Mf\|_{L^p(w)}$ and then using the boundedness of $M_{\mathrm{c},w}$, we obtain

$$\begin{aligned}
\|Mf\|_{L^p(w)} &\lesssim_{p,n} [w]_{A_p}^{\frac{1}{p-1}} \|M_{\mathrm{c},w}[(M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}])^{p-1}w^{-1}]^{\frac{1}{p-1}}\|_{L^p(w)}\\
&\le ([w]_{A_p}\|M_{\mathrm{c},w}[(M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}])^{p-1}w^{-1}]\|_{L^{p'}(w)})^{\frac{1}{p-1}}\\
&\lesssim_p ([w]_{A_p}\|(M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}])^{p-1}w^{-1}\|_{L^{p'}(w)})^{\frac{1}{p-1}}\\
&= ([w]_{A_p}\|(M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}])^{p-1}\|_{L^{p'}(\sigma)})^{\frac{1}{p-1}}\\
&= [w]_{A_p}^{\frac{1}{p-1}}\|M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}]\|_{L^p(\sigma)}.
\end{aligned}$$

If we use the boundedness of $M_{\mathrm{c},\sigma}$, then we have

$$\|Mf\|_{L^p(w)} \lesssim_{p,n} [w]_{A_p}^{\frac{1}{p-1}}\|M_{\mathrm{c},\sigma}[f\cdot\sigma^{-1}]\|_{L^p(\sigma)} \lesssim_p [w]_{A_p}^{\frac{1}{p-1}}\|f\cdot\sigma^{-1}\|_{L^p(\sigma)} = \|f\|_{L^p(w)}$$

for all $f \in L^0(\mathbb{R}^n)$. Thus, we obtain the desired $L^p(w)$-estimate.

Example 120. We dualize Theorem 290. Let $1 < p < \infty$ and $w \in A_p$. Then $\|w^{-1}M[wh]\|_{L^{p'}(w)} = \|M[wh]\|_{L^{p'}(w^{-\frac{1}{p-1}})} \lesssim ([w^{-\frac{1}{p-1}}]_{A_{p'}})^{\frac{1}{p'-1}}\|h\|_{L^{p'}(w)} = [w]_{A_p}\|h\|_{L^{p'}(w)}$ for all $h \in L^0(\mathbb{R}^n)$.

As the following example shows, the power in $[w]_{A_p}$ is best possible:

Example 121. Let $W(x) \equiv |x|^{(p-1)(n-\delta)}$, $x \in \mathbb{R}^n$, where $1 < p < \infty$ and $0 < \delta \le n$. Then $f \equiv \chi_{B(1)}|\cdot|^{-n+\delta} \in L^p(W)$. Meanwhile $Mf \gtrsim \delta^{-1}f$. Thus, we cannot improve the power a in the bound $\|Mf\|_{L^p(W)} \lesssim ([W]_{A_p})^a\|f\|_{L^p(W)}$ with $a = \dfrac{1}{p-1}$.

A doubling weight w is a weight whose corresponding weighted measure $w\mathrm{d}x = w(x)\mathrm{d}x$ is a doubling measure. We will use the *strong doubling property* of A_p weights. For the proof we use the weighted weak maximal inequality.

Corollary 291. *Let $w \in A_p$, $p \ge 1$, and let $Q \in \mathcal{Q}$. Then the strong doubling property* $\left(\dfrac{|S|}{|Q|}\right)^p w(Q) \lesssim [w]_{A_p} w(S)$ *holds for any measurable set $S \subset Q$.*

Proof Using Theorem 289, we have

$$w(Q) \le w(\{x\in\mathbb{R}^n : M\chi_S(x) \ge |Q|^{-1}|S|\}) \lesssim [w]_{A_p}(|S|^{-1}|Q|)^p w(S).$$

Let $\mathbf{a} \in \{0, 1/3\}^n$ be fixed. Suppose that we have a family $\mathcal{S}_j = \{Q_k^j\}_{k\in K_j}$ of disjoint cubes in $\mathfrak{D}_{\mathbf{a}}$ for each $j \in \mathbb{N}_0$. Recall that $\mathcal{S} \equiv \bigcup_{j=0}^{\infty} \mathcal{S}_j$ is an η-sparse

family with a level structure if $\bigcup_{k\in K_{j+1}} Q_k^{j+1} \subset \bigcup_{k\in K_j} Q_k^j$ and $\sum_{k\in K_{j+1}} |Q_k^{j+1}\cap R| \le \eta|R|$ for all $R \in \mathcal{S}_j$, $j \in \mathbb{N}$. We also remark that a weight w belongs to A_2 if and only if w^{-1} belongs to A_2. Hence, the class A_2 is special. Due to this special property we can prove the following boundedness. We note that the estimate $[w]_{A_2}$ is optimal in view of the A_2-theorem, which we consider later.

Lemma 292. *Let $\mathbf{a} \in \{0, 1/3\}^n$. Let $\{Q_k^j\}_{j\in\mathbb{N},k\in K_j} \subset \mathfrak{D}_{\mathbf{a}}$ be a sparse family of generalized dyadic cubes with a level structure. As long as the sum makes sense, define*

$$\mathcal{A}f \equiv \sum_{j=1}^{\infty}\sum_{k\in K_j} m_{Q_k^j}(f)\chi_{Q_k^j}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $\|\mathcal{A}f\|_{L^2(w)} \lesssim [w]_{A_2}\|f\|_{L^2(w)}$ for all $f \in L^2(w)$ and $w \in A_2$.

Before the proof, a helpful remark may be in order. As is easily seen from the inequality $\|f\chi_Q\|_{L^1} \le \|f\|_{L^2(w)}\|\chi_Q\|_{L^2(w^{-1})}$, if $f \in L^2(w)$ and $w \in A_2$, then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, at least the definition of $m_{Q_{kj}}(f)$ in $\mathcal{A}f$ makes sense if $f \in L^2(w)$.

For the proof we use the following notation. For a weight v, $f \in L^0(\mathbb{R}^n)$ and $\mathbf{a} \in \{0, 1/3\}^n$, we define

$$M_v^{\mathfrak{D}_{\mathbf{a}}} f(x) \equiv \sup_{Q\in\mathfrak{D}_{\mathbf{a}}} \frac{\chi_Q(x)}{v(Q)}\int_Q |f(y)|v(y)\mathrm{d}y \quad (x \in \mathbb{R}^n).$$

Set $\Omega_j \equiv \bigcup_{k\in K_j} Q_k^j$ and $E_k^j \equiv Q_k^j \setminus \Omega_{j+1}$, so that $|E_k^j| \gtrsim |Q_k^j|$.

Proof Write $\sigma \equiv w^{-1}$. We dualize the left-hand side: Matters are reduced to showing

$$\mathrm{I} \equiv \int_{\mathbb{R}^n} \mathcal{A}f(x)g(x)\mathrm{d}x \lesssim [w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(\sigma)}$$

for all $f \in L^2(w) \cap \mathbb{M}^+(\mathbb{R}^n)$ and $g \in L^2(\sigma) \cap \mathbb{M}^+(\mathbb{R}^n)$. By the truncation, we may also assume that $f, g \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. If we insert the definition of $\mathcal{A}$ into the left-hand side, then we obtain $\mathrm{I} = \sum_{j=1}^{\infty}\sum_{k\in K_j} m_{Q_k^j}(f)m_{Q_k^j}(g)|Q_k^j|$. Thus,

$$\mathrm{I} \lesssim \sum_{j=1}^{\infty}\sum_{k\in K_j} m_{Q_k^j}(f)m_{Q_k^j}(g)|E_k^j|. \tag{7.6}$$

Since $w \in A_2$, we have

$$\begin{aligned} &m_{Q_k^j}(f)m_{Q_k^j}(g) \\ &\le [w]_{A_2}\cdot\frac{1}{\sigma(Q_k^j)}\int_{Q_k^j} f(x)w(x)\sigma(x)\mathrm{d}x\cdot\frac{1}{w(Q_k^j)}\int_{Q_k^j} g(x)\sigma(x)w(x)\mathrm{d}x. \end{aligned}$$

By the definition of the maximal operator $M_\sigma^{\mathfrak{D}_\mathbf{a}}$ and $M_w^{\mathfrak{D}_\mathbf{a}}$, we obtain

$$m_{Q_k^j}(f)m_{Q_k^j}(g) \le [w]_{A_2} \inf_{x\in Q_k^j} M_\sigma^{\mathfrak{D}_\mathbf{a}}[f\cdot w](x)\cdot \inf_{x\in Q_k^j} M_w^{\mathfrak{D}_\mathbf{a}}[g\cdot\sigma](x).$$

If we insert this estimate into (7.6), then

$$\mathrm{I} \lesssim [w]_{A_2}\|M_\sigma^{\mathfrak{D}_\mathbf{a}}[f\cdot w]\|_{L^2(\sigma)}\|M_w^{\mathfrak{D}_\mathbf{a}}[g\cdot\sigma]\|_{L^2(w)}.$$

It remains to use the universal estimate, Theorem 257.

We end Section 7.1.2 with the following two observations on the classes A_1, A_p and A_r with $1<r\le p<\infty$ and with $1\le p\le r$.

Lemma 293. *Let $1<r\le p<\infty$. If $w\in A_p$ and $W\in\mathbb{M}^+(\mathbb{R}^n)$ satisfy $Ww\in A_1$, then $[W^{\frac{p-r}{p-1}}w]_{A_r}\le([Ww]_{A_1})^{\frac{p-r}{p-1}}([w]_{A_p})^{\frac{r-1}{p-1}}$.*

If $p=\infty$, Lemma 293 formally says that $A_1\subset A_r$. Although the proof is simple, we need monkish patience.

Proof Let $Q\in\mathcal{Q}$ be fixed. We have to show that

$$\begin{aligned}\mathrm{I}&\equiv m_Q(W^{\frac{p-r}{p-1}}w)\left(\frac{1}{|Q|}\int_Q (W(x)^{\frac{p-r}{p-1}}w(x))^{-\frac{1}{r-1}}\mathrm{d}x\right)^{r-1}\\&\le([Ww]_{A_1})^{\frac{p-r}{p-1}}([w]_{A_p})^{\frac{r-1}{p-1}}.\end{aligned}$$

Since $W^{\frac{p-r}{p-1}}w=(Ww)^{\frac{p-r}{p-1}}w^{\frac{r-1}{p-1}}$, we have

$$\begin{aligned}\mathrm{I}&=m_Q(W^{\frac{p-r}{p-1}}w)\left(\frac{1}{|Q|}\int_Q (W(x)w(x))^{-\frac{p-r}{(p-1)(r-1)}}w(x)^{-\frac{p'}{p}}\mathrm{d}x\right)^{r-1}\\&\le m_Q(W^{\frac{p-r}{p-1}}w)m_Q(w^{-\frac{1}{p-1}})^{r-1}\left(\left\|\frac{1}{Ww}\right\|_{L^\infty(Q)}\right)^{\frac{p-r}{p-1}}.\end{aligned}$$

By Hölder's inequality, $m_Q(W^{\frac{p-r}{p-1}}w)\le m_Q(Ww)^{\frac{p-r}{p-1}}m_Q(w)^{\frac{r-1}{p-1}}$. Thus,

$$\mathrm{I}\le([Ww]_{A_1})^{\frac{p-r}{p-1}}m_Q(w^{-\frac{1}{p-1}})^{r-1}m_Q(w)^{\frac{r-1}{p-1}}\le([Ww]_{A_1})^{\frac{p-r}{p-1}}([w]_{A_p})^{\frac{r-1}{p-1}}.$$

This completes the proof.

Lemma 294. *Let $1\le p\le r<\infty$. Also let $w\in A_1$ and $v\in A_p$. Then we have $v^{-r+p}w\in A_r$. More precisely, $[v^{-r+p}w]_{A_r}\le[v]_{A_1}{}^{r-p}[w]_{A_r}$.*

Once again we need to prove Lemma 294 with monkish patience.

Proof Let $Q\in\mathcal{Q}$ be fixed. Then,

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q v(x)^{-r+p}w(x)\mathrm{d}x\right)m_Q(w^{-\frac{1}{r-1}}v^{\frac{p-r}{r-1}})^{r-1}\\&\le\|v^{-r+p}\|_{L^\infty(Q)}m_Q(w)m_Q(w^{-\frac{1}{r-1}}v^{\frac{p-r}{r-1}})^{r-1}\end{aligned}$$

by the definition of the $L^\infty(Q)$-norm. We remark that $\dfrac{p-1}{r-1}+\dfrac{r-p}{r-1}=1$. By Hölder's inequality

$$m_Q(w^{-\frac{1}{r-1}}v^{\frac{p-r}{r-1}}) \le m_Q(w^{-\frac{1}{p-1}})^{\frac{p-1}{r-1}} m_Q(v)^{\frac{p-r}{r-1}}.$$

If we insert this estimate into the above expression, we obtain the desired result.

Example 122. Let $1 < q \le p < \infty$. Then M is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$. In fact, for $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, letting $\theta = \frac{q}{2p}$, we obtain

$$\|Mf\|_{\mathcal{M}^p_q} \sim \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|Mf\|_{L^q(M^{(\frac{1}{1-\theta})}\chi_Q)}$$

thanks to Proposition 285. Meanwhile, since $M^{(\frac{1}{1-\theta})}\chi_Q \in A_1$, we have

$$\sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|Mf\|_{L^q(M^{(\frac{1}{1-\theta})}\chi_Q)} \lesssim \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(M^{(\frac{1}{1-\theta})}\chi_Q)}.$$

If we use Proposition 285 once again, $\|Mf\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}$. This gives another proof of Theorem 382 to follow. This simple proof seems to be applicable to many things. However, as it turns out, in some generalized cases, we need different discussions.

To conclude this section, we remark that we can consider the local counterpart of the results in this section. The details are omitted.

7.1.3 The class A_∞

Since M is always bounded on $L^\infty(w)$, it does not seem to make sense to introduce the class A_∞. However, since $[w]_{A_p} \ge [w]_{A_q}$ if $1 \le p \le q < \infty$, it seems important that we take the limit as $p \to \infty$ of $[w]_{A_p}$ to define A_∞. This attempt makes the following definition natural:

Definition 72. A weight w is an *A_∞-weight*, if $0 < w < \infty$ almost everywhere, and $[w]_{A_\infty} \equiv \sup_{Q\in\mathcal{Q}} m_Q(w)\exp(-m_Q(\log w)) < \infty$. The quantity $[w]_{A_\infty}$ is referred to as the *A_∞-constant* or the *A_∞-characteristic*. The class A_∞ collects all weights w for which it satisfies $0 < w < \infty$ and $[w]_{A_\infty}$ is finite.

We can extend Remark 13 as follows:

Example 123. It is easy to check $A_1 \subset A_p \subset A_q \subset A_\infty \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by Hölder's inequality whenever $1 \le p \le q \le \infty$. Thus $A_\infty \supset \bigcup_{1\le p<\infty} A_p$. Later we will check that the last inclusion is actually an equality.

For the time being we investigate the property of the class A_∞. Let w be a weight. If we use Hölder's inequality, then $m_Q^{(1+\varepsilon)}(w) \geq m_Q(w)$ for all weights w. However, since $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ does not imply $w \in L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $\varepsilon > 0$, the reverse inequality $m_Q^{(1+\varepsilon)}(w) \leq m_Q(w)$ and its weaker version

$$m_Q^{(1+\varepsilon)}(w) \lesssim m_Q(w) \tag{7.7}$$

fails.

However, if the weight belongs to the class A_∞, then the reverse estimate can be obtained in the following sense:

Theorem 295 (Reverse Hölder inequality)**.** *Let $w \in A_\infty$. We abbreviate $\varepsilon \equiv \dfrac{1}{2^{n+3}[w]_{A_\infty}} > 0$. Then $m_Q^{(1+\varepsilon)}(w) \leq 2m_Q(w)$ for all cubes $Q \in \mathcal{Q}$.*

Proof By the dilation and the translation, we may assume that $Q \in \mathcal{D}(\mathbb{R}^n)$. We will use cubes in $\mathcal{D}(Q)$. Thus, we may assume that $w \in L^\infty(\mathbb{R}^n)$ by approximating w with functions of the form $\sum_{R \in \mathcal{D}_j(\mathbb{R}^n)} m_R(w)\chi_R$ for some $j \in \mathbb{Z}$. Let $E_{Q,\lambda} \equiv \{x \in Q : M^{\mathcal{D}(Q)}[\chi_Q w](x) > \lambda\}$ for $\lambda > 0$. We use the Layer-Cake formula to obtain

$$\mathrm{I} \equiv \int_Q M^{\mathcal{D}(Q)}[\chi_Q w](x)^\varepsilon w(x)\mathrm{d}x = \varepsilon \int_0^\infty \lambda^{\varepsilon-1} w(E_{Q,\lambda})\mathrm{d}\lambda.$$

Let $\lambda > m_Q(w)$. Consider the set $\{Q_j\}_{j \in J(\lambda)} \subset \mathcal{D}(Q)$ of all maximal dyadic cubes in the set $E_{Q,\lambda}$ whose average of w exceeds λ. Since $\lambda > m_Q(w)$, each Q_j is a proper subset of Q. So thanks to the maximality of each Q_j, we have

$$w(E_{Q,\lambda}) = \sum_{j \in J(\lambda)} w(Q_j) \leq 2^n \lambda \sum_{j \in J(\lambda)} |Q_j| = 2^n \lambda |E_{Q,\lambda}|.$$

Thus

$$\begin{aligned}
\mathrm{I} &\leq \varepsilon \int_0^{m_Q(w)} \lambda^{\varepsilon-1} w(E_{Q,\lambda})\mathrm{d}\lambda + 2^n \varepsilon \int_{m_Q(w)}^\infty \lambda^\varepsilon |E_{Q,\lambda}|\mathrm{d}\lambda \\
&\leq m_Q(w)^{1+\varepsilon}|Q| + \frac{2^n \varepsilon}{1+\varepsilon} \int_Q M^{\mathcal{D}(Q)}[\chi_Q w](x)^{1+\varepsilon}\mathrm{d}x.
\end{aligned}$$

Since $w \in A_\infty$, $m_R(w) \leq [w]_{A_\infty} \exp\left(m_R(\log w)\right)$ for all $R \in \mathcal{D}(\mathbb{R}^n)$. This implies that

$$M^{\mathcal{D}(Q)} w(x) \leq [w]_{A_\infty} M^{\mathcal{D}(Q)}_{\log} w(x) = [w]_{A_\infty} \sup_{R \in \mathcal{D}(Q)} \chi_R(x) \exp\left(m_R(\log w)\right)$$

for all $x \in Q$. Thus

$$\mathrm{I} \leq m_Q(w)^{1+\varepsilon}|Q| + \frac{2^n \varepsilon ([w]_{A_\infty})^{1+\varepsilon}}{1+\varepsilon} \int_Q M^{\mathcal{D}(Q)}_{\log}[\chi_Q w](x)^{1+\varepsilon}\mathrm{d}x.$$

Due to the Lebesgue differentiation theorem, we have

$$\int_Q w(x)^{1+\varepsilon}\mathrm{d}x \le m_Q(w)^{1+\varepsilon}|Q| + \frac{2^n\varepsilon([w]_{A_\infty})^{1+\varepsilon}}{1+\varepsilon}\int_Q M_{\log}^{\mathcal{D}(Q)}[\chi_Q w](x)^{1+\varepsilon}\mathrm{d}x.$$

By virtue of (6.16), we have

$$\int_Q w(x)^{1+\varepsilon}\mathrm{d}x \le m_Q(w)^{1+\varepsilon}|Q| + \frac{2^n e\varepsilon}{1+\varepsilon}([w]_{A_\infty})^{1+\varepsilon}\int_Q w(x)^{1+\varepsilon}\mathrm{d}x.$$

Note that

$$\frac{2^n e\varepsilon[w]_{A_\infty}}{1+\varepsilon}([w]_{A_\infty})^{\varepsilon} = \frac{2^n e[w]_{A_\infty}([w]_{A_\infty})^{\frac{1}{2^{n+3}[w]_{A_\infty}}}}{2^{n+3}[w]_{A_\infty}+1} \le \frac{e}{8}\exp\left(\frac{1}{16e}\right) \le \frac{1}{2},$$

since $e^a \le 1+2a$ for $0<a<0.1$. Thus, it follows that

$$\int_Q w(x)^{1+\varepsilon}\mathrm{d}x \le m_Q(w)^{1+\varepsilon}|Q| + \frac{1}{2}\int_Q w(x)^{1+\varepsilon}\mathrm{d}x.$$

Since w is assumed to be bounded, we obtain the desired result.

The following corollary can be located as the weak version of Corollary 291.

Lemma 296. *Let E be a Lebesgue measurable set contained in a cube Q. Also let $w \in A_\infty$ and $a > 2^{n+1+2^{n+4}[w]_{A_\infty}}$. Assume that $|Q| \le 2^n a^{-1}|E|$. Then $w(Q) \lesssim_{[w]_{A_\infty}} w(E)$.*

Proof Since $w(Q) = w(E) + w(Q\setminus E)$, it suffices to show that

$$\frac{w(Q\setminus E)}{w(Q)} \le 2\left(\frac{1}{2^{1+2^{n+4}[w]_{A_\infty}}}\right)^{\frac{1}{1+2^{n+3}[w]_{A_\infty}}} = 2^{-\frac{2^{n+3}[w]_{A_\infty}}{1+2^{n+4}[w]_{A_\infty}}} (<1). \quad (7.8)$$

Let $\varepsilon \equiv \dfrac{1}{2^{n+3}[w]_{A_\infty}}$ as before. First, observe that $|Q\setminus E| \le 2^n a^{-1}|Q|$. As a result, by Theorem 295 we obtain

$$\begin{aligned} w(Q\setminus E) &\le \left(\int_Q w(x)^{1+\varepsilon}\mathrm{d}x\right)^{\frac{1}{1+\varepsilon}} |Q\setminus E|^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le 2\frac{|Q|^{\frac{1}{1+\varepsilon}}}{w(Q)} m_Q(w)\left(\frac{2^n}{a}|Q|\right)^{\frac{\varepsilon}{1+\varepsilon}} w(Q) \\ &= 2\left(\frac{2^n}{a}\right)^{\frac{\varepsilon}{1+\varepsilon}} w(Q) \\ &\le 2\left(\frac{1}{2^{1+2^{n+4}[w]_{A_\infty}}}\right)^{\frac{1}{1+2^{n+3}[w]_{A_\infty}}}. \end{aligned}$$

Thus we obtain (7.8).

Another direct consequence of Theorem 295 is that the class A_p with $1 < p < \infty$ enjoys the *openness property*. We will characterize the class A_p as follows:

Theorem 297. *If $w \in A_p$ with $1 < p \le \infty$, then $w \in A_q$ for some $1 \le q < p$. In other words $A_p = \bigcup_{q \in [1,p)} A_q$.*

Proof According to Remark 13, we see that $A_v \subset A_u \subset A_\infty$ whenever $1 \le v \le u \le \infty$. So, one inclusion is clear.

We now distinguish two cases: $p < \infty$ and $p = \infty$.

Suppose $w \in A_p$ and $p < \infty$. Then $w^{-\frac{1}{p-1}} \in A_{p'} \subset A_\infty$. Let ε be the constant satisfying $m_Q^{(1+\varepsilon)}(w^{-\frac{1}{p-1}}) \le 2m_Q(w^{-\frac{1}{p-1}})$ for all cubes Q, whose existence is guaranteed by Theorem 295. Thus if we set $q \equiv \dfrac{p+\varepsilon}{1+\varepsilon}$, then we have $m_Q(w^{-\frac{1}{q-1}})^{q-1} \le 2^{p-1} m_Q(w^{-\frac{1}{p-1}})^{p-1}$, implying that $w \in A_q$.

Suppose instead that $w \in A_\infty$ (and $p = \infty$). Use Theorem 295 once again. Then according to reverse Hölder's inequality, $\dfrac{w(E)}{w(Q)} \le 2\left(\dfrac{|E|}{|Q|}\right)^{\frac{\varepsilon}{1+\varepsilon}}$, whenever E is a subset of a cube Q. Consequently, if we let $0 < \alpha_0 \ll 1$ and $0 < \beta_0 \equiv 2\alpha_0{}^{\frac{\varepsilon}{1+\varepsilon}}$, then $w(E) \le \beta_0 w(Q)$ for all measurable sets E and Q such that Q is a cube containing E and that $|E| \le \alpha_0 |Q|$. Set $\alpha \equiv 1 - \beta_0 \in (0,1)$ and $\beta \equiv 1 - \alpha_0 \in (0,1)$. If we contrapose this fact, then

$$|E| < \beta |Q| \tag{7.9}$$

whenever measurable sets E and Q satisfy that Q is a cube containing E $w(E) < \alpha w(Q)$. This in particular implies that $w(B(x,2r)) \lesssim w(B(x,r))$ for all $x \in \mathbb{R}^n$ and $r > 0$.

To prove $w \in A_u$ for some $u \in (1,\infty)$, fix a cube Q and set $\lambda_k \equiv \left(\frac{D}{\alpha}\right)^k \lambda_0$ for each $k \in \mathbb{N}_0$, where $D \gg 1$ and $\lambda_0 \equiv \dfrac{|Q|}{w(Q)}$. Denote by $M_w^{\mathcal{D}(Q)}$ the dyadic maximal operator with respect to Q and the weight w:

$$M_w^{\mathcal{D}(Q)} f(x) \equiv \sup_{R \in \mathcal{D}(Q)} \frac{\chi_R(x)}{w(R)} \int_R |f(y)| w(y) \mathrm{d}y \quad (x \in \mathbb{R}^n).$$

Keeping in mind the fact that $D \gg 1$, we split $\Omega_k \equiv \{x \in Q : M_w^{\mathcal{D}(Q)}[w^{-1}](x) > \lambda_k\}$ into a disjoint union of maximal dyadic cubes $\{Q_k^j\}_j$ such that

$$\lambda_k < \frac{|Q_k^j|}{w(Q_k^j)} < D\lambda_k = \alpha \lambda_{k+1}. \tag{7.10}$$

Fix $k \in \mathbb{N}_0$. By the Lebesgue differentiation theorem, $w(x) \ge \dfrac{1}{\lambda_k}$ for a.e. $x \in Q \setminus \Omega_k$.

Let k, j_0 be fixed. Then

$$w(\Omega_{k+1} \cap Q_{k,j_0}) \le \sum_{j;Q_{k+1,j} \in \mathcal{D}(Q_{k,j_0})} w(Q_{k+1,j}).$$

Recall that two dyadic cubes are disjoint unless one is contained in the other. Therefore we have

$$w(\Omega_{k+1} \cap Q_{k,j_0}) \le \sum_{j;Q_{k+1,j} \in \mathcal{D}(Q_{k,j_0})} \frac{|Q_{k+1,j}|}{\lambda_{k+1}} \le \frac{|Q_{k,j_0}|}{\lambda_{k+1}}.$$

If we use (7.10) once again, then $w(\Omega_{k+1} \cap Q_{k,j_0}) \le \dfrac{|Q_{k,j_0}|}{\lambda_{k+1}} \le \alpha w(Q_{k,j_0})$. We can pass the above inequality to the unweighted one: $|\Omega_{k+1} \cap Q_{k,j_0}| \le \beta |Q_{k,j_0}|$. Adding the above inequality over j_0, we obtain

$$|\Omega_{k+1}| \le \beta |\Omega_k|, \tag{7.11}$$

yielding that $\bigcap_{k=0}^{\infty} \Omega_k = 0$ is almost equal to an empty set. We write $\Omega_{-1} \equiv Q$. Then $\displaystyle\int_Q \frac{\mathrm{d}x}{w(x)^\tau} = \sum_{k=-1}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} \frac{\mathrm{d}x}{w(x)^\tau}$. Invoking (7.10)–(7.11), we obtain

$$\begin{aligned}
\int_Q \frac{\mathrm{d}x}{w(x)^\tau} &\le \sum_{k=-1}^{\infty} {\lambda_{k+1}}^\tau |\Omega_k| \\
&\lesssim {\lambda_0}^\tau \sum_{k=-1}^{\infty} \left(\frac{D}{\alpha}\right)^{k\tau} \beta^k |\Omega_0| \\
&\le {\lambda_0}^\tau |Q| \sum_{k=-1}^{\infty} \left(\frac{D}{\alpha}\right)^{k\tau} \beta^k.
\end{aligned}$$

Therefore, if $\tau > 0$ is sufficiently small, then the series in the above inequality converges. Thus, we obtain $\dfrac{1}{|Q|} \displaystyle\int_Q \frac{\mathrm{d}x}{w(x)^\tau} \lesssim \frac{|Q|^\tau}{w(Q)^\tau}$. Hence, $w \in A_{1+\tau^{-1}}$.

Example 124. From (7.9), we obtain $w(2^k Q) \ge \kappa w(2^{k-1} Q)$ for all $k \in \mathbb{N}$ and for all cubes Q, where $\kappa > 1$ is independent of Q and k. In particular, $\displaystyle\sum_{k=1}^{\infty} w(2^k Q)^{-\theta} \lesssim w(Q)^{-\theta}$ for any cube Q and $\theta > 0$.

Let $0 < p < \infty$ and $0 < q \le \infty$. Given a sequence of measurable functions $\{f_j\}_{j=0}^{\infty}$, we define

$$\|\{f_j\}_{j=0}^{\infty}\|_{\ell^q(L^p(w))} \equiv \left\{\sum_{j=0}^{\infty}(\|f_j\|_{L^p(w)})^q\right\}^{\frac{1}{q}},$$

$$\|\{f_j\}_{j=0}^{\infty}\|_{L^p(w,\ell^q)} \equiv \left\|\left(\sum_{j=0}^{\infty}|f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}.$$

Define $\ell^q(L^p(w))$ and $L^p(w,\ell^q)$, called *weighted vector-valued Lebesgue spaces*, to be the collection of all sequences $\{f_j\}_{j=0}^{\infty}$ of measurable functions for which the norms $\|\{f_j\}_{j=0}^{\infty}\|_{\ell^q(L^p(w))}$ and $\|\{f_j\}_{j=0}^{\infty}\|_{L^p(w,\ell^q)}$ are finite, respectively. As applications of Theorem 270, we can prove the following inequality:

Theorem 298 (Weighted vector-valued inequality for M)**.** *Let $1 < p, q < \infty$, $1 < r \le \infty$ and $w \in A_p$. Then $\|\{Mf_j\}_{j=0}^{\infty}\|_{L^p(w,\ell^q)} \lesssim_{p,q,r} \|\{f_j\}_{j=0}^{\infty}\|_{L^p(w,\ell^q)}$ for all $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$.*

Proof Since $w \in A_{\tilde{p}}$ for some $\tilde{p} \in (1,p)$ by virtue of Theorem 295, we see that $Mf_j \lesssim M_{22,\mathrm{c},\mu}[|f_j|^{\frac{p}{\tilde{p}}}]^{\frac{\tilde{p}}{p}}$, where $\mu \equiv w\mathrm{d}x$ and $M_{22,\mathrm{c},\mu}$ is defined in Section 6.2.1 with $k = 22$. It remains to combine Theorem 270 with this pointwise estimate.

7.1.4 The class $A_{p,q}$

We are now oriented to the class of (couples of) weights adapted to fractional integral operators or fractional maximal operators. Weak type estimates are easy to obtain. Hence, we formulate the results in full generality.

Definition 73 ($A_{p,q}^{\alpha}$)**.** Given $1 < p \le q < \infty$ and $0 \le \alpha < n$, $A_{p,q}^{\alpha}$ collects all the pairs (u,σ) of weights if $[u,\sigma]_{A_{p,q}^{\alpha}} \equiv \sup_{Q\in\mathcal{Q}} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} m_Q(u)^{\frac{1}{q}} m_Q(\sigma)^{\frac{1}{p'}} < \infty$. The class $A_{p,q}^{\alpha}$ collects all pairs (u,σ) of weights for which $[u,\sigma]_{A_{p,q}^{\alpha}}$ is finite.

Example 125. Let $1 < p \le q < \infty$. Also let w be a weight. Write $\alpha = \frac{n}{p} - \frac{n}{q}$.

(1) We claim that $(u,\sigma) \equiv \left((Mw)^{-\frac{q}{p'}}, w\right) \in A_{p,q}^{\alpha}$. In fact, $m_Q(u)^{\frac{1}{q}} m_Q(\sigma)^{\frac{1}{p'}} = m_Q((Mw)^{-\frac{q}{p'}})^{\frac{1}{q}} m_Q(w)^{\frac{1}{p'}}$. If $x \in Q$, then $Mw(x) \ge m_Q(w)$. Therefore, we have

$$m_Q(u)^{\frac{1}{q}} m_Q(\sigma)^{\frac{1}{p'}} \le m_Q(w)^{-\frac{1}{p'}} m_Q(w)^{\frac{1}{p'}} = 1.$$

(2) Likewise, we can check that $(u,\sigma) = (w,(Mw)^{-\frac{p'}{q}}) \in A_{p,q}^{\alpha}$.

If the weights are radial and monotone, we can calculate the quantity with ease.

Example 126. Let $p, q > 1$ and $0 \le \alpha < n$. Also let v, w be weights. Assume that there exist $\hat{w} \in \mathbb{M}^{\uparrow}[0,\infty)$ and $\hat{v} \in \mathbb{M}^{\downarrow}[0,\infty)$ such that $w(x) = \hat{w}(|x|)$, $v(x) = \hat{v}(|x|)$ for all $x \in \mathbb{R}^n$. Define $\sigma \equiv w^{-\frac{1}{p-1}}$ and $\hat{\sigma} \equiv w^{-\frac{1}{p-1}}$. We compare $\mathcal{I} \equiv \sup_{R>0} R^{\alpha-n} \left(\|\hat{\sigma}\chi_{(0,R)}\|_{L^1(t^{n-1})}\right)^{\frac{1}{q'}} \left(\|\hat{v}\chi_{(0,R)}\|_{L^1(t^{n-1})}\right)^{\frac{1}{p}}$ and

$$\mathcal{J} \equiv \sup_{x\in\mathbb{R}^n, R>0} R^{\alpha-n}\sigma(B(x,R))^{\frac{1}{q'}} v(B(x,R))^{\frac{1}{p}}.$$

Since $\hat{v} \in \mathbb{M}^{\downarrow}(0,\infty)$, $v(B(x,r)) \le v(B(r))$ for any $x \in \mathbb{R}^n$ and $r > 0$ and, since $\hat{w} \in \mathbb{M}^{\uparrow}(0,\infty)$, we have $\sigma(B(x,r)) \le \sigma(B(r))$. Hence, $\mathcal{J} \sim \mathcal{I}$.

For the class $A^{\alpha}_{p,q}$, we have a weak type estimate.

Theorem 299. *Let $1 < p \le q < \infty$ and $0 \le \alpha < n$. Then for any pair of weights $(u,\sigma) \in A^{\alpha}_{p,q}$ we have*

$$\sup_{\lambda>0} \lambda(u\{x \in \mathbb{R}^n \,:\, M_\alpha f(x) > \lambda\})^{\frac{1}{q}} \lesssim [u,\sigma]_{A^{\alpha}_{p,q}} \|f\|_{L^p(\sigma^{1-p})} \tag{7.12}$$

for all $f \in L^p(\sigma^{1-p})$.

Note that (7.12) is necessary for $(u,\sigma) \in A^{\alpha}_{p,q}$ since $M_\alpha f(x) \ge \chi_Q(x)\ell(Q)^\alpha m_Q(|f|), x \in Q$ for all cubes Q and $f \in L^1(\mathbb{R}^n)$. Note also that

$$[u,\sigma]_{A^{\alpha}_{p,q}} = \inf\{D > 0 \,:\, \ell(Q)^\alpha m_Q(|f|)u(Q)^{\frac{1}{q}} \le D\|f\|_{L^p(\sigma^{1-p})}\}.$$

Proof We use Theorem 179. Let Q^0 be a fixed dyadic cube. By the monotone convergence theorem, we have only to work in Q^0. Set

$$E \equiv \left\{ x \in Q^0 \,:\, \sum_{Q\in\mathcal{S}} \ell(Q)^\alpha m_{3Q}(|f|)\chi_{E_Q}(x) > \lambda \right\}$$

for $\lambda > 0$ when we have a sparse family $\mathcal{S}$, where $E_Q \equiv Q \setminus \bigcup_{R\in\mathcal{S}\cap\mathcal{D}(Q)\setminus\{Q\}} R$ for each $Q \in \mathcal{S}$. We use Hölder's inequality to have

$$\sup_{Q\in\mathcal{Q}^\sharp(Q^0)} \ell(Q)^\alpha m_Q(|f|)\left(u(Q^0)\right)^{\frac{1}{q}} \lesssim [u,\sigma]_{A^{\alpha}_{p,q}} \|f\|_{L^p(\sigma^{1-p})}.$$

It thus remains to show that

$$\sup_{\lambda>0} \lambda\,(u(E))^{\frac{1}{q}} \lesssim [u,\sigma]_{A^{\alpha}_{p,q}} \|f\|_{L^p(\sigma^{1-p})}.$$

We concentrate on the first term. We fix $\lambda > 0$ and calculate

$$u(E) = \sum_{Q\in\mathcal{S}} \chi_{(\lambda,\infty)}(\ell(Q)^\alpha m_Q(|f|))u(E_Q).$$

We let $\mathcal{S}^*$ be the maximal disjoint family of cubes $Q \in \mathcal{S}$ satisfying $\ell(Q)^\alpha m_{3Q}(|f|) > \lambda$. Then

$$\lambda^q u(E) \le \sum_{Q\in\mathcal{S}^*} (\ell(Q)^\alpha m_{3Q}(|f|))^q u(Q).$$

By Hölder's inequality we have $m_{3Q}(|f|) \le m_{3Q}^{(p)}(f\cdot\sigma^{\frac{1-p}{p}}) m_{3Q}(\sigma)^{\frac{1}{p'}}$. As a result, we obtain

$$\begin{aligned}\lambda^q u(E) &\lesssim \sum_{Q\in\mathcal{S}^*} \left[\ell(Q)^{\alpha+\frac{n}{p'}} m_{3Q}^{(p)}(f\cdot\sigma^{\frac{1-p}{p}})\sigma(3Q)^{\frac{1}{p'}}\right]^q u(Q)\\ &\lesssim [u,\sigma]_{A_{p,q}^\alpha}{}^q \sum_{Q\in\mathcal{S}^*} \left(|Q| m_{3Q}^{(p)}(f\cdot\sigma^{\frac{1-p}{p}})\right)^q\\ &\lesssim [u,\sigma]_{A_{p,q}^\alpha}{}^q \left(\sum_{Q\in\mathcal{S}^*} |Q| m_{3Q}^{(p)}(f\cdot\sigma^{\frac{1-p}{p}})^p\right)^{\frac{q}{p}}.\end{aligned}$$

Here, for the last line, we used $q \ge p$. Since $\mathcal{S}^*$ is disjoint, we are in the position of adding this inequality over $Q \in \mathcal{S}^*$. Thus, $\lambda\,(u(E))^{\frac{1}{q}} \lesssim [u,\sigma]_{A_{p,q}^\alpha} \|f\|_{L^p(\sigma^{1-p})}$.

We consider a special case in the light of the classical Hardy–Littlewood–Sobolev theorem.

Definition 74 ($A_{p,q}$)**.** Let $1 < p \le q < \infty$. Let w be a weight. One says that $w \in A_{p,q}$ if $[w]_{A_{p,q}} \equiv \sup_{Q\in\mathcal{Q}} m_Q^{(q)}(w) m_Q^{(p')}(w^{-1}) < \infty$. The class $A_{p,q}$ collects all weights w for which $[w]_{A_{p,q}}$ is finite.

We consider the strong type inequalites.

Example 127. Let $1 < p, q < \infty$ and $\alpha \in \mathbb{R}$. Then $|\cdot|^\alpha \in A_{p,q}$ if and only if $|\cdot|^{\alpha q} \in A_{1+\frac{q}{p'}}$, or equivalently $-\frac{n}{q} < \alpha < \frac{n}{p'}$.

We characterize the class of weights for which the operator $f \in L^p(\mathbb{R}^n) \mapsto wI_\alpha[w^{-1}f] \in L^q(\mathbb{R}^n)$ is bounded.

Theorem 300. *Let $1 < q \le p < \infty$ and $0 < \alpha < n$. If $w \in A_{p,q}^\alpha$, then $\|I_\alpha f\|_{L^q(w^q)} \lesssim \|f\|_{L^p(w^p)}$ for all $f \in L^p(w^p)$.*

Note that there is no other condition on p, q and α.

For the proof we use the following notation: We define

$$M_w^{\mathcal{D}} f(x) \equiv \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} \frac{\chi_Q(x)}{w(Q)} \int_Q |f(y)| w(y)\mathrm{d}y \quad (x \in \mathbb{R}^n)$$

for a weight w and $f \in L^0(\mathbb{R}^n)$.

Proof Let $\beta \equiv \frac{n}{p} - \frac{n}{q}$. We also set $\sigma \equiv w^{-\frac{p}{p-1}}$ and $v \equiv w^q$. By duality and a sparse version of I_α (see Corollary 189), we have only to show

$$\mathrm{I} \equiv \sum_{Q \in \mathcal{S}} \ell(Q)^\alpha m_Q(f) \int_Q g(x)\mathrm{d}x \lesssim \|f\|_{L^p(w^p)} \|g\|_{L^{q'}(w^{-q'})}$$

for all $f \in L^p(w^p) \cap \mathbb{M}^+(\mathbb{R}^n)$ and $g \in L^{q'}(w^{-q'}) \cap \mathbb{M}^+(\mathbb{R}^n)$ and all sparse families $\mathcal{S}$. Although we need to take into account the vector $\vec{e}$, due to similarity, we assume $\vec{e} = 0$ and hence $\mathcal{S} \subset \mathcal{D}(\mathbb{R}^n)$.

By Hölder's inequality, we obtain

$$\begin{aligned}
\mathrm{I} &= \sum_{Q \in \mathcal{S}} \ell(Q)^{\alpha-n} \sigma(Q) v(Q)^{1-\frac{\beta}{n}} \frac{1}{\sigma(Q)} \int_Q f(x)\mathrm{d}x \cdot v(Q)^{\frac{\beta}{n}} \frac{1}{v(Q)} \int_Q g(x)\mathrm{d}x \\
&\le \sum_{Q \in \mathcal{S}} \ell(Q)^{\alpha-n} \sigma(Q) v(Q)^{1-\frac{\beta}{n}} \inf_{x \in Q} M_\sigma^{\mathcal{D}}[f \cdot \sigma^{-1}](x) \inf_{x \in Q} M_{v,\beta}^{\mathcal{D}}[g \cdot v^{-1}](x).
\end{aligned}$$

Arithmetic shows that $1 - \frac{\beta}{n} = \frac{1}{p'} + \frac{1}{q}$. If we insert the definition of $[w]_{A_{p,q}}^\alpha$ into the above expression, then

$$\mathrm{I} \lesssim \sum_{Q \in \mathcal{S}} \sigma(Q)^{\frac{1}{p}} v(Q)^{\frac{1}{p'}} \inf_{x \in Q} M_\sigma^{\mathcal{D}}[f \cdot \sigma^{-1}](x) \inf_{x \in Q} M_{v,\beta}^{\mathcal{D}}[g \cdot v^{-1}](x).$$

Since $\mathcal{S}$ is disjoint, we have

$$\mathrm{I} \lesssim \left(\int_{\mathbb{R}^n} M_\sigma^{\mathcal{D}}[f \cdot \sigma^{-1}](x)^p \sigma(x)\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} M_{v,\beta}^{\mathcal{D}}[g \cdot v^{-1}](x)^{p'} v(x)\mathrm{d}x \right)^{\frac{1}{p'}}.$$

By Theorem 257, the universal estimates of $M_\sigma^{\mathcal{D}}$ and $M_{v,\beta}^{\mathcal{D}}$, we obtain

$$\begin{aligned}
\mathrm{I} &\lesssim \left(\int_{\mathbb{R}^n} f(x)^p \sigma(x)^{-p} \sigma(x)\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} g(x)^{q'} v(x)^{-q'} v(x)\mathrm{d}x \right)^{\frac{1}{q'}} \\
&= \|f\|_{L^p(w^p)} \|g\|_{L^{q'}(w^{-q'})}.
\end{aligned}$$

Since $M_\alpha \lesssim I_\alpha$, we can transplant Theorem 300 to M_α.

Theorem 301. *Let $1 < q \le p < \infty$ and $0 < \alpha < n$. If $w \in A_{p,q}^\alpha$, then $\|M_\alpha f\|_{L^q(w^q)} \lesssim \|f\|_{L^p(w^p)}$ for all $f \in L^0(\mathbb{R}^n)$.*

7.1.5 Extrapolation

We have established and will establish that the fundamental operators in harmonic analysis behave well with respect to these classes of weights; see Theorems 280, 289, 290 and 298 for the Hardy–Littlewood maximal operator and Theorem 305 to follow for singular integral operators. When we consider estimates of A_p-weights, we face the following type of estimate:

$$\|Th\|_{L^p(W)} \le N([W]_{A_p}) \|h\|_{L^p(W)} \quad (h \in L^p(W), W \in A_p), \tag{7.13}$$

where T is a mapping from $L^p(W)$ to $L^0(\mathbb{R}^n)$ and $N \in \mathbb{M}^{\uparrow}[1,\infty)$. Extrapolation is a technique to expand the validity of (7.13) for all $1 < p < \infty$ based on the validity of (7.13) for some p_0.

The following proposition describes more precisely how large the constant should be when $N(t) = t^\theta$, $t \geq 0$. We obtain a technique to create new inequalities with the ingredient of weighted estimates.

Theorem 302. *Let $1 < r < \infty$ and $\theta \geq 0$. Let $\mathfrak{F} \subset \mathbb{M}^+(\mathbb{R}^n)^2$ be the set of all pairs (f,g) for which the inequality $\|g\|_{L^r(v)} \leq ([v]_{A_r})^\theta \|f\|_{L^r(v)}$ holds for all $v \in A_r$. Let $w \in A_p$ with $1 < p < \infty$.*

(1) *For $1 < p \leq r$ and $(f,g) \in \mathfrak{F}$, $\|g\|_{L^p(w)} \leq C_{p,r}([w]_{A_p})^{\frac{r-1}{p-1}\theta}\|f\|_{L^p(w)}$.*

(2) *For $r \leq p < \infty$ and $(f,g) \in \mathfrak{F}$, $\|g\|_{L^p(w)} \leq C_{p,r}([w]_{A_p})^{\theta}\|f\|_{L^p(w)}$.*

Proof If $f = 0$, then $g = 0$, so that there is nothing to prove in this case. Thus, we may assume $f \neq 0$. We may also suppose $p \neq r$ since the conclusion is nothing but the assumption when $p = r$.

(1) We use Theorem 284. If we choose

$$v \equiv |f| + \sum_{k=1}^{\infty} \frac{M^k f}{2^k (\|M\|_{L^p(w)\to L^p(w)})^k},$$

then $Mv \leq 2\|M\|_{L^p(w)\to L^p(w)} v$ thanks to the sublinearity of M, so that $[v]_{A_1} \leq 2\|M\|_{L^p(w)\to L^p(w)}$. Using Hölder's inequality and the decomposition $g = gv^{-1+\frac{p}{r}} \cdot v^{1-\frac{p}{r}}$, we have

$$\|g\|_{L^p(w)} \leq \left(\int_{\mathbb{R}^n} v(x)^p w(x)\mathrm{d}x\right)^{\frac{1}{p}-\frac{1}{r}} \|g\|_{L^r(v^{-r+p}w)}.$$

We will estimate each term. Since M is assumed on $L^p(w)$,

$$\left(\int_{\mathbb{R}^n} v(x)^p w(x)\mathrm{d}x\right)^{\frac{1}{p}-\frac{1}{r}} \leq (2\|f\|_{L^p(w)})^{1-\frac{p}{r}}.$$

Meanwhile, if $w \in A_p$ and $v \in A_1$, then $v^{-r+p}w \in A_r$ with the estimate $[v^{-r+p}w]_{A_r} \leq [v]_{A_1}{}^{r-p}[w]_{A_r}$ thanks to Lemma 294. By the property of $\mathfrak{F}$ and Theorem 290,

$$\begin{aligned}
\|g\|_{L^r(v^{-r+p}w)} &\leq ([v^{-r+p}w]_{A_r})^\theta \|f\|_{L^r(v^{-r+p}w)} \\
&\lesssim (\|M\|_{L^p(w)\to L^p(w)})^{(r-p)\theta} ([w]_{A_p})^\theta \|f\|_{L^r(v^{-r+p}w)} \\
&\lesssim ([w]_{A_p})^{\frac{r-p}{p-1}\theta} ([w]_{A_p})^\theta \|f\|_{L^r(v^{-r+p}w)} \\
&= ([w]_{A_p})^{\frac{r-1}{p-1}\theta} \|f\|_{L^r(v^{-r+p}w)}.
\end{aligned}$$

Since $|v| \geq |f|$, we have $\|f\|_{L^r(v^{-r+p}w)} \leq (\|f\|_{L^p(w)})^{\frac{p}{r}}$. Thus,

$$\|g\|_{L^r(v^{-r+p}w)} \lesssim ([w]_{A_p})^{\frac{r-1}{p-1}\theta} (\|f\|_{L^p(w)})^{\frac{p}{r}}.$$

If we combine these estimates, then we obtain the desired result.

