Spectral Geometry of Partial Differential Operators

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Michael Ruzhansky
Makhmud Sadybekov
Durvudkhan Suragan

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Preface

This book is an attempt to collect a number of properties emerging in recent research describing certain features of the theory of partial differential equations that can be attributed to the field of spectral geometry. Both being vast fields, *our attempt is not to give a comprehensive account of the whole theory, but to provide the reader with a quick introduction to a number of its important aspects.*

The topic of spectral geometry is a broad research area appearing in different mathematical subjects. As such, it allows one to compare spectral information associated with various objects over different domains with selected geometric properties. For example, when the area of the domain is fixed, one often talks of the isoperimetric inequalities in such context. The purpose of this book is to highlight one direction of such research aimed at the understanding of spectral properties of partial differential operators as well as of the related integral operators.

An indispensable language of the area is that of functional analysis and, aimed also at the student readership, we give a basic introduction to the theory. In general, the functional analysis can be viewed as a powerful collection of mathematical tools allowing one to obtain significant generalisations of various effects detected in the investigation of one concrete problem. At present, this is a vast mathematical field with numerous investigations and excellent monographs readily available.

Thus, in the first chapters of this book we give a brief account of the basics of functional analysis aiming at consequent applications in the theory of differential equations. Chapter 1 deals with the basic notions of function spaces, Chapter 2 is devoted to the foundation of the theory of linear operators, and Chapter 3 discusses the basics of the spectral theory of differential operators. These are aimed to provide the reader with a quick introduction to the subject. As there are many detailed and comprehensive monographs already available, we omit many proofs of basic results that can be easily found in a variety of sources. From this point of view, this book is only a “guidebook” indicating the main directions and necessary basic facts. The general course of the functional analysis contains a large number of various definitions and facts. It is clear that it is impossible to cover and understand all these concepts even briefly. Therefore, in the present book we introduce only those concepts which are necessary (but, of course, not sufficient) for a beginner wanting to do research on spectral problems for the differential operators.

We introduce only the main concepts of the functional analysis, and only those on which we will lean in the further exposition. This presentation is therefore far from being complete, and there are many other concepts and ideas widely used in the theory. We also do not dwell on detailed justifications of introduced concepts and their general properties, as the proof of those (standard) facts goes beyond the scope of the present book. The choice of the exposition objects is stipulated only by opinions of the authors and their own experience in using the introduced concepts of the functional analysis for analysing concrete problems appearing in the theory of differential equations.

Surely, the functional analysis contains much more general concepts and has numerous important methods successfully applied to a wide range of mathematical problems. In this work we dwell only on the illustration of some concrete concepts by means of the simplest examples related to the subject of differential operators. Our first goal here is to provide a *simplest exposition of the used concepts* to move on to the spectral geometry questions.

Thus, in our exposition we also provide the reader with a collection of concrete examples of the simplest operators. Our goal is to demonstrate, on one side, advantages appearing in using the general methods for solving concrete problems and, on the other side, the fact that these methods are not complicated and very soon lead to a number of concrete applications.

In Chapter 4 we review another important ingredient often playing a crucial role in the subject of the spectral geometry of differential and integral operators: the symmetric decreasing rearrangement. This is a basic tool to allow one to compare integral expressions over different domains provided the functions under the integral are also rearranged in an appropriate way. Since this is a well-known subject already treated in much detail in many excellent books, we touch upon it only briefly, emphasising different applications of such methods, and preparing the scenery for the results in the following chapter.

Finally, in Chapter 5 our exposition culminates in the core subject of this monograph: geometric spectral inequalities for a collection of most important differential and integral operators. Here is where the background material presented in previous chapters comes into play, to allow one to compare spectral information for various operators over different domains. In particular, we treat in detail the logarithmic, Riesz and Bessel potential operators and the corresponding boundary value problems, also extending the analysis to the Riesz operators in spherical and hyperbolic geometries.

Subsequently, we concentrate on several cases of non-selfadjoint operators, the case that is much less understood. Here we discuss different versions of the isoperimetric inequalities for the singular numbers, for the heat operators of different types: higher-order heat operators, as well as the heat operators with the Cauchy-Dirichlet, Cauchy-Robin, Cauchy-Neumann and Cauchy-Dirichlet-Neumann boundary conditions. Part of the presentation in this chapter is based on the authors' recent research in the area.

It is our pleasure to thank Junqing Huang for help in producing the picture for the cover of the book in Mathematica, representing the ball with respect to the Carnot-Carathéodory distance on the Heisenberg group.

Michael Ruzhansky
London (UK) & Ghent (Belgium)
Makhmud Sadybekov
Almaty (Kazakhstan)
Durvudkhan Suragan
Astana (Kazakhstan)

Authors

Michael Ruzhansky is a Senior Full Professor of Mathematics at Ghent University, Belgium, and a Professor of Mathematics at the Queen Mary University of London, United Kindgdom. He is currently also an Honorary Professor of Pure Mathematics at Imperial College London, where he has been working from 2000–2018. His research is devoted to different topics in the analysis of partial differential equations, harmonic and non-harmonic analysis, spectral theory, microlocal analysis, as well as the operator theory and functional inequalities on groups. His research was recognised by the ISAAC Award 2007, Daiwa Adrian Prize 2010, as well as by the Ferran Sunyer I Balaguer Prizes in 2014 and 2018.

Makhmud Sadybekov is a Kazakhstani mathematician who graduated from the Kazakh State University (Almaty, Kazakhstan) in 1985 and received his doctorate in physical-mathematical sciences in 1993. He is a specialist in the field of Ordinary Differential Equations, Partial Differential Equations, Equations of Mathematical Physics, Functional Analysis, and Operators Theory. Currently, he is Director General at the Institute of Mathematics and Mathematical Modeling in Almaty, Kazakhstan.

Durvudkhan Suragan an Associate Professor at Nazarbayev University. He won the Ferran Sunyer i Balaguer Prize in 2018. He has previously worked in spectral geometry, and in the theory of subelliptic inequalities at Imperial College London as a research associate and as a leading researcher in the Institute of Mathematics and Mathematical Modeling.

Chapter 1

Functional spaces

This chapter contains a brief and basic introduction to the part of the functional analysis dealing with function spaces. This chapter, as well as the next two, are aimed to serve as a "guidebook" indicating the main directions and necessary facts. As there is a large variety of sources available, and as the material of the first three chapters can be readily found in numerous monographs with much detail, we avoid giving technical proofs but restrict ourselves here to giving an exposition of main ideas, concepts, and their main properties.

Thus, as a rule, main theorems are introduced without proofs, which can be found in the extensive mathematical literature. Here, our goal is the explanation of introduced concepts and properties in concrete simple examples.

There are numerous excellent books containing very detailed and rigorous exposition of the material of the first two or three chapters on the basics of the functional analysis. We can only mention a few, such as the books by Reed and Simon [93, 94], Davies [32], Gohberg and Kreĭn [44], Lax [72], and many others. For a more informal introduction to basic and more advanced analysis and measure theory we can also recommend [106], with the additional emphasis put on the Fourier analysis aspects of the operator theory.

However, one distinguishing feature of our presentation is the particular emphasis put on many examples related to the theory of differential equations.

The following conventions will apply to the material throughout the whole book. New terms appearing in the text are ***in italics and in bold***. For the convenience of the reader, the logical completion of a separate idea, justification of an approval, discussions of theorems, lemmas and remarks, consideration of an example, proofs, etc., are denoted by the symbol □.

1.1 Normed spaces

We start with the concept of a ***linear space*** (or a ***vector space***) which is the basic notion of the (linear) functional analysis. A collection X of elements is called a linear space if any linear combination of them still belongs to X. The rigorous definition of this concepts looks like:

Definition 1.1 *A* ***linear space*** *over the field* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C})$, *or a* ***vector space***, *is a nonempty set* $X \neq \emptyset$, *equipped with two fixed operations:*

(1) addition of set elements,

(2) multiplication of set elements by a scalar,

such that the following properties hold true:

- *X is a group with respect to the addition, that is:*

$$x + y = y + x \quad \forall x, y \in X;$$

$$(x + y) + z = x + (y + z) \quad \forall x, y, z \in X;$$
$$\exists 0 \in X: \quad x + 0 = x \quad \forall x \in X;$$
$$\forall x \in X \quad \exists(-x) \in X: \quad x + (-x) = 0;$$

- *and also axioms for the scalar multiplication are satisfied:*

$$(\alpha\beta)x = \alpha(\beta x), \quad \alpha(x + y) = \alpha x + \alpha y, \quad 1 \cdot x = x, \quad (\alpha + \beta)x = \alpha x + \beta x.$$

A rather general concept of spaces appearing in the functional analysis are linear (vector) ***topological spaces***. These spaces are linear spaces X over a field of complex numbers $\mathbb{C}$ (or real numbers $\mathbb{R}$) which are at the same time also topological spaces, that is, the linear operations from Definition 1.1 are continuous in the topology of the space.

A more particular, but very important setting appears when one can introduce a norm (length) of vectors in the linear space X, with properties mimicking the length properties of vectors in the standard Euclidean space. Namely, ***the norm*** of the element $x \in X$ is a real number $\|x\|$ such that we always have

$$\|x\| \geq 0, \text{ and } \|x\| = 0 \text{ if and only if } x = 0;$$

$$\|\lambda x\| = |\lambda| \cdot \|x\|, \ \forall \lambda \in \mathbb{C}, \ \forall x \in X,$$

and "*the triangle inequality*" is satisfied:

$$\|x + y\| \leq \|x\| + \|y\|.$$

A linear space equipped with the norm introduced on it is called a ***normed space***.

The ***convergence*** in X can be introduced as

$$x_n \to x, \text{ if } \|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Suppose we have two norms $\|x\|^{(1)}$ and $\|x\|^{(2)}$ in a normed space X. Then the norms $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ are called ***equivalent***, if there exist numbers $\alpha > 0$, $\beta > 0$ such that for all elements $x \in X$ we have

$$\alpha\|x\|^{(1)} \leq \|x\|^{(2)} \leq \beta\|x\|^{(1)}.$$

It follows that two norms in a linear space are equivalent if and only if each of them is subordinated to another. Thus, if for a linear space X two equivalent norms are given, and we denote by X_1 and X_2 the corresponding normed spaces, then any sequence converging in one of these spaces also converges in another, moreover, to the same limit. This fact allows us to choose one of the equivalent norms, which may be more convenient to work with in the linear space X.

In the case when the considered space X is finite-dimensional, it turns out that any choice of a norm leads to an equivalent normed space. More precisely: *In a finite-dimensional linear space all the norms are equivalent.*

Example 1.2 (Euclidean space $\mathbb{R}^n$) Let $\mathbb{E}^n$ be a linear space consisting of n-dimensional vectors $x = (x_1, x_2, \ldots, x_n)$, with $x_k \in \mathbb{R}$ for all $k = 1, \ldots, n$. If in $\mathbb{E}^n$ we introduce one of the following norms:

$$\|x\|_\infty := \max_{1 \le k \le n} |x_k| \quad \text{or} \quad \|x\|_p := \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}, \ 1 \le p < \infty, \tag{1.1}$$

then the obtained normed space is called the ***Euclidean space*** $\mathbb{R}^n$. Checking the axioms of the norm is straightforward. Here, the triangle inequality for the second type of the p-norms is a consequence of the well-known ***Minkowski inequality*** for finite sums:

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p \right)^{1/p} \le \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{1/p}. \tag{1.2}$$

If the "coordinates" of a vector are complex numbers, then the linear space consisting of complex columns $x = (x_1, x_2, \ldots, x_n)$ with the norm (1.1) (where for $a \in \mathbb{C}$, $|a| = \sqrt{(\mathrm{Re}(a))^2 + (\mathrm{Im}(a))^2}$), is a normed space, denoted by $\mathbb{C}^n$. $\square$

A point $x_0 \in X$ is called a ***limit point*** (or a limiting point) of the set $M \subset X$, if any neighbourhood of the point x_0 has at least one point of the set M, different from x_0. In other words, x_0 is a limit point of M, if in any ***ball*** $B_r(x_0) = \{x \in X : \|x - x_0\| < r\}$ there always exists some element $x \in M$, $x \ne x_0$. A necessary and sufficient condition for the point $x_0 \in X$ to be a limit point of the set $M \subset X$ is the existence of a sequence $\{x_k\} \subset M$ converging to x_0, where also $x_k \ne x_0$, $k = 1, 2, \ldots$.

Let $M \subset X$, and let M' be the set of the limit points of M. Then the set

$$\overline{M} = M \bigcup M'$$

is called ***the closure*** of the set M. In other words, $\overline{M}$ is the smallest set containing M and all of its limit points. The set M, for which we have $\overline{M} = M$, is called ***closed***. In other words, a set is closed if it contains all of its limit points.

A set $\widetilde{X}$ in a linear space X is called ***a linear subspace***, if for any $x, y \in \widetilde{X}$ and for any numbers α, β (from $\mathbb{K}$), their linear combination satisfies $\alpha x + \beta y \in \widetilde{X}$. Note that since $\widetilde{X}$ is a subset of the linear space X, it follows that $\widetilde{X}$ is also a linear space.

We should pay attention to the fact that, generally speaking, such a linear space $\widetilde{X}$ need not be closed with respect to the norm of the normed space X.

A linear subspace $\widetilde{X}$ of the normed space X $(\widetilde{X} \subset X)$ is called ***dense*** in X, if for any $x \in X$ and any $\varepsilon > 0$ there exists an element $\widetilde{x} \in \widetilde{X}$ such that $\|x - \widetilde{x}\| < \varepsilon$. Thus, if $\widetilde{X}$ is dense in X, then for any $x \in X$ there exists a sequence $\{x_k\} \subset \widetilde{X}$ such that $x_k \to x$ as $k \to \infty$.

Comparing the definitions of the closure and the density, we see that the assertion "$\widetilde{X}$ is dense in X", $\widetilde{X} \subset X$, $\widetilde{X} \neq X$, means that the closure of the linear subspace $\widetilde{X}$ with respect to the norm of X coincides with X. Then one also says that the space X is the ***completion*** of the linear subspace $\widetilde{X}$ with respect to the norm X. Each linear normed space X has the completion and this completion is unique up to an *isometric* (i.e. norm preserving) mapping, mapping X into itself.

Similar to linear subspaces, a general subset $\widetilde{X}$ of the topological space X is called dense (in X) if every point $x \in X$ either belongs to $\widetilde{X}$ or is a limit point of $\widetilde{X}$. That is, for every point in X, the point is either in $\widetilde{X}$ or it is arbitrarily "close" to some element in $\widetilde{X}$. For instance, every real number is either a rational number or has one arbitrarily close to it. Thus, the set of rational numbers is dense in the space of real numbers.

A set $\widetilde{X}$ is called ***dense everywhere*** (in X) if it is dense in X. It can be readily seen that a set $\widetilde{X}$ is dense in X if and only if its closure $\widetilde{X}$ contains X, that is, $\overline{\widetilde{X}} \supset X$. In particular, $\widetilde{X}$ is dense everywhere in X if $\overline{\widetilde{X}} = X$.

One of the central questions of the spectral theory is the property of completeness of the system of eigenfunctions (sometimes complemented by the so-called associated functions) in the linear space under consideration. In many cases the proof of the completeness of a system $\{u_k\}$ in the space X is based on the density everywhere in X of the linear subspace ***spanned*** by the vectors $\{u_k\}$, that is, of the set of all linear combinations of the vectors $\{u_k\}$.

As a visual demonstration of how "frequent" or "rare" the elements must be in order for their linear span to be dense in the space under consideration, we mention the following theorem being, generally speaking, a generalisation of the Weierstrass theorem on polynomial approximations of continuous functions.

Theorem 1.3 (Muntz) *Let $n_0 = 0$, $n_1 < n_2 < \ldots \in \mathbb{R}$. A linear span of power functions $\{x^{n_k}\}_{k=0}^{\infty}$ is dense in $C[a,b]$, $b > a \geq 0$, if and only if the series*

$$\sum_{k=1}^{\infty} \frac{1}{n_k} = +\infty$$

diverges.

Thus, for example, the linear span of the system of the power functions $\{x^n\}_{k=0}^{\infty}$ is dense in $C[a,b]$, while the linear span of the functions $\left\{x^{n^2}\right\}_{k=0}^{\infty}$ is not dense in $C[a,b]$.

The following lemma is useful for understanding the completeness of normed spaces. For this, recall that ***a subsequence*** of the sequence $\{x_k\}_{k=1}^{\infty}$ is a subset $\{x_{k_j}\}$

such that $k_{j+1} > k_j,\ j = 1, 2, \ldots$, that is, in $\{x_{k_j}\}$ the sequential order of elements of $\{x_k\}_{k=1}^{\infty}$ is preserved.

A ***Cauchy sequence*** is a sequence whose elements become arbitrarily close to each other as the sequence progresses. More precisely, given any positive $\varepsilon > 0$, all but a finite number of elements of the sequence are at distance $< \varepsilon$ from each other: there exists $N > 0$ such that for all $k, m > N$ we have $\|x_k - x_m\| < \varepsilon$.

Lemma 1.4 (On convergence of sequences) *Let X be a normed (not necessarily complete) linear space. For any sequence $\{x_k\}_{k=1}^{\infty}$, $x_k \in X$, the following statements are equivalent:*

1) the sequence $\{x_k\}_{k=1}^{\infty}$ converges;

2) any subsequence $\{x_{k_j}\}$ of the sequence $\{x_k\}_{k=1}^{\infty}$ converges;

3) the sequence $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence and any subsequence $\{x_{k_j}\}$ converges;

4) the sequence $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence and it has some converging subsequence $\{x_{k_j}\}$;

5) the series $\sum\limits_{k=1}^{\infty} (x_{k+1} - x_k)$ converges.

It is usual to define a ***complete normed space*** X by requiring the property that every Cauchy sequence of points in X converges to some element of X.

1.2 Hilbert spaces

In a large number of problems one deals with a more particular case when in the linear space X one can introduce an inner product which is a generalisation of the ordinary inner product in the Euclidean space. Namely, the ***inner product*** of the elements $x, y \in X$ is a complex number denoted by $\langle x, y \rangle$, such that

- we always have $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;
- $\langle x, x \rangle = \overline{\langle x, x \rangle}$;
- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for any numbers $\alpha, \beta \in \mathbb{C}$.

A real number $\sqrt{\langle x, x \rangle}$ satisfies all axioms of the norm and, therefore, can be chosen as a norm of the element x:

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

Such space is called a ***pre-Hilbert*** space. For developing rich functional analysis it is important for spaces under consideration to be ***complete*** (that is, any Cauchy sequence of elements of the space converges to some element of this space; in other words, from the fact that $\|x_n - x_m\| \to 0$ as $n,m \to \infty$ for $x_m, x_n \in X$, it follows that the limit $\lim\limits_{n\to\infty} x_n = x$ exists and that $x \in X$).

Complete linear normed and complete pre-Hilbert spaces are called ***Banach*** and ***Hilbert*** spaces, respectively. For non-complete spaces, the well-known completion procedure of a metric space (analogous to the transition from rational numbers to real ones) in the case of the linear normed (pre-Hilbert) space leads to the Banach (Hilbert) space, respectively.

If in a linear space the norm is generated by an inner product $\|x\| = \sqrt{\langle x,x\rangle}$, then the ***parallelogram law*** is valid:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.3}$$

The ordinary Euclidean space is one of the simplest examples of the (real) Hilbert space. The space of complex vectors $\mathbb{C}^n$ is also a Hilbert space, with the inner product defined by the formula

$$\langle x,x\rangle := \sum_{k=1}^{n} x_k \overline{y_k}, \quad \forall x,y: \; x = (x_1,x_2,...,x_n) \in \mathbb{C}^n, \; y = (y_1,y_2,...,y_n) \in \mathbb{C}^n.$$

However, the infinite dimensional spaces, that is, the spaces having an infinite number of linearly independent vectors, play the main role in the functional analysis. In the next section we recall some examples of such spaces.

1.3 Examples of basic functional spaces

Thus, in this section we recall several examples of the most commonly encountered functional spaces.

Example 1.5 Consider the Banach space $C[a,b]$, the space of all continuous complex-valued (i.e. with values in $\mathbb{C}$) functions $f$on the closed interval $[a,b]$, with the norm

$$\|f\|_\infty := \max_{x\in[a,b]} |f(x)|.$$

It is well-known from any general course of real analysis that the convergence in the space $C[a,b]$ with respect to this norm is the uniform convergence of functions.

Example 1.6 Consider the Banach space $C^k[a,b]$ consisting of all complex-valued functions f which are k-times continuously differentiable on the closed interval $[a,b]$, with the norm

$$\|f\|_{C^k} := \max_{x\in[a,b]} \left(|f(x)| + |f'(x)| + \cdots + |f^{(k)}(x)|\right),$$

where $f^{(k)}(x)$ is the derivative of the function f of order k. The convergence of the sequence $\{f_j\} \subset C^k[a,b]$ is the ***uniform convergence*** on $[a,b]$ of the sequences $\{f_j^{(i)}\}$, $i = 0, 1, \ldots, k$.

Example 1.7 Consider the Banach space $L^p(a,b)$ $(1 \le p < \infty)$ of all (measurable) functions on (a,b) with the integrable p-th power, with the finite norm

$$\|f\|_p := \left(\int_a^b |f(x)|^p dx \right)^{1/p}.$$

The convergence of a sequence in the norm of the space $L^1(a,b)$ is also called the ***convergence in mean***, and the convergence in the norm of the space $L^2(a,b)$ is sometimes called the ***mean-square convergence***.

Example 1.8 Consider the Banach space ℓ^p $(1 \le p < \infty)$ of all sequences $\{x_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}} |x_k|^p < \infty$, with the norm

$$\|x\|_p := \left(\sum_{k\in\mathbb{Z}} |x_k|^p \right)^{1/p},$$

where $\mathbb{Z}$ is the set of integers.

Example 1.9 In the case $p = 2$ the spaces ℓ^2 and $L^2(a,b)$ are Hilbert spaces. For example, in $L^2(a,b)$ the inner product is defined by

$$\langle f, g \rangle := \int_a^b f(x)\overline{g(x)}dx.$$

It is easy to see that the spaces ℓ^p and $L^p(a,b)$ are not Hilbert spaces for $p \neq 2$, since the parallelogram identity (1.3) is not satisfied for their norms. All these spaces are *infinite dimensional*. It is most easily seen for ℓ^p: it is clear that the set consisting of linearly independent vectors

$$e_j = (\underbrace{0, ..., 0}_{j-1}, 1, 0, ...)$$

is countable.

1.4 The concept of Lebesgue integral

Here and in the sequel, all integrals are understood in the Lebesgue sense. Only in this case the spaces introduced above will be complete. But how to understand the Lebesgue integral? The general theory answers this question quite accurately and in depth by using the notion of the Lebesgue measure. For beginners the following simple understanding of such an integral is sufficient.

The set $M \subset [a,b]$ has ***measure zero***, if for every $\varepsilon > 0$ there exists a finite or countable collection of the intervals $[\alpha_n, \beta_n]$, such that $M \subset \bigcup_n [\alpha_n, \beta_n]$, and

$$\sum_n (\beta_n - \alpha_n) < \varepsilon.$$

If for the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ there exists a limit equal to f everywhere on $[a,b]$ with a possible exception of a set of measure zero, then we say that f_n converges to f ***almost everywhere*** on $[a,b]$, and this is written as

$$\lim_{n\to\infty} f_n(x) \overset{\text{a.e.}}{=} f(x).$$

The function f is called ***Lebesgue integrable*** on $[a,b]$, if there exists a Cauchy sequence with respect to the norm $\|f\|_{L^1} := \int_a^b |f(x)|dx$ of the functions $\{f_n\}_{n\in\mathbb{N}}$, continuous on the closed interval $[a,b]$, such that

$$\lim_{n\to\infty} f_n(x) \overset{\text{a.e.}}{=} f(x)$$

exists. Here the integral in the definition of the norm is meant in the usual Riemannian sense as the integral of the continuous function. Then the number

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

is called the ***Lebesgue integral*** of the function f over the interval $[a,b]$.

Thus, in Example 1.7 the elements of the function space $L^1(a,b)$ are functions for which the Lebesgue integral $\int_a^b |f(x)|dx < \infty$ is finite, and the elements of the space $L^p(a,b)$ are measurable functions $f(x)$, for which the Lebesgue integral $\int_a^b |f(x)|^p dx < \infty$ is finite.

1.5 Lebesgue spaces

We will use the following result from the functional analysis.

Theorem 1.10 *Any normed space X can be considered as a linear space which is dense in some Banach space $\widetilde{X}$. Then $\widetilde{X}$ is called the* **completion of the space** X.

By $L^p(a,b)$ $(1 \leq p < \infty)$ we denote the Banach space of functions, obtained by the completion of continuous functions on $[a,b]$, with respect to the norm

$$\|f\|_{L^p} \equiv \|f\|_p := \left(\int_a^b |f(x)|^p dx \right)^{1/p}.$$

Thus, the limits (with respect to the norm $\|\cdot\|_p$) of the Cauchy sequences of continuous functions on $[a,b]$ are the elements of the space $L^p(a,b)$.

For $p=2$ the space becomes a Hilbert space with the inner product

$$\langle f,g\rangle := \int_a^b f(x)\overline{g(x)}dx. \tag{1.4}$$

Remark 1.11 Let us consider a more general case of a measure space (X,μ), which is a set X with the measure μ such that $\mu > 0$, μ is countably additive, $\mathrm{Dom}(\mu)$ is a σ-algebra subordinate to X, such that μ is complete in the following sense: if $E \in \mathrm{Dom}(\mu)$ and $\mu(E)=0$, then $\forall E' \subseteq E$ we have $E' \in \mathrm{Dom}(\mu)$. Then the Lebesgue space $L^p(X,\mu)$ is defined as the space

$$\left\{[f] \;\middle|\; f: X \to \mathbb{C},\ \ \mathrm{Re}(f)\ \text{ and }\ \mathrm{Im}(f)\ \text{ are measurable },\ \int_X |f|^p d\mu < \infty\right\},$$

where $[f]$ is the equivalence class of functions coinciding with f almost everywhere.

Thus, since in the Lebesgue integration the sets of measure zero can be neglected, the elements of the space $L^p(X,\mu)$ are the classes of equivalent functions $[f]$ differing from each other on the sets of measure zero (that is, coinciding almost everywhere).

If now as a space X we choose the set of all integers $\mathbb{Z}$, and the measure is defined as

$$\mu(E) \stackrel{def}{=} cardE,\ \ Dom(\mu) \stackrel{def}{=} \{E \subset \mathbb{Z}\},$$

then the space $L^p(X,\mu)$ in this case will coincide with the space ℓ^p. Indeed, $f:\mathbb{Z}\to\mathbb{R}$ is measurable if the integral

$$\int_{\mathbb{Z}} |f|d\mu = \sum_{n\in\mathbb{Z}} \int_{\{n\}} |f|d\mu = \sum_{n\in\mathbb{Z}} |f(n)|d\mu(\{n\}) = \sum_{n\in\mathbb{Z}} |f(n)| < \infty$$

is finite. Therefore, we have that

$$f \in L^p(X,\mu) \Longleftrightarrow \sum_{n\in\mathbb{Z}} |f(n)|^p < \infty \Longleftrightarrow \{f(n)\}_{n\in\mathbb{Z}} \in \ell^p.$$

That is, the space ℓ^p is a particular case of L^p.

This remark illustrates the initial depth of the general theory of the Lebesgue integral and of the Lebesgue spaces with the general norm. However, in the majority of applications a more simple Lebesgue measure allowing one to understand the definition of the integral in the sense introduced in Section 1.4 is applied for dealing with problems related to the differential equations.

Further, for simplicity, we will assume all the considered domains $\Omega \subset \mathbb{R}^n$ to be bounded. The set of all functions measurable in Ω, whose modulus to the p-th degree is integrable over Ω, will be denoted by $L^p(\Omega)$, $1 \le p < \infty$. This set with the norm

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx\right)^{1/p} \tag{1.5}$$

is a Banach space. Here the integral is meant in the Lebesgue sense. Moreover, we identify the functions different from each other on the set of measure zero. That is, the functions coinciding with each other almost everywhere will be assumed to be the same element of the space $L^p(\Omega)$.

By $L^\infty(\Omega)$ we denote the Banach space of measurable functions with the norm based on the essential supremum

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| \equiv \inf\big\{a>0:\ \mu\{|f(x)|>a\}=0\big\}.$$

As in the one-dimensional case, the space $L^p(\Omega)$ can be obtained as the completion of functions continuous in $\overline{\Omega}$ with respect to the norm (1.5). And this completion can be carried out in a constructive way.

We now briefly review the averaging construction due to S. L. Sobolev. The function

$$\omega_1(t) := \begin{cases} c_n \exp\left\{-\dfrac{1}{1-t^2}\right\}, & |t|<1, \\ 0, & |t|\geq 1, \end{cases}$$

is called an *"averaging kernel"*, with the constant c_n depending on the dimension of the space $\mathbb{R}^n$ and computed from the condition

$$\int_{\mathbb{R}^n} \omega_1(|x|)dx = 1.$$

Let $h>0$. The function

$$\omega_h(|x|) := \frac{1}{h^d}\omega_1(|x|/h), \ \ x\in\mathbb{R}^n,$$

is also often called an *"averaging kernel"*. One can readily check that this kernel has the following properties:

- $\omega_h \in C^\infty(\mathbb{R}^n)$ and $\omega_h(|x|)\geq 0$ in $\mathbb{R}^n$;
- $\omega_h(|x|)\equiv 0$ for $|x|\geq h$;
- $\int_{\mathbb{R}^n}\omega_h(|x|)dx = 1$.

For a measurable function f, for any $h>0$, the function

$$f_h(x) := \int_\Omega f(\xi)\omega_h(|x-\xi|)d\xi, \ \ x\in\mathbb{R}^n,$$

is called an ***averaged function*** for the function f. It readily follows that $f_h\in C^\infty(\mathbb{R}^n)$.

In the theory of Sobolev spaces the following result has an important value.

Theorem 1.12 *If $f\in L^p(\Omega)$, then $f_h(x)\to f(x)$ in $L^p(\Omega)$ as $h\to 0$.*

Consequently, as in the one-dimensional case, the space $L^p(\Omega)$ can be obtained as the completion of continuous functions in $\overline{\Omega}$, with respect to the norm (1.5).

Example 1.13 The set $C_0^\infty(\Omega)$ of smooth compactly supported functions in Ω is everywhere dense in $L^p(\Omega)$.

For the proof, first of all, we define the distance function to the boundary of the domain $\partial\Omega$: for each point $x \in \overline{\Omega}$ we set

$$r(x) := \min_{y \in \partial\Omega} |x - y|,$$

which is well-defined since Ω is bounded. For any $\delta > 0$, by Ω_δ we denote the set of the points $\{x \in \Omega : \ r(x) > \delta\}$.

Now, for each $f \in L^p(\Omega)$ we define the corresponding family of functions with compact support in Ω by

$$f^\delta(x) := \begin{cases} f(x), & x \in \Omega_\delta, \\ 0, & x \notin \Omega_\delta. \end{cases}$$

It is clear that for any small $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f - f^\delta\|_{L^p(\Omega)} < \varepsilon/2$. For small $h < \delta/2$, the averaged function f_h^δ will be a function with compact support in Ω. That is, $f_h^\delta \in C_0^\infty(\Omega)$. In view of Theorem 1.12 for small enough $h < \delta/2$, we will have $\|f_h^\delta - f^\delta\|_{L^p(\Omega)} < \varepsilon/2$.

Thus, for each $f \in L^p(\Omega)$, for any small $\varepsilon > 0$ there exists a differentiable function $f_h^\delta \in C_0^\infty(\Omega)$ with compact support in Ω such that $\|f_h^\delta - f\|_{L^p(\Omega)} < \varepsilon$. That is, $C_0^\infty(\Omega)$ is everywhere dense in $L^p(\Omega)$.

1.6 Sobolev spaces

In this section we recall the notion of Sobolev spaces.

Example 1.14 For $1 \le p < \infty$ and $k \in \mathbb{N}$, we denote by $L_k^p(a,b)$ the Banach space of functions obtained by the completion of the set of k times continuously differentiable functions on $[a,b]$, with respect to the norm

$$\|f\| = \left(\int_a^b \left[|f(x)|^p + |f^{(k)}(x)|^p \right] dx \right)^{1/p}.$$

For $p = 2$, this space becomes a Hilbert space with the inner product

$$\langle f, g \rangle = \int_a^b \left[f(x)\overline{g(x)} + f^{(k)}(x)\overline{g^{(k)}}(x) \right] dx.$$

In fact, while here the Lebesgue integral exists everywhere and it is not immediately clear what will happen after such a completion, these spaces do not contain too "extravagant" functions. Thus, for example, elements composing the space $L_1^2(a,b)$

can be identified with absolutely continuous functions f on $[a,b]$ (see Remark 1.15) having an ordinary derivative $f'(x)$ almost everywhere on $[a,b]$, for which the Lebesgue integral

$$\int_a^b |f'(x)|^2\, dx < \infty$$

is finite.

Here we must note that if the space dimension of variables is greater than one, that is, $x \in \mathbb{R}^n$ for $n > 1$, then this (absolute continuity) is not always true. Moreover, the functions from the Sobolev space can be discontinuous. For example, in a two-dimensional domain $\Omega = \{x \in \mathbb{R}^2 : \ |x| < 1/2\}$ the function $f(x) = \ln\big|\ln|x|\big|$ belongs to the space $L_1^2(\Omega)$, but it has a discontinuity of the second kind at the point x. □

Remark 1.15 Since in the sequel we will often use the Sobolev spaces, let us discuss in more detail the concept of absolutely continuous functions.