(2) It suffices to show that

$$\int_{\mathbb{R}^n} g(x)h(x)w(x)\mathrm{d}x \le C_{p,r}([w]_{A_p})^{\theta}\|f\|_{L^p(w)}\|h\|_{L^{p'}(w)}$$

for all $h \in (L^{p'}(w) \cap \mathbb{M}^+(\mathbb{R}^n)) \setminus \{0\}$. With Example 120 in mind, let us set

$$H \equiv |h| + \sum_{k=0}^{\infty} \frac{w^{-1}M^k[wh]}{(D[w]_{A_p})^k}.$$

Here, $D \gg_p 1$. Since $h \le H$ and

$$H = H^{1-\frac{p'}{r'}}H^{\frac{p'}{r'}}, \quad r - r\frac{p'}{r'} = r - p'(r-1) = r - \frac{p(r-1)}{p-1} = \frac{p-r}{p-1},$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(x)h(x)w(x)\mathrm{d}x \\
&\le \int_{\mathbb{R}^n} g(x)H(x)^{1-\frac{p'}{r'}}H(x)^{\frac{p'}{r'}}w(x)\mathrm{d}x \qquad (7.14)\\
&\le \left(\int_{\mathbb{R}^n} g(x)^r H(x)^{\frac{p-r}{p-1}}w(x)\mathrm{d}x\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} H(x)^{p'}w(x)\mathrm{d}x\right)^{\frac{1}{r'}}.
\end{aligned}$$

Thanks to Theorem 290

$$[H\cdot w]_{A_1} \lesssim \|M\|_{L^{p'}(w^{-\frac{1}{p-1}})\to L^{p'}(w^{-\frac{1}{p-1}})} \sim ([w^{-\frac{1}{p-1}}]_{A_{p'}})^{\frac{1}{p'-1}} = [w]_{A_p}.$$

Consequently, using Lemma 293, we deduce

$$[H^{\frac{p-r}{p-1}}w]_{A_r} \lesssim ([H\cdot w]_{A_1})^{\frac{p-r}{p-1}}([w]_{A_p})^{\frac{r-1}{p-1}} \lesssim [w]_{A_p}.$$

As a result, if we use the fact that $(f,g) \in \mathfrak{F}$, then

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} g(x)^r H(x)^{\frac{p-r}{p-1}}w(x)\mathrm{d}x\right)^{\frac{1}{r}} \\
&\lesssim ([H^{\frac{p-r}{p-1}}w]_{A_r})^{\theta}\left(\int_{\mathbb{R}^n} f(x)^r H(x)^{\frac{p-r}{p-1}}w(x)\mathrm{d}x\right)^{\frac{1}{r}} \qquad (7.15)\\
&\lesssim ([w]_{A_p})^{\theta}\left(\int_{\mathbb{R}^n} f(x)^r H(x)^{\frac{p-r}{p-1}}w(x)\mathrm{d}x\right)^{\frac{1}{r}}.
\end{aligned}$$

If we use Hölder's inequality, we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} f(x)^r H(x)^{\frac{p-r}{p-1}}w(x)\mathrm{d}x\right)^{\frac{1}{r}} \\
&= \left\{\int_{\mathbb{R}^n} \left(f(x)H(x)^{\frac{p-r}{r(p-1)}}\right)^r w(x)\mathrm{d}x\right\}^{\frac{1}{r}} \qquad (7.16)\\
&\le \|f\|_{L^p(w)}\left(\int_{\mathbb{R}^n} H(x)^{p'}w(x)\mathrm{d}x\right)^{\frac{1}{r}-\frac{1}{p}}.
\end{aligned}$$

Since $D \gg 1$,

$$\left(\int_{\mathbb{R}^n} H(x)^{p'} w(x) \mathrm{d}x \right)^{\frac{1}{r'}} \lesssim (\|h\|_{L^{p'}(w)})^{\frac{p'}{r'}}. \tag{7.17}$$

Combining (7.14)–(7.17) gives the desired result.

We now apply Theorem 302 to obtain the vector-valued inequalities for $L^p(w, \ell^q)$ once again.

Theorem 303. *Let $1 < p < \infty$, $1 < q \le \infty$ and let $w \in A_p$. Then $\|\{Mf_j\}_{j=1}^\infty\|_{L^p(w,\ell^q)} \lesssim \|\{f_j\}_{j=1}^\infty\|_{L^p(w,\ell^q)}$ for all $\{f_j\}_{j=1}^\infty \in L^p(w, \ell^q)$.*

Proof When $p = q$ or $q = \infty$, the result is clear from the Fatou lemma: We may assume $1 < p, q < \infty$. Let $D \gg 1$. Thus, setting

$$\mathcal{F}_J \equiv \left\{ \left(\left(\sum_{j=1}^J |f_j|^q \right)^{\frac{1}{q}}, \frac{1}{D} \left(\sum_{j=1}^J M f_j{}^q \right)^{\frac{1}{q}} \right) : f_1, f_2, \dots, f_J \in L^\infty_{\mathrm{c}}(\mathbb{R}^n) \right\},$$

we apply Theorem 302 to $\mathcal{F} \equiv \mathcal{F}_J$ for each $J \in \mathbb{N}$ and then apply the monotone convergence theorem. This is possible, because we have $\|g\|_{L^q(w)} \le \|f\|_{L^q(w)}$ for all $(f, g) \in \mathcal{F}_J$ and $w \in A_q$ as long as $D \gg 1$.

We present another application. As we did in Lemma 292, define

$$\mathcal{A}f \equiv \sum_{j=1}^\infty \sum_{k \in K_j} m_{Q_k^j}(f) \chi_{Q_k^j}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as long as the sum makes sense.

Corollary 304. *Let $\{Q_k^j\}_{j \in \mathbb{N}, k \in K_j}$ be a sparse family of generalized dyadic cubes with a level structure. Also let $1 < p < \infty$ and $w \in A_p$. Then for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, $\|\mathcal{A}f\|_{L^p(w)} \lesssim ([w]_{A_p})^{\max\left(1, \frac{1}{p-1}\right)} \|f\|_{L^p(w)}$.*

Proof Simply combine Lemma 292 and Theorem 302.

7.1.6 A_2-theorem

So far we have seen how the constant behaves in the weighted maximal estimate in Theorems 280, 289 and 290. We will consider the boundedness property of singular integral operators. The behavior of the constant had been unknown and it is known as the A_2-problem/A_2-conjecture. This was settled down by Petermichl [355, 356] and then was expanded by Hytonen and Lerner [207, 270]. Nowadays, this theorem is called the A_2-theorem. We establish the following A_2-theorem here.

Theorem 305 (A_2-theorem). *Let T be a singular integral operator. Then $\|Tf\|_{L^2(w)} \lesssim [w]_{A_2}\|f\|_{L^2(w)}$ for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.*

We will prove Theorem 305 in a couple of steps.

Combining Theorems 302 and 305, we obtain the A_p-version.

Theorem 306 (A_p-theorem). *Let $1 < p < \infty$. Let T be a singular integral operator. Then $\|Tf\|_{L^p(w)} \lesssim [w]_{A_p}{}^{\max\left(1,\frac{1}{p-1}\right)}\|f\|_{L^p(w)}$ for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.*

Earlier it was known that $\|Tf\|_{L^p(w)} \lesssim_{[w]_{A_p}} \|f\|_{L^p(w)}$ for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ [205].

For the proof of Theorem 305, we need to set up notation. For a sparse family $\mathcal{S} \equiv \{U^j_k\}_{j\in\mathbb{N},k\in K_j} \subset \mathfrak{D} \equiv \mathcal{D}\big(\prod_{j=1}^n [a_j, b_j)\big)$ with a level structure and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ write $\mathcal{A}_{\mathfrak{D},\mathcal{S}} f \equiv \sum_{j=1}^\infty \sum_{k\in K_j} m_{U^j_k}(f)\chi_{U^j_k}$. To prove Theorem 305, we use the following estimate together with Lemma 218:

Lemma 307. *Let $m, N \in \mathbb{N}$. Suppose that $\mathcal{L} \equiv \{Q^j_k\}_{j\in\mathbb{N},k\in K_j} \subset \mathcal{D}(\mathbb{R}^n)$ is a sparse family with a level structure and that $K_0 = \{k\}$ is a singleton with $Q^0_k = [-2^N, 2^N)^n$ for $k \in K_0$. Let $\mathcal{L}' \equiv \{R^j_k\}_{j\in\mathbb{N},k\in K_j} \subset \mathcal{D}(\mathbb{R}^n)$ be a family such that $Q^j_k \subset 3R^j_k$ and that $R^j_k \subset 2^m Q^j_k$ for all $j \in \mathbb{N}$ and $k \in K_j$. Write $L' \equiv R^j_k$ if $L = Q^j_k$. Fix $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$. Define*

$$\mathcal{A}_{\mathcal{L},\mathcal{L}';m} f \equiv \sum_{L\in\mathcal{L}} m_{L'}(f)\chi_L.$$

(1) *We have $\|\mathcal{A}_{\mathcal{L},\mathcal{L}';m} f\|_{L^2} \lesssim \|f\|_{L^2}$ and $\|\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f\|_{L^2} \lesssim \|f\|_{L^2}$, where the implicit constants are independent of f and m.*

(2) *There exists a constant $c(n)$ that depends only on n such that*

$$\lambda|\{x \in \mathbb{R}^n : \mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f(x) > \lambda\}| \le c(n)m\|f\|_{L^1}$$

for all $\lambda > 0$.

(3) *For all cubes $Q \in \mathcal{D}(\mathbb{R}^n)$, $\omega_{2^{-n-2}}(\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f; Q) \lesssim m \cdot m_Q(f)$, where the implicit constants are independent of f and m.*

(4) *For any $N \in \mathbb{N}$ and real-valued $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ there exists another sparse family $\mathcal{S} \equiv \{U^j_k\}_{j\in\mathbb{N},k\in K_j} \subset \mathcal{D}(\mathbb{R}^n)$ which depends on f and N such that the pointwise estimate $\chi_{[-2^N,2^N)^n}|\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f - \mathrm{Med}(\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f; [-2^N, 2^N)^n)| \lesssim Mf + m\mathcal{A}_{\mathcal{D},\mathcal{S}} f$ holds.*

Proof The most serious part of the proof is to obtain a reasonable control of the weak-$(1,1)$ constant of the adjoint operator.

(1) We concentrate on the first formula, since the second one can be handled completely in a similar manner or by the duality argument. To prove the first formula, we dualize the left-hand side: Matters are reduced to the inequality

$$\int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{L},\mathcal{L}';m} f(x) \cdot g(x) \mathrm{d}x \lesssim \|f\|_{L^2} \|g\|_{L^2}$$

for all $f, g \in L^\infty_c(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$. Written out in full, the left-hand side becomes

$$\int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{L},\mathcal{L}';m} f(x) \cdot g(x) \mathrm{d}x = \sum_{L \in \mathcal{L}} m_{L'}(f) m_L(g) |L|.$$

We define $E_L \equiv L \setminus \bigcup_{k' \in K_{j+1}} Q^j_{k'}$, if $L = Q^j_k$. Then keeping in mind that $\{E_L\}_{L \in \mathcal{L}}$ is disjoint and that $2|E_L| \geq |L|$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{L},\mathcal{L}';m} f(x) \cdot g(x) \mathrm{d}x &\leq 2 \sum_{L \in \mathcal{L}} m_{2^m L}(f) m_L(g) |E_L| \\ &\leq 2 \int_{\mathbb{R}^n} Mf(x) Mg(x) \mathrm{d}x \\ &\leq 2 \|Mf\|_{L^2} \|Mg\|_{L^2}. \end{aligned}$$

It remains to utilize the $L^2(\mathbb{R}^n)$-boundedness of M.

(2) We ignore the contribution of $[-2^N, 2^N)^n \in \{Q^0_k\}_{k \in K_0}$ because this term can be handled with ease. Form the Calderón–Zygmund decomposition of f as in Example 95. In particular, we obtain a family $\mathcal{N} \subset \mathcal{D}(\mathbb{R}^n)$ satisfying

$$\left| \bigcup_{Q \in \mathcal{N}} Q \right| \lesssim \frac{1}{\lambda} \|f\|_{L^1}.$$

Then it is easy to see that the good part g can be handled easily because we already know that $\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m}$ is bounded on $L^2(\mathbb{R}^n)$. To handle the bad part b, we have only to show

$$\lambda \left| \left\{ x \in \mathbb{R}^n \setminus \bigcup_{Q \in \mathcal{N}} 15Q \,:\, |\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} b(x)| > \lambda \right\} \right| \leq c(n) m \|f\|_{L^1}.$$

We decompose

$$\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} b = \sum_{Q \in \mathcal{N}} \mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} b_Q = \sum_{Q \in \mathcal{N}} \sum_{L \in \mathcal{L}} \frac{1}{|L'|} \left(\int_L b_Q(y) \mathrm{d}y \right) \chi_{L'}.$$

Here and below we fix $x \in \mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{N}} 15Q$. Let $Q \in \mathcal{N}$ and $L \in \mathcal{D}(\mathbb{R}^n)$. If $L \cap Q = \emptyset$ or $L \supset Q$, then $m_L(b_Q) = 0$. Thus

$$\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} b(x) = \sum_{Q\in\mathcal{N}} \sum_{L\in\mathcal{L}\cap\mathcal{D}(Q)} \frac{1}{|L'|} \left(\int_L b_Q(y)\mathrm{d}y \right) \chi_{L'}(x).$$

Suppose that $x \in L'$ and that $L \in \mathcal{L}$ and $Q \in \mathcal{N}$ satisfy $L \subsetneq Q$. Observe that L' is not contained in $15Q$; otherwise

$$x \in L' \subset 15Q, \quad Q \in \mathcal{N}, \quad x \in \mathbb{R}^n \setminus \bigcup_{U\in\mathcal{N}} 15U,$$

which is a contradiction. Note also that both $3L'$ and Q meet at L by assumption. Thus, as is seen from the covering of $3L'$ by 3^n cubes in $\mathcal{D}(\mathbb{R}^n)$ with the same size as L', $L \subsetneq Q \subset 3L'$ since L' is not contained in $15Q$. Since $\ell(L') \le 2^m \ell(L)$, we have

$$|\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} b(x)| \le \sum_{Q\in\mathcal{N}} \sum_{\substack{L\in\mathcal{L}\cap\mathcal{D}(Q) \\ |L|\ge 2^{-(m+3)n}|Q|}} \frac{1}{|L'|} \left(\int_L |b_Q(y)|\mathrm{d}y \right) \chi_{L'}(x).$$

Consequently, by considering cubes L' satisfying the condition in the summand

$$\begin{aligned} \|\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} b\|_{L^1(\mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{N}} 15Q)} &\le \sum_{Q\in\mathcal{N}} \sum_{\substack{L\in\mathcal{L}\cap\mathcal{D}(Q) \\ |L|\ge 2^{-(m+3)n}|Q|}} \int_L |b_Q(y)|\mathrm{d}y \\ &\le (m+3) \sum_{Q\in\mathcal{N}} \int_Q |b_Q(y)|\mathrm{d}y \\ &\le 2(m+3)\|f\|_{L^1}. \end{aligned}$$

It remains to use the Chebyshev inequality.

(3) Since $\chi_Q(x)|\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f(x) - c| \le \mathcal{A}^*_{\mathcal{L},\mathcal{L}';m}[|f|\chi_Q](x)$, we may assume $m_Q(|f|) > 0$. Let $c \equiv \sum_{L\in\mathcal{L},Q\subset L} \frac{1}{|L'|} \int_L f(z)\mathrm{d}z$. Note that the sum defining c is convergent since $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Let $\lambda_0 >$ be a constant that is specified shortly. Then we have

$$|\{x \in Q : |\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f(x) - c| > \lambda_0\}| \lesssim \frac{m}{\lambda_0} \|f\chi_Q\|_{L^1},$$

as is seen from the above pointwise estimate and the previous part. Let $D \gg 1$, which will be specified shortly. If we take $\lambda_0 = Dm \cdot m_Q(|f|)$, then $\lambda_{\chi_Q|\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f - c|}(\lambda_0) \le 2^{-n-2}|Q|$. As a result, $\omega_{2^{-n-2}}(\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f; Q) \le 2\lambda_0 \sim m \cdot m_Q(|f|)$.

(4) Let Q be a fixed cube. By the Lerner–Hytönen formula, we have a sparse family $\mathcal{S} \equiv \{[-2^N, 2^N)^n\} \cup \{U_k^j\}_{j\in\mathbb{N}, k\in K_j} \subset \mathcal{D} \cup \{[-2^N, 2^N)^n\}$ which depends on f such that

$$\begin{aligned}
&\chi_{[-2^N,2^N)^n} |\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f - \operatorname{Med}(\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f; [-2^N, 2^N)^n)| \\
&\lesssim Mf + \sum_{j=1}^{\infty} \sum_{k\in K_j} \omega_{2^{-n-2}}(\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f; U_k^j) \chi_{U_k^j} \\
&\lesssim Mf + m \sum_{j=1}^{\infty} \sum_{k\in K_j} m_{U_k^j}(f) \chi_{U_k^j} \\
&= Mf + m\mathcal{A}_{\mathcal{D},\mathcal{S}} f.
\end{aligned}$$

By the use of Lemmas 218, we obtain the weighted estimate of the operator $\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m}$.

Proposition 308. *Maintain the notation in Lemma 307. Let $w \in A_2$. Then*

$$\|\mathcal{A}_{\mathcal{L},\mathcal{L}';m} f\|_{L^2(w)} \lesssim m[w]_{A_2} \|f\|_{L^2(w)}, \quad \|\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f\|_{L^2(w)} \lesssim m[w]_{A_2} \|f\|_{L^2(w)}$$

for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

Proof We may assume that $\mathcal{L}$ is a finite set by the use of the monotone convergence theorem and that $f \in \mathbb{M}^+(\mathbb{R}^n)$. It suffices to prove the second formula thanks to the duality $L^2(w^{-1})$-$L^2(w)$ and the fact that $w^{-1} \in A_2$ with $[w^{-1}]_{A_2} = [w]_{A_2}$. Then we calculate

$$\begin{aligned}
&\|\chi_{[-2^N,2^N)^n} (\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f - \operatorname{Med}(\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f; [-2^N, 2^N)^n))\|_{L^2(w)} \\
&\lesssim \|Mf\|_{L^2(w)} + m\|\mathcal{A}_{\mathcal{D},\mathfrak{S}} f\|_{L^2(w)} \\
&\lesssim m[w]_{A_2} \|f\|_{L^2(w)}
\end{aligned}$$

thanks to Lemmas 218 and 307. We note $\operatorname{Med}(\mathcal{A}^*_{\mathcal{L},\mathcal{L}';m} f; [-2^N, 2^N)^n) \to 0$ as $N \to \infty$, since $\mathcal{L}$ is finite. Thus, we are in the position of using Fatou's lemma.

We transform Proposition 308 into the form which we use.

Corollary 309. *Let $\mathfrak{D}$ be a generalized dyadic grid. Also let $\mathcal{L}$ be a sparse family and $m \in \mathbb{N}$. Then for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$,*

$$\left\| \sum_{L\in\mathcal{L}} m_{2^m L}(f) \chi_L \right\|_{L^2(w)} \lesssim m[w]_{A_2} \|f\|_{L^2(w)}.$$

Proof For each $L \in \mathcal{L}$, there exist disjoint 2^n-dyadic cubes $L^1, L^2, \dots, L^{2^n}$ such that $\ell(L^k) = 2^m \ell(L)$ for each $k = 1, 2, \dots, 2^n$ and that $2^m L \subset \bigcup_{k=1}^{2^n} L^k$.

Thanks to Proposition 308 we have

$$\left\| \sum_{L\in\mathcal{L}} m_{L^{(k)}}(f)\chi_L \right\|_{L^2(w)} \lesssim m[w]_{A_2}\|f\|_{L^2(w)}$$

for each $k = 1, 2, \dots, 2^n$. If we add this estimate over k, we obtain the desired result.

Before we prove the A_2-conjecture/theorem, we prove that the exponent is sharp using the first Riesz transform R_1.

Proposition 310. *Let $1 < p < \infty$. Then the exponent $\max\left(1, \dfrac{1}{p-1}\right)$ in the inequality $\|R_1 f\|_{L^p(w)} \lesssim_n ([w]_{A_p})^{\max\left(1,\frac{1}{p-1}\right)}\|f\|_{L^p(w)}$, $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, is sharp.*

Proof Let $1 < p \le 2$. We use the weight $w(x) = |x|^{(p-1)(n-\delta)}$, $x \in \mathbb{R}^n$. Define $f(x) \equiv |x|^{-n+\delta}\chi_{B(1)}(x)$, $x \in \mathbb{R}^n$. If $x_1 \ge \max(2, x_2, x_3, \dots, x_n)$, then

$$R_1 f(x) = \int_{\mathbb{R}^n} \frac{x_1 - y_1}{|x-y|^{n+1}} f(y)\mathrm{d}y \gtrsim \frac{1}{\delta|x|^n}.$$

As a result,

$$\|R_1 f\|_{L^p(w)} \gtrsim \left(\int_{x_1 \ge \max(2,x_2,x_3,\dots,x_n)} \frac{|x|^{(p-1)(n-\delta)}}{|x|^{pn}} \mathrm{d}x \right)^{\frac{1}{p}} \gtrsim \delta^{-\frac{1}{p}-1}.$$

Meanwhile

$$\|f\|_{L^p(w)} = \left(\int_{B(1)} |x|^{-n+\delta} \mathrm{d}x \right)^{\frac{1}{p}} \sim \delta^{-\frac{1}{p}}.$$

Since $[w]_{A_p} \sim \delta^{-p+1}$, it follows that the exponent $\frac{1}{p-1}$ is sharp.

Let $2 < p < \infty$ and assume that we have

$$\|R_1 f\|_{L^p(w)} \lesssim_n ([w]_{A_p})^A \|f\|_{L^p(w)} \quad (f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n), w \in A_p).$$

By duality

$$\|R_1 g\|_{L^{p'}(w^{-\frac{1}{p-1}})} \lesssim_n ([w]_{A_p})^A \|g\|_{L^{p'}(w^{-\frac{1}{p-1}})} \quad (g \in L^\infty_{\mathrm{c}}(\mathbb{R}^n), w \in A_p),$$

or equivalently,

$$\|R_1 g\|_{L^{p'}(\sigma)} \lesssim_n ([\sigma]_{A_{p'}})^{\frac{A}{p'-1}} \|g\|_{L^{p'}(\sigma)} \quad (g \in L^\infty_{\mathrm{c}}(\mathbb{R}^n), \sigma \in A_{p'}).$$

This forces $A \ge 1$, since $p' \in (1, 2)$.

We now conclude the proof of Theorem 305. We may assume that Tf is real-valued because we can consider $\mathrm{Re}(Tf)$ and $\mathrm{Im}(Tf)$ separately. We have only to establish

$$\|\chi_{Q^0}(Tf - \mathrm{Med}(Tf; Q^0))\|_{L^2(w)} \lesssim [w]_{A_2}\|f\|_{L^2(w)}$$

for all cubes Q^0, where the implicit constant is independent of Q^0, since $\mathrm{Med}(Tf; [-2^N, 2^N)) \to 0$ as $N \to \infty$. At this step, we obtained a good cancellation of the function Tf.

By the Lerner formula, we have

$$\chi_{Q^0}(x)|Tf(x) - \mathrm{Med}(Tf; Q^0)| \lesssim \sum_{j=1}^{\infty} \sum_{k \in K_j} \omega(Tf; Q_k^j)\chi_{Q_k^j}(x)$$

for almost every $x \in \mathbb{R}^n$, where $\{Q_k^j\}_{j \in \mathbb{N}, k \in K_j}$ is a sparse family with a level structure. By Lemma 217, we have

$$\chi_{Q^0}(x)|Tf(x) - \mathrm{Med}(Tf; Q^0)| \lesssim \sum_{l=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k \in K_j} 2^{-l} m_{2^l Q^0}(|f|)\chi_{Q_k^j}(x)$$

for almost every $x \in \mathbb{R}^n$. At this step, the singular integral operator T was controlled by a positive operator. Thus, by Corollary 309,

$$\begin{aligned}
\|\chi_{Q^0}(Tf - \mathrm{Med}(Tf; Q^0))\|_{L^2(w)} &\lesssim \sum_{l=0}^{\infty} \left\| \sum_{j=1}^{\infty} \sum_{k \in K_j} 2^{-l} m_{2^l Q^0}(|f|)\chi_{Q_k^j} \right\| \\
&\lesssim \sum_{l=0}^{\infty} l \cdot 2^{-l} [w]_{A_2}\|f\|_{L^2(w)} \\
&\sim [w]_{A_2}\|f\|_{L^2(w)}.
\end{aligned}$$

Thus, the A_2-conjecture is solved.

We end this section with the boundedness of singular integral operators in the vector-valued setting:

Theorem 311 (Weighted vector-valued boundedness of singular integral operators)**.** *Let $1 < q, r < \infty$ and suppose $w \in A_q$. Also let T be a singular integral operator. Then for all $\{f_j\}_{j=1}^{\infty} \subset L_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$),*

$$\|\{Tf_j\}_{j=0}^{\infty}\|_{L^p(w, \ell^q)} \lesssim_{q,r} \|\{f_j\}_{j=0}^{\infty}\|_{L^p(w, \ell^q)}.$$

Here to ignore completely the definability of Tf with $f \in L^q(w)$, we suppose $f \in L_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$.

Proof Simply apply Theorems 303 and 306.

A direct consequence of the A_p-theorem is that we can extend T to a bounded linear operator on $L^p(w)$.

7.1.7 Exercises

Exercise 97. Go back to the definition of the $A_p/A_{p'}$-characteristics to show Lemma 287.

Exercise 98. Let w be a weight on $\mathbb{R}^n$. Consider the uncentered maximal operator M generated by cubes.

(1) Let $f \in L^0(\mathbb{R}^n)$. Then show that $f \in L^1(w)'$ if and only if $w^{-1}f \in L^\infty$. Furthermore, in this case, $\|w^{-1}f\|_{L^1(w)'} = \|w^{-1}f\|_{L^\infty}$.

(2) Show that M maps $L^1(w)'$ boundedly if and only if $w \in A_1$. Furthermore, in this case, find the operator norm of M.

Exercise 99. Consider the uncentered maximal operator M generated by cubes.

(1) Let $f \in L^0(\mathbb{R}^n)$. Let $\lambda > 0$ and define $\Omega \equiv \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$. Show that, for any compact set K contained in Ω, there exists a finite collection $\{Q_j\}_{j=1}^J$ of cubes such that $m_{Q_j}(|f|) > \lambda$ for all $j = 1, 2, \dots, J$ and that $K \subset \bigcup_{j=1}^J Q_j$.

(2) Prove Theorem 289 for $p > 1$ using Hölder's inequality and the $5r$-covering lemma.

Exercise 100. [135, 450] Let w be a weight. Then show that $w \in A_\infty$ if and only if $\int_Q M[w\chi_Q](x)\mathrm{d}x \lesssim w(Q)$ for any cube Q.

7.2 Two-weight norm inequality

So far, we considered weighted norm inequalities of operators whose range and domains are the same. Here, we generalize the situation in this direction: we consider two different weights or two different measures and consider the boundedness of operators from a weighted Lebesgue space to another weighted Lebesgue space. Of course it is impossible to do so only for a limited class of operators. The Hardy operator $\mathcal{H}$ defined by (1.1) is one of such operators.

Section 7.2.1 deals with the Hardy operator. Section 7.2.2 considers fractional maximal operators, while we consider singular integral operators in Section 7.2.3.

7.2.1 Weighted estimates for the Hardy operator

We recall that the Hardy operator is given by

$$\mathcal{H}f(t) = \frac{1}{t}\int_0^t f(s)\mathrm{d}s \quad (t>0).$$

See Section 1.4.3. We will consider the weighted norm estimates for this operator. Since we can consider a more general setting of Radon measures instead of weighted measures, we let μ be a Borel measure on $(0,\infty)$. To avoid some technical difficulty, we consider continuous Borel measures. We are now interested in the following type of estimate:

$$\left\{\int_0^\infty \left(\int_0^t f(s)\mathrm{d}s\right)^q \mathrm{d}\mu(t)\right\}^{\frac{1}{q}} \le C\|f\cdot w\|_{L^p(0,\infty)},$$

or equivalently

$$\|\mathcal{H}f\|_{L^p(t^p\mathrm{d}\mu)} \le C\|f\|_{L^p(w^p)},$$

where $f\in\mathbb{M}^+(0,\infty)$. If $p=q$, thanks to Theorem 38 we have rich examples of monomial weights. In the upper triangle case, $1<p\le q<\infty$, we can completely characterize the boundedness of the integral operators.

Theorem 312. *Let $1<p\le q<\infty$. Suppose that $w\in\mathbb{M}^+(0,\infty)$ and that μ is a continuous Borel measure on $(0,\infty)$. Then*

$$\left\{\int_0^\infty \left(\int_0^v f(t)\mathrm{d}t\right)^q \mathrm{d}\mu(v)\right\}^{\frac{1}{q}} \lesssim \|f\cdot w\|_{L^p(0,\infty)}$$

holds for any $f\in\mathbb{M}^+(0,\infty)$ if and only if

$$K \equiv \sup_{r>0} \sqrt[q]{\mu(r,\infty)}\left(\int_0^r w(v)^{-p'}\mathrm{d}v\right)^{\frac{1}{p'}} < \infty.$$

We prove Theorem 312 after we prove a lemma. The following lemma can be located as a sort of integration by parts.

Lemma 313. *Let $1<p<\infty$ and μ be a continuous Borel measure. Then*

$$\int_{(t,\infty)\cap\mathrm{supp}(\mu)} \frac{1}{\sqrt[p']{\mu(s,\infty)}}\mathrm{d}\mu(s) \le p\sqrt[p]{\mu(t,\infty)}.$$

Proof Let $s>t>0$ and $N\in\mathbb{N}$. Then define $t_N(s)\equiv 2^N t\left[\frac{s}{2^N t}\right]$. By the monotone convergence theorem,

$$\begin{aligned}\int_{(t,\infty)\cap\mathrm{supp}(\mu)} \frac{1}{\sqrt[p']{\mu(s,\infty)}}\mathrm{d}\mu(s) &= \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\int_t^\infty \frac{1}{\sqrt[p']{\mu(-\varepsilon+t_N(s),\infty)}}\mathrm{d}\mu(s)\\ &\le \lim_{N\to\infty}\sum_{k=1}^\infty \frac{\mu(t+2^{-N}(k-1)t, t+2^{-N}kt)}{\sqrt[p']{\mu[t+2^{-N}kt,\infty)}},\end{aligned}$$

for each fixed $\varepsilon > 0$ since if $\varepsilon > 2^{-N}$ for N large enough. By the mean-value theorem,

$$\begin{aligned}
&\int_{(t,\infty)\cap \mathrm{supp}(\mu)} \frac{1}{\sqrt[p']{\mu(s,\infty)}} \mathrm{d}\mu(s) \\
&\le \lim_{N\to\infty} \sum_{k=1}^{\infty} p \left(\sqrt[p]{\mu(t+2^{-N}kt,\infty)} - \sqrt[p]{\mu(t+2^{-N}(k-1)t,\infty)} \right) \\
&= p \sqrt[p]{\mu(t,\infty)}.
\end{aligned}$$

We prove Theorem 312.

Proof The "only if" part is easy to check; simply use that

$$\int_0^r f(t)\mathrm{d}t \cdot \sqrt[q]{\mu(r,\infty)} = \left\{ \int_r^{\infty} \left(\int_0^r f(t)\mathrm{d}t \right)^q \mathrm{d}\mu(v) \right\}^{\frac{1}{q}} \lesssim \|f\cdot w\|_{L^p(0,\infty)}$$

for all $r > 0$ and resort to duality. Consequently, we concentrate on the "if" part.

Let $h(t) \equiv \left(\int_0^t w(s)^{-p'} \mathrm{d}s \right)^{\frac{1}{p'p}}$ for $t \in (0,\infty)$. Then for all $s > 0$,

$$\frac{1}{h(s)^{p'} w(s)^{p'}} = \frac{1}{h(s)^{p'}} \frac{d}{\mathrm{d}s} h(s)^{pp'} = p'ph'(s)h(s)^{p'p-p'-1} = p' \frac{d}{\mathrm{d}s} h(s)^p.$$

Thus,

$$\int_0^t \frac{1}{h(s)^{p'} w(s)^{p'}} \mathrm{d}s = p'h(t)^p = p' \left(\int_0^t w(s)^{-p'} \mathrm{d}s \right)^{\frac{1}{p'}} \le \frac{Kp'}{\sqrt[q]{\mu(t,\infty)}}.$$

Meanwhile, by Hölder's inequality, we have

$$\begin{aligned}
&\int_0^{\infty} \left(\int_0^v f(t)\mathrm{d}t \right)^q \mathrm{d}\mu(v) \\
&= \int_0^{\infty} \left(\int_0^v f(t)h(t)w(h)h(t)^{-1}w(t)^{-1}\mathrm{d}t \right)^q \mathrm{d}\mu(v) \\
&\le \int_0^{\infty} \left(\int_0^s f(t)^p h(t)^p w(t)^p \mathrm{d}t \right)^{\frac{q}{p}} \left(\int_0^s \frac{1}{h(t)^{p'} w(t)^{p'}} \mathrm{d}t \right)^{\frac{q}{p'}} \mathrm{d}\mu(s).
\end{aligned}$$

Recall that we are assuming that $p \le q$. By virtue of Minkowski's inequality which is applied to the weighted measure $\left(\int_0^s \frac{\mathrm{d}t}{h(t)^{p'} w(t)^{p'}} \right)^{\frac{q}{p'}} \mathrm{d}\mu(s)$, we have

$$\begin{aligned}
&\left\{ \int_0^{\infty} \left(\int_0^s f(t)\mathrm{d}t \right)^q \mathrm{d}\mu(s) \right\}^{\frac{1}{q}} \\
&\le \left\{ \int_0^{\infty} f(t)^p h(t)^p w(t)^p \left[\int_t^{\infty} \left(\int_0^s \frac{\mathrm{d}v}{h(v)^{p'} w(v)^{p'}} \right)^{\frac{q}{p'}} \mathrm{d}\mu(s) \right]^{\frac{p}{q}} \mathrm{d}t \right\}^{\frac{1}{p}}.
\end{aligned}$$

As a result,

$$\left\{\int_0^\infty \left(\int_0^s f(t)\mathrm{d}t\right)^q \mathrm{d}\mu(s)\right\}^{\frac{1}{q}}$$
$$\leq K^{\frac{1}{p'}} \left\{\int_0^\infty f(t)^p h(t)^p w(t)^p \left(\int_{(t,\infty)\cap \mathrm{supp}(\mu)} \frac{\mathrm{d}\mu(s)}{\sqrt[p']{\mu(s,\infty)}}\right)^{\frac{p}{q}} \mathrm{d}t\right\}^{\frac{1}{p}}.$$

Thus, from the definition of K and Lemma 313,

$$\int_{(t,\infty)\cap \mathrm{supp}(\mu)} \frac{\mathrm{d}\mu(s)}{\sqrt[p']{\mu(s,\infty)}} \lesssim \sqrt[p]{\mu(t,\infty)} \leq K^{\frac{q}{p}} h(t)^{-q}.$$

Consequently,

$$\left\{\int_0^\infty \left(\int_0^t f(s)\mathrm{d}s\right)^q \mathrm{d}\mu(t)\right\}^{\frac{1}{q}} \lesssim K\|f\cdot w\|_{L^p(0,\infty)}.$$

We compare Theorem 38 with Theorem 312.

Example 128. Let $1 < p < \infty$, and let w be a weight on $(0,\infty)$. Then thanks to Theorem 312

$$\left\{\int_0^\infty \left(\int_0^t f(s)\mathrm{d}s\right)^p w(t)^p \mathrm{d}t\right\}^{\frac{1}{q}} \lesssim \|f\cdot w\|_{L^p(0,\infty)}$$

for all $f \in \mathcal{M}^+(0,\infty)$, if and only if

$$\sup_{r>0} \|w\|_{L^p(0,r)} \|w^{-1}\|_{L^{p'}(r,\infty)} < \infty.$$

If furthermore $w(t) = t^\alpha, t > 0$, this inequality holds if and only if $-\frac{1}{p} < \alpha$.

It should be noted that we can handle the operator $\mathfrak{H}$. By a change of variables we can handle another integral operator. Theorem 314 can be located as the dual of Theorem 312.

Theorem 314. *Let $1 < p \leq q < \infty$. Suppose that $w \in \mathbb{M}^+(0,\infty)$ and that μ is a continuous Borel measure on $(0,\infty)$. Then*

$$\left\{\int_0^\infty \left(\int_t^\infty f(t)\mathrm{d}t\right)^q \mathrm{d}\mu(t)\right\}^{\frac{1}{q}} \lesssim \|f\cdot w\|_{L^p(0,\infty)} \tag{7.1}$$

holds for $f \in \mathbb{M}^+(0,\infty)$ if and only if

$$K \equiv \sup_{r>0} \sqrt[q]{\mu(0,r)} \left(\int_r^\infty w(t)^{-p'}\mathrm{d}t\right)^{\frac{1}{p'}} < \infty.$$

As is seen by the proof below, Theorems 312 and 314 are equivalent.

Proof Let ν be a measure satisfying $\nu(E) = \mu\{t > 0 : t^{-1} \in E\}$ for all Borel sets E. Then

$$\begin{aligned}\int_0^\infty \left(\int_t^\infty f(s)\mathrm{d}s\right)^q \mathrm{d}\mu(t) &= \int_0^\infty \left(\int_{t^{-1}}^\infty f(s)\mathrm{d}s\right)^q \mathrm{d}\nu(t)\\ &= \int_0^\infty \left(\int_0^t s^{-2} f(s^{-1})\mathrm{d}s\right)^q \mathrm{d}\nu(t).\end{aligned}$$

Meanwhile the assumption reads as

$$K = \sup_{r>0} \sqrt[q]{\nu(r^{-1},\infty)} \left(\int_r^\infty w(t)^{-p'}\mathrm{d}t\right)^{\frac{1}{p'}}.$$

Consequently, we have

$$\left\{\int_0^\infty \left(\int_0^t g(s)\mathrm{d}s\right)^q \mathrm{d}\nu(t)\right\}^{\frac{1}{q}} \lesssim \left(\int_0^\infty (g(t)t^{\frac{2}{p'}} w(t^{-1}))^p \mathrm{d}t\right)^{\frac{1}{p}}$$

for all $g \in \mathbb{M}^+(0,\infty)$. In particular,

$$\left\{\int_0^\infty \left(\int_0^t s^{-2} f(s^{-1})\mathrm{d}s\right)^q \mathrm{d}\nu(t)\right\}^{\frac{1}{q}} \lesssim \left\{\int_0^\infty (f(t^{-1})t^{-\frac{2}{p}} w(t^{-1}))^p \mathrm{d}t\right\}^{\frac{1}{p}}.$$

By the change of variables once again, we conclude

$$\left\{\int_0^\infty \left(\int_t^\infty f(s)\mathrm{d}s\right)^q \mathrm{d}\mu(t)\right\}^{\frac{1}{q}} \lesssim \|f \cdot w\|_{L^p(0,\infty)}.$$

We have a complete charecterization in the case of $0 < p \le 1$ and $0 < p \le q \le \infty$ once we give up considering all non-negative measurable functions: We can drop the assumption $1 < p$. For $f \in L^0(0,\infty)$, we write $\|f\|_{L^p(0,\infty;\mu)}$ for the $L^p(\mu)$-norm. It might be strange to restrict monotone functions when we consider the boundendness of operators. However, there are some cases where the norm is a composition of another norm and a monotone function. For example, the local Morrey-type space, which we are going to define later in this book, is made up of the norm $\|\cdot\|_{\Phi_{\lambda,q}(0,\infty)}$ and a monotone function. We state the result in full generality.

Theorem 315. *Suppose $0 < p \le 1$ and $0 < p \le q < \infty$. Let $v, w : (0,\infty) \to [0,\infty)$ be locally integrable functions. Let μ, ν be σ-finite Borel measures on $(0,\infty)$ charging no point in $(0,\infty)$. Then $\|\mathcal{H}f\|_{L^q(0,\infty;\mu)} \lesssim \|f\|_{L^p(0,\infty;\nu)}$ for all $f \in \mathbb{M}^\downarrow(0,\infty)$ with $\lim_{t\to\infty} f(t) = 0$ if and only if $\mathcal{A} \equiv \sup_{s>0} \left(\int_0^\infty s^q \max(s,t)^{-q}\mathrm{d}\mu(t)\right)^{\frac{1}{q}} \left(\int_0^t \mathrm{d}\nu(t)\right)^{-\frac{1}{p}} < \infty$.*

The proof of Theorem 315 will be given after we prove two lemmas. First, under our assumption on the function f, we estimate the Hardy operator $\mathcal{H}$ so that we can understand why max in the definition of $\mathcal{A}$ comes into play.

Lemma 316. *Let $0 < p \le 1$. Also let $f \in C^1(0,\infty) \cap \mathbb{M}^{\downarrow}(0,\infty)$ satisfy $\lim_{t\to\infty} f(t) = 0$. Then $\mathcal{H}f(t) \lesssim f(t) + \frac{1}{t}\left(-\int_0^t u^p f(u)^{p-1} f'(u)\mathrm{d}u\right)^p$ for all $t > 0$.*

Proof Thanks to the fundamental theorem of calculus and the fact that $\lim_{t\to\infty} f(t) = 0$, we have $\mathcal{H}f(t) \simeq \frac{1}{t}\int_0^t \left(-\int_s^\infty f(u)^{p-1} f'(u)\mathrm{d}u\right)^{\frac{1}{p}} \mathrm{d}s$. We split the integral into two parts:

$$\begin{aligned}&\mathcal{H}f(t)\\ &\lesssim \frac{1}{t}\int_0^t \left(-\int_s^t f(u)^{p-1} f'(u)\mathrm{d}u\right)^{\frac{1}{p}} \mathrm{d}s + \frac{1}{t}\int_0^t \left(-\int_t^\infty f(u)^{p-1} f'(u)\mathrm{d}u\right)^{\frac{1}{p}} \mathrm{d}s.\end{aligned}$$

We calculate the second integral in the right-hand side to have

$$\begin{aligned}\mathcal{H}f(t) &\sim \frac{1}{t}\int_0^t \left(-\int_s^t f(u)^{p-1} f'(u)\mathrm{d}u\right)^{\frac{1}{p}} \mathrm{d}s + f(t)\\ &= \frac{1}{t}\int_0^t \left(-\int_0^t \chi_{(0,u)}(s) f(u)^{p-1} f'(u)\mathrm{d}u\right)^{\frac{1}{p}} \mathrm{d}s + f(t). \qquad (7.2)\end{aligned}$$

We consider

$$\mathrm{I} \equiv \left\{\int_0^t \left(-\int_0^t \chi_{(0,u)}(s) f(u)^{p-1} f'(u)\mathrm{d}u\right)^{\frac{1}{p}} \mathrm{d}s\right\}^p.$$

Recall that $0 < p \le 1$. By Minkowski's inequality, we have

$$\begin{aligned}\mathrm{I} &\le \int_0^t \left(\int_0^t \left(-\chi_{(0,u)}(s) f(u)^{p-1} f'(u)\right)^{\frac{1}{p}} \mathrm{d}s\right)^p \mathrm{d}u\\ &= -\int_0^t f(u)^{p-1} f'(u) \left(\int_0^t \chi_{(0,u)}(s)\mathrm{d}s\right)^p \mathrm{d}u\\ &= -\int_0^t u^p f(u)^{p-1} f'(u)\mathrm{d}u.\end{aligned}$$

It remains to insert this inequality into (7.2).

Proof of Theorem 315 Once we assume (7.1), we readily have $\mathcal{A} < \infty$; simply let $f = \chi_{(0,t)}$ for each $t > 0$ and take the supremum. We consider the sufficiency part. By the montone convergence theorem we may assume that

$f \in L^\infty(0,\infty)$ and that $f(t) = 0$ for $t \gg 1$. By mollification, we can also suppose that $f \in C^\infty(0,\infty)$. Consequently from Lemma 316, we have

$$\left(\int_0^\infty \mathcal{H}f(t)^q \mathrm{d}\mu(t)\right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_0^\infty f(t)^q \mathrm{d}\mu(t)\right)^{\frac{1}{q}} + \left\{\int_0^\infty \left(-\int_0^t f(s)^{p-1} f'(s) s^p \mathrm{d}s\right)^{\frac{q}{p}} t^{-q} \mathrm{d}\mu(t)\right\}^{\frac{1}{q}}.$$

For the first term, we use Lemma 13. Recall that $0 < p \le q < \infty$. Applying Minkowski's inequality to the weighted measures $f(s)^{p-1} f'(s) s^p \mathrm{d}s$ and $t^{-q}\mathrm{d}\mu(t)$, we find

$$\left\{\int_0^\infty \left(-\int_0^t f(s)^{p-1} f'(s) s^p \mathrm{d}s\right)^{\frac{q}{p}} t^{-q} \mathrm{d}\mu(t)\right\}^{\frac{p}{q}}$$
$$\le -\int_0^\infty f(s)^{p-1} f'(s) s^p \left(\int_s^\infty t^{-q} \mathrm{d}\mu(t)\right)^{\frac{p}{q}} \mathrm{d}s.$$

If we use the definition of $\mathcal{A}$ and Fubini's theorem once again, then

$$\left\{\int_0^\infty \left(-\int_0^t f(s)^{p-1} f'(s) s^p \mathrm{d}s\right)^{\frac{q}{p}} t^{-q} \mathrm{d}\mu(t)\right\}^{\frac{p}{q}}$$
$$\le -\mathcal{A}^p \int_0^\infty f(t)^{p-1} f'(t) \nu(0,t) \mathrm{d}t$$
$$= -\mathcal{A}^p \int_0^\infty f(t)^{p-1} f'(t) \left(\int_0^t \mathrm{d}\nu(s)\right) \mathrm{d}t$$
$$\sim \mathcal{A}^p \int_0^\infty f(t)^p \mathrm{d}\nu(t).$$

Thus, the estimate of the second term is valid.

The following theorem supplements Theorem 315. We can consider $q = \infty$.

Theorem 317. *Let $0 < p \le 1$, $v \in L^1_{\mathrm{loc}}([0,\infty)) \cap \mathbb{M}^+([0,\infty))$, and let $w : (0,\infty) \to (0,\infty)$ be a continuous function. Then $\sup_{t>0} w(t)\mathcal{H}f(t) \lesssim \|f\|_{L^p(v)}$ for all $f \in \mathbb{M}^\downarrow(0,\infty)$ with $\lim_{t\to\infty} f(t) = 0$ if and only if $\mathcal{A} \equiv \sup_{T,t>0} \frac{w(T)\min(T,t)}{T v(0,t)^{\frac{1}{p}}} < \infty$.*

Proof The necessity part is trivial again: Simply test the conclusion on $f = \chi_{(0,T)}$, $T > 0$.

We concentrate on the sufficiency part. We can assume that f is smooth by mollification and that f is bounded. Thanks to Lemma 316

$$\sup_{t>0} w(t)\mathcal{H}f(t) \lesssim \sup_{t>0} f(t)w(t) + \sup_{t>0} \frac{w(t)}{t} \left(-\int_0^t u^p f(u)^{p-1} f'(u) \mathrm{d}u\right)^{\frac{1}{p}}.$$

For the first term, we use (1.8). Applying the assumption $\mathcal{A} < \infty$, we find

$$\begin{aligned}
&\sup_{t>0} \frac{w(t)}{t} \left(- \int_0^t f(u)^{p-1} f'(p)(u) u^p \mathrm{d}u \right)^{\frac{1}{p}} \\
&\leq \sup_{t>0} \left(- \int_0^t f(u)^{p-1} f'(p)(u) u^p \left(\sup_{s \geq u} s^{-1} w(s) \right)^p \mathrm{d}u \right)^{\frac{1}{p}} \\
&\leq \mathcal{A} \left(- \int_0^\infty f(y)^{p-1} f'(y) \left(\int_0^y v(s) \mathrm{d}s \right) \mathrm{d}y \right)^{\frac{1}{p}} \\
&\lesssim \mathcal{A} \| f \|_{L^p(v)}.
\end{aligned}$$

Finally we will list some sufficient conditions for which the Hardy operator is bounded.

Theorem 318. *Let $0 < \theta_2 \leq \infty$, and let $v_1, v_2 \in \mathbb{M}^+(0,\infty)$.*

(1) *Let $1 < \theta_1 \leq \infty$. Then*

$$\left\{ \int_0^\infty \left(v_2(t) \int_0^t f(s) v_1(s) \mathrm{d}s \right)^{\theta_1} \mathrm{d}t \right\}^{\frac{1}{\theta_1}} \lesssim \| f \|_{L^{\theta_1}(0,\infty)} \tag{7.3}$$

for all $f \in \mathbb{M}^+(0,\infty)$ if $\mathcal{A} \equiv \int_0^\infty v_2(t)^{\theta_2} \left(\int_0^t v_1(s)^{\theta_1'} \mathrm{d}s \right)^{\frac{\theta_2}{\theta_1'}} \mathrm{d}t < \infty$.

(2) *Let $\theta_1 = 1$. Then inequality (7.3) holds if*

$$\int_0^\infty v_2(t)^{\theta_2} \left(\sup_{s \in [0,1]} v_1(s)^{\theta_2} \right) \mathrm{d}t < \infty. \tag{7.4}$$

(3) *Let $0 < \theta_1 \leq \min(1, \theta_2)$ and suppose $\int_0^\infty v_2(t)^{\theta_2} \left(\sup_{s \in [0,t]} \frac{v_1(s)^{\theta_2}}{s^{\theta_2 \frac{1-\theta_1}{\theta_1}}} \right) \mathrm{d}t < \infty$. Then inequality (7.3) holds for all $f \in \mathbb{M}^\uparrow(0,\infty)$.*

(4) *If $\theta_1 = \infty$, then condition (7.4) is also necessary for (7.3).*

Proof Let $f \in \mathbb{M}^+(0,\infty)$.

(1) By Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \left(v_2(t) \int_0^t v_1(s) f(s) \mathrm{d}s \right)^{\theta_2} \mathrm{d}t \\
&\leq \int_0^\infty v_2(t)^{\theta_2} \left(\int_0^t f(s)^{\theta_1} \mathrm{d}s \right)^{\frac{\theta_2}{\theta_1}} \left(\int_0^t v_1(s)^{\theta_1'} \mathrm{d}s \right)^{\frac{\theta_2}{\theta_1'}} \mathrm{d}t \\
&\leq \mathcal{A} (\| f \|_{L^{\theta_1}(0,\infty)})^{\theta_2}.
\end{aligned}$$

(2) Simply mimic the previous case; we leave the proof for interested readers as Exercise 101.

(3) Thanks to Lemma 14,

$$\int_0^\infty \left(v_2(t) \int_0^t g(s)\mathrm{d}s \right)^{\theta_2} \mathrm{d}t \le \int_0^\infty v_2(t)^{\theta_2} \left(\int_0^t s^{\theta_1 - 1} g(s)^{\theta_1} \mathrm{d}s \right)^{\frac{\theta_2}{\theta_1}} \mathrm{d}t.$$

So, we apply (2).

(4) Simply test the boundedness on $g = {v_1}^{\theta_1}$.

We apply the one dimensional results to the boundedness of operators acting on functions defined in $\mathbb{R}^n$. We consider the following examples:

Definition 75 (Generalized Hardy operator)**.** Let $\varphi \in \mathbb{M}^+(0,\infty)$ and $0 < \alpha < n$. The *generalized Hardy operators* $\mathcal{H}_\varphi^\alpha$ and $\mathfrak{H}_\varphi^\alpha$ are defined by

$$\mathcal{H}_\varphi^\alpha f(x) \equiv |x|^{\alpha-n} \varphi(|x|) \int_{B(|x|)} \frac{f(t)}{\varphi(|t|)} \mathrm{d}t \quad (x \in \mathbb{R}^n)$$

and

$$\mathfrak{H}_\varphi^\alpha f(x) \equiv |x|^{\alpha-n} \varphi(|x|) \int_{\mathbb{R}^n \setminus B(|x|)} \frac{f(t)}{\varphi(|t|)} \mathrm{d}t \quad (x \in \mathbb{R}^n).$$

We do not deal with these operators, since these operators go beyond the coverage of this book, although there are many research papers on them.

7.2.2 Two-weight norm inequality for fractional maximal operators

Let $1 < p \le q < \infty$, and let v, w be weights. We now seek a characterization for the weighted inequality $\|M_\alpha f\|_{L^q(v)} \lesssim \|f\|_{L^p(w)}$, $f \in L^0(\mathbb{R}^n)$ to hold. We can reduce matters to the estimate $\|M_\alpha^{\mathfrak{D}_\mathbf{a}} f\|_{L^q(v)} \lesssim \|f\|_{L^p(w)}$, $f \in L^0(\mathbb{R}^n)$ and $\mathbf{a} \in \{0, 1/3\}^n$. For simplicity we suppose $\mathbf{a} = (0, 0, \dots, 0)$, since other cases can be handled in a similar manner.