The function f is called ***absolutely continuous*** on the interval $[a,b]$, if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that we have

$$\sum_{k=1}^{m} |f(x_k^{'}) - f(x_k^{''})| < \varepsilon$$

for any set of intervals $[x_k^{'}, x_k^{''}] \subseteq [a,b]$ of the total length less than δ:

$$\sum_{k=1}^{m} |x_k^{'} - x_k^{''}| < \delta.$$

If a function f is absolutely continuous on $[a,b]$, then it is differentiable almost everywhere and $f' \in L^1(a,b)$. An inverse statement is also true: if $g \in L^1(a,b)$, then the function $G(x) := \int\limits_a^x g(t)dt$ is absolutely continuous on $[a,b]$, and almost everywhere on this interval we have $G'(x) = g(x)$. □

To consider the Sobolev space in multi-dimensional domains it is necessary to explain the concept of a generalised derivative. Thus, let f be a function defined on the set $\Omega \subset \mathbb{R}^n$.

If f has a partial derivative $f_{x_i} \equiv \partial_{x_i} f$ continuous at Ω, then for any $g \in C_0^1(\Omega)$ we have the equality

$$\int_\Omega f(x) g_{x_i}(x) dx = -\int_\Omega f_{x_i}(x) g(x) dx. \tag{1.6}$$

Moreover, this equality completely determines the derivative f_{x_i} of the function f. Indeed, if for the function $f \in C^1(\Omega)$ there exists a function $\varphi \in C(\Omega)$ such that for any $g \in C_0^1(\Omega)$ we have

$$\int_\Omega f(x) g_{x_i}(x) dx = -\int_\Omega \varphi(x) g(x) dx,$$

then comparing it with (1.6), we get

$$\int_{\Omega} [f_{x_i}(x) - \varphi(x)]\, g(x)dx = 0. \tag{1.7}$$

Since $C_0^{\infty}(\Omega)$ is everywhere dense in $L^1(\Omega)$ (see Example 1.13), and $C_0^{\infty}(\Omega) \subset C_0^1(\Omega)$, then $C_0^1(\Omega)$ is also everywhere dense in $L^1(\Omega)$. Consequently, (1.7) implies that $f_{x_i}(x) = \varphi(x)$ for all $x \in [a,b]$, since the functions are continuous.

If now in Eq. (1.6) we abandon the continuity of the involved functions, and instead require their integrability, then we arrive at the concept of a generalised derivative introduced by S. L. Sobolev.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a multi-index with nonnegative integer components, and we denote $|\alpha| := \alpha_1 + \cdots + \alpha_n$. For a function $g \in C^{|\alpha|}(\Omega)$, by D^{α} we denote the classical derivative

$$D^{\alpha} g(x) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} g(x).$$

The function, which (by analogue with the classical derivative) is denoted as $D^{\alpha} f \in L^1(\Omega)$, is called ***a generalised derivative of order*** α of the function $f \in L^1(\Omega)$, if the equality

$$\int_{\Omega} D^{\alpha} f(x) g(x) dx = (-1)^{|\alpha|} \int_{\Omega} f(x) D^{\alpha} g(x) dx \tag{1.8}$$

holds for all $g \in C_0^{|\alpha|}(\Omega)$.

As in the case with the continuous functions, it is easy to show that equality (1.8) uniquely defines the generalised derivative (if it exists).

This definition of the generalised derivative is essentially the same as the definition of the derivative of a generalised function. Moreover, the definition above is a particular case of the situation when both the function f and its derivative $D^{\alpha} f$ are *regular* generalised functions.

Example 1.16 Let us compare the classical derivative, the generalised derivative, and the derivative in the sense of generalised functions in an example. In the unit ball $\Omega = \{x \in \mathbb{R}^n : \, |x| < 1\}$, consider the function $u(x) = |x_1|$. It is clear that the classical derivative with respect to the variable x_1 does not exist at $x_1 = 0$.

At the same time u has the first generalised derivative with respect to any variable: $u_{x_1} = \operatorname{sgn} x_1$, and $u_{x_k} = 0$ for $k = 2, \dots, n$. In turn, the function $\operatorname{sgn} x_1$ does not have the generalised derivative with respect to the variable x_1, but its generalised derivatives with respect to other variables are all equal to zero.

If $u(x) = |x_1|$ is considered as a generalised function, then it has all derivatives of any order. In particular, its second derivative with respect to the variable x_1 gives the Dirac delta function: $u_{x_1 x_1} = 2\delta(x_1)$. As is well-known, the Dirac delta function is not a regular generalised function. Therefore, it cannot be taken as the generalised derivative of the function $u(x) = |x_1|$.

We must also note that unlike the classical derivative, the generalised derivative $D^{\alpha} f$ is defined directly for the order α, without the assumption of the existence of corresponding lower-order derivatives. For example, for the function $u(x) = \operatorname{sgn} x_1 + \operatorname{sgn} x_2$ the generalised derivatives of first order u_{x_1} and u_{x_2} in Ω do not exist. At the same time, the second mixed generalised derivative does exist: $u_{x_1x_2} = 0$. □

Let us now move directly to the definition of the Sobolev space. The set of functions $f \in L^p(\Omega)$, whose generalised derivatives up to k-th order inclusively belong to the space $L^p(\Omega)$, form the ***Sobolev space*** $L^p_k(\Omega)$. Here, similar to the definition of the space $L^p(\Omega)$, the functions from $L^p_k(\Omega)$ different on the set of measure zero, are identified with each other. For $p = 2$, the notation $H^k(\Omega)$ is also used.

The space $L^p_k(\Omega)$, $1 \leq p < \infty$, is a Banach space with the norm

$$\|f\|_{L^p_k(\Omega)} := \left(\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha f(x)|^p dx\right)^{1/p} \equiv \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p}, \tag{1.9}$$

and the space $H^k(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{H^k(\Omega)} := \int_\Omega \sum_{|\alpha|\leq k} D^\alpha f(x)\overline{D^\alpha g(x)}dx \equiv \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}.$$

For $L^p_k(\Omega)$ an analogue of Example 1.13 holds:

Theorem 1.17 *Let $\partial\Omega \in C^k$ for $k \geq 1$. Then the set $C^\infty\left(\overline{\Omega}\right)$ is everywhere dense in $L^p_k(\Omega)$.*

A function from $L^p_k(\Omega)$ for any p and k is defined up to an arbitrary set of measure zero. It follows that each function from $L^p_k(\Omega)$ can be arbitrarily changed on any set of measure zero still being the same element of this space. Since the boundary $\partial\Omega$ has measure zero, a question arises: *how can we understand the value of the functions from $L^p_k(\Omega)$ on $\partial\Omega$?* The concept of a trace answers this question.

Without getting into too much detail, we introduce the general scheme of the concept of the trace of functions from $H^1(\Omega)$. First, we assume that $f \in C^1\left(\overline{\Omega}\right)$. For such functions one can justify the inequality

$$\int_{\partial\Omega} |f(s)|^2 dS \leq C\|f\|^2_{H^1(\Omega)}, \tag{1.10}$$

with the constant C independent of f.

Now let $f \in H^1(\Omega)$ with $H^1(\Omega) = L^2_1(\Omega)$ being the Sobolev space of order 1 over $L^2(\Omega)$. In view of Theorem 1.17, there exists a sequence $f_1, f_2, ...$ of the functions from $C^1\left(\overline{\Omega}\right)$, converging to f in the norm of $H^1(\Omega)$. From (1.10), for the elements of this sequence we have

$$\|f_k - f_j\|^2_{L^2(\partial\Omega)} \leq C\|f_k - f_j\|^2_{H^1(\Omega)} \to 0, \text{ as } k, j \to \infty.$$

This means that the sequence of values $f_k\big|_{\partial\Omega}$ of the functions f_k on the surface $\partial\Omega$ is the Cauchy sequence with respect to the norm of $L^2(\partial\Omega)$. Consequently, there exists a function $f\big|_{\partial\Omega} \in L^2(\partial\Omega)$, to which this sequence converges in the norm of $L^2(\partial\Omega)$.

It can also be proved that the function $f\big|_{\partial\Omega}$ does not depend on the choice of the sequence $f_1, f_2, \ldots$, approximating the function f. This function $f\big|_{\partial\Omega}$ is called ***the trace*** on $\partial\Omega$ of the function $f \in H^1(\Omega)$.

By the density of $C^1(\overline{\Omega})$ in $H^1(\Omega)$, inequality (1.10) remains true for all functions $f \in H^1(\Omega)$. Thus, we take $f(s) = f\big|_{\partial\Omega}(s)$ for $s \in \partial\Omega$.

1.7 Subspaces

From the geometric point of view, the simplest functional spaces are the Hilbert spaces H, because their properties resemble the properties of finite-dimensional Euclidean spaces most. In particular, two vectors $x, y \in H$ are called ***orthogonal*** (this is written as in the finite-dimensional case as $x \perp y$), if $\langle x, y\rangle = 0$.

Example 1.18 In the space $L^2(0, 2\pi)$ with respect to the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(t)\overline{g(t)}dt,$$

the functions $e_k = \frac{1}{\sqrt{2\pi}} e^{ikt}$ $(k \in \mathbb{Z})$ are ***orthonormal***, that is

$$(e_k, e_j) = \delta_{kj} \equiv \begin{cases} 0, & \text{for } k \neq j, \\ 1, & \text{for } k = j. \end{cases}$$

This fact is easily checked by a direct calculation. Here δ_{kj} is the so-called Kronecker delta, i.e. a function of two variables k and j, usually defined for nonnegative integers k and j. This function is equal to 1 if the variables are equal, and is 0 otherwise.

A closed linear subset of the space H is called its ***subspace***. For any $x \in H$ one can define its ***projection*** to an arbitrary subspace F as the vector x_F such that $x - x_F \perp f$ for any $f \in F$. Due to this fact, a large number of geometric constructions, holding in the Euclidean space, is transferred to the abstract setting of Hilbert spaces, where such constructions often take an analytical character.

So, for example, an ordinary procedure of orthogonalisation (for example, the Gram-Schmidt orthogonalisation process in the proof of Theorem 1.24 in Section 1.9) leads to the existence of an orthonormal basis in H. That is, it leads to an infinite sequence of the vectors $\{e_k\}_{k\in\mathbb{Z}}$ from H such that $\|e_k\| = 1$, $e_k \perp e_j$ for $k \neq j$, and for any element $x \in H$ a "coordinate-wise" expansion is valid:

$$x = \sum_{k\in\mathbb{Z}} x_k e_k. \tag{1.11}$$

Here $x_k = \langle x, e_k\rangle$ and $\|x\| = \left(\sum_{k\in\mathbb{Z}} |x_k|^2\right)^{1/2}$ (for simplicity H is assumed to be separable (see Section 1.7), that is, there exists a countable dense set in H).

If we take $H = L^2(0,2\pi)$ and denote $e_k(t) := \frac{1}{\sqrt{2\pi}} e^{ikt}$, $k = 0, \pm 1, \pm 2, ...$, then the formula (1.11) gives an expansion of a function $x \in L^2(0,2\pi)$ into its ***Fourier series*** converging in mean square:

$$x(t) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} x_k e^{ikt}, \quad \text{where } x_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x(t) e^{-ikt} dt. \tag{1.12}$$

Furthermore, the relation (1.11) shows the existence of a one-to-one correspondence (bijection) between the abstract separable Hilbert spaces H and ℓ^2: to each element $x \in H$ there corresponds a unique element $\widehat{x} \in \ell^2$, $\widehat{x} = \{x_k\}_{k \in \mathbb{Z}}$. And vice versa, to each element $\widehat{x} \in \ell^2$, $\widehat{x} = \{x_k\}_{k \in \mathbb{Z}}$, there corresponds a unique element $x \in H$ and this element is given by the formula (1.11).

Example 1.19 The set of measurable functions $f \in L^2(a,b)$, for which $\int\limits_a^b f(x)dx = 0$, forms a subspace of the space $L^2(a,b)$.

Taking into account the inner product (1.4), we can describe this subspace as the functions $f \in L^2(a,b)$, for which $f \perp 1$. Moreover, this subspace can be considered as a linear space of functions with the finite norm

$$\|f\| = \left[\int_a^b |f(x)|^2 dx + \left| \int_a^b f(x)dx \right|^2 \right]^{1/2}.$$

This space will be a Hilbert space with the inner product given by the formula

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx + \left(\int_a^b f(x)dx \right) \left(\int_a^b \overline{g(x)}dx \right).$$

Example 1.20 In the Hilbert space of Sobolev $L_1^2(a,b)$, with the inner product

$$\langle f, g \rangle = \int_a^b \left[f(x)\overline{g(x)} + f'(x)\overline{g'(x)} \right] dx, \tag{1.13}$$

consider the linear space M_0 of functions vanishing at some point $x_0 \in [a,b]$:

$$M_0 := \left\{ f \in L_1^2(a,b) : f(x_0) = 0 \right\}. \tag{1.14}$$

Let us show that M_0 forms a linear subspace in $L_1^2(a,b)$, that is, a closed linear subset with respect to the norm of $L_1^2(a,b)$. For this it is sufficient to show the existence of a function $g \in L_1^2(a,b)$ such that the linear space M_0 coincides with a subspace orthogonal to g, that is

$$\exists g \in L_1^2(a,b) : \quad M_0 \equiv \left\{ f \in L_1^2(a,b) : \langle f, g \rangle = 0 \right\}. \tag{1.15}$$

Taking into account the representation of the inner product by the formula (1.13), comparing (1.14) and (1.15), we obtain that for any $f \in L_1^2(a,b)$, we have the relation

$$f(x_0) = \int\limits_a^b \left[f(x)\overline{g(x)} + f'(x)\overline{g'(x)} \right] dx. \tag{1.16}$$

We will look for g in the class $C[a,b]\cap C^2[a,x_0]\cap C^2[x_0,b]$, i.e. in the class of more regular functions. Since it can still happen that the function g' is not continuously differentiable at the point $x_0 \in [a,b]$, in order to apply the integration by parts to the second summand in (1.16) it is necessary to divide the interval into two parts: $[a,b]=[a,x_0]\cup[x_0,b]$. Then

$$f(x_0)=\int_a^b \left[f(x)\overline{g(x)}\right]dx-\int_a^{x_0} f(x)\overline{g''(x)}dx-\int_{x_0}^b f(x)\overline{g''(x)}dx \qquad (1.17)$$
$$+f(b)\overline{g'(b)}-f(a)\overline{g'(a)}+f(x_0)[\overline{g'(x_0-0)}-\overline{g'(x_0+0)}].$$

Therefore,

$$f(x_0)\left(1-\left[\overline{g'(x_0-0)-g'(x_0+0)}\right]\right)=\int_a^{x_0} f(x)\left[\overline{g(x)-g''(x)}\right]dx$$
$$+\int_{x_0}^b f(x)\left[\overline{g(x)-g''(x)}\right]dx+f(b)\overline{g'(b)}-f(a)\overline{g'(a)}. \qquad (1.18)$$

It is easy to check that formula (1.18) will hold for any function $f\in L_1^2(a,b)$ provided that the function g satisfies the following conditions:

1) the function belongs to the class

$$g\in C[a,b]\cap C^2[a,x_0]\cap C^2[x_0,b]; \qquad (1.19)$$

2) on the intervals (a,x_0) and (x_0,b), the function is a solution of the differential equation

$$g''(x)=g(x),\ a<x<x_0,\ x_0<x<b; \qquad (1.20)$$

3) the function satisfies the conditions

$$g'(a)=0,\ \ g'(b)=0,\ \ g'(x_0-0)-g'(x_0+0)=1. \qquad (1.21)$$

Let us show that such function g exists.

Thus, it is necessary to find a solution of the ordinary differential equation (1.20), satisfying "the boundary conditions" (1.21). Although Eq. (1.20) is a second-order differential equation, and the number of the boundary conditions equals three, this problem is solvable. The point is that the last of the conditions (1.21) is not a boundary condition in the usual sense, but is a so-called "internal" condition. Here, for solving Eq. (1.20) one assumes a discontinuity of the first derivative (jump) at the inner point $x_0\in[a,b]$, and the solution itself is continuous, since it belongs to the space $L_1^2(a,b)$. The condition of continuity can be written in the form

$$g(x_0-0)-g(x_0+0)=0. \qquad (1.22)$$

Thus, for Eq. (1.20) we have already obtained *four* boundary conditions: (1.21) and (1.22). But this equation should be understood as *two separate equations* of the second order: one equation is on the interval (a,x_0) and the second one is on (x_0,b). Therefore this problem is solvable.

Let us check this by a direct calculation.

The general solution of Eq. (1.20) in the class of functions (1.19) on each of the intervals (a,x_0) and (x_0,b) has the form

$$g(x) = \begin{cases} C_{11}e^x + C_{12}e^{-x}, \; a \le x \le x_0, \\ C_{21}e^x + C_{22}e^{-x}, \; x_0 \le x \le b, \end{cases} \tag{1.23}$$

where C_{ij}, $(i,j=1,2)$ are arbitrary constants.

By the function (1.23) to satisfy the conditions (1.21) and (1.22) for defining the constants C_{ij} we get the linear system of equations

$$\begin{cases} C_{11}e^a & -C_{12}e^{-a} & & & =0, \\ & & C_{21}e^b & -C_{22}e^{-b} & =0, \\ C_{11}e^{x_0} & +C_{12}e^{-x_0} & -C_{21}e^{x_0} & -C_{22}e^{-x_0} & =0, \\ C_{11}e^{x_0} & -C_{12}e^{-x_0} & -C_{21}e^{x_0} & +C_{22}e^{-x_0} & =1. \end{cases} \tag{1.24}$$

The determinant of this system equals $\triangle = 2e^{a-b} + 2e^{b-a} > 0$. Therefore, the system (1.24) has the unique solution.

Since the function g is represented by the formula (1.23), it is clear that $g \in L_1^2(a,b)$. Thus, we have proved the existence of a function g, for which (1.15) holds. Therefore, the linear space M_0 is a (closed) subspace of $L_1^2(a,b)$.

Example 1.21 (The space $\mathring{H}^1(\Omega)$) Denote by $\mathring{H}^1(\Omega)$ the set of functions from $H^1(\Omega) = L_1^2(\Omega)$ for which the trace to the boundary $\partial\Omega$ is equal to zero. It is clear that $\mathring{H}^1(\Omega)$ is a linear subset of the space $H^1(\Omega)$. Therefore it is the Hilbert space with respect to the inner product of the space $H^1(\Omega)$.

Indeed, let us show that $\mathring{H}^1(\Omega)$ is a (closed) subspace of $H^1(\Omega)$. For this, we need to show that the set $\mathring{H}^1(\Omega)$ is closed with respect to the norm $H^1(\Omega)$.

Consider a sequence $f_1, f_2, \ldots$ of functions from $\mathring{H}^1(\Omega)$, converging to f in the norm of $H^1(\Omega)$. Let us show that the trace of f on the boundary $\partial\Omega$ is equal to zero. From (1.10), for the elements of this sequence we have

$$\|f\|^2_{L^2(\partial\Omega)} \le C\|f_k - f\|^2_{H^1(\Omega)} \to 0, \;\text{ as }\; k \to \infty.$$

Consequently, $f\big|_{\partial\Omega} = 0$. Thus, the set $\mathring{H}^1(\Omega)$ is closed with respect to the norm of $H^1(\Omega)$. Since in $H^1(\Omega)$ there is an element $f \equiv 1$ not belonging to $\mathring{H}^1(\Omega)$, it follows that $\mathring{H}^1(\Omega)$ is a *proper* closed subspace of $H^1(\Omega)$.

1.8 Initial concept of embedding of spaces

The concept of embedding of linear spaces is helpful in situations when one is using several spaces at the same time. A linear normed space X is said to be

embedded into a linear normed space Y, if there exists a linear mapping $J : X \to Y$, bijective on the domain of J, for which there exists a constant $\alpha > 0$ such that for all $x \in X$ we have the inequality

$$\|Jx\|_Y \leq \alpha \|x\|_X,$$

that is, the mapping J is bounded and the domain of J is $D(J) = X$. The mapping J is called ***an embedding operator***. It is clear that from the definition that an embedding operator is always bounded. The fact of embedding is often indicated by the use of a "hooked arrow": $X \hookrightarrow Y$.

Frequently one considers the embeddings when X is a subset of the space Y. Then the identity mapping $J(x) = x$ is often chosen as an embedding mapping. In this case the normed linear space X is called ***embedded*** into the normed linear space Y (indicated by $X \hookrightarrow Y$), if X is the subset of the space Y and the inequality

$$\|x\|_Y \leq C\|x\|_X$$

holds for all $x \in X$. Here the constant C should not depend on $x \in X$. In this case the identity operator acting from the space X into the space Y, mapping each element $x \in X$ to the (same) element x of the space Y, is called the ***embedding operator***. It is clear that such an embedding operator is a bounded linear operator. The theorems establishing the fact of an embedding of functional spaces are called embedding theorems.

Example 1.22 The following simple embedding theorems hold (here for simplicity, an interval $[a,b]$ is considered to be finite $-\infty < a < b < +\infty$ and the set $\Omega \subset \mathbb{R}^n$ is assumed to be bounded):

- $E^n \hookrightarrow E^m$ for $n \leq m$, where E^d stands for the d-dimensional Euclidean space;
- $C^{k+1}\left(\overline{\Omega}\right) \hookrightarrow C^k\left(\overline{\Omega}\right)$ for all nonnegative integers k, for spaces of functions continuously differentiable in $\overline{\Omega}$ up to the k-th order;
- $C^{k+\nu}\left(\overline{\Omega}\right) \hookrightarrow C^{k+\mu}\left(\overline{\Omega}\right)$ for all nonnegative integers k and $\nu > \mu$, for the spaces $C^{k+\mu}\left(\overline{\Omega}\right)$ of functions satisfying together with all their derivatives up to the k^{th} order inclusively the Hölder condition with the index μ:

$$|D^\alpha f(x) - D^\alpha f(y)| \leq C|x-y|^\mu \quad \forall x, y \in \overline{\Omega},\ \forall \alpha : |\alpha| = k;$$

- $C^{k+\nu}\left(\overline{\Omega}\right) \hookrightarrow C^k(\overline{\Omega})$ for any nonnegative integer k and any $\nu > 0$;
- $L^q(\Omega) \hookrightarrow L^p(\Omega)$ for any $q > p \geq 1$ for the Lebesgue space $L^p(\Omega)$ over a bounded set Ω;
- $C\left(\overline{\Omega}\right) \hookrightarrow L^p(\Omega)$ for any $p \geq 1$;
- $L^p_{k+1}(\Omega) \hookrightarrow L^p_k(\Omega)$ for any $n \geq 0$ and $p \geq 1$, for the Sobolev spaces $L^p_k(\Omega)$ of functions having all generalised derivatives up to the k^{th} order inclusively belonging to $L^p(\Omega)$;

- $L_k^q(\Omega) \hookrightarrow L_k^p(\Omega)$ for any $k \geq 0$ and $q > p \geq 1$;
- $L_n^1(\Omega) \hookrightarrow C(\overline{\Omega})$, if the set Ω is an n-dimensional parallelepiped.

For more complicated cases of embeddings we need several definitions. For some $x \in \mathbb{R}^n$, let B_1 be an open ball with centre at the point x, and let B_2 be an open ball not containing x. The set

$$C_x = B_1 \cap \{(1-\alpha)x + \alpha y : \ \forall y \in B_2, \forall \alpha > 0\}$$

is called ***a finite cone*** with the top at the point x.

We say that the domain $\Omega \subset \mathbb{R}^n$ has ***a cone property*** if there exists a finite cone C such that each point $x \in \Omega$ is the top of some finite cone C_x congruent to C, and completely contained in Ω. Roughly speaking, it means that at each point of the boundary, we can fit in a cone of a fixed angle inside the domain.

Theorem 1.23 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the cone property, let $p > 1$ and $m > 1$. Then the embedding*

$$L_m^p(\Omega) \hookrightarrow L^q(\Omega)$$

holds for $mp < n$ for all $p \leq q \leq \frac{np}{n-mp}$, while for $mp = n$ it holds for all $p \leq q$.

For $p = 1$, the embedding

$$L_n^1(\Omega) \hookrightarrow C_B(\Omega)$$

holds true.

Under some additional assumptions on the smoothness of the boundary $\partial\Omega$ of the domain Ω, the embedding

$$L_m^p(\Omega) \hookrightarrow L^q(\partial\Omega) \tag{1.25}$$

for $mp < n$ holds true for all $p \leq q \leq \frac{(n-1)p}{n-mp}$, while for $mp = n$ it holds true for all $p \leq q < \infty$.

Here $C_B(\Omega)$ denotes the space of functions, continuous in Ω and bounded in $\overline{\Omega}$, with the norm $\|f\| = \sup\limits_{x \in \Omega} |f(x)|$. Note that $C_B(\Omega) \subset C(\Omega)$ but the space $C_B(\Omega)$ is not contained in $C\left(\overline{\Omega}\right)$. So, for example, the function $\sin(1/x)$ belongs to $C_B(0,1)$, but does not belong to $C[0,1]$.

The embedding (1.25) is understood in the sense that the restriction operator to the boundary $\partial\Omega$ is bounded from $L_m^p(\Omega)$ to $L^q(\partial\Omega)$.

1.9 Separable spaces

Let X be an infinite-dimensional Banach space. A sequence $\{e_k\}_{k=1}^{\infty}$ of elements $e_k \in X$ is called a ***basis of the space*** X if every $x \in X$ *can be uniquely represented* in the form of a converging series

$$x = \sum_{k=1}^{\infty} x_k e_k. \tag{1.26}$$

Here the numbers $x_k \in \mathbb{C}$ are called ***coordinates*** of the element $x \in X$ with respect to the basis $\{e_k\}$.

Note that this definition has two conditions: first, any element *can be* represented in the form of the series (1.26). And second, such representation must be *unique*, that is, the set of the numbers $x_k \in \mathbb{C}$ in such a representation must be unique.

The infinite system of elements $\{e_k\}_{k=1}^{\infty}$ is called ***linearly independent*** if for any $n \in \mathbb{N}$ the finite system $\{e_k\}_{k=1}^{n}$ is linearly independent.

In particular, it follows from the uniqueness of the representation (1.26) that any finite set of elements of the basis will be a linearly independent system. Consequently, *any basis is a linearly independent system*.

The concept of the basis of an infinite-dimensional space introduced in this way is a natural generalisation of the same concept in the finite-dimensional case. The majority of the spaces used in practice have a basis.

A normed space X is called ***separable*** if it contains a countable, everywhere dense set. The majority of the spaces used in practice are separable.

In particular, *a Banach space with a countable basis is separable*. Indeed, if $\{e_k\}$ is a basis of the space X, then the set of all possible linear combinations $\sum\limits_{k=1}^{n} \xi_k e_k$ (where $n \in \mathbb{N}$, $\xi_k \in \mathbb{C}$, $\mathrm{Re}(\xi_k)$ and $\mathrm{Im}(\xi_k)$ are rational numbers) forms a countable and dense set in X.

The inverse statement is also true. It can be more simply formulated for the Hilbert spaces.

Theorem 1.24 *In any separable infinite-dimensional Hilbert space H there exists an orthogonal basis of a countable number of elements.*

Recall that a system $\{e_k\}$ in a Hilbert space H is called ***orthogonal*** if $e_k \neq 0$ and $\langle e_k, e_j \rangle = 0$ for $k \neq j$. Here one often uses the notation $e_k \perp e_j$. If additionally $\langle e_k, e_k \rangle = 1$ (that is, $\|e_k\| = 1$ for all k), then the system $\{e_k\}$ is called ***orthonormal***.

Let us give a proof of Theorem 1.24 since there one can use an important *process of Gram-Schmidt orthogonalisation* that will be useful also in the sequel.

Let H be a separable Hilbert space. Then it has a countable, everywhere dense set. Let us take all nonzero elements from this set. Since this set is countable, its elements can be numbered as $\{u_k\}_{k=1}^{\infty}$. Since this set is numbered, it can be interpreted as a sequence.

Let us extract from this sequence a linearly independent system $\{\omega_k\}_{k=1}^{\infty}$ in the following way.

Let $\omega_1 := u_1$.

Then let ω_2 be the first element of the sequence $\{u_k\}_{k=1}^{\infty}$ which is linearly independent with ω_1. Let it be the element u_j. Then we set $\omega_2 := u_j$.

Consider now a sequence $u_{j+1}, u_{j+2}, u_{j+3}, \ldots$ and denote by ω_3 the first element from it that is linearly independent with ω_1 and ω_2.

Continuing this process, we get an infinite (since the space is infinite-dimensional) system $\{\omega_k\}_{k=1}^{\infty}$ of linearly independent elements. Since its ***linear span*** L (that is, the set of all possible finite linear combinations $\sum\limits_{k=1}^{n} c_k \omega_k$ for various $n \in \mathbb{N}$) contains the system $\{u_k\}_{k=1}^{\infty}$, the set L is dense in H.

We now show that given any linearly independent system $\{\omega_k\}_{k=1}^{\infty}$ one can construct the orthogonal system $\{e_k\}_{k=1}^{\infty}$ using the following so-called ***Gram-Schmidt orthogonalisation process***.

Let us assume $e_1 = \omega_1$. Note that $e_1 \neq 0$ since the sequence $\{u_k\}_{k=1}^{\infty}$ contains only nonzero elements.

Further, we look for an element e_2 in the form $e_2 = \omega_2 - \alpha_{21} e_1$, where the number α_{21} is chosen so that e_1 and e_2 are orthogonal, that is, so that $\langle e_1, e_2 \rangle = 0$. From here we get that $(e_1, \omega_2 - \alpha_{21} e_1) = 0$. Therefore, $\alpha_{21} = \frac{\langle e_1, \omega_2 \rangle}{\langle e_1, e_1 \rangle}$. It is easy to see that $e_2 \neq 0$ since otherwise ω_1 and ω_2 would be linearly dependent.

Continuing this process further, we obtain at the k^{th} step the orthogonal system $e_1, e_2, \ldots, e_{k-1}$ which has been already constructed. Then we look for the next element e_k in the form

$$e_k = \omega_k - \sum_{j=1}^{k-1} \alpha_{kj} e_j.$$

We are looking for the coefficients α_{kj} to satisfy the orthogonality condition $e_k \perp e_j$ $(j = 1, 2, \ldots, k-1)$. Since first $k-1$ of elements are already mutually orthogonal, we must have $\alpha_{kj} = \frac{\langle e_j, \omega_k \rangle}{\langle e_j, e_j \rangle}$. Thus, $e_k \neq 0$. Continuing this process we obtain the orthogonal system $\{e_k\}_{k=1}^{\infty}$.

Since the linear spans of the systems $\{\omega_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$ coincide (they coincide with L), and L is dense in H, the system $\{e_k\}_{k=1}^{\infty}$ is the required orthogonal basis of the space H.

Note that if we take $v_k := \frac{e_k}{\|e_k\|}$, then the system $\{v_k\}$ will be an orthonormal basis of the space H. Thus, we have shown that any separable infinite-dimensional Hilbert space H has an *orthonormal* basis consisting of a countable number of elements, completing the proof of the theorem. □

Let us give some examples of separable spaces:

- the real line $\mathbb{R}$ is separable since rational numbers form a countable and dense set in $\mathbb{R}$;

- any finite-dimensional space is separable since for an arbitrary basis it is sufficient to consider the set of its linear combinations with rational coefficients;

- the space $C\left(\overline{\Omega}\right)$ is separable since the set of polynomials with rational coefficients is dense in it;

- the spaces $C^k\left(\overline{\Omega}\right)$, $L^p(\Omega)$, $1 \le p < \infty$, and $L^p_k(\Omega)$, $1 \le p < \infty$, $k \ge 1$, are separable.

Separability of a space is not applied just by itself. It is important that in a separable space (in view of Theorem 1.24) there is always an orthogonal (and even orthonormal) basis. It is sufficient to understand once that the space under consideration is separable, and then to use the possibilities of working with its basis.

Chapter 2

Foundations of the linear operator theory

The operator theory is the study of linear operators on functional spaces, beginning with differential and integral operators. The operators may be presented abstractly by their characteristics, such as bounded linear operators or closed operators, and consideration may be given to nonlinear operators. The operator theory, which depends heavily on the topology of the functional spaces, is a branch of the functional analysis. In this chapter we review the basic notions of the theory of linear operators and functionals, with the aim of discussing properties of the adjoint operators and adjoint boundary value problems. The main emphasis in our exposition is the detailed treatment of examples that show that there may be certain hidden obstacles when one is dealing with different classes of boundary value problems, and we aim at showing effective ways of overcoming them.

2.1 Definition of operator

Concept of an operator (one of the most general mathematical concepts) is a particular case of the general concept of the mapping of one set into another.

Let A and B be two sets. Suppose that a definite element b contained in the set B, by the rule

$$b = f(a)$$

is put into correspondence to each element a of the set A. In this case one says that it defines the ***mapping*** f of the set A into the set B, which can be briefly written in the following way:

$$f : A \to B.$$

Let X, Y be linear spaces. Let, further, a subset $D \subseteq X$ be selected in X. If on the space X there is given a mapping F, under which a definite element $y \in Y$ is put into correspondence to each element $x \in D$, then one says that ***there is given an operator***

$$y = F(x).$$

Herewith the set D is called a ***domain*** or domain of definition of the operator F and is denoted as $D(F)$. In turn, the set

$$R = R(F) = \{y \in Y : y = F(x) \text{ for some } x \in D\}$$

is called the ***range*** or domain of values of the operator F.

Schematically the correspondence given by the operator F can be demonstrated as follows:

$$X \supseteq D(F) \xrightarrow{F} R(F) \subseteq Y,$$

or briefly $F : X \to Y$.

Thus, for defining an operator A it is necessary to fix the spaces X and Y, to give its domain $D(A) \subset X$ and the law under which its action on the elements $x \in D(A)$ is defined.

The simplest example of an operator is an ordinary real function defined on the space $\mathbb{R}$ with the domain of values in $\mathbb{R}$.

When studying questions of a geometric nature, one usually uses the term "mapping" instead of "operator". Due to this, one also uses the following "geometric" terminology: the element y is called the ***image*** of the element x, and the element x is called the ***preimage*** of the element y.

In a particular case when Y is a space of scalars ($\mathbb{R}$ or $\mathbb{C}$), the operator $F : X \to Y$ is called a ***functional***.

2.2 Linear operators

An operator $A : X \to Y$ is called ***linear***, if

1) its ***domain*** $D(A)$ is a linear space (see Section 1.1),

2) for all $x, y \in D(A)$, and all numbers $\alpha, \beta \in \mathbb{C}$ the equality

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \tag{2.1}$$

holds.

Condition (2.1) is a combination of two conditions:

- ***additivity*** of the operator: $A(x+y) = Ax + Ay$, $\forall x, y \in D(A)$;
- ***homogeneity*** of the operator: $A(\alpha x) = \alpha Ax$, $\forall x \in D(A)$, where $\alpha \in \mathbb{C}$.

The concept of a linear operator generalises the concept of a linear function $y = ax$ defined on the real axis. We note that in the standard function theory, a linear function usually means functions of the form $y = ax + b$. However, in the case of operators it is necessary to have $b = 0$ in order for the second condition in the definition of the linear operator to hold (see Example 2.2). In analogy to the theory of functions, the notation Ax is usually used for linear operators instead of $A(x)$.

We defined the ***range of the operator*** A as the set of its values:

$$R(A) := \{Ax : x \in D(A)\}. \tag{2.2}$$

Example 2.1 In the space $C^2[0,1]$ of twice continuously differentiable functions, we consider an operator $L: C^2[0,1] \to C^2[0,1]$ with the domain

$$D(L) = \left\{ u \in C^2[0,1] : u(0) = a,\, u(1) = b \right\},\ (a, b \text{ are constants}),$$

acting by

$$Lu(x) = -u''(x) + q(x)u(x),\ q \in C[0,1]. \tag{2.3}$$

It is easy to see that this operator is defined on all functions $u \in D(L)$. The range $R(L)$ does not coincide with the whole space $C^2[0,1]$. For example, the function $u_0(x) = a + (b-a)\sqrt{x^5}$ belongs to the domain of the operator L, but its image $Lu_0(x) = -\frac{15}{4}(b-a)\sqrt{x} + q(x)u_0(x)$ does not have to be continuously differentiable.

This example shows that for the operators $A: X \to Y$, the range $R(A)$ does not necessarily coincide with the whole space Y.

Let us now consider the question of the linearity of the operator L in (2.3). It is easy to see that for the domain $D(L)$ of the operator to be a linear space (see Section 1.1) it is necessary and sufficient that $a = b = 0$. Indeed, if $u_1, u_2 \in D(L)$, then $u = u_1 + u_2$ satisfies boundary conditions $u(0) = 2a$, $u(1) = 2b$, and for $u \in D(L)$ it is necessary and sufficient to have that $a = b = 0$.

Therefore, when studying boundary value problems by methods of the theory of linear operators, mostly the problems with homogeneous boundary conditions are considered. However, as is known from the general theory of linear differential equations (or partial differential equations), a boundary value problem with inhomogeneous boundary conditions can be always reduced to studying a problem with homogeneous boundary conditions. $\square$

Example 2.2 In the space of continuous functions, consider an operator $A: C[0,1] \to C[0,1]$ defined by

$$Au(x) = au(x) + b;\ (a, b \text{ are constants}).$$

The domain of the operator A is the whole space, that is $D(A) = C[0,1]$. Therefore it is a linear space.

Let us now check the second condition in the definition of linearity. For $u_1, u_2 \in D(A)$, we have

$$A(\alpha u_1 + \beta u_2) = a(\alpha u_1 + \beta u_2) + b = \alpha a u_1 + \alpha b + \beta a u_2 + \beta b + b - \alpha b - \beta b =$$

$$= \alpha A u_1 + \beta A u_2 + b(1 - \alpha - \beta).$$

In turn, it is easy to see that in order to satisfy condition (2.1) in the definition of a linear operator for all $\alpha, \beta \in \mathbb{C}$, it is necessary and sufficient to have $b = 0$.

Thus, the operator A will be linear, if its action is given by the formula

$$Au(x) = au(x).$$

This example of an operator A is a direct generalisation of the notion of a linear function $y = ax$ defined on the real axis. However, in the theory of functions, the linearity of a function usually means that we have $y = ax + b$. However, in the case of operators, it is necessary to have $b = 0$ in order to fulfil condition (2.1) in the definition of a linear operator. □

Example 2.3 In the space of continuous functions, consider an operator $A: C[0,1] \to C[0,1]$ defined by

$$Au(x) = \overline{u(x)}, \tag{2.4}$$

where the operation $\bar{z} = \overline{(\mathrm{Re}z + i\mathrm{Im}z)} = \mathrm{Re}z - i\mathrm{Im}z$ is the complex conjugation.

The domain of the operator A is the whole space, that is $D(A) = C[0,1]$ and, therefore, it is a linear space.

Let us check the second condition (2.1) in the definition of linearity. For $u_1, u_2 \in D(A)$, we have

$$A(u_1 + u_2) = \overline{(u_1 + u_2)} = \overline{u_1} + \overline{u_2} = Au_1 + Au_2,$$

and the operator A "looks like" a linear operator. More precisely, the operator A is additive. However, for constants $\alpha, \beta \in \mathbb{C}$, we have

$$A(\alpha u_1 + \beta u_2) = \overline{[\alpha u_1 + \beta u_2]} = \overline{\alpha u_1} + \overline{\beta u_2} = \overline{\alpha}(\overline{u_1}) + \overline{\beta}(\overline{u_2}) = \overline{\alpha} A u_1 + \overline{\beta} A u_2,$$

which means that condition (2.1) is satisfied only for real numbers $\alpha, \beta \in \mathbb{R}$, which is not sufficient for the linearity of the operator.

Thus, the operator A defined by formula (2.4) is not linear because it is not homogeneous. □

The general theory of operators is much more developed for the linear operators; therefore, in the sequel in this monograph *we will suppose that all considered operators are linear.*

Two cases of linear operators are very important in practice and applications:

- the case when $D(A) = X$, that is, the operator A is defined on the whole space X;
- the case when $\overline{D(A)} = X$, that is, the domain $D(A)$ is dense in the space X.

We also note that for the linear operators *the range* $R(A)$ is always a *linear space.*

2.3 Linear bounded operators

Let X, Y be Banach spaces. A linear operator $A : X \to Y$ with domain $D(A) = X$, is called ***continuous at the point*** $x_0 \in X$, if

$$Ax \to Ax_0 \ \text{ whenever } \ x \to x_0.$$

Here and further such notation means that

$$\|Ax - Ax_0\|_Y \to 0 \ \text{ for } \ \|x - x_0\|_X \to 0.$$

From the linearity properties of an operator it follows that if the operator A is continuous at the point $0 \in X$, then it is continuous at every point $x_0 \in X$. The linear operator continuous at $0 \in X$ is called ***continuous***.

A wide class of linear operators is the so-called *bounded operators*. In the theory of functions, a function $f = f(x)$ is called bounded on the interval $[a,b]$, if there exists a constant M such that $|f(x)| \leq M \ \ \forall x \in [a,b]$. However, in the theory of linear operators it is not possible to draw such an analogy.

Indeed, in view of the linearity of the operator A, its domain $D(A)$ is a linear space, and along with each element $x \in D(A)$ it contains also the element kx, for an arbitrarily large number k. Therefore, even if the value of the operator A at the element $x \in D(A)$, that is, $\|Ax\|$, is bounded by some constant M, if it is nonzero, we can always choose the number k sufficiently large so that the value of the operator A at the element $kx \in D(A)$, that is, $\|kAx\| = k\|Ax\|$, exceeds M. Therefore, unlike in the theory of functions, in the theory of operators the values of a bounded operator are, generally speaking, not uniformly bounded.

A linear operator $A : X \to Y$ with domain $D(A) = X$ is called ***bounded***, if it maps bounded sets from X into bounded sets in Y. Such operator $A : X \to Y$, $D(A) = X$ is bounded if and only if there exists a constant M such that

$$\|Ax\|_Y \leq M\|x\|_X, \ \ \forall x \in X. \tag{2.5}$$

The smallest constant M satisfying condition (2.5), is called ***the norm of the operator*** and is denoted by $\|A\|$. Therefore, the norm of the bounded operator is expressed by the formula

$$\|A\| = \sup_{x \in X} \frac{\|Ax\|_Y}{\|x\|_X}. \tag{2.6}$$

Moreover, the norm of a bounded operator $A : X \to Y,\ D(A) = X$ (where X and Y are normalised spaces) is "attained" on the unit sphere, that is,

$$\|A\| = \sup_{x \in X} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\| \leq 1} \|Ax\|_Y = \sup_{\|x\|_X = 1} \|Ax\|_Y.$$

Indeed, this is the direct consequence of the chain of equalities

$$\sup_{x \in X} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{x \in X} \left\| \frac{Ax}{\|x\|_X} \right\|_Y = \sup_{x \in X} \left\| A\left(\frac{x}{\|x\|_X}\right) \right\|_Y = \sup_{\|x\| = 1} \|Ax\|_Y$$

and evident inequalities

$$\sup_{\|x\| = 1} \|Ax\|_Y \leq \sup_{\|x\| \leq 1} \|Ax\|_Y \leq \sup_{\|x\| \leq 1} \frac{\|Ax\|_Y}{\|x\|_X} \leq \sup_{x \in X} \frac{\|Ax\|_Y}{\|x\|_X}.$$

Here we must note that (since the concept "supremum" is used) the norm of an operator is calculated on elements on the unit sphere $\|x\| = 1$, but it is not necessarily attained on some elements of this unit sphere.

We also note that if $x = 0$, we have $\|x\|_X = 0$ and $\|Ax\|_Y = 0$, so that in the notation (2.6), in order to avoid adding $x \neq 0$ in too many formulae, one can agree to consider that the quotient $0/0$ is also 0, thus not contributing to taking the supremum.

The immediate consequence of formulae (2.5) and (2.6) is the inequality

$$\|Ax\|_Y \leq \|A\| \, \|x\|_X, \; \forall x \in X. \tag{2.7}$$

The following theorem substantiates the equivalence of concepts of linear continuous and linear bounded operators.

Theorem 2.4 *Let X and Y be Banach spaces and let $A : X \to Y$ be a linear operator with domain $D(A) = X$. Then, for the operator A to be continuous it is necessary and sufficient that it is bounded.*

Moreover, the linear bounded operator is not only continuous but also satisfies an analogue of the Lipshitz conditions from the theory of functions:

$$\|Ax - Ay\|_Y \leq \|A\| \cdot \|x - y\|_X, \; \forall x, y \in X.$$

The linear operators acting in finite-dimensional spaces have the simplest general form.

Example 2.5 Let X and Y be finite-dimensional spaces of dimensions n and m, respectively. Let us denote by $\{u_j\}_{j=1}^n$ a basis of the space X. An arbitrary element $x \in X$ is represented in the form of a linear combination with respect to this basis:

$$x = \sum_{j=1}^{n} x_j u_j, \; x = (x_1, x_2, ..., x_n). \tag{2.8}$$

If we denote by $\{v_i\}_{i=1}^m$ a basis in Y, then the image of an operator T acting on the basis $\{u_j\}_{j=1}^n$ admits the expansion in Y:

$$Tu_j = \sum_{i=1}^{m} a_{ij} \, v_i, \;\; \forall j = 1, \ldots, n. \tag{2.9}$$

Then from (2.8) and (2.9), by the linearity of the operator T, we get

$$\begin{aligned} Tx = T\left(\sum_{j=1}^{n} x_j \, u_j\right) = \sum_{j=1}^{n} x_j T u_j = \\ = \sum_{j=1}^{n} \; \sum_{i=1}^{m} x_j a_{ij} v_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \; x_j a_{ij}\right) v_i. \end{aligned} \tag{2.10}$$

In the space Y, any element $y \in Y$ has "coordinate" expansion in the basis $\{v_i\}_{i=1}^m$: $y = (y_1, y_2, ..., y_m)$ or $y = \sum_{i=1}^{m} y_i v_i$. Then for the image $Tx = y = (y_1, y_2, ..., y_m)$ from (2.10) we find that

$$y_i = \sum_{j=1}^{n} a_{ij} x_j, \quad \forall i = 1, \ldots, m. \tag{2.11}$$

Thus, the action of the linear operator T from the finite-dimensional space X (of dimension n) into the finite-dimensional space Y (of dimension m) is completely described with the help of a matrix (a_{ij}) of the size $m \times n$.

With this, the operator equation $Tx = y$ can be written in a matrix form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_m \end{pmatrix}. \tag{2.12}$$

The elements of the matrix (a_{ij}) for each operator are uniquely fixed by choosing the bases in the spaces X and Y.

Note that all the above arguments are valid not only for ordinary Euclidean spaces, but also for any finite-dimensional spaces.

From representations (2.10), (2.11) and (2.12), it is easily seen that *all the linear operators in the finite-dimensional spaces are bounded*. Indeed, let us choose some bases in the spaces X and Y to be normalised (i.e. $\|u_j\| = \|v_i\| = 1$). We will use the norms given in the following way (this does not reduce the generality, since all the norms in the finite-dimensional spaces are equivalent):

$$\|x\| = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2}, \ \forall x = (x_1, x_2, ..., x_n) \in X,$$

$$\|y\| = \left(\sum_{i=1}^{m} |y_i|^2 \right)^{1/2}, \ \forall y = (y_1, y_2, ..., y_m) \in Y.$$

We use the well-known ***Hölder's inequality for finite sums***:

$$\sum_{k=1}^{N} |\xi_k \eta_k| \leq \left(\sum_{k=1}^{N} |\xi_k|^p \right)^{1/p} \left(\sum_{k=1}^{N} |\eta_k|^q \right)^{1/q},$$

where $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then from (2.10) we calculate:

$$\|Tx\| = \|y\| = \left(\sum_{i=1}^{m} |y_i|^2 \right)^{1/2} = \left(\sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|^2 \right)^{1/2}$$

$$\leq \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij} x_j| \right)^2 \right)^{1/2} \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \sum_{k=1}^{n} |x_k|^2 \right)^{1/2}$$

$$= \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \cdot \|x\|.$$

Therefore, $\|T\| \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$.

Consequently, the operator T is bounded. □

The considered example demonstrates that all the linear operators acting in the finite-dimensional spaces are bounded. In fact, there exists a more general fact that any linear *operator*, *acting from a finite-dimensional space* X into a Banach space Y (which can be infinite-dimensional) is *bounded*. This is the consequence of the fact that in this case, even in the infinite-dimensional space Y, the image still will be a finite-dimensional linear space.

The situation becomes significantly more complicated in the case of infinite-dimensional spaces. One of the simplest operators in an infinite-dimensional space is the operator of multiplication by a function.

Example 2.6 In the space $C[a,b]$ with the norm $\|f(x)\| = \max_{x\in[a,b]} |f(x)|$, consider an operator defined on the whole space by the formula

$$Tf(x) = G(x) \cdot f(x), \ \forall f \in C[a,b],$$

where $G(x)$ is a given function continuous on the closed interval $[a,b]$.

Let us show that the operator T is bounded. Indeed, it is easily seen that for all $x \in [a,b]$ we have

$$|G(x)f(x)| \leq \|G\| \cdot |f(x)| \leq \|G\| \cdot \|f\|.$$

Consequently,

$$\|Gf\| \leq \|G\| \cdot \|f\|.$$

Therefore $\|Tf\| \leq \|G\| \cdot \|f\|$ and by the definition of the operator norm we have

$$\|T\| \leq \|G\|. \tag{2.13}$$

On the other hand, consider an action of the operator T on a "test function" $f_0(x) = 1$. It is clear that $f_0 \in C[a,b]$ and $\|f_0\| = 1$.

Then

$$\|T\| = \sup_{\|f\|=1} \|Tf\| \geq \|Tf\|_{f=f_0} = \|G\|, \ \text{so that} \ \|T\| \geq \|G\|.$$

Comparing the last inequality and (2.13), we obtain that the operator $T: \ C[a,b] \to C[a,b]$ is bounded and its norm is $\|T\| = \|G\|$. □

Example 2.7 Now consider the same action of an operator given in the Banach space $L^p(a,b)$ with the norm $\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$.

The action of the operator is given by the formula

$$Tf(x) = G(x) \cdot f(x),\ \forall f \in L^p(a,b),$$

where $G = G(x)$ is a given function continuous on the closed interval $[a,b]$.

Let us show that the operator $T : L^p(a,b) \to L^p(a,b)$ is bounded. Indeed, denoting $G_0 = \max\limits_{x\in[a,b]} |G(x)|$, it is easy to see that for all $x \in [a,b]$ we have

$$|G(x)f(x)| \leq G_0 \cdot |f(x)|.$$

Then

$$\|Tf\|_p = \left(\int_a^b |G(x)f(x)|^p dx\right)^{1/p} \leq G_0 \left(\int_a^b |f(x)|^p dx\right)^{1/p} = G_0\|f\|_p.$$

So we have $\|Tf\|_p \leq G_0 \cdot \|f\|_p$ for all $f \in L^p(a,b)$. Therefore the operator T is bounded in $L^p(a,b)$ and, by the definition of the operator norm,

$$\|T\| \leq G_0. \tag{2.14}$$

On the other hand, since the function $G = G(x)$ is continuous on the closed interval $[a,b]$, then there exists a point $x_0 \in [a,b]$ such that

$$G_0 = \max_{x\in[a,b]} |G(x)| = |G(x_0)|.$$

Now we choose a "test function". Let for any $0 < \varepsilon \leq 1,\quad [a_\varepsilon, b_\varepsilon] \subseteq [a,b]$ be an arbitrary interval of the length $\varepsilon = b_\varepsilon - a_\varepsilon$ containing the point $x_0 \in [a,b]$. Let us use the function $f_\varepsilon(x)$ given by the formula

$$f_\varepsilon(x) = \begin{cases} 0, & x \notin [a_\varepsilon, b_\varepsilon], \\ \varepsilon^{-1/p}, & x \in [a_\varepsilon, b_\varepsilon]. \end{cases} \tag{2.15}$$

The simple calculations show that $f_\varepsilon \in L^p(a,b)$ and $\|f_\varepsilon\|_p = 1$:

$$\|f_\varepsilon\|_p = \left(\int_{a_\varepsilon}^{b_\varepsilon} \left|\varepsilon^{-1/p}\right|^p dx\right)^{1/p} = \varepsilon^{-1/p}\left(\int_{a_\varepsilon}^{b_\varepsilon} dx\right)^{1/p} = \varepsilon^{-1/p}(b_\varepsilon - a_\varepsilon)^{1/p} = 1.$$

Consider now the action of the operator T on the test function $f_\varepsilon(x)$. We have

$$\|Tf_\varepsilon\|_p = \left(\int_{a_\varepsilon}^{b_\varepsilon} \left|G(x)\varepsilon^{-1/p}\right|^p dx\right)^{1/p} = \varepsilon^{-1/p}\left(\int_{a_\varepsilon}^{b_\varepsilon} |G(x)|^p dx\right)^{1/p}$$

$$\geq \varepsilon^{-1/p} \cdot \min_{[a_\varepsilon, b_\varepsilon]} |G(x)| \cdot \left(\int_{a_\varepsilon}^{b_\varepsilon} dx\right)^{1/p} = \min_{[a_\varepsilon, b_\varepsilon]} |G(x)|.$$

Therefore from the definition of the operator norm we have

$$\|T\| = \sup_{\|f\|=1} \|Tf\|_p \geq \|Tf_\varepsilon\| \geq \min_{[a_\varepsilon, b_\varepsilon]} |G(x)|.$$

From this and from (2.15) it follows that

$$\min_{[a_\varepsilon,b_\varepsilon]}|G(x)| \leq \|T\| \leq G_0. \tag{2.16}$$

Since $G = G(x)$ is continuous on the closed interval $[a,b]$, we have

$$\lim_{\varepsilon\to 0}\min_{[a_\varepsilon,b_\varepsilon]}|G(x)| = G(x_0) = G_0.$$

Therefore from (2.16), by the arbitrariness of $\varepsilon > 0$, it follows that $\|T\| = G_0$. □

The studied Examples 2.6 and 2.7 show that the operators given by the same formula $Tf(x) = G(x)\cdot f(x)$ but considered in different spaces $C[a,b]$ and $L^p(a,b)$, turn out to be simultaneously bounded and even the norms of these operators turn out to be equal. However, this is not true in general. That is, there exist the operators, actions of which are given by the same formula, but (depending on the spaces in which they are considered) they can be both bounded or unbounded. The following example illustrates this fact.

Example 2.8 Consider an ***operator of taking a trace***, acting by the formula

$$Pf(x) = f(x_0), \tag{2.17}$$

where $x_0 \in [a,b]$ is some fixed point.

If we consider the operator P as acting by formula (2.17) in the space of continuous functions $P : C[a,b] \to C[a,b]$, then it is easy to see that the operator is bounded. Indeed, for all $f \in C[a,b]$ we have

$$\|Pf\| = \max_{x\in[a,b]}|Pf(x)| \equiv |f(x_0)| \leq \max_{x\in[a,b]}|f(x)| = \|f\|.$$

Hence we have $\|P\| \leq 1$ and, consequently, the operator $P : C[a,b] \to C[a,b]$ is bounded.

Consider now the operator P as acting by the same formula (2.17) in the Banach space of functions $P:\ L^p(a,b) \to L^p(a,b)$. We now show that in this case the operator is not bounded.

As a test function we take the function $f_\varepsilon(x)$ from Example 2.7, defined by formula (2.15). We can calculate the norm of the image $Pf_\varepsilon(x)$:

$$\|Pf_\varepsilon\|_p = \|f_\varepsilon(x_0)\|_p = \left\|\varepsilon^{-1/p}\right\|_p = \left(\int_a^b \left|\varepsilon^{-1/p}\right|^p dx\right)^{1/p} =$$

$$= \varepsilon^{-1/p}\cdot(b-a)^{1/p} \to \infty, \text{ for } \varepsilon \to 0.$$

Since $\|f_\varepsilon\|_p = 1$ and $\lim\limits_{\varepsilon\to\infty}\|Pf_\varepsilon(x)\|_p = \infty$, then from the definition of the operator norm it follows that the norm of the operator P is not bounded and, consequently, P is an unbounded operator in $L^p(a,b)$. □

The considered Example 2.8 illustrates that the operator, given by the same action (by the same formula) in different normed spaces, can be bounded in one of them, and can be unbounded in another space.

The same effect can occur if one considers an operator acting between different spaces. Let us show this by the following examples.

Example 2.9 Let us consider an operator acting by the formula

$$L_1 u(x) = \frac{d}{dx}u(x), \ \ a < x < b. \tag{2.18}$$

Let us look at the question of the boundedness of the operator L_1 acting in different spaces:

1) Consider the operator L_1 as $L_1 : C^1[a,b] \to C[a,b]$. It is easy to see that in this case the operator L_1 is defined on the whole space, that is, $D(L_1) = C^1[a,b]$ and the operator is bounded. Indeed, recalling the formula by which we have defined the norm in $C^1[a,b]$ (see Example 1.6), it is easy to see that for all $u \in D(L_1)$ the inequality

$$\|L_1 u\|_{C[a,b]} = \max_{x\in[a,b]} \left|\frac{d}{dx}u(x)\right| \leq \max_{x\in[a,b]} \left(|u(x)| + \left|\frac{d}{dx}u(x)\right|\right) = \|u\|_{C^1[a,b]}$$

holds. Consequently, the operator L_1 is bounded, and by the definition of the operator norm we have $\| L_1 \|_{C^1[a,b]\to C[a,b]} \leq 1$.

2) Consider now the operator L_1 defined by the same formula (2.18), acting in the space of continuous functions $C[a,b]$, that is, $L_1 : \ C[a,b] \to \ C[a,b]$. Here, in order to correctly define the operator (since its action by formula (2.18) is not well defined for all functions from $C[a,b]$), we choose its domain $D(L_1) \equiv C^1[a,b] \subset C[a,b]$. Let us show that in this case the operator L_1 is unbounded. As a test function we take

$$u_\varepsilon(x) = \sqrt{\frac{x-a+\varepsilon}{b-a+\varepsilon}}, \ \ \varepsilon > 0.$$

It is clear that for all $\varepsilon > 0$ the test function belongs to the domain of the operator: $u_\varepsilon \in \ D(L_1)$ and

$$\|u_\varepsilon\| = \max_{x\in[a,b]} \sqrt{\frac{x-a+\varepsilon}{b-a+\varepsilon}} = 1, \ \ \forall\varepsilon > 0.$$

Let us calculate the norm of the image of this function:

$$\|L_1 u_\varepsilon\| = \left\|\frac{d}{dx}u_\varepsilon\right\| = \left\|\frac{d}{dx}\sqrt{\frac{x-a+\varepsilon}{b-a+\varepsilon}}\right\|$$

$$= \max_{x\in[a,b]} \left|\frac{1}{2\sqrt{(x-a+\varepsilon)(b-a+\varepsilon)}}\right| = \frac{1}{2\sqrt{\varepsilon(b-a+\varepsilon)}}.$$

It is easily seen that for $\varepsilon \to 0$ the norm $\|L_1 u_\varepsilon\|$ becomes arbitrarily large while the norm of the function itself is $\|u_\varepsilon\| = 1$. That is, the operator L_1 is unbounded.

We can also mention here *another, a more wide-spread in the literature example* of the test function for proving the unboundedness of the operator L_1. Consider the sequence of functions

$$u_k(x) = \sin kx, \ k = k_0, k_0+1, \dots,$$

where we choose the integer k_0 large enough in order to have $(b-a)k_0 \geq \pi$. Then, on the interval $[a,b]$, for each $k \in \mathbb{N}$, $k \geq k_0$, there exists at least one solution of the equation $|\sin kx| = 1$, and therefore $\|u_k\| = 1$.

Now we calculate the norm of the image of this test function:

$$\|L_1 u_k\| = \left\| \frac{d}{dx} u_k \right\| = \left\| \frac{d}{dx} \sin kx \right\| = \max_{x\in[a,b]} |k \cos kx| = k, \ \forall k \geq 2k_0.$$

Now it is easily seen that for $k \to \infty$ the norm $\|L_1 u_k\|$ becomes arbitrarily large while the norm of the function itself is $\|u_k\| = 1$, that is the operator L_1 is unbounded. □

The considered example clearly illustrates that the linear operator given by formula (2.18) turns out to be bounded, if it is considered as acting from $C^1[a,b]$ into $C[a,b]$, and turns out to be unbounded as the operator acting from $C[a,b]$ into $C[a,b]$.

The similar fact also holds in Lebesgue and Sobolev spaces:

Example 2.10 Let, as in Example 2.9, the action of an operator be given by formula (2.18):

$$L_1 u(x) = \frac{d}{dx} u(x), \ a < x < b.$$

Let us consider the question of the boundedness of this operator in the Lebesgue and Sobolev spaces.

1) Consider L_1 as the operator acting as $L_1 : L_1^p(a,b) \to L^p(a,b)$. It is easy to see that in this case the operator L_1 is defined on the whole space, that is, we can choose $D(L_1) = L_1^p(a,b)$, and the operator is bounded. Indeed, recalling the formula by which we have defined the norm in $L_1^p(a,b)$ (see Example 1.14), it is readily seen that for all $u \in D(L_1)$ we have

$$\|L_1 u\|_{L^p(a,b)} = \left(\int_a^b \left| \frac{d}{dx} u(x) \right|^p dx \right)^{1/p} \leq \left(\int_a^b \left[|u(x)|^p + \left| \frac{d}{dx} u(x) \right|^p \right] dx \right)^{1/p}$$
$$= \|u\|_{L_1^p(a,b)}.$$

Consequently, the operator L_1 is bounded and by the definition of the operator norm we have $\| L_1 \|_{L_1^p(a,b) \to L^p(a,b)} \leq 1$.

2) Now consider the operator L_1 defined by the same formula (2.18), as the operator in the space $L^p(a,b)$, $p \geq 1$, that is, $L_1 : L^p(a,b) \to L^p(a,b)$. Here, in order to correctly define the operator (since its action by formula (2.18) is not well-defined on all functions from $L^p(a,b)$), we choose its domain to be $D(L_1) = L_1^p(a,b) \subset L^p(a,b)$.

Let us show that in this case the operator L_1 is unbounded. As a test function we take $u_\varepsilon(x) = m_\varepsilon (x-a+\varepsilon)^{1-1/p}$, $\varepsilon > 0$, where for short we have introduced the notation $m_\varepsilon = p^{1/p}[(b-a+\varepsilon)^p - \varepsilon^p]^{-1/p}$. It is clear that for all $\varepsilon > 0$ the test function belongs to the domain of the operator: $u_\varepsilon \in D(L_1)$ and

$$\|u_\varepsilon\|_p = \left(\int_a^b |u_\varepsilon(x)|^p\, dx\right)^{1/p} = m_\varepsilon \left(\int_a^b (x-a+\varepsilon)^{p-1} dx\right)^{1/p} = 1,\ \forall \varepsilon > 0.$$

Let us calculate the norm of the image of this test function:

$$L_1 u_\varepsilon(x) = \frac{p-1}{p} m_\varepsilon (x-a+\varepsilon)^{-1/p},$$

$$\|Lu_\varepsilon\|_p = \left(\int_a^b |Lu_\varepsilon(x)|^p\, dx\right)^{1/p} = m_\varepsilon \frac{p-1}{p}\left(\int_a^b (x-a+\varepsilon)^{-1} dx\right)^{1/p}$$
$$= m_\varepsilon \frac{p-1}{p}\left(\ln \frac{b-a+\varepsilon}{\varepsilon}\right)^{1/p}.$$

Now it is easily seen that for $\varepsilon \to 0$ the norm $\|L_1 u_\varepsilon\|$ becomes arbitrarily large while the norm of the function itself is $\|u_\varepsilon\| = 1$. That is, the operator L_1 is unbounded as the operator acting from $L^p(a,b)$ into $L^p(a,b)$. $\square$

Examples 2.8, 2.9 and 2.10 show how important the choice of spaces is when we define an operator and its domain.

Some of the important and widely used operators are integral operators. We now consider the following simplest example.

Example 2.11 Let the action of an operator K be given by the formula

$$Kf(x) = \int_a^b k(x,t) f(t) dt, \tag{2.19}$$

where $k = k(x,t)$ is a function on a closed rectangle $[a,b] \times [a,b]$. The linear operators given by a formula of the form (2.19) are called ***integral operators***, and the function $k(x,t)$ is called a ***kernel of the integral operator***. Let us look at the question of the boundedness of the operator K in spaces $C[a,b]$ and $L^2(a,b)$.

1) First, consider the operator in the space $C[a,b]$ and let the kernel of the operator K be a continuous function $k = k(x,t) \in C([a,b] \times [a,b])$. Then there exists a constant M such that $|k(x,t)| \le M$ for all $x,t \in [a,b]$. Therefore

$$\|Kf\| = \max_{x\in[a,b]} \left|\int_a^b k(x,t) f(t) dt\right| \le \max_{x\in[a,b]} \int_a^b |k(x,t)||f(t)| dt$$

$$\le M \int_a^b |f(t)| dt \le M(b-a) \max_{t\in[a,b]} |f(t)|,$$

that is, $\|Kf\| \le M(b-a)\|f\|$, $\forall f \in C[a,b]$. Consequently, the operator K is defined on the whole space $C[a,b]$, is bounded on it, and $\|K\| \le M(b-a)$.

2) Now consider the integral operator K in the Hilbert space $L^2(a,b)$ and let now the kernel $k(x,t)$ of the integral operator K be a measurable function integrable according to Lebesgue with a square: $k = k(x,t) \in L^2((a,b)\times(a,b))$. We will use the ***integral Hölder inequality***

$$\int_a^b |f(t)g(t)|dx \le \left(\int_a^b |f(t)|^p\,dt\right)^{1/p}\left(\int_a^b |g(t)|^q\,dt\right)^{1/q},$$
$$\text{where } \frac{1}{p}+\frac{1}{q}=1. \quad (2.20)$$

For the case $p=q=2$, for all $f\in L^2(a,b)$ we easily obtain

$$\|Kf\|^2 = \int_a^b \left|\int_a^b k(x,t)f(t)dt\right|^2 dx \le \int_a^b \left(\int_a^b |k(x,t)f(t)|dt\right)^2 dx$$
$$\le \int_a^b \left(\left(\int_a^b |k(x,t)|^2 dt\right)^{1/2}\left(\int_a^b |f(t_1)|^2 dt_1\right)^{1/2}\right)^2 dx$$
$$= \int_a^b\int_a^b |k(x,t)|^2 dxdt\cdot\int_a^b |f(t_1)|^2 dt_1 = \int_a^b\int_a^b |k(x,t)|^2 dxdt\cdot\|f\|^2.$$

Consequently, the operator K is defined on the whole space $L^2(a,b)$, is bounded on it, and the norm of the operator can be estimated as

$$\|K\|\le k_0, \text{ where } k_0 = \left(\int_a^b\int_a^b |k(x,t)|^2 dxdt\right)^{1/2}.$$

One should pay attention to the fact that the considered operators *are defined on the whole space* and are bounded. □

2.4 Space of bounded linear operators

Let X and Y be linear normed spaces (both are complex), and let A and B be linear bounded operators defined on the whole space X with values in Y. In a natural way we introduce the concept of ***addition of operators*** A and B by

$$(A+B)x = Ax+Bx,\ \forall x\in X,$$

and the concept of ***multiplication of a linear operator by a scalar*** by $\forall\lambda\in\mathbb{C}$:

$$(\lambda A)x = \lambda(Ax),\ \forall x\in X.$$

It is clear that $(A+B)$ and λA are linear bounded operators and $A+B: X\to Y$, $\lambda A: X\to Y$.

Thus, on the linear set of the bounded operators mapping the linear normed space X into Y, we introduced operations of summation and multiplication by a scalar, therefore, obtaining the space (elements of which are the linear bounded operators), which is called the ***space of linear bounded operators*** and is denoted by $\mathscr{L}(X,Y)$. This space becomes normed if as a norm of the space one takes the standard norm of the bounded operator, and it will be a Banach space if the considered spaces X and Y are Banach spaces.