Define $\mathcal{Q}^\sharp(Q^0) \equiv \{Q \in \mathcal{Q} : Q \supset Q^0\}$ for a cube $Q^0 \in \mathcal{D}(\mathbb{R}^n)$. We recall that thanks to Remark 7 for any $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$, there exists a sparse family $\mathcal{S} \subset \mathcal{D}(Q^0)$ such that, for all $x \in Q^0$,

$$M_\alpha^{\mathcal{D}} f(x) \lesssim L_\alpha^{\mathcal{D};\mathcal{S}} f(x) + c_{\infty,\alpha}^{\mathcal{D}}(f),$$

where $E_Q \equiv Q \setminus \bigcup_{R \in \mathcal{S}, R \subsetneq Q} R$,

$$L_\alpha^{\mathcal{D};\mathcal{S}} f \equiv \sum_{Q \in \mathcal{S}} |Q|^{\frac{\alpha}{n}} m_Q(f) \chi_{E_Q} \text{ and } c_{\infty,\alpha}^{\mathcal{D}}(f) \equiv \sup_{Q \in \mathcal{D}^\sharp(Q^0)} |Q|^{\frac{\alpha}{n}} m_Q(f).$$

Although we are mainly concerned with the case $\mathbf{a} = (0, 0, \dots, 0)$, we state our main result in terms of M_α instead of $M_\alpha^{\mathcal{D}}$.

Theorem 319 (Cube testing for M_α)**.** *Let v, w be weights, and let $0 \le \alpha < n$ and $1 < p \le q < \infty$. Define $\sigma \equiv w^{-\frac{1}{p-1}}$, the dual weight of w. Then*

$$\|M_\alpha f\|_{L^q(v)} \lesssim \|f\|_{L^p(w)} \tag{7.5}$$

holds for all $f \in L^\infty_{\rm c}(\mathbb{R}^n)$ if and only if

$$\|M_\alpha[\sigma\chi_{Q^0}]\|_{L^q(v)} \lesssim \sigma(Q^0)^{\frac{1}{p}} \tag{7.6}$$

for all cubes Q^0.

Furthermore, conditions (7.5) *and* (7.6) *both imply*

$$|Q|^{\frac{\alpha}{n}-1}\sigma(Q)^{\frac{1}{p'}} \cdot v(Q)^{\frac{1}{q}} \lesssim 1. \tag{7.7}$$

In particular, condition (7.5) *implies that $w(x) \neq 0$ for almost all $x \in \mathbb{R}^n$ and that $\sigma, v \in L^1_{\rm loc}(\mathbb{R}^n)$.*

Here, we tolerate the case $\alpha = 0$. It is important that the fundamental operator M_α is essentially replaced by a somewhat strange operator $L^{\mathcal{S}}_\alpha$, whose role in other fields of mathematics is yet unknown. Also, it matters that $L^{\mathcal{S}}_\alpha$ depends on f.

Proof The "only if" part is easy to check. Simply test (7.5) on $f = \chi_{Q^0}\sigma$. Once we have (7.5), then (7.7) follows immediately from $\ell(Q)^\alpha m_Q(|f|)\chi_Q \le M_\alpha f$ for all $f \in L^0(\mathbb{R}^n)$.

Let us check the "if" part. We will use the fractional maximal operator $M^{\mathcal{D}}_\alpha$ defined in Remark 7. Fix a cube $Q^0 \in \mathcal{D}(\mathbb{R}^n)$ to show that $\|\chi_{Q^0}M^{\mathcal{D}}_\alpha f\|_{L^q(v)} \lesssim \|f\|_{L^p(w)}$ with the constant independent of Q^0. As we mentioned, although we need to consider all families $\mathfrak{D}_{\mathbf{a}}$ with $\mathbf{a} \in \{0, 1/3\}^n$, we consider $\mathcal{D} = \mathfrak{D}_{(0,0,\dots,0)}$ only for simplicity. We may as well assume that $f \in \mathbb{M}^+(\mathbb{R}^n)$. We consider $L^{\mathcal{D};\mathcal{S}}_\alpha$: $L^{\mathcal{D};\mathcal{S}}_\alpha f = \sum\limits_{Q\in\mathcal{S}} |Q|^{\frac{\alpha}{n}} m_Q(f)\chi_{E_Q}$ and $c^{\mathcal{D}}_{\infty,\alpha}$ based on Theorem 179, where $\mathcal{S} \subset \mathcal{D}(\mathbb{R}^n)$ is a sparse family. We write

$$V \equiv \sum_{Q\in\mathcal{S}} \ell(Q)^{\alpha q} \left(m_Q(\sigma)\right)^q \chi_{E_Q} v.$$

Note that V is almost everywhere positive on Q^0, so that we can suppose $L^q(V)$ is a linear subspace of $L^0(Q^0)$. Thus, it suffices to show the boundedness from $L^p(\sigma)$ to $L^q(V)$ of the operator given by

$$m^{\mathcal{S}} f \equiv \sum_{Q\in\mathcal{S}} \left(\frac{1}{\sigma(Q)}\int_Q f(y)\sigma(y)\mathrm{d}y\right)\chi_{E_Q}. \tag{7.8}$$

It is easy to check that $m^{\mathcal{S}}$ is bounded from $L^\infty(\sigma) = L^\infty(\mathbb{R}^n)$ to $L^\infty(V) = L^\infty(\mathbb{R}^n)$; see Exercise 102. Consequently, it remains to show that $m^{\mathcal{S}}$ is

bounded from $L^1(\sigma)$ to $\mathrm{W}L^{\frac{q}{p}}(V)$ according to Example 67. To this end, we will fix $\lambda > 0$ and $h \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. If we set $\mathcal{S}_\lambda \equiv \{Q \in \mathcal{S} : |m_Q(h)| > \lambda\}$, then

$$\{x \in \mathbb{R}^n : |m^{\mathcal{S}} h(x)| > \lambda\} = \bigcup_{Q \in \mathcal{S}_\lambda} E_Q.$$

Recall that $\mathcal{S}$ and $\mathcal{S}_\lambda$ are subsets of $\mathcal{D}(\mathbb{R}^n)$. We define $\mathcal{S}_\lambda^{\max}$ to be the set of all maximal cubes in $\mathcal{S}_\lambda$ keeping this in mind. Then

$$\{x \in \mathbb{R}^n : |m^{\mathcal{S}} h(x)| > \lambda\} \subset \bigcup_{Q \in \mathcal{S}_\lambda^{\max}} Q.$$

Here, the right-hand side is actually a disjoint union, Thus, we have

$$\begin{aligned} V\{x \in \mathbb{R}^n : |m^{\mathcal{S}} h(x)| > \lambda\} &\le \sum_{Q \in \mathcal{S}_\lambda^{\max}} \ell(Q)^{\alpha q} \left(m_Q(\sigma)\right)^q v(Q) \\ &\le \sum_{Q \in \mathcal{S}_\lambda^{\max}} \int_Q M_\alpha(\chi_Q \sigma)(x)^q v(x) \mathrm{d}x \\ &\le \left(\|M_\alpha[\chi_Q \sigma]\|_{L^q(v)}\right)^q. \end{aligned}$$

We now invoke the assumption to obtain

$$V\{x \in \mathbb{R}^n : |m^{\mathcal{S}} h(x)| > \lambda\} \lesssim \sum_{Q \in \mathcal{S}_\lambda^{\max}} \sigma(Q)^{\frac{q}{p}} \le \left[\sigma\left(\bigcup_{Q \in \mathcal{S}_\lambda^{\max}} Q\right)\right]^{\frac{q}{p}},$$

where we used $q \ge p$ for the second inequality. Since we know that $\lambda \sigma(Q) \le \|h \chi_Q\|_{L^1(\sigma)}$ for all $Q \in \mathcal{S}_\lambda^{\max}$ by the definition of $\mathcal{S}_\lambda$, it follows from the disjointness of $\mathcal{S}_\lambda^{\max}$ that

$$V\{x \in \mathbb{R}^n : |m^{\mathcal{S}} h(x)| > \lambda\} \lesssim \left(\frac{\|h\|_{L^1(\sigma)}}{\lambda}\right)^{\frac{q}{p}}.$$

Next, we will show that $c^{\mathcal{D}}_{\infty,\alpha}(f) \|\chi_{Q^0}\|_{L^q(v)} \lesssim \|f\|_{L^p(w)}$. To this end, we need to choose a cube $Q \in \mathcal{D}$ which contains Q^0 and we need to show that

$$|Q|^{\frac{\alpha}{n}} m_{3Q}(|f|) \|\chi_{Q^0}\|_{L^q(v)} \lesssim \|f\|_{L^p(w)}.$$

To this end, since $Q^0 \subset Q$, it is enough to show that $|3Q|^{\frac{\alpha}{n}} m_{3Q}(|f|) \|\chi_{3Q}\|_{L^q(v)} \lesssim \|f\|_{L^p(w)}$. However, this is already done by the previous estimation. In fact, the family $\{3Q\}$ is a sparse family.

Thus,

$$\begin{aligned} c^{\mathcal{D}}_{\infty,\alpha}(f) \|\chi_{Q^0}\|_{L^q(v)} + \|\chi_{Q^0} m^{\mathcal{S}}[f \cdot \sigma^{-1}]\|_{L^q(V)} &\lesssim \|f\|_{L^p(w)} + \|f \cdot \sigma^{-1}\|_{L^p(\sigma)} \\ &\sim \|f\|_{L^p(w)}. \end{aligned}$$

Thus, the proof is complete.

Concerning condition (7.7), we have a clarifying remark.

Remark 14. Assume that $w, v \in \mathbb{M}^+(\mathbb{R}^n) \setminus \{0\}$. Then by the Lebesgue differentiation theorem there exists $x_0 \in \mathbb{R}^n$ such that

$$\lim_{r\downarrow 0} m_{B(x_0,r)}(\sigma) > 0, \quad \lim_{r\downarrow 0} m_{B(x_0,r)}(v) > 0.$$

By condition (7.7), $\lim_{r\downarrow 0} |B(x_0,r)|^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}} m_{B(x_0,r)}(\sigma)^{\frac{1}{p'}} m_{B(x_0,r)}(v)^{\frac{1}{q}} \le \mathcal{J} < \infty$. Hence, the condition $\alpha \ge \dfrac{n}{p} - \dfrac{n}{q}$ is necessary for the validity of (7.5).

7.2.3 Two-weight norm inequality for singular integral operators

We next investigate the necessary and sufficient conditions on W_1 and W_2 ensuring the validity of the inequality

$$\|Tf\|_{L^q(W_2{}^q)} \lesssim \|f\|_{L^q(W_1{}^q)}, \tag{7.9}$$

where W_1 and W_2 are radial functions non-negative and measurable on $\mathbb{R}^n$ and the implicit constant is independent of $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. Under some conditions (7.10) or (7.11), we can also investigate the boundedness of singular integral operators.

Theorem 320. *Let $1 < q < \infty$. Let W_1, W_2 be radial weights satisfying*

$$\sup_{y\in B(4|x|)\setminus B(\frac{1}{4}|x|)} W_2(y) \lesssim W_1(x) \quad \text{for } x \in \mathbb{R}^n \tag{7.10}$$

or

$$W_2(x) \lesssim \inf_{y\in B(4|x|)\setminus B(\frac{1}{4}|x|)} W_1(y) \quad \text{for } x \in \mathbb{R}^n. \tag{7.11}$$

Then

$$\|Tf\|_{L^q(W_2{}^q)} \le D\|f\|_{L^q(W_1{}^q)} \quad (f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n))$$

holds if the following two conditions are satisfied:

$$\mathcal{J}_1 \equiv \sup_{r>0} r^{-n} \|W_2\|_{L^q(B(r))} \, \|M\chi_{B(r)} {W_1}^{-1}\|_{L^{q'}(\mathbb{R}^n\setminus B(2r))} < \infty \tag{7.12}$$

and

$$\mathcal{J}_2 \equiv \sup_{r>0} r^{-n} \|{W_1}^{-1}\|_{L^{q'}(B(r))} \, \|M\chi_{B(r)} W_2\|_{L^q(\mathbb{R}^n\setminus B(2r))} < \infty. \tag{7.13}$$

Moreover, the sharp constant D^* in the above satisfies $D^* \lesssim \mathcal{J}_1 + \mathcal{J}_2$, where the implicit constants are independent of W_1 and W_2.

Proof We may assume that $f \in L^q(W_1{}^q)$ satisfies $\|f\|_{L^q(W_1{}^q)} = 1$ by normalization. We decompose

$$\int_{\mathbb{R}^n} |Tf(x)|^q W_2(x)^q \mathrm{d}x = \sum_{j=-\infty}^{\infty} \int_{B(2^{j+1})\setminus B(2^j)} |Tf(x)|^q W_2(x)^q \mathrm{d}x.$$

Thus, it suffices to show that

$$\sum_{j=-\infty}^{\infty}\int_{B(2^{j+1})\setminus B(2^j)}|T[\chi_{B(2^{j+2})\setminus B(2^{j-1})}f](x)|^qW_2(x)^q\mathrm{d}x\lesssim 1, \tag{7.14}$$

$$\sum_{j=-\infty}^{\infty}\int_{B(2^{j+1})\setminus B(2^j)}|T[\chi_{\mathbb{R}^n\setminus B(2^{j+2})}f](x)|^qW_2(x)^q\mathrm{d}x\lesssim 1 \tag{7.15}$$

and

$$\sum_{j=-\infty}^{\infty}\int_{B(2^{j+1})\setminus B(2^j)}|T[\chi_{B(2^{j-1})}f](x)|^qW_2(x)^q\mathrm{d}x\lesssim 1. \tag{7.16}$$

Assume for the time being (7.10). As for (7.14), we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\int_{B(2^{j+1})\setminus B(2^j)}|T[\chi_{B(2^{j+2})\setminus B(2^{j-1})}f](x)|^qW_2(x)^q\mathrm{d}x\\
&\le\sum_{j=-\infty}^{\infty}\sup_{x\in B(2^{j+1})\setminus B(2^j)}W_2(x)^q\cdot\int_{B(2^{j+1})\setminus B(2^j)}|T[\chi_{B(2^{j+2})\setminus B(2^{j-1})}f](x)|^q\mathrm{d}x\\
&\lesssim\sum_{j=-\infty}^{\infty}\sup_{x\in B(2^{j+1})\setminus B(2^j)}W_2(x)^q\cdot\int_{B(2^{j+2})\setminus B(2^{j-1})}|f(x)|^q\mathrm{d}x\\
&\le\sum_{j=-\infty}^{\infty}\int_{B(2^{j+2})\setminus B(2^{j-1})}|f(x)|^q\sup_{\frac{1}{4}|x|\le|z|\le 4|x|}W_2(z)^q\mathrm{d}x\\
&\lesssim\sum_{j=-\infty}^{\infty}\int_{B(2^{j+2})\setminus B(2^{j-1})}|f(x)|^qW_1(x)^q\mathrm{d}x=3.
\end{aligned}$$

Next we assume (7.11). Then, we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\int_{B(2^{j+1})\setminus B(2^j)}|T[\chi_{B(2^{j+2})\setminus B(2^{j-1})}f](x)|^qW_2(x)^q\mathrm{d}x\\
&\lesssim\sum_{j=-\infty}^{\infty}\inf_{x\in B(2^{j+1})\setminus B(2^j)}W_1(x)^q\cdot\int_{B(2^{j+1})\setminus B(2^j)}|T[\chi_{B(2^{j+2})\setminus B(2^{j-1})}f](x)|^q\mathrm{d}x\\
&\lesssim\sum_{j=-\infty}^{\infty}\inf_{x\in B(2^{j+1})\setminus B(2^j)}W_1(x)^q\cdot\int_{B(2^{j+2})\setminus B(2^{j-1})}|f(x)|^q\mathrm{d}x\\
&=\sum_{j=-\infty}^{\infty}\int_{B(2^{j+2})\setminus B(2^{j-1})}|f(x)|^q\inf_{z\in B(2^{j+1})\setminus B(2^j)}W_1(z)^q\mathrm{d}x\\
&\le 3.
\end{aligned}$$

Thus, if either (7.10) or (7.11) holds, then we have (7.14).

We establish (7.15). By the size condition it suffices to show that

$$\sum_{j=-\infty}^{\infty} \int_{B(2^{j+1})\setminus B(2^j)} \left(\int_{\mathbb{R}^n \setminus B(2^{j+2})} |f(y)| \frac{\mathrm{d}y}{|y|^n} \right)^q W_2(x)^q \mathrm{d}x \lesssim 1. \tag{7.17}$$

We set $v_1(r) \equiv W_1(r, 0, \dots, 0)$ and $v_2(r) \equiv W_2(r, 0, \dots, 0)$ for $r \geq 0$. Let μ be a measure on $(0, \infty)$ given by

$$\mu(E) \equiv \sum_{j=-\infty}^{\infty} \chi_E(2^j) {W_2}^q(B(2^{j+1}) \setminus B(2^j)).$$

Then (7.17) reads

$$\begin{aligned} &\int_0^\infty \left(\int_{2r}^\infty \frac{1}{R} \left(\int_{S^{n-1}} |f(R\omega)| \mathrm{d}\sigma(\omega) \right) \mathrm{d}R \right)^q \mathrm{d}\mu(t) \\ \lesssim\ &\int_0^\infty r^{n-1} v_1(r)^q \left(\int_{S^{n-1}} |f(r\omega)|^q \mathrm{d}\sigma(\omega) \right) \mathrm{d}r. \end{aligned}$$

Assumption (7.12) reads

$$\sup_{r>0} \mu((0,r])^{\frac{1}{q}} \ \| |\cdot|^{\frac{n-1}{q'} - n} {v_1}^{-1} \|_{L^{q'}(2r,\infty)} < \infty.$$

Hence we have

$$\int_0^\infty \left(\int_r^\infty F(s) \mathrm{d}s \right)^q \mathrm{d}\mu(s) \lesssim \int_0^\infty s^{q+n-1} F(2^{-1}s)^q v_1(s) \mathrm{d}s$$

for all $F \in \mathbb{M}^+(\mathbb{R}^n)$. Or equivalently we have

$$\int_0^\infty \left(\int_{2r}^\infty s^{-1} F(s) \mathrm{d}s \right)^q \mathrm{d}\mu(s) \lesssim \int_0^\infty s^{n-1} F(s)^q v_1(s) \mathrm{d}s$$

for all $F \in \mathbb{M}^+(\mathbb{R}^n)$. We note that (7.15) is a special case of

$$F(r) \equiv \int_{S^{n-1}} |f(r\omega)|^q \mathrm{d}\sigma(\omega) \quad (r > 0).$$

Thus, we obtain (7.15).

The proof of (7.16) is almost the same. By the size condition it suffices to show that

$$\sum_{j=-\infty}^{\infty} \int_{B(2^{j+1})\setminus B(2^j)} \left(\frac{1}{|x|^n} \int_{B(2^{j-1})} |f(y)| \mathrm{d}y \right)^q W_2(x)^q \mathrm{d}x \lesssim 1. \tag{7.18}$$

We define the measue μ^* by

$$\mu^*(E) \equiv \sum_{j=-\infty}^{\infty} \chi_E(2^j) \int_{B(2^{j+1})\setminus B(2^j)} |x|^{-q} W_2(x)^q \mathrm{d}x.$$

Then (7.18) reads

$$\int_0^\infty \left(\int_{B(\frac{1}{2}t)} |f(y)| \mathrm{d}y \right)^q \mathrm{d}\mu^*(t) \lesssim 1.$$

By the polar coordinate we can rephrase (7.18) as

$$\begin{aligned} &\int_0^\infty \left\{ \int_0^{\frac{1}{2}t} \left(r^{n-1} \int_{S^{n-1}} |f(r\omega)| \mathrm{d}\sigma(\omega) \right) \mathrm{d}r \right\}^q \mathrm{d}\mu^*(t) \\ &\lesssim \int_0^\infty \left(\int_{S^{n-1}} |f(r\omega)|^q \mathrm{d}\sigma(\omega) \right) r^{n-1} v_1(r)^q \mathrm{d}r. \end{aligned}$$

Hence, it suffices to show that

$$\begin{aligned} &\int_0^\infty \left\{ \int_0^{\frac{1}{2}t} \left(r^{n-1} \int_{S^{n-1}} |f(r\omega)| \mathrm{d}\sigma(\omega) \right) \mathrm{d}r \right\}^q \mathrm{d}\mu^*(t) \\ &\lesssim \int_0^\infty \left(\int_{S^{n-1}} |f(r\omega)| \mathrm{d}\sigma(\omega) \right)^q r^{n-1} v_1(r)^q \mathrm{d}r. \end{aligned}$$

Note that our assumption (7.13) reads

$$\int_0^\infty \left(\int_0^{\frac{1}{2}t} F(s) \mathrm{d}s \right)^q \mathrm{d}\mu^*(t) \lesssim \left(\int_0^\infty F(t)^q t^{q\frac{n-1}{q'}} v_1(t)^q \mathrm{d}t \right)^{\frac{1}{q}}$$

for all $F \in \mathbb{M}^+(\mathbb{R}^n)$, or equivalently,

$$\int_0^\infty \left(\int_0^{\frac{1}{2}t} s^{-n+1} F(s) \mathrm{d}s \right)^q \mathrm{d}\mu^*(t) \lesssim \left(\int_0^\infty F(t)^q t^{q\frac{n-1}{q'} - q(n-1)} v_1(t)^q \mathrm{d}t \right)^{\frac{1}{q}},$$

which yields (7.18).

7.2.4 Exercises

Exercise 101. Prove Theorem 318(2) using $\|f\|_{L^1(0,t)} \le \|f\|_{L^1(0,\infty)}$ for all $f \in \mathbb{M}^+(\mathbb{R}^n)$.

Exercise 102. Using the disjointness of $\{E_Q\}_{Q \in \mathcal{S}}$, prove that $m^{\mathcal{S}}$, given by (7.8), is bounded on $L^\infty(\mathbb{R}^n)$.

7.3 Notes

Section 7.1

General remarks and textbooks in Section 7.1

The A_p condition initially appeared, in a somewhat different form, in [369]. On the real line, Muckenhoupt characterized the characterization of the weights for which the maximal operator M is bounded on $L^p(w)$ if $p > 1$ and is weak $L^p(w)$-bounded if $p \geq 1$ [324]. For singular integral operators, Muckenhoupt, Rochberg and Wheenden obtained the same characterization [205]. The same attempt was done for fractional integral operators in [326].

See standard textbooks [104, Chapter 7], [112], [146, Chapter IV], [147], [156, Chapter 7], [231], [293, §1.4], [416, Chapter V], [382, §6.1.2] and [432] for the class A_p. See [293, §3.4] for the class $A_{p,q}$.

See the textbooks [104, Chapter 7, §3], [146, Chapter IV, §5], [89] and [382, §6.1.2.6] for the extrapolation. Especially we can find an exhaustive account for the recent extrapolation theory in [89].

Section 7.1.1

Muckenhoupt initialized the theory of the A_1 class [324]. Example 113 is from [322, Remark 2.2] and [332, Example 2.3].

Theorem 280, which gives the information on the growth of the operator norm, is from [206]. Theorem 284 is recorded in [206, Lemma 2.1].

Finally, Example 115 comes from [105, Corollary 4.4].

Section 7.1.2

Muckenhoupt initialized the theory of the A_p class [324]. Theorem 289 can be found in [39, (2.6)]. See also the proof of [39, Theorem 2.5], where Buckley applies the weak estimate to obtain the sharp strong estimate, which in turn proves the sharpness of his weak estimate. See [39, 206, 324] for Theorem 290. The proof of Theorem 290 in this book hinges on the idea of Lerner [268].

Example 120 is due to Lerner; see [274, Lemma 2.3].

Soria and Weiss showed that more general operators are bounded in [412]. This is a generalization of Theorem 288.

Section 7.1.3

Muckenhoupt initialized the theory of the A_∞ class [324]. Theorem 298 is due to Andersen and John [19, Theorem 3.1].

The reverse Hölder inequality for A_p, $1 < p < \infty$ is proved by Coifman and Fefferman in [82, Theorem IV].

The openness property for A_p, Theorem 297, is shown by Coifman and Fefferman in [82, Lemma II] (the case of $1 < p < \infty$) and [82, Theorem V] (the case of $p = \infty$).

We refer to [209, Theorem 2.3] for Theorem 295, the reverse Hölder inequality with precise constant.

Section 7.1.4

Muckenhoupt and Wheeden considered the boundedness of I_α and M_α from $L^p(v^p)$ to $L^q(v^q)$. See [326] for Definition 74 and Theorem 300. Cruz-Uribe and Perez considered the weak $L^p(v^p)$-$L^p(u^p)$ boundedness of fractional integral operators in [93, Theorem 1.1]. See Theorem 299.

Section 7.1.5

Curbera, García-Cuerva, Martell and Pérez proved the extrapolation theorem, Theorem 302 [94].

Section 7.1.6

Coifman and Fefferman showed that $\|Tf\|_{L^p(w)} \lesssim_{[w]_{A_p}} \|f\|_{L^p(w)}$ for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$; see [82, Theorems I and III]. Later on, the estimate was improved. The solution of this conjecture motivated the study of sharp dependence on the A_p constants. Petermichl studied the Hilbert transform [355, 356]. Later on, Hytönen and Lerner considered the decomposition of measurable functions based on Fujii's idea in [136]. See [86, 210, 262, 272, 273, 275] for more recent advances for this field.

We followed [270] for the proof of Lemma 307. Lemma 307(1),(2) and (3) correspond to [270, Proposition 3.1], [270, Lemma 3.2] and [270, Lemma 3.3]. Based on these steps, Lerner concluded his proof using Lemma 307(4) as we did in the proof of Theorem 305. See [208] for more about this direction. See [19, Theorem 5.2] for Theorem 311. See [262] for a more recent approach for the A_2-theorem.

In Section 7.1.6, for $1 < p < \infty$ we showed that $w \in A_p$ is sufficient for singular integral operators to be $L^p(\mathbb{R}^n)$-bounded. See [82, Theorem II] for the converse.

Section 7.2

General remarks and textbooks in Section 7.2

See the standard textbooks and lecture notes [112, Chapter 2], [146, Chapter VI, §6], [147, Chapters 3, 4 and 9], [223, Chapter 2, §I], [231] and [235, Chapters 4 and 6].

Section 7.2.1

Muckenhoupt considered the weighted inequality for Hardy operators; see [325]. Theorems 312 and 314 are well known; see [35, 230, 309, 422, 437] for Theorems 312 and 314, the case $1 < p \le q < \infty$. Our formulation of Theorem

312 is exactly [112, Lemma 2.7.1]. See [47, Lemma 5] for Theorem 318(1), (2), (3) and [45, Lemma 6] for Theorem 318(4).

Section 7.2.2

Kerman and Sawyer obtained the testing condition in [227, 385].

See [227, Theorem 2.3] and [400, Theorem 1] for the counterpart to fractional integral operators of Theorem 319. Hunt, Kurtz and Neugebauer pointed out that this theorem characterizes the A_p-class in [204].

Section 7.2.3

See [47, Theorem 3] for Theorem 320, where they also proved the necessity of the conditions for the Hilbert transform to be bounded.

We list the definition of some related operators.

Example 129. For suitable $f \in \mathbb{M}^+(\mathbb{R}^n)$, the weighted Hardy operator U_ψ is defined by

$$U_\psi f(x) \equiv \int_0^1 f(sx)\psi(s)\mathrm{d}s \quad (x \in \mathbb{R}^n). \tag{7.1}$$

Example 130. For suitable $f \in \mathbb{M}^+(\mathbb{R}^n)$, the weighted Cesàro operator V_ψ is defined by

$$V_\psi f(t) \equiv \int_0^1 f\left(\frac{t}{s}\right) s^{-n}\psi(s)\mathrm{d}s \quad (x \in \mathbb{R}^n). \tag{7.2}$$

Note that when $\psi \equiv 1$ and $n = 1$, the operator is reduced to the classical Cesàro operator.

We can consider the commutator generated by V_ψ and a function $b \in \mathrm{BMO}(\mathbb{R}^n)$:

$$V_\psi^b f \equiv V_\psi(b \cdot f) - bV_\psi f. \tag{7.3}$$

Likewise, we can consider the commutator generated by U_ψ and a function $b \in \mathrm{BMO}(\mathbb{R}^n)$:

$$U_\psi^b f \equiv U_\psi(b \cdot f) - bU_\psi f. \tag{7.4}$$

for the operator U_ψ given by (7.1).

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Chapter 8

Approximations in Morrey spaces

Chapter 8 defines several closed subspaces, two of which coincide. This implies that Morrey functions are not easy to approximate with nice functions.

Section 8.1 shows that closure in a metric space generally amounts to measuring the size of the sets which can be approximated in the topology in question. Section 8.2 gives closed subspaces in terms of closure for actual approximations in Morrey spaces as well as canonical approximations.

8.1 Various closed subspaces of Morrey spaces

Next, we consider various subspaces. Considering the subspaces makes sense unlike Lebesgue spaces, since there are many non-dense subspaces. It could not be better if we can approximate all Morrey functions as we did for the functions in Lebesgue spaces. Unfortunately, this is not the case. Here, we classify Morrey functions based on various approximations. More and more attention has been paid to the closed subspaces of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q < p < \infty$. In particular, the closed subspaces $\widehat{\mathcal{M}}^p_q(\mathbb{R}^n)$ and $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ arise naturally. We define some closed subspaces in Section 8.1.1, all of which enjoy the lattice property. Section 8.1.2 considers a special subspace which fails to enjoy the lattice property. Section 8.1.3 investigates the mutual relationship between them.

8.1.1 Closed subspaces generated by linear lattices

We deal with closed subspaces of Morrey spaces. As we will see, we can generate closed subspaces starting from linear subspaces satisfying the lattice property. The following is the key ingredient for us to create closed subspaces.

Definition 76 ($U\mathcal{M}^p_q(\mathbb{R}^n)$)**.**

(1) A linear subspace $U(\mathbb{R}^n) \subset L^0(\mathbb{R}^n)$ enjoys the *lattice property* if $g \in U$ if $f \in U$ and $|g| \le |f|$.

(2) Let $U(\mathbb{R}^n) \subset L^0(\mathbb{R}^n)$ be a linear space with lattice property. For $0 < q \le p < \infty$, define $U\mathcal{M}^p_q(\mathbb{R}^n) \equiv \overline{U(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)}^{\mathcal{M}^p_q(\mathbb{R}^n)}$.

The spaces of U we envisage are the following. Due to its importance we state them as the definitions.

Definition 77 ($\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$, $\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, $\mathcal{M}^p_q(\Omega)$)**.** Let $0 < q \le p < \infty$.

(1) The case where $U(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$: The *bar subspace* $\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ stands for the closure of $L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$ with respect to $\mathcal{M}^p_q(\mathbb{R}^n)$.

(2) The case where $U(\mathbb{R}^n) = L^0_{\rm c}(\mathbb{R}^n) \equiv \{f \in L^0(\mathbb{R}^n) : \mathrm{supp}(f) \text{ is compact }\}$: The *star subspace* $\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ stands for the closure in $\mathcal{M}^p_q(\mathbb{R}^n)$ of $L^0_{\rm c}(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$.

(3) The case where $U(\mathbb{R}^n) = L^\infty_{\rm c}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \cap L^0_{\rm c}(\mathbb{R}^n)$: Denote by $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, the *tilde subspace*, the $\mathcal{M}^p_q(\mathbb{R}^n)$-closure of $L^\infty_{\rm c}(\mathbb{R}^n) = L^\infty_{\rm c}(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$.

(4) Let $\Omega \subset \mathbb{R}^n$ be a measurable set. Define $L^0(\Omega)$ to be the subset of $L^0(\mathbb{R}^n)$ which consists of all measurable functions which vanish almost everywhere outside Ω. Let $U(\mathbb{R}^n) = L^0(\Omega)$. Then we obtain the closed subspace $\mathcal{M}^p_q(\Omega)$ of all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ for which f vanishes almost everywhere outside Ω.

Needless to say, $L^\infty_{\rm c}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $0 < p < \infty$. Hence, the above definition is significant only when $0 < q < p < \infty$.

Example 131. Let $0 < q \le \tilde{q} \le p < \infty$. We may take $U(\mathbb{R}^n) = \mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$. Note that $U\mathcal{M}^p_q(\mathbb{R}^n)$ differs from $\mathcal{M}^p_q(\mathbb{R}^n)$ thanks to Theorem 20.

Recall that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $0 < q \le p < \infty$ is defined as the set of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ for which $\|f\|_{\mathcal{M}^p_q} \equiv \sup_{r>0} m(f,p,q;r) < \infty$, where

$$m(f,p,q;r) \equiv \sup_{x\in\mathbb{R}^n} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q \mathrm{d}y \right)^{\frac{1}{q}} \quad (r>0).$$

Using $m(f,p,q;r)$, we can consider the following subspaces.

Definition 78 $(V_0\mathcal{M}^p_q(\mathbb{R}^n),\ V_\infty\mathcal{M}^p_q(\mathbb{R}^n),\ V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n),\ V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n))$**.** Let $0 < q \le p < \infty$.

(1) The *vanishing Morrey space* $V_0\mathcal{M}^p_q(\mathbb{R}^n)$ *at* 0 is defined by

$$V_0\mathcal{M}^p_q(\mathbb{R}^n) \equiv \{f \in \mathcal{M}^p_q(\mathbb{R}^n) : \lim_{r\downarrow 0} m(f,p,q;r) = 0\}.$$

(2) The *vanishing Morrey space* $V_\infty\mathcal{M}^p_q(\mathbb{R}^n)$ *at infinity* is defined by

$$V_\infty\mathcal{M}^p_q(\mathbb{R}^n) \equiv \left\{ f \in \mathcal{M}^p_q(\mathbb{R}^n) : \lim_{r\to\infty} m(f,p,q;r) = 0 \right\},$$

(3) The space $V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$ is defined as the set of all functions $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ such that $\lim_{N\to\infty}\left(\sup_{x\in\mathbb{R}^n}\int_{B(x,1)}|f(y)|^q\chi_{\mathbb{R}^n\setminus B(N)}(y)\mathrm{d}y\right)=0$.

(4) The space $V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)$ is defined as the intersection of these three spaces.

We will show that $V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)$ coincides with $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ in Theorem 326.

There are many closed subspaces. Here, we see that $L^p(\mathbb{R}^n)$ is contained in any of them.

Proposition 321. *Let $0<q\le p<\infty$. Then*

$$L^p(\mathbb{R}^n)\subset V_0\mathcal{M}^p_q(\mathbb{R}^n)\cap V_\infty\mathcal{M}^p_q(\mathbb{R}^n)\cap V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n).$$

In particular, $L^p(\mathbb{R}^n)\subset V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)=\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)=\overline{\mathcal{M}}^p_q(\mathbb{R}^n)\cap\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Proof For example, to check that $L^p(\mathbb{R}^n)\subset V_0\mathcal{M}^p_q(\mathbb{R}^n)$, we argue as follows: Let $f\in L^p(\mathbb{R}^n)$. Then $m(f,p,q;r)\le m(f,p,p;r)$ by Hölder's inequality. Since $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we have

$$\begin{aligned}m(f,p,p;r)&\le m(f-g,p,p;r)+m(g,p,p;r)\\&\le\|f-g\|_{L^p}+m(g,p,p;r)\\&\le\|f-g\|_{L^p}+|B(x,r)|^{\frac{1}{p}}\|g\|_{L^\infty}.\end{aligned}$$

If we let $r\downarrow 0$, then we obtain $\limsup_{r\downarrow0}m(f,p,p;r)\le\|f-g\|_{L^p}$. Since $g\in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is arbitrary, we obtain $\lim_{r\downarrow0}m(f,p,p;r)=0$.

The closed subspaces of local Morrey spaces can be defined analogously to global Morrey spaces. Here, we content ourselves with definitions.

Definition 79 ($\mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$, $\mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, $\mathrm{L}\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$)**.** Let $0<q\le p<\infty$.

(1) The *local bar subspace* $\mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ denotes the closure of $L^\infty(\mathbb{R}^n)\cap\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ in $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.

(2) The *local star subspace* $\mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ stands for the closure in $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ of $L^0_{\mathrm{c}}(\mathbb{R}^n)\cap\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.

(3) Denote by $\mathrm{L}\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, the *local tilde subspace*, the $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$-closure of $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

What we can say for the closed subspaces in Morrey spaces can be applied to the closed subspaces in local Morrey spaces.

Example 132. Let $0<q<p<\infty$, and let $f(x)=|x|^{-\frac{n}{p}}, x\in\mathbb{R}^n$.

(1) Let E be the set defined in Example 11. Then $\chi_E \in \mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n) \setminus \mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

(2) $\chi_{B(1)} f \in \mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n) \setminus \mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(3) $f \in \mathcal{M}^p_q(\mathbb{R}^n) \setminus (\mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n) \cup \mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n))$.

8.1.2 Closed subspaces generated by the translation

As we will see, all these spaces above have the lattice property. Here, we list an important closed subspace which does not enjoy the lattice property. The Zorko subspace $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ is one of such spaces and is used for the boundary value problem in partial differential equations.

Definition 80 ($\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$)**.** Let $0 < q \le p < \infty$. Then define the *Zorko subspace* $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ by

$$\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n) \equiv \left\{ f \in \mathcal{M}^p_q(\mathbb{R}^n) \,:\, \lim_{y\to 0} f(\cdot + y) = f \text{ in } \mathcal{M}^p_q(\mathbb{R}^n) \right\}.$$

The Zorko subspace is also called the *diamond subspace*.

As before, we are interested in the case of $p \neq q$. If $q \ge 1$, then we have the following characterization:

Theorem 322. *Let $1 \le q \le p < \infty$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ belongs to $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ if and only if f belongs to the closure with respect to $\mathcal{M}^p_q(\mathbb{R}^n)$ of the set of all functions $g \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\partial^\alpha g \in \mathcal{M}^p_q(\mathbb{R}^n)$ for all $\alpha \in {\mathbb{N}_0}^n$.*

Proof We start with a setup: Choose $\psi \in C^\infty_c(B(1)) \cap \mathbb{M}^+(\mathbb{R}^n)$ with $\|\psi\|_{L^1} = 1$ and define $\psi_j(y) \equiv j^n \psi(jy)$ for $y \in \mathbb{R}^n$.

If $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfies $\partial^\alpha f \in \mathcal{M}^p_q(\mathbb{R}^n)$ for all multi-indexes α with $|\alpha| = 1$, then by the mean-value theorem

$$f(x+y) - f(x) = \int_0^1 y \cdot \mathrm{grad} f(x+ty)\mathrm{d}t \quad (x, y \in \mathbb{R}^n).$$

Hence $\|f(\cdot + y) - f\|_{\mathcal{M}^p_q} \le |y| \cdot \|\mathrm{grad} f\|_{\mathcal{M}^p_q}$. Thus, if f is a member in the closure of the set above, then $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$. Conversely if $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, then

$$\begin{aligned} \|f - f * \psi_j\|_{\mathcal{M}^p_q} &= \left\| f - \int_{\mathbb{R}^n} f(\cdot - y)\psi_j(y)\mathrm{d}y \right\|_{\mathcal{M}^p_q} \\ &= \left\| f \int_{\mathbb{R}^n} \psi_j(y)\mathrm{d}y - \int_{\mathbb{R}^n} f(\cdot - y)\psi_j(y)\mathrm{d}y \right\|_{\mathcal{M}^p_q}. \end{aligned}$$

Thus, $\|f - f * \psi_j\|_{\mathcal{M}^p_q} \le \sup_{y \in B(j^{-1})} \|f(\cdot + y) - f\|_{\mathcal{M}^p_q} \to 0$. As a result, f belongs to the closure with respect to $\mathcal{M}^p_q(\mathbb{R}^n)$ of the set of all functions $\varphi \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\partial^\alpha \varphi \in \mathcal{M}^p_q(\mathbb{R}^n)$ for all multi-indexes α.

8.1.3 Inclusions in closed subspaces of Morrey spaces

Knowing that $\mathcal{M}^p_q(\mathbb{R}^n)$ does not contain $L^\infty_{\rm c}(\mathbb{R}^n)$ as a dense subspace, we are oriented to the difference between $\mathcal{M}^p_q(\mathbb{R}^n)$ and $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$. We now discuss the difference of these subspaces we defined in Sections 8.1.1 and 8.1.2. Here is an example of what we want to do.

Theorem 323. *Let $0 < q \le p < \infty$. Then $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \subset \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.*

Proof Let $f \in L^\infty_{\rm c}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. Then $f(\cdot + y) \to f$ in $L^p(\mathbb{R}^n)$, as $y \to 0$. Hence, $f(\cdot + y) \to f$ in $\mathcal{M}^p_q(\mathbb{R}^n)$ as $y \to 0$, since $C^\infty_{\rm c}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Consequently, we deduce $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Later we show that $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is a proper subspace of $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$. We now compare $\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ with $V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

Proposition 324. *Let $0 < q \le p < \infty$. Then $\overline{\mathcal{M}}^p_q(\mathbb{R}^n) \subset V_0\mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof Simply observe that $L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n) \subset V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

We compare $\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ with $V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$.

Proposition 325. *Let $0 < q \le p < \infty$. Then $\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n) \subset V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof Simply observe that $L^0_{\rm c}(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n) \subset V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$.

We will show that $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) = V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)$.

Theorem 326. *Let $0 < q \le p < \infty$. Then $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) = V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof Since $L^\infty_{\rm c}(\mathbb{R}^n) \subset V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)$, we have $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \subset V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)$ by taking the closure with respect to $\mathcal{M}^p_q(\mathbb{R}^n)$. To verify the converse inclusion, we let $f \in V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n)$. Fix $R_1, R_2, S > 0$ so that $R_1 < R_2$. Then

$$\begin{aligned}
&\|f - \chi_{[0,R]}(|f|)\chi_{B(R)}f\|_{\mathcal{M}^p_q} \\
&\le \sup_{x\in\mathbb{R}^n, r\ge R_2} m(f,p,q;r) + \sup_{x\in\mathbb{R}^n, r\le R_1} m(f,p,q;r) \\
&\quad + \sup_{x\in\mathbb{R}^n\setminus B(S), R_1\le r\le R_2} m(f,p,q;r) \\
&\quad + \sup_{x\in B(S), R_1\le r\le R_2} m(f - \chi_{[0,R]}(|f|)\chi_{B(R)}f,p,q;r).
\end{aligned}$$

We note that

$$\begin{aligned}
&\sup_{x\in B(S), R_1\le r\le R_2} m(f - \chi_{[0,R]}(|f|)\chi_{B(R)}f,p,q;r) \\
&\le C(R_1,R_2,S,r)\|f - \chi_{[0,R]}(|f|)\chi_{B(R)}f\|_{L^q(B(S+R_2))}.
\end{aligned}$$

Thus, by the Lebesgue convergence theorem

$$\begin{aligned}&\limsup_{R\to\infty}\|f-\chi_{[0,R]}(|f|)\chi_{B(R)}f\|_{\mathcal{M}^p_q}\\&\le \sup_{\substack{x\in\mathbb{R}^n\\ r\ge R_2}} m(f,p,q;r)+\sup_{\substack{x\in\mathbb{R}^n\\ r\le R_1}} m(f,p,q;r)+\sup_{\substack{x\in\mathbb{R}^n\setminus B(S)\\ R_1\le r\le R_2}} m(f,p,q;r).\end{aligned}$$

If we let $R_1\downarrow 0$, $R_2\to\infty$ and $S\to\infty$, we have

$$\limsup_{R\to\infty}\|f-\chi_{[0,R]}(|f|)\chi_{B(R)}f\|_{\mathcal{M}^p_q}=0.$$

Since $\chi_{[0,R]}(|f|)\chi_{B(R)}f\in L^\infty_{\mathrm{c}}(\mathbb{R}\ n)$, we obtain the desired result.

8.1.4 Exercises

Exercise 103. Let $0<q<p<\infty$, and let $f(x)\equiv|x|^{-\frac{n}{p}}, x\in\mathbb{R}^n$.

(1) Disprove that $f=\lim_{k\to\infty}\chi_{B(k)}f$ holds in $\mathcal{M}^p_q(\mathbb{R}^n)$.

(2) Disprove that $\lim_{k\to\infty}\chi_{B(k^{-1})}f=0$ holds in $\mathcal{M}^p_q(\mathbb{R}^n)$.

(3) Conclude that an analogue of the Lebesgue convergence theorem fails for Morrey spaces.

Exercise 104. Let $0<q<p<\infty$. Show that $\mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ and $\mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ are different in the sense that one is not contained in another. Hint: Let $f(x)\equiv\chi_{B(1)}(x)|x|^{-\frac{n}{p}}, x\in\mathbb{R}^n$ and $g(x)\equiv\chi_{\mathbb{R}^n\setminus B(1)}(x)|x|^{-\frac{n}{p}}, x\in\mathbb{R}^n$. Show that $f\in\mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)\setminus\mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ and that $g\in\mathrm{L}\overline{\mathcal{M}}^p_q(\mathbb{R}^n)\setminus\mathrm{L}\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

8.2 Approximation in Morrey spaces

Here, we consider approximation in Morrey spaces. There are several standard approximations of functions; truncation of functions using compact sets, truncation of functions using level sets, mollifications and so on. Here, we characterize our closed subspaces using these approximations.

Section 8.2.1 characterizes $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ defined in Definition 77. Section 8.2.2 is what we want to do here. We investigate how we can approximate Morrey functions by using other function spaces. With concrete examples of functions Section 8.2.3 shows that the subspaces defined in Sections 8.2.1 and 8.2.2 are all different.

8.2.1 Characterization of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$

It is convenient to have equivalent characterizations of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ for $0 < q \leq p < \infty$ keeping in mind that

$$f = \lim_{R\to\infty} \chi_{(0,R)}(|f|)\chi_{B(R)}f \tag{8.1}$$

takes place in $L^p(\mathbb{R}^n)$ and hence in $\mathcal{M}^p_q(\mathbb{R}^n)$ for any $f \in L^p(\mathbb{R}^n)$. When we consider the boundedness of operators, the following observation is useful.

Proposition 327. *Let $0 < q \leq p < \infty$. Then $L^p(\mathbb{R}^n)$ is dense in $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.*

Proof Since $L^\infty_{\rm c}(\mathbb{R}^n)$ is dense in $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ and $L^\infty_{\rm c}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, it follows that $L^p(\mathbb{R}^n)$ is densely contained in $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

We may replace $L^\infty_{\rm c}(\mathbb{R}^n)$ with $C^\infty_{\rm c}(\mathbb{R}^n)$.

Proposition 328. *Let $0 < q \leq p < \infty$. Then, $C^\infty_{\rm c}(\mathbb{R}^n)$ is densely contained in $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.*

Proof We have only to show that any function in $L^\infty_{\rm c}(\mathbb{R}^n)$ lies in the $\mathcal{M}^p_q(\mathbb{R}^n)$-closure of $C^\infty_{\rm c}(\mathbb{R}^n)$. Let $f \in L^\infty_{\rm c}(\mathbb{R}^n)$. Then since $C^\infty_{\rm c}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, f can be approximated by functions in $C^\infty_{\rm c}(\mathbb{R}^n)$ in the topology of $L^p(\mathbb{R}^n)$ and hence $\mathcal{M}^p_q(\mathbb{R}^n)$.

We also have the following characterization of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, namely any function in $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ can be approximated by a linear combination of characteristic functions over sets of finite measure. Denote by $\mathrm{Simple}(\mathbb{R}^n)$ the set of all such functions.

Proposition 329. *For $0 < q \leq p < \infty$, $\mathrm{Simple}(\mathbb{R}^n)$ is dense in $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.*

Proof Let $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, and let $\varepsilon > 0$. Then, from the definition of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ there exists $g \in L^\infty_c(\mathbb{R}^n)$ such that $\|f-g\|_{\mathcal{M}^p_q} < \varepsilon$. Let $E = \mathrm{supp}(g)$. Choose a sequence $\{g_k\}_{k=1}^\infty$ of simple functions such that $\lim_{k\to\infty} \|g-g_k\|_{L^\infty} = 0$. Fix $k \in \mathbb{N}$, and write $g_k = \sum_{l=1}^k a_l \chi_{E_l}$. Define $f_k \equiv g_k\chi_E = \sum_{l=1}^k a_l \chi_{E\cap E_l}$. Since $\mathrm{supp}(f_k) \subseteq E$, we have $\|g-f_k\|_{L^\infty} = \|(g-g_k)\chi_E\|_{L^\infty} \leq \|g-g_k\|_{L^\infty}$. By using the inequality

$$\|g-f_k\|_{\mathcal{M}^p_q} \leq \|g-g_k\|_{L^\infty}\|\chi_E\|_{\mathcal{M}^p_q} \leq \|g-g_k\|_{L^\infty}|E|^{\frac{1}{p}}$$

and taking $k \to \infty$, we obtain $\lim_{k\to\infty} \|g-f_k\|_{\mathcal{M}^p_q} = 0$. Since

$$\|f-f_k\|_{\mathcal{M}^p_q} \lesssim \|f-g\|_{\mathcal{M}^p_q} + \|g-f_k\|_{\mathcal{M}^p_q} < \varepsilon + \|g-f_k\|_{\mathcal{M}^p_q},$$

we have $\limsup_{k\to\infty} \|f - f_k\|_{\mathcal{M}^p_q} \lesssim \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{k\to\infty} \|f - f_k\|_{\mathcal{M}^p_q} = 0$, as desired.

We characterize the space $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ for $0 < q \le p < \infty$. We will use $\{E_k\}_{k=1}^\infty$ to denote an arbitrary sequence of measurable subsets of $\mathbb{R}^n$. We will write $E_k \to \emptyset$ a.e., if the characteristic functions χ_{E_k} converge to 0 pointwise a.e. Namely, $E_k \to \emptyset$ a.e. if and only if $\lim_{k\to\infty} \chi_{E_k}(x) = 0$ for a.e. $x \in \mathbb{R}^n$. Notice that the sets E_k are not required to have finite measure.

Definition 81 (Absolutely continuous norm)**.** Let $0 < q \le p < \infty$. A function f in $\mathcal{M}^p_q(\mathbb{R}^n)$ has *absolutely continuous norm* in $\mathcal{M}^p_q(\mathbb{R}^n)$ if $\|f\chi_{E_k}\|_{\mathcal{M}^p_q} \to 0$ for every sequence $\{E_k\}_{k=1}^\infty$ satisfying $E_k \to \emptyset$ a.e., in the sense that $\lim_{k\to\infty} \chi_{E_k}(x) = 0$ for almost all $x \in \mathbb{R}^n$.

Here are functions having or not having the absolutely continuous norm.

Example 133. Let $0 < q < p < \infty$.

(1) The function $f = \chi_{[0,1]^n}$ has the absolutely continuous norm. In fact simply notice that $\|\chi_{E_j} f\|_{\mathcal{M}^p_q} \le \|\chi_{E_j} f\|_{L^p}$ for any decreasing sequence $\{E_j\}_{j=1}^\infty$ of measurable sets such that $\chi_{E_j} \downarrow 0$ almost everywhere.

(2) The function $f \equiv |\cdot|^{-\frac{n}{p}} \chi_{B(1)}$ fails to have the absolutely continuous norm. This can be checked by using the sequence $\{B(j^{-1})\}_{j=1}^\infty$.

(3) The function $f \equiv |\cdot|^{-\frac{n}{p}} \chi_{\mathbb{R}^n \setminus B(1)}$, $x \in \mathbb{R}^n$, does not have the absolutely continuous norm. This can be checked by using the sequence $\{\mathbb{R}^n \setminus B(j)\}_{j=1}^\infty$.

We can also characterize $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ in terms of the absolute continuity of the norm.

Theorem 330. *Let $0 < q \le p < \infty$ and $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Then f has the absolutely continuous norm if and only if $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.*

Proof We begin with a preliminary observation. Let $\{F_k\}_{k=1}^\infty$ be an arbitrary sequence for which $F_k \to \emptyset$ a.e. Then it follows from the Lebesgue dominated convergence theorem that

$$\|\chi_E \chi_{F_k}\|_{\mathcal{M}^p_q} \le \|\chi_E \chi_{F_k}\|_{L^p} \to 0 \text{ as } k \to \infty$$

as long as E is a set of finite measure.

Let $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ now. Then we can find a sequence of simple functions $\{f_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} \|f_k - f\|_{\mathcal{M}^p_q} = 0$. Let $\{F_k\}_{k=1}^\infty$ be an arbitrary sequence for which $F_k \to \emptyset$ a.e. Then

$$\|\chi_{F_j} f\|_{\mathcal{M}^p_q} \le \|f_k - f\|_{\mathcal{M}^p_q} + \|\chi_{F_j} f_k\|_{\mathcal{M}^p_q} \le \|f_k - f\|_{\mathcal{M}^p_q} + \|\chi_{F_j} f_k\|_{L^p}.$$

Letting $j \to \infty$, we obtain

$$\limsup_{j\to\infty} \|\chi_{F_j} f\|_{\mathcal{M}^p_q} \le \|f_k - f\|_{\mathcal{M}^p_q}.$$

Since k is arbitrary, we have $\lim_{j\to\infty} \|\chi_{F_j} f\|_{\mathcal{M}^p_q} = 0$. To prove the reverse inclusion, we assume that $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ has the absolutely continuous norm. From the definition of the absolute continuity and the fact that

$$\{|f| > R\} \cup (\mathbb{R}^n \setminus B(R)) \to \emptyset \quad (R \to \infty),$$

we have $\lim_{R\to\infty} \|f - \chi_{\{|f|\le R\}\cap B(R)} f\|_{\mathcal{M}^p_q} = \lim_{R\to\infty} \|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(R))} f\|_{\mathcal{M}^p_q} = 0$. We thus conclude $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

8.2.2 Approximations and closed subspaces

Given a Morrey function, we discuss whether the function is (canonically) approximated in the topology of Morrey spaces. To discuss such a kind of thing, we take up once again several subspaces. Here, we consider the relation between various approximations and closed subspaces. The following proposition relates the truncation by balls centered at the origin and $\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Proposition 331. *Let $0 < q \le p < \infty$. Then*

$$\overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n) = \left\{ f \in \mathcal{M}^p_q(\mathbb{R}^n) \,:\, f = \lim_{R\to\infty} \chi_{B(R)} f \text{ in } \mathcal{M}^p_q(\mathbb{R}^n) \right\}.$$

Proof Assume that $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(R)} f\|_{\mathcal{M}^p_q} = 0$. Define $f_R \equiv \chi_{B(R)} f$. Then $f_R \in L^0_{\mathrm{c}}(\mathbb{R}^n)$ and that $\lim_{R\to\infty} \|f - f_R\|_{\mathcal{M}^p_q} = 0$. This shows that $f \in \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Conversely, assume that $f \in \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$. Given $\varepsilon \in (0,1)$, there exist $g_\varepsilon \in L^0_{\mathrm{c}}(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$ such that $\|f - g_\varepsilon\|_{\mathcal{M}^p_q} < \varepsilon$. Choose $R_\varepsilon > 0$ such that $\operatorname{supp}(g_\varepsilon) \subset B(R_\varepsilon)$. For any $R > R_\varepsilon$, we have

$$|\chi_{\mathbb{R}^n\setminus B(R)} f| \le |\chi_{\mathbb{R}^n\setminus B(R)} g_\varepsilon| + |\chi_{\mathbb{R}^n\setminus B(R)}(f - g_\varepsilon)| \le |f - g_\varepsilon|.$$

Consequently, for all $R > R_\varepsilon$, $\|\chi_{\mathbb{R}^n\setminus B(R)} f\|_{\mathcal{M}^p_q} \le \|f - g_\varepsilon\|_{\mathcal{M}^p_q} < \varepsilon$. This shows that $\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(R)} f\|_{\mathcal{M}^p_q} = 0$.

The following proposition relates the truncation by the level set of functions and $\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Proposition 332. *Let $0 < q \le p < \infty$. Then*

$$\overline{\mathcal{M}}^p_q(\mathbb{R}^n) = \left\{ f \in \mathcal{M}^p_q(\mathbb{R}^n) \,:\, f = \lim_{R\to\infty} \chi_{[0,R]}(|f|) f \text{ in } \mathcal{M}^p_q(\mathbb{R}^n) \right\}.$$

Proof One inclusion is clear from the definition of $\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$. Let us concentrate on another inclusion. Let $f \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$. Choose $\{f_j\}_{j=1}^\infty \subseteq \mathcal{M}^p_q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\lim_{j\to\infty} \|f - f_j\|_{\mathcal{M}^p_q} = 0$. Then, there exists a subsequence $\{f_{j_m}\}_{m=1}^\infty \subseteq \{f_j\}_{j=1}^\infty$ such that

$$\lim_{m\to\infty} f_{j_m}(x) = f(x) \tag{8.2}$$

for a.e. $x \in \mathbb{R}^n$.

We may assume $f \geq 0$ by decomposing f suitably. Accordingly, we may assume that $f_j \geq 0$. By replacing f_j with $\min(f_j, f)$, we may assume $f_j \leq f$. If $\sup_{j\in\mathbb{N}} \|f_j\|_{L^\infty} < \infty$, then $f \in L^\infty(\mathbb{R}^n)$ and there is nothing to prove. Assume otherwise; $\sup_{j\in\mathbb{N}} \|f_j\|_{L^\infty} = \infty$. Since $f_j \leq \min(f, \|f_j\|_{L^\infty}) \leq f$, we have $\min(f, \|f_j\|_{L^\infty}) \to f$ in $\mathcal{M}^p_q(\mathbb{R}^n)$. Consequently, since $f - \chi_{[0,2\|f_j\|_{L^\infty}]}(|f|)f \leq 2(f - \min(f, \|f_j\|_{L^\infty}))$, we have $\chi_{[0,2\|f_j\|_{L^\infty}]}(|f|)f \to f$ in $\mathcal{M}^p_q(\mathbb{R}^n)$.

We have another expression of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Proposition 333. *Let $0 < q \leq p < \infty$. Then*

$$\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) = \left\{ f \in \mathcal{M}^p_q(\mathbb{R}^n) \,:\, f = \lim_{R\to\infty} \chi_{B(R)\cap\{|f|\leq R\}} f \text{ in } \mathcal{M}^p_q(\mathbb{R}^n) \right\}. \tag{8.3}$$

Proof Denote by A the right-hand side of (8.3). As before $A \subset \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ from the definition of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Conversely, suppose that $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$. Then we can find a sequence $\{f_j\}_{j=1}^\infty \subset L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $f_j \to f$ in $\mathcal{M}^p_q(\mathbb{R}^n)$. Fix $j \in \mathbb{N}$. By the quasi-triangle inequality, we have

$$\begin{aligned}\|f - \chi_{\{|f|\leq R\}\cap B(R)} f\|_{\mathcal{M}^p_q} &\lesssim_q \|f - f_j\|_{\mathcal{M}^p_q} + \|f_j - \chi_{\{|f|\leq R\}\cap B(R)} f_j\|_{\mathcal{M}^p_q} \\ &\lesssim_q \|f - f_j\|_{\mathcal{M}^p_q} + \|f_j - \chi_{\{|f|\leq R\}\cap B(R)} f_j\|_{L^p}.\end{aligned}$$

Thanks to the dominated convergence theorem $\|f_j - \chi_{\{|f|\leq R\}\cap B(R)} f_j\|_{L^p} \downarrow 0$, as $R \to \infty$. We thus obtain $f \in A$.

By combining Propositions 331–333, we obtain a simple but useful characterization.

Corollary 334. *Let $0 < q \leq p < \infty$. Then $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) = \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n) \cap \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.*

The diamond subspace $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ is important because we can obtain the natural initial condition of the partial differential equation if we use the functions in $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ as initial data.

Theorem 335. *Let $1 \leq q \leq p < \infty$ and $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Then*

$$e^{t\Delta} f \equiv \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\cdot - y|^2}{4t}\right) f(y)\mathrm{d}y$$

converges back to f in $\mathcal{M}^p_q(\mathbb{R}^n)$ as $t \downarrow 0$ if and only if $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Proof The "if" part is clear because $e^{t\Delta}f$ is smooth and satisfies $\partial^\alpha[e^{t\Delta}f] \in \mathcal{M}^p_q(\mathbb{R}^n)$ for any multi-index α thanks to the property of convolution. Mimic the proof of Theorem 322 for the "only if" part. See Exercise 108 for more details.

The first part of the following corollary, whose proof we omit, is an analogy to

Corollary 336. *Let $1 \le q \le p < \infty$.*

(1) *Let $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, then $f * \psi \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.*

(2) *Let $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ with $\|\psi\|_{L^1} = 1$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ belongs to $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ if and only if $\psi_j * f \to f$ in $\mathcal{M}^p_q(\mathbb{R}^n)$ as $j \to \infty$, where $\psi_j(x) \equiv j^n\psi(jx)$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$.*

(3) *If $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.*

Proof

(1) Simply use $\partial^\alpha(f * \psi) = f * \partial^\alpha\psi$.

(2) Go through a similar argument to Theorem 335 using (1).

(3) Since $\|f * g\|_{\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^p_q}\|g\|_{L^1}$, we may assume that $g \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. In this case, the assertion follows immediately from (1).

8.2.3 Examples of functions in closed subspaces

We have defined many closed subspaces in Morrey spaces. Here, we aim to detect the difference between them using many functions constructed earlier. We start with a function which belongs to all of our closed subspaces.

Example 134. Let $0 < q \le p < \infty$. Then since $\chi_{Q(1)} \in L^p(\mathbb{R}^n)$,

$$\chi_{Q(1)} \in V_0\mathcal{M}^p_q(\mathbb{R}^n) \cap V_\infty\mathcal{M}^p_q(\mathbb{R}^n) \cap V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n) \cap \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n).$$

In particular, $\chi_{Q(1)} \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) = V^{(*)}_{0,\infty}\mathcal{M}^p_q(\mathbb{R}^n) = \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n) \cap \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Next, we check which subspace $|\cdot|^{-\frac{n}{p}}$ belongs to.

Example 135. Let $0 < q < p < \infty$, and let $f \equiv |\cdot|^{-\frac{n}{p}} \in \mathcal{M}^p_q(\mathbb{R}^n)$.

(1) We check $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$. Let $g \in L^0_{\mathrm{c}}(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$. Suppose that $\operatorname{supp}(g)$ is contained in $B(R)$. Then from the definition of f we have

$$\|f\|_{\mathcal{M}^p_q} = \lim_{r\to\infty} |B(r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B(r)\setminus B(R))}$$

and hence

$$\|f\|_{\mathcal{M}^p_q} \le \lim_{r\to\infty} |B(r)|^{\frac{1}{p}-\frac{1}{q}}\|f-g\|_{L^q(B(r))} \le \|f-g\|_{\mathcal{M}^p_q}.$$

(2) We check $f \notin \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$. Let $g \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$. Then

$$|B(r)|^{\frac{1}{p}-\frac{1}{q}}\|g\|_{L^q(B(r))} \le |B(r)|^{\frac{1}{p}}\|g\|_{L^\infty}.$$

Thus, it follows that $\lim_{r\downarrow 0}|B(r)|^{\frac{1}{p}-\frac{1}{q}}\|g\|_{L^q(B(r))} = 0$. If we mimic the argument above, then we obtain

$$\sup_{(x,r)\in\mathbb{R}^{n+1}_+}|B(x,r)|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B(x,r)}|f(y)-g(y)|^q\mathrm{d}y\right)^{\frac{1}{q}} \ge \|f\|_{\mathcal{M}^p_q}.$$

This implies that $f \in \mathcal{M}^p_q(\mathbb{R}^n) \setminus \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(3) Since $f \in \mathcal{M}^p_q(\mathbb{R}^n) \setminus \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$, $f \notin \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(4) Since $|B(r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B(r))} = |B(1)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B(1))}$ for $r > 0$, $f \notin V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

(5) Likewise $f \notin V_\infty\mathcal{M}^p_q(\mathbb{R}^n)$.

(6) Since $f(x) \to 0$ as $x \to \infty$, $f \in V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$.

(7) Since $\|f(y+\cdot)-f\|_{\mathcal{M}^p_q} \gtrsim 1$ for any $y \in \mathbb{R}^n \setminus \{0\}$, $f \notin \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

In Example 135 we considered the power function $|\cdot|^{-\frac{n}{p}}$. Let us see what happens if we truncated the function by the unit ball $B(1)$.

Example 136. Let $0 < q < p < \infty$. Define $f(x) \equiv |x|^{-\frac{n}{p}}\chi_{B(1)}(x)$ for $x \in \mathbb{R}^n$.

(1) $f \in \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ since f is compactly supported.

(2) $f \notin \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ since f behaves like $|\cdot|^{-\frac{n}{p}}$ near the origin.

(3) $f \notin \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ since $f \notin \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(4) $f \notin V_0\mathcal{M}^p_q(\mathbb{R}^n)$ since f behaves like $|\cdot|^{-\frac{n}{p}}$ near the origin.

(5) $f \in V_\infty\mathcal{M}^p_q(\mathbb{R}^n)$ since f is compactly supported.

(6) $f \in V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$ since f is compactly supported.

(7) $f \notin \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ since $f \notin \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

In Example 135 we considered the power function $|\cdot|^{-\frac{n}{p}}$. Let us see what happens if we remove the singularity at $x = 0$. Firstly, we consider the smooth truncation.

Example 137. Let $0 < q \le p < \infty$, and let $f(x) \equiv (|x|^2+1)^{-\frac{n}{2p}}$ for $x \in \mathbb{R}^n$.

(1) Since f behaves like $|\cdot|^{-\frac{n}{p}}$ away from the origin, $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

(2) Since $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$, $f \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(3) Since $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, $f \notin \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(4) Since $f \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$, $f \in V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

(5) Since f behaves like $|\cdot|^{-\frac{n}{p}}$ away from the origin, $f \notin V_\infty\mathcal{M}^p_q(\mathbb{R}^n)$.

(6) Since $\lim_{x\to\infty} f(x) = 0$, $f \in V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$.

(7) Since $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and satisfies $\partial^\alpha f \in \mathcal{M}^p_q(\mathbb{R}^n)$ for all multi-indexes α, $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

We also consider the truncation of $|\cdot|^{-\frac{n}{p}}$ using the indicator function $\chi_{\mathbb{R}^n\setminus B(1)}$.

Example 138. Let $0 < q < p < \infty$. Define $f(x) \equiv |x|^{-\frac{n}{p}}\chi_{\mathbb{R}^n\setminus B(1)}(x)$ for $x \in \mathbb{R}^n$. Then $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

(1) Since f behaves like $|\cdot|^{-\frac{n}{p}}$ away from the origin, $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

(2) Since $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$, $f \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(3) Since $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, $f \notin \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(4) Since $f \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$, $f \in V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

(5) Since f behaves like $|\cdot|^{-\frac{n}{p}}$ away from the origin, $f \notin V_\infty\mathcal{M}^p_q(\mathbb{R}^n)$.

(6) Since $f(x) \to 0$ as $x \to \infty$, $f \in V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$.

(7) If $\psi \in C^\infty_{\rm c}(\mathbb{R}^n)$ satisfies $\chi_{B(1)} \le \psi \le \chi_{B(2)}$, then $f - k \in L^p(\mathbb{R}^n) \subset \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, where $k(x) \equiv |x|^{-\frac{n}{p}}(1-\psi(x))$ for $x \in \mathbb{R}^n$. Since $k \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, we have $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Let $0 < q < p < \infty$. The indicator function of $G \equiv \bigcup_{k=1}^{\infty} (k-1+(k!)^{\frac{p}{p-q}}, k+(k!)^{\frac{p}{p-q}})$ belongs to $\mathcal{M}^p_q(\mathbb{R})$ similar to Example 12. Let us see which subspace χ_G belongs to.

Example 139. Let $0 < q < p < \infty$. Let $G \subset \mathbb{R}$ be as in Example 12. According to Example 12, we learn that χ_{G^n} belongs to $\mathcal{M}^p_q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

(1) Using the fact that G^n is made up of infinitely many cubes of volume 1, we will establish that χ_{G^n} does not belong to $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, showing that $\mathcal{M}^p_q(\mathbb{R}^n)$ is similar to $L^\infty(\mathbb{R}^n)$ in some sense. In fact, let $f \in L^0_{\mathrm{c}}(\mathbb{R}) \cap \mathcal{M}^p_q(\mathbb{R}^n)$. Then $\|f - \chi_{G^n}\|_{\mathcal{M}^p_q} \geq 1$, since

$$\operatorname{supp}(f) \cap (k - 1 + (k!)^{\frac{p}{p-q}},\, k + (k!)^{\frac{p}{p-q}})^n \neq \emptyset$$

for large $k \in \mathbb{N}$. In fact, taking such large k, we have

$$|f - \chi_{G^n}| \geq \chi_{(k-1+(k!)^{\frac{p}{p-q}},k+(k!)^{\frac{p}{p-q}})^n}.$$

Hence $\|f - \chi_{G^n}\|_{\mathcal{M}^p_q} \geq \|\chi_{(k-1+(k!)^{\frac{p}{p-q}},k+(k!)^{\frac{p}{p-q}})^n}\|_{\mathcal{M}^p_q} = 1$, which shows that $\chi_{G^n} \notin \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(2) Since $\chi_{G^n} \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$, $\chi_{G^n} \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(3) Since $\chi_{G^n} \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(4) Since $\chi_{G^n} \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$, $f \in V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

(5) Since $k!$ grows faster than any polynomial as $k \to \infty$, $\chi_{G^n} \in V_\infty \mathcal{M}^p_q(\mathbb{R}^n)$.

(6) $\chi_{G^n} \notin V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$ because G^n is the disjoint union of infinitely many cubes of volume 1.

(7) A geometric observation shows that $\lim_{y\to 0} \chi_{G^n}(\cdot + y) = \chi_{G^n}$ in $\mathcal{M}^p_q(\mathbb{R}^n)$. Thus, $\chi_{G^n} \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Example 139 is a positive function but here we consider the case of real functions.

Example 140. Let

$$H(t) = \sum_{k=1}^{\infty} \chi_{[0,1]}(t - k + 1 - k^{\frac{p}{p-q}}) \sin^3(2^{k+1}\pi(t - k + 1 - k^{\frac{p}{p-q}})) \quad (t \in \mathbb{R}).$$

Let $G \subset \mathbb{R}$ be the set given by (1.14).

(1) $H \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R})$, since $|H|$ and χ_G behave almost similarly.

(2) Since $|H| \leq \chi_G \in \overline{\mathcal{M}}^p_q(\mathbb{R})$, we see that $H \in \overline{\mathcal{M}}^p_q(\mathbb{R})$.

(3) Since $H \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R})$, $H \notin \widetilde{\mathcal{M}}^p_q(\mathbb{R})$.

(4) Since $H \in \overline{\mathcal{M}}^p_q(\mathbb{R})$, $H \in V_0\mathcal{M}^p_q(\mathbb{R})$.

(5) Since $|H|$ and χ_G behave almost similarly, $H \notin V_\infty \mathcal{M}^p_q(\mathbb{R})$.

(6) Since $|H|$ and χ_G behave almost similarly, $H \notin V^{(*)}\mathcal{M}^p_q(\mathbb{R})$.

(7) Set $S(t) \equiv \displaystyle\int_0^t \sin^3 s\,\mathrm{d}s$ for $t \in \mathbb{R}$ Then

$$H(t) = \frac{d}{\mathrm{d}t}\sum_{k=1}^{\infty} 2^{-k}\chi_{[0,1]}(t-k+1-k^{\frac{p}{p-q}})S(2^k(t-k+1-k^{\frac{p}{p-q}})) \quad (t \in \mathbb{R}).$$

If $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ satisfy $\chi_{(-1,1)} \le \psi \le \chi_{(-2,2)}$, then we learn $|\psi_j * H(t)| \le \mathrm{O}(2^{j-k})$ using integration by parts if $t \in [k-1+k^{\frac{p}{p-q}}, k+k^{\frac{p}{p-q}}]$ and $k > j$. Thus $\|H - \psi_j * H\|_{\mathcal{M}^p_q} \gtrsim 1$ for all $j \in \mathbb{N}$. Thus $H \notin \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R})$.

We use the self-similar increasing sequence for $\mathcal{M}^p_q(\mathbb{R}^n)$ considered in Example 11.

Example 141. Let $0 < q < p < \infty$. Let

$$F \equiv \{y + Ra_1 + R^2a_2 + \cdots : \{a_j\}_{j=1}^\infty \in \{0,1\}^{\mathbb{N}} \cap \ell^1(\mathbb{N}), y \in [0,1]^n\},$$

where $R > 1$ solves $(1+R)^{\frac{n}{p}-\frac{n}{q}}2^{\frac{n}{q}} = 1$, so that each connected component of F is a closed cube with volume 1.

(1) $\chi_F \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$, because F is the disjoint union of cubes of volume 1.

(2) $\chi_F \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ since $\chi_F \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$.

(3) $\chi_F \notin \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ since $\chi_F \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

(4) $\chi_F \in V_0\mathcal{M}^p_q(\mathbb{R}^n)$ since $\chi_F \in \overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

(5) $\chi_F \notin V_\infty\mathcal{M}^p_q(\mathbb{R}^n)$ thanks to the choice of R.

(6) $\chi_F \notin V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$, since F is the disjoint union of cubes of volume 1.

(7) $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$. In fact, for $|y| < \min(1, R-1)$, by Hölder's inequality

$$\begin{aligned}&\|\chi_F(\cdot+y) - \chi_F\|_{\mathcal{M}^p_q}\\ &\lesssim \sup_{\substack{Q\in\mathcal{Q}\\|Q|=1}} \|\chi_F(\cdot+y)-\chi_F\|_{L^q(Q)} + \sup_{\substack{Q\in\mathcal{Q}\\|Q|\le 1}} |Q|^{\frac{1}{p}-\frac{1}{q}}\|\chi_F(\cdot+y)-\chi_F\|_{L^q(Q)}.\end{aligned}$$

Let $|y| \ll 1$. If $|Q| = 1$, then

$$\|f(\cdot+y) - f\|_{L^q(Q)} \le \|\chi_{[0,1]^n}(\cdot+y)f(\cdot+y) - \chi_{[0,1]^n}f\|_{L^q} = \mathrm{o}(1)$$

as $y \to 0$. If $|Q| \le 1$, then

$$\|\chi_F(\cdot+y) - \chi_F\|_{L^q(Q)} \lesssim |Q|^{\frac{1}{q}}|(y+[0,1]^n) \setminus [0,1]^n|^{\frac{1}{q}} = |Q|^{\frac{1}{q}}\mathrm{o}(1).$$

Thus, $\lim_{y\to 0}\chi_F(\cdot+y) = \chi_F$ in $\mathcal{M}^p_q(\mathbb{R}^n)$.

8.2.4 Exercises

Exercise 105. See what happens if we use Example 15. Let $0 < q < p < \infty$, and let $a \in \left(-\frac{q}{p-q}, -\frac{q}{p}\right)$. Also let $f = \otimes^n \left(\sum_{j=1}^{N} \chi_{(j-1,j-1+j^a)}\right)$ be a function in Example 15.

(1) Show that $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

(2) Show that $f \in \overline{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

(3) Show that $f \notin \widetilde{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

(4) By proving $\|f\chi_Q\|_{\mathcal{M}^p_q} \leq N^{-\frac{1}{p}}$ for all cubes Q with $\ell(Q) \leq N^{-1}$, show that $f \in V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

(5) Show that $f \notin V_\infty \mathcal{M}^p_q(\mathbb{R}^n)$.

(6) Prove that $f \notin V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$.

(7) Prove that $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$.

Exercise 106. In Example 135 alternatively we can verify that $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ as follows: Fix $R > 0$. Define $B \equiv B((2R, 0, \ldots, 0), R)$.

(1) Show that $B \subseteq \mathbb{R}^n \setminus B(R)$ and that $|x| \leq 3R$ whenever $x \in B$.

(2) Establish that $|B|^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |\chi_{\mathbb{R}^n \setminus B(R)}(x) f(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} \gtrsim 1$ for all $R > 0$ and hence $\liminf_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(R)} f\|_{\mathcal{M}^\varphi_q} \gtrsim 1$

(3) Use Proposition 331 to conclude that $f \notin \overset{*}{\mathcal{M}}{}^\varphi_q(\mathbb{R}^n)$.

Exercise 107. Let $0 < q < p < \infty$, and let $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ be such that $\chi_{B(1)} \leq \psi \leq \chi_{B(2)}$. Let $f(x) \equiv (1 - \psi(x))|x|^{-\frac{n}{p}}$ for $x \in \mathbb{R}^n$.

(1) Show that $f \notin \overset{*}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$: Hint: f behaves like $|\cdot|^{-\frac{n}{p}}$.

(2) Show that $f \in \overline{\mathcal{M}}{}^p_q(\mathbb{R}^n)$: Hint: Check that $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$.

(3) Show that $f \notin \widetilde{\mathcal{M}}{}^p_q(\mathbb{R}^n)$. Hint: We showed $f \in \overline{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ in part (1).

(4) Show that $f \in V_0\mathcal{M}^p_q(\mathbb{R}^n)$ using $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{M}^p_q(\mathbb{R}^n)$.

(5) Show that $f \notin V_\infty \mathcal{M}^p_q(\mathbb{R}^n)$. Hint: f behaves like $|\cdot|^{-\frac{n}{p}}$.

(6) Show that $f \in V^{(*)}\mathcal{M}^p_q(\mathbb{R}^n)$ since $f(x) = \mathrm{O}(|x|^{-|\alpha|-\frac{n}{p}})$ as $|x| \to \infty$.

(7) Show that $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$. Hint: f is smooth and $\partial^\alpha f(x) = \mathrm{O}(|x|^{-|\alpha|-\frac{n}{p}})$ for all $\alpha \in \mathbb{N}_0$ as $|x| \to \infty$.

Exercise 108. Let $1 \leq q \leq p < \infty$ and $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Assume $\partial^\gamma f \in \mathcal{M}^p_q(\mathbb{R}^n)$ for all multi-indexes γ. Let $0 < t \leq 1$ for the purpose of considering the limit as $t \downarrow 0$.

(1) Assume for the time being $f \in C^\infty(\mathbb{R}^n)$.

(a) Show that $\left(1 - e^{t\Delta}\right) f(x) = \displaystyle\int_0^t e^{s\Delta} \Delta f(x) \mathrm{d}s$.

(b) Verify that $\dfrac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\dfrac{|\cdot|^2}{t}\right) \in L^1(\mathbb{R}^n)$ for all $t > 0$.

(c) Show that $\left\| \displaystyle\int_0^t e^{s\Delta} \Delta f \mathrm{d}s \right\|_{\mathcal{M}^p_q} \leq t \|\Delta f\|_{\mathcal{M}^p_q}$ using the fact that $e^{s\Delta}$ is the convolution operator.

(d) Conclude that $e^{t\Delta} f$ converges back to f in $\mathcal{M}^p_q(\mathbb{R}^n)$ as $t \downarrow 0$.

(2) For general $f \in \overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ and for any $k \in \mathbb{N}$, we have $f_k \in C^\infty$ such that for all multi-indexes γ, $\partial^\gamma f_k \in \mathcal{M}^p_q(\mathbb{R}^n)$ and that

$$\|f - f_k\|_{\mathcal{M}^p_q} < 2^{-k}. \tag{8.4}$$

(a) Find a small number $T(k) > 0$ such that for all $t \in (0, T(k))$,

$$\left\|f_k - e^{t\Delta} f_k\right\|_{\mathcal{M}^p_q} < 2^{-k} \tag{8.5}$$

holds.

(b) Show that $\left\|f - e^{t\Delta} f\right\|_{\mathcal{M}^p_q} \leq 2\|f - f_k\|_{\mathcal{M}^p_q} + \left\|f_k - e^{t\Delta} f_k\right\|_{\mathcal{M}^p_q}$

(c) By combining this fact with (8.4) and (8.5), show that

$$\left\|f - e^{t\Delta} f\right\|_{\mathcal{M}^p_q} < 2^{2-k},$$

which implies that $e^{t\Delta} f$ goes to f in $\mathcal{M}^p_q(\mathbb{R}^n)$ as t tends to zero.

8.3 Notes

Section 8.1

General remarks and textbooks in Section 8.1

As we have seen, there are many non-trivial and important non-dense subspaces. This aspect seems to have been untouched.

Section 8.1.1

The definition of vanishing Morrey spaces at 0 goes back to the work of Vitanza [444]; see Definition 78. We refer to [361] for commutators acting on these spaces. See [177, Definition 1.1], [467, Definition 2.23] and [398, Definition 4.5] for bar subspaces, star subspaces and tilde subspaces, respectively. Density of simple functions in $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ can be found in [65, Remark 3.2]; see Proposition 329. See [65, Lemmas 3.3 and 6.4] for different characterizations of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$. As an application of the theory of closed subspaces in Morrey spaces, we can obtain some characterization of closed subpsaces; see [65, Lemmas 4.1, 4.2 and 6.2]. We refer to [467, Theorem 2.29] for more. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ does not have $C^\infty_{\rm c}(\mathbb{R}^n)$ as a dense closed subspace; see [438, Proposition 2.16].

Vanishing Morrey spaces at 0 are studied from various aspects. See [14] and [15] for fractional integral operators and Marcinkiewicz intergrals associated with Schrödinger operators, respectively. Cao and Chen handled Toeplitz-type operators in [61].

Section 8.1.2

Zorko explained that the set of smooth functions in Morrey spaces cannot approximate Morrey spaces [472, p. 587]. Actually as in Theorem 322, she pointed out in [472, Proposition 3] that the translation and the convolution are closely related. We refer to [467, Definition 2.23] for $\overset{\diamond}{\mathcal{M}}{}^p_q(\mathbb{R}^n)$ and to [398, Theorem 4.3] for $\widehat{\mathcal{M}}^p_q(\mathbb{R}^n)$. See [65, Lemma 2.1] as well as [180, Theorem 6] for Theorem 322. The space $\tilde{L}^{p,\alpha}(\mathbb{R}^n)$ in [34] is nothing but the diamond subspace. See [33, 56] for more.

Section 8.1.3

We refer to [180, Lemma 3] for Proposition 324, where we showed $\overline{\mathcal{M}}^p_q(\mathbb{R}^n) \subset V_0\mathcal{M}^p_q(\mathbb{R}^n)$.

Section 8.2

General remarks and textbooks in Section 8.2

We can find an exhaustive account for absolutely continuous norms in [29, Chapter 3]. Adams pointed out that any element in diamond subspaces can be approximated by the smooth functions in his textbook [6].

Section 8.2.1

See [21, Theorem 1.1] for the approximation of Morrey functions by means of the Fourier-Jacobi expansions. Note that such an expansion goes back to the original function in the norm topology of Morrey spaces if and only if

the original function belongs to closed subspaces of Morrey spaces. See [398, Theorem 4.6] for Theorem 330.

Section 8.2.2

Recall that we have charecterized $\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$ in Theorem 332. See [65, Lemma 3.1] for a different characterization of $\overline{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Israfilov and Tozman considered the approximation by means of Morrey-Smilnov classes [213].

Section 8.2.3

Example 140 can be found [181, Example 1.4]. See [277] for analytic Morrey spaces.

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Chapter 9

Predual of Morrey spaces

This chapter is devoted to considering the predual. The dual of Morrey spaces contains an element which cannot be expressed in terms of locally integrable functions. However, we show that Morrey spaces are realized as the dual of Banach function spaces. Furthermore these Banach function spaces are again realized as the dual of Banach function spaces. One of the reasons why such a situation occurs is that Morrey spaces are wider than Lebesgue spaces. Hence, there is difficulty in defining some linear operators.

Given a Banach space $\mathcal{X}$, recall that the *dual space* $\mathcal{X}^*$ is made up of all bounded linear functionals. For $x^* \in \mathcal{X}^*$, its *dual norm* is expressed by $\|x^*\|_{\mathcal{X}^*} \equiv \sup\{|x^*(x)| \in \mathbb{R} : x \in \mathcal{X}, \|x\|_{\mathcal{X}} = 1\}$ by definition. The space $\mathcal{X}^*$ $\|\cdot\|_{\mathcal{X}^*}$ is a Banach space under the dual norm. Thus, the notation $\mathcal{X}^{**}$ always makes sense. Given $x \in X$, we can define $Q(x) \in \mathcal{X}^{**}$ by $Q(x)(x^*) \equiv x^*(x)$ for $x^* \in \mathcal{X}^*$. According to the Hahn–Banach theorem, Theorem 87, the canonical mapping $Q : x \in X \mapsto Q(x) \in \mathcal{X}^{**}$ is isometrc. If this isometry Q is surjective, then we say that $\mathcal{X}$ is reflexive. See [310, 1.11.6] for example.

Chapter 9 is interested in predual spaces of Morrey spaces. In functional analysis, duality plays a key role in justifying the convergence of sequences. However, the dual space of the Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q < p < \infty$ cannot be described via coupling. The dual space differs from the Köthe dual. This is a big difference from Lebesgue spaces.

Section 9.1 introduces the predual of Morrey spaces. The predual of Morrey spaces has some equivalent characterizations, which are investigated in Section 9.2.

9.1 Predual of Morrey spaces

Having investigated functions which belong to Morrey spaces, we are now oriented to their dual spaces and their reflexivity. As Example 148 shows, the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q < p < \infty$ is not reflexive. Consequently, we are more interested in the functionals on $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q < p < \infty$. Like a certain functional of $\ell^\infty(\mathbb{N})$ known as the Banach limit, we can find a similar functional in the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q < p < \infty$.

We start with defining the block spaces in Section 9.1.1 and then we introduce predual spaces. Section 9.1.2 mainly considers subspaces in predual spaces. Section 9.1.3 deals with duality. The Fatou property of the function spaces is one of the important properties. As it turns out, among others it is difficult to check the Fatou property of predual spaces. We will actually show that predual spaces enjoy this property in Section 9.1.4. The Fatou property of predual spaces has an application. We can specify the Köthe dual of Morrey spaces in Section 9.1.5. We consider the theory of decomposition in Section 9.1.6. In particular, as an application of the (pre)duality, we develop a technique to control the Morrey norm in Section 9.1.6.

9.1.1 Definition of block spaces and examples

Section 9.1.1 seeks to show that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q \leq p < \infty$ can be realized as the dual of a certain Banach space. The space will be called the block space or the Zorko space and we investigate the lattice property and a standard expression of functions in this space.

Before we come to the precise definition of the Zorko space, let us recall the norm expression of $L^p(\mu)$ for $1 \leq p \leq \infty$ over a σ-finite measure space $(X, \mathcal{B}, \mu)$, which we restate; see Theorem 9 as well.

Theorem 337 (Norm expression of $L^p(\mu)$)**.** *Assume that $(X, \mathcal{B}, \mu)$ is a σ-finite measure space. Let f be a function that is integrable on any set of finite measure. Then, for all $1 \leq p \leq \infty$, we have*

$$\|f\|_{L^p(\mu)} = \sup_g \frac{1}{\|g\|_{L^{p'}(\mu)}} \left| \int_X f(x)g(x)\mathrm{d}\mu(x) \right|,$$

where g runs over all $g \in L^\infty(\mu) \cap L^{p'}(\mu) \setminus \{0\}$ such that $\mu\{g \neq 0\} < \infty$.

We omit its proof, since there are many textbooks; see [319, Theorem 12.7], for example.

We state a direct consequence of Theorems 9 and 89 together with the Banach Alaoglu theorem.

Corollary 338. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $1 < p < \infty$ and $\{f_j\}_{j=1}^\infty$ be a bounded sequence in $L^p(\mu)$. Namely, we suppose that there exists $M > 0$ such that $\|f_j\|_{L^p(\mu)} \leq M$. Then we can find $f \in L^p(\mu)$ and a subsequence $\{f_{j_k}\}_{k=1}^\infty$ such that*

$$\lim_{k\to\infty} \int_X f_{j_k}(x)g(x)\mathrm{d}\mu(x) = \int_X f(x)g(x)\mathrm{d}\mu(x)$$

for all $g \in L^{p'}(\mu)$.

We intend to consider the counterpart of Theorems 337 to Morrey spaces. We have by definition

$$\|f\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |Q(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q(x,r)} |f(y)|^q \mathrm{d}y \right)^{\frac{1}{q}}.$$

Of course we can start with the definition using balls. However, for the sake of simplicity in later discussions, we use cubes. Note that this can be written as follows:

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} \sup_A \int_{\mathbb{R}^n} |f(x)A(x)|\mathrm{d}x,$$

where A moves over all $L^{q'}(\mathbb{R}^n)$-functions supported on a cube $Q(x,r)$ and satisfying $\|A\|_{L^{q'}} \le |Q(x,r)|^{\frac{1}{p}-\frac{1}{q}}$. Our observation justifies the following definition.

Definition 82 ((p,q)-block)**.** Let $1 \le q \le p < \infty$. A function $A \in L^0(\mathbb{R}^n)$ is a (p,q)-*block* if there exists a compact cube Q that supports A and

$$\|A\|_{L^{q'}} \le |Q|^{\frac{1}{q'}-\frac{1}{p'}}. \tag{9.1}$$

If we need to specify Q, then we say that b is a (p,q)-*block supported on* Q.

Some prefer to call (p,q)-blocks (p',q')-blocks. Here we will need to consider the predual space, so we have to adopt the name "(p,q)-block".

Example 142. Let Q be a cube, and let $1 \le q \le p < \infty$. Then $|Q|^{-\frac{1}{p'}}\chi_Q$ is a (p,q)-block.

Other examples of (p,q)-blocks, more precisely how to create (p,q)-blocks in a standard manner, will be presented below.

By definition if Q supports A, then $A(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus Q$.

Let $1 \le q \le p < \infty$. As is easily verified by Hölder's inequality, any (p,q)-block has $L^{p'}(\mathbb{R}^n)$-norm less than 1:

$$\|A\|_{L^{p'}} \le 1 \tag{9.2}$$

for all blocks A. The set of constant multiples of all blocks forms a linear space. However, this set is unnatural due to the freedom of the choice of cubes in the definition of (p,q)-blocks. We are thus interested in a natural way to allow us to consider that the set of all (p,q)-blocks is a Banach space.

Definition 83 ($\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$)**.** Let $1 \le q \le p < \infty$ and define the *block space* (the *Zorko space*) $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ as the set of all $f \in L^{p'}(\mathbb{R}^n)$ for which f is realized as the sum $f = \sum_{j=1}^{\infty} \lambda_j A_j$ with some $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ and a sequence $\{A_j\}_{j=1}^{\infty}$ of (p,q)-blocks. Define the norm $\|f\|_{\mathcal{H}^{p'}_{q'}}$ for $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ as

$$\|f\|_{\mathcal{H}^{p'}_{q'}} \equiv \inf_{\lambda} \|\lambda\|_{\ell^1}, \tag{9.3}$$

where $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ runs over all admissible expressions:

$$f = \sum_{j=1}^{\infty} \lambda_j A_j,\ \{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}),\ A_j \text{ is a } (p,q)\text{-block for all } j \in \mathbb{N}. \tag{9.4}$$

A direct consequence of (9.2) is that the series $f = \sum_{j=1}^{\infty} \lambda_j A_j$ in (9.4) converges in the topology of $L^{p'}(\mathbb{R}^n)$. Thus, the postulate that $f = \sum_{j=1}^{\infty} \lambda_j A_j$ converges in $L^{p'}(\mathbb{R}^n)$ is natural as we will see.

Proposition 339. *Let $1 < p < \infty$. Then $\mathcal{H}^{p'}_{p'}(\mathbb{R}^n) = L^{p'}(\mathbb{R}^n)$ with coincidence of norms.*

Proof Let $f \in L^{p'}(\mathbb{R}^n)$. Fix $\varepsilon > 0$. Then there is a decomposition

$$f = \chi_{Q_1} f + \sum_{j=2}^{\infty} \chi_{Q_j \setminus Q_{j-1}} f,$$

where $\{Q_j\}_{j=1}^{\infty}$ is an increasing sequence of cubes centered at the origin satifying

$$\|\chi_{\mathbb{R}^n \setminus Q_k} f\|_{L^{p'}} \leq \frac{\varepsilon}{2^k}$$

for all $k \in \mathbb{N}$. Then using this decomposition, we learn $\|f\|_{\mathcal{H}^{p'}_{p'}} \leq \|f\|_{L^{p'}} + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}$. Thus, $f \in \mathcal{H}^{p'}_{p'}(\mathbb{R}^n)$.

Meanwhile, if $f \in \mathcal{H}^{p'}_{p'}(\mathbb{R}^n)$, then $f \in L^{p'}(\mathbb{R}^n)$ as we have seen.

We give an example of a function in $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Example 143. Let $1 < q \leq p < \infty$. Define

$$f(x) \equiv \frac{1}{1 + \max(|x_1|, |x_2|, \ldots, |x_n|)^M} \quad (x \in \mathbb{R}^n).$$

Then $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ if and only if $Mp' > n$. Indeed, if $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, then $f \in L^{p'}(\mathbb{R}^n)$, which forces $Mp' > n$. Suppose instead that $Mp' > n$. Define

$$b_0 \equiv \frac{1}{2^n} \chi_{Q(1)} \cdot f,\ b_j \equiv 2^{-j\frac{n}{p'} - \frac{n}{p'} + jM - M} \chi_{Q(2^j) \setminus Q(2^{j-1})} \cdot f \quad (j \in \mathbb{N}).$$

Then, $\|b_0\|_{L^\infty} = 2^{-n}$ and for each $j \in \mathbb{N}$

$$\|b_j\|_{L^\infty} = \frac{2^{-j\frac{n}{p'} - \frac{n}{p'} + jM - M}}{1 + 2^{(j-1)M}} < 2^{-j\frac{n}{p'} - \frac{n}{p'}} \leq |Q(2^j)|^{-\frac{1}{p'}}.$$

Hence, each b_j, $j \in \mathbb{N}_0$ is a (p, q)-block modulo a multiplicative constant by Hölder's inequality or by Example 142. Observe also that

$$f = 2^n b_0 + \sum_{j=1}^{\infty} 2^{j\frac{n}{p'} + \frac{n}{p'} - jM + M} b_j.$$

Assuming that $Mp' > n$, we have $2^n + \sum_{j=1}^{\infty} 2^{j\frac{n}{p'}+\frac{n}{p'}-jM+M} < \infty$. Hence, it follows that $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Below we collect some fundamental properties of block spaces.

Lemma 340. *Let $1 < q \le p \le \infty$. If A is a (p,q)-block, then $\|A\|_{\mathcal{H}^{p'}_{q'}} \le 1$.*

Proof In (9.4), we choose

$$A_1 = A, A_2 = A_3 = \cdots = 0, \lambda_1 = 1, \lambda_2 = \lambda_3 = \cdots = 0.$$

Then, (9.3) gives the desired result.

The following lemma indicates how to generate blocks.

Lemma 341. *Let $1 < q \le p \le \infty$. Let A be an $L^{q'}(\mathbb{R}^n)$ function supported on a cube Q. Then $\|A\|_{\mathcal{H}^{p'}_{q'}} \le \|A\|_{L^{q'}} |Q|^{-\frac{1}{p}+\frac{1}{q}}$.*

Proof Set $B \equiv \frac{|Q|^{\frac{1}{p}-\frac{1}{q}}}{\|A\|_{L^{q'}}} A$, assuming that A is not zero almost everywhere. Then B is supported on the cube Q and, by virtue of the fact that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$,

$$\|B\|_{L^{q'}} = |Q|^{\frac{1}{p}-\frac{1}{q}} = |Q|^{-\frac{1}{p'}+\frac{1}{q'}}.$$

Hence, B is a (p,q)-block. This implies $\|B\|_{\mathcal{H}^{p'}_{q'}} \le 1$ according to Lemma 340. Equating this inequality, we obtain the desired result.

We have the lattice property of block spaces.

Lemma 342 (Lattice property)**.** *Let $1 < q \le p \le \infty$. Then, a function f belongs to $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ if and only if there exists $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ such that $|f(x)| \le g(x)$ for a.e. $x \in \mathbb{R}^n$.*

Proof Suppose that $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Then there exists a sequence $\{\lambda_k\}_{k=1}^{\infty} \in l^1$ and a (p,q)-block b_k such that $f = \sum_{k=1}^{\infty} \lambda_k b_k$. Letting $g \equiv \sum_{k=1}^{\infty} |\lambda_k||b_k|$, we have $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ and $|f| \le g$. Conversely, suppose that there exists $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ that satisfies $|f(x)| \le g(x)$ for a.e. $x \in \mathbb{R}^n$. Decompose g as $g = \sum_{k=1}^{\infty} \lambda'_k b'_k$ where $\{\lambda'_k\}_{k=1}^{\infty} \in \ell^1(\mathbb{N})$ and b'_k is a (p,q)-block. Then we see that

$$\chi_{\{y:\, g(y)\neq 0\}}(x) = \sum_{k=1}^{\infty} \lambda'_k \frac{1}{g(x)} b'_k(x)$$

and, hence, $f(x) = \sum_{k=1}^{\infty} \lambda'_k \frac{f(x)}{g(x)} b'_k(x)$ for almost all $x \in \mathbb{R}^n$. Since $\frac{|f(x)|}{g(x)} \le 1$ for a.e. $x \in \mathbb{R}^n$, the function $(f/g)b'_k$ becomes a (p,q)-block. This proves the lemma.

In the next lemma, we can say that each $b_k(Q)$ is supported on $3Q$, while, in Definition 83, each A_j has its support in a cube Q_j but there is no information on the cube Q_j.

Lemma 343. *Let $1 < q \le p \le \infty$ and $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{H}^{p'}_{q'}} < 1$. Then f can be decomposed as $f = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda(Q)b(Q)$ in the topology of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, where $\{\lambda(Q)\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ satisfies*

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda(Q) \le 3^n \tag{9.5}$$

and $b(Q)$ is a (p,q)-block supported in $3Q$.

Proof We may as well assume that $f \neq 0$. First, decompose f as

$$f = \sum_{k\in K} \lambda_k b_k$$

where $K \subset \mathbb{N}$ is an index set, the coefficients $\{\lambda_k\}_{k\in K}$ satisfy

$$0 < \sum_{k\in K} |\lambda_k| < 1 \tag{9.6}$$

and each b_k is a (p,q)-block. We divide K into the disjoint sets $K(Q) \subset \mathbb{N}$, $Q \in \mathcal{D}(\mathbb{R}^n)$, as

$$K = \bigcup_{Q\in\mathcal{D}(\mathbb{R}^n)} K(Q)$$

and $K(Q)$ fulfills $\mathrm{supp}(b_k) \subset 3Q$ and $|Q_k| \ge |Q|$ when $k \in K(Q)$. We do this as follows: Let $\mathcal{D} = \{Q^{(j)}\}_{j=1}^{\infty}$ be an enumeration of $\mathcal{D}(\mathbb{R}^n)$. For each $k \in K$, we define

$$j_k \equiv \min\{j : \mathrm{supp}(b_k) \subset 3Q^{(j)}, |Q_k| \ge |Q^{(j)}|\}$$

for each k. We write $K(Q^{(j)}) \equiv \{k \in K : j_k = j\}$. We set

$$\lambda(Q) \equiv 3^n \sum_{k\in K(Q)} |\lambda_k|, \quad b(Q) \equiv \begin{cases} \dfrac{1}{\lambda(Q)} \displaystyle\sum_{k\in K(Q)} \lambda_k b_k & (\lambda(Q) \neq 0), \\ 0 & (\lambda(Q) = 0). \end{cases}$$

We now rewrite f as

$$\begin{aligned} f &= \sum_{k\in K} \lambda_k b_k = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \left(\sum_{k\in K(Q)} \lambda_k b_k \right) \\ &= \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \left\{ 3^n \sum_{k\in K(Q)} |\lambda_k| \right\} \cdot \left\{ \left(3^n \sum_{k\in K(Q)} |\lambda_k| \right)^{-1} \sum_{k\in K(Q)} \lambda_k b_k \right\}. \end{aligned}$$

By (9.6), we have

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda(Q)=3^n\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\left(\sum_{k\in K(Q)}|\lambda_k|\right)=3^n\sum_{k\in K}|\lambda_k|<3^n,$$

which implies (9.5). Since each b_k is a (p,q)-block, we obtain

$$\begin{aligned}
&\left(3^n\sum_{k\in K(Q)}|\lambda_k|\right)^{-1}\left\|\sum_{k\in K(Q)}\lambda_k b_k\right\|_{L^{q'}}\\
&\le\left(3^n\sum_{k\in K(Q)}|\lambda_k|\right)^{-1}\sum_{k\in K(Q)}|\lambda_k|\cdot\|b_k\|_{L^{q'}}\\
&\le\left(3^n\sum_{k\in K(Q)}|\lambda_k|\right)^{-1}|Q|^{\frac{1}{p}-\frac{1}{q}}\sum_{k\in K(Q)}|\lambda_k|\\
&\lesssim_{p,q}|3Q|^{\frac{1}{p}-\frac{1}{q}},
\end{aligned}$$

which implies that $b(Q)$ is a (p,q)-block supported in $3Q$ modulo a multiplicative constant. These complete the proof.

By decomposing $b(Q)$ further, we obtain an equivalent expression of the space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Theorem 344 is useful in that we can remove the duplicate of the cubes; it can happen that the cubes for A_j and $A_{j'}$ are identical for some $1\le j<j'<\infty$ in Definition 83.

Theorem 344. *Let $1<q\le p\le\infty$ and $f\in\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{H}^{p'}_{q'}}<1$. Then f can be decomposed as $f=\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda(Q)b(Q)$, where each $\lambda(Q)\ge 0$ for each Q, each $b(Q)$ is a (p,q)-block supported in Q, and $\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda(Q)\le 9^n$.*

The block space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is designed to study the duality of $\mathcal{M}^p_q(\mathbb{R}^n)$. To study a predual space of $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ we can extend our definition naturally. For example, the notion of local blocks can be defined in a similar fashion.

Definition 84. Let $1\le p\le q<\infty$. A function $A\in L^0(\mathbb{R}^n)$ is a *local (p,q)-block* if there exists a cube $Q=Q(r)$ that is centered at the origin, supports A and satisfies (9.1).