In the same natural way we can introduce the concept of multiplication of operators. Let linear bounded operators $A: X \to Y$ and $B: Y \to Z$ be given. Then the ***multiplication (or composition) of the linear bounded operators*** B and A is defined according to the rule

$$(BA)x = B(Ax), \ \forall x \in X.$$

It is easy to see that the operator BA is a linear bounded operator $BA: X \to Z$, and $\|BA\| \leq \|B\| \cdot \|A\|$. One should pay attention to the fact that the operators BA and AB do not necessarily coincide even in the case when all the spaces are equal, $X = Y = Z$. The simplest examples of this fact are finite-dimensional operators $A: \mathbb{R}^n \to \mathbb{R}^n$, since the multiplication of matrices is not commutative.

Similar operations of addition, multiplication by a scalar, and multiplication of operators are introduced in a natural way also for operators that are not necessarily bounded. The only difference is that it is necessary to accurately combine required domains and values of the operators. Considering that this procedure is natural, we do not dwell on it in detail.

In a particular case when $X = Y$, the space $\mathscr{L}(X,X)$ of the linear bounded operators in X becomes an algebra with a unit element, where the unit element is the ***identity operator*** $I: X \to X$, acting according to the formula

$$Ix = x, \quad \forall x \in X.$$

Similarly in the space $\mathscr{L}(X,Y)$ we introduce concepts of limit and convergence for sequences of the operators. We do not dwell on these concepts in detail, however, we discuss the important property of extention of the operators with respect to continuity.

2.5 Extension of a linear operator with respect to continuity

Let X and Y be normed spaces, and let $A: X \to Y$ be a linear operator with the domain $D(A) \neq X$, $\overline{D(A)} = X$, which is dense in X. One says that the linear operator A is ***bounded on*** $D(A)$, if its norm is finite

$$\|A\| := \sup_{x \in D(A), \|x\| \leq 1} \|Ax\| < +\infty.$$

As in the case of the bounded linear operators defined on the whole space X, in this case the ***norm of the operator*** is the least of all constants $M > 0$, satisfying the condition

$$\|Ax\| \le M\|x\|, \quad \forall x \in D(A).$$

The following important statement is true.

Theorem 2.12 (On extension of a linear operator with respect to continuity) *Let X be a linear normed space and let Y be a Banach space. Let $A : X \to Y$ be a linear operator with the domain $D(A) \neq X$ which is dense in X, $\overline{D(A)} = X$, and let the operator A be bounded on $D(A)$. Then there exists a linear bounded operator $\widehat{A} \in \mathscr{L}(X,Y)$ such that $\widehat{A}x = Ax$ for all $x \in D(A)$, and $\|\widehat{A}\| = \|A\|$.*

*In other words, any linear bounded operator with values in the Banach space Y, **given on the linear space** $D(A)$ **dense in the linear normed space** X, can be continued onto the whole space with preservation of the value of its norm.*

The indicated process of continuation is called the ***extension with respect to continuity***. If the operator is not bounded (this case is essentially more complicated), then its continuation in the general case is one of the variants of *extension of the operator*. The theory of extensions constitutes an independent and interesting part of the theory of operators. The exact definition and some examples will be given later in Section 2.6.

The simplest examples of an extension of the operator with respect to continuity are situations when the initial operator is given on a domain which is "unnatural" (so to speak, too "narrow") for it.

Example 2.13 In the space of continuous functions $C[a,b]$, consider the operator of taking a trace $P : C[a,b] \to C[a,b]$, acting according to the formula $Pf(x) = f(x_0)$ (where $x_0 \in [a,b]$ is some fixed point), given on the domain $D(P) = C^1[a,b]$. It is easy to see that the operator P is bounded on its domain. Indeed, for all $f \in D(P) = C^1[a,b]$ we have

$$\|Pf\| = \max_{x\in[a,b]} |Pf(x)| = |f(x_0)| \le \max_{x\in[a,b]} |f(x)| = \|f\|,$$

so that

$$\|P\| = \sup_{f\in D(P),\|f\|\le 1} \|Pf\| \le \sup_{f\in D(P),\|f\|\le 1} \|f\| \le 1.$$

We now consider the question of the density of the domain $D(P) = C^1[a,b]$ in the space $C[a,b]$. From the course of analysis it is known that by the Weierstrass theorem one can approximate any continuous function $f = f(x)$ by a sequence of polynomials $f_n(x) = \sum\limits_{k=1}^{n} c_k x^k$. Since $f_n \in C^\infty[a,b]$, then the density of the space $C^\infty[a,b]$ in $C[a,b]$ follows. And since the space of infinitely differentiable on the closed interval $[a,b]$ functions is embedded in the space of continuous functions $C[a,b]$, that is, $C^\infty[a,b] \subset C^1[a,b] \subset C[a,b]$, then it is clear that $C^1[a,b]$ is also dense in $C[a,b]$. And this fact proves that $\overline{D(P)} = C[a,b]$.

Thus all the conditions of Theorem 2.12 are fulfilled. Consequently, there exists the linear bounded operator $\widehat{P} \in \mathscr{L}(C[a,b], C[a,b])$, such that $\widehat{P}f = Pf$, $\forall f \in D(P) = C^1[a,b]$ and $\|\widehat{P}\| = \|P\|$. Earlier, in Example 2.8, we have shown that the operator $P : C[a,b] \to C[a,b]$ given on the whole space $C[a,b]$ is bounded and $\|P\| \leq 1$. That is, as the continuous continuation in this case one can take an operator given by the same formula but already on the whole space. In other words, in this case the action of the operator can be continued onto the set $C[a,b] \backslash C^1[a,b]$ not only preserving the norm value but also "preserving" the formula. □

The cases of such continuation of the bounded operators are precisely the ones most often encountered in practice. The following example also demonstrates this fact.

Example 2.14 In the Hilbert space $L^2(a,b)$, consider the integral operator $K : L^2(a,b) \to L^2(a,b)$ given on the domain $D(K) = C[a,b]$ by formula (2.19), as in Example 2.11:

$$Kf(x) = \int_a^b k(x,t) f(t) dt,$$

where the kernel $k = k(x,t)$ of the integral operator K is a measurable Lebesgue square integrable function $k \in L^2((a,b) \times (a,b))$. To apply Theorem 2.12 on continuous extension it is necessary to show the density of the domain $D(K)$ in $L^2(a,b)$, and the boundedness of the operator on its domain.

From the course of Analysis (the theory of Fourier series, see also Example 1.18, formula (1.12)) it is known that one can approximate any function $f \in L^2(a,b)$ in the norm of $L^2(a,b)$ by a sequence of trigonometric polynomials $f_n = f_n(x)$ being partial sums of the trigonometric Fourier series. Since $f_n \in C[a,b]$, it is clear that $C[a,b]$ is dense in $L^2(a,b)$ and, consequently, $\overline{D(K)} = L^2(a,b)$.

The boundedness of the operator K on $D(K)$ is a consequence of the boundedness of the operator in $L^2(a,b)$, which we have shown in the second part of Example 2.11. There we have also proved the estimate $\|K\| \leq k_0$, where $k_0 = \left(\int_a^b \int_a^b |k(x,t)|^2 dt dx\right)^{1/2}$.

Thus all the conditions of Theorem 2.12 are satisfied. Consequently, there exists the linear bounded operator $\widehat{K} \in \mathscr{L}(L^2(a,b), L^2(a,b))$ such that both $\widehat{K}f = Kf$, $\forall f \in D(K) = C[a,b]$ and $\|\widehat{K}\| = \|K\|$. Earlier, in Example 2.11, we have shown that the operator $K : L^2(a,b) \to L^2(a,b)$ given on the whole space $L^2(a,b)$ is bounded and $\|K\| \leq k_0$. That is, as the continuous extension in this case one can take the operator given by the same formula but already on the whole space.

In other words, in this case the action of the operator can be extended onto the set $L^2(a,b) \backslash C[a,b]$ not only preserving the norm value but also "preserving" formula (2.19). □

2.6 Linear functionals

As mentioned at the end of Section 2.1, the functionals are a particular case of the linear operators. They constitute a large and important class of operators. Let us dwell on their properties in detail.

Let X be a linear normed space. Any linear operator $F : X \to Y$, where $Y = \mathbb{R}$ or $Y = \mathbb{C}$, is called a ***linear functional***. The value of the functional F on the element $x \in X$ is denoted by $F(x)$. Thus the functional maps the linear space into a field of coefficients, that is, the functional is a numerical function defined on the elements $x \in X$.

Since a linear functional is a particular case of a linear operator, concepts of the continuity, boundedness and norm, and also all other properties of the linear operators remain valid for it.

The simplest examples of linear functionals are linear functions acting in a real Euclidean space.

In particular, if we consider the operator from Example 2.8 as an operator acting from $C[a,b]$ into the field of complex numbers $\mathbb{C}$ by the formula

$$Pf(x) = f(x_0), \text{where } x_0 \in [a,b] \text{ is some fixed point,}$$

then the operator P is a functional, that is, an operator matching some complex number $f(x_0)$ to each function $f \in C[a,b]$. $\square$

As in the case of linear operators (see the beginning of Section 2.3), *the image of the linear functional* (which is even bounded) *is not a bounded set*. More precisely, if the not-identically-zero linear functional $F : X \to \mathbb{C}$ is defined on the whole space X, then the image of the functional F is all $\mathbb{C}$.

The important fact in using the linear functionals is an opportunity of describing their general form in specific spaces.

The general form of the linear functionals in Hilbert spaces is established by the following

Theorem 2.15 (F. Riesz). *Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. For any linear bounded functional F defined on the whole space H, there exists a unique element $\sigma \in H$ such that $F(x) = \langle x, \sigma \rangle$ for all $x \in H$. Then we also have $\|F\| = \|\sigma\|$.*

This theorem establishes the general form of the linear bounded functional in a Hilbert space. Let us consider this in a specific case.

Example 2.16 Let the Hilbert space be $H = L^2(a,b)$. The usual inner product in this space is given by the formula

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx. \tag{2.21}$$

Taking this into account, by the Riesz theorem 2.15, if F is a linear bounded functional defined everywhere on $L^2(a,b)$, then there exists a function $\sigma = \sigma(x) \in L^2(a,b)$ such that

$$F(f) = \int_a^b f(x)\overline{\sigma(x)}dx, \ \forall f \in L^2(a,b), \tag{2.22}$$

with $\|F\| = \|\sigma\|_{L^2(a,b)}$.

Formula (2.22) sets ***the general form of the linear bounded functional in the space*** $L^2(a,b)$.□

The linear bounded functionals can be set by a formula analogous to (2.22) also in other (not necessarily Hilbert) spaces.

Example 2.17 Let $X = L^p(a,b)$, $p > 1$. On this space let us set a linear functional by the formula

$$F(f) = \int_a^b f(x)\overline{\sigma(x)}dx, \ \forall f \in L^p(a,b). \tag{2.23}$$

Lemma 2.18 *The linear functional F defined on the entire space $L^p(a,b)$ by formula* (2.23) *is bounded if and only if $\sigma \in L^q(a,b)$, where $\frac{1}{p}+\frac{1}{q}=1$, and we have*

$$\|F\| = \|\sigma\|_{L^q(a,b)}.$$

Let us show here only the sufficiency. Let $\sigma \in L^q(a,b)$, where $\frac{1}{p}+\frac{1}{q}=1$. Then using the Hölder integral inequality (2.20), we get

$$|F(f)| = \left|\int_a^b f(x)\overline{\sigma(x)}dx\right| \le \int_a^b |f(x)\overline{\sigma(x)}|dx$$

$$\le \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |\sigma(x)|^q dx\right)^{1/q} = \|f\|_p \cdot \|\sigma\|_q.$$

Therefore the functional is bounded and

$$\|F\| \le \|\sigma\|_{L^q(a,b)}. \tag{2.24}$$

To prove that in fact in (2.24) the equality is achieved, assuming that $\sigma \not\equiv 0$, as a test function we take the function $f_\sigma(x) = \overline{\sigma(x)} \cdot |\ \sigma(x)|^{q-2} \cdot \|\sigma\|^{1-q}$. By a direct calculation we obtain

$$\|f_\sigma\|_p = \left(\int_a^b |f_\sigma(x)|^p dx\right)^{1/p} = \left(\int_a^b \left||\sigma(x)|^{q-1} \cdot \|\sigma\|^{1-q}\right|^p dx\right)^{1/p} =$$

$$= \|\sigma\|^{1-q} \left(\int_a^b |\sigma(x)|^{p(q-1)} dx\right)^{1/p}.$$

From this, taking into account that $p(q-1)=q,\ 1-q+\frac{q}{p}=0$, we have

$$\|f_\sigma\|_p = \|\sigma\|^{1-q}\left(\int_a^b |\sigma(x)|^q dx\right)^{1/p} = \|\sigma\|^{1-q+\frac{q}{p}} = \|\sigma\|^0 = 1.$$

Therefore, $f_\sigma \in L^p(a,b)$ and $\|f_\sigma\|_p = 1$.

Consider now the action of the functional F on the test function f_σ :

$$F(f_\sigma) = \int_a^b f_\sigma(x)\overline{\sigma(x)}dx = \int_a^b |\sigma(x)|^{2+q-2}\cdot\|\sigma\|^{1-q}dx =$$

$$= \|\sigma\|^{q+1-q} = \|\sigma\|.$$

Thus for the test function f_σ we get

$$\|f_\sigma(x)\|_p = 1 \text{ and } F(f_\sigma) = \|\sigma\|.$$

Then $|F(f_\sigma)| = \|\sigma\|$ and, consequently, $\|F\| \geq \|\sigma\|_{L^q(a,b)}$.

Comparing this inequality with (2.24), we obtain that $\|F\| = \|\sigma\|_{L^q(a,b)}$. The sufficiency of the conditions of Lemma 2.18 is proved. We will not dwell on the proof of necessity of the conditions. □

As already mentioned in Section 1.1, the proof of the completeness of the system $\{u_k\}$ in X is substantiated by means of proving everywhere density in X of all linear combinations of these vectors $\{u_k\}$, that is, of the density in X of the linear space spanned by the vectors $\{u_k\}$.

The following theorem substantiates the research method of the density in X of a linear subspace by using the linear functionals.

Theorem 2.19 *For a linear subspace M to be dense in H, it is necessary and sufficient that the linear functional F vanishing on all elements $x \in M$ is identically zero.*

In the case when H is the Hilbert space, taking into account the Riesz theorem 2.15, we obtain the important corollary which is often used in the spectral theory of operators.

Corollary 2.20 *For the completeness of the system $\{u_k\}$ in the Hilbert space H (that is, for the density in H of the linear space M spanned by the vectors $\{u_k\}$), it is necessary and sufficient that the equalities $\langle u_k, \sigma\rangle = 0$, $\forall u_k \in M$, imply that $\sigma = 0$.*

In other words, using the terminology of Section 1.6, for the completeness of the system $\{u_k\}$ in the Hilbert space H (that is, for the density in H of the linear space M spanned by the vectors $\{u_k\}$), it is necessary and sufficient that $\sigma = 0$ follows from the condition $u_k \perp \sigma\ \ \forall u_k \in M$.

This means that the absence of the nonzero element in the Hilbert space H which is orthogonal to all elements of the system $\{u_k\}$ is necessary and sufficient for the completeness of the system $\{u_k\}$ in H. □

2.7 Inverse operators

Generally speaking, the whole theory of operators is developed for "solving" some equations (and for studying their various properties) which in the operator language can be written in the form

$$Ax = y,\ x \in D(A), \tag{2.25}$$

where $A : X \to Y$ is an operator, y is a given element from Y, and x is an unknown desired element of the space X. Writing equations in the operator form (2.25) allows one to detract from specific and partial difficulties inherent to each particular problem by focussing on more general patterns.

If we somehow try to solve equation (2.25), then we get some mapping under which some *particular* element x of the space X is put in correspondence to the element y of the space Y. This mapping generates some operator called an ***inverse*** operator.

More precisely, if to two different elements from $D(A)$ the operator $A : X \to Y$ puts in correspondence different elements from $R(A)$, then A has an ***inverse operator***, which to the elements from $R(A)$ puts in correspondence the elements from $D(A)$. The inverse operator is denoted by the symbol A^{-1}.

Thus, the solution of Eq. (2.25) is written in the form $x = A^{-1}y$, and herewith $D(A^{-1}) = R(A)$ and $R(A^{-1}) = D(A)$.

The operator A having an inverse operator defined on $R(A)$ is called ***invertible***.

If this operator A is linear, then the inverse operator A^{-1} is also linear.

Thus, the question of the solvability of Eq. (2.25) reduces to finding conditions under which the inverse operator A^{-1} exists, and the question of properties of solutions to Eq. (2.25) reduces to studying the properties of the operators A and A^{-1}.

Since A is an operator, to each element $x \in D(A)$ there corresponds some *specific* element $y \in R(A)$. For the inverse correspondence $Y \to X$ to be also an operator, it is necessary that to each element $y \in R(A)$ there corresponds *the specific* element $x \in R(A)$. Thus, the existence of an inverse operator effectively means that the operator A defines a *one-to-one* correspondence between $D(A)$ and $R(A)$. Therefore the inverse operator is denoted by A^{-1}, since $AA^{-1} = I$ and $A^{-1}A = I$, where I is the identity operator acting in the first case from $R(A)$ to $R(A)$, and in the second case from $D(A)$ to $D(A)$. Recall that the concept of multiplication of the operators has been discussed earlier, at the beginning of Section 2.4.

Example 2.21 Consider the operator of "taking a trace" $P :\ C[a,b] \to C[a,b]$, introduced in Example 2.8, acting according to the formula $Pf(x) = f(x_0)$, where $x_0 \in [a,b]$ is some fixed point. Here the domain of the operator coincides with the whole space: $D(P) = C[a,b]$, and the image coincides with the whole complex plane: $R(P) = \mathbb{C}$. It is easily seen that the operator P *does not have an inverse operator*, since there is no one-to-one correspondence between $D(P)$ and $R(P)$. Indeed, to two various functions $f_1(x) \neq f_2(x)$ there corresponds a single number in the image of the operator P, as soon as these functions coincide at one point: $f_1(x_0) = f_2(x_0)$. □

The question of when A defines a one-to-one correspondence between $D(A)$ and $R(A)$ is answered by the following theorem.

Let us consider a linear space

$$\ker(A) = \{x \in D(A) : Ax = 0\},$$

called a ***set of zeros*** or an (operator) ***kernel*** of the operator A. The kernel of the operator is always nonempty, since $0 \in \ker(A)$.

Theorem 2.22 *The operator A is one-to-one from $D(A)$ to $R(A)$ if and only if* $\ker(A) = \{0\}$*, that is, if the kernel of the operator consists of only the element* 0.

Corollary 2.23 *An inverse operator A^{-1} defined on $R(A)$ exists if and only if* $\ker(A) = \{0\}$.

Let us return again to Example 2.21. It is easy to see that the kernel of the operator P contains all functions vanishing at the point x_0:

$$\ker P = \{f \in C[a,b] : f(x_0) = 0\}.$$

Since this set consists of more than the single element 0, then by Theorem 2.22 the inverse operator to P does not exist. □

As is known from the theory of linear equations (for example, linear differential equations or integral equations), the proof of the uniqueness of solution is equivalent to the proof of the absence of nonzero solutions of the corresponding homogeneous equation. Thus, from the point of view of the operator equation (2.25), *the existence of the inverse operator A^{-1}*, defined on $R(A)$, *is equivalent to the uniqueness of the equation's solution*, if it exists.

The question of the existence of *a bounded* inverse operator is answered by the following:

Theorem 2.24 *The operator A^{-1} exists and is bounded on $R(A)$ if and only if for some constant $m > 0$ and for all $x \in D(A)$ the inequality*

$$\|Ax\| \geq m\|x\| \tag{2.26}$$

holds.

Inequality (2.26) for Eq. (2.25) can be rewritten in the form

$$\|x\| \leq C\|y\|,$$

where the constant C does not depend on x and y. The inequalities of such type are usually called ***a priori estimates***. A priori is Latin for "from before" and refers to the fact that the estimate for the solution is derived before the solution is known to exist.

In solving Eq. (2.25), the "good" case is when the equation's solution exists for all $y \in Y$. An operator $A : X \to Y$ is called ***everywhere solvable*** (see also Example 2.21), if $R(A) = Y$. In the case when $R(A) \neq Y$, but the image $R(A)$ of the operator A is closed, the operator is called ***normally solvable*** (see also Example 2.37).

The operator $A : X \to Y$ is called ***well-posedly solvable***, if it is invertible and A^{-1} is bounded (see the first part of Example 2.27). If the operator is well-posedly solvable, then for all $y_1, y_2 \in R(A)$ there exist corresponding unique solutions $x_1 = A^{-1}y_1$, $x_2 = A^{-1}y_2$ of Eq. (2.25) and (by the boundedness and linearity of A^{-1}) they satisfy the inequality $\|x_1 - x_2\| \leq \|A^{-1}\| \|y_1 - y_2\|$. This means that a slight change in the right-hand side y leads to a slight change in the solution x. In this case one also says that the solution continuously depends on the right-hand side of the equation.

The problem expressed by the operator equation (2.25) is called ***well-posed***, if its solution exists for any right-hand side $y \in Y$, is unique, and continuously depends on the right-hand side. Correspondingly, we can introduce a similar concept for the operators. The operator $A : X \to Y$ is called ***well-posed***, if the inverse operator A^{-1} exists, is defined on the whole space Y, and is bounded.

By using the earlier introduced terms we can say that the operator is well-posed, if it is well-posedly and anywhere solvable. The well-posed operators are also called ***continuously invertible***. Taking into account this terminology, we have the following consequence of Theorem 2.24:

Theorem 2.25 *The operator A is continuously invertible (i.e. is well-posed) if and only if $R(A) = Y$, and for some constant $m > 0$ and $\forall x \in D(A)$ the inequality* (2.26) *holds true:*

$$\|Ax\| \geq m\|x\|.$$

Let us make up the table of correspondence for the terms of invertibility of the operator and the solvability of Eq. (2.25):

The solution of Eq. (2.25) *is unique*	$\Leftrightarrow$	The inverse operator A^{-1} exists, that is, *the operator A* is *invertible*
The solution of Eq. (2.25) *exists for any right-hand side* $y \in Y$	$\Leftrightarrow$	The image of the operator coincides with the entire space: $R(A) = Y$, that is, *the operator A is everywhere solvable.*

The solution of Eq. (2.25) *continuously depends on the right-hand side* $y \in Y$	$\Leftrightarrow$	The operator A^{-1} is *bounded on* $R(A)$.
The solution of Eq. (2.25) *exists for any right-hand side* $y \in Y$, *uniquely and continuously depends on the right-hand side* $y \in Y$	$\Leftrightarrow$	*The operator* A is *well-posed, that is, the inverse operator* A^{-1} *exists, is defined on the whole space* Y, *and is bounded. It is also said that the operator* A *is boundedly invertible on the whole space* Y.

Thus, to prove that the initial *problem* expressed in the form of the operator equation (2.25) *is well-posed*, it is necessary and sufficient to prove that the *operator A* corresponding to it *is well-posed*, that is, the inverse operator A^{-1} exists, is defined on the whole space Y, and is bounded.

Theorems 2.22, 2.24 and 2.25 formulated above hold for arbitrary linear operators. In the case when the operator A is defined on the whole space ($D(A) = X$) and is bounded ($\|A\| < +\infty$), a significant result in the well-posedness theory is the ***Banach theorem on the inverse operator***.

Theorem 2.26 (S. Banach) *If a linear bounded operator $A : X \to Y$ maps the Banach space X onto the Banach space Y and is one-to-one, then A has the inverse operator A^{-1}, and the operator A^{-1} is bounded.*

In other words, the Banach theorem means that if one has the existence and uniqueness of the solution to the equation

$$Ax = y$$

for any right-hand side $y \in Y$, then it implies also the continuous dependence of the solution $x = A^{-1}y$ on the right-hand side y. We note that this fact is valid only for the case of linear operators. Thus, the linear operator *is well-posed* $\Leftrightarrow$ the operator *is everywhere solvable and invertible*.

2.8 Examples of invertible operators

We now consider some examples illustrating introduced terms and theorems.

Example 2.27 The inverse operator to the operator of multiplication by a function, considered in Examples 2.6 and 2.7 can be constructed rather simply. Let the action of an operator be defined by the formula

$$Tf(x) = G(x) \cdot f(x),$$

where $G = G(x)$ is a continuous function defined on the closed interval $[a,b]$. Consider the question of the invertibility of this operator in the space $C[a,b]$.

From the results of Example 2.6 it follows that the operator $T : C[a,b] \to C[a,b]$ is defined on the whole space $C[a,b]$ and is bounded. The invertibility of the operator T is equivalent to the fact that the solution of the equation

$$G(x) \cdot f(x) = F(x), \ a \leq x \leq b, \tag{2.27}$$

exists for any function $F \in C[a,b]$, is unique, and depends continuously on F. Formally, the solution of Eq. (2.27) is the function

$$f(x) = \frac{F(x)}{G(x)}, \ a \leq x \leq b, \tag{2.28}$$

and this solution is unique. It would seem that the solution is unique and exists for any right-hand part $F(x)$. However, we have to remember that the obtained solution has to belong to the considered space. It is easy to see that the solution (2.28) of Eq. (2.27) will belong to the space of continuous functions $C[a,b]$ for any right-hand side $F \in C[a,b]$ if and only if

$$|G(x)| > 0, \ \ \forall x \in [a,b]. \tag{2.29}$$

Now consider the same question but from the point of view of the operators. According to the corollary from Theorem 2.22 for the existence of the inverse operator T^{-1}, it is necessary and sufficient to show that the kernel of the operator T has no nonzero elements. The kernel of the operator is the functions $f = f(x)$ being the solutions of the equation $Tf = 0$, that is, the solution of the equation

$$G(x) \cdot f(x) = 0, \ a \leq x \leq b. \tag{2.30}$$

It is clear that Eq. (2.30) has the nonzero solution in the space of the continuous functions $C[a,b]$ if and only if there is an interval $[a_1,b_1] \subseteq [a,b]$ of the nonzero length $b_1 - a_1 > 0$ such that $G(x) \equiv 0$ for all $a_1 \leq x \leq b_1$. Then the solutions of Eq. (2.30) will be all the continuous functions which are equal to zero outside of the interval $[a_1,b_1]$, and in this case the inverse operator T^{-1} does not exist. If such interval $[a_1,b_1]$ does not exist, then Eq. (2.30) has only zero solution and, consequently, the kernel of the operator T consists only of the element 0. In this case the inverse operator T^{-1} exists. In comparison with the solution of Eq. (2.27), the existence of such interval $[a_1,b_1]$ means that in this case the formal expression of the solution by formula (2.28) is ill-posed for $x \in [a_1,b_1]$, since, as one says, "one cannot divide by zero".

Suppose now that such interval $[a_1, b_1]$ does not exist. Then the operator T^{-1} exists, but is defined only on the range $R(T)$ of the operator T, so that its domain does not necessarily coincide with the whole space $C[a,b]$.

For example, for $G(x) = (x-a)^2$ the image of the operator T contains only functions vanishing at the point $x = a$ and, consequently, cannot coincide with the whole space $C[a,b]$. In this case condition (2.29) does not hold and it is easy to show that the operator T^{-1} defined on $R(T)$ is not bounded. By the terminology introduced earlier the operator T in such case *is not well-posedly solvable*.

If condition (2.29) holds, then there exists a number $m > 0$ such that

$$|G(x)| \geq m \;\; \forall x \in [a,b].$$

Therefore, $|Tf(x)| = |G(x)||f(x)| \geq m|f(x)| \;\; \forall x \in [a,b]$. By passing to maximum in this inequality, we obtain

$$\|Tf\| \geq m\|f\|, \; m > 0, \; \forall f \in C[a,b].$$

This inequality is inequality (2.26) from Theorems 2.22 and 2.24. Consequently, the inverse operator T^{-1} exists, is defined on the whole space $C[a,b]$ and is bounded, that is, the operator T is well-posed. □

Example 2.28 In the space of continuous functions $C[a,b]$, consider the operator acting by the formula

$$L_1 u(x) = \frac{d}{dx} u(x), \; a < x < b,$$

defined on the domain $D(L_1) = C^1[a,b] \subset C[a,b]$. We have considered this operator in the second part of Example 2.9. Let us show that in this case the operator L_1 is not well-posed, since it does not have an inverse operator.

Let us describe the kernel of the operator L_1. Its elements are all functions $u \in D(L_1) = C^1[a,b]$, for which $L_1 u(x) = 0$, that is, all solutions of the differential equation $u'(x) = 0$, $a < x < b$. From the theory of ordinary differential equations it is known that all continuously differentiable solutions of this equation have the form $u(x) = Const$, $a \leq x \leq b$. Consequently, $\ker L_1 = \{u(x) : u(x) = Const \; \forall x \in [a,b]\}$. It is clear that the kernel of the operator L_1 consists not only of a zero element and, consequently, according to the corollary from Theorem 2.22 the inverse operator to the operator L_1 does not exist.

The problem corresponding to finding an inverse operator in this case is the problem of finding a continuously differentiable solution to the differential equation

$$\frac{d}{dx} u(x) = f(x). \tag{2.31}$$

From the course on Ordinary Differential Equations it is known that all continuously differentiable solutions of these equations have the form

$$u(x) = \int_a^x f(t)dt + Const, \; a \leq x \leq b.$$

Consequently, the solution of Eq. (2.31) is not unique. According to the correspondence table of the terms of the operator invertibility and the equation solvability this means that the operator L_1 *does not have the inverse one*. However from the previous discussions it follows that the solution exists for any right-hand side $f \in C[a,b]$, that is, $R(L_1) = C[a,b]$. By the terminology introduced earlier this means that the operator L_1 *is everywhere solvable*.

Once more we pay attention to the fact that the operator can be everywhere solvable and does not have the inverse one. □

The considered example shows that the operator does not have the inverse one, since the solution of the corresponding problem is not unique. In the course of differential equations, additional conditions, for example boundary conditions, are set for defining a unique solution of the differential equation. In the following example we consider an operator corresponding to this problem.

Example 2.29 In the space of continuous functions $C[a,b]$, consider the operator acting by the formula

$$Lu(x) = \frac{d}{dx}u(x), \ a < x < b,$$

given on the domain $D(L) = \{u \in C^1[a,b] : u(a) = 0\}$.

Recall that in order for the operator L to be linear, we cannot consider boundary conditions of the kind $u(a) = u_0$, $u_0 \neq 0$ (see Example 2.1). This operator, in contrast to the operator considered in Example 2.28, has a "smaller" domain.

Let us describe the kernel of the operator L. Its elements are all functions $u \in D(L)$, for which $Lu(x) = 0$, that is, all continuously differentiable solutions of the differential equation $u'(x) = 0$, $a < x < b$, for which $u(a) = 0$. All solutions of this differential equation have the form $u(x) = Const$, $a \leq x \leq b$. Since $u(a) = 0$, then, consequently, $u(x) \equiv 0$ and $\ker L = \{0\}$. According to the corollary from Theorem 2.22, it means that the inverse operator to the operator L exists.

The problem corresponding to finding the inverse operator in this case is a problem of finding a continuously differentiable solution of the differential equation $u'(x) = f(x)$ satisfying the boundary condition $u(a) = 0$. Since all continuously differentiable solutions of this equation have the form

$$u(x) = \int_a^x f(t)dt + Const, \ a \leq x \leq b, \tag{2.32}$$

then, satisfying equality (2.32) with the boundary condition $u(a) = 0$, we find that $Const = 0$. Consequently, the unique solution of the problem has the form

$$u(x) = \int_a^x f(t)dt, \ a \leq x \leq b, \tag{2.33}$$

for all $f \in C[a,b]$. Formula (2.33) actually gives the representation of the inverse operator as

$$L^{-1}f(x) = \int_a^x f(t)dt, \ \forall f \in C[a,b]. \tag{2.34}$$

It is clear that the operator L^{-1} is defined on the whole space $C[a,b]$. From (2.34) for all $f \in C[a,b]$ we get

$$\|L^{-1}f\| = \max_{x\in[a,b]} |L^{-1}f(x)| = \max_{x\in[a,b]} \left|\int_a^x f(t)dt\right| \leq \max_{x\in[a,b]} \int_a^x |f(t)|dt$$
$$\leq (b-a) \max_{x\in[a,b]} |f(t)| = (b-a)\|f\|.$$

That is, the operator L^{-1} is bounded and $\|L^{-1}\| \leq b-a$.

Thus, we have shown that the inverse operator L^{-1} exists, is defined on the whole space $C[a,b]$ and is bounded, that is, the operator L is well-posed. □

Example 2.30 In the linear space of vectors $\mathbb{R}^n$, consider a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ acting by the formula

$$A(x) = (a_{ij})x,$$

where (a_{ij}) is a matrix of the size $n \times n$. If the vector x has "coordinate" representation in the form of a column $x = (x_1, x_2, \dots, x_n)$, then the image $y = Ax$ also has "coordinate" representation $y = (y_1, y_2, \dots, y_n)$, and

$$y_i = \sum_{j=1}^{n} a_{ij}x_j, \ \forall i = 1, \dots, n. \tag{2.35}$$

From the theory of system of linear equations it is known that system (2.35) has a unique solution if and only if the determinant of this system $\det(a_{ij}) \neq 0$. In this case the solution of system (2.35) is given by the matrix

$$x = (a_{ij})^{-1}y$$

which is inverse to (a_{ij}). In this case the inverse operator A^{-1} exists and is defined on the whole space $\mathbb{R}^n$. Since in the finite-dimensional space all operators are bounded (see Example 2.5), the well-posedness of the operator A for $\det(a_{ij}) \neq 0$ follows. □

Example 2.31 In the space of continuous functions $C[0,1]$, consider the integral operator acting by the formula

$$Tu(x) = u(x) - \int_0^1 xt\,u(t)dt, \ \ 0 < x < 1.$$

The operator T is the sum of the identity operator I, $Iu = u$, $\forall u \in C[0,1]$, and the integral operator S is defined by

$$Su(x) = -\int_0^1 xt\,u(t)dt, \ \forall u \in C[0,1].$$

Since the kernel of the integral operator S given by the formula $k(x,t) = xt$ satisfies the condition $k \in C([0,1] \times [0,1])$, then, as is shown in the first part of Example 2.11, the operator S is bounded on the space $C[0,1]$. Consequently, the operator $T = I + S$ is also defined on the whole space $C[0,1]$ and is bounded.