Example 144. Let $1\le q\le p<\infty$. For $r>0$, $|Q(r)|^{\frac{1}{p}-\frac{1}{q}}\chi_{Q(r)}$ is a local (p,q)-block.

We now present the definition of local block spaces.

Definition 85 ($\mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$)**.** Let $1 \le p \le q < \infty$ and define the *local block space* $\mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ as the set of all $f \in L^{p'}(\mathbb{R}^n)$ for which f is realized as the sum $f = \sum_{j=1}^{\infty} \lambda_j A_j$ converges in $L^{p'}(\mathbb{R}^n)$ with some $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ and a sequence $\{A_j\}_{j=1}^{\infty}$ of local (p,q)-blocks. Define the norm $\|f\|_{\mathrm{L}\mathcal{H}^{p'}_{q'}}$ for $f \in \mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ by

$$\|f\|_{\mathrm{L}\mathcal{H}^{p'}_{q'}} \equiv \inf_{\lambda} \|\lambda\|_{\ell^1},$$

where $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ runs over all admissible expressions

$$f = \sum_{j=1}^{\infty} \lambda_j A_j,\ \{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}),\ A_j \text{ is a } (p,q)\text{-local block for all } j \in \mathbb{N}.$$

We remark that an analogy to usual Morrey spaces is available, whose details we omit.

9.1.2 Finite decomposition and a dense subspace

Here, we consider a dense subspace of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ for $1 < q \le p < \infty$ together with a decomposition result, which will be useful when we consider the boundedness property of Morrey spaces. It is noteworthy that the space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ has $L^{q'}_{\mathrm{c}}(\mathbb{R}^n)$ as a dense space, while the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ does not.

Theorem 345. *Let $1 < q \le p < \infty$. Then $L^{q'}_{\mathrm{c}}(\mathbb{R}^n)$ is dense in $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. In particular, by mollification, $C^{\infty}_{\mathrm{c}}(\mathbb{R}^n)$ is dense.*

Proof Since $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, there exists a non-negative summable sequence $\{\lambda_j\}_{j=1}^{\infty}$ and a collection $\{b_j\}_{j=1}^{\infty}$ of (p,q)-blocks such that $g = \sum_{j=1}^{\infty} \lambda_j b_j$. Define $g_N \equiv \sum_{j=1}^{N} \lambda_j b_j$. Then $\|g - g_N\|_{\mathcal{H}^{p'}_{q'}} \le \sum_{j=N+1}^{\infty} |\lambda_j| \to 0$ as $N \to \infty$.

Consequently, when we investigate $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, the space $L^{q'}_{\mathrm{c}}(\mathbb{R}^n)$ is a useful space.

We will be interested in linear operators defined and continuous on $L^q(\mathbb{R}^n)$. Consequently, it will be helpful to have a finite decomposition in $L^{q'}_{\mathrm{c}}(\mathbb{R}^n)$. What differs from Definition 83 is that we are considering finite admissible expressions. That is, the sum is finite.

Theorem 346. *Let $1 < q \le p < \infty$. Then any $f \in L^{q'}_{\mathrm{c}}(\mathbb{R}^n)$ admits the finite decomposition $f = \sum_{j=1}^{N} \lambda_j b_j$, where $\lambda_1, \lambda_2, \dots, \lambda_N \ge 0$ and each b_j is a (p,q)-block. Furthermore, $\|f\|_{\mathcal{H}^{p'}_{q'}} \sim \inf_{\lambda}(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_N|)$, where $\lambda = \{\lambda_j\}_{j=1}^{N}$*

runs over all finite admissible expressions:

$$f=\sum_{j=1}^{N}\lambda_j b_j,\quad \lambda_1,\lambda_2,\dots,\lambda_N\geq 0,\ b_j \text{ is a } (p,q)\text{-block for all } j=1,2,\dots,N.$$

Proof Let $f\in L^{q'}_{\mathrm{c}}(\mathbb{R}^n)\subset \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. We may assume that $\|f\|_{\mathcal{H}^{p'}_{q'}}<1$ and that $\mathrm{supp}(f)\subset Q^0$ for some dyadic cube Q^0, since we can decompose f into 2^n sums, each of which is supported on a closure of a quadrant. Then according to Theorem 344 f can be decomposed as

$$f=\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda(Q)b(Q)$$

where

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda(Q)\leq 9^n.$$

We may assume that each $b(Q)$ is non-negative and that $\lambda(Q)$ is a non-negative real number, since $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ enjoys the lattice property. Furthermore, we may assume that $\mathrm{supp}(b(Q))\subset Q^0$. Thus f is non-negative. By the monotone convergence theorem and the fact that $\mathrm{supp}(f)\subset Q^0$, we have

$$f=\lim_{\varepsilon\downarrow 0}\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\chi_{(\varepsilon,\varepsilon^{-1})}(|Q|)\lambda(Q)b(Q)\chi_{Q^0}$$

in $L^{q'}(\mathbb{R}^n)$. Thus, if we set

$$g\equiv\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}(1-\chi_{(\varepsilon,\varepsilon^{-1})}(|Q|))\lambda(Q)b(Q)\chi_{Q^0},$$

then since $f\in L^{q'}(\mathbb{R}^n)$, we deduce that g is a (p,q)-block as long as ε is small, so that

$$f=g+\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\chi_{(\varepsilon,\varepsilon^{-1})}(|Q|)\lambda(Q)b(Q)\chi_{Q^0}$$

is the desired finite decomposition.

Since the norm of g can be made as small as we wish, we have the desired norm equivalence: $\|f\|_{\mathcal{H}^{p'}_{q'}}\sim\inf_{\lambda}(|\lambda_1|+|\lambda_2|+\cdots+|\lambda_N|)$.

9.1.3 Duality–block spaces and Morrey spaces

We show that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is realized by the dual space of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Theorem 347 (Duality of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$)**.** *Suppose that* $1<q\leq p<\infty$.

(1) *Any* $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ *defines a continuous functional* $L_f : g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \in \mathbb{C}$.

(2) *Conversely, every continuous functional* L *on* $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ *can be realized with* $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

(3) *The correspondence* $f \in \mathcal{M}^p_q(\mathbb{R}^n) \mapsto L_f \in (\mathcal{H}^{p'}_{q'}(\mathbb{R}^n))^*$ *is an isomorphism. Furthermore*

$$\|f\|_{\mathcal{M}^p_q} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n), \|g\|_{\mathcal{H}^{p'}_{q'}} = 1\right\} \tag{9.7}$$

and

$$\|g\|_{\mathcal{H}^{p'}_{q'}} = \max\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : f \in \mathcal{M}^p_q(\mathbb{R}^n), \|f\|_{\mathcal{M}^p_q} = 1\right\}. \tag{9.8}$$

Proof The proof of the boundedness of L_f is simple so we omit it. In fact it is straightforward to show that $\|L_f\|_* \le \|f\|_{\mathcal{M}^p_q}$.

To prove that every continuous functional L on $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ can be realized with $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, we let $Q_j = 2^j Q(1)$. For the sake of the simplicity we denote $L^{q'}(Q_j)$ by the set of $L^{q'}(\mathbb{R}^n)$-functions supported on Q_j. In view of Lemma 341 the functional $g \mapsto L(g)$ is well defined and bounded on $L^{q'}(Q_j)$. Thus, by Theorem 9 there exists $f_j \in L^q(Q_j)(\subset L^0(\mathbb{R}^n))$ such that

$$L(g) = \int_{Q_j} f_j(x)g(x)\mathrm{d}x$$

for all $g \in L^q(Q_j)$. By the uniqueness of this theorem we can find an $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ function f such that $f = f_j$ a.e. on Q_j for any j.

We will prove $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. For this purpose, we take an arbitrary Q and estimate

$$\mathrm{I} \equiv |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}. \tag{9.9}$$

For a fixed cube Q and a fixed function f we set $g \equiv \chi_Q \mathrm{sgn}(f)|f|^{q-1}$. Notice that the function $\frac{|Q|^{\frac{1}{p}-\frac{1}{q}}}{\|g\|_{L^{q'}}} g$ is a (p,q)-block by virtue of Lemma 341. Thus, we can write

$$\mathrm{I} = |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q f(x)g(x)\mathrm{d}x\right)^{\frac{1}{q}} = |Q|^{\frac{1}{p}-\frac{1}{q}} \left(L(g)\right)^{\frac{1}{q}}. \tag{9.10}$$

Hence, we have $|L(g)| \le \|L\|_* |Q|^{-\frac{1}{p}+\frac{1}{q}} \|g\|_{L^{q'}}$. As a result we have $I \le \|L\|_*$. This is the desired result. The proof of (9.7) and (9.8) is included in what we have proven.

As the following example shows, $\ell^1(\mathbb{Z})$ is embedded into $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Example 145. Let $1 < q < p < \infty$.

(1) Let $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Then $f*g \in L^\infty(\mathbb{R}^n)$. Since $C^\infty_{\rm c}(\mathbb{R}^n)$ is dense, $f*g$ is continuous.

(2) Let $f \equiv \sum_{j=-\infty}^{\infty} 2^{-\frac{jn}{p}} a_j \chi_{B(2^{j+1})\setminus B(2^j)}$, where $\{a_j\}_{j=-\infty}^{\infty}$ is a summable complex sequence. Then

$$\|f\|_{\mathcal{H}^{p'}_{q'}} \sim \sum_{j=-\infty}^{\infty} |a_j|.$$

In fact, since $2^{-\frac{jn}{p}} \chi_{B(2^{j+1})\setminus B(2^j)}$ is a (p,q)-block modulo a multiplicative constant, we have $\|f\|_{\mathcal{H}^{p'}_{q'}} \lesssim \sum_{j=-\infty}^{\infty} |a_j|$. To check the opposite inequality, we use $\|f\cdot g\|_{L^1} \le \|g\|_{\mathcal{M}^p_q}\|f\|_{\mathcal{H}^{p'}_{q'}}$, where $g(x) \equiv |x|^{-\frac{n}{p}}$ for $x \in \mathbb{R}^n$. If we use this inequality, then $\|f\|_{\mathcal{H}^{p'}_{q'}} \gtrsim \sum_{j=-\infty}^{\infty} |a_j|$.

9.1.4 Fatou property of block spaces

We cannot hope that the block space is a Banach function space because the Morrey space, which is its dual, is not. However, we do have the Fatou property as the following theorem shows:

Theorem 348. *Let $1 < q \le p < \infty$. Suppose that f and f_k, $(k = 1, 2, \ldots)$, are non-negative, that each $f_k \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, that $\|f_k\|_{\mathcal{H}^{p'}_{q'}} \le 1$ and that $f_k \uparrow f$ a.e.. Then $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{H}^{p'}_{q'}} \le 1$.*

It is significant that we do not assume $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, so that there is no guarantee that f admits the decomposition into the sum of blocks. Before we prove Theorem 348, we consider its important corollary.

Corollary 349. *Let $1 < q \le p < \infty$. Suppose that we have $g \in L^0(\mathbb{R}^n)$ such that $f \cdot g \in L^1(\mathbb{R}^n)$ for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Then $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.*

Proof By the closed graph theorem, $\|f\cdot g\|_{L^1} \le L\|f\|_{\mathcal{M}^p_q}$ for some $L > 0$ independent of $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. If we set $g_M \equiv \chi_{B(M)}\chi_{[0,M]}(|g|)g$ for each $M \in \mathbb{N}$, then $g_M \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ according to Lemma 341. If we use Theorem 347, then $\|g_M\|_{\mathcal{H}^{p'}_{q'}} \le L$. Consequently by Theorem 348, we see that $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Proof of Theorem 348 We may assume that $1 < q < p < \infty$ as we remarked just below the statement. By Theorem 344 f_k can be decomposed as

$$f_k = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_k(Q) b_k(Q),$$

where $\lambda_k(Q)$ is a non-negative number with

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_k(Q) \le 2\cdot 9^n \tag{9.11}$$

and $b_k(Q)$ is a (p,q)-block supported in Q and

$$\|b_k(Q)\|_{L^{q'}} \le |Q|^{\frac{1}{p}-\frac{1}{q}}. \tag{9.12}$$

In particular,

$$0 \le \lambda_k(Q) \le 2\cdot 9^n \tag{9.13}$$

Noticing (9.12), (9.13) and the weak-compactness of the Lebesgue space $L^{q'}(Q)$ (see Corollary 338), we now apply a diagonalization argument or the Tikhonov compact theorem and, hence, we can select an infinite subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ that satisfies the following:

$$f_{k_j} = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_{k_j}(Q) b_{k_j}(Q), \quad \lim_{j\to\infty} \lambda_{k_j}(Q) = \lambda(Q),$$

$$\lim_{j\to\infty} b_{k_j}(Q) = b(Q) \text{ in the weak-topology of } L^{q'}(Q),$$

where $b(Q)$ is a (p,q)-block supported in Q. We set $f_0 \equiv \sum\limits_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda(Q)b(Q)$. Then, by Fatou's lemma and (9.11),

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda(Q) \le \liminf_{j\to\infty} \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_{k_j}(Q) \le 2\cdot 9^n, \tag{9.14}$$

which implies $f_0 \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

We will verify that

$$\lim_{j\to\infty} \int_{Q_0} f_{k_j}(x)\mathrm{d}x = \int_{Q_0} f_0(x)\mathrm{d}x \tag{9.15}$$

for all $Q_0 \in \mathcal{D}(\mathbb{R}^n)$. Once (9.15) is established, we will see that $f = f_0$ and $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ by virtue of the Lebesgue differentiation theorem because at least we know that f_0 locally in $L^{q'}(\mathbb{R}^n)$. By the definition of f_{k_j} and the fact that the sum defining f_{k_j} converges in $L^{p'}(\mathbb{R}^n)$, we have

$$\int_{Q_0} f_{k_j}(x)\mathrm{d}x = \sum_{l=-\infty}^{\infty} \sum_{Q\in\mathcal{D}(Q_0)\cup\mathcal{D}^\sharp(Q_0),\, |Q|=2^{ln}|Q_0|} \lambda_{k_j}(Q) \int_{Q_0} b_{k_j}(Q)(x)\mathrm{d}x.$$

Note that

$$\left|\int_{Q_0} b_{k_j}(Q)(x)\mathrm{d}x\right| \le \|b_{k_j}(Q)\|_{L^1} \le \|b_{k_j}(Q)\|_{L^{q'}} |Q|^{\frac{1}{q}} \le |Q|^{\frac{1}{p}-\frac{1}{q}} |Q\cap Q_0|^{\frac{1}{q}}$$

for any cube Q contained in Q_0. Since

$$\sum_{l=-\infty}^{\infty} |Q|^{\frac{1}{p}-\frac{1}{q}} |Q \cap Q_0|^{\frac{1}{q}} < \infty,$$

we are in the position of Lebesgue's convergence theorem to obtain

$$\begin{aligned}&\lim_{j\to\infty} \int_{Q_0} f_{k_j}(x)\mathrm{d}x \\ &= \sum_{l=-\infty}^{\infty} \left(\lim_{j\to\infty} \sum_{Q\in\mathcal{D}(Q_0)\cup\mathcal{D}^\sharp(Q_0),\, |Q|=2^{ln}|Q_0|} \lambda_{k_j}(Q) \int_{Q_0} b_{k_j}(Q)(x)\mathrm{d}x \right).\end{aligned}$$

Since

$$\sum_{Q\in\mathcal{D}(Q_0)\cup\mathcal{D}^\sharp(Q_0),\, |Q|=2^{ln}|Q_0|}$$

is the symbol of summation over a finite set for each l, we have

$$\lim_{j\to\infty} \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_{k_j}(Q) \int_{Q_0} b_{k_j}(Q)(x)\mathrm{d}x = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda(Q) \int_{Q_0} b(Q)(x)\mathrm{d}x. \quad (9.16)$$

Thus, we obtain (9.15).

Since $f_k \uparrow f$ a.e., we must have $\displaystyle\int_{Q_0} f(x)\mathrm{d}x = \int_{Q_0} f_0(x)\mathrm{d}x$ for all $Q_0 \in \mathcal{D}(\mathbb{R}^n)$ by (9.15). This together with the Lebesgue differential theorem yields $f = f_0$ a.e. and hence $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Since we have verified $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, it follows that

$$\|f\|_{\mathcal{H}^{p'}_{q'}} = \sup\left\{ \left| \int_{\mathbb{R}^n} f_k(x)g(x)\mathrm{d}x \right| : k = 1, 2, \ldots, \|g\|_{\mathcal{M}^p_q} \le 1 \right\} \le 1.$$

This completes the proof of the theorem.

Similar to Theorem 348, we can prove the Fatou property of local block spaces and its consequence.

Theorem 350. *Let $1 < q \le p \le \infty$.*

(1) *Let $f \in \mathbb{M}^+(\mathbb{R}^n)$ and $f_k \in \mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$, $k \in \mathbb{N}$. Suppose that $\|f_k\|_{\mathrm{L}\mathcal{H}^{p'}_{q'}} \le 1$ and that $f_k \uparrow f$ a.e.. Then $f \in \mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ and $\|f\|_{\mathrm{L}\mathcal{H}^{p'}_{q'}} \le 1$.*

(2) *The dual of $\mathrm{L}\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is canonically identified with $\mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.*

9.1.5 Köthe dual of Morrey spaces

We will characterize the predual of block spaces in terms of the Köthe dual. The result will be an application of the Fatou property. Let us start with a general fact. We give a general definition.

Definition 86 (ρ')**.** If ρ is a ball Banach function norm, its "associate norm" ρ' is defined in $\mathbb{M}^+(\mathbb{R}^n)$ by

$$\rho'(g) \equiv \sup\left\{\|f \cdot g\|_{L^1} \,:\, f \in \mathbb{M}^+(\mathbb{R}^n), \rho(f) \leq 1\right\} \quad (g \in \mathbb{M}^+(\mathbb{R}^n)). \tag{9.17}$$

Likewise we can consider the Köthe dual for Banach lattices.

Example 146. Let $1 \leq p \leq \infty$. Then $L^p(\mathbb{R}^n)' = L^{p'}(\mathbb{R}^n)$. It is worth comparing with this $L^p(\mathbb{R}^n)^* = L^{p'}(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Let $f, g \in \mathbb{M}^+(\mathbb{R}^n)$. A trivial consequence of the related definitions is;

$$\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \leq \rho(f)\rho'(g). \tag{9.18}$$

By no means do we have a guarantee that any functional in Morrey spaces can be described in terms of an integral like $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Consequently, we here want to consider functionals in Morrey spaces that can be described in terms of integrals.

The dual of a ball Banach function norm is again a ball Banach function norm.

Theorem 351. *Let ρ be a ball Banach function norm. Then the associate norm ρ' is a ball Banach function norm.*

Proof We need to verify the properties (P1)–(P5) of Banach function norms.

By virtue of the Lebesgue differentiation theorem, we have $g = 0$ a.e. whenever $g \in \mathbb{M}^+(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x = 0$$

for all $f \in \mathbb{M}^+(\mathbb{R}^n)$ such that $\rho(f) \leq 1$. Other two properties of (P1) are clear from the corresponding properties of integrals. Consequently, (P1) is verified. By the monotonicity of integral (P2) is also trivial. Let us check (P3). To this end, we take a non-negative increasing sequence $\{g_j\}_{j=1}^{\infty}$; $0 \leq g_j \uparrow g$. If $\rho(g) = 0$, then there is nothing to prove. Hence, assume $\rho'(g) > 0$. Choose any non-negative number $M \in [0, \rho'(g))$.

Then by the definition of $\rho'(g)$, we can find $f \in \mathbb{M}^+(\mathbb{R}^n)$ with $\rho(f) \leq 1$ such that

$$M < \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \leq \rho'(g).$$

By virtue of the monotone convergence theorem, if j is large enough

$$M < \int_{\mathbb{R}^n} f(x)g_j(x)\mathrm{d}x \leq \rho'(g_j) \leq \rho'(g).$$

Thus, $\rho'(g_j) \uparrow \rho'(g)$.

The property (P4) follows from (P5) for (X, ρ);

$$\int_Q f(x)\mathrm{d}x \lesssim_Q 1$$

for all $f \in \mathbb{M}^+(\mathbb{R}^n)$ with $\rho(f) \leq 1$.

Finally the property (P5) follows from (P4) for (X, ρ) and (9.18);

$$\int_Q g(x)\mathrm{d}x \leq \rho(\chi_Q)$$

for all $g \in \mathbb{M}^+(\mathbb{R}^n)$ with $\rho'(g) \leq 1$.

We define the associated space with respect to ρ. It should be noted that in general the associate space differs from the dual space as the example of Morrey spaces.

Definition 87 ($\mathcal{X}(\rho')$, $\mathcal{X}'(\rho)$)**.** Let ρ be a ball Banach function norm, and let $\mathcal{X} = \mathcal{X}(\rho)$ be the ball Banach function space determined by ρ as in Definition 13. Let ρ' be the associate norm of ρ. The Banach function space $\mathcal{X}(\rho') = \mathcal{X}'(\rho)$ determined by ρ' is called the *associate space* or the *Köthe dual* of $\mathcal{X}$.

At this moment, we content ourselves with a simple example of Lebesgue spaces.

Example 147. Let $1 \leq p \leq \infty$ and p' be its conjugate, and let $(X, \mathcal{B}, \mu)$ be a measure space. Then the associate space of $L^p(\mu)$ is $L^{p'}(\mu)$. It matters that we tolerate the case $p = \infty$.

As the example of $\mathcal{X} = \ell^1(\mathbb{N})$ shows, the bidual $\mathcal{X}^{**}$ of a Banach space $\mathcal{X}$ can be strictly bigger than $\mathcal{X}$ itself. However, unlike the dual spaces of Banach spaces, the associate space of the associate space of any ball Banach function space $E(\mathbb{R}^n)$ is $E(\mathbb{R}^n)$ itself.

Theorem 352. *Every ball Banach function space $E(\mathbb{R}^n)$ coincides with its second associate space $E''(\mathbb{R}^n)$. In other words, a function f belongs to $E(\mathbb{R}^n)$ if and only if it belongs to $E''(\mathbb{R}^n)$, and in that case $\|f\|_E = \|f\|_{E''}$ or all $f \in E(\mathbb{R}^n) = E''(\mathbb{R}^n)$.*

We have an analogy to Banach function spaces; see [29]. Keeping in mind that Morrey spaces are main function spaces in this book, we consider ball Banach function spaces.

Proof It is trivial from (9.18) that $E(\mathbb{R}^n) \subset E''(\mathbb{R}^n)$ and that $\rho''(f) \leq \rho(f)$ for any $f \in \mathbb{M}^+(\mathbb{R}^n)$.

To prove the reverse inclusion, we let $h \in E''(\mathbb{R}^n)$. Since we have (P2), (P3) and (P4)', we can suppose that $h \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Set

$$S \equiv \{f \in L^1(\mathbb{R}^n) : \mathrm{supp}(f) \subset \mathrm{supp}(h), \rho(|f|) \leq 1\}.$$

Let us check that S is a closed convex set in $L^1(\mathbb{R}^n)$. Let $f_1, f_2 \in S$ and $a \in (0,1)$. Then $\rho(|af_1+(1-a)f_2|) \le a\rho(|f_1|)+(1-a)\rho(|f_2|) \le 1$ by virtue of (P1) and $\operatorname{supp}(af_1+(1-a)f_2) \subset \operatorname{supp}(f_1) \cup \operatorname{supp}(f_2) \subset \operatorname{supp}(h)$. Thus, S is a convex set. To prove that S is a closed set, we take a sequence $\{f_j\}_{j=1}^{\infty}$ in S that converges to $f \in L^1(\mathbb{R}^n)$ in the $L^1(\mathbb{R}^n)$ topology. If necessary, by passing to a subsequence, we can assume that $\{f_j\}_{j=1}^{\infty}$ converges almost everywhere to f. Thus, $\operatorname{supp}(f)$ is contained in $\operatorname{supp}(h)$. In addition, by (P3), the Fatou property,

$$\rho(|f|) = \rho\left(\lim_{k\to\infty}\inf_{j\ge k}|f_j|\right) = \lim_{k\to\infty}\rho\left(\inf_{j\ge k}|f_j|\right) \le 1.$$

Thus, S is a closed set.

Let $\lambda > 1$. Then $g \equiv \frac{\lambda}{\|h\|_E}h$ does not belong to S, so that, by the Hahn–Banach theorem of geometric form (see Theorem 88) and the duality $L^1(\mathbb{R}^n)$-$L^\infty(\mathbb{R}^n)$, we can find $\varphi \in L^\infty(\mathbb{R}^n)$ such that

$$\operatorname{Re}\left(\int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x\right) < \frac{\lambda}{\|h\|_E}\int_{\mathbb{R}^n} h(x)\varphi(x)\mathrm{d}x \quad (\in \mathbb{R})$$

for any $f \in S$. From the definition of $E''(\mathbb{R}^n)$, we have

$$\operatorname{Re}\left(\int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x\right) < \frac{\lambda}{\|h\|_E}\rho''(h)\rho'(\varphi).$$

If we take supremum over all $f \in S$, then

$$\rho'(\varphi) \le \frac{\lambda}{\|h\|_E}\rho''(h)\rho'(\varphi).$$

Since $\lambda > 1$ is arbitrary, we see that $\|h\|_E \le \rho''(h) = \|h\|_{E''}$.

We will show that ball Banach function spaces are closed under the operation of taking the Köthe dual.

Proposition 353. *Let $E(\mathbb{R}^n)$ be a ball Banach function space. Then its associate $E'(\mathbb{R}^n)$ is also a ball Banach function space.*

Proof To show that $E'(\mathbb{R}^n)$ is a ball Banach function space, we need to show (i)–(v) below:

(i) $\|f\|_{E'} = 0$ implies that $f = 0$ almost everywhere;

(ii) $|g| \le |f|$ almost everywhere implies that $\|g\|_{E'} \le \|f\|_{E'}$;

(iii) $0 \le f_n \uparrow f$ almost everywhere implies that $\|f_n\|_{E'} \uparrow \|f\|_{E'}$;

(iv) $B \in \mathcal{B}$ implies that $\chi_B \in E(\mathbb{R}^n)$;

(v) for any $B \in \mathcal{B}$, there exists $C_{(B)} > 0$ such that $\|f\|_{L^1(B)} \le C_{(B)}\|f\|_{E'}$ for all $f \in E'(\mathbb{R}^n)$.

We concentrate on (iii) since the remaining assertions are proven easily.

By Lebesgue's differentiation theorem, Theorem 12, we conclude that (i) holds true. From the definition of $E'(\mathbb{R}^n)$, it follows that (ii) holds true. Next, we obtain (iv). Moreover, from the fact that $\chi_B \in E(\mathbb{R}^n)$ for all $B \in \mathcal{B}$ and the definition of $E'(\mathbb{R}^n)$, we deduce that (v) holds true.

Finally, we prove (iii). Let $\{f_m\}_{m=1}^{\infty} \subset E'(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ and $f \in E'(\mathbb{R}^n)$ satisfy that $f_m \uparrow f$ almost everywhere, and let $A \in (0, \|f\|_{E'})$. Then, by the definition of $\|f\|_{E'}$, we know that there exists a real-valued function $g \in E(\mathbb{R}^n)$ with $\|g\|_E = 1$ such that $A < \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \leq \int_{\mathbb{R}^n} f(x)|g(x)|\mathrm{d}x$, which, combined with the monotone convergence theorem, implies that there exists $N \in \mathbb{N}$ such that $A < \int_{\mathbb{R}^n} f_N(x)|g(x)|\mathrm{d}x$. Thus, $\|f_N\|_{E'} > A$, which, together with the arbitrariness of $A \in (0, \|f\|_{E'})$, implies that (iii) holds true. This finishes the proof of Proposition 353.

Remark that a similar theory can be developed for Banach lattices. For example, we can extend Theorem 352 as follows:

Theorem 354. *Let $E(\mu)$ be a Banach lattice over a measure space $(X, \mathcal{B}, \mu)$. Then $E(\mu)$ has the Fatou property if and only if $E(\mu) = E''(\mu)$.*

Proof If $E(\mu)$ has the Fatou property, then we can mimic the proof of Theorem 352. See Exercise 117. Conversely, suppose $E(\mu) = E''(\mu)$. Then since $F'(\mu)$ has the Fatou property for any Banach lattice $F(\mu)$, we see that $E(\mu)$ has the Fatou property.

We characterize the Köthe dual of Morrey spaces.

Theorem 355. *Let $1 < q \leq p < \infty$. Then the associate space $(\mathcal{M}^p_q)'(\mathbb{R}^n)$ as a ball Banach function space coincides with the block space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.*

By no means does Theorem 355 give any complete information on the dual space of $\mathcal{M}^p_q(\mathbb{R}^n)$.

Proof The proof is a mere combination of Theorems 348 and 354. Alternatively, we can argue directly as follows: As we have seen in Theorem 347, $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \subset (\mathcal{M}^p_q)'(\mathbb{R}^n)$. Hence, we will verify the converse. Suppose that $f \in L^0(\mathbb{R}^n)$ satisfies

$$\sup\left\{\|f \cdot g\|_{L^1} : \|g\|_{\mathcal{M}^p_q} \leq 1\right\} \leq 1. \tag{9.19}$$

Then we first see that $|f(x)| < \infty$, (a.e. $x \in \mathbb{R}^n$). Splitting f into its real and imaginary parts and each of these into its positive and negative parts, we may assume without loss of generality that $f \in \mathbb{M}^+(\mathbb{R}^n)$. For $k = 1, 2, \ldots$, set $f_k \equiv \min(f, k)\chi_{Q(k)}$. We notice that $f_k \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ and $\|f_k\|_{\mathcal{H}^{p'}_{q'}} \leq 1$ by Lemma 342 and (9.19). Since $f_k \uparrow f$ a.e., it follows that $f \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{H}^{p'}_{q'}} \leq 1$ by virtue of Theorem 348. This proves the theorem.

Theorems 352 and 355 yield $\mathcal{H}^{p'}_{q'}{}'(\mathbb{R}^n) = (\mathcal{M}^p_q)''(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$. Furthermore, from the fact that $(\mathcal{M}^p_q)''(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$, we are able to characterize the predual of block spaces.

Recall that we defined $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ in Definition 77. The block spaces are realized as the dual of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Theorem 356. *Let $1 < q \le p < \infty$. Then a predual space of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ in the following sense*:

If $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, then $L_g : f \mapsto \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x$ is an element of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*$. Moreover, for any $L \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*$, there exists $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ such that $L = L_g$.

Proof The first assertion is clear, since $\mathcal{M}^p_q(\mathbb{R}^n)$ is the dual of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Hence, we will prove that $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^* \subset \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Thanks to Theorem 355, we need only show $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^* \subset (\mathcal{M}^p_q)'(\mathbb{R}^n)$. Suppose that L belongs to $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*$. We will exhibit a function g in $(\mathcal{M}^p_q)'(\mathbb{R}^n)$ such that

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \quad (f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)). \tag{9.20}$$

If we let $Q_m \equiv m + [0,1)^n$, $m \in \mathbb{Z}^n$, then the sequence $\{Q_m\}_{m\in\mathbb{Z}^n}$ partitions $\mathbb{R}^n$, each of which has measure one and whose union is all of $\mathbb{R}^n$. For each $i \in \mathbb{Z}^n$, let $\mathcal{A}_i$ denote the Lebesgue measurable subsets of Q_i and define a set-function λ_i on $\mathcal{A}_i$ by

$$\lambda_i(A) \equiv L(\chi_A), \quad (A \in \mathcal{A}_i).$$

Notice that $\lambda_i(A)$ is well defined for all $A \in \mathcal{A}_i$ because χ_A belongs to $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

We claim that λ_m is countably additive on $\mathcal{A}_m$ for any $m \in \mathbb{Z}^n$. Indeed, let $(A_k)_{k=1}^\infty$ be a sequence of disjoint sets from $\mathcal{A}_m$, and let

$$B_l \equiv \bigcup_{k=1}^{l} A_k, \quad (l = 1, 2, \ldots), \qquad A \equiv \bigcup_{k=1}^{\infty} A_k = \bigcup_{l=1}^{\infty} B_l.$$

It follows from the Lebesgue dominated convergence theorem that

$$\|\chi_A - \chi_{B_l}\|_{\mathcal{M}^p_q} \le \|\chi_A - \chi_{B_l}\|_{L^p(Q_m)} \downarrow 0 \text{ as } l \to \infty.$$

The continuity and linearity of L give

$$\lambda_m(A) = L(\chi_A) = \lim_{l\to\infty} L(\chi_{B_l}) = \lim_{l\to\infty} \sum_{k=1}^{l} L(\chi_{A_k}) = \sum_{k=1}^{\infty} \lambda_m(A_k),$$

which establishes the claim.

Since $|\lambda_m(A)| \le \|L\|_{(\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n))^*}\|\chi_A\|_{\widetilde{\mathcal{M}}^p_q}$ for all $A \in \mathcal{A}_i$ and $\lambda_m(A) = 0$ for all $A \in \mathcal{A}_m$ such that $|A| = 0$, by the Radon–Nikodym theorem (see [236, Theorem 15.41], for example), there uniquely exists $g_m \in L^0(Q_m)$ such that

$$L(\chi_A) = \lambda_m(A) = \int_{\mathbb{R}^n} \chi_A(x) g_m(x) \mathrm{d}x, \quad (A \in \mathcal{A}_m).$$

Since the sets Q_m are disjoint we may define a function g on all of $\mathbb{R}^n$ by setting $g \equiv g_m$ on each Q_m. Clearly,

$$L(\chi_E) = \int_{\mathbb{R}^n} \chi_E(x) g(x) \mathrm{d}x \tag{9.21}$$

for all characteristic functions of sets of finite measure χ_E.

We will show that g belongs to $(\mathcal{M}^p_q)'(\mathbb{R}^n)$. Choose and fix f in $\mathcal{M}^p_q(\mathbb{R}^n)$. Let $f_l \equiv \sum_{k=1}^{4^l} 2^{-l} k \chi_{F_{k,l}}$ for $l = 1, 2, \ldots$, where

$$F_{k,l} \equiv \left\{ x \in B(2^l) : 2^{-l}k \le |f(x)| < 2^{-l}(k+1) \right\}.$$

If we suppose for the moment that g is real-valued, then $f_l \cdot \mathrm{sgn}(g)$ becomes a finite linear combination of characteristic functions of sets of finite measure. Hence, we may apply (9.21) and use the linearity of L to obtain

$$\int_{\mathbb{R}^n} f_l(x)|g(x)|\mathrm{d}x = L(f_l \cdot \mathrm{sgn}(g)) \le \|L\|_{\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*}\|f_l\|_{\mathcal{M}^p_q}.$$

Letting $l \to \infty$, we have $\|f \cdot g\|_{L^1} \le \|L\|_{\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*}\|f\|_{\mathcal{M}^p_q}$ from the monotone convergence theorem and the Fatou property of Morrey norm. This means that g belongs to $(\mathcal{M}^p_q)'(\mathbb{R}^n)$. If g is complex-valued, then the same argument applied separately to the real and imaginary parts of g shows that each of these is in $(\mathcal{M}^p_q)'(\mathbb{R}^n)$. Hence, that g again belongs to $(\mathcal{M}^p_q)'(\mathbb{R}^n)$.

Write, for a function f which can be written as a finite linear combination of characteristic functions of sets of finite measure,

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x) \mathrm{d}x$$

and observe the continuity of both sides on $\mathcal{M}^p_q(\mathbb{R}^n)$. Then we conclude from Theorem 330 that (9.20) holds. This completes the proof of the theorem.

We conclude Section 9.1.5 with an example showing that the dual space of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q < p < \infty$ does not coincide with its Köthe dual. In particular, $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q < p < \infty$ is not reflexive.

Example 148. Let $1 < q < p < \infty$ and $L : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathbb{C}$ be a bounded linear functional. Then in view of the embedding $L^p(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_q(\mathbb{R}^n)$, one has a function $g \in L^{p'}(\mathbb{R}^n)$ such that

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x) \mathrm{d}x, \quad (f \in L^p(\mathbb{R}^n)).$$

However, it can happen that $L \neq 0$ even when $g \equiv 0$; one can show this by an example. Recall the set G considered in Example 12. Set

$$I_k \equiv (k - 1 + k^{\frac{p}{p-q}},\, k + k^{\frac{p}{p-q}}).$$

for $k \in \mathbb{N}$. Then,

$$G = \bigcup_{k=1}^{\infty} I_k, \quad \lim_{k\to\infty} \int_{I_k} \chi_G(x)\mathrm{d}x = 1.$$

With this in mind, let us define a closed subspace H of $\mathcal{M}^p_q(\mathbb{R}^n)$ by

$$H \equiv \left\{ f \in \mathcal{M}^p_q(\mathbb{R}) : \lim_{k\to\infty} \int_{I_k} f(x)\mathrm{d}x \text{ exists } \right\}.$$

Then, from the definition of the norm, we have

$$\lim_{k\to\infty} \left| \int_{I_k} f(x)\mathrm{d}x \right| \leq \|f\|_{\mathcal{M}^p_q(\mathbb{R})}.$$

Consequently, it follows from the Hahn–Banach theorem, Theorem 86, that mapping

$$f \in H \mapsto \lim_{k\to\infty} \int_{I_k} f(x)\mathrm{d}x \in \mathbb{R}$$

extends to a certain continuous linear functional L on $\mathcal{M}^p_q(\mathbb{R})$ satisfying $|L(f)| \leq \|f\|_{\mathcal{M}^p_q}$. Observe that $L(\chi_G) = 1$. Hence $L \neq 0$. Meanwhile, L annihilates any compactly supported function in $\mathcal{M}^p_q(\mathbb{R})$ because such a function belongs to H. Therefore, if one considers a function g satisfying

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x$$

for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ and hence all $f \in L^p(\mathbb{R}^n)$, then one obtains $g \equiv 0$ by virtue of the Lebesgue dominated convergence theorem.

9.1.6 Decomposition and averaging technique in Morrey spaces

One of the fundamental techniques in the theory of function spaces is to decompose functions into some elementary pieces. As we witnessed in the Calderón–Zygmund decomposition, this is useful when we analyze linear operators. Here we provide a standard decomposition of functions in Morrey spaces.

Theorem 357. *Let $1 < q \leq p < \infty$. Then any $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ admits a decomposition*:

$$f = \sum_{Q\in\mathfrak{A}} \lambda_Q a_Q,$$

where $\mathfrak{A}$ *is sparse,* $a_Q \in \mathcal{P}_L(\mathbb{R}^n)^\perp$ *and* $|a_Q| \le \chi_Q$ *for all* $Q \in \mathfrak{A}$ *and* $\{\lambda_Q\}_{Q\in\mathfrak{A}}$ *satisfies*

$$\left\| \sum_{Q\in\mathfrak{A}} \lambda_Q \chi_Q \right\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}.$$

For the proof of Theorem 357, we need to recall some elementary notions.

Let Q be a cube and let $L \in \mathbb{N}$. For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ there uniquely exists $P^L_Q(f) \in \mathcal{P}_L(\mathbb{R}^n)$ such that $\displaystyle\int_Q f(x)g(x)\mathrm{d}x = \int_Q P^L_Q f(x)g(x)\mathrm{d}x$ for all $g \in \mathcal{P}_L(\mathbb{R}^n)$ since $\{x^\alpha\}_{|\alpha|\le k}$ is independent in $L^2(Q)$. The polynomial $P^L_Q f$ is called the *Gram–Schmidt polynomial* of order L for Q. We have

$$P^L_{Q(1)} f(x) = \int_{Q(1)} R(x,y) f(y) dy$$

for some polynomial R. The scaling law $P^L_{Q(x_0,r)} f = r^{-n} P^L_{Q(1)} f\left(\dfrac{\cdot - x_0}{r}\right)$ holds (see Proposition 117).

Let $A > 4^n$. We can find a set $\{Q^k_j\}_{j\in J_k}$ of dyadic cubes and a collection $\{g\} \cup \{b^k_j\}_{j\in J_k}$ of countable collections of $L^1(\mathbb{R}^n)$-functions such that:

(1) (Partition of the set $\{M^{\mathcal{D}} f > A^k\}$) The family $\{Q^k_j\}_{j\in J_k}$ is disjoint and $\Omega^k \equiv \{\, M^{\mathcal{D}} f > A^k \,\} = \displaystyle\sum_{j\in J_k} Q^k_j$. Furthermore,

$$A^k < m_{Q^k_j}(|f|) \le 2^n A^k.$$

(2) f admits the following decomposition: $f = g^k + \displaystyle\sum_{j\in J_k} b^k_j$. Here

$$g^k = f\chi_{\mathbb{R}^n\setminus\Omega^k} + \sum_{j\in J_k} P_{Q^k_j}(f)\chi_{Q^k_j}.$$

(3) (The $L^\infty(\mathbb{R}^n)$-condition) The function g^k is $L^\infty(\mathbb{R}^n)$-bounded: More precisely, $\|g^k\|_{L^\infty} \le 2^n A^k$.

(4) (The support condition) The function b^k_j is supported on the closure of Q^k_j; namely $\mathrm{supp}(b^k_j) \subset \overline{Q^k_j}$.

(5) (The moment condition) $b^k_j \perp \mathcal{P}_l(\mathbb{R}^n)$.

In the above collection of cubes, we have the sparseness property.

Proposition 358. *If* $A > 4^n$*, then* $\mathfrak{A} = \{Q^k_j\}_{k\in\mathbb{Z}, j\in J_k}$ *is sparse.*

Proof Since the proof is similar to Theorem 172, we omit the detail.

We have the following property:

Proposition 359. *For all $k \in \mathbb{Z}$, $|g^k| \lesssim A^k$.*

Proof If $x \in \mathbb{R}^n \setminus \Omega^k$, then this is clear from the Lebesgue differentiation theorem. If $x \in Q_j^k$ for some $j \in J_k$, we have

$$|g^k(x)| \le \frac{1}{|Q_j^k|} \int_{Q_j^k} \left| R\left(\frac{x - c(Q_j^k)}{\ell(Q_j^k)}, \frac{y - c(Q_j^k)}{\ell(Q_j^k)} \right) \right| |f(y)| dy \lesssim m_{Q_j^k}(|f|) \lesssim A^k.$$

Corollary 360. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $\{M^{\mathcal{D}} f > \lambda\}$ never contains any quadrant for any $\lambda > 0$. Then $f = \lim_{K \to \infty} (g^K - g^{-K+1})$ in the sense of almost everywhere convergence.*

Proof Let $K \in \mathbb{N}$. Since $f - g^K$ is supported on $\{M^{\mathcal{D}} f > A^K\}$, $f = \lim_{K \to \infty} g^K$. Likewise $|g^{-K}| \le A^{-K}$ almost everywhere, $f = \lim_{K \to \infty} (g^K - g^{-K+1})$.

We set

$$a_j^k \equiv \chi_{Q_j^k}(g^k - g^{k+1}) \quad (k \in \mathbb{Z}, j \in J_k).$$

Proposition 361. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let $k \in \mathbb{Z}, j \in J_k$.*

(1) $|a_j^k| \lesssim A^k \chi_{Q_j^k}$.

(2) $a_j^k \in \mathcal{P}_L^{\perp}(\mathbb{R}^n)$.

Proof

(1) This is clear from Proposition 359.

(2) We calculate

$$\begin{aligned}
\int_{\mathbb{R}^n} x^\beta a_j^k(x) dx &= \int_{Q_j^k} x^\beta P_{Q_j^k} f(x) dx - \int_{Q_j^k} x^\beta \chi_{Q_j^k \setminus \Omega^{k+1}}(x) f(x) dx \\
&\quad - \sum_{j' \in J_{k+1}} \int_{Q_j^k \cap Q_{j'}^{k+1}} x^\beta P_{Q_{j'}^{k+1}} f(x) dx \\
&= \int_{Q_j^k} x^\beta f(x) dx - \int_{Q_j^k} x^\beta \chi_{Q_j^k \setminus \Omega^{k+1}}(x) f(x) dx \\
&\quad - \sum_{j' \in J_{k+1}} \int_{Q_j^k \cap Q_{j'}^{k+1}} x^\beta f(x) dx \\
&= 0.
\end{aligned}$$

Here for the third line we used the fact that Q_j^k contains $Q_{j'}^{k+1}$ or that $Q_j^k \cap Q_{j'}^{k+1} = \emptyset$ for any $k \in \mathbb{Z}$, $j \in J_k$ and $j' \in J_{k+1}$.

We prove Theorem 357 after we translate it into the following form:

Theorem 362. *Let $1 < q \le p < \infty$. Then for any $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ there is a decomposition*

$$f = \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} a_j^k$$

satisfying $a_j^k \in \mathcal{P}_L(\mathbb{R}^n)^\perp$,

$$|a_j^k| \lesssim A^k \chi_{Q_j^k} \quad (k \in \mathbb{Z}, j \in J_k), \qquad \left\| \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} A^k \chi_{Q_j^k} \right\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}$$

for some sparse collection $\{Q_j^k\}_{k \in \mathbb{Z}, j \in J_k}$ of dyadic cubes.

Proof Note that $M^{\mathcal{D}}$ is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$, so that we are in the position of using the previous observation. What is not contained in the previous observations is the norm estimate. We do this as follows: First, observe that $A^k \chi_{\Omega^k} \le M^{\mathcal{D}} f$. Consequently, since the Ω^k's are nested,

$$\begin{aligned} \left\| \sum_{k=-\infty}^{\infty} \sum_{j \in J_k} A^k \chi_{Q_j^k} \right\|_{\mathcal{M}^p_q} &= \left\| \sum_{k=-\infty}^{\infty} A^k \chi_{\Omega^k} \right\|_{\mathcal{M}^p_q} \\ &\lesssim \left\| \sup_{k \in \mathbb{Z}} A^k \chi_{\Omega^k} \right\|_{\mathcal{M}^p_q} \\ &\le \|M^{\mathcal{D}} f\|_{\mathcal{M}^p_q} \\ &\lesssim \|f\|_{\mathcal{M}^p_q}. \end{aligned}$$

Here for the last line we used the boundedness of $M^{\mathcal{D}}$ on $\mathcal{M}^p_q(\mathbb{R}^n)$.

There is another side in the theory of decomposition. As an application of the duality, we obtain the averaging technique. As we saw, in some cases, operators can be decomposed into the sum $\sum_{Q \in \mathcal{U}} \lambda_Q(f) \chi_{F_Q}$, where $\mathcal{U}$ is a family of cubes and F_Q is a measurable subset of Q. Hence, we are interested in the general estimate of the function of the form

$$\sum_{Q \in \mathcal{U}} a_Q,$$

where $a_Q \in L^0_{\mathrm{c}}(Q)$. We have partial information on a_Q, which we think has the singularity at least locally. We will try to remove the singularity. The next theorem, which guarantees that this is possible as long as the size of a_j is not too large, is a standard result for Morrey spaces.

Theorem 363. *Suppose that the parameters p, q, s and t satisfy*

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad q < t, \quad p < s.$$

Assume that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}^s_t(\mathbb{R}^n)$ *and* $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ *fulfill the support condition* $\operatorname{supp}(a_j) \subset Q_j$, *the size condition* $\|a_j\|_{\mathcal{M}^s_t} \le |Q_j|^{\frac{1}{s}}$ *and the norm condition* $\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}^p_q} < \infty$. *Then* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{\mathcal{M}^p_q} \lesssim_{p,q,s,t} \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}^p_q}. \tag{9.22}$$

Proof By decomposing Q_j, we may suppose each Q_j is dyadic.

To prove this, we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. Let us assume in addition that $a_j \ge 0$ for each j. Fix a (p,q)-block $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$ with the associated cube Q. Then we claim

$$\|f \cdot g\|_{L^1} \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}^p_q}. \tag{9.23}$$

By Theorem 344, we may assume that Q is dyadic.

Assume first that each Q_j contains Q as a proper subset. If we group j's such that Q_j are identical, we can assume that $|Q_j| = 2^{jn}|Q|$ for each $j \in \mathbb{N}$. Then

$$\|f \cdot g\|_{L^1} = \sum_{j=1}^{\infty} \lambda_j \int_Q a_j(x) g(x) \mathrm{d}x \le \sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^q(Q)} \|g\|_{L^{q'}(Q)}$$

from $f = \sum_{j=1}^{\infty} \lambda_j a_j$. By the size condition of a_j and g, we obtain

$$\|f \cdot g\|_{L^1} \le \sum_{j=1}^{\infty} \lambda_j |Q|^{\frac{1}{q}-\frac{1}{s}} |Q_j|^{\frac{1}{s}} |Q|^{\frac{1}{q'}-\frac{1}{p'}} = \sum_{j=1}^{\infty} \lambda_j |Q|^{\frac{1}{p}-\frac{1}{s}} |Q_j|^{\frac{1}{s}}.$$

Note that

$$\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}^p_q} \ge \left\|\lambda_{j_0} \chi_{Q_{j_0}}\right\|_{\mathcal{M}^p_q} = |Q_{j_0}|^{\frac{1}{p}} \lambda_{j_0}$$

for each j_0. Consequently, it follows from the condition $p < s$ that

$$\|f \cdot g\|_{L^1} \le \sum_{j=1}^{\infty} |Q|^{\frac{1}{p}-\frac{1}{s}} |Q_j|^{\frac{1}{s}-\frac{1}{p}} \left\|\sum_{k=1}^{\infty} \lambda_k \chi_{Q_k}\right\|_{\mathcal{M}^p_q} \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}^p_q}.$$

Conversely, assume that Q contains each Q_j. Then

$$\|f\cdot g\|_{L^1}=\sum_{j=1}^{\infty}\lambda_j\int_{Q_j}a_j(x)g(x)\mathrm{d}x\le\sum_{j=1}^{\infty}\lambda_j\|a_j\|_{L^t(Q_j)}\|g\|_{L^{t'}(Q_j)}.$$

Due to the condition of a_j, $\|f\cdot g\|_{L^1}\le\sum_{j=1}^{\infty}\lambda_j|Q_j|^{\frac{1}{t}-\frac{1}{s}}|Q_j|^{\frac{1}{s}}\|g\|_{L^{t'}(Q_j)}$. Thus, in terms of the Hardy–Littlewood maximal operator M, we obtain

$$\begin{aligned}\|f\cdot g\|_{L^1}&\le\sum_{j=1}^{\infty}\lambda_j|Q_j|\times\inf_{y\in Q_j}M[|g|^{t'}](y)^{\frac{1}{t'}}\\&\le\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}(y)\right)M[|g|^{t'}](y)^{\frac{1}{t'}}\mathrm{d}y\\&\le\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}(y)\right)\chi_Q(y)M[|g|^{t'}](y)^{\frac{1}{t'}}\mathrm{d}y.\end{aligned}$$

If we let κ be the operator norm of the maximal operator M on $L^{\frac{q'}{t'}}(\mathbb{R}^n)$, then we obtain $\kappa^{-1/t'}\chi_QM[|g|^{t'}]^{\frac{1}{t'}}$ is a (p,q)-block thanks to Lemma 341. Hence, we obtain (9.23). This is the desired result.

When $q=1$, we may assume $t=1$ in Theorem 364.

Theorem 364. *Suppose that the parameters p and s satisfy*

$$1<p<\infty,\quad 1<s<\infty,\quad p<s.$$

Assume that $\{Q_j\}_{j=1}^{\infty}\subset\mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty}\subset\mathcal{M}^s_1(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^{\infty}\subset[0,\infty)$ fulfill the support condition $\operatorname{supp}(a_j)\subset Q_j$, *the size condition $\|a_j\|_{\mathcal{M}^s_1}\le|Q_j|^{\frac{1}{s}}$ and the norm condition $\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_1}<\infty$. Then $f=\sum_{j=1}^{\infty}\lambda_ja_j$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies*

$$\|f\|_{\mathcal{M}^p_1}\lesssim_{p,s}\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_1}.$$

Proof We may assume $f\in\mathbb{M}^+(\mathbb{R}^n)$. The proof resembles that of Theorem 363. What differs from Theorem 363 is that we can take $g\equiv|Q|^{\frac{1}{p}-1}\chi_Q$ in Theorem 364. We need to change the proof of Theorem 363 in particular, the case of $Q_j\subset Q$ since we need to use $q<t$ there. However, since $g=|Q|^{\frac{1}{p}-1}\chi_Q$, we have

$$\|f\cdot g\|_{L^1}=|Q|^{\frac{1}{p}-1}\sum_{j=1}^{\infty}\lambda_j\int_{Q_j}a_j(x)\mathrm{d}x\le|Q|^{\frac{1}{p}-1}\sum_{j=1}^{\infty}\lambda_j\|a_j\|_{L^1(Q_j)}.$$

If we use the condition on each a_j, we obtain

$$\|f\cdot g\|_{L^1}\le\sum_{j=1}^{\infty}\lambda_j\frac{|Q_j|}{|Q|^{1-\frac{1}{p}}}=|Q|^{\frac{1}{p}-1}\int_Q\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}(x)\mathrm{d}x\le\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_1}.$$

9.1.7 Exercises

Exercise 109. Let $1<q\le p<\infty$. Then show that $\mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)\subset\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Exercise 110. [472, Proposition 4] Let $1<q\le p<\infty$. Then show that $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is a Banach space by using (9.2) if necessary.

Exercise 111. [218, Example 2.5] Let $1<q\le p<\infty$ and $a>0$. Define

$$f(x)\equiv\frac{1}{a+|x|^M}\quad(x\in\mathbb{R}^n).$$

Then similar to Example 143, show that $f\in\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ if and only if $Mp'>n$.

Exercise 112. Let $1\le p\le q<\infty$. Let b_1,b_2 be (p,q)-blocks supported on cubes Q_1 and Q_2, respectively. Then show that $D^{-1}(b_1+b_2)$ is a (p,q)-block supported on a bigger cube Q as long as $D>0$ is large enough.

Exercise 113. Let Q be a fixed cube, and let $1<p\le q<\infty$. Suppose that we have a sequence $\{b_k\}_{k=1}^{\infty}$ of (p,q)-blocks supported on Q. Assume in addition that $\{b_k\}_{k=1}^{\infty}$ converges weakly to b. Then show that b is a (p,q)-block.

Exercise 114. Let $1<q\le p<\infty$, and let $f\in\mathcal{M}^p_q(\mathbb{R}^n)$. For each $j\in\mathbb{N}$ we set $f_j\equiv f\chi_{B(j)}\chi_{[0,j]}(|f|)$. Then show that $f_j\to f$ as $j\to\infty$ in the weak-* topology. Hint: Although we need to consider the pairing of f and any $g\in\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, we can justify that g is a (p,q)-block.

Exercise 115. [95, Lemma 1], [282, Lemma 4.6] Let $0<\alpha<n$ and $1<q\le p<\frac{n}{\alpha}$. Suppose that we have a bounded sequence $\{f_j\}_{j=1}^{\infty}$ in $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ such that $I_\alpha f_j(x)$ converges to $F(x)$ for almost all $x\in\mathbb{R}^n$ and that $\{f_j\}_{j=1}^{\infty}$ converges to f in the weak-$*$ topology of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ via Theorem 356.

(1) Let $g_\varepsilon(x)\equiv\varepsilon^{-\alpha}|x|^{\alpha-n}\chi_{B(\varepsilon)}(x)$ and $h_\varepsilon(x)\equiv\varepsilon^{-\alpha}|x|^{\alpha-n}\chi_{\mathbb{R}^n\setminus B(\varepsilon)}(x)$ for $x\in\mathbb{R}^n$ and $\varepsilon>0$. Use the weak-$*$ convergence of $\{f_j\}_{j=1}^{\infty}$ to show that $F(x)-I_\alpha f(x)=\lim_{j\to\infty}\varepsilon^\alpha g_\varepsilon*(f_j-f)(x)$ for almost all $x\in\mathbb{R}^n$.

(2) Prove that $\sup_{j\in\mathbb{N}}\|g_\varepsilon*(f_j-f)\|_{\mathcal{H}^{p'}_{q'}}<\infty$.

(3) Conclude that $F(x)=I_\alpha f(x)$ for almost all $x\in\mathbb{R}^n$.

Exercise 116. For a Banach lattice $E(\mu)$ over a measure space $(X,\mathcal{B},\mu)$ and $a>0$, the $a\,E(\mu)$, a times $E(\mu)$, is given by $(E(\mu),a\|\cdot\|_{E(\mu)})$. Then show that $(aE(\mu))'=a^{-1}E'(\mu)$. Hint:$f\in E(\mu)\mapsto af\in E(\mu)$ is an isomorphism.

Exercise 117. Let $E(\mu)$ be a Banach lattice over a measure space $(X, \mathcal{B}, \mu)$. By reexamining the proof of Theorem 352, prove that $E(\mu) = E''(\mu)$ if $E(\mu)$ has the Fatou property.

Exercise 118. [158, Lemma 4.6] Let $0 < p < \infty$ and $\alpha < \max(1, p)$. Assume that $q \in (p, \infty] \cap [1, \infty]$. Suppose that we have a sequence $\{Q_k\}_{k=1}^{\infty}$ of cubes and a sequence $\{F_k\}_{k=1}^{\infty} \subset L^q(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$. Then show that

$$\left\| \sum_{k=1}^{\infty} M^{(\alpha)} \chi_{Q_k} \cdot F_k \right\|_{L^p} \lesssim \left\| \sum_{k=1}^{\infty} \chi_{Q_k} M^{(q)} F_k \right\|_{L^p}.$$

9.2 Choquet integral and predual spaces

There are couple of integration theories, Lebesgue integral, Riemannian integral, Henstock integral and so on. Choquet integral is one of the integration theories. Here, we are interested in other representations of predual spaces of Morrey spaces.

To solidify our idea, we first introduce the notion of Hausdorff capacity in Section 9.2.1 and then develop an integration theory in Section 9.2.2. We obtain some equivalent characterizations in Section 9.2.3.

9.2.1 Hausdorff capacity

Here, we aim to rewrite the predual space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. To this end, we need to consider integration against the Hausdorff capacity. There are several definitions of the capacities. However, some of them do not have good properties, although they are equivalent pointwise. Hence, first we clarify what is correct for these capacities and then we investigate integration. Based on these preliminaries, we will consider characterizing the dual spaces.

Since we want to collect some more preliminary facts on measure theory, recall the theory of capacity briefly.

Definition 88 (Hausdorff capacity)**.** Let $0 \le d \le n$. The *d-dimensional Hausdorff capacity* of the set $E \subset \mathbb{R}^n$ is defined by setting

$$H^d(E) \equiv \inf \left\{ \sum_{j=1}^{\infty} |B(x_j, r_j)|^{\frac{d}{n}} : E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j) \right\},$$

where the infimum takes over all covers $\{B(x_j, r_j)\}_{j=1}^{\infty}$ of E by countable families of open balls.

Here, E need not be measurable.

We can show that the set function H^d is monotone, countably subadditive and vanishes on empty sets. Moreover, the notion of H^d can be extended to $d = 0$; for any subset $E \subset \mathbb{R}^n$, define

$$H^0(E) \equiv \inf\left\{ N \in \mathbb{N} : \ E \subset \bigcup_{j=1}^{N} B(x_j, \, r_j) \right\},$$

where the infimum takes over all covers $\{B(x_j, \, r_j)\}_{j=1}^N$ of E by at most countable families of open balls.

As is the case with the Lebesgue integral, it is difficult to calculate $H^d(E)$ for general sets. Here is a special example of E for which we can find $H^d(E)$ exactly.

Example 149. Let us show that $H^d(B(1)) = |B(1)|^{\frac{d}{n}}$ for $0 < d \leq n$. It follows from the definition that $H^d(B(1)) \leq |B(1)|^{\frac{d}{n}}$. Thus, we need to show $H^d(B(1)) \geq |B(1)|^{\frac{d}{n}}$. Let $\{B_j\}_{j=1}^{\infty}$ be a ball covering of $B(1)$. Then

$$\sum_{j=1}^{\infty} |B_j|^{\frac{d}{n}} \geq \left(\sum_{j=1}^{\infty} |B_j| \right)^{\frac{d}{n}} \geq |B(1)|^{\frac{d}{n}}.$$

We want to define a functional by the Hausdorff capacity. This functional is known to be nonsublinear as Example 150 shows, so occasionally we need to use an equivalent integral with respect to the d-dimensional modified dyadic Hausdorff capacity $\tilde{H}^d$.

Definition 89 (Modified dyadic Hausdorff capacity)**.** Let $E \subset \mathbb{R}^n$ and $0 < d \leq n$. One defines *d-dimensional modified dyadic Hausdorff capacity* by

$$\tilde{H}_0^d(E) \equiv \inf\left\{ \sum_{j=1}^{\infty} \ell(Q_j)^d \right\},$$

where $\{Q_j\}_{j=1}^{\infty}$ runs over all countable collections of dyadic cubes satisfying

$$E \subset \operatorname{Int}\left(\bigcup_{j=1}^{\infty} Q_j \right). \tag{9.1}$$

Example 150. Let $n = 2$ and $a \in \mathbb{R}$. Then $\tilde{H}_0^1([0,1) \times \{a\}) = 1$. In fact, $\tilde{H}_0^1([0,1) \times \{a\}) \leq 1$ since $[0,1) \times \{a\} \subset Q_{0m}$ for some $m \in \mathbb{Z}^2$. Let us prove the reverse conclusion. Let $\{Q_\lambda\}_{\lambda \in \Lambda}$ be a covering of $[0,1) \times \{a\}$. By Lemma 136 we may assume that $\{Q_\lambda\}_{\lambda \in \Lambda}$ is disjoint. Then denoting by $\ell([x_0, x_1) \times \{a\}) = x_1 - x_0$ for $-\infty < x_0 < x_1 < \infty$, we obtain

$$\ell([0,1) \times \{a\}) = \sum_{\lambda \in \Lambda} \ell(([0,1) \times \{a\}) \cap Q_\lambda) \leq \sum_{\lambda \in \Lambda} \ell(Q_\lambda),$$

showing $\tilde{H}_0^1([0,1)\times\{a\}) \geq 1$. Likewise, we can show $\tilde{H}_0^1([0,1)\times\{3,\pi\}) = 1(<2)$.

Based on a fundamental geometric observation, we see that H^d and $\tilde{H}_0^d$ are not so different.

Proposition 365. *Let $0 < d \leq n$. Then $\tilde{H}_0^d \sim H^d$.*

Proof Let $E \subset \mathbb{R}^n$. Let $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$ be a countable collection satisfying (9.1). We write $B_j \equiv B(c(Q_j), 2n\ell(Q_j))$. Then $E \subset \mathrm{Int}\left(\bigcup_{j=1}^\infty Q_j\right) \subset \bigcup_{j=1}^\infty Q_j \subset \bigcup_{j=1}^\infty B_j$. As a result, we see that $H^d \lesssim \tilde{H}_0^d$. To end the proof, we may thus assume $H^d(E) < \infty$.

When we have a ball $B \equiv B(x,r)$, we can find 6^n dyadic cubes $Q_1, Q_2, \ldots, Q_{6^n}$ such that

$$\frac{1}{2} \leq \frac{\ell(Q_j)}{r} \leq 2, \quad B(x,r) \subset \bigcup_{j=1}^{6^n} Q_j. \tag{9.2}$$

Thus, we see that $B(x,r) = \mathrm{Int}(B(x,r)) \subset \mathrm{Int}\left(\bigcup_{j=1}^{6^n} Q_j\right)$. With this in mind, we suppose that $M \in (H^d(E), \infty)$. Then we have a covering $\{B(x_j,r_j)\}_{j=1}^\infty$ such that $M > \sum_{j=1}^\infty |B(x_j,r_j)|^{\frac{d}{n}}$ and that $E \subset \bigcup_{j=1}^\infty B(x_j,r_j)$. Then we can find dyadic cubes $\{Q_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{N}\cap[1,6^n]}$ such that

$$B(x_j,r_j) \subset \mathrm{Int}\left(\bigcup_{k=1}^{6^n} Q_{j,k}\right), \quad \frac{1}{2} \leq \frac{\ell(Q_{j,k})}{r_j} \leq 2, \qquad (j \in \mathbb{N}).$$

As a result, we have $M \gtrsim \tilde{H}_0^d(E)$, as was to be shown.

From the definition of the covering, it is easy to see that

$$\tilde{H}_0^d(E_1 \cup E_2) \leq \tilde{H}_0^d(E_1) + \tilde{H}_0^d(E_2). \tag{9.3}$$

According to Example 150, we do not have $\tilde{H}_0^d(E_1 \cup E_2) + \tilde{H}_0^d(E_1 \cap E_2) = \tilde{H}_0^d(E_1) + \tilde{H}_0^d(E_2)$. But we have the following weaker version.

Theorem 366 (Strong subadditivity)**.** *We have $\tilde{H}_0^d(E_1\cup E_2)+\tilde{H}_0^d(E_1\cap E_2) \leq \tilde{H}_0^d(E_1) + \tilde{H}_0^d(E_2)$ for any sets $E_1, E_2 \subset \mathbb{R}^n$ and $0 < d \leq n$.*

Proof From (9.3), we may assume that $\tilde{H}_0^d(E_1) + \tilde{H}_0^d(E_2) < \infty$ and that $E_1 \cap E_2 \neq \emptyset$ without any loss of generality. From the definition of the $\tilde{H}_0^d(E_1)$ and $\tilde{H}_0^d(E_2)$, we have only to establish that

$$\tilde{H}_0^d(E_1 \cup E_2) + \tilde{H}_0^d(E_1 \cap E_2) \leq \sum_{j=1}^\infty \ell(Q_{1,j})^d + \sum_{k=1}^\infty \ell(Q_{2,k})^d, \tag{9.4}$$

whenever we have two collections $\{Q_{1,j}\}_{j=1}^{\infty}$ and $\{Q_{2,k}\}_{k=1}^{\infty}$ of dyadic cubes satisfying

$$E_1 \subset \operatorname{Int}\left(\bigcup_{j=1}^{\infty} Q_{1,j}\right), \quad E_2 \subset \operatorname{Int}\left(\bigcup_{k=1}^{\infty} Q_{2,k}\right).$$

Observe first that

$$E_1 \cap E_2 \subset \operatorname{Int}\left(\bigcup_{j=1}^{\infty} Q_{1,j}\right) \cap \operatorname{Int}\left(\bigcup_{k=1}^{\infty} Q_{2,k}\right) = \operatorname{Int}\left(\bigcup_{j,k=1}^{\infty} Q_{1,j} \cap Q_{2,k}\right) \quad (9.5)$$

and that

$$E_1 \cup E_2 \subset \operatorname{Int}\left(\bigcup_{j=1}^{\infty} Q_{1,j}\right) \cup \operatorname{Int}\left(\bigcup_{k=1}^{\infty} Q_{2,k}\right) = \operatorname{Int}\left(\bigcup_{j,k=1}^{\infty} Q_{1,j} \cup Q_{2,k}\right). \quad (9.6)$$

Let

$$\mathfrak{A}_1 \equiv \{(j,k) : Q_{1,j} \subsetneq Q_{2,k}\}, \quad \mathfrak{A}_2 \equiv \{(j,k) : Q_{1,j} \supsetneq Q_{2,k}\},$$
$$\mathfrak{A}_3 \equiv \{(j,k) : Q_{1,j} = Q_{2,k}\}, \quad \mathfrak{A}_4 \equiv \mathbb{N} \times \mathbb{N} \setminus \bigcup_{l=1}^{3} \mathfrak{A}_l.$$

Then

$$\bigcup_{j,k=1}^{\infty} Q_{1,j} \cup Q_{2,k} = \bigcup_{(j,k)\in\mathfrak{A}_2\cup\mathfrak{A}_3\cup\mathfrak{A}_4} Q_{1,j} \cup \bigcup_{(j,k)\in\mathfrak{A}_1\cup\mathfrak{A}_4} Q_{2,k} \quad (9.7)$$

and

$$\bigcup_{j,k=1}^{\infty} Q_{1,j} \cap Q_{2,k} = \bigcup_{(j,k)\in\mathfrak{A}_1} Q_{1,j} \cup \bigcup_{(j,k)\in\mathfrak{A}_2\cup\mathfrak{A}_3} Q_{2,k}. \quad (9.8)$$

Consequently, we deduce from (9.6) and (9.7)

$$E_1 \cup E_2 \subset \operatorname{Int}\left(\bigcup_{(j,k)\in\mathfrak{A}_2\cup\mathfrak{A}_3\cup\mathfrak{A}_4} Q_{1,j} \cup \bigcup_{(j,k)\in\mathfrak{A}_1\cup\mathfrak{A}_4} Q_{2,k}\right)$$

and from (9.5) and (9.8)

$$E_1 \cap E_2 \subset \operatorname{Int}\left(\bigcup_{(j,k)\in\mathfrak{A}_1} Q_{1,j} \cup \bigcup_{(j,k)\in\mathfrak{A}_2\cup\mathfrak{A}_3} Q_{2,k}\right).$$

Thus,

$$\tilde{H}_0^d(E_1 \cup E_2) \le \sum_{(j,k)\in\mathfrak{A}_2\cup\mathfrak{A}_3\cup\mathfrak{A}_4} |Q_{1,j}|^{\frac{d}{n}} + \sum_{(j,k)\in\mathfrak{A}_1\cup\mathfrak{A}_4} |Q_{2,k}|^{\frac{d}{n}}$$

and

$$\tilde{H}_0^d(E_1 \cap E_2) \le \sum_{(j,k)\in\mathfrak{A}_1} |Q_{1,j}|^{\frac{d}{n}} + \sum_{(j,k)\in\mathfrak{A}_2\cup\mathfrak{A}_3} |Q_{2,k}|^{\frac{d}{n}}.$$

Therefore, (9.4) follows.

The capacity $\tilde{H}_0^d$ is compatible with decreasing sequences of compact sets.

Theorem 367. *Let $0 < d \le n$. If $\{K_j\}_{j=1}^\infty$ is a sequence of compact sets decreasing to K, then* $\lim_{j\to\infty} \tilde{H}_0^d(K_j) = \tilde{H}_0^d\left(\bigcap_{j=1}^\infty K_j\right)$.

Proof We have only to show that

$$\lim_{j\to\infty} \tilde{H}_0^d(K_j) \le \tilde{H}_0^d(K). \tag{9.9}$$

To this end, we choose a sequence of dyadic cubes $\{Q_j\}_{j=1}^\infty$ arbitrarily so that $K \subset \mathrm{Int}\left(\bigcup_{j=1}^\infty Q_j\right)$. Since K is compact, we can choose $\varepsilon > 0$ so that

$$K^\varepsilon \equiv \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(K,x) < \varepsilon\} \subset \mathrm{Int}\left(\bigcup_{j=1}^\infty Q_j\right). \tag{9.10}$$

Thus, there exists $j_0 \in \mathbb{N}$ such that $K_l \subset \mathrm{Int}\left(\bigcup_{j=1}^\infty Q_j\right)$ for $l \ge j_0$. As a result, $\lim_{j\to\infty} \tilde{H}_0^d(K_j) \le \sum_{j=1}^\infty \ell(Q_j)^d$. Since the sequence $\{Q_j\}_{j=1}^\infty$ is arbitrary, we obtain (9.9).

Like the Lebesgue measure, essentially corresponding to the case where $d = n$, the compactness of the sets is absolutely necessary as an example similar to Example 150 shows.

Lemma 368 (Increasing property for open sets)**.** *Let $0 < d \le n$, and let $\{O_j\}_{j=1}^\infty$ be a sequence of open sets in $\mathbb{R}^n$ expanding to O. Then* $\lim_{j\to\infty} \tilde{H}_0^d(O_j) = \tilde{H}_0^d\left(\bigcup_{j=1}^\infty O_j\right)$.

Proof We have only to show that

$$\lim_{j\to\infty} \tilde{H}_0^d(O_j) \ge \tilde{H}_0^d(O). \tag{9.11}$$

To this end, we may assume that $\sup_{j\in\mathbb{N}} \tilde{H}_0^d(O_j) < \infty$; otherwise (9.11) is trivial. Let $\varepsilon > 0$ be fixed. We choose $\{Q_{j,k}\}_{k=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^n)$ so that

$$O_j \subset \operatorname{Int}\left(\bigcup_{k=1}^{\infty} Q_{j,k}\right) \tag{9.12}$$

and

$$\sum_{j=1}^{\infty} \ell(Q_{j,k})^d \le \tilde{H}_0^d(O_j) + 2^{-j}\varepsilon. \tag{9.13}$$

Note that $\sup_{j,k\in\mathbb{N}} \ell(Q_{j,k})^d \le \varepsilon + \sup_{j\in\mathbb{N}} \tilde{H}_0^d(O_j)$. This means that the size of $Q_{j,k}$ is bounded by a constant independent of j and k. Denote by $\{Q_i\}_{i\in I}$ the disjoint maximal cubes in $\{Q_{j,k}\}_{j,k\in\mathbb{N}}$.

Then $O = \bigcup_{j=1}^{\infty} O_j \subset \operatorname{Int}\left(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} Q_{j,k}\right) = \operatorname{Int}\left(\bigcup_{i\in I} Q_i\right)$ from (9.12). Let $m \in \mathbb{N}$ be fixed. We write

$$\{Q_i\}_{i\in I} \cap \{Q_{1,j}\}_{j=1}^{\infty} = \{Q_i^{(1)}\}_{i\in I_1}. \tag{9.14}$$

Define

$$O_{1,m} \equiv O_m \cap \operatorname{Int}\left(\bigcup_{i\in I_1} Q_i^{(1)}\right) \quad (m \in \mathbb{N}).$$

We set $\{Q_{m,k}^{(1)}\}_{k\in K_{1,m}} \equiv \{Q_{m,k} : k \in \mathbb{N}, O_{1,m} \cap Q_{m,k} \neq \emptyset\}$. Then

$$O_{1,m} \subset \bigcup_{k\in K_{1,m}} Q_{m,k}^{(1)}. \tag{9.15}$$

For each $k \in K_{1,m}$, there exists $i \in I_1$ such that $Q_i^{(1)} \cap Q_{m,k}^{(1)} \neq \emptyset$. Since $Q_i^{(1)}$ is maximal in inclusion among $\{Q_{j,k}\}_{j,k\in\mathbb{N}}$ and $Q_{m,k}^{(1)} = Q_{J,K}$ for some $J, K \in \mathbb{N}$, we have $Q_{m,k}^{(1)} \subset Q_i^{(1)}$. As a result, we have

$$\bigcup_{k\in K_{1,m}} Q_{m,k}^{(1)} \subset \bigcup_{i\in I_1} Q_i^{(1)}. \tag{9.16}$$

Let

$$\{Q_{1,k}\}_{k=1}^{\infty} \setminus \{Q_i^{(1)}\}_{i\in I_1} = \{Q_{1,k,m}^{*}\}_{k\in J_1}. \tag{9.17}$$

From (9.12) with $j = 1$, (9.14) and (9.17), we learn

$$O_1 = \left(O_1 \cap O_m \cap \bigcup_{k\in I_1} Q_i^{(1)}\right) \cup \left(O_1 \cap \bigcup_{k\in J_1} Q_{1,k,m}^{*}\right).$$

Hence

$$O_1 = \left(O_1 \cap O_m \cap \operatorname{Int} \left(\bigcup_{k \in I_1} Q_i^{(1)} \right) \right) \cup \left(O_1 \cap \bigcup_{k \in J_1} Q_{1,k,m}^* \right), \tag{9.18}$$

since we are assuming that O_1 is open. From (9.15) and (9.18), we have

$$O_1 \subset \bigcup_{k \in K_{1,m}} Q_{m,k}^{(1)} \cup \bigcup_{k \in J_1} Q_{1,k,m}^*. \tag{9.19}$$

As a result, we have

$$\tilde{H}_0^d(O_1) \le \sum_{k \in K_{1,m}} \ell(Q_{m,k}^{(1)})^d + \sum_{k \in J_1} \ell(Q_{1,k,m}^*)^d. \tag{9.20}$$

Note that (9.13) with $j = 1$ reads as:

$$\sum_{i \in I_1} \ell(Q_i^{(1)})^d + \sum_{i \in J_1} \ell(Q_{1,k,m}^*)^d \le \tilde{H}_0^d(O_1) + \frac{\varepsilon}{2}, \tag{9.21}$$

by virtue of (9.14) and (9.17). By combining (9.20) and (9.21), we obtain

$$\sum_{i \in I_1} \ell(Q_i^{(1)})^d \le \tilde{H}_1^d(O_1) + \frac{\varepsilon}{2} - \sum_{k \in J_1} \ell(Q_{1,k,m}^*)^d \le \sum_{k \in K_{1,m}} \ell(Q_{m,k}^{(1)})^d + \frac{\varepsilon}{2}. \tag{9.22}$$

For $j = 2$, we mimic the argument above to have a sequence $\{Q_{m,k}^{(2)}\}_{k \in K_{2,m}}$ with properties similar to (9.19) and (9.22) above in addition to the property

$$\{Q_{m,k}^{(1)}\}_{k \in K_{1,m}} \cap \{Q_{m,k}^{(2)}\}_{k \in K_{2,m}} = \emptyset. \tag{9.23}$$

More precisely, we set

$$(\{Q_i\}_{i \in I} \setminus \{Q_{m,k}^{(1)}\}_{k \in K_{1,m}}) \cap \{Q_{2,k}\}_{k=1}^{\infty} = \{Q_{i,m}^{(2)}\}_{i \in I_{2,m}}. \tag{9.24}$$

Define

$$O_{2,m} \equiv O_m \cap \operatorname{Int} \left(\bigcup_{i \in I_{2,m}} Q_{i,m}^{(2)} \right).$$

We write $\{Q_{m,k}^{(2)}\}_{k \in K_{2,m}} \equiv \{Q_{m,k} : O_{2,m} \cap Q_{m,k} \neq \emptyset\}$. As before, for any $k \in K_{2,m}$, $Q_{m,k}^{(2)}$ intersects $Q_{i,m}^{(2)}$ for some $i \in I_2$. In view of (9.24), we have (9.23). Rearrange the cubes to obtain

$$\{Q_{2,k}\}_{k=1}^{\infty} \setminus \{Q_{i,m}^{(2)}\}_{i \in I_{2,m}} = \{Q_{2,k}^*\}_{k \in J_1}.$$

Going through an argument above, we have $O_2 \subset \bigcup_{k \in K_{2,m}} Q_{m,k}^{(2)} \cup \bigcup_{k \in J_2} Q_{2,k}^*$ corresponding to (9.19), and $\sum_{i \in I_{2,m}} \ell(Q_i^{(2)})^d \le \sum_{k \in K_{2,m}} \ell(Q_{m,k}^{(2)})^d + \frac{1}{4}\varepsilon$, corresponding to (9.22).

Continuing this procedure, we can find a collection $\{Q_{m,k}^{(j)}\}_{k\in I_{j,m}}$ for $j = 1, 2, \ldots, m$ and a sequence of subsets $\{J_j\}_{j=1}^m$ of $\mathbb{N}$ such that $\{Q_{m,k}^{(j_1)}\}_{k\in I_{j_1,m}} \cap \{Q_{m,k}^{(j_2)}\}_{k\in I_{j_2,m}} = \emptyset$ for any $j_1 < j_2 \le m$, that $O_2 \subset \bigcup_{k\in K_{j,m}} Q_{m,k}^{(j)} \cup \bigcup_{k\in J_j} Q_{j,k}$, and that

$$\sum_{i\in I_{j,m}} \ell(Q_i^{(j)})^d \le \sum_{k\in K_{j,m}} \ell(Q_{m,k}^{(j)})^d + 2^{-j}\varepsilon.$$

Notice that $\sum_{j=1}^m \sum_{i\in I_{j,m}} \ell(Q_i^{(j)})^d \le 2\varepsilon + \tilde{H}_0^d(O_m)$. Observe that, for any finite set I_0, there exists an integer $M = M(I_0)$ such that $\{Q_i\}_{i\in I_0} \subset \{Q_{m,k} : m = 1, 2, \ldots, M, k = 1, 2, \ldots\}$. Thus, if we let $m \to \infty$, we have $\sum_{i\in I} \ell(Q_i)^d \le \lim_{m\to\infty} \tilde{H}_0^d(O_m) + 2\varepsilon$, as was to be shown.

The capacity $\tilde{H}_0^d$ enjoys the monotone property.

Theorem 369 (Monotone property of $\tilde{H}_0^d$)**.** *Let $0 < d \le n$. Whenever we have an increasing sequence $\{E_j\}_{j=1}^\infty$ of subsets in $\mathbb{R}^n$, we have*

$$\lim_{j\to\infty} \tilde{H}_0^d(E_j) = \tilde{H}_0^d\left(\bigcup_{j=1}^\infty E_j\right).$$

Proof Again, we need to prove

$$\lim_{j\to\infty} \tilde{H}_0^d(E_j) \ge \tilde{H}_0^d\left(\bigcup_{j=1}^\infty E_j\right). \tag{9.25}$$

To this end, we may assume that $\sup_{j\in\mathbb{N}} \tilde{H}_0^d(E_j) < \infty$. We fix $\varepsilon > 0$. Then we can choose an open set O_j such that $E_j \subset O_j$ and that $\tilde{H}^d(O_j) \le \tilde{H}_0^d(E_j) + 2^{-j}\varepsilon$. Since

$$\tilde{H}_0^d(O_1 \cup O_2) + \tilde{H}_0^d(O_1 \cap O_2) \le \tilde{H}_0^d(O_1) + \tilde{H}_0^d(O_1) \le \tilde{H}_0^d(E_1) + \tilde{H}_0^d(E_2) + \frac{3}{4}\varepsilon,$$

we have $\tilde{H}^d(O_1 \cup O_2) \le \tilde{H}_0^d(E_2) + \frac{3}{4}\varepsilon$. Inducting on j, we obtain $\tilde{H}^d(O_1 \cup O_2 \cup \cdots \cup O_j) \le \tilde{H}_0^d(E_j) + (1 - 2^{-j})\varepsilon$. Letting $j \to \infty$, we obtain

$$\tilde{H}_0^d\left(\bigcup_{j=1}^\infty E_j\right) \le \tilde{H}_0^d\left(\bigcup_{j=1}^\infty O_j\right) = \lim_{J\to\infty} \tilde{H}_0^d\left(\bigcup_{j=1}^J O_j\right) \le \lim_{J\to\infty} \tilde{H}_0^d(E_J) + \varepsilon$$

thanks to Lemma 368, proving (9.25).

9.2.2 Choquet integral

Motivated by the Layer-Cake formula, Theorem 5, we define the Choquet integral against H^d [78]. We adopt the idea in Theorem 5 because H^d is not additive. As is the case with the Lebesgue integral, we need to consider $f^{-1}(\lambda,\infty]=\{x\in\mathbb{R}^n:\ f(x)>\lambda\}$ for $f\in\mathbb{M}^+(\mathbb{R}^n)$; see Theorem 5.

Definition 90 (Choquet integral against H^d)**.** For any function $f:\mathbb{R}^n\mapsto[0,\infty]$, its *Choquet integral* $\int\limits_{\mathbb{R}^n} f(x)\mathrm{d}H^d(x)$ *against* H^d is defined by

$$\int_{\mathbb{R}^n} f(x)\mathrm{d}H^d(x)\equiv\int_0^\infty H^d(f^{-1}(\lambda,\infty])\mathrm{d}\lambda,$$

where the right-hand side is understood as the Riemannian integral. Likewise, for a set E and a function $f:E\mapsto[0,\infty]$ $\int\limits_E f(x)\mathrm{d}H^d(x)$ against H^d is defined by

$$\int_E f(x)\mathrm{d}H^d(x)\equiv\int_0^\infty H^d(f^{-1}(\lambda,\infty])\mathrm{d}\lambda,$$

where in this case $f^{-1}(\lambda,\infty]=\{x\in E:\ f(x)>\lambda\}$.

We do not have to assume that f is measurable in some sense.

Example 151. Let $d\in(0,n]$, $N\geq 0$ and $a>0$. Define $f(x)\equiv(a+|x|)^{-N}$ for $x\in\mathbb{R}^n$. Then $f^{-1}(\lambda,\infty]=B((\lambda^{-1/N}-a)_+)$. Here, it will be understood that $B(0)=\emptyset$. Hence $|f^{-1}(\lambda,\infty]|\sim((\lambda^{-1/N}-a)_+)^n$. If we integrate this over $(0,\infty)$, we see that

$$\begin{aligned}\int_{\mathbb{R}^n}(a+|x|)^{-N}\mathrm{d}H^d(x)&=\int_0^\infty H^d(\{f>\lambda\})\mathrm{d}\lambda\\&\sim\int_0^\infty((\lambda^{-1/N}-a)_+)^d\mathrm{d}\lambda\\&=\int_0^{a^{-N}}(\lambda^{-1/N}-a)^d\mathrm{d}\lambda\\&=a^{d-N}\int_0^1(\lambda^{-1/N}-1)^d\mathrm{d}\lambda\\&=C_{N,d}a^{d-N}.\end{aligned}$$

Here, $C_{N,d}$ is a constant such that $C_{N,d}$ is finite if and only if $d<N$. Likewise, we can check that $\|\chi_B\|_{L^1(H^d)}\sim|B|^{\frac{d}{n}}$ for any ball B.

One considers the Choquet integral against $\tilde{H}^d_0$ analogously to that against H^d keeping in mind that H^d and $\tilde{H}^d_0$ are equivalent according to Proposition 365.

Definition 91 (Choquet integral against $\tilde{H}_0^d$)**.** For any function $f : \mathbb{R}^n \mapsto [0, \infty]$, its *Choquet integral* $\int\limits_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x)$ *against* $\tilde{H}_0^d$ is defined by

$$\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) \equiv \int_0^\infty \tilde{H}_0^d(f^{-1}(\lambda, \infty])\mathrm{d}\lambda,$$

where the right-hand side is understood as the Riemann integral. Likewise, for a set E and a function $f : E \mapsto [0, \infty]$ we can define $\int\limits_E f(x)\mathrm{d}H^d(x)$.

As before it is difficult to find the precise value of the integral. Here, we content ourselves with such an example.

Example 152. Let E be a set. Then

$$\int_{\mathbb{R}^n} \chi_E(x)\mathrm{d}\tilde{H}_0^d(x) \equiv \tilde{H}_0^d(E).$$

As is the case with the Lebesgue integral, we are not interested in the value of the integral itself. Let $f : \mathbb{R}^n \mapsto [0, \infty]$ be any function, and let $a > 0$, We mention that we readily obtain

$$\int_{\mathbb{R}^n} a \cdot f(x)\mathrm{d}\tilde{H}_0^d(x) = a\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) \tag{9.26}$$

by a change of variables.

The following formulas are easy to verify:

Proposition 370. *For any increasing sequences* $\{f_k\}_{k=1}^\infty$ *satisfying* $0 \leq f_k \leq f_{k+1}$*,*

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} f_k(x)\mathrm{d}\tilde{H}_0^d(x) = \int_{\mathbb{R}^n} \lim_{k\to\infty} f_k(x)\mathrm{d}\tilde{H}_0^d(x).$$

Proof By the monotone convergence theorem and Theorem 369, we have

$$\begin{aligned}
\lim_{k\to\infty}\int_{\mathbb{R}^n} f_k(x)\mathrm{d}\tilde{H}_0^d(x) &= \lim_{k\to\infty}\int_0^\infty \tilde{H}_0^d\left({f_k}^{-1}(\lambda, \infty]\right)\mathrm{d}\lambda \\
&= \int_0^\infty \lim_{k\to\infty} \tilde{H}_0^d\left({f_k}^{-1}(\lambda, \infty]\right)\mathrm{d}\lambda \\
&= \int_0^\infty \tilde{H}_0^d\left\{x \in \mathbb{R}^n : \lim_{k\to\infty} f_k(x) > \lambda\right\}\mathrm{d}\lambda \\
&= \int_{\mathbb{R}^n} \lim_{k\to\infty} f_k(x)\mathrm{d}\tilde{H}_0^d(x),
\end{aligned}$$

as was to be shown.

One of the important properties of the Choquet integral is the subadditivity given in the next theorem:

Theorem 371 (Subadditivity)**.** *Let $0 < d \le n$. Then for any functions $f, g : \mathbb{R}^n \mapsto [0, \infty]$,*

$$\int_{\mathbb{R}^n} (f(x) + g(x)) \mathrm{d}\tilde{H}_0^d(x) \le \int_{\mathbb{R}^n} f(x) \mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x) \mathrm{d}\tilde{H}_0^d(x). \tag{9.27}$$

We prove Theorem 371 step by step. The next lemma, showing that additivity is available for some special cases, is a key to our observation:

Lemma 372. *Let $E_1 \supset E_2 \supset \cdots \supset E_N$ be a finite decreasing sequence of sets in $\mathbb{R}^n$. Set $f \equiv \sum_{k=1}^{N} \chi_{E_k}$ and $h \equiv \sum_{k=1}^{N-1} \chi_{E_k} = f - \chi_{E_N}$. Then*

$$\int_{\mathbb{R}^n} f(x) \mathrm{d}\tilde{H}_0^d(x) = \int_0^\infty \tilde{H}_0^d(h^{-1}(\lambda, \infty]) \mathrm{d}\lambda + \tilde{H}_0^d(E_N).$$

Proof Keeping in mind that $f^{-1}((\lambda, \infty]) = \emptyset$ if $\lambda \ge N$, we decompose the integral in the definition of $\int_{\mathbb{R}^n} f(x) \mathrm{d}\tilde{H}_0^d(x)$ into three parts to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x) \mathrm{d}\tilde{H}_0^d(x) \\
&= \int_0^\infty \tilde{H}_0^d(f^{-1}(\lambda, \infty]) \mathrm{d}\lambda \\
&= \int_0^{N-1} \tilde{H}_0^d(h^{-1}(\lambda, \infty]) \mathrm{d}\lambda + \int_{N-1}^{N} \tilde{H}_0^d(E_N) \mathrm{d}\lambda + \int_N^\infty \tilde{H}_0^d(\emptyset) \mathrm{d}\lambda \\
&= \int_0^\infty \tilde{H}_0^d(h^{-1}(\lambda, \infty]) \mathrm{d}\lambda + \tilde{H}_0^d(E_N),
\end{aligned}$$

as was to be shown.

Lemma 373. *Inequality* (9.27) *remains true when f and g assume their value in $\mathbb{N}_0$ and $\sup g \le 1$.*

Proof In view of Proposition 370, we may assume that f is bounded. Consequently, we can assume that $f = \sum_{k=1}^{N} \chi_{E_k}$ and $g = \chi_{F_1}$, where $E_1 \supset E_2 \supset \cdots \supset E_N$. If $N = 1$, then we readily have (9.27) thanks to Theorem 366:

$$\begin{aligned}
\int_{\mathbb{R}^n} (f(x) + g(x)) \mathrm{d}\tilde{H}_0^d(x) &= \tilde{H}_0^d(E_1 \cup F_1) + \tilde{H}_0^d(E_1 \cap F_1) \\
&\le \tilde{H}_0^d(E_1) + \tilde{H}_0^d(F_1) \\
&= \int_{\mathbb{R}^n} f(x) \mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x) \mathrm{d}\tilde{H}_0^d(x).
\end{aligned}$$

Suppose that (9.27) is true for $N = m - 1 \geq 1$. Set $f \equiv \sum_{k=1}^{N-1} \chi_{E_k}$. Then according to Lemma 372, we have

$$\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x) = \int_{\mathbb{R}^n} h(x)\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}^d(F_1) + \tilde{H}_0^d(E_N).$$

We have

$$\begin{aligned}&\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x)\\ &= \int_{\mathbb{R}^n} h(x)\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}^d(F_1 \cup E_N) + \tilde{H}_0^d(F_1 \cap E_N)\end{aligned}$$

by virtue of Theorem 366. Thus,

$$\begin{aligned}&\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x)\\ &\geq \int_{\mathbb{R}^n} (h(x) + \chi_{F_1 \cup E_N}(x))\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}_0^d(F_1 \cap E_N)\end{aligned}$$

from the induction assumption to the function h, which satisfies $\sup h \neq N-1$. Finally, using Lemma 372, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x)\\ &\geq \int_{\mathbb{R}^n} (h(x) + \chi_{F_1 \cup E_N}(x) + \chi_{F_1 \cap E_N}(x))\mathrm{d}\tilde{H}_0^d(x)\\ &\geq \int_{\mathbb{R}^n} (h(x) + \chi_{F_1}(x) + \chi_{E_N}(x))\mathrm{d}\tilde{H}_0^d(x)\\ &\geq \int_{\mathbb{R}^n} (f(x) + g(x))\mathrm{d}\tilde{H}_0^d(x),\end{aligned}$$

which proves (9.27).

We will complete the proof of the subadditivity of the Choquet integral after another auxiliary step proposed by Akihiko Miyachi.

Lemma 374. *Proposition* 370 *remains true when f and g assume its value in $\mathbb{N}_0$.*

Proof In view of Proposition 370, we may assume that f and g are bounded. Consequently, we can assume that

$$f = \sum_{k=1}^{N} \chi_{E_k}, \quad g = \sum_{k=1}^{N} \chi_{F_k}, \tag{9.28}$$

where $E_1 \supset E_2 \supset \cdots \supset E_N$ and $F_1 \supset F_2 \supset \cdots \supset F_N$.

We prove (9.27) by the induction on N. If $N = 1$, then (9.27) readily follows from Lemma 373. Suppose that (9.27) is true for $N = m - 1 \geq 1$. Consider a function given by (9.28). Set $h \equiv \sum_{k=1}^{N-1} \chi_{E_k}$, and $k \equiv \sum_{k=1}^{N-1} \chi_{F_k}$. Then according to Lemma 372 and Theorem 366, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x) \\
&= \int_{\mathbb{R}^n} h(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} k(x)\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}_0^d(E_N) + \tilde{H}_0^d(F_N) \\
&\geq \int_{\mathbb{R}^n} h(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} k(x)\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}_0^d(E_N \cup F_N) + \tilde{H}_0^d(E_N \cap F_N).
\end{aligned}$$

We now invoke the induction assumption to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x) \\
&\geq \int_{\mathbb{R}^n} (h(x) + k(x))\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}_0^d(E_N \cup F_N) + \tilde{H}_0^d(E_N \cap F_N) \\
&= \int_{\mathbb{R}^n} (h(x) + k(x))\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} \chi_{E_N \cup F_N}(x)\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}_0^d(E_N \cap F_N).
\end{aligned}$$

Next, by using Lemma 373 twice, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) + \int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x) \\
&\geq \int_{\mathbb{R}^n} (h(x) + k(x) + \chi_{E_N \cup F_N}(x))\mathrm{d}\tilde{H}_0^d(x) + \tilde{H}_0^d(E_N \cap F_N) \\
&\geq \int_{\mathbb{R}^n} (h(x) + k(x) + \chi_{E_N \cup F_N}(x) + \chi_{E_N \cap F_N}(x))\mathrm{d}\tilde{H}_0^d(x) \\
&\geq \int_{\mathbb{R}^n} (f(x) + g(x))\mathrm{d}\tilde{H}_0^d(x),
\end{aligned}$$

which proves (9.27).

Finally, we prove (9.27) for the general case. Set

$$f_N \equiv 2^N \min([2^{-N} f], 2^{-N} N), \quad g_N \equiv 2^N \min([2^{-N} g], 2^{-N} N).$$

Then according to (9.26) and Proposition 370, we have

$$\int_{\mathbb{R}^n} (f(x) + g(x))\mathrm{d}\tilde{H}_0^d(x) = \lim_{N\to\infty} \int_{\mathbb{R}^n} (f_N(x) + g_N(x))\mathrm{d}\tilde{H}_0^d(x)$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\mathrm{d}\tilde{H}_0^d(x) &= \lim_{N\to\infty} \int_{\mathbb{R}^n} f_N(x)\mathrm{d}\tilde{H}_0^d(x), \\
\int_{\mathbb{R}^n} g(x)\mathrm{d}\tilde{H}_0^d(x) &= \lim_{N\to\infty} \int_{\mathbb{R}^n} g_N(x)\mathrm{d}\tilde{H}_0^d(x).
\end{aligned}$$

This proves (9.27).

9.2.3 Predual spaces of Morrey spaces by way of the Choquet integral

We present a method of expressing the predual in terms of the Choquet integral.

Definition 92 ($L^1(\tilde{H}_0^d)$)**.** Let $0 \leq d \leq n$. The space $L^1(\tilde{H}_0^d)$ denotes the closure of $C_{\rm c}(\mathbb{R}^n)$ with respect to the norm $\|f\|_{L^1(\tilde{H}_0^d)} \equiv \displaystyle\int_0^\infty \tilde{H}_0^d(\{|f| > \lambda\}){\rm d}\lambda$.

Example 153. Let E be a subset of $\mathbb{R}^n$. Then $\|\chi_E\|_{L^1(\tilde{H}_0^d)} = \tilde{H}_0^d(E)$ since $\tilde{H}_0^d(\{\chi_E > \lambda\}) = \tilde{H}_0^d(E)\chi_{(0,1)}(\lambda)$.

Let μ be a signed measure. We recall Definition 9; $\mu \in \mathbb{M}_1^p(\mathbb{R}^n)$ if and only if $|\mu|(B(x,r)) \lesssim r^{n-\frac{n}{p}}$ for all balls $B(x,r)$, where $|\mu|$ denotes the total variation of μ. The following is the fundamental duality result:

Lemma 375. *Let $1 \leq p < \infty$. Write $d \equiv n - \frac{n}{p}$. Let $\mu \in \mathbb{M}_1^p(\mathbb{R}^n)$ and $f \in C_{\rm c}(\mathbb{R}^n)$. Then*

$$\left|\int_{\mathbb{R}^n} f(x){\rm d}\mu(x)\right| \lesssim \|\mu\|_{\mathbb{M}_1^p}\|f\|_{L^1(\tilde{H}_0^d)}.$$

Proof By the triangle inequality and the Layer-Cake formula

$$\left|\int_{\mathbb{R}^n} f(x){\rm d}\mu(x)\right| \leq \int_{\mathbb{R}^n} |f(x)|\,d|\mu|(x) = \int_0^\infty |\mu|(\{|f| > \lambda\}){\rm d}\lambda. \tag{9.29}$$

Let $\lambda > 0$ be fixed. Let $\{B(x_j, r_j)\}_{j=1}^\infty$ be a covering of the set $\{|f| > \lambda\}$. Then

$$|\mu|(\{|f| > \lambda\}) \leq \sum_{j=1}^\infty |\mu|(B(x_j, r_j)) \leq \|\mu\|_{\mathbb{M}_1^p} \sum_{j=1}^\infty {r_j}^d.$$

As a result, taking the infimum of the ball covers $\{B(x_j, r_j)\}_{j=1}^\infty$ of E, we conclude

$$|\mu|(\{|f| > \lambda\}) \lesssim \|\mu\|_{\mathbb{M}_1^p} H^d(\{|f| > \lambda\}). \tag{9.30}$$

Inserting (9.30) to (9.29) and using Proposition 365, we obtain

$$\left|\int_{\mathbb{R}^n} f(x){\rm d}\mu(x)\right| \lesssim \|\mu\|_{\mathbb{M}_1^p} \int_0^\infty \tilde{H}_0^d(\{|f| > \lambda\}){\rm d}\lambda = \|\mu\|_{\mathbb{M}_1^p}\|f\|_{L^1(\tilde{H}_0^d)},$$

as was to be shown.

If we choose d suitably, $L^1(\tilde{H}_0^d)$ can be considered as the predual of $\mathbb{M}_1^p(\mathbb{R}^n)$. Unlike the duality $\mathcal{H}_{q'}^{p'}(\mathbb{R}^n)$–$\mathcal{M}_q^p(\mathbb{R}^n)$, the characterization is somewhat indirect because we need to depend on a dense subspace of $L^1(\tilde{H}_0^d)$.

Proposition 376. *Let $1 < p < \infty$. Write $d \equiv n - \frac{n}{p}$. Then a predual space of $\mathbb{M}^p_1(\mathbb{R}^n)$ is given by $L^1(\tilde{H}^d_0)$. More precisely, for $\mu \in \mathbb{M}^p_1(\mathbb{R}^n)$, there exists a unique linear functional L_μ on $L^1(\tilde{H}^d_0)$ such that*

$$L_\mu(\omega) = \int_{\mathbb{R}^n} \omega(x) \mathrm{d}\mu(x) \tag{9.31}$$

for all $\omega \in C_{\mathrm{c}}(\mathbb{R}^n)$, and conversely any continuous linear functional L on $L^1(\tilde{H}^d_0)$ is realized as $L = L_\mu$ for some $\mu \in \mathbb{M}^p_1(\mathbb{R}^n)$.

Proof According to Lemma 375, we see that such L_μ exists. Conversely, for any linear functional L, we consider the composition $\iota : C_{\mathrm{c}}(\mathbb{R}^n) \to L^1(\tilde{H}^d_0)$ and $L : L^1(\tilde{H}^d_0) \to \mathbb{C}$ to obtain a linear functional $L \circ \iota : C_{\mathrm{c}}(\mathbb{R}^n) \to \mathbb{C}$. By the Riesz representation theorem [118, §1.8], there exists a Radon measure μ such that (9.31) holds for all $f \in C_{\mathrm{c}}(\mathbb{R}^n)$. Let us show $\mu \in \mathbb{M}^p_1(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|\mu\|(B(x,r)) &\lesssim \sup\{|L(f)| \,:\, f \in C_{\mathrm{c}}(B(x,2r)),\, \|f\|_{L^\infty} \le 1\} \\
&\le \sup\left\{\|L\|_{(L^1(\tilde{H}^d_0))^*}\|f\|_{\tilde{H}^d_0} \,:\, f \in C_{\mathrm{c}}(B(x,2r)),\, \|f\|_{L^\infty} \le 1\right\} \\
&\lesssim \|L\|_{(L^1(\tilde{H}^d_0))^*} r^d
\end{aligned}$$

thanks to Example 153, as was to be shown.

We see once again that $\mathcal{M}^p_q(\mathbb{R}^n)$ is realized as the dual of a Banach space.