The problem corresponding to finding an inverse operator T^{-1} in this case is the problem of finding a continuous solution of the integral equation

$$u(x) - \int_0^1 xt\,u(t)dt = f(x),\ 0 < x < 1. \tag{2.36}$$

Analysing Eq. (2.36), it is easy to see that all its solutions have the form

$$u(x) = f(x) + cx, \text{ where } c = \int_0^1 tu(t)dt. \tag{2.37}$$

Multiplying this equality by x and integrating the obtained result over the interval $[0,1]$, we find

$$c = \int_0^1 xu(x)dx = \int_0^1 xf(x)dx + c\int_0^1 x^2dx = \int_0^1 xf(x)dx + \frac{c}{3} \Rightarrow c = \frac{3}{2}\int_0^1 tf(t)dt.$$

Consequently, the solution of Eq. (2.36) exists for any right-hand side $f \in C[0,1]$, is unique, and is represented in the form

$$u(x) = f(x) + \frac{3}{2}\int_0^1 xtf(t)dt.$$

In fact this formula proves the existence of the inverse operator and gives its representation in an explicit form as

$$T^{-1}f(x) = f(x) + \frac{3}{2}\int_0^1 xtf(t)dt.$$

Paying attention to the fact that $T^{-1} = I + \frac{3}{2}S$, from the boundedness of the operator S it follows that the inverse operator T^{-1} is defined on the whole space and is bounded. Consequently, the operator T is well-posed. □

2.9 The contraction mapping principle

In all the considered Examples 2.27 - 2.31 the basis of the well-posedness proof of the initial operator is the explicit construction of an inverse operator. There, for proving the well-posedness of the problem we have not used any operator methods, but made the conclusion on the well-posedness of the operator based on the well-posedness of the corresponding problem. The following theorem allows us to make the conclusion on the well-posedness of the operator without having an explicit solution of the corresponding problem.

Theorem 2.32 (The contraction mapping principle) *Let X be a Banach space and let a bounded operator $A : X \to X$ satisfy $\|A\| < 1$. Then the operator $I - A$ is well-posed, that is, is boundedly invertible on the whole space X, and the estimate*

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$$

is valid.

This theorem is also called the ***Banach Fixed Point Theorem***.

A bounded operator having the property $\|A\| < 1$ is also called a ***contraction***, however this terminology is often also used in the theory of nonlinear operators.

The following example demonstrates one of the simplest applications of Theorem 2.32.

Example 2.33 Let us return to the integral operator case considered in Example 2.11. Let the action of an operator be given by formula (2.19):

$$Kf(x) = \int_a^b k(x,t)f(t)dt,$$

where $k = k(x,t)$ is a function defined on the closed rectangle $[a,b] \times [a,b]$. We consider the question of the well-posedness of the operator $I - \lambda K$ in spaces $C[a,b]$ and $L^2(a,b)$.

1) Case of the space $C[a,b]$. Let the kernel of the operator K be a continuous function $k = k(x,t) \in C([a,b] \times [a,b])$. Then, as is shown in Example 2.11, the operator K if defined on the whole space $C[a,b]$, is bounded on it, and

$$\|K\| \leq M(b-a).$$

Here, as in Example 2.11, we used the notation $M = \max\limits_{x,t \in [a,b]} |k(x,t)|$.

From this, Theorem 2.32 implies the well-posedness of the operator $I - \lambda K : C[a,b] \to C[a,b]$ for all $\lambda \in \mathbb{C}$ such that

$$|\lambda| < \frac{1}{M(b-a)}. \tag{2.38}$$

Indeed, if condition (2.38) holds, then the norm of the operator λK is less than one:

$$\|\lambda K\| = |\lambda| \cdot \|K\| \leq |\lambda| \cdot M \cdot (b-a) < 1,$$

which allows us to use Theorem 2.32.

From the well-posedness of the operator we can make the conclusion on the unique solvability of the corresponding problem:

Conclusion 2.34 Under condition (2.38), the integral equation

$$f(x) - \lambda \int_a^b k(x,t)f(t)dt = F(x), \quad a \leq x \leq b, \tag{2.39}$$

with the continuous kernel $k = k(x,t) \in C([a,b] \times [a,b])$ has a unique solution for any right-hand side $F \in C[a,b]$. This solution belongs to the space $C[a,b]$ and continuously depends on the right-hand side of Eq. (2.39).

2) Case of the space $L^2(a,b)$. Let now the kernel $k(x,t)$ of the integral operator K be a Lebesgue square integrable function, $k = k(x,t) \in L^2((a,b) \times (a,b))$. Then, as shown in Example 2.11, the operator K is defined on the whole space $L^2(a,b)$, is bounded on it, and $\|K\| \leq k_0$, where $k_0 = \left(\int_a^b \int_a^b |k(x,t)|^2 dt dx\right)^{1/2}$. As in the first part of this example, by using Theorem 2.32, we conclude that the operator $I - \lambda K : L^2(a,b) \to L^2(a,b)$ is well-posed for all $\lambda \in \mathbb{C}$ satisfying the inequality

$$|\lambda| < \frac{1}{k_0}. \tag{2.40}$$

From the well-posedness of the operator we can conclude the unique solvability of the corresponding problem:

Conclusion 2.35 Under condition (2.40), the integral equation (2.39) with the kernel $k \in L^2((a,b) \times (a,b))$ has a unique solution for any right-hand side $F \in L^2(a,b)$. This solution belongs to the space $L^2(a,b)$ and continuously depends on the right-hand side of (2.39).

The above example is interesting in the sense that one can conclude the existence and uniqueness of the problem's solution (in this case of the integral equation) without resorting to specific calculations of solutions, only on the basis of the general theorems of functional analysis. From this point of view, the specific form of the integral kernel of the operator has no influence for conclusions; the main feature is only its "size" (that is, the norm of the integral kernel $k = k(x,t)$ in the spaces $C([a,b] \times [a,b])$ or $L^2((a,b) \times (a,b))$, respectively). And if such norm of the integral kernel is finite, then the well-posedness of the operator $I - \lambda K$ can be always ensured by choosing the number λ small enough.

Let us give now an example illustrating the terminology introduced earlier of the normally solvable operator (see definitions after Theorem 2.24).

Example 2.36 In the space of square summable functions, consider the operator $L : L^2(a,b) \to L^2(a,b)$ defined by the differential expression

$$Lu(x) = \frac{d}{dx}u(x), \ \ a < x < b,$$

on the domain $D(L) = \left\{u \in L_1^2(a,b) : u(a) = u(b) = 0\right\}$.

We have considered the similar operator in the Lebesgue spaces but with a larger domain in Example 2.10. In this case, as in Example 2.10, it is easy to see that the operator L is linear, is defined on all functions from its domain, and is bounded as the operator from the space $L^2(a,b)$ into $L^2(a,b)$.

The problem corresponding to finding an inverse operator L^{-1} in this case is the problem of finding a solution $u \in L_1^2(a,b)$ to the differential equation

$$u'(x) = f(x), \ a < x < b, \ \ f \in L^2(a,b), \tag{2.41}$$

satisfying the boundary conditions

$$u(a) = 0 \text{ and } u(b) = 0. \tag{2.42}$$

A general solution of (2.41) in the class $u \in L_1^2(a,b)$ is represented in the form

$$u(x) = \int_a^x f(t)dt + C,\ C = Const,\ \ a \le x \le b.$$

This together with the first of conditions (2.42) implies $C = 0$. Consequently, any solution of Eq. (2.41) in the class $u \in L_1^2(a,b)$ satisfying the first of the boundary conditions (2.42) has the form

$$u(x) = \int_a^x f(t)dt,\ a \le x \le b. \tag{2.43}$$

From this it is readily seen that in order for this solution to satisfy also the second of the boundary conditions (2.42), it is necessary and sufficient that the condition

$$\int_a^b f(x)dx = 0 \tag{2.44}$$

holds.

Thus, the range $R(L)$ of the operator L coincides with the linear space of functions from the space $L^2(a,b)$ satisfying condition (2.44). As it was shown earlier in Example 1.19, this space is the subspace of the space $L^2(a,b)$, and, consequently, is a closed set. That is, the range $R(L)$ of the operator L is closed and therefore the operator is *normally solvable*.

Thus, it is clear that the operator L *is not well-posed* since it *is not everywhere solvable*, although the inverse operator exists and is bounded on $R(L)$. □

2.10 Normally solvable operators

Recall that the operator $A : X \to Y$ is called ***everywhere solvable*** (see also Example 2.21), if $R(A) = Y$. In the case when $R(A) \neq Y$, but the image $R(A)$ of the operator A is closed, the operator is called ***normally solvable*** (see also Example 2.37).

The normal solvability of an operator is a necessary condition for its well-posedness. We should pay attention to the normal solvability when introducing an operator corresponding to the problem. Conditionally saying, *when introducing the operator* corresponding to the problem, its domain must be defined (given) in such a correct way that the operator is normally solvable. That is, *the image of the operator must be a closed set*.

Consider the following examples of operators with “unnatural” domains.

Example 2.37 In the space of continuous functions $C[a,b]$, consider the operator defined by

$$Lu(x) = \frac{d}{dx}u(x),\ a < x < b,$$

given on the domain $D(L) = \{u \in C^2[a,b] :\ u(a) = 0\}$. This operator differs from the one considered in Example 2.29 by the property that the domain in the previous example consisted of the continuously differentiable functions, but in this example we require that any function from $D(L)$ is twice continuously differentiable.

As in Example 2.29 it is easy to show that the kernel of the operator consists only of the zero element 0, that is, $\ker L = \{0\}$. It means that by the corollary of Theorem 2.22 the inverse operator to the operator L exists, that is, the operator L is invertible. Its inverse operator L^{-1} is defined on $R(L)$ and is expressed by the formula

$$u(x) = L^{-1}f(x) = \int_a^x f(t)dt,\ a \leq x \leq b,\ \ \forall f \in R(L).$$

Further, for all $f \in R(L)$ we get

$$\|L^{-1}f\| = \max_{x\in[a,b]} |L^{-1}f(x)| = \max_{x\in[a,b]} \left|\int_a^x f(t)dt\right| \leq \max_{x\in[a,b]} \int_a^x |f(t)|dt$$
$$\leq (b-a) \max_{x\in[a,b]} |f(t)| = (b-a)\|f\|,$$

that is, the inverse operator L^{-1} is *bounded on* $R(L)$. Consequently, the operator L *is well-posedly solvable*.

However, in spite of the fact that the operator L is invertible and well-posedly solvable, it *is not well-posed*, since its image $R(L)$ does not coincide with the whole considered space $C[a,b]$, since it contains only continuously differentiable functions (the image of twice continuously differentiable functions after one differentiation). That is, the operator L *is not everywhere solvable*.

Analysing the obtained result, we can come to the conclusion that the operator L has, so to speak, "the unnatural domain", which causes problems with the solvability everywhere. As Example 2.29 shows, the most "natural" domain of this operator is

$$D(L) = \left\{u \in C^1[a,b] : u(a) = 0\right\}, \tag{2.45}$$

that is, the domain consisting of functions of $C^1[a,b]$, but not of $C^2[a,b]$. □

The following example shows that if we consider the same operator with the "natural" domain but already in another space, then it can again lead to the ill-posedness of the operator.

Example 2.38 In the space of square integrable functions consider the operator $L : L^2(a,b) \to L^2(a,b)$ with the domain (2.45), given by

$$Lu(x) = \frac{d}{dx}u(x),\ \ a < x < b.$$

As in the previous case it is easy to show that the kernel of the operator consists only of the zero element 0, that is $\ker L = \{0\}$. It means that by the corollary of Theorem 2.22 an inverse operator to the operator L exists, that is, the operator L is invertible. Its inverse operator L^{-1} is defined on $R(L)$ and is expressed by the formula

$$u(x) = L^{-1}f(x) = \int_a^x f(t)dt, \ a \leq x \leq b, \ \ \forall f \in R(L). \tag{2.46}$$

Further, for all $f \in R(L)$, as in Example 2.11, by using the integral Hölder inequality we get

$$\|L^{-1}f\|^2 = \int_a^b \left| \int_a^x f(t)dt \right|^2 dx \leq \int_a^b \left(\int_a^x |f(t)|dt \right)^2 dx = \Big| f(t) \equiv f(t) \cdot 1 \Big|$$

$$\leq \int_a^b \left[\left(\int_a^x |f(t)|^2 dt \right)^{1/2} \left(\int_a^x 1^2 dt \right)^{1/2} \right]^2 dx$$

$$\leq \int_a^b \left[\left(\int_a^b |f(t)|^2 dt \right)^{1/2} (x-a)^{1/2} \right]^2 dx = \frac{(b-a)^2}{2} \int_a^b |f(t)|^2 dt.$$

Therefore

$$\|L^{-1}f\| \leq \frac{b-a}{\sqrt{2}} \|f\|, \quad \forall f \in R(L), \tag{2.47}$$

that is, the inverse operator L^{-1} *is bounded on* $R(L)$. Consequently, the operator L *is well-posedly solvable*.

Let us step back here a little from the main topic and show that inequality (2.47) can be also obtained from more general facts. Earlier, in Example 2.11, we have considered the integral operator (2.19) with the kernel of general kind $k = k(x,t) \in L^2((a,b) \times (a,b))$. Assume that $k(x,t) = \theta(x-t)$, where $\theta(x)$ is the Heaviside step function given by

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

It is easy to see that this $k = k(x,t) = \theta(x-t) \in L^2([a,b] \times [a,b])$ and

$$\begin{aligned} k_0 &\equiv \left(\int_a^b \int_a^b |k(x,t)|^2 dt dx \right)^{\frac{1}{2}} \equiv \left(\int_a^b \int_a^b |\theta(x-t)|^2 dt dx \right)^{\frac{1}{2}} = \\ &= \left(\int_a^b \left(\int_a^x dt \right) dx \right)^{1/2} = \frac{b-a}{\sqrt{2}}. \end{aligned} \tag{2.48}$$

Therefore the operator given by formula (2.46) is a particular case of the general integral equation of the type (2.19), and its boundedness in the space $L^2(a,b)$, and also inequality (2.47), are consequences of the results of Example 2.11 and of equality (2.48).

However, in spite of the fact that the operator L is invertible and well-posedly solvable, it *is not well-posed*, since its image $R(L)$ does not coincide with the whole considered space $L^2(a,b)$, since it contains only continuous functions (the image of continuously differentiable functions constituting the domain of the operator, under operation of one differentiation). That is, the operator L *is not everywhere solvable*. It is caused, as in the previous example, by the fact that the operator L has an "unnatural" domain. $\square$

Let us compare now the results obtained in Example 2.36 with the results of Examples 2.37 and 2.38. In all three cases the considered operators are not well-posed. However, in the first case (Example 2.36) the image $R(L)$ of the operator L is closed and forms a closed subspace of the considered space. In the other two cases (Examples 2.37 and 2.38) the image $R(L)$ of the operator L is not closed (the reader can easily see it) and constitutes a linear subspace which is dense in the considered space. That is, one can say, in Example 2.36 the image $R(L)$ of the operator L is essentially "narrower" than the entire considered space, and in the last two examples the image $R(L)$ of the operator L "almost coincides" with the whole space.

It makes one think that in Examples 2.37 and 2.38 the image of the operator should be "slightly" extended so that it will coincide with the whole considered space. This can be done only by "slightly" increasing the domain of the operator. For example, "to extend" the domain of Example 2.37 to the domain of the form (2.45), would lead (see Example 2.29) to the well-posedness of the considered operator.

Thus we come to *the necessity* of some ("natural") *extension of the domain* of the operator. Such operation is analogous to the closure operation for a set in the linear normed space and, by analogy, is called *a closure operation of the operator*.

2.11 Restrictions and extensions of linear operators

Let us introduce new definitions which we will widely use in what follows.

If the domain $D(A)$ of a linear operator A is wider than the domain $D(B)$ of a linear operator B (that is, $D(B) \subset D(A)$) and on $D(B)$ the actions of the operators coincide (that is, $Ax = Bx$, $\forall x \in D(B)$), then the operator A is called an ***extension*** of the operator B and we denote it by $B \subset A$. In turn, the operator B is called a ***restriction*** of the operator A. In analogy with the embedding of sets, one says that the operator B is ***embedded*** into the operator A.

Example 2.39 In the space of continuous functions $C[0,1]$, consider the linear operator $T_0 : C[0,1] \to C[0,1]$ on the domain

$$D(T_0) = \{f \in C^1[0,1] : f(0) = 0, f(1) = 0\},$$

given by

$$T_0 f(x) = a(x) f(x), \ \ \forall f \in D(T_0), \tag{2.49}$$

where $a \in C[0,1]$ is some given function. It is easy to see that the operator T_0 is linear. Also note (though it is not important for further discussions) that the operator is bounded.

To construct an extension of the operator, it is necessary to set an operator with a wider domain, in a way that its action on $D(T_0)$ coincides with the action of T_0, that is, is expressed by formula (2.49). The set $D(T_0)$ can be extended by choosing a class of functions wider than $C^1[0,1]$ (but, staying within the space $C[0,1]$); we can extend this set by means of eliminating one (or all) of the boundary conditions.

The following operators are the extensions of the operator T_0:

- $T_1 f(x) = a(x)f(x),\ D(T_1) = \{f \in C[0,1] : f(0) = 0, f(1) = 0\};$

- $T_2 f(x) = a(x)f(x),\ D(T_2) = \{f \in C^1[0,1] : f(0) = 0\};$

- $T_3 f(x) = a(x)f(x),\ D(T_3) = C[0,1];$

- $T_4 f(x) = a(x)f(x),\ D(T_4) = \{f \in C^1[0,1] : \alpha f(0) + \beta f(1) = 0\},\ \alpha, \beta \in \mathbb{C};$

- $T_5 f(x) = a(x)f(x) + b(x)f(1),\ D(T_5) = \{f \in C[0,1] : f(0) = 0\},\ b \in C[0,1];$

- $T_6 f(x) = a(x)f(x) + b(x)f(1) + c(x)f(0), D(T_5) = C[0,1], b, c \in C[0,1].$

All the indicated operators are the extensions of the initial operator T_0, that is $T_0 \subset T_k$, $k = 1, \ldots, 6$.

Let us dwell on each of these extensions in detail:

- $T_0 \subset T_1$, since the actions of the operators coincide, and the domain T_1 has become wider by means of the class extension (lowering of required smoothness) of the functions from the domain;
- $T_0 \subset T_2$, since the actions of the operators coincide, and the domain T_2 has become wider due to eliminating one of the boundary conditions in the domain;
- $T_0 \subset T_3$, since the actions of the operators coincide, and the domain T_3 has become wider both due to the lowering of the smoothness requirements for the functions from the domain, and due to eliminating both boundary conditions;
- The fourth example shows that the boundary conditions in the domain can be not just eliminated but can be also modified. It is easy to see that $D(T_0) \subset D(T_4)$ and, since the actions of the operators coincide, we have $T_0 \subset T_4$;
- In the fifth example the domain of the operator T_5 is wider than that of the operator T_0, but their actions are already given by different formulae. In spite of this, since on all the functions $f \in D(T_0)$ the actions of the operators T_5 and T_0 coincide (due to $f(1) = 0$), we have $T_0 \subset T_5$;
- $T_0 \subset T_6$ is justified similarly to the previous one;
- By similar discussions we show the embedding of the operators between each other: $T_1 \subset T_3$, $T_2 \subset T_3$, $T_4 \subset T_3$, $T_1 \subset T_5$, $T_1 \subset T_6$, $T_5 \subset T_6$.

However, it is not always the case that one can have the embedding relations between two operators. So, for example, neither is the operator T_1 the extension of T_2, nor is the operator T_2 the extension of T_1, although their actions coincide. $\square$

The introduced example shows that an extension of an operator, although having something in common with the original operator, is actually a fundamentally different operator, which can differ both by the domain and by its action. Therefore, an extension of the operator can have fundamentally different properties (including well-posedness properties, spectral properties, etc.) which differ significantly from the properties of the original operator.

Remark 2.40 By Theorem 2.12 (see Section 2.5), any linear bounded operator A given on the linear space $D(A)$ which is dense in the linear space X, with values in a Banach space Y, can be extended to the whole space X with preservation of its norm value. That is, there exists a linear bounded operator $\widehat{A} \in \mathscr{L}(X,Y)$ such that $\widehat{A}x = Ax \;\; \forall x \in D(A)$ and $\|\widehat{A}\| = \|A\|$. With the newly introduced definitions, the operator $\widehat{A}$ is an extension of the operator A. $\square$

2.12 Closed operators

First of all we will need some definitions.

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Their ***direct sum*** is defined as the Banach space (denoted by $X \oplus Y$, whose elements are pairs $z = (x;y)$, where $x \in X, y \in Y$), with operations

- addition: $z_1 + z_2 = (x_1 + x_2; y_1 + y_2)$, where $z_1 = (x_1;y_1), z_2 = (x_2;y_2)$;
- multiplication by a scalar: $\lambda z = (\lambda x; \lambda y)$;

and the norm $\|z\| := \|x\|_X + \|y\|_Y$.

The simplest example of the direct sum of two spaces is the finite-dimensional space $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$, since in this space the operations of addition and multiplication by a scalar are done “coordinate-wise”.

Another example is the space of polynomials of degree not larger than n, which can be represented in the form of the direct sum

$$M_0 \oplus M_1 \oplus \cdots M_{n-1} \oplus M_n,$$

where M_k is the space of homogeneous polynomials of degree k.

For the Banach spaces X and Y, let $A : X \to Y$ be a linear operator with the domain $D(A) \subset X$ and the range $R(A) \subset Y$. The collection of pairs

$$(x;Ax) \in Z = X \oplus Y,$$

where x runs along the whole domain is called the ***graph of the linear operator*** A and is denoted by G_A. Defining the graph of the operator in this way is in good agreement with the usual definition of the graph of functions.

A linear operator $A : X \to Y$ is called ***closed***, if its graph G_A is a closed set in the (normed) space $X \oplus Y$. Recalling the definition of the closed set (see Section 1.1), we see that the closedness of the operator A means:

$$\textit{if } x_k \in D(A) \textit{ and } (x_k; Ax_k) \to (x; y) \quad \textit{for } k \to \infty, \quad \textit{then } x \in D(A) \textit{ and } Ax = y.$$

Since the convergence in $X \oplus Y$ is given by the norm $\|z\| = \|x\|_X + \|y\|_Y$, the definition of closedness can be rewritten as:

for any sequence $\{x_k\}_{k=1}^{\infty} \in D(A)$, the convergence of two sequences $x_k \to x$ and $Ax_k \to y$ implies that $x \in D(A)$ and $Ax = y$.

Remark 2.41 Comparing the above definitions of the closedness and the definition of continuity of an operator (see Section 2.3), we can make the following conclusion. The difference between the closedness and the continuity is in the fact that if an operator A is continuous, then the existence of the limit of the sequence $\{x_k\}, (x_k \in D(A))$, necessarily implies the existence of the limit of the sequence $\{Ax_k\}$; if the operator A is only closed, then from the convergence of the sequence

$$x_1, x_2, x_3, \ldots \quad (x_k \in D(A)) \tag{2.50}$$

the convergence of the sequence

$$Ax_1, Ax_2, Ax_3, \ldots \tag{2.51}$$

does not necessarily follow. The only thing is that two sequences of type (2.51) are not allowed to converge to different limits, if the corresponding sequences (2.50) converge to the same limit. □

An operator $\overline{A}$ is called the ***closure*** of the operator A, if its graph is the closure of the graph of the operator A:

$$G_{\overline{A}} = \overline{G_A}.$$

Not all operators can have the closure. An operator having the closure is called ***closeable*** or ***admitting the closure***.

It would be natural to construct the closure of an operator by means of the closure of its graph. However, the graph closure of a linear operator does not always produce the graph of some operator. The following two examples illustrate this fact.

Example 2.42 In the Hilbert space ℓ^2 of all infinite square summable sequences $\{x_k\}_{k=1}^{\infty}$ we consider the linear subspace M of all *finite sequences*: $\{x_k\} \in M$, if $\exists N :$ $x_k = 0$ for all $k > N$.

Let us denote by A the linear operator with the domain $D(A) = M$, the action of which on basis elements (sequences)

$$e_k = (\underbrace{0, 0, \ldots, 0, 1}_{k}, 0, \ldots), \; \|e_k\| = 1,$$

is given by the formula $Ae_k = k \cdot e_1$.

It is clear that this operator is defined on all sequences from $D(A) = M$ and is linear. The range of this operator is the one-dimensional space of vectors of the kind $(x_1, 0, 0, ...)$. The operator A transfers any element $x = (x_1, x_2, ..., x_N, 0, 0, 0, ...)$ from the domain into the element $Ax = (x_1 + 2x_2 + ... + Nx_N, 0, 0, ...)$.

It is easy to see that the graph G_A of the operator A contains points of the kind $\left(\frac{1}{k}e_k; e_1\right)$ for all k. Consequently, firstly, the graph G_A of the operator A is not a closed set and therefore *the operator A is not closed.*

Secondly, letting $k \to \infty$, we see that the point $(0; e_1)$ belongs to the graph closure of the operator A: $(0; e_1) \in \overline{G_A}$. Therefore, $\overline{G_A}$ is not the graph of any operator, since under linear mappings the zero element must be mapped into zero. It means that *the operator A does not admit a closure.* $\square$

Example 2.43 In the Hilbert space $L^2(0,1)$, consider the linear operator B with the domain $D(B) = C[0,1]$, defined by

$$Bf(x) = xf(0), \ \ \forall f \in D(B).$$

It is clear that this operator is defined for all functions from $D(B) = C[0,1]$ and is linear. The range of this operator is the one-dimensional space of functions of the type ax, where $a = const$.

Let us show that the operator B is not closed and does not admit a closure. To do this we construct a sequence of functions

$$f_n(x) = \begin{cases} 1 - nx, & \text{for } 0 \le x \le 1/n, \\ 0, & \text{for } 1/n \le x \le 1. \end{cases}$$

It is clear that the function f_n is continuous on the interval $[0,1]$ and, consequently, belongs to the domain of the operator B. Here, $f_n(0) = 1$ and

$$\|f_n\| = \left(\int_0^1 |f_n(x)|^2 dx\right)^{\frac{1}{2}} = \left(\int_0^{\frac{1}{n}} |1 - nx|^2 dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3n}} \to 0, \ n \to \infty.$$

Consequently, the sequence of the functions f_n converges to zero in the space $L^2(0,1)$: $f_n \to 0$, $n \to \infty$. However the sequence of images $Bf_n(x) = xf_n(0) \equiv x$ does not converge to zero as $n \to \infty$.

We choose the second sequence of functions $g_n \in D(B)$ as the zero sequence: $g_n(x) \equiv 0$ for $\forall n$. Here, the sequence of images $Bg_n(x) = xg_n(x) \equiv 0$ converges to zero at $n \to \infty$.

Thus, we have two sequences of functions: $f_n \to 0$, $g_n \to 0$ converging in $L^2(0,1)$ to zero as $n \to \infty$, for which the sequences of their images converge to different limits: $Bf_n(x) \to x$ and $Bg_n(x) \to 0$, as $n \to \infty$. Therefore, by Remark 2.41, *the operator B is not closed.*

The graph G_B of the operator B contains points (functions) of the kind $(f_n(x); x)$, $\forall n \in \mathbb{N}$. Since the sequence of the functions $f_n(x)$ converges to zero in $L^2(0,1)$: $f_n \to 0$, $n \to \infty$, the closure $\overline{G_B}$ of the graph of the operator B contains the point (function) $(0,x)$. Therefore, $\overline{G_B}$ cannot be the graph of any operator, since the linear operator would map the zero element to the zero one. It means that *the operator B does not admit a closure*. □

Since $G_A \subset \overline{G_A}$, then, taking into account the terminology of Section 2.5, the closure $\overline{A}$ is *the extension* of the operator A. Thus, *the closure $\overline{A}$* (if it exists) *is the minimal closed extension* of the operator A.

The following theorems indicate wide classes of closed operators and their basic properties.

Theorem 2.44 *If $D(A) = X$ and the operator A is bounded (that is, $A \in \mathscr{L}(X,Y)$), then A is closed.*

Theorem 2.45 *If A is closed and A^{-1} exists, then A^{-1} is closed.*

Corollary 2.46 *If $A \in \mathscr{L}(X,Y)$ and A^{-1} exists, then A^{-1} is closed.*

Corollary 2.47 *If A^{-1} exists and $A^{-1} \in \mathscr{L}(X,Y)$, then A is closed.*

Corollary 2.48 *If $\overline{R(A)} = R(A)$ and there exists a number $m > 0$ such that*

$$\|Ax\| \geq m\|x\|$$

for all $x \in D(A)$, then the operator A is closed.

Corollary 2.49 *If A^{-1} exists and $\overline{R(A)} = R(A)$, then A is closed.*

It should be kept in mind that if an operator A is closed, then $D(A)$ will not necessarily be a closed set and $R(A)$ will not necessarily be a closed set. However, the following fact holds.

Theorem 2.50 *If A is bounded, then it is closed if and only if $D(A)$ is closed.*

Remark 2.51 From these theorems we can make *a conclusion which is important for the theory of well-posedness* of operators. Recall that the operator A is *well-posed*, if the inverse operator A^{-1} exists, is defined on the whole space Y, and is bounded. Thus, *the closedness of the operator is the necessary condition for its well-posedness*. Consequently, if an operator is not closed, then it is automatically not well-posed. Usually in such cases everyone immediately proceeds to investigating the well-posedness of the operator's closure, if it exists. Therefore, *when constructing an operator corresponding to the problem which is investigated for well-posedness, it is necessary to correctly set up the operator in such a way that it is closed.* □

The following theorem belongs to S. Banach and is the converse to Theorem 2.44.

Theorem 2.52 (Banach's closed graph theorem) *If A is a closed linear operator from a Banach space X to a Banach space Y and $D(A) = X$, then A is bounded.*

Thus, from Theorems 2.4, 2.44 and 2.52 it follows that *the concepts of boundedness, continuity and closedness* for *operators defined* (given) *on the whole space, coincide*. Consequently, all the examples introduced above of the bounded operators defined on the whole space are examples of closed operators.

However, we should not think that any closed operator will be automatically bounded. From Corollary 2.47 of Theorem 2.45 it follows that the examples introduced earlier of linear (unbounded) well-posed operators are examples of unbounded closed operators. Consequently, *the closed operators are not necessarily bounded.*

The previous discussions have provided the closedness of an unbounded operator based on the fact that it has a bounded inverse operator defined on the whole space. However, we should not think that any closed operator is well-posed. The following examples illustrate this.

Example 2.53 Let us show that there exist unbounded closed operators, whose inverse operators are also closed and unbounded.

In the Banach space $C[0,1]$, consider the operator A with the domain

$$D(A) = \left\{ u \in C[0,1] : \frac{u(x)}{x} \in C[0,1] \right\},$$

given by

$$Au(x) = \frac{1-x}{x} u(x).$$

This operator is linear, and all functions from its domain satisfy the inequality

$$|u(x)| \leq Cx$$

for some C.

Note that the introduced operator A is one of the variants of the multiplication operator by a function: $Tf(x) = G(x) \cdot f(x)$, considered earlier in Example 2.6. But in this case the function $G(x)$ is not bounded, and has an essential discontinuity (first-order singularity) for $x \to 0$.

First we show that the operator A is not bounded. Take the following sequence of continuous functions $u_k(x) = \frac{(1+k)x}{1+kx}$, $k = 1, 2, \ldots$. It is clear that $u_k \in D(A)$, since for any fixed k the function $\frac{u_k(x)}{x} = \frac{1+k}{1+kx}$ is continuous on $[0,1]$. We estimate the norms of these functions as

$$\|u_k\| = \max_{0 \leq x \leq 1} |u_k(x)| = \max_{0 \leq x \leq 1} \frac{(1+k)x}{1+kx} = \frac{1+k}{1+k} = 1.$$

Calculating the norm of the images of functions u_k, we get

$$\|Au_k\| = \max_{0 \leq x \leq 1} \frac{(1+k)(1-x)}{1+kx} = (1+k) \max_{0 \leq x \leq 1} \frac{(1-x)}{1+kx} = 1+k.$$

Consequently,

$$\|A\| = \sup_{u\in D(A),\|u\|=1} \|Au\| \geq \sup_{k\in N} \|Au_k\| = +\infty.$$

Therefore the operator A is not bounded.

Let us show now that A is closed. Let u_k be an arbitrary converging sequence from the domain of the operator such that Au_k also converges. That is, in $C[0,1]$, we have

$$u_k(x) \to u(x),\ Au_k(x) = \frac{1-x}{x} u_k(x) \to \upsilon(x),\ k\to\infty.$$

Summing these sequences, we get that

$$\frac{1}{x} u_k(x) \to u(x) + \upsilon(x),\ \ k\to\infty.$$

The convergence is meant with respect to the norm of the space $C[0,1]$. Therefore, for any $\varepsilon > 0$ there exists a number k_0 such that $\left\|\frac{1}{x}u_k - [u+\upsilon]\right\| < \varepsilon$ for all $k \geq k_0$. Consequently, we have $\left|\frac{1}{x}u_k(x) - [u(x)+\upsilon(x)]\right| < \varepsilon$, for all $x \in [0,1]$ and $k \geq k_0$.

Hence we easily get that

$$|u_k(x) - x[u(x)+\upsilon(x)]| < \varepsilon x \leq \varepsilon,\ \forall k \geq k_0.$$

That is, the sequence $u_k(x)$ converges to $x[u(x)+\upsilon(x)]$.

But $u_k \to u$, therefore, $u(x) = x[u(x)+\upsilon(x)]$. Thus we have $\upsilon(x) = \frac{1-x}{x}u(x)$, that is,

$$\upsilon(x) = Au(x). \tag{2.52}$$

Since the sequence Au_k converges in the space $C[0,1]$, we have $\upsilon \in C[0,1]$. Therefore, $\frac{u(x)}{x} \in C[0,1]$, which means that $u \in D(A)$. This fact together with (2.52), taking into account the arbitrariness of the sequence u_k, proves the closedness of the operator A by one of the equivalent criteria formulated after the definition of a closed operator.