Theorem 377. *Let $1 \le q \le p < \infty$. Set $d \equiv n\left(1 - \frac{q}{p}\right)$. Then $\|f\|_{\mathcal{M}^p_q} \sim \sup\left\{\|f\|_{L^q(w)} \,:\, w \in \mathbb{M}^+(\mathbb{R}^n), \quad \|w\|_{L^1(\tilde{H}^d_0)} \le 1\right\}$ for any $f \in L^0(\mathbb{R}^n)$.*

Proof For a ball B, by letting $w \equiv |B|^{-1+\frac{q}{p}}\chi_B$, we obtain

$$\|f\|_{\mathcal{M}^p_q} \lesssim \sup\left\{\|f\|_{L^q(w)} \,:\, w \in \mathbb{M}^+(\mathbb{R}^n), \quad \|w\|_{L^1(\tilde{H}^d_0)} \le 1\right\}$$

from the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$. To prove the converse relation, we fix $w \in \mathbb{M}^+(\mathbb{R}^n)$ with $\|w\|_{L^1(\tilde{H}^d_0)} \le 1$ and we set $\mu(E) \equiv \displaystyle\int_E |f(x)|^q \mathrm{d}x$ for a Borel set E. Then μ is a measure. We observe

$$\int_{\mathbb{R}^n} |f(x)|^q w(x) \mathrm{d}x \lesssim \|\mu\|_{\mathbb{M}^p_1} \sim (\|f\|_{\mathcal{M}^p_q})^q$$

from Lemma 375. Combining these inequalities completes the proof.

We are now interested in another expression of a predual space of $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q < p < \infty$.

Definition 93. Let $1 < u < v < \infty$. Define $d \equiv n\left(1 - \frac{v'}{u'}\right) \in (0, n)$. Let $g \in L^0(\mathbb{R}^n)$. Then define

$$\|g\|_{V^u_v} \equiv \inf_w \|g\|_{L^v(w^{1-v})}, \tag{9.32}$$

where w runs over all elements in $\mathcal{B}(\mathbb{R}^n)$ such that

$$\{w = 0\} \subset \{g = 0\}, \quad \|w\|_{L^1(\mathring{H}^d_0)} \le 1.$$

The space $V^u_v(\mathbb{R}^n)$ denotes the linear space of $g \in L^0(\mathbb{R}^n)$ for which there exists $w \in L^1(\mathring{H}^d_0) \cap \mathcal{B}_+(\mathbb{R}^n)$ which does not vanish almost everywhere on $\{g \neq 0\}$.

Example 154. Let $1 < u < v < \infty$. If $g \in L^v_{\mathrm{c}}(\mathbb{R}^n)$, then $g \in V^u_v(\mathbb{R}^n)$. In fact, we can take $w(x) \equiv \kappa e^{-|x|^2}$, $x \in \mathbb{R}^n$ with some suitable $\kappa > 0$ in Definition 93. If in addition g is supported on a ball B, then $\|g\|_{V^u_v} \le \frac{\|\chi_B g\|_{L^v}}{\|\chi_B\|_{L^1(\mathring{H}^d_0)}} \lesssim |B|^{-\frac{d}{nv'}} \|\chi_B g\|_{L^v}$.

We show that $V^u_v(\mathbb{R}^n)$ is a ball Banach function space.

Lemma 378. *Let $1 < u \le v < \infty$. Then $V^u_v(\mathbb{R}^n)$ is a ball Banach function space.*

Proof We content ourselves with the completeness of $V^u_v(\mathbb{R}^n)$: Let $\{g_j\}_{j=1}^\infty \subset V^u_v(\mathbb{R}^n)$ satisfy $\sum_{j=1}^\infty \|g_j\|_{V^u_v} < \infty$. Write $\lambda_j \equiv \|g_j\|_{V^u_v}$ for $j \in \mathbb{N}$. By the definition of λ_j there exists $w_j \in \mathcal{B}_+(\mathbb{R}^n)$ such that

$$\{w_j = 0\} \subset \{g_j = 0\}, \quad \|w_j\|_{L^1(\mathring{H}^d_0)} \le 1,$$

$$\left(\int_{\{w_j \neq 0\}} |g_j(x)|^v w_j(x)^{1-v} \mathrm{d}x\right)^{\frac{1}{v}} \lesssim \lambda_j.$$

Let $W \equiv \sum_{j=1}^\infty \lambda_j w_j$.

By Hölder's inequality, we have

$$\left(\sum_{j=1}^\infty |g_j|\right)^q \le \left(\sum_{j=1}^\infty \lambda_j w_j\right)^{q-1} \left(\sum_{j=1}^\infty \lambda_j |g_j| {w_j}^{1-q}\right),$$

so that

$$\left(\sum_{j=1}^\infty |g_j|\right)^q W^{1-q} \le \sum_{j=1}^\infty \lambda_j |g_j| {w_j}^{1-q}.$$

If we integrate this inequality over $\{W \neq 0\}$, we obtain

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty |g_j(x)|\right)^q W(x)^{1-q} \mathrm{d}x \lesssim \sum_{j=1}^\infty \lambda_j.$$

Thus, since $\|W\|_{L^1(\tilde{H}_0^d)} \le 1$, $\sum_{j=1}^{\infty} |g_j| \in V_v^u(\mathbb{R}^n)$. Hence $\sum_{j=1}^{\infty} g_j \in V_v^u(\mathbb{R}^n)$. If we set $W^k \equiv \sum_{j=k}^{\infty} w_j$, then

$$\int_{\mathbb{R}^n} \left| \sum_{j=k}^{\infty} g_j(x) \right|^q W^k(x)^{1-q} \mathrm{d}x \lesssim \sum_{j=k}^{\infty} \lambda_j.$$

Thus, $\sum_{j=1}^{\infty} g_j = g$ in the topology of $V_v^u(\mathbb{R}^n)$.

So far we showed that $V_{q'}^{p'}(\mathbb{R}^n)$ is a ball Banach function space. We will establish that $V_{q'}^{p'}(\mathbb{R}^n)$ and $\mathcal{H}_{q'}^{p'}(\mathbb{R}^n)$ are the same.

Theorem 379. *Let $1 < q \le p < \infty$. Then a predual space of $\mathcal{M}_q^p(\mathbb{R}^n)$ is $V_{q'}^{p'}(\mathbb{R}^n)$ with the equivalence of norms under the following pairing*:

$$\langle f, g \rangle \equiv \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \quad (f \in \mathcal{M}_q^p(\mathbb{R}^n),\ g \in V_{q'}^{p'}(\mathbb{R}^n)).$$

Moreover, for all $f \in \mathcal{M}_q^p(\mathbb{R}^n)$

$$\|f\|_{\mathcal{M}_q^p} \sim \sup \left\{ \|f \cdot g\|_{L^1} \,:\, g \in V_{q'}^{p'}(\mathbb{R}^n),\ \|g\|_{V_{q'}^{p'}} \le 1 \right\}.$$

Proof Let $g \in V_{q'}^{p'}(\mathbb{R}^n)$ and $f \in \mathcal{M}_q^p(\mathbb{R}^n)$. Then by Hölder's inequality

$$\|f \cdot g\|_{L^1} \le \left(\int_{\mathbb{R}^n} |f(x)|^q w(x) \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |g(x)|^{q'} w(x)^{1-q'} \mathrm{d}x \right)^{\frac{1}{q'}}$$

for all Borel functions w that are allowed to vanish only on the set $\{g = 0\}$. Therefore, it follows from Theorem 377 that

$$\|f \cdot g\|_{L^1} \lesssim \|f\|_{\mathcal{M}_q^p} \left(\int_{\mathbb{R}^n} |g(x)|^{q'} w(x)^{1-q'} \mathrm{d}x \right)^{\frac{1}{q'}}$$

for all such Borel functions w. Thus, from (9.32), we obtain the desired equality.

Conversely, suppose that we have a linear functional L on $V_{q'}^{p'}(\mathbb{R}^n)$. Then for all balls B and a function $g \in L^{q'}(\mathbb{R}^n)$ with support B, we have

$$|L(g)| \le \|g\|_{V_{q'}^{p'}} \lesssim |B|^{-\frac{1}{q'}\left(1-\frac{q'}{p'}\right)} \|g\|_{L^{q'}} = |B|^{\frac{1}{q}-\frac{1}{p}} \|g\|_{L^{q'}},$$

thanks to Example 154. Thus, we can mimic the proof of Theorem 347.

As a consequence, we obtain an equivalent expression of the Köthe dual of $\mathcal{M}_q^p(\mathbb{R}^n)$.

Corollary 380. *Let $1 < q \le p < \infty$. Then the Köthe dual of $V^{p'}_{q'}(\mathbb{R}^n)$ is $\mathcal{M}^p_q(\mathbb{R}^n)$, or equivalently the Köthe dual of $\mathcal{M}^p_q(\mathbb{R}^n)$ is $V^{p'}_{q'}(\mathbb{R}^n)$.*

Thanks to Theorem 355 and Corollary 380, we see that two spaces $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ and $V^{p'}_{q'}(\mathbb{R}^n)$ are the same.

Theorem 381. *Let $1 < q \le p < \infty$. Then the block space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ coincides with $V^{p'}_{q'}(\mathbb{R}^n)$ with equivalence of norms.*

Proof Simply combine Theorem 355 and Corollary 380.

9.2.4 Exercises

Exercise 119. By a change of variables, prove (9.26).

Exercise 120. For all nonempty sets $E \subset \mathbb{R}^n$, show that $H^0(E) \ge 1$, and that $H^0(E) = 1$ if and only if E is bounded. Hint: When $H^0(E) = 1$, then we can find a ball that covers E.

Exercise 121. Let $0 < d \le n$, and let $E \subset \mathbb{R}^n$. Then show that $\tilde{H}^d_0(E) = \inf\{\tilde{H}^d_0(O) : E \subset O,\ O \text{ is open}\}$.

9.3 Notes

Section 9.1

General remarks and textbooks in Section 9.1

Zorko [472], Adams [4] and Kalita [224] studied preduals of Morrey spaces. Motivated by the work [221], Zorko considered the atomic space [472, p. 589] unlike what we did in this book. Adams and Xiao investigated the relation between them [8]. See [298] for the duality of central Herz–Morrey–Musielak–Orlicz spaces of variable exponents.

See [457] for the duality in terms of the heat kernel.

Section 9.1.1

The definition of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ in Definition 83, originally denoted by $H^{q,\varphi}$ in [472] in the framework of generalized Morrey spaces (see Section 12.1), is due to Zorko. Examples of functions in $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ can be found in [218]; see [218, Example 2.4] for Example 143. Example 142, which can be used to calculate the norm $\|\chi_Q\|_{\mathcal{H}^{p'}_{q'}}$, is due to Komori and Mizuhara [252, Lemma 1]. Although the position of the cubes in the definition of blocks is not specified, we can

use the dyadic grid $\mathcal{D}(\mathbb{R}^n)$ for this purpose; see [217, Lemma 2.4] and [397, Lemma 2.2] for Theorem 344. Note that Theorem 348 goes back to a work by Izumi, Sato and Yabuta [217, Lemma 2.5] as well as the one by Sawano and Tanaka [398, Theorem 1.3]. The lattice property, Lemma 342, is [396, Proposition 3.3].

Theorem 351 is [29, Theorem 2.2] when $E(\mu)$ is a Banach function space; let ρ be a function norm. Then the associate norm ρ' is itself a function norm. Theorem 352 is [29, Theorem 2.7] when $E(\mu)$ is a Banach function space. More precisely, every Banach function space $E(\mu)$ coincides with its second associate space $E''(\mu)$. In other words, a function f belongs to $E(\mu)$ if and only if it belongs to $E''(\mu)$, and in that case

$$\|f\|_E(\mu) = \|f\|_{E''(\mu)}, \quad (f \in E(\mu) = E''(\mu)).$$

Theorem 347 was investigated by Zorko [472]; see [224] as well. We refer to [8] and [151] for more recent characterizations.

Section 9.1.2

We showed in Theorem 345 that $L^{q'}_{\mathrm{c}}(\mathbb{R}^n)$ is dense in $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$; see [341, Lemma 2.6]. If the function is compactly supported, then we have an equivalent expression of the norm in $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$; see [398, Proposition 5.3] for Theorem 346.

Section 9.1.3

Zorko characterized the predual in [472, Proposition 5]; see Theorem 347.

Section 9.1.4

Kantorovich and Akilov proved that the Fatou property of Banach lattices is a key for the "bi"-Köthe dual to be back to the original space. [226]; see Theorem 354.

Section 9.1.5

Köthe and Toeplitz began the study of certain pairs of subspaces of the space of all real sequences [239, 240, 242]. Their theory has been generalized by Dieudonné, Cooper and Lorentz and Wertheim [87, 103, 290]. Theorem 352 can be located as a counterpart to ball Banach function spaces of the result by Lorentz and Luxemberg. We essentially followed the argument used in [29, Theorem 4.1] to specify a predual space of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Yang and Yuan showed that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is not reflexive. We followed [398, Example 5.2] for the proof. See [438, Corollary 2.20] for Theorem 356. The case where $q = 1$ can be covered; see [303, Theorem 4.1].

Section 9.1.6

We refer to [420, Chapter 8, Lemma 5] for the averaging technique for Lebesgue spaces. We followed the averaging technique used in [212]; see Theorem 363. See [388] for the applications to elliptic differential operators and [180] for the applications to bilinear fractional integral operators. See [315] for the case of Herz spaces.

See [255] for another type of decomposition of Morrey spaces.

Section 9.2

General remarks and textbooks in Section 9.2

See standard textbooks [6, Chapter 3], [112], [231], [382, §1.1.4] for the Choquet integral. In particular, we can find Proposition 9.31 in [6, Theorem 5.1]. See [6, Theorem 5.2] for Lemma 375. In [6, Section 5.2] Adams showed that the various predual spaces defined in this book coincide.

See [156, Appendix J], [415, Chapter I, §3] for the Whitney decomposition of the domain. Using the capacity, we can consider the notion of quasi-continuity.

Definition 94 ($\tilde{H}_0^d$-quasi continuous)**.** A function f is *$\tilde{H}_0^d$-quasi continuous*, if for all $\varepsilon > 0$ there exists an open set such that $\tilde{H}_0^d(O) < \varepsilon$ and that f is continuous outside O.

Section 9.2.1

Yang and Yuan showed that $\tilde{H}$ is subadditive in [462]; see Theorem 366.

Section 9.2.2

The theory of the Choquet integral goes back to Choquet [78]. Orobitg and Verdera applied the Choquet integral to the boundedness of the Hardy–Littlewood maximal operators [349]. See [426, Theorem 1] for a generalization to fractional maximal operators. Kuznetsov considered a vector-valued inequality in [261]. Essoh, Fofana and Koua passed this result to Morrey spaces [114]. Example 151 comes from [349, Lemma. 1]. See the lecture note [4] for more. It may be interesting that the Hardy-Littlewood maximal operator is L^1-bounded if the integral is considered in the sense of Choquet; see [4, Theorem A].

Section 9.2.3

Adams gave a characterization of a predual space of $\mathbb{M}^p_1(\mathbb{R}^n)$; see [4, Proposition 1]. The definition of the space $V^u_v(\mathbb{R}^n)$ is due to Adams and Xiao [8, p.1632]; see Definition 93. Adams and Xiao proved Theorems 377, 379 and 381 in [8, Proposition 1 and Theorem 2.2], [8, Theorem 2.2] and [8, Theorem 3.3], respectively.

Chapter 10

Linear and sublinear operators in Morrey spaces

One of the important problems in the theory of function spaces is the boundedness property of operators. This is important because it yields many applications in various fields of mathematics such as partial differential equations and potential theory. There is a standard technique which is a local/global strategy. However, unlike Lebesgue spaces, the operators need to be carefully defined in some cases. For example, singular integral operators, dealt with in Section 10.4, must be handled with care because they are defined by an approximation of integral kernels. Chapter 10 presents methods to define operators in Morrey spaces. Likewise, care must be taken when handling commutators in Section 10.5. The Hardy–Littlewood maximal operator (including the sharp maximal operator) and fractional integral operators are discussed in Sections 10.1 and 10.3, respectively.

Basically, the boundedness property of operators can be proved by the local estimate initiated by Burenkov and Guliyev. However, to make the proof self-contained we consider a direct proof which still recasts the flavor of the local estimates.

10.1 Maximal operators in Morrey spaces

Having set down the structure of Morrey spaces and block spaces in Chapter 9, we are now oriented to the boundedness property of operators. We continue to work on $\mathbb{R}^n$ equipped with the Lebesgue measure $\mathrm{d}x$ as before. We will consider the boundedness properties of Morrey spaces and local Morrey spaces in Sections 10.1.1 and 10.2.2, respectively.

10.1.1 Maximal operator in Morrey spaces

Our main interest here is the Hardy–Littlewood maximal operator M. We now present a typical argument about the proof of the boundedness of operators in Morrey spaces. As the proof shows, we depend heavily upon the

so called local/global strategy. We fix a cube or a ball and decompose functions according to its five times expansion.

Theorem 382. *Let $1 < q \le p < \infty$. Then $\|Mf\|_{\mathcal{M}^p_q} \lesssim_q \|f\|_{\mathcal{M}^p_q}$ whenever $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.*

Before we come to the proof of Theorem 382, it is noted that Lemma 130 is a key tool for the proof. The proof will be a model case of the proof of the boundedness of operators acting on Morrey spaces.

Proof We will take the local/global strategy. We have only to show, from the definition, that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}^p_q} \quad (Q \in \mathcal{Q}). \tag{10.1}$$

Write $f = f_1 + f_2$, where $f_1 = f$ on $5Q$ and $f_2 = f$ outside $5Q$. The estimate of (10.1) can be split into the local estimate

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_1(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}^p_q} \tag{10.2}$$

and the global estimate

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_2(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}^p_q}. \tag{10.3}$$

We know that M is $L^q(\mathbb{R}^n)$-bounded. Thus, by expanding the integration domain and inserting the definition of f_1 into what we have obtained from the $L^p(\mathbb{R}^n)$-boundedness, we obtain

$$\left(\int_Q Mf_1(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \le \left(\int_{\mathbb{R}^n} Mf_1(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \left(\int_{5Q} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

From the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$,

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_1(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim |5Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{5Q} |f(y)|^q \mathrm{d}y\right)^{\frac{1}{q}} \le \|f\|_{\mathcal{M}^p_q}.$$

Thus (10.2) is proved.

It remains to prove (10.3). If we use Lemma 130, then we obtain

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_2(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim |Q|^{\frac{1}{p}} \sup_{R \in \mathcal{Q}^\sharp(Q)} m_R(|f|).$$

Taking into account $\mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_1(\mathbb{R}^n)$, we see that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_2(y)^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \sup_{R \in \mathcal{Q}} |R|^{\frac{1}{p}-1}\int_R |f(y)|\mathrm{d}y = \|f\|_{\mathcal{M}^p_1} \le \|f\|_{\mathcal{M}^p_q}.$$

Consequently (10.3) is proven.

We obtain the vector-valued Morrey-boundedness of the Hardy–Littlewood maximal operator by taking the local/global strategy again. We will write

$$\left\|\{f_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}^p_q(\ell^u)} \equiv \left\|\|\{f_j\}_{j=1}^{\infty}\|_{\ell^u}\right\|_{\mathcal{M}^p_q}$$

for every sequence of $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$.

Theorem 383. *Suppose that the parameters p, q and u satisfy $1 < q \le p < \infty$ and $1 < u \le \infty$. Then $\left\|\{Mf_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}^p_q(\ell^u)} \lesssim \left\|\{f_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}^p_q(\ell^u)}$ for every sequence of $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$.*

Proof We modify the proof of Theorem 382. Instead of (10.1), we have to show that

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \|\{Mf_j(y)\}_{j=1}^{\infty}\|_{\ell^u}{}^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \left\|\{f_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}^p_q(\ell^u)} \quad (Q \in \mathcal{Q}).$$

We consider a similar decomposition as before; write $f_j = f_{1,j} + f_{2,j}$, where $f_{j,1} \equiv f$ on $5Q$ and $f_{j,2} \equiv f$ outside $5Q$. Estimates (10.2) and (10.3) correspond to

$$\mathrm{I} \equiv |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \|\{Mf_{j,1}(y)\}_{j=1}^{\infty}\|_{\ell^u}{}^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \left\|\{f_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}^p_q(\ell^u)}, \tag{10.4}$$

$$\mathrm{II} \equiv |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \|\{Mf_{j,2}(y)\}_{j=1}^{\infty}\|_{\ell^u}{}^q \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \left\|\{f_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}^p_q(\ell^u)}, \tag{10.5}$$

respectively. For (10.4), we use Theorem 270 and we proceed as in (10.2): We first expand the integration domain.

$$\mathrm{I} \le |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} \|\{Mf_{j,1}(y)\}_{j=1}^{\infty}\|_{\ell^u}{}^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

Next, we use the vector-valued boundedness of M to have

$$\mathrm{I} \lesssim |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} \|\{f_{j,1}(y)\}_{j=1}^{\infty}\|_{\ell^u}{}^q \mathrm{d}y\right)^{\frac{1}{q}} = |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{5Q} \|\{f_j\}_{j=1}^{\infty}\|_{\ell^u}{}^q \mathrm{d}y\right)^{\frac{1}{q}}.$$

By the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$, $\mathrm{I} \lesssim \left\|\{f_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}^p_q(\ell^u)}$. Thus, the estimate of I is valid.

For the estimate of II, recall Lemma 130. By Lemma 130 with f replaced by f_j, we have $Mf_{j,2}(y) \lesssim \sum_{k=1}^{\infty} m_{2^kQ}(|f_j|)$. By Minkowski's inequality, we have

$$\|\{Mf_j(y)\}_{j=1}^{\infty}\|_{\ell^u} \lesssim \sum_{k=1}^{\infty} \frac{1}{|2^kQ|} \int_{2^kQ} \|\{f_j(z)\}_{j=1}^{\infty}\|_{\ell^u} \mathrm{d}z \quad (y \in Q).$$

Consequently,

$$|Q|^{\frac{1}{p}-1}\int_Q \|\{Mf_j(y)\}_{j=1}^\infty\|_{\ell^u}\mathrm{d}y \lesssim \sum_{k=1}^\infty |Q|^{\frac{1}{p}}\frac{1}{|2^kQ|}\int_{2^kQ}\|\{f_j(z)\}_{j=1}^\infty\|_{\ell^u}\mathrm{d}z.$$

Going through an argument similar to (10.3), we obtain (10.5).

The space $\mathcal{M}_1^p(\mathbb{R}^n)$ is a nasty space as the following example shows:

Example 155. Let $1<p<\infty$. Let $0<r<1/2$. Define $f_0(t)\equiv rt$ and $f_1(t)\equiv 1-r+rt$ so that f_0 and f_1 are affine mappings from $[0,1]$ to $[0,1]$. Define inductively $E_0\equiv[0,1]$, $E_{k+1}\equiv f_0(E_k)\cup f_1(E_k)$ for $k\in\mathbb{N}$. Choose $r>0$ so that $r^{\frac{1}{p}}=2r$. Then we have $\|\chi_{E_k}\|_{\mathcal{M}_1^p(\mathbb{R})}\sim\|\chi_{[0,r^k]}\|_{L^p}=r^{\frac{k}{p}}=(2r)^k$. Meanwhile, we have the lower bound of $M\chi_{E_k}$ from below:

$$M\chi_{E_k}\geq\chi_{E_k}+2r\chi_{E_{k-1}\setminus E_k}+\cdots+(2r)^l\chi_{E_{k-l}\setminus E_{k-l+1}}+\cdots+(2r)^k\chi_{E_0\setminus E_1}.$$

As a result,

$$\begin{aligned}\|M\chi_{E_k}\|_{\mathcal{M}_1^p(\mathbb{R})}&\geq(2r)^k+2r\cdot(2r)^{k-1}(1-2r)+\cdots+(2r)^k(1-2r)\\&=k(1-2r)(2r)^k+(2r)^{k+1}.\end{aligned}$$

This implies that $\|M\chi_{E_k}\|_{\mathcal{M}_1^p(\mathbb{R})}\geq k(1-2r)\|\chi_{E_k}\|_{\mathcal{M}_1^p(\mathbb{R})}$. Since k is arbitrary, this relation shows that M is NOT bounded on $\mathcal{M}_1^p(\mathbb{R})$.

A natural question appears: what we can say if $f\in L^0(\mathbb{R}^n)$ satisfies $Mf\in\mathcal{M}_1^p(\mathbb{R}^n)$? Since $\mathcal{M}_1^p(\mathbb{R}^n)$ contains $\mathcal{M}_q^p(\mathbb{R}^n)$ and M is bounded on $\mathcal{M}_q^p(\mathbb{R}^n)$ if $1<q\leq p<\infty$, $Mf\in\mathcal{M}_1^p(\mathbb{R}^n)$ does not imply $f=0$. Meanwhile, if $f\in L^0(\mathbb{R}^n)$ satisfies $Mf\in L^1(\mathbb{R}^n)$, then $f=0$. So one may guess that there is something different from Lebesgue spaces. We will completely characterize the functions $f\in L^0(\mathbb{R}^n)$ for which $Mf\in\mathcal{M}_1^p(\mathbb{R}^n)$ in the second book.

Using the weak-boundedness of operators, we can prove various results. For example, we can supplement Theorem 382.

Theorem 384. *For $1\leq p<\infty$, M is bounded from $\mathcal{M}_1^p(\mathbb{R}^n)$ to $\mathrm{W}\mathcal{M}_1^p(\mathbb{R}^n)$.*

Proof We have only to show, from the definition, that

$$t|Q|^{\frac{1}{p}-1}\lambda_{Mf}(2t)=t|Q|^{\frac{1}{p}-1}|\{x\in Q:Mf(x)>2t\}|\lesssim\|f\|_{\mathcal{M}_1^p}\quad(Q\in\mathcal{Q}) \tag{10.6}$$

for all $t>0$. Write $f=f_1+f_2$, where $f_1=f$ on $5Q$ and $f_2=f$ outside $5Q$. The estimate of (10.6) can be split into:

$$t|Q|^{\frac{1}{p}-1}\lambda_{Mf_1}(t)=t|Q|^{\frac{1}{p}-1}|\{x\in Q:Mf_1(x)>t\}|\lesssim\|f\|_{\mathcal{M}_1^p}, \tag{10.7}$$

$$t|Q|^{\frac{1}{p}-1}\lambda_{Mf_2}(t)=t|Q|^{\frac{1}{p}-1}|\{x\in Q:Mf_2(x)>t\}|\lesssim\|f\|_{\mathcal{M}_1^p}. \tag{10.8}$$

We know that M is weak $L^1(\mathbb{R}^n)$-bounded. Thus, (10.7) can be shown easily;

$$t|Q|^{\frac{1}{p}-1}\lambda_{Mf_1}(t)\lesssim|Q|^{\frac{1}{p}-1}\int_{5Q}|f(y)|\mathrm{d}y\simeq|5Q|^{\frac{1}{p}-1}\int_{5Q}|f(y)|\mathrm{d}y\leq\|f\|_{\mathcal{M}_1^p}.$$

It remains to prove (10.8). Note that $\{x \in Q : Mf_2(x) > t\} = \emptyset$, if $t \gtrsim m_R(|f|)$ for any cube R. Consequently, we may assume otherwise. Then we obtain

$$t|Q|^{\frac{1}{p}-1}\lambda_{Mf_2}(t) \lesssim |Q|^{\frac{1}{p}} \sup_{R\in\mathcal{Q}^\sharp(Q)} m_R(|f|)$$

from Lemma 130. Thus, $t|Q|^{\frac{1}{p}-1}\lambda_{Mf_2}(t) \lesssim \sup_{R\in\mathcal{Q}} |R|^{\frac{1}{p}} m_R(|f|) = \|f\|_{\mathcal{M}^p_1}$. Consequently (10.8) is proven.

The following remark shows that the weak Morrey spaces are not so artificial.

Remark 15. It is noteworthy that $\|Mf\|_{\mathrm{W}\mathcal{M}^p_1} \sim \|f\|_{\mathcal{M}^p_1}$ for all $f \in \mathcal{M}^p_1(\mathbb{R}^n)$. In fact, given a cube Q by letting $\lambda_0 \equiv m_Q(|f|)$, we obtain $\lambda_0|Q|^{\frac{1}{p}-1}|\{x \in Q : Mf(x) \geq \lambda_0\}| = |Q|^{\frac{1}{p}}\lambda_0$.

We end this section with the boundedness of the Hardy–Littlewood maximal operator in predual spaces.

Theorem 385. *Let $1 < q \leq p < \infty$. Then M is bounded on $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.*

Proof It suffices to show that M maps (p,q)-blocks to a bounded set in $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Let a be a (p,q)-block supported on Q. This follows from the decomposition $Ma = \chi_{2Q}Ma + \sum_{k=1}^{\infty} \chi_{2^kQ\setminus 2^{k-1}Q}Ma$.

10.1.2 Maximal operator in local Morrey spaces

If we mimic the proof above, then we obtain the boundedness of the Hardy–Littlewood maximal operator in local Morrey spaces.

Theorem 386. *Let $1 < q \leq p < \infty$. Then $\|Mf\|_{\mathrm{L}\mathcal{M}^p_q} \lesssim_q \|f\|_{\mathrm{L}\mathcal{M}^p_q}$ for all $f \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof We omit the proof for interested readers; see Exercise 123.

We also have the vector-valued maximal inequality for local Morrey spaces. We will write

$$\big\|\{f_j\}_{j=1}^\infty\big\|_{\mathrm{L}\mathcal{M}^p_q(\ell^u)} \equiv \big\|\|\{f_j\}_{j=1}^\infty\|_{\ell^u}\big\|_{\mathrm{L}\mathcal{M}^p_q}$$

for every sequence of $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$.

Theorem 387. *Suppose that the parameters p, q and u satisfy $1 < q \leq p < \infty$ and $1 < u \leq \infty$. Then $\big\|\{Mf_j\}_{j=1}^\infty\big\|_{\mathrm{L}\mathcal{M}^p_q(\ell^u)} \lesssim \big\|\{f_j\}_{j=1}^\infty\big\|_{\mathrm{L}\mathcal{M}^p_q(\ell^u)}$ for every sequence of $\{f_j\}_{j=1}^\infty \subset L^0(\mathbb{R}^n)$.*

Proof We omit the proof; see Exercise 123.

We avoided defining $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ for $-\infty < p < 0 < q < \infty$. One of the reasons is that M fails to be bounded as the following example shows.

Example 156. Let $-\infty < p < 0 < q < \infty$. Define the local Morrey norm $\|\cdot\|_{\mathrm{L}\mathcal{M}^p_q}$ formally by (1.1). Then since $|\cdot|^\theta \chi_{\mathbb{R}^n\setminus B(1)} \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ whenever $0 < \theta < -\frac{n}{p}$, the maximal operator M does not map $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ boundedly.

10.1.3 Exercises

Exercise 122. For a cube $Q \in \mathcal{Q}$, we define $\mathcal{Q}_1(Q) \equiv \{R \in \mathcal{Q} : R \cap Q \neq \emptyset, R \cap (\mathbb{R}^n \setminus 8Q) \neq \emptyset\}$. For a function $f \in \mathcal{M}^p_1(\mathbb{R}^n)$, define $\Lambda \equiv \sup_{R\in\mathcal{Q}_1(Q)} m_R(|f|)$. Let us write $E_\lambda \equiv \{x \in Q : Mf(x) > \lambda\}$ for $\lambda > \Lambda$. Then for any $x \in E_\lambda$, show that we can find a cube Q_0 such that Q_0 contains x, that $m_Q(|f|) > \lambda$ and that $Q_0 \subset 8Q$. Prove also that $|Q|^{\frac{1}{p}}\Lambda \lesssim \|f\|_{\mathcal{M}^p_1}$.

Exercise 123.

(1) Reexamine the proof of Theorem 382 to prove Theorem 386.

(2) Prove Theorem 387 by mimicking the proof of Theorem 383.

10.2 Sharp maximal operators in Morrey spaces

One of the prominent roles of Morrey spaces is that Morrey spaces can describe the local regularity and the global regularity more precisely than Lebesgue spaces. This aspect can be seen from the sharp maximal inequality. As is seen from the definition of the sharp maximal operator, it annihilates the constant function 1. This means that it annihilates the global regularity. To recover the global regularity, we can use Morrey spaces. Section 10.2.2 defines the sharp maximal operator and then formulates the sharp maximal inequality. As an application, we will prove the boundedness of commutators, which is left open in Section 5.2.

To control the singularity of operators, we use the sharp maximal operators following the idea of Lerner and Hytönen. Section 10.2.2. considers the counterpart to local Morrey spaces

10.2.1 Sharp maximal inequalities for Morrey spaces

As an application of Theorem 174, following Definition 51, we will prove the following sharp maximal inequality for the operator $M^{\sharp,\mathcal{D}}_{2^{-n-2}}$ given by

$$M^{\sharp,\mathcal{D}}_{2^{-n-2}} f(x) \equiv \sup_{Q\in\mathcal{Q}} \chi_Q(x) M^{\sharp,\mathcal{D}}_{2^{-n-2},Q} f(x) \quad (x \in \mathbb{R}^n).$$

Theorem 388 (Sharp-maximal inequality for Morrey spaces)**.** *Let $0 < \bar{q} < q \le p < \infty$. Then $\|M^{\sharp,\mathcal{D}}_{2^{-n-2}} f\|_{\mathcal{M}^p_q} + \|f\|_{\mathcal{M}^p_{\bar{q}}} \sim \|f\|_{\mathcal{M}^p_q}$ for all $f \in L^0(\mathbb{R}^n)$.*

Proof Since $M^{\sharp,\mathcal{D}}_{2^{-n-2}} f \lesssim \mathfrak{M} f$, where $\mathfrak{M}$ is the maximal operator given by

$$\mathfrak{M}_\tau f(x) = \mathfrak{M} f(x) \equiv \sup_{Q \in \mathcal{Q}} \chi_Q(x) (f\chi_Q)^*(\tau|Q|) \quad (x \in \mathbb{R}^n), \tag{10.1}$$

one inequality $\lesssim$ is clear. Let us prove the reverse inequality $\gtrsim$.

Let Q^0 be a fixed cube. Then we need to estimate $|Q^0|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q^0)}$. By Theorem 174, we have a decomposition: for almost every $x \in Q^0$,

$$|f(x) - \mathrm{Med}(f; Q^0)| \le 2 \sum_{j=1}^{\infty} \sum_{k \in K_j} \omega_{2^{-n-2}}(f; Q^j_k) \chi_{Q^j_k}(x),$$

so that we obtain

$$\begin{aligned} |Q^0|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q^0)} &\le |Q^0|^{\frac{1}{p}} |\mathrm{Med}(f; Q^0)| \\ &\quad + 2|Q^0|^{\frac{1}{p}-\frac{1}{q}} \left\| \sum_{j=1}^{\infty} \sum_{k \in K_j} \omega_{2^{-n-2}}(f; Q^j_k) \chi_{Q^j_k} \right\|_{L^q(Q^0)}. \end{aligned}$$

Set $E^j_k \equiv Q^j_k \setminus \bigcup_{k' \in K_{j+1}} Q^{j+1}_{k'}$. Then we have $\chi_{Q^j_k}(x) \le 2M\chi_{E^j_k}(x)$, since $2|E^j_k| \ge |Q^j_k|$. Consequently, by letting $u > (\bar{q})^{-1}$, we obtain

$$\begin{aligned} |Q^0|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q^0)} &\lesssim |Q^0|^{\frac{1}{p}} |\mathrm{Med}(f; Q^0)| \\ &\quad + |Q^0|^{\frac{1}{p}-\frac{1}{q}} \left\| \sum_{j=1}^{\infty} \sum_{k \in K_j} \omega_{2^{-n-2}}(f; Q^j_k) (M\chi_{E^j_k})^u \right\|_{L^q(Q^0)}. \end{aligned}$$

If we use the Fefferman-Stein vector-valued inequality for Morrey spaces (Theorem 145), then

$$\begin{aligned} |Q^0|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q^0)} &\lesssim |Q^0|^{\frac{1}{p}} |\mathrm{Med}(f; Q^0)| \\ &\quad + |Q^0|^{\frac{1}{p}-\frac{1}{q}} \left\| \sum_{j=1}^{\infty} \sum_{k \in K_j} \omega_{2^{-n-2}}(f; Q^j_k) \chi_{E^j_k} \right\|_{L^q(Q^0)}. \end{aligned}$$

Thus, we obtain

$$|Q^0|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q^0)} \lesssim |Q^0|^{\frac{1}{p}} |\mathrm{Med}(f; Q^0)| + |Q^0|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q^0} M^{n,\sharp}_{2^{-n-2}}(x)^q \mathrm{d}x \right)^{\frac{1}{q}}.$$

From the definition of the Morrey norm, we conclude $\|f\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_{\bar{q}}} + \|M^{\sharp,\mathcal{D}}_{2^{-n-2}} f\|_{\mathcal{M}^p_q}$. Thus, we obtain the desired result.

Similar to Theorem 166, we can prove the following result:

Theorem 389 (Sharp maximal inequality for Morrey spaces)**.** *Let $1 < q \le p < \infty$. Then $\|f\|_{\mathcal{M}^p_q} \lesssim_p \|M^\sharp f\|_{\mathcal{M}^p_q}$ for all $f \in L^0(\mathbb{R}^n)$ with $\min(Mf, 1) \in \mathcal{M}^p_q(\mathbb{R}^n)$.*

We have to be careful since f does not always satisfy (4.2) as the example of $f = \chi_F$ shows, where F is the set defined in Example 11.

Proof Although f fails (4.2), we will prove $\|f \cdot g\|_{L^1} \le 6\|M^\sharp f \cdot M^{\mathcal{D}} g\|_{L^1}$ for any (p,q)-blocks g arguing as in Theorem 165. Once this is done, we can use the boundedness of $M^{\mathcal{D}}$ on a predual space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, Theorem 385.

We may assume that $g \in L^\infty_{\rm c}(\mathbb{R}^n)$ by the truncation. We indicate the change. We use the same idea assuming that $f \in L^\infty(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$, so that $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ by assumption. The major change from Theorem 165 is the control of I_k. Note that

$$|G^k(x)| \le Mg(x) \lesssim 2^{k\theta} Mg(x)^{1-\theta} \quad (x \in \mathbb{R}^n).$$

Observe that $(Mg)^{1-\theta} \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ as long as θ is sufficiently small and note that $f \in \mathcal{M}^p_q(\mathbb{R}^n)$; see Example 143.

Thus, $\mathrm{I}_k = \mathrm{O}(2^{k\theta})$ as $k \to -\infty$. If we go through the same argument as before, we obtain the desired result.

10.2.2 Sharp maximal inequalities for local Morrey spaces

Similar to Theorem 388, we can prove the following theorem:

Theorem 390 (Sharp-maximal inequality for local Morrey spaces)**.** *Let* $0 < r < q \le p < \infty$. *Then* $\|M^{\sharp,\mathcal{D}}_{2^{-n-2}} f\|_{\mathrm{L}\mathcal{M}^p_q} + \|f\|_{\mathrm{L}\mathcal{M}^p_r} \sim \|f\|_{\mathrm{L}\mathcal{M}^p_q}$ *for all* $f \in L^0(\mathbb{R}^n)$.

Proof The proof is omitted; see Exercise 124.

10.2.3 Exercises

Exercise 124. Reexamine the proof of Theorem 388, to prove Theorem 390.

Exercise 125. Let $1 < q \le p < \infty$.

(1) Show that $\|f\|_{\mathrm{L}\mathcal{M}^p_q} \lesssim_p \|M^\sharp f\|_{\mathrm{L}\mathcal{M}^p_q}$ for $f \in L^0(\mathbb{R}^n)$ with $\min(Mf, 1) \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ by reexamining the proof of Theorem 389.

(2) Using (1) and the translation, show that $\|f\|_{\mathcal{M}^p_q} \lesssim_p \|M^\sharp f\|_{\mathcal{M}^p_q}$ for $f \in L^0(\mathbb{R}^n)$ with $\min(Mf, 1) \in \mathcal{M}^p_q(\mathbb{R}^n)$. This will reprove Theorem 389.

10.3 Fractional integral operators in Morrey spaces

One of the important properties in Morrey spaces is that we can describe the boundedness property of fractional integral operators more precisely than

Lebesgue spaces because Morrey spaces are equipped with two parameters. We will consider the boundedness properties of Morrey spaces and local Morrey spaces in Sections 10.3.1 and 10.3.2. One of the important differences between the boundedness properties of Morrey spaces and local Morrey spaces is the difference of the range of the parameter t for which the fractional integral operator I_α is bounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ and I_α is bounded from $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathrm{L}\mathcal{M}^s_t(\mathbb{R}^n)$.

10.3.1 Fractional integral operators in Morrey spaces

Recall that the Riesz potential I_α of order α and the fractional maximal operator M_α of order α are defined by

$$I_\alpha f \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|\cdot - y|^{n-\alpha}}\mathrm{d}y, \quad M_\alpha f \equiv \sup_{Q\in\mathcal{Q}} \frac{\chi_Q}{|Q|^{1-\alpha}} \int_Q |f(y)|\mathrm{d}y,$$

respectively. Define the modified fractional integral operator by

$$\tilde{I}_\alpha f \equiv \int_{\mathbb{R}^n} \left(\frac{1}{|\cdot - y|^{n-\alpha}} - \frac{\chi_{Q_0{}^c}(y)}{|x_0 - y|^{n-\alpha}} \right) f(y)\mathrm{d}y,$$

where Q_0 is a fixed cube centered at x_0.

Morrey spaces, the BMO space and Hölder–Zygmund spaces stand in a line. This fact can be described by using the (modified) fractional integral operator $\tilde{I}_\alpha$. More precisely, we formulate this as follows:

Theorem 391. *Suppose that* $1 \le q \le p < \infty$ *and* $0 < \alpha < n$.

(1) (Subcritical case) *The Adams theorem* [2]*: Let* $1 < q \le p < \dfrac{n}{\alpha}$. *Assume that the parameters* s *and* t *satisfy*

$$1 < t \le s < \infty, \quad \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}. \tag{10.1}$$

Then

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \lesssim \|f\|_{\mathcal{M}^p_q} \tag{10.2}$$

for every $f \in \mathcal{M}^p_q(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$.

(2) (Critical case) *Assume that* $1 \le q \le p = \dfrac{n}{\alpha}$. *Then*

$$\|\tilde{I}_\alpha f\|_* \lesssim \|f\|_{\mathcal{M}^p_q} \tag{10.3}$$

for every $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, *where* $\|\cdot\|_*$ *denotes the* $\mathrm{BMO}(\mathbb{R}^n)$*-norm.*

(3) (Supercritical case) *Assume that* $0 < \alpha - \dfrac{n}{p} < 1$ *and that* $1 \le q \le p < \infty$. *Then*

$$\left\|\tilde{I}_\alpha f\right\|_{\mathrm{Lip}^{\alpha-\frac{n}{p}}} \lesssim \|f\|_{\mathcal{M}^p_q} \tag{10.4}$$

for every $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

Here, we content ourselves with proving (10.3). We leave the interested readers for the proof of (10.2) and (10.4) as exercises; see Exercise 126.

Proof We have to prove $m_Q(|\tilde{I}_\alpha f - m_Q(\tilde{I}_\alpha f)|) \lesssim \|f\|_{\mathcal{M}^{\frac{n}{\alpha}}_q}$ for all cubes Q. For this purpose, we may assume that $q < \dfrac{n}{\alpha}$ because we always have $\|f\|_{\mathcal{M}^{\frac{n}{\alpha}}_q} \le \|f\|_{\mathcal{M}^{\frac{n}{\alpha}}_{\frac{n}{\alpha}}} = \|f\|_{L^{\frac{n}{\alpha}}}$ for any $f \in \mathcal{M}^{\frac{n}{\alpha}}_{\frac{n}{\alpha}}(\mathbb{R}^n) = L^{\frac{n}{\alpha}}(\mathbb{R}^n)$. We decompose f according to $2Q$ as usual. That is, we split $f = f_1 + f_2$ with $f_1 \equiv \chi_{2Q} \cdot f$ and $f_2 \equiv f - f_1$. By virtue of the triangle inequality our present task is partitioned into proving:

$$m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) \lesssim \|f\|_{\mathcal{M}^{\frac{n}{\alpha}}_q} \tag{10.5}$$

and

$$m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \lesssim \|f\|_{\mathcal{M}^{\frac{n}{\alpha}}_q}. \tag{10.6}$$

Then the estimate (10.5) for f_1 is simple. Indeed, to estimate f_1, we define an auxiliary index $w \in (q, \infty)$ by $\dfrac{1}{w} = \dfrac{1}{q} - \dfrac{\alpha}{n}$. We may replace $\tilde{I}_\alpha$ with I_α, since $f_1 \in L^1(\mathbb{R}^n)$ and we are considering $I_\alpha f_1 - m_Q(I_\alpha f_1)$. By the triangle inequality and Hölder's inequality, we have

$$m_Q(|I_\alpha f_1 - m_Q(I_\alpha f_1)|) \le 2m_Q(|I_\alpha f_1|) \le 2m^{(w)}_Q(|I_\alpha f_1|).$$

By using the $L^q(\mathbb{R}^n)$-$L^w(\mathbb{R}^n)$ boundedness of the fractional integral operator of I_α, which follows from (10.2), we obtain

$$m_Q(|I_\alpha f_1 - m_Q(I_\alpha f_1)|) \lesssim |Q|^{-\frac{1}{w}} \|I_\alpha f_1\|_{L^w} \lesssim |Q|^{\frac{n}{\alpha}} m_{2Q}(|f|^q)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}^{\frac{n}{\alpha}}_q}.$$

For the proof of the second inequality, we write the left-hand side out in full:

$$\begin{aligned} &m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \\ &= \frac{1}{|Q|^2} \int_Q \left| \iint_{Q \times (\mathbb{R}^n \setminus 2Q)} \left(\frac{f(z)}{|x-z|^{n-\alpha}} - \frac{f(z)}{|y-z|^{n-\alpha}} \right) \mathrm{d}y \mathrm{d}z \right| \mathrm{d}x. \end{aligned}$$

First, we bound the right-hand side with the triangle inequality. Then the right-hand side is majorized by

$$\frac{1}{|Q|^2} \iiint_{Q \times Q \times (\mathbb{R}^n \setminus 2Q)} |f(z)| \cdot \left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right| \mathrm{d}x \mathrm{d}y \mathrm{d}z. \tag{10.7}$$

By virtue of the mean-value theorem, we have

$$\left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right| \lesssim \frac{|x-y|}{|z-c(Q)|^{n-\alpha+1}} \lesssim \frac{\ell(Q)}{|z-c(Q)|^{n-\alpha+1}} \tag{10.8}$$

for all $x, y \in Q$ and $z \in \mathbb{R}^n \setminus 2Q$. Thus inserting (10.8) into (10.7) gives us

$$\begin{aligned} m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) &\lesssim \ell(Q) \int_{\mathbb{R}^n \setminus 2Q} \frac{|f(z)|}{|z - c(Q)|^{n-\alpha+1}} \mathrm{d}z \\ &\leq \ell(Q) \int_{\mathbb{R}^n \setminus B(c(Q),\ell(Q))} \frac{|f(z)|}{|z - c(Q)|^{n-\alpha+1}} \mathrm{d}z. \end{aligned}$$

Lemma 160 yields

$$\frac{1}{|z - c(Q)|^{n-\alpha+1}} = (n - \alpha + 1) \int_0^\infty \frac{\chi_{\mathbb{R}^n \setminus B(c(Q),\ell)}(z) d\ell}{\ell^{n-\alpha+2}}.$$

Hence, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(c(Q),\ell(Q))} \frac{|f(z)|}{|z - c(Q)|^{n-\alpha+1}} \mathrm{d}z. \\ &= (n - \alpha + 1) \int_0^\infty \left(\int_{B(c(Q),\ell) \setminus B(c(Q),\ell(Q))} \frac{|f(z)|}{\ell^{n-\alpha+2}} \mathrm{d}\ell \right) \mathrm{d}z \\ &\leq (n - \alpha + 1) \int_{\ell(Q)}^\infty \left(\int_{B(c(Q),\ell)} \frac{|f(z)|}{\ell^{n-\alpha+2}} \mathrm{d}\ell \right) \mathrm{d}z. \end{aligned}$$

Thus the integral of the right-hand side is bounded by

$$\int_{\ell(Q)}^\infty \left(\int_{B(c(Q),\ell)} |f(z)| \mathrm{d}z \right) \frac{\mathrm{d}\ell}{\ell^{n-\alpha+2}} \lesssim \|f\|_{\mathcal{M}_1^{\frac{n}{\alpha}}} \int_{\ell(Q)}^\infty \frac{d\ell}{\ell^2} \leq \frac{\|f\|_{\mathcal{M}_q^{\frac{n}{\alpha}}}}{\ell(Q)}.$$

Thus, the proof of (10.6) is complete. Hence the proof of (10.3) is concluded.

We will discuss certain inequalities for the fractional maximal operator, which is defined by

$$M_\alpha f(x) \equiv \sup_{r>0} \frac{r^\alpha}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| \mathrm{d}y.$$

In particular, when $\alpha = 0$, $M_\alpha = M$ is the (centered) Hardy–Littlewood maximal operator. Here, we summarize the boundedness property of M_α in Morrey spaces.

Corollary 392. *Let $0 \leq \alpha < n$. Let $1 < q \leq p < \dfrac{n}{\alpha}$. Assume that the parameters s and t satisfy* (10.1)*. Then M_α is bounded from $\mathcal{M}_q^p(\mathbb{R}^n)$ to $\mathcal{M}_t^s(\mathbb{R}^n)$.*

Proof If $\alpha = 0$, then we go back to Theorem 382. If $0 < \alpha < n$, then we use the Adams theorem, Theorem 391, for I_α together with the pointwise estimate $M_\alpha f \lesssim I_\alpha[|f|]$ valid for all $f \in L^0(\mathbb{R}^n)$.

We discuss the necessity of the parameter t: It must be small enough.

Proposition 393. *Let $0 < \alpha < n$, $1 \le q \le p < \infty$ and $1 \le t \le s < \infty$. If there exists a constant $C > 0$ such that $\|I_\alpha f\|_{\mathrm{W}\mathcal{M}^s_t} \le C\|f\|_{\mathcal{M}^p_q}$ for all $f \in \mathbb{M}^+(\mathbb{R}^n)$, it is necessary that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} \le \frac{q}{p}. \tag{10.9}$$

Proof We prove $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$. From Theorem 17

$$\|g(\lambda\cdot)\|_{\mathrm{W}\mathcal{M}^s_t} = \lambda^{-\frac{n}{s}}\|g\|_{\mathrm{W}\mathcal{M}^s_t},\ \|f(\lambda\cdot)\|_{\mathcal{M}^p_q} = \lambda^{-\frac{n}{p}}\|f\|_{\mathcal{M}^p_q},$$

for $\lambda > 0$, and arithmetic shows $I_\alpha[f(\lambda\cdot)] = \lambda^{-\alpha} I_\alpha f(\lambda\cdot)$ for all $f, g \in \mathbb{M}^+(\mathbb{R}^n)$. Consequently, we obtain $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$. We may assume that $q < p$ for the purpose of establishing $\dfrac{t}{s} = \dfrac{q}{p}$. Let $R > 1$ be the solution of $(1+R)^{-\frac{1}{p}} = 2^{\frac{1}{q}}(1+R)^{-\frac{1}{q}}$. Then for the set E_j in Example 9, we have $\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim (1+R)^{-j\frac{n}{p}}$.

Let R be any one of the connected components of E_j. Then

$$I_\alpha \chi_{E_j}(x) \ge I_\alpha \chi_R(x) \sim \ell(R)^\alpha \chi_R(x) = (1+R)^{-j\alpha}\chi_R(x) \quad (x \in \mathbb{R}^n).$$

Thus, $I_\alpha \chi_{E_j}(x) \gtrsim (1+R)^{-j\alpha}\chi_{E_j}(x)$. Also, from the definition of $\mathcal{M}^s_t$, we have $(1+R)^{-\frac{jnq}{pt}} = \|\chi_{E_j}\|_{L^t} \le \|\chi_{E_j}\|_{\mathcal{M}^s_t}$ from the definition of the Morrey norm. Consequently, it follows that

$$(1+R)^{-\frac{jnq}{pt}} \lesssim (1+R)^{j\alpha}\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim (1+R)^{j\left(\alpha - \frac{n}{p}\right)} = (1+R)^{-\frac{jn}{s}}.$$

Since $j \in \mathbb{N}$ is arbitrary, it follows that $t \le \dfrac{qs}{p}$.

Example 157. We work in $\mathbb{R}$. Suppose that the parameters p, s, t and α satisfy

$$1 < p < \infty, \quad 0 < \alpha < 1$$

and

$$\frac{1}{s} = \frac{1}{p} - \alpha, \quad t = \frac{s}{p}.$$

Let $E_k \subset [0,1]$ be the same set as Example 155. As before, choose $r > 0$ so that $r^{\frac{1}{p}} = 2r$. Then the fractional maximal operator M_α, given by

$$M_\alpha f(x) \equiv \sup_{R>0} \frac{1}{R^{1-\alpha}} \int_{[-R,R]} |f(x-y)|\mathrm{d}y$$

for $f \in L^0(\mathbb{R})$, satisfies

$$M_\alpha \chi_{E_k}(x) \gtrsim \sum_{l=0}^{k} r^{(k-l)\alpha}(2r)^l \chi_{E_{k-l}\setminus E_{k-l+1}}.$$

Consequently,

$$\begin{aligned}(\|M_\alpha \chi_{E_k}\|_{L^t})^t &\gtrsim \sum_{l=1}^{k} r^{(k-l)t\alpha}(2r)^{tl}(2r)^{k-l} \\ &\geq \sum_{l=1}^{k} r^{(k-l)t\alpha} r^{tl/p} r^{(k-l)/p} \\ &= k r^{kt/p}.\end{aligned}$$

since

$$\alpha + \frac{1}{tp} = \alpha + \frac{1}{s} = \frac{1}{p},\ \frac{lt}{p} - lt\alpha - \frac{l}{p} = lt\left(\frac{1}{p} - \alpha - \frac{1}{pt}\right) = lt\left(\frac{1}{p} - \alpha - \frac{1}{s}\right) = 0.$$

As before, this shows that M_α is not bounded from $\mathcal{M}^p_1(\mathbb{R})$ to $\mathcal{M}^s_t(\mathbb{R})$. Since $I_\alpha |f| \gtrsim M_\alpha f$, it follows that the Hardy–Littlewood-Sobolev inequality fails; I_α is not bounded from $\mathcal{M}^p_1(\mathbb{R})$ to $\mathcal{M}^s_t(\mathbb{R})$.

One important observation is that we did not use the Hardy–Littlewood–Sobolev theorem, which asserts I_α maps $L^u(\mathbb{R}^n)$ to $L^v(\mathbb{R}^n)$ boundedly as long as $1 < u < v < \infty$ and $\frac{1}{v} = \frac{1}{u} - \frac{\alpha}{n}$, to prove the boundedness from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$. Instead, we used the $\mathcal{M}^p_q(\mathbb{R}^n)$-boundedness of the Hardy–Littlewood maximal operator.

To clarify the terminology which we use in the sequel, we recall some terminologies. Estimate (10.2), the boundedness of fractional integral operators on the (classical) Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ was studied by Spanne, Adams [2], Chiarenza and Frasca [75] etc. Basically there are two results on the Morrey boundedness of fractional integral operators. The first result is due to Spanne, but was communicated by Peetre in [353, Theorem 5.4].

Theorem 394. *Let $1 < q \leq p < \dfrac{n}{\alpha}$. Assume that the parameters s and u satisfy*

$$1 < u \leq s < \infty,\ \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n},\ \frac{1}{u} = \frac{1}{q} - \frac{\alpha}{n}. \tag{10.10}$$

Then

$$\|I_\alpha f\|_{\mathcal{M}^s_u} \lesssim \|f\|_{\mathcal{M}^p_q} \tag{10.11}$$

for every $f \in \mathcal{M}^p_q(\mathbb{R}^n) \cap \mathbb{M}^+(\mathbb{R}^n)$.

Before the proof, let us review Theorem 391 due to Adams. In 1975, as we saw in Theorem 391, Adams [2] proved that

$$\|I_\alpha f\|_{\mathcal{M}^s_u} \lesssim \|f\|_{\mathcal{M}^p_q} \quad (f \in \mathcal{M}^p_q(\mathbb{R}^n)) \tag{10.12}$$

as long as q satisfies (10.1).

Proof Arithmetic shows that u defined by (10.10) is less than q defined by (10.1). Therefore, as was mentioned in [75, Corollary, p. 277], in view of the embedding in Theorem 19 (together with the sharpness proven in Proposition 393) (10.12) is a better estimate than (10.11).

10.3.2 Fractional integral operators in local Morrey spaces

We prove the boundedness of the fractional integral operator I_α in local Morrey spaces. To this end, we need a pointwise estimate different from the Hedberg inequality used for Morrey spaces.

We will prove the Hardy–Littlewood Sobolev theorem for local Morrey spaces. We obtain a result of Spanne-type. An outstanding difference between local Morrey spaces and Morrey spaces lies in the assumption postulated on the parameter t; compare Theorems 391 and 395.

Theorem 395. *Let $0 < \alpha < n$, $1 < q \le p < \infty$ and $1 < t \le s < \infty$ satisfy*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}. \tag{10.13}$$

Then I_α is bounded from $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathrm{L}\mathcal{M}^s_t(\mathbb{R}^n)$.

Proof Let $B = B(x, r)$ be a ball. Note that our assumption reads $\frac{1}{p} + \frac{1}{t} = \frac{1}{s} + \frac{1}{q}$. Thus, thanks to Lemma 187 and the embedding $\mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_1(\mathbb{R}^n)$,

$$\begin{aligned} r^{\frac{n}{s}-\frac{n}{t}} \|I_\alpha f\|_{L^t(B)} &\lesssim r^{\frac{n}{p}-\frac{n}{q}} \|f\|_{L^q(2B)} + r^{\frac{n}{p}-\frac{n}{q}+\frac{n}{t}} \int_r^\infty s^{\alpha-n} \|f\|_{L^1(B(x,s))} \frac{\mathrm{d}s}{s} \\ &\lesssim r^{\frac{n}{p}-\frac{n}{q}} \|f\|_{L^q(2B)} + r^{\frac{n}{p}-\frac{n}{q}+\frac{n}{t}} \int_r^\infty s^{-\frac{n}{p}+\alpha} \|f\|_{\mathrm{L}\mathcal{M}^p_1} \frac{\mathrm{d}s}{s} \\ &\lesssim \|f\|_{\mathrm{L}\mathcal{M}^p_q}. \end{aligned}$$

As a corollary of the above result, we recover the boundedness of I_α of Spanne-type.

Corollary 396. *Let $0 < \alpha < n$, $1 < q \le p < \infty$ and $1 < t \le s < \infty$ satisfy* (10.13). *Then I_α is bounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.*

Proof Simply observe that I_α commutes with translation. See Exercise 129.

Recall once again that we have the Adams theorem for Morrey spaces; see Theorem 391. An example shows that the Adams type boundedness of fractional integral operators in local Morrey spaces fails.

Example 158. Let $1 < q \le p < \infty$, $1 < t \le s < \infty$ and $0 < \alpha < n$. Assume (10.13). Let $\mathbf{e}_1 \equiv (1, 0, \dots, 0)$ be the unit vector in the x_1 direction. Let $0 < u < \infty$ and define

$$f(x) \equiv |x - 3\mathbf{e}_1|^{-\frac{n}{u}} \chi_{B(3\mathbf{e}_1, 2)}(x) \quad (x \in \mathbb{R}^n).$$

Then f belongs to $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $u < q$ according to Example 25. Since I_α maps $|\cdot|^{-n/u}$ to $c_0|\cdot|^{\alpha-n/u}$, where $c_0 \neq 0$, we see that I_α maps f to $c_0|\cdot - 3\mathbf{e}_1|^{\alpha-n/u} + g$, where g is an $L^\infty(\mathbb{R}^n)$-function. We also remark that we can use the scaling argument. Thus, I_α maps $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathrm{L}\mathcal{M}^u_v(\mathbb{R}^n)$ if and only if $u = s$ and $v \le t$.

10.3.3 Exercises

Exercise 126. Let $0 < \alpha < n$, $1 \le q \le p < \frac{n}{\alpha}$.

(1) Define $s > 1$ by

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

Show that

$$|I_\alpha f(x)| \lesssim Mf(x)^{\frac{p}{s}} (\|f\|_{\mathcal{M}^p_q})^{1-\frac{p}{s}} \tag{10.14}$$

for a.e. $x \in \mathbb{R}^n$ by mimicking the proof of Lemma 181.

(2) Prove (10.2) and (10.4).

Exercise 127. Let $0 < \alpha < n$.

(1) Show that M_α maps $\mathcal{M}^{n/\alpha}_1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

(2) Suppose that a ball Banach space $E(\mathbb{R}^n)$ is mapped by M_α to $L^\infty(\mathbb{R}^n)$. Then show that $E(\mathbb{R}^n)$ is embedded into $\mathcal{M}^{n/\alpha}_1(\mathbb{R}^n)$.

(3) Disprove that I_α maps $\mathcal{M}^{n/\alpha}_1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. Hint: Disprove that I_α maps $L^{n/\alpha}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Exercise 128. Let $1 < q \le p < \infty$ and $0 < \alpha < n$. If $\dfrac{1}{S} < \dfrac{1}{p} - \dfrac{\alpha}{n}$ and $\dfrac{T}{S} = \dfrac{q}{p}$, then disprove that I_α maps $\mathcal{M}^p_q(\mathbb{R}^n)$ to $L^T_{\mathrm{loc}}(\mathbb{R}^n)$ using the proof of Proposition 393.

Exercise 129. Let $0 < \alpha < n$.

(1) Show that $I_\alpha[f(\cdot + x_0)] = I_\alpha f(\cdot + x_0)$ for $x_0 \in \mathbb{R}^n$ and $f \in \mathbb{M}^+(\mathbb{R}^n)$.

(2) Prove Corollary 396.

Exercise 130. [390, Proposition 3.8] Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$ and $1 < r \le r_0 < \infty$. Suppose that $\frac{n}{p_0} > \alpha$, $\frac{1}{r_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{r}{r_0} = \frac{p}{p_0}$.

(1) Show that $\|I_\alpha f\|_{\mathcal{H}^{p_0'}_{p'}} \lesssim \|f\|_{\mathcal{H}^{r_0'}_{r'}}$ for all $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

(2) Using the Fatou property of $\mathcal{H}^{r_0'}_{r'}(\mathbb{R}^n)$, show that $\|I_\alpha f\|_{\mathcal{H}^{p_0'}_{p'}} \lesssim \|f\|_{\mathcal{H}^{r_0'}_{r'}}$ for all $f \in \mathcal{H}^{r_0'}_{r'}(\mathbb{R}^n)$.

10.4 Singular integral operators in Morrey spaces

One of the successful achievements in the theory of Morrey spaces is that we could nicely loosen the integrability assumption of the gradient to have the Lipschitz continuity. This fact was initially applied by Morrey. Morrey used this observation to elliptic differential equations. Later some important formulas which can be expressed in terms of singular integral operators came about. These expressions are used to consider the boundedness of the solution operator for the elliptic differential operators. Consequently, the action of singular integral operators in Morrey spaces is interesting in application. However, there is a problem. Due to the singularity of the integral kernel, it is difficult to define singular integral operators in Morrey spaces. We present some methods to overcome this problem here. In principle, Sections 10.4.1 and 10.4.2 are parallel, where we develop the idea presented above. We will present 4 methods: the use of the A_1-weighted Lebesgue spaces, the use of the duality, the use of the biduality and the use of the estimate of the integral kernel.

10.4.1 Singular integral operators in Morrey spaces

We will now show that singular integral operators T defined and investigated in Section 4.5 are bounded in Morrey spaces. Recall that singular integral operators are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and satisfy the weak-$(1,1)$ estimate. Note that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ contains $L^p(\mathbb{R}^n)$. Therefore, it is yet for T to be defined in $\mathcal{M}^p_q(\mathbb{R}^n)$. Namely, since the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is strictly larger than $L^p(\mathbb{R}^n)$, we have to define Tf carefully for $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q \le p < \infty$ and a singular integral operator T.

Here, we consider four different methods. The first but simplest one is to resort to Example 115. That is, since $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is dense in $L^q(w)$ for any $w \in A_1$, we can extend the domain of operators T up to $L^q(w)$. Then by restricting T to $L^q((M\chi_{B(1)})^\theta)$ with $1 - \frac{q}{p} < \theta < 1$, we can define T over $\mathcal{M}^p_q(\mathbb{R}^n)$ although $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$.

We use the following lemma to define the operator T on $\mathcal{M}^p_q(\mathbb{R}^n)$. In many methods we will present, the following estimate is fundamental:

Lemma 397. *Let $f \in L^0(\mathbb{R}^n)$. Then for any ball $B \equiv B(a,r)$, we have*

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)|\,\mathrm{d}y \lesssim \sum_{k=1}^{\infty} m_{2^k B}(|f|) \tag{10.1}$$

for all $x \in B$.

Inequality (10.1) corresponds to Lemma 130 for the maximal operator.

Proof By the size condition and the dyadic decomposition of $\mathbb{R}^n \setminus 2B$, we obtain

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)|\,\mathrm{d}y \lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y = \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y.$$

Observe that for $y \in 2^{k+1}B \setminus 2^k B$ and $x \in B$, we have

$$|x-y| \geq |y-a| - |x-a| > (2^k-1)r \geq 2^{k-1}r \sim |2^{k+1}B|^{1/n}.$$

Hence, for all $x \in B$, we have

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)|\mathrm{d}y \lesssim \sum_{k=1}^{\infty} m_{2^{k+1}B}(|f|),$$

as was to be shown modulo shifting the index k.

We verify that Morrey spaces fall under the scope of Lemma 397.

Corollary 398. *Let $1 \leq q \leq p < \infty$. If $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)|\,\mathrm{d}y \lesssim |B|^{-\frac{1}{p}} \|f\|_{\mathcal{M}^p_q}$$

for all balls B and $x \in B$.

Proof Simply use $m_{2^{k+1}B}(|f|) \lesssim |2^k B|^{-\frac{1}{p}} \|f\|_{\mathcal{M}^p_q}$ for each $k \in \mathbb{N}$ and Lemma 397.

We start with a simple method to define singular integral operators T on Morrey spaces. A direct consequence is that T, which is defined initially on $L^q(w)$ for any $w \in A_1$, can be restricted to a linear operator from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $L^0(\mathbb{R}^n)$. If we argue as we did for the Hardy–Littlewood maximal operator using Corollary 398, we can show that T is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$.

We move on to the second definition of singular integral operators. Keeping the above auxiliary estimate of singular integral operators above in mind, we indicate how to define singular integral operators in Morrey spaces. This is the second method; other techniques will be obtained later in Section 10.4.1.

Definition 95 (Singular integral operators in Morrrey spaces I)**.** Let T be a singular integral operator. Define T as

$$Tf(x) \equiv T[f\chi_{2B}](x) + \int_{\mathbb{R}^n \setminus 2B} K(x,y)f(y)\mathrm{d}y, \quad (x \in B) \tag{10.2}$$

for functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for which the left-hand side of (10.1) converges for every ball B.

We need to check that (10.2) makes sense despite ambiguity of the choice of the ball B containing x. We check this as follows:

Proposition 399. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that the left-hand side of* (10.1) *converges for every ball B. If B_1 and B_2 are balls and $x \in B_1 \cap B_2$, then*

$$\begin{aligned} &T[f\chi_{2B_1}](x) + \int_{\mathbb{R}^n \setminus 2B_1} K(x,y)f(y)\mathrm{d}y \\ &= T[f\chi_{2B_2}](x) + \int_{\mathbb{R}^n \setminus 2B_2} K(x,y)f(y)\mathrm{d}y. \end{aligned}$$

Proof Actually, letting B_3 be a ball such that $B_1 \cup B_2 \subset B_3$, we have, for $x \in B_1 \cap B_2$,

$$\begin{aligned} &T[f\chi_{2B_j}](x) + \int_{\mathbb{R}^n \setminus 2B_j} K(x,y)f(y)\mathrm{d}y \\ &\qquad - \left(T[f\chi_{2B_3}](x) + \int_{\mathbb{R}^n \setminus 2B_3} K(x,y)f(y)\mathrm{d}y \right) \\ &= T[f\chi_{2B_j} - f\chi_{2B_3}](x) + \int_{2B_3 \setminus 2B_j} K(x,y)f(y)\mathrm{d}y = 0. \end{aligned}$$

Therefore, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $Tf(x)$ is well defined for all $x \in \mathbb{R}^n$ since the right-hand side of (10.1) converges for every ball B. In this case we also write

$$T[f\chi_{\mathbb{R}^n \setminus 2B}](x) = \int_{\mathbb{R}^n \setminus 2B} K(x,y)f(y)\mathrm{d}y \quad (x \in B).$$

We now extend the domain of singular integral operators to Morrey spaces via another method: To do this, we use duality. We propose two different methods here. Although the structure of the dual space of $\mathcal{M}^p_q(\mathbb{R}^n)$ is complicated, the predual, whose dual space is $\mathcal{M}^p_q(\mathbb{R}^n)$, is well understood because it is included in $L^{p'}(\mathbb{R}^n)$. Note that the operation taking dual reverses inclusion: $\mathcal{X} \subset \mathcal{Y}$ implies $\mathcal{X}^* \supset \mathcal{Y}^*$.

Keep in mind that the domain of a singular integral operator T defined in Definition 57 contains $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$; $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is a subset of $L^{p'}(\mathbb{R}^n)$.

Theorem 400. *Let $1 < p \le q < \infty$. Let T be a singular integral operator. Then T, which is defined a priori on $L^p(\mathbb{R}^n)$, is bounded on $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.*

Proof We have only to prove the assertion in the block level. That is, given a block a, which is supported on a cube Q and satisfies $\|a\|_{L^q} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$, we need only show $\|Ta\|_{\mathcal{H}^{p'}_{q'}} \lesssim 1$.

Note that T is a bounded operator on $L^q(\mathbb{R}^n)$. Hence, $\chi_{2Q} \cdot Ta$ is a (p,q)-block modulo a multiplicative constant. Consequently,

$$\|\chi_{2Q} \cdot Ta\|_{\mathcal{H}^{p'}_{q'}} \lesssim 1. \tag{10.3}$$

Furthermore, by forming a dyadic decomposition away from the singularity, for all $x \in \mathbb{R}^n$, we have

$$|\chi_{\mathbb{R}^n \setminus 2Q}(x) \cdot Ta(x)| \lesssim \frac{|Q|^{1-\frac{1}{p}}\chi_{\mathbb{R}^n \setminus 2Q}(x)}{|x - c(Q)|^n} \lesssim \sum_{j=1}^{\infty} 2^{-j\left(n-\frac{n}{p}\right)}\chi_{2^jQ}(x) \cdot |2^jQ|^{-\frac{1}{p}},$$

which together with Lemma 342 and (10.3), yields $\|Ta\|_{\mathcal{H}^{p'}_{q'}} \lesssim 1$.

In (4.7), we defined the adjoint T^*. Here, we consider the "formal" adjoint T^*, since $\mathcal{M}^p_q(\mathbb{R}^n)$ does not carry the structure of a Hilbert space. Here is a precise definition.

Definition 96 (Singular integral operators in Morrrey spaces II)**.** Let $1 < q \leq p < \infty$. Given a singular integral operator T, extend it to $\mathcal{M}^p_q(\mathbb{R}^n)$ so that

$$\int_{\mathbb{R}^n} Tf(x) \cdot g(x) \mathrm{d}x = \int_{\mathbb{R}^n} f(x) \cdot T^* g(x) \mathrm{d}x \tag{10.4}$$

holds for all $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, where T^* is a formal adjoint whose kernel is given by $(x, y) \mapsto K(y, x)$.

Example 159. Although $\mathcal{M}^p_q(\mathbb{R}^n)$ is larger than the Lebesgue space $L^p(\mathbb{R}^n)$, by using the Lebesgue differentiation theorem, we can verify that the formal adjoint $R_j^* : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n)$ of $R_j : \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \to \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ in the above sense is given by $R_j^* = -R_j : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n)$.

Consequently, as we will see, $Tf \in \mathcal{M}^p_q(\mathbb{R}^n)$ is determined indirectly from Theorem 347 and (10.4).

Theorem 401. *Suppose that $1 < q \leq p < \infty$. Let T be a singular integral operator. Then T, which is initially defined in $L^p(\mathbb{R}^n)$, can be extended to a bounded linear operator on $\mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof Let $f \in L^p(\mathbb{R}^n)$. Then consider the functional

$$g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x) \cdot T^* g(x) \mathrm{d}x \in \mathbb{C}.$$

Note that this functional is bounded on $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ by Theorem 400. Therefore, thanks to Theorem 347, there exists a unique element $Tf \in \mathcal{M}^p_q(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} Tf(x) \cdot g(x) \mathrm{d}x = \int_{\mathbb{R}^n} f(x) \cdot T^* g(x) \mathrm{d}x \tag{10.5}$$

for all $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. We claim that the mapping $f \in \mathcal{M}^p_q(\mathbb{R}^n) \mapsto Tf \in \mathcal{M}^p_q(\mathbb{R}^n)$ is the desired extension.

By the Hahn–Banach theorem, Theorem 87, for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, we can find $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ with $\|g\|_{\mathcal{H}^{p'}_{q'}} \leq 2$ such that

$$\|Tf\|_{\mathcal{M}^p_q} = \int_{\mathbb{R}^n} Tf(x) \cdot g(x) \mathrm{d}x.$$

By (10.5), we obtain $\|Tf\|_{\mathcal{M}^p_q} \leq \|f\|_{\mathcal{M}^p_q} \|T^* g\|_{\mathcal{H}^{p'}_{q'}}$. Now that T^* is bounded on $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, we see, taking into account that g has unit norm, that $\|Tf\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}$, which is the desired result.

There is another way to define singular integral operators via duality. Let $1 < q \leq p < \infty$ as before. Suppose that we have a linear operator $S : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ such that $\|Sf\|_{\mathcal{M}^p_q} \leq C\|f\|_{\mathcal{M}^p_q}$ for all $f \in L^p(\mathbb{R}^n)$. Then thanks to Theorem 356 by density we can extend S to a bounded linear

operator $T : \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \to \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ thanks to Proposition 321. By duality we have $T^* : \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \to \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. Again by duality we have $T^{**} : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n)$. We claim that T^{**} is the extension of T.

Lemma 402. *We have* $T^{**}|\tilde{\mathcal{M}}^p_q(\mathbb{R}^n) = T|\tilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Proof Let Q be a cube. Then for all $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$,

$$\begin{aligned}\int_Q T^{**}f(x)\mathrm{d}x &= \int_{\mathbb{R}^n} T^{**}f(x)\chi_Q(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)T^*\chi_Q(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} Tf(x)\chi_Q(x)\mathrm{d}x.\end{aligned}$$

Consequently, for $S = T^{**}$, we have

$$\int_Q Sf(x)\mathrm{d}x = \int_Q Tf(x)\mathrm{d}x. \tag{10.6}$$

Thus, by the Lebesgue differentiation theorem, we obtain $Tf = T^{**}f$.

Note that Definitions 95 and 96 coincide since (10.6) holds with S replaced by the operator in Definition 96.

We give another definition of singular integral operators in Morrey spaces.

Definition 97 (Singular integral operators in Morrrey spaces III)**.** Let $1 < q \le p < \infty$. Suppose that we have a linear operator $S : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ such that $\|Sf\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}$ for all $f \in L^p(\mathbb{R}^n)$. Then the *natural extension* $T : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n)$ of S is defined as S^{**} given as above.

Example 160. Let $1 < q \le p < \infty$. In view of the Lebesgue differentiation theorem, we see that the formal adjoint of $R_j : \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \to \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is given by the same formula as the one for Lebesgue spaces:

$$R_j^* f(x) = -\int_{\mathbb{R}^n} \frac{(x_j - y_j)f(y)}{|x-y|^{n+1}}\mathrm{d}y \quad (x \in \mathbb{R}^n \setminus \mathrm{supp}(f))$$

for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

We define T to be T^{**} on $\mathcal{M}^p_q(\mathbb{R}^n)$. Let us see that the kernel expression is still valid.

Proposition 403. *For all* $f \in \mathcal{M}^p_q(\mathbb{R}^n) \cap L^0_{\mathrm{c}}(\mathbb{R}^n)$*, we have* (4.1).

Proof Let $x \in \mathbb{R}^n \setminus \mathrm{supp}(f)$. Let $B(y,r)$ be a ball that never intersects $\mathrm{supp}(f)$. Then for all $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ supported on $B(y,r)$,

$$\int_{\mathbb{R}^n} Tf(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)T^*g(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\left(\int_{\mathbb{R}^n} K(y,x)g(y)\mathrm{d}y\right)\mathrm{d}x.$$

Since the kernel of T^* is $(x,y)\mapsto K(y,x)$, we obtain

$$T^*g(x)=\int_{\mathbb{R}^n}K(y,x)g(y)\mathrm{d}y$$

for almost all $x\in\mathbb{R}^n\setminus\mathrm{supp}(g)$. Thus,

$$\int_{\mathbb{R}^n}Tf(x)g(x)\mathrm{d}x=\int_{\mathbb{R}^n}g(y)\left(\int_{\mathbb{R}^n}K(x,y)f(x)\mathrm{d}x\right)\mathrm{d}y.$$

Consequently, since g is arbitrary, (4.1) follows.

Remark that Definitions 96 and 97 coincide because the adjoint operator $T:\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)\to\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ coincides with the restriction of T^* from $L^{p'}(\mathbb{R}^n)$ to $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Here, we list some examples and a counterexample for which we can or cannot extend the domain of operators.

Example 161. Let $1<q\le p<\infty$. Let T be a linear operator defined in $L^p(\mathbb{R}^n)$. Assume that there exists a constant $c_0>0$ such that, for all $f\in L^1_{\mathrm{c}}(\mathbb{R}^n)$,

$$|Tf(x)|\le c_0\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\mathrm{d}y,\quad x\notin\mathrm{supp}(f),\tag{10.7}$$

where $\Omega\in L^0(\mathbb{R}^n)$ is homogeneous of degree zero, in the sense that $\Omega(t\cdot)=\Omega$ for all $t>0$ and $\Omega\in L^{\tilde{q}}(S^{n-1})$ for some $\tilde{q}\in[q',\infty]$. Assume also that T is $L^q(\mathbb{R}^n)$-bounded, in the sense that

$$\|Tf\|_{L^q}\le C_q\|f\|_{L^q}\tag{10.8}$$

for all $f\in L^p(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$. Then $\|Tf\|_{\mathcal{M}^p_q}\lesssim\|f\|_{\mathcal{M}^p_q}$ for all $f\in L^p(\mathbb{R}^n)$. To prove this, we have only to show that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\|T[\chi_{3Q}f]\chi_Q\|_{L^q}+|Q|^{\frac{1}{p}-\frac{1}{q}}\|T[f-\chi_{3Q}f]\chi_Q\|_{L^q}\lesssim\|f\|_{\mathcal{M}^p_q}.$$

The first estimate follows readily from (10.8). Meanwhile, for almost all $x\in Q$, we deduce from (10.7)

$$\begin{aligned}
&|Tf(x)|\\
&\lesssim\sum_{l=1}^{\infty}\frac{1}{|B(c(Q),2^l\ell(Q))|}\int_{B(c(Q),2^l\ell(Q))}|\Omega(x-y)|\,|f(y)|\mathrm{d}y\\
&\lesssim\sum_{l=1}^{\infty}\frac{1}{|B(2^{l+n}\ell(Q))|}\left(\int_{B(2^l\ell(Q))}|\Omega(y)|^{\tilde{q}}\mathrm{d}y\right)^{\frac{1}{\tilde{q}}}\left(\int_{B(c(Q),2^l\ell(Q))}|f(y)|^{\tilde{q}'}\mathrm{d}y\right)^{\frac{1}{\tilde{q}'}}\\
&\lesssim\sum_{l=1}^{\infty}\left(\frac{1}{|B(c(Q),2^{l+n}\ell(Q))|}\int_{B(c(Q),2^l\ell(Q))}|f(y)|^{\tilde{q}'}\mathrm{d}y\right)^{\frac{1}{\tilde{q}'}}
\end{aligned}$$

by Hölder's inequality. Since $\tilde{q}' \geq q$, if we integrate this inequality we obtain

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\|T[f-\chi_{3Q}f]\chi_Q\|_{L^q} \lesssim \|f\|_{\mathcal{M}^p_{\tilde{q}'}} \leq \|f\|_{\mathcal{M}^p_q}.$$

Thus, as a result we obtain the desired result.

Example 162. Let T be a sublinear operator defined in $L^\infty_{\rm c}(\mathbb{R}^n)$. That is, the operator T satisfies that, for all $f, g \in L^1_{\rm c}(\mathbb{R}^n)$, $k \in \mathbb{C}$ and for a.e. $x \in \mathbb{R}^n$, $|T[f+g](x)| \leq |Tf(x)|+|Tg(x)|$ and $|T[kf](x)| = |k||Tf(x)|$. We also assume that

$$|Tf(x) - Tg(x)| \lesssim |T(f-g)(x)| \tag{10.9}$$

Assume that there exists a constant $c_0 > 0$ such that, for all $f \in L^\infty_{\rm c}(\mathbb{R}^n)$, (10.7) holds, where $\Omega \in L^0(\mathbb{R}^n)$ is homogeneous of degree zero and $\Omega \in L^{\tilde{p}}(S^{n-1})$ for some $\tilde{p} \in [1,\infty]$. Assume also that T is $L^q(\mathbb{R}^n)$-bounded, in the sense that $\|Tf\|_{L^p} \leq C_p\|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. Then $\|Tf\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}$ for all $f \in L^1_{\rm c}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap L^0_{\rm c}(\mathbb{R}^n)$.

Example 163. Let $1 \leq q < p < \infty$. Then the operator $Tf \equiv e^{i\|f\|_{L^q}}Mf$ does not satisfy (10.9) and cannot be extended to Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$.

One important observation is that we used the $L^q(\mathbb{R}^n)$-boundedness of singular integral operators to prove the $\mathcal{M}^p_q(\mathbb{R}^n)$-boundedness. So one of the advantages of using Morrey spaces is that Morrey spaces can describe the boundedness property more precisely than Lebesgue spaces.

10.4.2 Singular integral operators in local Morrey spaces

We now define singular integral operators in local Morrey spaces. To this end, it is helpful to note the following fact:

Proposition 404. *Let $1 \leq q \leq p < \infty$. Then $L^p(\mathbb{R}^n)$ is dense in $\mathrm{L}\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.*

Proof Simply mimic the proof of Proposition 327; see Exercise 131.

Lemma 405. *Let $1 < q \leq p < \infty$, and let T be a singular integral operator. Then $\|Tf\|_{\mathrm{L}\mathcal{M}^p_q} \lesssim \|f\|_{\mathrm{L}\mathcal{M}^p_q}$ for all $f \in L^p(\mathbb{R}^n)$.*

Proof Reexamine the proof of Theorem 401, where Lemma 397 is used; see Exercise 132.

We can take the same strategy as the definition of singular integral operators acting on Morrey spaces. What is useful in Theorem 406 is that we can define singular integral operators on the function space $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ much larger than $\mathcal{M}^p_q(\mathbb{R}^n)$.

Theorem 406. *Let $1 < q \leq p < \infty$. Then any singular integral operator T, defined initially on $L^p(\mathbb{R}^n)$, extends to a bounded linear operator on $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.*

Proof Mimic Definition 97 as before. See Exercise 133.

An important conclusion of Theorem 406 is that the boundedness of singular integral operators acting on Morrey spaces can be "decomposable" in the sense that a counterpart to local Morrey spaces together with the translation readily gives the boundedness on Morrey spaces. This is a good contrast to the fractional integral operators. Actually, the Adams-type boundedness is no longer available from the boundedness of I_α on local Morrey spaces as Example 158 shows.

10.4.3 Exercises

Exercise 131. Mimic the proof of Proposition 327 to prove Proposition 404.

Exercise 132. Use Lemma 397 to prove Lemma 405.

Exercise 133. Prove Theorem 406 based on Definition 97.

Exercise 134. Let $1 < q \le p < \infty$. Suppose that we have a linear operator $T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ satisfying $\|T\|_{L^q \to L^q} = \sup\{\|Tf\|_{L^q} : \|f\|_{L^q} = 1\} < \infty$. Assume in addition that there exists $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|K(x,y)| \lesssim |x-y|^{-n}$$

and that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \mathrm{d}y \quad (\text{a.e. } x \notin \operatorname{supp}(f))$$

for all $f \in L^q_{\mathrm{c}}(\mathbb{R}^n) \equiv L^q(\mathbb{R}^n) \cap L^0_{\mathrm{c}}(\mathbb{R}^n)$. Then show that

$$\|Tf\|_{\mathcal{M}^p_q} \lesssim_{p,q} (1 + \|T\|_{L^q \to L^q}) \|f\|_{\mathcal{M}^p_q} \quad (f \in \mathcal{M}^p_q(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)).$$

10.5 Commutators in Morrey spaces

Section 10.5 investigates the boundedness of commutators generated by singular integral operators and $\mathrm{BMO}(\mathbb{R}^n)$-functions. Recall first that $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$, if b satisfies;

$$\|b\|_* \equiv \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q |b(x) - m_Q(b)| \, \mathrm{d}x < \infty.$$

Let T be a singular integral operator in Definition 57. In view of the John–Nirenberg inequality, for any $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, we can define $[T,b]f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$[T,b]f(x) \equiv T[b \cdot f](x) - b(x) Tf(x). \tag{10.1}$$

Note that $b \cdot f \in L^q_{\mathrm{c}}(\mathbb{R}^n)$ for all $q \in (1,\infty)$. Hence, the definition (10.1) above makes sense. We define and investigate commutators acting on Morrey spaces and local Morrey spaces in Sections 10.5.1 and 10.5.2, respecitvely.

10.5.1 Commutators in Morrey spaces

As a continuation of Section 5.2.1, we consider the commutator $[a,T]$ generated by $a \in \mathrm{BMO}(\mathbb{R}^n)$ and a singular integral operator T:

$$[a,T]f(x) \equiv \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} (a(x)-a(y))K(x,y)f(y)\mathrm{d}y \quad (x \in \mathbb{R}^n). \tag{10.2}$$

Keeping in mind the proof of the boundedness of singular integral operators, we start with the boundedness of commutators on predual spaces.

Proposition 407. *Let $1 < q \leq p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a singular integral operator. Then, the commutator $[a,T]$, which is defined initially on $L^p(\mathbb{R}^n)$, is $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$-bounded if it is restricted to $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.*

Proof Mimic the proof of Theorem 400.

As we did in the case of singular integral operators, we define the commutator $[a,T]$ on Morrey spaces via duality.

Definition 98. Let $1 < q \leq p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a singular integral operator. Then, the *commutator* $[a,T] : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n)$ is defined by way of the duality $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$-$\mathcal{M}^p_q(\mathbb{R}^n)$; for $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, $[a,T]f \in \mathcal{M}^p_q(\mathbb{R}^n)$ is a unique element $h \in \mathcal{M}^p_q(\mathbb{R}^n)$ such that for all $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} h(x)g(x)\mathrm{d}x = -\int_{\mathbb{R}^n} f(x)[a,T]g(x)\mathrm{d}x.$$

Arguing as in Theorem 401, we automatically obtain the boundedness of commutators.

Theorem 408. *Let $1 < q \leq p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a singular integral operator. Then, the commutator $[a,T]$ is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$.*

Theorem 236 also carries over to Morrey spaces.

Theorem 409. *Let $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < p < \infty$ and $a \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the following conditions*:

(1) *$K(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for all $x \in \mathbb{R}^n$;*

(2) *$K(x,tz) = t^{-n}K(x,z)$ for every $t > 0$ and $(x,z) \in \mathbb{R}^n \times S^{n-1}$;*

(3) *$\displaystyle\int_{S^{n-1}} K(x,z)\mathrm{d}\sigma(z) = 0$ for all $x \in \mathbb{R}^n$;*

(4) $\displaystyle\max_{|\alpha| \leq 2n} \left\| \frac{\partial^\alpha K}{\partial z^\alpha} \right\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} < \infty.$

For any $\varepsilon > 0$, we set

$$[a,T]_\varepsilon f(x) \equiv \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,x-y)(a(x)-a(y))f(y)\mathrm{d}y \quad (f \in \mathcal{M}^p_q(\mathbb{R}^n)).$$

Then for any $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, there exist $[a,T]f \in \mathcal{M}^p_q(\mathbb{R}^n)$ such that $\|[a,T]f\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}$.

Proof This is a general fact on singular integral operators together with the expansion formula used in the proof of Theorem 236. In fact, since $q > 1$, we see that the limit

$$\begin{aligned}[a,T]_\varepsilon f(x) &= [a,T]_\varepsilon[\chi_{B(2R)}f](x) + [a,T]_\varepsilon[\chi_{\mathbb{R}^n\setminus B(2R)}f](x)\\ &= [a,T]_\varepsilon[\chi_{B(2R)}f](x) + [a,T][\chi_{\mathbb{R}^n\setminus B(2R)}f](x)\end{aligned}$$

exists as $\varepsilon \downarrow 0$ for almost all $x \in B(R)$ if $R > 0$. Since $R > 0$ is arbitrary, we obtain the desired result.

10.5.2 Commutators in local Morrey spaces

As we did in the case of commutator operators acting on Morrey spaces, we define the commutator $[a,T]$ on local Morrey spaces via duality.

Definition 99. Let $1 < q \le p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a singular integral operator. Then, the *commutator* $[a,T] : \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n) \to \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ is defined by way of the duality $\mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$-$\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$; for $f \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$, $[a,T]f \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ is a unique element $h \in \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} h(x)g(x)\mathrm{d}x = -\int_{\mathbb{R}^n} f(x)[a,T]g(x)\mathrm{d}x$$

for all $g \in \mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

We can also take the same strategy of Morrey spaces and the related spaces when we consider the boundedness property of commutators in local Morrey spaces.

Proposition 410. *Let $1 < q \le p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a singular integral operator. Then, the commutator $[a,T]$, which is defined initially on $L^p(\mathbb{R}^n)$, is $\mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$-bounded if it is restricted to $\mathrm{L}\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.*

Proof The proof is left as an exercise; see Exercise 135.

Arguing as in Theorem 401, we automatically obtain the following conclusion:

Theorem 411. *Let $1 < q \le p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a singular integral operator. Then, the commutator $[a,T]$ is bounded on $\mathrm{L}\mathcal{M}^p_q(\mathbb{R}^n)$.*

10.5.3 Exercises

Exercise 135. Prove Proposition 410 by mimicking the proof of Theorem 400.

Exercise 136. Prove Theorem 401 by mimicking the proof of Theorem 408.

10.6 Notes

Section 10.1

See the survey [362].

General remarks and textbooks in Section 10.1

See the textbook [6, Chapter 6] for the boundedness property of the Hardy–Littlewood maximal operator, where we can find two different proofs; one is the same as our proof of Theorem 382 and the other hinges on Proposition 285 and Theorem 290.

Section 10.1.1

Chiarenza and Frasca showed that the Hardy–Littlewood maximal operator is weakly and strongly bounded on the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ [75, Theorem 1(1)] and [75, Theorem 1(2)] when $1 \leq q \leq p$ and when $1 < q \leq p < \infty$, respectively. See [75, Theorem 1] for Theorem 382 as well as the survey paper [391, Theorem 3.1]. The proof hinges upon the Stein inequality for the Lebesgue measure Corollary 272 and Proposition 271. Remark 15, concerning the weak-boundedness of M acting on Morrey spaces, comes from [381].

Theorem 145 is from [394, Theorem 2.2] and [428, Lemma 2.5]. Example 155, asserting that M fails to be $\mathcal{M}^p_1(\mathbb{R}^n)$-bounded, comes from [328, Corollary 2.5]. The proof used here comes from [304, Lemma 4.6(i) and Lemma 4.7]. See [194] for the action of Hardy operators on Morrey spaces.

See [202, Lemma 3.5] for the sharp maximal inequality as well as the application to singular integral operators.

Di Fazio and Ragusa considered the boundedness of the sharp maximal operators in [123]. See [247, Theorem 12], [283, Lemma 2.5] and [395, Theorems 1.3 and 1.4] as well. See [383] for weak Morrey spaces.

Section 10.1.2

See [22] for the case of Carleson curves.

Section 10.2

General remarks and textbooks in Section 10.2

It seems that there is no textbook which specilizes in sharp maximal inequality on Morrey spaces.

Section 10.2.1

Di Fazio and Ragusa initially considered the sharp maximal inequality for Morrey spaces in connection with elliptic differential equations [123]. See [247, 333, 395, 440, 441] for more.

Section 10.2.2

See [169, Theorem 4.2] for the sharp maximal inequality for local Morrey spaces.

Section 10.3

See the textbook [6, Chapter 7], where we can also find an application of closed subspaces and preduals to the Morrey-boundedness of fractional integral operators. See also the survey [362].

We refer to [6, Lemma 9.1] for the critical case of Theorem 391. In the supercritical case, we considered the difference of the kernel and the constant (average). We can consider the difference of the kernel and the Taylor polynomial; see [282] together with its application to fractional Laplace equations.

General remarks and textbooks in Section 10.3

See standard textbooks [112], [147], [231], [293] and [319, Chapter 2].

Section 10.3.1

The boundedness of fractional integral operators on the (classical) Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ was studied by Spanne, Adams [2], Chiarenza and Frasca [75] etc. See [2] for the subcritical case of Theorem 391. See [3, Theorem 4.1 and Remark 4.1] for the critical case of Theorem 391. See [10] for the supercritical case of Theorem 391. We refer to [402, Theorem 1.1] for the case of bounded domains, where Serrin gave an explicit bound for the operator norms. We followed the survey paper [391, Theorem 3.3] for the proof of Theorem 391. Olsen showed that we can no longer improve the Adams theorem for the Morrey boundedness of fractional integral operators. See [342, Theorem 10] for Proposition 393 with $n = 3$. His proof works for general dimensions. See also the survey paper [391, Proposition 2.2].

Adams used the fractional maximal operator M_α to investigate the boundedness of the fractional integral operator I_α [2, Proposition 3.1 and Theorem 3.1]. Adams employed a technique obtained by Hedberg in 1972; see [2, p. 768]. Hedberg pointed out that

$$I_\alpha f \lesssim M f^{\frac{p}{s}} \|f\|_{L^p}^{1-\frac{p}{s}} \quad (f \in \mathbb{M}^+(\mathbb{R}^n)) \tag{10.1}$$

in [190, (3)]. Chiarenza and Frasca improved the method; in fact Chiarenza and Frasca refined (10.1) to:

$$I_\alpha f \lesssim M f^{\frac{p}{s}} (\|f\|_{\mathcal{M}^p_q})^{1-\frac{p}{s}} \quad (f \in \mathbb{M}^+(\mathbb{R}^n))$$

in [75, p. 277]. Chiarenza and Frasca reproved the weak/strong boundedness of I_α, which was initially obtained in [2, Theorems 5.1 and 3.1], respectively; see [75, Theorem 2].

Let $1 < q \le p < \infty$, $1 < t \le s < \infty$ satisfy $\frac{t}{s} = \frac{q}{p}$ and $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$. We can extend result (10.12) to the vector-valued setting:

$$\left\| \left(\sum_{j=1}^\infty |I_\alpha f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^s_u} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q} \quad (\{f_j\}_{j=1}^\infty \in \mathcal{M}^p_q(\ell^r)).$$

We can assume that $1 \le r \le \infty$, due to the positivity of the integral kernel; see [379, Theorem 2] for this technique. Note that the vector-valued inequality of the Hardy–Littlewood maximal operator fails for $r = 1$ [416].

Adams used the fractional maximal operator M_α to investigate the boundedness of the fractional integral operator I_α [2, Proposition 3.1 and Theorem 3.1]. Adams employed a technique obtained by Hedberg in 1972. More precisely, he obtained an estimate in Lemma 183 and its generalization with $\alpha_1 = 0$; see [2, p. 768]. Hedberg pointed out Lemma 181 in [190, (3)]. It seems that the method was improved by Chiarenza and Frasca; in fact Chiarenza and Frasca pointed out (10.14) in [75, p. 277]. Chiarenza and Frasca reproved the weak/strong boundedness of I_α, which was initially obtained in [2, Theorems 5.1 and 3.1], respectively; see [75, Theorem 2].

We can pass to this result (10.12) to the vector valued setting;

$$\left\| \left(\sum_{j=1}^\infty I_\alpha f_j{}^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^s_t} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}.$$

It counts that we can assume that $1 \le r \le \infty$ due to the positivity of the integral kernel; see [379, Theorem 2] for this technique.

Komori-Furuya and Sato gave an example showing that the Adams type boundedness of fractional integral operators in local Morrey spaces fails. See [253] for Example 158.

Section 10.3.2

Fu, Li and Lu [133] and Li and Yang [278] proved the boundedness of fractional integral operators in Herz spaces. See also [306].

Section 10.4

General remarks and textbooks in Section 10.4

We refer to the textbook [6, Chapter 8], where Adams used Proposition 285 as well as Theorem 209. See the survey [264] for applications of Morrey spaces to Navier–Stokes equations, where singular integral operators come into play. The strategy taken to define singular integrals via duality is due to Rosenthal and Schmeisser [370].

Section 10.4.1

Let us start to recall the results on singular integral operators acting on Morrey spaces. Peetre considered the boundedness of singular integral operators in [353]; see Theorem 401. There are several methods of the definition of singular integral operators but the definition given here is close to that in [327, (3.1)] and [198, §3]. There is another definition by means of the predual [396, p. 483]. We refer to [438, Proposition 2.25] for the boundedness of singular integral operators. In particular, the boundedness of the Riesz transform is discussed in [438, Corollary 2.27]; see [371, 438] for more recent information. The boundedness of singular integrals is also proven there [75, Theorem 3], where Chiarenza and Frasca employed the A_1-weight technique for the boundedness.

We move on to recall the results of singular integral operators acting on related function spaces. Komori-Furuya investigated the boundedness of singular integral operators acting on block spaces in [245, Theorem]; see Theorem 400. See [372] for the concrete case of singular integral operators; they investigated the Cauchy integral in the predual of Morrey spaces.

Finally, we consider operators related to singular integral operators. Fu and Lu calculated the operator norm in Morrey spaces of the operators given by (7.1) and (7.2) [134, Theorem 1.3]. Fu and Lu also characterized a condition under which U_ψ^b is bounded in Morrey spaces for all $b \in \mathrm{BMO}$ in terms of ψ [134, Theorem 1.6]. A similar result for V_ψ^b is obtained in [134, Theorem 1.6]. A passage to the higher order commutator is done in [134, Theorem 3.1].

See [286, Theorem 1.3] for the boundendness of the Bochner–Riesz operator.

The Schrödinger operator generates singular integral operators; see [75]. Kurata and Sugano investigated such operators acting on Morrey spaces in [259, 260]; see [285, Theorem 1] for its subsequent result and [285, Theorem 2] for a passage to commutators.

Chen and Jin obtained the boundedness of Marcinkiewicz integrals associated with the Schrödinger operator on Morrey spaces [67].

Ho dealt with singular integral operators with rough kernel [197].

We can consider "more singular" integral operators in Morrey spaces as well. The boundedness of the Riesz transform with respect to elliptic differential operators with non-smooth coefficients can be found in [153, 388]. Nakai and Sobukawa constructed an example showing that the $L^p(\mathbb{R}^n)$-boundedness is not enough to extend the operators to Morrey spaces [330, Remark 5]; see Example 163.

There are not so many works dealing with the convolution as an extension of Young's inequality. See the paper by Peetre [352].

Section 10.4.2

Hu, Lu and Yang obtained the boundedness of singular integral operators on Herz spaces [200]. Later on, Balakishiyev, Burenkov, Guliyev, Gurbuz, Tararykova and Serbetchi considered singular integral operators in local Morrey spaces [23, 48, 47]. Deringoz, Guliyev and Samko essentially considered the boundedness of singular integral operators in generalized local Morrey spaces [99, Theorem 5.5]. See also [132, 301] as well.

Section 10.5

See the survey [362]. Many authors investigated the boundedness property of commutators on Morrey spaces; see [11, 97, 123, 395].

General remarks and textbooks in Section 10.5

See the textbook [6, Chapter 12] for the boundedness property of commutators handled in this book.

Section 10.5.1

Many authors investigated the boundedness property of commutators given by (10.2); see [102, 252, 317, 318, 389, 395] for more details.

Di Fazio and Ragusa considered commutators generated by BMO functions and singular integral operators. See [123, Theorem 2.1] for Theorems 408 and 409. Shirai considered commutators generated by BMO functions and the Riesz potential [408]. See [469] for the application of Theorem 409 to parabolic differential equations. Zhang, Jiang, Sheng and Zhou discussed the uniqueness and existence of solution.

There are many generalizations. A passage to higher orders is done by Pen and Liu in [351]. See [201, 286] for commutators generated by the Bochner–Riesz operator and Lipschitz functions.

The compactness of commutators is also investigated. See [389] for the sufficient conditions and [71, 72, 73, 74] for the necessary and sufficient conditions. See [20, 341, 343, 430] for more generalization of [71, 72, 73, 74]. Concerning the commutator of the form $[b, T]$, based on the result by Janso

and Paluszynski [219, 350], Shi and Lu considered the case when b is a Lipschitz function or a function in Morrey–Campanato spaces [406, 407]. We can consider commutators generated by BMO and the maximal operators; see [97, 456].

Section 10.5.2

Yu and Tao obtained the boundedness of commutators in local Morrey spaces [468]. See [98, Theorem 3.1] for the boundedness of commutators in Morrey-type spaces. Guliyev, Rahimova and Omarova considered the boundedness of commutators generated by the Littlewood–Paley functions and BMO [171, Theorem 1–3]. Guliyev, Omarova, Ragusa and Scapellato worked in the setting of generalized Morrey spaces. [169, Theorem 4.7]. The work by Hasanov, Muradova and Guliyev [188] and the one by Eroglu, Hasanov Omarova [113] considered the commutators in the multilinear and generalized setting. See also [25, 96, 165, 166, 170, 403].

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