For the invertibility of A, since from equality $\frac{1-x}{x}u(x) = 0$ it follows that $u(x) \equiv 0$, the kernel of the operator A consists only of the zero element 0. Therefore, an inverse operator A^{-1} exists. It is easy to see that its action is given by the formula

$$A^{-1}\upsilon(x) = \frac{x}{1-x}\upsilon(x),$$

and its domain is

$$D(A^{-1}) = \left\{ \upsilon \in C[0,1] : \frac{\upsilon(x)}{1-x} \in C[0,1] \right\}.$$

The operator A^{-1} has the same structure as A, with changing the places of the points $x=0$ and $x=1$. Therefore, it can be proved in a similar way that the operator A^{-1} is closed, but not bounded. $\square$

The considered example shows that *there exist unbounded closed operators, for which the inverse operators are also closed and unbounded*. Let us now also show that there exist closed operators which are not invertible, that is, do not have an inverse operator.

Example 2.54 Let us return to the differentiation operator $L_1 : C[a,b] \to C[a,b]$, considered in Examples 2.9 and 2.28. The domain of this operator is $D(L_1) = C^1[a,b]$, and its action is given by the formula

$$L_1 u(x) = \frac{d}{dx} u(x), \ a < x < b.$$

In the second part of Example 2.9 we have shown that the operator L_1 is unbounded, and in Example 2.28 we have proved that it does not have the inverse operator. Let us show that inspite of this fact, the operator is closed.

Let u_k be an arbitrary converging sequence from the domain of the operator such that $L_1 u_k$ converges. Since the convergence in the space $C[a,b]$ is the uniform convergence, then as $k \to \infty$ we have that

$u_k(x) \to u(x)$ uniformly on $[a,b]$;

$u_k'(x) \to f(x)$ uniformly on $[a,b]$.

From the basic course of Analysis, according to the well-known theorem on differentiating uniformly converging sequence of functions, the function u must be continuously differentiable (that is, $u \in D(L_1)$) and $u'(x) = f(x)$ (that is, $L_1 u(x) = f(x)$). Consequently, the operator L_1 is closed. $\square$

The considered example shows that *there exist closed operators which are not invertible, that is, they do not have the inverse operator.* Let us show now that there exist closed invertible operators, which are not everywhere solvable, that is, whose range does not coincide with the whole space.

Example 2.55 Let us return to the differential operator $L : L^2(a,b) \to L^2(a,b)$, considered in Example 2.36. The domain of this operator is

$$D(L) = \{u \in L_1^2(a,b) : u(0) = u(1) = 0\},$$

and its action is given by the formula

$$Lu(x) = \frac{d}{dx} u(x), \ a < x < b.$$

In Example 2.36 we have shown that the operator L is linear, is defined on all functions from its domain, is unbounded as the operator from $L^2(a,b)$ to $L^2(a,b)$, and is not well-posed (since it is not everywhere solvable, although the inverse operator exists and is bounded on $R(L)$). Let us now show that this operator is closed in the space $L^2(a,b)$.

In Example 2.9 we have shown that the inverse operator L^{-1} exists, is given by formula (2.43):

$$u(x) = L^{-1}f(x) = \int_a^x f(t)dt, \ a \leq x \leq b, \tag{2.53}$$

and is defined on all functions from the image of the operator L, forming a subspace of the space $L^2(a,b)$:

$$R(L) = \left\{ f \in L^2(a,b) : \int_a^b f(x)dx = 0 \right\}.$$

Consequently, the operator L is invertible (that is, $\exists\ L^{-1}$, defined on $R(L)$) and is normally solvable (the image of the operator is closed: $\overline{R(L)} = R(L)$). Therefore, by Corollary 2.49 of Theorem 2.45, the operator L is closed in the space $L^2(a,b)$.

Note also that this result follows from Theorem 2.50. Indeed, since $\overline{R(L)} = R(L)$ and the operator L^{-1} is bounded on $R(L)$, then it is closed. Then by Theorem 2.45 the operator $L = (L^{-1})^{-1}$ is also closed. □

The considered example shows that *there exist invertible closed operators which are not everywhere solvable*. Let us now introduce another result which makes it much easier to prove the closedness of a particular operator.

Theorem 2.56 *Let $A, B : X \to Y$ be linear operators, moreover let A be closed, and let B be bounded, and let $D(A) \subset D(B)$. Then the operator $A+B$ with the domain $D(A)$, given by the action*

$$(A+B)x = Ax + Bx \quad \forall x \in D(A),$$

is closed.

This theorem helps one to justify the closedness of an operator proving only the closedness of its "main part". Addition of "smaller terms" (such as the bounded operator B), as a rule, does not destroy the closedness property.

Example 2.57 Consider the operator $L_q : L^2(a,b) \to L^2(a,b)$ which is the generalisation of the operator L considered in the previous Example 2.55. The domain of this operator is

$$D(L_q) = D(L) \equiv \{u \in L_1^2(a,b) : u(0) = u(1) = 0\},$$

and its action is given by the formula

$$L_q u(x) = \frac{d}{dx}u(x) + q(x)u(x), \ a < x < b,$$

where $q(x)$ is a given continuous function on $[a,b]$.

The operator L_q can be represented as a sum of two operators: $L_q = L + Q$, where L is the operator considered in Example 2.55, and Q is the linear operator of the multiplication by a function, acting by the formula $Qu(x) = q(x)u(x)$, defined in the whole space $L^2(a,b)$. That is, $D(Q) = L^2(a,b)$.

The operator L is closed (see Example 2.55), the operator Q is bounded (see Example 2.7) and $D(L) \subset D(Q)$. Consequently, by Theorem 2.56, the operator L_q is closed. □

2.13 Closure of differential operators in $L^2(a,b)$

When studying boundary value problems for differential equations by the operator methods, the first step is the construction of operators corresponding to the formulated boundary value problems. As has been mentioned in Remark 2.51, when constructing an operator corresponding to the problem and investigating its well-posedness, it is necessary to set up an operator in a way that *it is closed*. Most convenient spaces to work in are the Hilbert spaces.

It would seem that it would be more convenient to consider differential operators in high-order Sobolev spaces, in which they are bounded. However, as is shown in Example 1.20, setting a boundary condition (the value of the unknown function at some point), we get that the domain of the operator becomes *not dense* in the considered Sobolev space.

As will be shown in what follows, the density of the domain of the operator is one of necessary conditions for applying general methods of the theory of operators (for example, for the existence of the *adjoint* operator). Therefore, most considerations for the differential operators are carried out in the Hilbert space $L^2(a,b)$ (when dealing with problems in one dimension, and with natural extensions to higher dimensions).

On the other hand, the spaces of continuous and differentiable (a number of times) functions are more natural for the differential equations. The operation of closure of the operator gives the possibility of passing from the consideration of problems for the differential equations in the space of continuously differentiable functions to the consideration of the differential operators in the Lebesgue spaces.

In itself, the proof of closedness of the operator and the construction of its closure is a procedure which is not complicated but rather cumbersome. The following theorems give an opportunity to avoid this feature.

Theorem 2.58 *If an operator A is densely defined, admits a closure and is invertible, then the inverse operator A^{-1} also admits the closure and $\overline{A^{-1}} = \left(\overline{A}\right)^{-1}$.*

This statement, which is useful in practice, is conveniently formulated by using "algebraic" language. Let us draw the following diagram:

$$\begin{array}{ccc} A & \overset{\text{closure}}{\Rightarrow} & \overline{A} \\ \Downarrow \text{inverse} & & \downarrow \text{inverse} \\ A^{-1} & \overset{\text{closure}}{\rightarrow} & \overline{A^{-1}} \end{array}$$

in which the horizontal lines represent the transition to the closure of an operator, and the vertical ones represent the transition to the inverse operator.

Now we can reformulate the statement of the above theorem in the following way: *If for a given densely defined operator A the operations drawn in the above diagram by double arrows are well-defined, then the single arrows are also well-defined, and they complement the diagram to the commutative one*.

Theorem 2.59 *A continuous densely defined operator always admits a closure. This closure is the extension with respect to the continuity on the entire space.*

These theorems are especially useful in cases where the construction of the closure $\overline{B^{-1}}$ of the inverse operator B^{-1} is simpler than the construction of the closure of an initial operator B; for example, when B is unbounded, and B^{-1} is the closed and densely defined operator. In this case we need to apply Theorem 2.58 to the operator $A = B^{-1}$.

Let us demonstrate the use of these theorems in the following simple example.

Example 2.60 Let us return to the consideration of the operator introduced in Example 2.38. In the space $L^2(a,b)$, consider the operator given by the differential expression

$$Lu(x) = \frac{d}{dx}u(x), \;\; a < x < b,$$

on the domain

$$D(L) = \left\{u \in C^1[a,b] : u(0) = 0\right\}.$$

This operator has been considered in Example 2.29 as acting in the space of continuous functions $C[a,b]$. It has been shown there that the inverse operator L^{-1} exists, is defined on the whole space $C[a,b]$, and is bounded, that is, L is a well-posed operator in the space $C[a,b]$.

Consider now the same operator in the space $L^2(a,b)$. It is clear that the range of the operator is $R(L) = C[a,b]$ and, consequently, does not coincide with the whole space $L^2(a,b)$. Therefore, it is obvious that the operator L is not well-posed since it *is not everywhere solvable*. However, as has been shown in Example 2.38, the inverse operator L^{-1} exists, *is defined on* $R(L)$ (that is, the operator L *is well-posedly solvable*), is given by the formula

$$L^{-1}f(x) = \int_a^x f(t)dt, \; a \leq x \leq b, \;\; \forall f \in R(L), \tag{2.54}$$

and admits the estimate (the corollary of formula (2.47) of Example 2.38):

$$\|L^{-1}\| \leq \frac{b-a}{\sqrt{2}}. \tag{2.55}$$

Since the domain of the inverse operator $D(L^{-1}) = R(L) = C[a,b]$ is dense in $L^2(a,b)$, according to Theorem 2.12 the operator L^{-1} can be continued onto the whole space $L^2(a,b)$ with the value preservation of its norm (2.55). The continuation of L^{-1} onto the whole space $L^2(a,b)$ is the operator bounded on $L^2(a,b)$, and according to Theorem 2.44 it is a closed operator in $L^2(a,b)$. By Theorem 2.59 this extension is the closure of the operator L^{-1}.

The operator L^{-1} given by formula (2.54) is one of the particular cases of the general integral operators considered in Example 2.14. Therefore, as has been shown there, it can be extended onto the whole space not only with the preservation of the norm value, but also with the "preservation of formula" (2.54). Thus, $\overline{L^{-1}}$ exists and is given by the formula

$$\overline{L^{-1}}f(x) = \int_a^x f(t)dt, \; a \leq x \leq b, \;\; \forall f \in L^2(a,b).$$

Now applying Theorem 2.58 to the operator $A = L^{-1}$, we get that the operator L is closeable and $\left(\overline{L}\right)^{-1} = \overline{L^{-1}}$. From this and from (2.47) we obtain the action of the operator which is the inverse of the closure:

$$\left(\overline{L}\right)^{-1} f(x) = \int_a^x f(t)dt, \;\; a \leq x \leq b, \;\; \forall f \in L^2(a,b). \tag{2.56}$$

Now using formula (2.56), we can describe exactly the domain of the operator $\overline{L}$:

$$D\left(\overline{L}\right) = R\left(\left(\overline{L}\right)^{-1}\right).$$

When the function f varies in the entire space $L^2(a,b)$, we get

$$D\left(\overline{L}\right) = \left\{u \in L_1^2(a,b) : u(0) = 0\right\}. \tag{2.57}$$

Since the closure $\overline{L}$ of the operator L is its extension, the actions of the operators $\overline{L}$ and L on the functions $u \in D(L)$ coincide and are given by the formula

$$\overline{L}u(x) = \frac{d}{dx}u(x), \; a < x < b. \tag{2.58}$$

The same formula can be also applied for the action of the operator $\overline{L}$ on the function $u \in D(\overline{L})$, $u \notin D(L)$. It is possible since there exists, almost everywhere, the derivative of any function from the class $L_1^2(a,b)$.

Thus, we have constructed the closure $\overline{L}$ of the operator L. This is a linear differential operator given by expression (2.58) on domain (2.57). This operator *is well-posed*, since $\left(\overline{L}\right)^{-1}$ exists, is defined on the whole space $L^2(a,b)$, and is bounded. □

The considered example shows that with the help of the closure operation of the operator we have managed to pass from the ill-posed operator (the operator L) to the well-posed operator (the operator $\overline{L}$).

2.14 General concept of strong solutions of the problem

In the considered Example 2.60, it turned out that the action $\overline{L}$ of the operator's closure coincides with the action of the original operator L. That is, the operator L can be closed "with preservation of the action". It turned out to be possible since the operation of differentiation can be understood almost everywhere for all functions from the domain of the operator's closure. However, in the general case this ("preservation of the action") is not always possible. In applications of the closed operators techniques to solving the problems for differential equations, there arises a question of how one can understand the action of the closure of an operator. The answer to this question is: *understanding the action of the closure of an operator is, generally speaking, impossible.*

Let us explain this idea. *For a better understanding, in subsequent discussions, we can think of the operator L from Example 2.60 .*

Let a linear (not closed) operator L be given, an action of which on functions $u \in D(L)$ can be understood in some way "in the ordinary sense". From the definition of the closure $\overline{L}$ of the operator L it follows that the graph of the operator $\overline{L}$ is the closure of the graph of L. Consequently, for any pair $(u;\overline{L}u) \in G_{\overline{L}}$ there exists a sequence of pairs $(u_k;Lu_k) \in G_L$, converging to it with respect to the graph norm. That is,

$$u_k \to u, \ \ Lu_k \to \overline{L}u \ \text{ as } k \to \infty.$$

With this, $u_k \in D(L)$, and the action Lu_k can be understood "in the ordinary sense".

Then the action of the operator $\overline{L}$ on elements $u \in D(\overline{L})$, $u \notin D(L)$ (that is, on elements on which the understanding "in the ordinary sense" is not possible) is defined as

$$\overline{L}u := \lim_{k\to\infty} Lu_k. \tag{2.59}$$

Formula (2.59) defines the action of the operator $\overline{L}$ and gives the answer to the question posed in the beginning of this section.

However, one should not think that such "complications" as the use of formulae of the form (2.59) always appear when using the concept of the operator's closure. As is shown in Example 2.54, the differential operator of that example is closed (that is, there is no need in the closure operation), and its action on all functions from the domain of the operator can be understood "in the ordinary sense" as the differentiation operation of a continuous differentiable function. Therefore, the necessity of the closure operation usually appears when investigating operators in the Lebesgue space $L^2(\Omega)$.

The concept of the operator's closure is closely related with the concept of *a strong solution* of a problem for a differential equation. The strong solution is one of the variants of a ***generalised solution***, which are introduced just in cases when the ordinary classical solution (or, as one says, *a regular* solution) does not exist. Generally speaking, the definition of the strong solution is introduced for each problem under consideration based on its specific features. Here we will try to give only *the general understanding* of this term.

For this, let us investigate the following boundary value problem for a linear second-order differential equation in $\Omega \subset \mathbb{R}^n$.

Problem. Find a solution for the equation

$$Lu = \sum_{|\alpha|\le 2} a_\alpha(x) D^\alpha u(x) = f(x), \;\; x = (x_1, x_2, ..., x_n) \in \Omega, \tag{2.60}$$

satisfying the boundary condition

$$Qu|_{\partial\Omega} = 0, \tag{2.61}$$

where Q is some linear boundary operator defined on traces of the function u and its first derivatives on the boundary $\partial\Omega$ of the domain Ω.

Let us denote by M the linear space of twice continuously differentiable functions in Ω, continuously differentiable including the boundary of Ω, and satisfying the boundary conditions (2.61):

$$M := \left\{u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right) : \; Qu|_{\partial\Omega} = 0\right\}.$$

It is clear that for all functions from M the action of the differential expression (2.60) and the boundary conditions (2.61) can be understood in the usual continuous sense. Therefore, the function $u = u(x)$ is called a ***regular solution*** (or a ***classical solution***) of problem (2.60), (2.61), if $u \in M$ and u satisfies Eq. (2.60) and the boundary conditions (2.61).

It is clear that the differential expression (2.60) maps all functions from M to the functions which are continuous in the domain Ω. Therefore, the regular solution of the problem can exist only for the right-hand sides f of Eq. (2.60) which are continuous in Ω. In the case when the right-hand side in Eq. (2.60) is not continuous, it is necessary to introduce the concept of *a generalised solution*. By the ***generalised solution*** of the problem we understand all solutions (introduced in some way) of the problem that are not regular.

We should immediately indicate that *one should not confuse the generalised solutions of a problem with solutions in the sense of generalised functions*, since the generalised functions are, generally speaking, distributions (that is, are linear continuous functionals on the space of infinitely differentiable compactly supported functions), and the generalised solutions are "ordinary" functions from the space $L^2(\Omega)$.

One of the variants of the generalised solutions is a *strong solution*.

Let now $f \in L^2(\Omega)$. A function $u \in L^2(\Omega)$ is called the ***strong solution*** of problem (2.60), (2.61), if there exists a sequence $u_k \in M$ such that $u_k \to u$ and $Lu_k \to f$ in the norm of the space $L^2(\Omega)$ as $k \to \infty$. Consequently, the boundary value problem (2.60), (2.61) is called ***strongly solvable***, if the strong solution of the problem exists for any right-hand side $f \in L^2(\Omega)$, and is unique.

Generally speaking, *the strong solutions* belong to the space $L^2(\Omega)$ and *may not have any differentiability*. Then in what sense do the strong solutions satisfy

Eq. (2.60)? The answer is that the strong solution $u \in L^2(\Omega)$ satisfies Eq. (2.60) in the sense that

$$Lu(x) \equiv \lim_{k\to\infty} Lu_k(x) = f(x),$$

where the sequence $u_k \in M$ is taken from the definition of the strong solution.

Consider now the operator corresponding to the boundary value problem (2.60), (2.61). Thus, we denote by L the linear operator given on the domain $D(L) = M$ by the differential expression (2.60). It is clear that the domain of this operator consists of all regular solutions of the problem (2.60), (2.61).

Assume now that this operator is closable. Then the domain $D\left(\overline{L}\right)$ of the closure $\overline{L}$ of the operator L consists of those functions $u \in L^2(\Omega)$, for which there exists a sequence of pairs $(u_k; Lu_k) \in G_L$ converging to $(u; Lu) \in G_{\overline{L}}$ with respect to the graph norm. That is, $u_k \to u, Lu_k \to \overline{L}u$ as $k \to \infty$, with respect to the norm of the space $L^2(\Omega)$.

Consequently, *the domain of the operator's closure consists of the strong solutions of the corresponding boundary value problem.* □

2.15 Compact operators

Let us now return to the consideration of the bounded operators. David Hilbert, for the first time, paid attention to one important class of linear bounded operators, which can be approximated with respect to the norm of the space $\mathscr{L}(X,Y)$ by finite-dimensional operators. This is the class of *compact operators*, sometimes also called the *completely continuous operators.*

Many problems lead to the necessity of studying the solvability of an equation of the kind $Ax = y$, where A is some operator, $y \in Y$ is a given element, and $x \in X$ is the unknown element. For example, if $X = Y = L^2(a,b)$,

$$A = I - K,$$

where K is an integral operator, and I is the identity operator, then we obtain the class of so-called *Fredholm integral equation of second kind.* If $A = L$ is a differential operator, then one has a differential equation, etc.

The class of compact operators appears as one of the important classes of operators resembling the operators acting on finite-dimensional spaces. For a compact operator A, there is a well-developed theory of solvability of the equation $x - Ax = y$, which is quite analogous to the finite-dimensional cases (containing, in particular, the theory of integral equations).

To start, we need to introduce some definitions from the theory of sets.

A set M in a linear normed space X is called ***compact***, if we can extract a Cauchy sequence from any infinite sequence $x_k \in M,\ k = 1,2,\ldots$. Note that if X is a Banach space, then every such Cauchy subsequence (because of the completeness of X) must converge to some element $x_0 \in X$. However, it is a-priori unclear whether $x_0 \in M$.

Not going into too many details for explaining this concept, we only say that if the set is compact, then it is bounded and separable. (***Separability of the set*** M, unlike the separability of the space, means that: it contains at most a countable set, whose closure contains M in the space X.) The inverse is true only for finite-dimensional sets. The important fact is that in *the infinite-dimensional space the unit ball is not a compact set*.

Let X and Y be linear normed spaces. An operator $A \in \mathscr{L}(X,Y)$ is called ***completely continuous*** if it maps the closed unit ball of the space X into a compact set of the space Y. The set of all completely continuous operators is denoted by

$$\sigma(X,Y).$$

The completely continuous operator maps any set bounded in X into a set which is compact in Y. Therefore the completely continuous operators are often called ***compact operators***.

However, generally speaking, these concepts are different. The definitions of completely continuous and of compact operators are equivalent in the case of a separable reflexive Banach space. In the general case, the complete continuity implies the compactness, but not vice versa. *But since we consider only the separable reflexive Banach spaces, then we will identify these concepts (the completely continuous and compact operators).*

Theorem 2.61 *The compact operators have the following properties:*

1. *If A_1 and A_2 are compact operators, then for any scalars α_1, α_2 the operator $\alpha_1 A_1 + \alpha_2 A_2$ is also a compact operator;*
2. *If A_1 is a compact operator, and A_2 is a bounded but not compact operator, then an operator $A_1 + A_2$ is a bounded but not compact operator;*
3. *If A is a compact operator and B is a bounded operator, then the composed operators AB and BA are compact operators;*
4. *If the spaces X or Y are finite-dimensional, then any bounded operator $A \in \mathscr{L}(X,Y)$ is a compact operator. In particular, in this case any linear bounded functional is a compact operator;*
5. *$\sigma(X,Y)$ is a subspace of $\mathscr{L}(X,Y)$. In particular, all compact operators are bounded; and the operator being the limit (in the sense of convergence with respect to the norm in the space $\mathscr{L}(X,Y)$) of the sequence of compact operators, is also a compact operator.*

We now consider some examples of compact operators.

Example 2.62 In the space ℓ^2 of square summable infinite sequences $x = (x_1, x_2, \ldots) = \{x_k\}_{k=1}^{\infty}$ with the norm

$$\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} < +\infty,$$

consider the operator A defined on the whole space ℓ^2 by the formula

$$Ax = y, \quad \text{where } \; y = \{y_k\}_{k=1}^{\infty}, \; y_k = \sum_{j=1}^{\infty} a_{kj} x_j, \; k = 1, 2, ... \tag{2.62}$$

Let the entries of the matrix (a_{kj}) be such that

$$\|A\|_2 = \left(\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |a_{kj}|^2 \right)^{1/2} < +\infty. \tag{2.63}$$

The value of (2.63) is called the ***Hilbert-Schmidt norm*** of operator (2.62), and operator (2.62) with the finite norm $\|A\|_2$ is called a ***matrix Hilbert-Schmidt operator***.

Linearity of operator (2.62) is clear. Let us show first its boundedness. Applying the ***Cauchy-Schwarz inequality***

$$\sum_{k=1}^{\infty} |\xi_k \eta_k| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |\eta_k|^2 \right)^{1/2},$$

we estimate the norm of the image:

$$\|y\|^2 = \sum_{k=1}^{\infty} |y_k|^2 = \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{kj} x_j \right|^2 \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{kj}|^2 \sum_{m=1}^{\infty} |x_m|^2 \right) = (\|A\|_2)^2 \, \|x\|^2.$$

It means that $\|Ax\| \leq \|A\|_2 \|x\|$ for all $x \in \ell^2$. Consequently, the operator A is bounded.

Now we show that the operator $A : \ell^2 \to \ell^2$ is a compact operator. We denote by A_n the finite-dimensional operators given on the whole space ℓ^2 by the formula

$$A_n x = y, \quad \text{where} \quad y = \begin{cases} \sum\limits_{j=1}^{\infty} a_{kj} x_j & \text{for } k = 1, \ldots, n, \\ 0 & \text{for } k > n. \end{cases}$$

Thus, for each element $x \in \ell^2$, the operator A_n sets in correspondence the element $(y_1, y_2, \ldots, y_n, 0, 0, \ldots)$.

Since the image of each operator A_n is finite-dimensional, then according to Theorem 2.61, Part 4, they are compact.

Let us now show that the operator A is a limit of the finite-dimensional operators A_n. Indeed, since for all $x \in \ell^2$ we have

$$\|(A - A_n)x\|^2 = \sum_{k=n+1}^{\infty} |y_k|^2 = \sum_{k=n+1}^{\infty} \left| \sum_{j=1}^{\infty} a_{kj} x_j \right|^2 \leq$$

$$\leq \sum_{k=n+1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{kj}|^2 \sum_{m=1}^{\infty} |x_m|^2 \right) = \sum_{k=n+1}^{\infty} \sum_{j=1}^{\infty} |a_{kj}|^2 \|x\|,$$

it follows that

$$\|A - A_n\| \leq \left(\sum_{k=n+1}^{\infty} \sum_{j=1}^{\infty} |a_{kj}|^2 \right)^{1/2} \to 0 \text{ as } n \to \infty.$$

Now the compactness of the operator A follows from Theorem 2.61, Part 5. □

The considered example shows that a compact operator can be represented by a limit of a converging sequence of finite-dimensional operators. This property distinguishes compact operators from the class of all other bounded operators.

Theorem 2.63 *The identity operator*

$$I : X \to X, \; Ix = x, \;\; \forall x \in X,$$

in an infinite-dimensional space X is a bounded but not compact operator.

Indeed, the linearity of the operator I is clear. Moreover, this operator maps the closed unit ball of the space X to itself. But the unit ball is compact in normed spaces only if the space is finite-dimensional. Therefore, I is a bounded but not compact operator. □

Corollary 2.64 *A compact operator in an infinite-dimensional space cannot have a bounded inverse operator.*

Indeed, if a compact operator A had a bounded inverse operator A^{-1}, then their composition $AA^{-1} = I$, according to Theorem 2.61, Part 3, would be a compact operator, which would contradict Theorem 2.63. □

Theorem 2.65 *Let the spaces X and Y be Banach spaces. If $A : X \to Y$ is a compact operator, then $R(A)$ is not closed in Y.*

Corollary 2.64 of Theorem 2.63 and Theorem 2.65 occur when investigating differential operators. Indeed, if $L : X \to X$ is an invertible operator, we apply Corollary 2.64 and Theorem 2.65 to the operator L^{-1}. Then, if L^{-1} is compact, then the operator L is unbounded and its domain $D(L) = R(L^{-1})$ cannot be closed in the space X. And indeed, as the considered earlier examples have shown, the differential operators are unbounded and their domains are dense in X. These facts confirm that expecting the compactness of the inverse operator L^{-1} is often natural.

To prove the compactness of particular operators one uses various methods. Not dwelling in detail on these methods, we now give some examples of compact and non-compact operators which will be necessary for us for further discussions related to the spectral theory.

Example 2.66 Consider the operator $Lu(x) = \frac{d}{dx}u(x)$, acting in various spaces below. Then for all integers $k \geq 0$, we have:

• $L : C^k[a,b] \to C^k[a,b]$ is unbounded;

- $L : C^{k+1}[a,b] \to C^k[a,b]$ is bounded, but not compact;
- $L : C^{k+2}[a,b] \to C^k[a,b]$ is compact;
- $L : L_k^2(a,b) \to L_k^2(a,b)$ is unbounded;
- $L : L_{k+1}^2(a,b) \to L_k^2(a,b)$ is bounded but not compact;
- $L : L_{k+2}^2(a,b) \to L_k^2(a,b)$ is compact.

Example 2.67 Let us return to studying the integral operator considered in Example 2.11. Let the action of the operator be given by

$$Kf(x) = \int_a^b k(x,t)f(t)dt,$$

where $k = k(x,t)$ is a function on the closed rectangle $[a,b] \times [a,b]$.

In Example 2.11 it has been shown that if $k = k(x,t) \in C([a,b] \times [a,b])$, then the operator $K : C[a,b] \to C[a,b]$ is bounded. In fact, a more precise statement holds: if $k = k(x,t) \in C([a,b] \times [a,b])$, then the integral operator $K : C[a,b] \to C[a,b]$ *is compact*. $\square$

In Example 2.62, a *matrix Hilbert-Schmidt operator* has been considered. This concept can be extended to general operators in Hilbert spaces. Let us denote by $e_k (k \in N)$ an orthonormal basis in the Hilbert space H. An operator A is called the ***Hilbert-Schmidt operator***, if

$$\|A\|_2 := \left(\sum_{k=1}^{\infty} \|Ae_k\|_H^2 \right)^{1/2} < \infty.$$

It can be shown that the value $\|A\|_2$ does not depend on the choice of the basis e_k and is called the ***Hilbert-Schmidt norm*** of the operator A. We always have $\|A\| \leq \|A\|_2$, that is, the Hilbert-Schmidt operators are bounded.

Moreover, *any Hilbert-Schmidt operator is compact*. The inverse is not true, that is, there exist compact operators which are not Hilbert-Schmidt operators. If A is a Hilbert-Schmidt operator and B is a bounded operator, then AB and BA are Hilbert-Schmidt operators and

$$\|AB\|_2 \leq \|A\|_2 \|B\| \text{ and } \|BA\|_2 \leq \|A\|_2 \|B\|.$$

Example 2.68 In the Hilbert space $L^2(a,b)$, consider the integral operator K given by

$$Kf(x) = \int_a^b k(x,t)f(t)dt, \tag{2.64}$$

where $k = k(x,t) \in L^2((a,b) \times (a,b))$. In this case the operator $K : L^2(a,b) \to L^2(a,b)$ is a Hilbert-Schmidt operator and therefore is called an ***integral Hilbert-Schmidt operator***.

Earlier, in Example 2.11 it has been shown that this operator is bounded. It turns out that it is a Hilbert-Schmidt operator and, as a consequence, it is a compact operator in $L^2(a,b)$.

The proof of the compactness of an integral operator in each particular case is a rather cumbersome problem. However, in many cases the appearing integral operators of the type (2.64) are Hilbert-Schmidt operators. Therefore, a sufficient (and, actually, also necessary) condition for the compactness of the integral operator $K : L^2(a,b) \to L^2(a,b)$ is the convergence of the integral

$$\int_a^b \int_a^b |k(x,t)|^2 dtdx < \infty. \ \square$$

However, the compact operators are not at all exhausted by the Hilbert-Schmidt operators. The size of the class of the Hilbert-Schmidt operators within the compact operators can be demonstrated by the introduction of the following terminology.

In the class $\sigma(H,H)$ of the compact operators on a Hilbert space H, we can introduce the following parametrisation. One says that an operator $A : H \to H$ belongs to the class $S_p(H)$, if

$$\|A\|_p := \left(\sum_{k=1}^{\infty} \|Ae_k\|_H^p \right)^{1/p} < \infty,$$

where $e_k (k \in \mathbb{N})$ is an orthonormal basis in H. The value $\|A\|_p$ does not depend on the choice of the orthonormal basis e_k. These classes are called the ***Schatten-von Neumann classes***. They are nested:

$$S_p(H) \subset S_q(H) \ \text{ for } \ p \leq q.$$

The operators from the class $S_1(H)$ are called ***trace class*** operators. The operator $A : H \to H$ will be trace class, if it can be represented in the form $A = BC$, where B and C are Hilbert-Schmidt operators. In this sense, the trace class operator is "the smallest" Schatten-von Neumann class among the compact operators. More details about the Schatten-von Neumann classes and some of their properties will be given in Section 3.9. $\square$

In Section 1.8 we have given the general definition of embeddings of the spaces and introduced some examples of embedding. In some cases this embedding is compact, that is, the operator $J : X \to Y$ realising the embedding of the space X into Y is compact.

Theorem 2.69 *The following embeddings of the spaces are compact:*

- *the embedding of all spaces $\mathbb{R}^n$ in $\mathbb{R}^m$ for $n < m$;*
- *the embedding of all spaces $C^k[a,b]$ in $C^m[a,b]$ for any integers $k > m \geq 0$;*
- *the embedding of the space $C[a,b]$ in $L^p(a,b)$ for any $p \geq 1$;*
- *the embedding of the Sobolev spaces $L_k^p(a,b)$ in $L_m^p(a,b)$ for $k > m, \ p \geq 1$;*

- *the embedding $L_k^p(a,b)$ in $C[a,b]$ for any $p \geq 1,\ k \geq 1$.*

The theorems on the compact embeddings of spaces are especially important for the theory of differential operators. This is due to the fact that, as a rule, inverse operators L^{-1} to the differential operators L have the property of "improving" the smoothness. That is, for example, often they have the property that $L^{-1}f \in L_k^2$ for $f \in L^2$. And, accordingly, the estimate of the form $\|L^{-1}f\|_{L_k^2} \leq \|f\|_{L^2}$ holds. Then L^{-1} can be represented as the composition of the bounded operator ($L^{-1} : L^2 \to L_k^2$) with a bounded operator (the embedding operator $I : L_k^2 \to L^2$). Therefore, the compactness of the operator L^{-1} follows from the compactness of the embedding operator $I : L_k^2 \to L^2$. That is, on the *basis of the smoothness of the solution of a problem one can conclude the compactness of the inverse operator.* $\square$

2.16 Volterra operators

An important class of the compact operators is the so-called *Volterra operators*.

A bounded operator $A : X \to X$ is called ***quasinilpotent***, if

$$\lim_{k\to\infty} \|A^k\|^{1/k} = 0, \quad \text{where } A^k = A\left(A^{k-1}\right).$$

A compact quasinilpotent operator is called a ***Volterra*** operator.

The Volterra operators are characterised by the fact that if $A :\ X \to X$ *is Volterra*, then for any number $\lambda \in \mathbb{C}$ *the operator* $I - \lambda A$ *is well-posed*. Consequently, the inverse operator $(I - \lambda A)^{-1}$ is compact. The compactness of the inverse operator is explained by the fact that in this case the operator $(I - \lambda A)^{-1}$ can be represented in the form of the limit of a sequence of compact operators:

$$S = \lim_{k\to\infty} S_k = \sum_{m=1}^{\infty} \lambda^m A^m, \tag{2.65}$$

where $S_k = \sum\limits_{m=1}^{k} \lambda^m A^m$ is compact as a finite sum of compact operators. The convergence of the series in the right-hand side (2.65) is ensured by the estimate

$$\|Sf\| = \left\| \sum_{m=1}^{\infty} \lambda^m A^m f \right\| \leq \sum_{m=1}^{\infty} \|\lambda^m A^m f\| \leq \left(\sum_{m=1}^{\infty} |\lambda|^m \|A^m\| \right) \cdot \|f\|$$

and the quasinilpotentness of the operator A, if the Cauchy criterion for the convergence of the sequences is applied:

$$\lim_{m\to\infty} \sqrt[m]{|a_m|} = |\lambda| \lim_{m\to\infty} \|A^m\|^{1/m} = 0.$$

Example 2.70 Let us return to the consideration of the integral operator already repeatedly met in the examples. In the space $L^2(a,b)$, consider the integral operator

$$Vf(x) = \int_a^x f(t)dt.$$

Note that here we choose the integral operator with *a variable upper limit*. The operator V is compact since it is a Hilbert-Schmidt operator (see Example 2.68). Let us show that it is quasinilpotent. For the powers of this operator we have

$$V^m f(x) = \int_a^x \left(V^{m-1}f\right)(x_1)dx_1 = \int_a^x dx_1 \int_a^x dx_2 ... \int_a^{x_{m-1}} f(x_m)dx_m$$
$$= \int_a^x \frac{(x-x_m)^{m-1}}{(m-1)!} f(x_m)dx_m.$$

Therefore, for all $f \in L^2(a,b)$, applying the Hölder inequality (2.20), we get

$$\|V^m f\|^2 \le \int_a^b \left(\int_a^x \left|\frac{(x-t)^{m-1}}{(m-1)!} f(t)\right| dt\right)^2 dx$$
$$\le \int_a^b \left(\int_a^x \left|\frac{(x-t)^{m-1}}{(m-1)!}\right|^2 dt\right)\left(\int_a^x |f(t)|^2 dt\right) dx$$
$$\le \frac{\|f\|^2}{[(m-1)!]^2} \int_a^b \left(\int_a^x (x-t)^{2m-2} dt\right) dx = \frac{(b-a)^{2m}}{[(m-1)!]^2 2m(2m-1)} \|f\|^2.$$

Consequently,

$$\|V^m\| \le \frac{(b-a)^m}{(m-1)!\sqrt{2m(2m-1)}}.$$

If we now use the ***asymptotic Stirling's formula*** (or Stirling's approximation)

$$m! = m^m \sqrt{2\pi m} \exp\left\{-m + \frac{1}{12m} + o\left(\frac{1}{m^2}\right)\right\},$$

then it is easy to obtain that $\lim_{m\to\infty} \|V^m\|^{1/m} = 0$. Consequently, the operator V is quasinilpotent and, therefore, is Volterra. □

In a more general case, the integral operator

$$Kf(x) = \int_a^x k(x,t)f(t)dt, \tag{2.66}$$

where $k = k(x,t) \in L^2((a,b)\times(a,b))$, is Volterra in $L^2(a,b)$. We will demonstrate this in Example 2.72. Therefore, the integral equation

$$f(x) - \lambda \int_a^x k(x,t)f(t)dt = g(x),$$

called the ***second kind integral Volterra equation***, *has a unique solution* in $L^2(a,b)$ for any $\lambda \in \mathbb{C}$.

Unlike the integral operators of the general form (2.64), the operators of the form (2.66) are called ***integral operators of a triangular kind***. This is due to the fact that on the coordinate plane (x,t), the support of the integral kernel of the integral operator is contained in the triangle $a \le t \le x \le b$.

However, in general, the Volterra operators are not necessarily operators of the triangular kind.

So, for example, the integral operator

$$K_\alpha f(x) = \alpha \int_a^x (x-t)f(t)dt - (1-\alpha)\int_x^1 (x-t)f(t)dt \tag{2.67}$$

is Volterra for all $\alpha \in \mathbb{C}$, although it does not have the triangular kind. We will show the Volterra property of the operator (2.67) in the sequel, when considering the spectral properties of well-posed problems for an ordinary differential equation (see Example 3.102). □

For proving the Volterra property of a wide class of the integral operators we may recommend the following criterion by A.B. Nersesyan [85] (1964). Although its formulation is rather cumbersome we will show with the example of operator (2.66) this criterion is quite operational.

To begin with, we introduce some definitions.

Let $S \subset \Omega \times \Omega$ $(\Omega \subset \mathbb{R}^n)$ be an open set. A function $K = K(x,y)$ of two variables $x,y \in \Omega$ is called an ***S-kernel*** if $K \in L^2(\Omega\times\Omega)$ and $K(x,y) = 0$ for $(x,y) \notin S$. Thus, the function $K \in L^2(\Omega\times\ \Omega)$ will be an S-kernel if it is equal to zero outside of the domain S. The open set $S \subset \Omega\times\Omega$ is called a ***set of type V*** if any integral operator with S-kernel is Volterra. That is, any integral operator $K : L^2(\Omega) \to L^2(\Omega)$ of the kind

$$Kf(x) = \int_\Omega K(x,t)f(t)dt, \tag{2.68}$$

where $K = K(x,y)$ is an S-kernel, will be Volterra. Moreover, the Volterra property of the operator does not depend on a particular kind of the function $K(x,y)$, but only on the set S, more precisely, on the set on which $K(x,y) = 0$.

Let us introduce provisional notations:

- we will write $x \xrightarrow{S} y$ if $(x,y) \in S$,
- we will write $x \xleftarrow{S} y$ if $(x,y) \notin S$.

Theorem 2.71 (A.B. Nersesyan [85]) *In order for the set $S \subset \Omega\times\Omega$ to be* a set of type V*, it is necessary and sufficient that for all $k \ge 1$, from the conditions*

$$x_1 \xrightarrow{S} x_2 \xrightarrow{S} x_3 \xrightarrow{S} \ldots \xrightarrow{S} x_k \tag{2.69}$$

it follows that

$$x_k \xleftarrow{S} x_1. \tag{2.70}$$

In other words, the meaning of conditions (2.69) and (2.70) consists in the property that from the conditions

$$(x_1,x_2) \in S,\ (x_2,x_3) \in S,\ \ldots, (x_{k-1},x_k) \in S, \tag{2.71}$$

one should get the condition

$$K(x_k,x_1) = 0. \tag{2.72}$$

For demonstrating an application of Theorem 2.71 we will prove the Volterra property of the integral operator (2.66). In Example 2.70, the Volterra property of the operator V, being a particular case of operator (2.66), has been proved. There, the representation of the operator V^m has been obtained in an explicit form and the estimate $\lim\limits_{m\to\infty} \|V^m\|^{1/m} = 0$ has been obtained. But in the case of the general kind of operator (2.66) it is impossible to calculate an explicit form of the operator K^m. Therefore, proving the Volterra property may be difficult if we rely on a direct estimation of the norm of K^m. However, Theorem 2.71 can make it easy to get around these difficulties.

Example 2.72 In the space $L^2(a,b)$, consider the integral operator (2.66):

$$Kf(x) = \int_a^x k(x,t) f(t)dt,$$

where $k = k(x,t) \in L^2((a,b)\times(a,b))$. As is shown in Example 2.68, this operator is compact. The operator (2.66) is a particular case of the general integral operators (2.64) and (2.68). The difference is that in (2.66) the integral is taken with the variable upper limit. The operator K can be represented in a general form

$$Kf(x) = \int_a^b K(x,t) f(t)dt, \tag{2.73}$$

where

$$K(x,t) = \theta(x-t)k(x,t) = \begin{cases} k(x,t), & x \geq t, \\ 0, & x < t. \end{cases}$$

Let us describe the set S, for which the kernel $K(x,t)$ of the integral operator (2.73) will be an S-kernel. Firstly, the set S is two-dimensional, that is, $S \subset \mathbb{R}^2$. Secondly, since $a \leq x,\ t \leq b$, we have $S \subset [a,b]\times[a,b] \subset \mathbb{R}^2$. Thirdly, since $K(x,t) = 0$ for $x < t$, we choose as S the set

$$S := \{(x,y) \in \mathbb{R}^2 : a < t < x < b\}. \tag{2.74}$$

This set S is the triangle in the plane $\mathbb{R}^2$.

Since $K(x,y) = 0$ for $(x,y) \notin S$, for the set S chosen in this way, the integral kernel (2.73) of the integral operator (2.66) will be the S-kernel.

Let us show now that S is *a set of type V*. Let us choose an arbitrary chain of points from the set S satisfying (2.71):

$$(x_1,x_2) \in S,\ (x_2,x_3) \in S,\ \ldots, (x_{k-1},x_k) \in S.$$

In view of (2.74) this means that points x_j are subject to the inequalities: $x_1 < x_2 < x_3 < ... < x_{k-1} < x_k$. Hence $x_1 < x_k$ and, therefore, the point (x_k, x_1) does not belong to the set S. Consequently, $K(x_k, x_1) = 0$, that is, (2.72) holds. By Theorem 2.71 the set S is then *a set of type* V. Therefore, the integral operator (2.66) will be *Volterra independently of a particular type of the function* $k = k(x,t) \in L^2((a,b)\times(a,b))$. $\square$

Note that Theorem 2.71 is a criterion in the sense that it provides the Volterra property of an operator only in terms of the set on which $K(x,y) = 0$, independently of the particular form of the function $K = K(x,y)$. However, there exist integral kernels $K(x,y)$ of a special type, for which the operator (2.68) will be Volterra, although its corresponding set S will not be a set of type V. An example of such an operator is the Volterra operator (2.67).

2.17 Structure of the dual space

Let us return to the concept of a linear functional introduced in Section 2.6. As we have shown, a functional is a particular case of an operator with the image being the complex plane $\mathbb{C}$. The value of a linear functional F on an element $x \in X$ is sometimes also denoted by $\langle x, F\rangle$.

Let X be a Banach space. Consider the space of linear bounded functionals $\mathscr{L}(X,\mathbb{C})$ defined from X to $\mathbb{C}$. This space is called ***dual*** to X and is denoted by X^*. Thus, $X^* = \mathscr{L}(X,\mathbb{C})$. The elements of the dual space X^* are the linear bounded functionals defined on X. The space X^* is infinite-dimensional if X is infinite-dimensional.

An important special case is the case of the Hilbert space H. In this case, by the Riesz theorem (Theorem 2.15) on the general form of linear continuous functionals in the Hilbert space, for any functional F there exists a unique element $\sigma \in H$, such that

$$\langle x, F\rangle = \langle x, \sigma\rangle \text{ for all } x \in H,$$

with $\|F\| = \|\sigma\|$. This indicates the possibility of setting a one-to-one correspondence between the spaces H and H^* preserving the norm. We can identify $H = H^*$ up to this one-to-one correspondence. That is, the space dual to the Hilbert space "coincides" with H. In this sense we can talk about ***self-duality of the Hilbert space***. This fact is especially important when considering later on the so-called *adjoint operators*.

If the space X is not a Hilbert space, then $X \neq X^*$. Since X^* is also a Banach space, one can introduce the space dual to X^* and denote it by $X^{**} = (X^*)^*$. An important class of spaces is ***reflexive spaces***, that is the spaces, for which $X^{**} = X$.

The self-dual spaces are reflexive. The Banach space is reflexive if and only if the space dual to it is reflexive.

For example, the space $L^p(a,b),\ p>1$, is reflexive. The space $L^q(a,b),\ \frac{1}{p}+\frac{1}{q}=1$, is dual to it. Therefore,

$$(L^p(a,b))^{**}=\left(L^{\frac{p}{p-1}}(a,b)\right)^*=L^p(a,b).$$

In particular, from this the self-duality of the Hilbert space $L^2(a,b)$ also follows.

2.18 Adjoint to a bounded operator

The simplest way to introduce the concept of an adjoint operator is for the case of *bounded operators*. Let X,Y be Banach spaces. For a linear bounded operator $A\in\mathscr{L}(X,Y)$ for each fixed $g\in Y^*$ we define a functional φ by the formula

$$\varphi(f):=\langle Af,g\rangle,\ \forall f\in X. \tag{2.75}$$

This functional has the properties:

- $D(\varphi)=X$;
- the functional $\varphi(f)$ is linear since for any (complex) scalars α and β we have

$$\varphi(\alpha f_1+\beta f_2)=\langle A(\alpha f_1+\beta f_2),g\rangle=\langle\alpha Af_1+\beta Af_2,g\rangle$$
$$=\alpha\langle Af_1,g\rangle+\beta\langle Af_2,g\rangle=\alpha\varphi(f_1)+\beta\varphi(f_2);$$

- the functional $\varphi(f)$ is bounded since

$$|\varphi(f)|=|\langle Af,g\rangle|\le\|Af\|\cdot\|g\|\le\|A\|\cdot\|g\|\cdot\|f\|. \tag{2.76}$$

Consequently, $\varphi(f)\in X^*$.

Thus, to each element $g\in Y^*$, we have put in correspondence the element $\varphi(f)\in X^*$. This element is *uniquely* defined by formula (2.75). This means that we are actually obtaining an operator from the space Y^* to the space X^*, which we denote by A^*. That is,

$$A^*:Y^*\to X^*,\quad A^*g=\varphi.$$

This operator is linear since for all $f\in X$ we have

$$\langle f,A^*(\alpha g_1+\beta g_2)\rangle=\langle Af,\alpha g_1+\beta g_2\rangle$$
$$=\overline{\alpha}\langle Af,g_1\rangle+\overline{\beta}\langle Af,g_2\rangle=\langle f,\alpha A^*g_1+\beta A^*g_2\rangle.$$

This operator is bounded in view of inequality (2.76).

So, we have obtained *the linear bounded operator* $A^*g=\varphi$. This operator $A^*\in\mathscr{L}(Y^*,X^*)$ is called the ***adjoint operator*** to the operator A. Here we note once again that the adjoint operator exists for any linear bounded operator $A\in\mathscr{L}(X,Y)$. In the case of unbounded operators the adjoint operator does not always exist.

The adjoint operators have equal norms:

$$\|A\| = \|A^*\|.$$

Moreover, the adjoint operators have many identical properties.

Theorem 2.73 (Schauder's theorem) *Let $A \in \mathscr{L}(X,Y)$. The operator A is compact if and only if A^* is compact.*

Theorem 2.74 *The operator $A \in \mathscr{L}(X,X)$ is Volterra if and only if A^* is Volterra.*

Theorem 2.75 *Let $A,B \in \mathscr{L}(X,Y)$, $\alpha,\beta \in \mathbb{C}$. Then $(\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$.*

Theorem 2.76 *If $A,B \in \mathscr{L}(X,X)$, then $(AB)^* = B^*A^*$.*

Theorem 2.77 *Let $A \in \mathscr{L}(X,Y)$. If A is a well-posed operator (that is, the inverse operator A^{-1} exists, is defined on the whole space Y, and is bounded), then A^* is continuously invertible and $(A^*)^{-1} = (A^{-1})^*$.*

Theorem 2.78 *If X is a reflexive space, Y is a linear normed space, $A \in \mathscr{L}(X,Y)$, then $(A^*)^* = A$.*

The simplest way to understand the concept of the adjoint operator is in the Hilbert spaces since they are self-adjoint.

According to the Riesz theorem 2.15, any linear bounded functional in the Hilbert space is represented in terms of an inner product. And in this case (2.75) can be written in the form

$$\varphi(f) = \langle f,\varphi\rangle = \langle Af,g\rangle, \text{ where } f \in H,\ g \in H. \tag{2.77}$$

For fixed $g \in H$ the expression $\langle Af,g\rangle$ is a linear functional applied to the element $f \in H$. By (2.76) this functional is bounded. By the Riesz theorem there exists a uniquely defined element $g^* \in H$ such that for any $g \in H$ the representation

$$\langle Af,g\rangle = \langle f,g^*\rangle \tag{2.78}$$

holds. Thus, to each element $g \in H$ one puts in correspondence the uniquely defined element $g^* \in H$. That is, in the space H there is an operator mapping $g \in H$ into $g^* \in H$. Denote it by A^*, so that $A^*g = g^*$. Substituting the obtained result into (2.78), we have $\langle Af,g\rangle = \langle f,A^*g\rangle$ for all $f \in H,\ g \in H$. Thus we have come to the definition of the adjoint operator in the Hilbert space:

The operator A^* is called the ***adjoint operator*** to the linear bounded operator $A : H \to H$, if for all $f,g \in H$ we have the equality

$$\langle Af,g\rangle = \langle f,A^*g\rangle. \tag{2.79}$$

Theorems 2.73–2.78 remain true for the adjoint operator in the Hilbert space.

An operator $A \in \mathscr{L}(H,H)$ in the Hilbert space H is called a ***self-adjoint operator***, if $A^* = A$, that is, if A coincides with its adjoint. According to this definition, A is self-adjoint if for all $f,g \in H$ the equality

$$\langle Af,g\rangle = \langle f,Ag\rangle$$

holds.

It is the possibility to "transfer" A from one entry to another in the inner product, that allows one to study the self-adjoint operators in more detail; this finds very wide application in various fields of mathematical sciences, in mechanics and in physics.

If $A,B \in \mathscr{L}(H,H)$ are self-adjoint operators, then for any real numbers $\alpha,\beta \in \mathbb{R}$ the operator $\alpha A + \beta B$ is self-adjoint in H.

An operator $A \in \mathscr{L}(H,H)$ in the Hilbert space H is called ***normal***, if

$$A^*A = AA^*,$$

that is, if A is commuting with its adjoint. It is clear that any *self-adjoint operator is normal*.

Let us now give some examples of the adjoint operators.

Example 2.79 Let us return to the operator of multiplication by a function considered earlier in Examples 2.7 and 2.27. Let the operator $T: L^2(a,b) \to L^2(a,b)$ be defined on a whole space $L^2(a,b)$ by the formula

$$Tf(x) = G(x)\cdot f(x),$$

where $G = G(x)$ is a given function continuous on the closed interval $[a,b]$. As has been shown earlier, this operator is bounded in $L^2(a,b)$. Since $D(T) = L^2(a,b)$, then the adjoint operator exists. Let us find it.

For all $f \in L^2(a,b)$ and all $g \in D(T^*)$ we have $\langle Tf,g\rangle = \langle f,T^*g\rangle$. That is,

$$\int_a^b G(x)f(x)\overline{g(x)}dx = \int_a^b f(x)\overline{T^*g(x)}dx,$$

or

$$\int_a^b f(x)\left[G(x)\overline{g(x)} - \overline{T^*g(x)}\right]dx = 0,$$

this equality holds for all $f \in L^2(a,b)$. Then we obtain (e.g. by Theorem 2.19) that $G(x)\overline{g(x)} - \overline{T^*g(x)} = 0$ or, that is the same, $T^*g(x) = \overline{G(x)}g(x)$. Since all the above calculations are true for all $g \in L^2(a,b)$, we have $D(T^*) = L^2(a,b)$.

So, to the operator $Tf(x) = G(x)\cdot f(x)$ defined on the whole space $L^2(a,b)$ the adjoint operator is the operator

$$T^*g(x) = \overline{G(x)}g(x),$$

which is also defined on the whole space $L^2(a,b)$.

It is clear that this operator is *self-adjoint*, if and only if $\overline{G(x)} = G(x)$, that is, when $G(x)$ is a real-valued function. However it is clear that for any $G(x)$ *the operator T is normal*. In particular, hence we also get an example of a non-self-adjoint operator being normal. □

Example 2.80 Let $X = Y = \mathbb{C}^n$ be the n-dimensional space of complex columns (see Example 1.2). Consider the linear operator $y = Ax$ given everywhere on $\mathbb{C}^n$ by the square matrix $A = (a_{ij})_{i,j=1}^n$, where the coefficients a_{ij} are complex numbers:

$$y_i = \sum_{j=1}^n a_{ij}x_j, \quad i = 1, \dots, n. \tag{2.80}$$

Here $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.

Since the space $\mathbb{C}^n$ is a Hilbert space, it is self-dual: $(\mathbb{C}^n)^* = \mathbb{C}^n$. Let $z = (z_1, z_2, ..., z_n)$ be a linear functional on $\mathbb{C}^n$. Since its action on the element Ax is expressed by their inner product (the Riesz theorem), we get

$$\langle Ax, z\rangle = \sum_{i=1}^n y_i\overline{z_i} = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j\right)\overline{z_i} = \sum_{j=1}^n x_j\overline{\left(\sum_{i=1}^n \overline{a_{ij}}z_i\right)} = \langle x, A^*z\rangle.$$

This defines the adjoint operator $\omega = A^*z$ acting by the formula

$$\omega_i = \sum_{j=1}^n \overline{a_{ji}}z_j, \quad i = 1, \dots, n. \tag{2.81}$$

Comparing (2.80) and (2.81), it is easy to see that the adjoint operator A^* is given by the matrix transposed to the matrix A, with complex-adjoint elements. It is clear that $A^{**} = (A^*)^* = A$, which we already know from the general theory.

The operator A will be *self-adjoint* (that is, $A^* = A$) if and only if $a_{ij} = \overline{a_{ji}}$ for all $i, j = 1, \dots, n$. In particular, we would have $a_{ii} = \overline{a_{ii}}$, that is, *only real coefficients are located on the main diagonal of the self-adjoint matrix.* The rest (outside the main diagonal) of the elements of the matrix can also be complex. So, for example, the 2×2 matrix

$$A_0 = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \tag{2.82}$$

is self-adjoint.

Consider now the question of when will a matrix operator A be normal?

For simplicity of calculations consider the two-dimensional space $\mathbb{C}^2$ and matrix operators in it, given by the square 2×2 matrices. Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}.$$

Calculate the multiplication of the matrices AA^* and A^*A:

$$AA^* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} = \begin{pmatrix} a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}} & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} & a_{21}\overline{a_{21}} + a_{22}\overline{a_{22}} \end{pmatrix},$$

$$A^*A = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \overline{a_{11}}a_{11} + \overline{a_{21}}a_{21} & \overline{a_{11}}a_{12} + \overline{a_{21}}a_{22} \\ \overline{a_{12}}a_{11} + \overline{a_{22}}a_{21} & \overline{a_{12}}a_{12} + \overline{a_{22}}a_{22} \end{pmatrix}.$$

The operator A will be normal if and only if $A^*A = AA^*$, that is, if the coefficients of the matrices AA^* and A^*A coincide. Comparing the above expressions for these matrices, we obtain conditions of normality:

(1) $|a_{11}|^2+|a_{12}|^2=|a_{11}|^2+|a_{21}|^2 \iff |a_{12}|=|a_{21}|$;

(2) $a_{11}\overline{a_{21}}+a_{12}\overline{a_{22}}=\overline{a_{11}}a_{12}+\overline{a_{21}}a_{22}$;

(3) $a_{21}\overline{a_{11}}+a_{22}\overline{a_{12}}=\overline{a_{12}}a_{11}+\overline{a_{22}}a_{21}$;

(4) $|a_{21}|^2+|a_{22}|^2=|a_{12}|^2+|a_{22}|^2 \iff |a_{12}|=|a_{21}|$.

One sees that conditions (1) and (4) above coincide, while equalities (2) and (3) are complex-adjoint and, consequently, also coincide. Thus, the operator A will be normal if and only if the following two conditions hold:

$$(a_{11}-a_{22})\overline{a_{21}}=\overline{(a_{11}-a_{22})}a_{12} \text{ and } |a_{12}|=|a_{21}|. \tag{2.83}$$

If the operator A is self-adjoint, that is, $a_{ij}=\overline{a_{ji}}$, then conditions (2.83) automatically hold.

However, *the class of the normal operators is wider than the class of the self-adjoint ones*. So, for example, the operator given by the matrix

$$A_1=\begin{pmatrix} i & -1 \\ 1 & 2i \end{pmatrix} \tag{2.84}$$

is normal (satisfies condition (2.83)), although it is not self-adjoint.

This example has the following generalisation. If a (nonzero) operator A is self-adjoint, then the operator $B=i\cdot A$ (where i is an imaginary unit) is not self-adjoint:

$$B^*=(iA)^*=\bar{i}\cdot A^*=-i\cdot A^*=-i\cdot A=-B\neq B,$$

although it is normal:

$$B^*B=(iA)^*(iA)=\bar{i}A^*iA=A^*A=AA^*=iA\bar{i}A^*=(iA)(iA)^*=BB^*.$$

In the above example (2.84), the operator A_1 is obtained from the self-adjoint operator (2.82) by multiplying by the imaginary unit: $A_1= \ i\cdot A_0$, and therefore is normal, although it is not self-adjoint. □

Example 2.81 Let $X=Y=L^2(a,b)$. Consider the integral operator introduced earlier in Examples 2.11 and 2.68:

$$Kf(x)=\int_a^b k(x,t)f(t)dt, \tag{2.85}$$

where $k=k(x,t)\in L^2((a,b)\times(a,b))$. As has been shown earlier, the operator K : $L^2(a,b)\to L^2(a,b)$ is a Hilbert-Schmidt operator and, therefore, it is defined on the whole space $L^2(a,b)$ and is compact. We will now find the operator adjoint to the operator K.

Since $L^2(a,b)$ is the Hilbert space, then for all $f,g\in L^2(a,b)$ we have the following chain of equalities

$$\langle Kf,g\rangle=\int_a^b Kf(x)\overline{g(x)}dx=\int_a^b\left\{\int_a^b k(x,t)f(t)dt\right\}\overline{g(x)}dx$$

$$= \int_a^b \left\{ \int_a^b k(x,t)\overline{g(x)}dx \right\} f(t)dt = \int_a^b f(t) \overline{\left\{ \int_a^b \overline{k(x,t)}g(x)dx \right\}} dt = \langle f, K^* g \rangle .$$

Consequently, the adjoint operator K^* is also the integral operator defined on the whole space $L^2(a,b)$ and is given by the formula

$$Kg(x) = \int_a^b \overline{k(t,x)}g(t)dt.$$

Herewith the kernel of the integral operator K^* is a function which is complex-conjugate and transposed to the kernel of the operator K. It is easy to see that K^*, as well as K, is the Hilbert-Schmidt operator. It is also clear that $K^{**} = (K^*)^* = K$, that demonstrates the conclusions of the general theory.

The operator K is a *self-adjoint* operator if and only if

$$k(x,t) = \overline{k(t,x)}, \quad \forall x,t \in [a,b].$$

Let us find normality conditions of the operator K. We calculate

$$K^*Kf(x) = \int_a^b \overline{k(t,x)}(Kf)(t)dt = \int_a^b \overline{k(t,x)} \left(\int_a^b k(t,s)f(s)ds \right) dt =$$

$$= \int_a^b \left(\int_a^b \overline{k(t,x)}k(t,s)dt \right) f(s)ds,$$

that is, the operator K^*K is also the integral operator

$$K^*Kf(x) = \int_a^b k_1(x,s)f(s)ds \text{ with the kernel } k_1(x,s) = \int_a^b \overline{k(t,x)}k(t,s)dt.$$

Calculating similarly, we find

$$KK^*f(x) = \int_a^b k_1^*(x,s)f(s)ds, \text{ where } k_1^*(x,s) = \int_a^b k(x,t)\overline{k(s,t)}dt.$$

The operator K is a normal operator if $k_1(x,s) \equiv k_1^*(x,s)$, that is, we obtain *the normality condition of the operator K*:

$$\int_a^b \overline{k(t,x)}k(t,s)dt = \int_a^b k(x,t)\overline{k(s,t)}dt, \quad \forall x,s \in [a,b]. \tag{2.86}$$

Condition (2.86) is satisfied for self-adjoint kernels $k(x,t) = \overline{k(t,x)}$. Therefore, any self-adjoint integral operator is automatically normal, that confirms the general theory. However not any normal operator is self-adjoint. A trivial example of this fact is the integral operator K_2 with the kernel $k_2(x,t) = i \cdot k(t,x)$, where $k(x,t)$ is a self-adjoint kernel. It is easy to see that K_2 is not self-adjoint, although it is a normal operator.

As we see, the class of normal operators is wider than the class of the self-adjoint ones. However these classes are sufficiently close to each other, which allows one to transfer many properties of the class of the self-adjoint operators to the normal operators. □

Example 2.82 Consider a particular case of operator (2.85), when an integral operator in $L^2(a,b)$ is a Volterra operator of the triangular form:

$$Kf(x) = \int_a^x k(x,t)f(t)dt, \text{ where } k \in L^2((a,b)\times(a,b)). \tag{2.87}$$

This operator has been considered earlier in Example 2.72. We have proved that it is a Hilbert-Schmidt operator and a Volterra operator. Let us find the operator adjoint to operator (2.87). Since this operator is represented by the integral with the upper limit variable, then for this we need to bring it to the form (2.85). Let us introduce notations

$$k_1(x,t) = \theta(x-t)k(x,t) = \begin{cases} k(x,t), & x \geq t, \\ 0, & x < t. \end{cases} \tag{2.88}$$

Then $k_1 \in L^2((a,b)\times(a,b))$ and the operator K is represented in the form

$$Kf(x) = \int_a^b k_1(x,t)f(t)dt$$

corresponding to the form (2.85).

The operator adjoint to the operator K has the form

$$K^*g(x) = \int_a^b \overline{k_1(t,x)}g(t)dt.$$

Using representation (2.88), we find

$$\overline{k_1(t,x)} = \theta(t-x)\overline{k(t,x)} = \begin{cases} 0, & x \geq t, \\ \overline{k(x,t)}, & x < t. \end{cases}$$

Therefore, the operator adjoint to the operator K has the form:

$$K^*g(x) = \int_x^b \overline{k(t,x)}g(t)dt.$$

This operator as well as (2.87) is integral (note that it is with variable *lower limit*), is defined on the whole space $L^2(a,b)$, and will be a Hilbert-Schmidt operator (consequently, it is also compact). Unlike the operators of the general form (2.85), in this case not only the kernel turns out to be complex conjugate and transposed, but also integration limits change. □

All the theorems and examples considered above apply only to the bounded operators defined on the whole space. The case when an operator is unbounded or $D(A) \neq X$, turns out to be rather complicated and requires a separate exposition.

2.19 Adjoint to unbounded operators

For the simplicity of exposition further considerations will be held only in Hilbert spaces. As has been indicated earlier, the Hilbert space is self-dual.

So, let a linear (bounded or unbounded) operator A with a domain $D(A) \subset H$ be given in a Hilbert space H. Assume that *the domain* of the operator *is dense* in the space H, that is, $\overline{D(A)} = H$. We will dwell on the importance of this a little later.

Fixing $D(A)$, we introduce the set $D^* \subset H$ consisting of such elements $v \in H$, for which there exists an element $\varphi \in H$ such that the equality

$$\langle Au, v\rangle = \langle u, \varphi\rangle, \ \forall u \in D(A) \tag{2.89}$$

holds. It is clear that the set D^* is not empty, since it contains at least the zero element.

Let us show that the set D^* is *a linear space* in H. Indeed, if $v_1, v_2 \in D^*$, then for them there exist corresponding $\varphi_1, \varphi_2 \in H$ such that (2.89) holds, that is,

$$\langle Au, v_1\rangle = \langle u, \varphi_1\rangle, \ \ \langle Au, v_2\rangle = \langle u, \varphi_2\rangle, \ \ \forall u \in D(A).$$

Then any of their linear combinations $\widehat{v} = \alpha v_1 + \beta v_2$ also belong to D^*, since for $\widehat{v}$ there exists $\widehat{\varphi} \in H$ such that (2.89) holds. This element is $\widehat{\varphi} = \alpha\varphi_1 + \beta\varphi_2$:

$$\langle Au, \alpha v_1 + \beta v_2\rangle = \overline{\alpha}\langle Au, v_1\rangle + \overline{\beta}\langle Au, v_2\rangle = \overline{\alpha}\langle Au, \varphi_1\rangle + \overline{\beta}\langle Au, \varphi_2\rangle = \langle Au, \alpha\varphi_1 + \beta\varphi_2\rangle.$$

Thus, we have shown that D^* is indeed a *linear space* in H.

So, the linear space D^* consists of those elements $v \in H$, for which *there exists* the element $\varphi \in H$ such that (2.89) holds.

Let us now show that if such element $\varphi \in H$ exists, then it is defined uniquely. Suppose the contrary: let there be two such elements $\varphi_1, \varphi_2 \in H$ corresponding to one element $v \in H$ in equality (2.89), that is, $\langle Au, v\rangle = \langle u, \varphi_1\rangle = \langle u, \varphi_2\rangle$. But then $\langle u, \varphi_1 - \varphi_2\rangle = 0$ for all $u \in D(A)$. That is, the element $\varphi_1 - \varphi_2 \in H$ has turned out to be orthogonal to all elements from $D(A)$. Since $\overline{D(A)} = H$, then by Theorem 2.19 we have $\varphi_1 - \varphi_2 = 0$, that is, $\varphi_1 = \varphi_2$.

Thus, *the density* in H *of the domain* of the operator *provides the uniqueness* of the choice of the element $\varphi \in H$.

In the end we have got that equality (2.89) gives to each element from the linear space D^* a uniquely determined element $\varphi \in H$. That is, in the space H some operator $A^* : H \to H$, $\varphi = A^* v$, with the domain $D(A^*) = D^*$ is given by equality (2.89). The operator A^* is called the ***adjoint operator*** to A and satisfies the equality

$$\langle Au, v\rangle = \langle u, A^* v\rangle, \ \ \forall u \in D(A) \text{ and } \forall v \in D(A^*). \tag{2.90}$$

This operator is linear. Indeed, for any linear combination of elements $v_1, v_2 \in D^*$ we have for all $u \in D(A)$ that

$$\langle u, A^*(\alpha v_1 + \beta v_2)\rangle = \langle Au, \alpha v_1 + \beta v_2\rangle$$

$$= \overline{\alpha}\langle Au, v_1\rangle + \overline{\beta}\langle Au, v_2\rangle = \langle u, \alpha A^* v_1 + \beta A^* v_2\rangle,$$

so that

$$A^*(\alpha v_1 + \beta v_2) = A^* v_1 + \beta A^* v_2.$$

All the above-mentioned properties can be summarised in the form of a theorem:

Theorem 2.83 *Let A be a linear operator in a Hilbert space H with the domain $D(A)$ dense in H. Then there exists a uniquely defined linear operator $A^* : H \to H$ adjoint to the operator A. For all $u \in D(A)$ and for all $v \in D(A^*)$, the equality* (2.90) *holds.*

An important difference in the definition of the adjoint operator in the general case from the definition of the adjoint operator for a bounded operator, is that equality (2.90) holds not for all $u, v \in H$, but only for all $u \in D(A)$ and for all $v \in D(A^*)$. The domain $D(A^*)$ should be understood as the largest possible linear space for which (2.90) holds.

In the particular case when the operator A is bounded and $D(A) = H$, both above definitions coincide. The definition of a *self-adjoint operator* has the same difference for the general and bounded cases.

Consider a linear operator A with a domain dense in H. Consequently, an adjoint operator A^* exists. If

$$D(A) \subset D(A^*) \text{ and } Au = A^*u \text{ for all } u \in D(A),$$

then the operator A is called ***symmetric***. That is, the operator A is called symmetric if A^* is an extension of the operator A.

Since for the symmetric operator $D(A) \subset D(A^*)$ and $\overline{D(A)} = H$, we also have $\overline{D(A^*)} = H$. Consequently, by Theorem 2.83, there exists an operator adjoint to A^*, that is, $A^{**} = (A^*)^*$. This fact is especially important, for example, for the theory of extensions of symmetric operators.

If $A = A^*$, then the operator A is called ***self-adjoint***. Unlike in the bounded operators case, in the general case it is required to have not only the equality of actions of the operators A and A^*, but also the equality of their domains:

$$D(A) = D(A^*).$$

We can also say that a symmetric operator will be self-adjoint if $D(A) = D(A^*)$.

The following important property of the adjoint operator is its closedness.

Theorem 2.84 *Let A be a linear operator with the domain dense in a Hilbert space H. Then:*

1. *The operator A^* adjoint to A is closed, although the operator A does not have to be closed;*

2. *If the operator A admits a closure $\overline{A}$, then $(\overline{A})^* = A^*$, that is, adjoint operators to the operator A and its closure coincide;*

3. *If the operator A^{**} exists, then $A \subset A^{**}$;*

4. *If the operator A admits a closure $\overline{A}$, then the operator A^{**} exists and is the closure of the operator A: $A^{**} = \overline{A}$. In particular, if A is closed and $\overline{D(A)} = H$, then $A^{**} = \overline{A}$;*

5. *If the operator A is invertible, $D(A)$ and $D(A^{-1})$ are dense in H (so that A^* and $(A^{-1})^*$ exist), then $(A^*)^{-1} = (A^{-1})^*$.*

In particular, from this theorem it follows that a self-adjoint operator is necessarily closed and for all $u, v \in D(A)$ the equality

$$\langle Au, v\rangle = \langle u, Av\rangle$$

holds. □

The fifth point of Theorem 2.84 is very useful in practice and one conveniently formulates it using the "algebraic" language. Let us draw the following diagram:

$$\begin{array}{ccc} A & \overset{*}{\Rightarrow} & A^* \\ \Downarrow (-1) & & \downarrow (-1) \\ A^{-1} & \overset{*}{\Rightarrow} & (A^{-1})^* \end{array}$$

in which the horizontal lines represent the transition to the adjoint operator, and the verticle ones represent the transition to the inverse operator.

Now the statement of Theorem 2.84, Part 5, can be reformulated in the following way: *If for a given operator A the operations drawn in the diagram by double arrows are well-defined, then the single arrow complements the diagram to the commutative one.* □

The following theorem indicates the influence of the boundedness of the operator on the domain of the adjoint one.

Theorem 2.85 *The equality $D(A^*) = H$ holds if and only if the operator A is bounded on $D(A)$. Then also $A^* \in \mathscr{L}(H,H)$ and $\|A^*\| = \|A\|$.*

Indeed, in this case it follows from the results of Section 2.18, there exists the operator adjoint to A^*, that is, $A^{**} = (A^*)^*$ and $A^{**} \in \mathscr{L}(H,H)$. However the operators A and A^{**} cannot coincide, since A^{**} is defined on the whole space H, and A is given only on $D(A)$. The thing is that the operator A^{**} will be necessarily closed, and the operator A (by virtue of the boundedness) will be closed if and only if $D(A) = H$. This is the condition for the coincidence of A and A^{**}. □

Let us introduce some more facts which will be important in the sequel.

Theorem 2.86 *Let A and B be linear operators in a Hilbert space H with domains dense in H. Then operators A^* and B^* exist, and if $B \subset A$ (that is, A is an extension of B), then $A^* \subset B^*$.*

Theorem 2.87 *If a linear operator A with a domain dense in H is well-posed (that is, A^{-1} exists, is defined on the whole space H, and is bounded), then $(A^*)^{-1}$ exists, is defined on the entire space H, and is bounded. That is, the operator A^* is also well-posed. Then also $(A^*)^{-1} = (A^{-1})^*$.*

For practical applications in investigating particular operators it is rather useful to introduce the following terminology.

Let an operator A with a domain dense in H be given in a Hilbert space H. An operator A^+ is called the ***formally adjoint operator*** to the operator A, if for all $u \in D(A)$ and $v \in D(A^+)$ the equality

$$\langle Au, v\rangle = \langle u, A^+ v\rangle$$

holds.

It is clear that the formally adjoint operator is not defined uniquely. Indeed, any restriction of the operator A^+ is also formally adjoint to the operator A. That is, the domain of the formally adjoint operator can turn out to be smaller than the linear space D^* from the definition of the adjoint operator. Now it is clear that the (real) adjoint operator A^* to A is the maximal extension among all adjoint operators $\{A^+\}$.

Since the operator A^* is also formally adjoint to A, the (real) adjoint operator A^* to A coincides with that formally adjoint operator which has a maximum domain.

Let us introduce now some examples of constructing adjoint operators to unbounded ones.

Example 2.88 Let us return to the operator considered earlier in Example 2.43. In the Hilbert space $L^2(0,1)$, consider the linear operator T, the action of which is given on the whole space $D(T) = C[0,1]$ by the formula

$$Tf(x) = xf(0), \; \forall f \in D(T).$$

As is shown in Example 2.43, *this operator is not closed and does not admit a closure*. Let us find the operator adjoint to T. Since the domain $D(T) = C[0,1]$ is a linear space dense in $L^2(0,1)$, the adjoint T^* exists.

By the definition of the adjoint operator for all functions $f \in D(T)$ and $g \in D(T^*)$ the equality $\langle Tf, g\rangle = \langle f, T^*g\rangle$ holds, that is,

$$f(0)\int_0^1 x\overline{g(x)}dx = \int_0^1 f(x)\overline{T^*g(x)}dx. \tag{2.91}$$

This equality should hold for all $f \in D(T)$, including those $f \in C[0,1]$, which have $f(0) = 0$. Therefore for such $f(x)$ from (2.91) we get

$$\int_0^1 f(x)\overline{T^*g(x)}dx = 0 \tag{2.92}$$

for all $g \in D(T^*)$ and for all $f \in C[0,1] \cap \{f(0) = 0\}$.

But the linear space $C[0,1] \cap \{f(0) = 0\}$ is dense in $L^2(0,1)$. Therefore from (2.92) we get

$$T^*g(x) = 0 \tag{2.93}$$

for all $g \in D(T^*)$. Thus, we *have found the action* of the operator T^*. Let us now define its domain.

Taking into account (2.93), from (2.91) we obtain $f(0)\int_0^1 x\overline{g(x)}dx = 0$ for all $f \in D(T)$, including for f such that $f(0) \neq 0$. Consequently, $\int_0^1 xg(x)dx = 0$ for all $g \in D(T^*)$. That is, the domain of the adjoint operator is

$$D(T^*) = \left\{ g \in L^2(0,1) : \int_0^1 xg(x)dx = 0 \right\}. \tag{2.94}$$

So, the operator $T^*g(x) = 0$ with the domain (2.94) is adjoint to the operator $Tf(x) = xf(0)$ with the domain $D(T) = C[0,1]$. However the operator T^* is not "zero" in the full sense of this word. Taking into account its domain, the action of the operator can be also written in the form

$$T^*g(x) = \int_0^1 xg(x)dx.$$

Note that the domain (2.94) of the adjoint operator T^* *is not dense* in the space $L^2(0,1)$, since all functions $g \in D(T^*)$ are orthogonal to the element $\varphi(x) = x \in L^2(0,1)$, that is, $g \perp \varphi$. Therefore, *an operator* T^{**} *does not exist*. But it should not exist, because, as shown in Example 2.43, the operator T is not closed and does not admit the closure. This illustrates item (4) of Theorem 2.84. □

Example 2.89 Let us return to the differentiation operator $L : L^2(a,b) \to L^2(a,b)$ considered in Examples 2.36 and 2.55. The domain of the operator is

$$D(L) = \left\{ u \in L_1^2(a,b) : \ u(a) = u(b) = 0 \right\},$$

and its action is given by the formula

$$Lu(x) = \frac{d}{dx}u(x), \ a < x < b.$$

Earlier it has been shown that the operator L is linear, is defined on all functions from its domain, is unbounded as an operator from the space $L^2(a,b)$ to $L^2(a,b)$, and is not well-posed (since it is not everywhere solvable, although the inverse operator exists and is bounded on $R(L)$), and also is closed in the space $L^2(a,b)$. Let us find the operator adjoint to L.

It is clear that $\overline{D(L)} = L^2(a,b)$ and, therefore, L^* exists. Since for all $u \in D(L)$ and for all $v \in L_1^2(a,b)$ we have

$$\langle Lu, v \rangle = \int_a^b u'(x)\overline{v(x)}dx = u(b)\overline{v(b)} - u(a)\overline{v(b)} - \int_a^b u(x)\overline{v'(x)}dx \equiv \left\langle u, -\frac{d}{dx}v \right\rangle,$$

any function $v \in L_1^2(a,b)$ belongs to $D(L^*)$, that is, $L_1^2(a,b) \subset D(L^*)$ and $L^*v = -\frac{d}{dx}v$. Let us show that in fact $L_1^2(a,b) = D(L^*)$.

Let $v \in D(L^*)$ be an arbitrary function. Denote $L^*v = v^*$. It is clear that for any constant C the identity

$$v^*(x) = \frac{d}{dx}\left\{ \int_a^x v^*(t)dt + C \right\}$$

holds. Then for all $u \in D(L)$ we have

$$\langle Lu, v\rangle = \langle u, v^*\rangle = \int_a^b u(x)\overline{v^*(x)}dx = \int_a^b u(x)\overline{\frac{d}{dx}\left\{\int_a^x v^*(t)dt + C\right\}}dx.$$

Integrating by parts and taking into account $u(a) = u(b) = 0$, we find

$$\langle Lu, v\rangle = -\int_a^b \left\{\frac{d}{dx}u(x)\right\}\overline{\left\{\int_a^x v^*(t)dt + C\right\}}dx.$$

Since $Lu(x) = \frac{d}{dx}u(x)$, we get from this that

$$\int_a^b \left\{\frac{d}{dx}u(x)\right\}\overline{\left\{v(x) + \int_a^x v^*(t)dt + C\right\}}dx = 0 \ \text{ for } \forall u \in D(L). \tag{2.95}$$

Choose C such that

$$\int_a^b \left\{v(x) + \int_a^x v^*(t)dt + C\right\} dx = 0.$$

Then the function

$$u_0(x) = \int_a^x \left\{v(s) + \int_a^s v^*(t)dt + C\right\} ds$$

belongs to $D(L)$.

Since (2.95) holds for all $u \in D(L)$, then for the function $u_0(x)$, (2.95) takes the form

$$\int_a^b \left|v(x) + \int_a^x v^*(t)dt + C\right|^2 dx = 0.$$

Consequently, $v(x) + \int_a^x v^*(t)dt + C = 0$, that is,

$$v(x) = -\int_a^x v^*(t)dt - C. \tag{2.96}$$

Therefore, any function $v \in D(L^*)$ belongs to $L_1^2(a,b)$, that is $L_1^2(a,b) \supset D(L^*)$. And since we have proved earlier that $L_1^2(a,b) \subset D(L^*)$, we have $L_1^2(a,b) = D(L^*)$.

From (2.96) we have that almost everywhere on (a,b) we have $\frac{d}{dx}v(x) = -v^*(x)$. Therefore, $L^*v = -\frac{d}{dx}v$.

So, we have shown that the differential operator $L^*v = -\frac{d}{dx}v$ with the domain $D(L^*) = L_1^2(a,b)$ is adjoint to the differential operator $Lu(x) = \frac{d}{dx}u(x)$ with the domain $D(L) = \{u \in L_1^2(a,b) : u(a) = u(b) = 0\}$.

According to Theorem 2.84, Part 4, there exists the operator $L^{**} = (L^*)^*$, and by the closedness of the operator L the equality $L^{**} = L$ holds. That is, the operator $Lu(x) = \frac{d}{dx}u(x)$ with the domain $D(L) = \{u(x) \in L_1^2(a,b) : u(a) = u(b) = 0\}$ is the operator adjoint to the operator $L^*v = -\frac{d}{dx}v$ with the domain $D(L^*) = L_1^2(a,b)$. $\square$

Here, a formally adjoint operator to the operator L will be, for example, any of operators $L_j^+ v = -\frac{d}{dx}v$ with the following domains:

- $D(L_1^+) = \{v \in L_1^2(a,b) : v(a) = v(b) = 0\}$;
- $D(L_2^+) = \{v \in L_1^2(a,b) : v(a) = 0\}$;
- $D(L_3^+) = \{v \in L_1^2(a,b) : \alpha v(a) + \beta v(b) = 0\}$;
- $D(L_4^+) = \{v \in L_k^2(a,b)\}$, $k \geq 1$.

This follows easily from the fact that $D(L_j^+) \subset D(L^*),\ j = 1,2,3,4$.

Example 2.90 Consider the operator $L : L^2(a,b) \to L^2(a,b)$ obtained from the operator of the previous example by multiplying by the imaginary unit:

$$Lu(x) = i\frac{d}{dx}u(x),$$

with the domain

$$D(L) = \{u(x) \in L_1^2(a,b) :\ u(a) = u(b) = 0\}.$$

Similarly to the previous example, it is easy to prove that the operator $L^*v(x) = i\frac{d}{dx}v(x)$ with the domain $D(L^*) = L_1^2$ is the operator adjoint to the operator L.

It is easy to see that the actions of the operators L and L^* coincide: $Lu = L^*u$ for all $u \in D(L)$. Since $D(L) \subset D(L^*)$, then the operator L is the restriction of L^*. Consequently, the operator L is *a symmetric* operator. Since $D(L) \neq D(L^*)$, *L is not a self-adjoint* operator.

The considered example shows the *difference between the concepts of symmetric and self-adjoint operators.* □

Example 2.91 Consider now the operator corresponding to a boundary value problem with general boundary conditions for the first-order differential equation:

$$Lu(x) = i\frac{d}{dx}u(x),\ D(L) = \{u \in L_1^2(a,b) :\ \alpha u(a) + (1-\alpha)u(b) = 0\},$$

where $\alpha \in \mathbb{C}$.

Since the domain of the operator is a linear space, and the action of the operator is linear, *the operator L is linear*.

By constructing a test function, similar to Example 2.10, it is easy to show that *the operator L is unbounded* as an operator on $L^2(a,b)$.

Let us find the operator adjoint to L. It is clear that $\overline{D(L)} = L^2(a,b)$, therefore, L^* exists. For all functions $u, v \in L_1^2(a,b)$, by integration by parts we get

$$\begin{aligned}
\langle Lu, v\rangle &= \int_a^b Lu(x)\cdot\overline{v(x)}dx = \int_a^b iu'(x)\cdot\overline{v(x)}dx = iu(x)\overline{v(x)}\Big|_a^b \\
&- \int_a^b iu(x)\cdot\overline{v'(x)}dx = u(b)\overline{v(b)} - u(a)\overline{v(a)} + \int_a^b u(x)\cdot\overline{iv'(x)}dx \qquad (2.97)\\
&= u(b)\overline{v(b)} - u(a)\overline{v(a)} + \langle u, i\frac{d}{dx}v(x)\rangle.
\end{aligned}$$

Let us introduce the operator

$$L^+v(x) = i\frac{d}{dx}v(x),\ D(L^+) = \left\{v \in L_1^2(a,b) : (1-\overline{\alpha})v(a) + \overline{\alpha}v(b) = 0\right\}.$$

Since for all $u \in D(L)$ and for all $v \in D(L^+)$ we have

$$\begin{cases} \alpha u(a)\overline{v(b)} + (1-\alpha)u(b)\overline{v(b)} = 0, \\ \alpha u(b)\overline{v(b)} + (1-\alpha)u(b)\overline{v(a)} = 0, \\ -\alpha u(a)\overline{v(a)} - (1-\alpha)u(b)\overline{v(a)} = 0, \\ -\alpha u(a)\overline{v(b)} - (1-\alpha)u(a)\overline{v(a)} = 0, \end{cases}$$

then summing these equations, we get $u(b)\overline{v(b)} - u(a)\overline{v(a)} = 0$. Therefore, from (2.97) it follows that $\langle Lu, v\rangle = \langle u, L^+v\rangle$ for all $u \in D(L)$ and for all $v \in D(L^+)$. That is, the operators L and L^+ are *formally adjoint*.

Let us show that L^+ is really the adjoint, that is, $L^+ = L^*$. To do this we show that the operators L and L^+ are well-posed and we will use Part 5 of Theorem 2.84.

A general solution of the differential equation

$$Lu(x) = i\frac{d}{dx}u(x) = f(x)$$

has the form

$$u(x) = -i\int_a^x f(t)dt + C, \text{ where } C \text{ is some constant.} \tag{2.98}$$

Using formula (2.98) with the boundary condition

$$\alpha u(a) + (1-\alpha)u(b) = 0,$$

we obtain the equation

$$0 = \alpha u(a) + (1-\alpha)u(b) = \alpha C + (1-\alpha)\left(-i\int_a^b f(t)dt + C\right),$$

to find the unknown constant C. Therefore, for any values of the complex coefficient α we find

$$C = i(1-\alpha)\int_a^b f(t)dt.$$

Hence and from (2.93) we find the explicit form of the inverse operator to the operator L,

$$u(x) = L^{-1}f(x) = -i\alpha\int_a^x f(t)dt + i(1-\alpha)\int_x^b f(t)dt. \tag{2.99}$$

Thus, we have shown that the operator L is well-posed and its inverse has the form (2.99).

Since the operator L^+ has the same structure as the operator L, then by the similar calculations one easily obtains that L^+ is well-posed and its inverse has the form

$$v(x) = (L^+)^{-1}g(x) = -i(1-\overline{\alpha})\int_a^x g(t)dt + i\overline{\alpha}\int_x^b g(t)dt. \tag{2.100}$$

Analysing formulae (2.99) and (2.100), it is easy to note that the operator L^{-1} is an integral operator with the integral kernel

$$k(x,t) = -i\alpha\theta(x-t) + i(1-\alpha)\theta(t-x),$$

and the operator $(L^+)^{-1}$ is an integral operator with the kernel $\overline{k(t,x)}$. Therefore, $(L^+)^{-1} = (L^{-1})^*$. That is, the operators $(L)^{-1}$ and $(L^+)^{-1}$ are adjoint.

Recall now the formulation of Part 5 of Theorem 2.84: *"If an operator A is invertible, and $D(A)$ and $D(A^{-1})$ are dense in H (so that A^* and $(A^{-1})^*$ exist), then $(A^*)^{-1} = (A^{-1})^*$"*.

As the operator A we choose the operator $(L)^{-1}$. Its domain coincides with the whole space, the domain of its inverse operator (that is, of the operator L) is dense in $L^2(a,b)$. Therefore from Part 5 of Theorem 2.84 it follows that $(A^*)^{-1} = (A^{-1})^*$, that is, $(L)^* = \left((L^{-1})^*\right)^{-1} = \left((L^+)^{-1}\right)^{-1} = L^+$. Thus, we have shown that the formally adjoint operator is really the adjoint, that is, $L^+ = L^*$.

The operator $(L^*)^{-1}$ has the same structure as the operator $(L)^{-1}$. Therefore, we can easily describe the action (differential expression) and the domain of the operator L^*. Comparing (2.99) and (2.100), we see that $D(L^*)$ is obtained from $D(L)$ by replacing α by $(1-\overline{\alpha})$. So, we have shown that the linear operator given by the differential expression $L^*v(x) = i\frac{d}{dx}v(x)$ on the domain

$$D(L^*) = \left\{v \in L_1^2(a,b) : (1-\overline{\alpha})u(a) + \overline{\alpha}u(b) = 0\right\}$$

is adjoint to L.

Comparing now the boundary conditions of the domains of the adjoint operator, it is easy to see that the operator L will be self-adjoint if and only if the equality

$$\alpha + \overline{\alpha} = 1 \text{ or, which is the same, } \operatorname{Re}(\alpha) = 1/2,$$

holds.

Note that a formally adjoint operator is defined in a non-unique way. The fact that in our case the formally adjoint operator and the real adjoint operator have coincided is caused by a special choice of the operator L^+. Under another choice of the operator L^+, it does not necessarily coincide with the operator L^*. □

The example considered earlier demonstrates the possibility of constructing the exact domain of the adjoint operator to a given well-posed operator. That is, in the case when the operator is well-posed it is often easier to find its adjoint operator. To do this, it is necessary to construct the inverse operator and to find the operator adjoint to it. The following example will also demonstrate a similar approach.

Example 2.92 Consider now the operator corresponding to a boundary value problem with boundary conditions of *Sturm type* for the second-order differential operator

$$Lu(x) = -\frac{d^2}{dx^2}u(x),$$

with the domain

$$D(L) = \left\{u \in L_2^2(a,b) :\ a_1 u'(a) + a_2 u(a) = 0,\ b_1 u'(b) + b_2 u(b) = 0\right\},$$

where $a_1, a_2, b_1, b_2 \in \mathbb{C}$. The boundary conditions of such kind are called ***conditions of Sturm type***, since in the first of conditions only values of the function and its derivative at the point a participate, and in the second one only values of the function and its derivative at the point b participate. Such boundary conditions are also called ***separated boundary conditions***.

Our goal now is to define the adjoint operator to the operator L. First, we note that the adjoint operator exists, since the domain $D(L)$ is dense in $L^2(a,b)$.

Second, we show that the operator is well-posed. A general solution of the differential equation

$$Lu(x) = -u''(x) = f(x)$$

has the form

$$u(x) = -\int_a^x (x-t) f(t)dt + C_1(x-a) + C_2, \tag{2.101}$$

where C_1, C_2 are some (yet) arbitrary constants.

Combining formula (2.101) with the boundary conditions

$$a_1 u'(a) + a_2 u(a) = 0,\ \ b_1 u'(b) + b_2 u(b) = 0,$$

we obtain the system of linear equations

$$\left\{ \begin{array}{l} a_1 C_1 + a_2 C_2 = 0, \\ [b_1 + b_2(b-a)]\, C_1 + b_2 C_2 = \int_a^b [b_1 + b_2(b-t)] f(t)dt, \end{array} \right. \tag{2.102}$$

to find the unknown constants C_1, C_2. This system has a unique solution if and only if its determinant is different from zero:

$$\triangle = \begin{vmatrix} a_1 & a_2 \\ b_1 + b_2(b-a) & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 - a_2 b_2 (b-a) \neq 0. \tag{2.103}$$

Condition (2.103) is a necessary and sufficient condition for the unique solvability of the system of linear equations (2.102). Everywhere in what follows we assume that this condition holds. Then the solution of system (2.102) can be written in the form

$$\left\{ \begin{array}{l} C_1 = -\frac{a_2}{\triangle} \int\limits_a^b [b_1 + b_2(b-t)] f(t)dt, \\ C_2 = \frac{a_1}{\triangle} \int\limits_a^b [b_1 + b_2(b-t)] f(t)dt. \end{array} \right.$$

Substituting now the obtained result into formula (2.101), we get the explicit form of the solution of our problem:

$$u(x) = -\int_a^x (x-t) f(t)dt - \frac{a_2(x-a) - a_1}{\triangle} \int_a^b [b_1 + b_2(b-t)] f(t)dt.$$

Consequently, the inverse operator has the form

$$L^{-1}f(x) = \int_a^b k(x,t)f(t)dt, \tag{2.104}$$

where the kernel of the integral operator is given by the formula

$$k(x,t) = -(x-t)\theta(x-t) - \frac{[a_2(x-a)-a_1][b_1+b_2(b-t)]}{\triangle}. \tag{2.105}$$

It is clear that the integral kernel (2.105) is a continuous and bounded function. Therefore, the operator (2.104) is defined on the whole space $L^2(a,b)$ and is bounded, that is, the operator L is *well-posed.*

Let us find the operator adjoint to the operator L^{-1}. The operator $(L^{-1})^*$ will be also the integral operator

$$(L^{-1})^* g(x) = \int_a^b \overline{k(t,x)}g(t)dt,$$

and its kernel is the function

$$\overline{k(t,x)} = -(t-x)\theta(t-x) - \frac{\left[\overline{b_1} + \overline{b_2}(b-x)\right]\left[\overline{a_2}(t-a) - \overline{a_1}\right]}{\overline{\triangle}}. \tag{2.106}$$

Therefore, the operator $(L^{-1})^*$ is represented in the form

$$(L^{-1})^* g(x) = \int_x^b (x-t)g(t)dt - \frac{\left[\overline{b_1} + \overline{b_2}(b-x)\right]}{\overline{\triangle}} \int_a^b \left[\overline{a_2}(t-a) - \overline{a_1}\right] g(t)dt.$$

By Part 5 of Theorem 2.84 the operators $(L^{-1})^*$ and $(L^*)^{-1}$ coincide. Consider now the function $v(x)$ which is an image under the action of the operator $(L^*)^{-1}$: $v(x) = (L^*)^{-1}g(x)$, that is,

$$v(x) = \int_x^b (x-t)g(t)dt - \frac{\left[\overline{b_1} + \overline{b_2}(b-x)\right]}{\overline{\triangle}} \int_a^b [\overline{a_2}(t-a) - \overline{a_1}]g(t)dt. \tag{2.107}$$

Here, when the functions $g(x)$ vary along the whole space $L^2(a,b)$, the functions $v(x)$ respectively vary along the whole range of the operator $(L^*)^{-1}$, that is, the whole domain of the operator L^*. Thus, to describe the operator L^* it is necessary to study the whole set of functions $v(x)$, when $g(x)$ varies over the whole space $L^2(a,b)$.

It is easy to see that the function $v(x)$ satisfies the differential equation

$$-\frac{d^2}{dx^2}v(x) = g(x). \tag{2.108}$$

Therefore, the action of the adjoint operator L^* is given by the differential expression

$$L^* v(x) = -\frac{d^2}{dx^2}v(x).$$

Let us describe the domain of the operator L^*. It is easy to see that $v \in L_2^2(a,b)$ holds for all $g \in L^2(a,b)$. Therefore, $D(L^*) \subset L_2^2(a,b)$.

Let us now find the boundary conditions for $v(x)$. To do this, in the expression (2.107) instead of the function $g(x)$ we substitute its value from (2.108). Then by a direct calculation we get

$$v(x) = \int_x^b (t-x)\frac{d^2}{dt^2}v(t)dt + \frac{[\overline{b_1}+\overline{b_2}(b-x)]}{\overline{\triangle}}\int_a^b [\overline{a_2}(t-a)-\overline{a_1}]\,\frac{d^2}{dt^2}v(t)dt$$

$$= \int_x^b d\left[(t-x)v'(t)-v(t)\right] + \frac{[\overline{b_1}+\overline{b_2}(b-x)]}{\overline{\triangle}}\int_a^b d\left([\overline{a_2}(t-a)-\overline{a_1}]\,v'(t)-\overline{a_2}v(t)\right)$$

$$= (b-x)v'(b)-v(b)+v(x)+\frac{[\overline{b_1}+\overline{b_2}(b-x)]}{\overline{\triangle}}\Big\{\,[\overline{a_2}(b-a)-\overline{a_1}]\,v'(b)$$

$$-\overline{a_2}v(b)+\overline{a_1}v'(a)+\overline{a_2}v(a)\Big\} = v(x)+\Big\{bv'(b)-v(b)$$

$$+\frac{[\overline{b}_1+\overline{b}_2 b]}{\overline{\triangle}}\left[[\overline{a_2}(b-a)-\overline{a_1}]\,v'(b)-\overline{a_2}v(b)+\overline{a_1}v'(a)+\overline{a_2}v(a)\right]\Big\}$$

$$-x\left\{v'(b)+\frac{\overline{b}_2}{\overline{\triangle}}\left[[\overline{a_2}(b-a)-\overline{a_1}]\,v'(b)-\overline{a_2}v(b)+\overline{a_1}v'(a)+\overline{a_2}v(a)\right]\right\}.$$

Note that in the obtained equality the function $v(x)$ cancels from both sides, and the remaining functions are first-order polynomials, so that the equality to zero for them means the equality to zero of the coefficients. Therefore, the obtained equality is equivalent to two equalities:

$$\begin{cases} bv'(b)-v(b)+\frac{[\overline{b}_1+\overline{b}_2 b]}{\overline{\triangle}}\left[[\overline{a_2}(b-a)-\overline{a_1}]v'(b)-\overline{a_2}v(b)+\overline{a_1}v'(a)+\overline{a_2}v(a)\right]=0, \\ \\ v'(b)+\frac{\overline{b}_2}{\overline{\triangle}}\left[[\overline{a_2}(b-a)-\overline{a_1}]v'(b)-\overline{a_2}v(b)+\overline{a_1}v'(a)+\overline{a_2}v(a)\right]=0. \end{cases}$$

Multiplying both equations by $\overline{\triangle} = \ a_1b_2-a_2b_1-a_2b_2(b-a)\neq 0$, the system can be written in the form

$$\begin{cases} \overline{a_2}[\overline{b_1}v'(b)+\overline{b_2}v(b)]-\overline{b_2}[\overline{a_1}v'(a)+\overline{a_2}v(a)]=0, \\ \\ (\overline{a_1}+\overline{a_2}a)[\overline{b_1}v'(b)+\overline{b_2}v(b)]-(\overline{b_1}+\overline{b_2}b)[\overline{a_1}v'(a)+\overline{a_2}v(a)]=0. \end{cases} \tag{2.109}$$

If one considers the obtained system as a system of linear equations with respect to unknowns $\left[\overline{b_1}v'(b)+\overline{b_2}v(b)\right]$ and $[\overline{a_1}v'(a)+\overline{a_2}v(a)]$, then its determinant coincides with $\overline{\triangle}$:

$$\begin{vmatrix} \overline{a_2} & -\overline{b_2} \\ (\overline{a_1}+\overline{a_2}a) & -(\overline{b_1}+\overline{b_2}b) \end{vmatrix} = \overline{a_1b_2}-\overline{a_2b_1}-\overline{a_2b_2}(b-a)=\overline{\triangle}\neq 0.$$

Therefore, the system (2.109) has only the zero solution, that is,

$$\overline{b_1}v'(b)+\overline{b_2}v(b)=0,\ \ \overline{a_1}v'(a)+\overline{a_2}v(a)=0. \tag{2.110}$$

These are the required boundary conditions, which any function from the domain of the operator L^* satisfies.

Thus, we have obtained that

$$D(L^*)\subset\left\{v\in L_2^2(a,b):\ \overline{a_1}v'(a)+\overline{a_2}v(a)=0,\ \overline{b_1}v'(b)+\overline{b_2}v(b)=0\right\}. \tag{2.111}$$

Let us show now that in fact in (2.111) the equality holds, not just an inclusion. To do this, we show that the operator given on the domain coinciding with the right-hand part (2.111), is the required operator L^*.

We denote by L^+ the linear operator given by the differential expression

$$L^+v(x)=-\frac{d^2}{dx^2}v(x)$$

on the domain

$$D(L^+)=\left\{v\in L_2^2(a,b):\ \overline{a_1}v'(a)+\overline{a_2}v(a)=0,\ \overline{b_1}v'(b)+\overline{b_2}v(b)=0\right\}.$$

In view of the embedding (2.111) it is clear that

$$L^*\subset L^+. \tag{2.112}$$

That is, the operator L^* is a restriction of the operator L^+.

On the other hand, for all $u\in D(L)$ and for all $v\in D(L^+)$ we have

$$\begin{aligned}\langle Lu,v\rangle &= \int_a^b Lu(x)\cdot\overline{v(x)}dx=-\int_a^b u''(x)\cdot\overline{v(x)}dx=-\int_a^b d\left[u'(x)\cdot\overline{v(x)}\right.\\ &\left.-u(x)\cdot\overline{v'(x)}\right]-\int_a^b u(x)\cdot\overline{v''(x)}dx=-\left[u'(b)\cdot\overline{v(b)}-u(b)\cdot\overline{v'(b)}\right]\\ &+\left[u'(a)\cdot\overline{v(a)}-u(a)\cdot\overline{v'(a)}\right]+\langle u,L^+v\rangle\overset{def}{=}B+A+\langle u,L^+v\rangle.\end{aligned} \tag{2.113}$$

Let us show that the term $B+A$ in (2.113) is equal to zero.

First we consider B. In the case when $b_1=0$, from the boundary conditions it follows that $u(b)=v(b)=0$ and, therefore, $B=0$. Let now $b_1\neq 0$. The boundary conditions at the point $x=b$ have the form

$$b_1u'(b)+b_2u(b)=0,$$

$$\overline{b_1}v'(b)+\overline{b_2}v(b)=0\ \text{ or }\ b_1\overline{v'(b)}+b_2\overline{v(b)}=0.$$

Multiplying the first equation by $\overline{v(b)}$, we subtract from it the second equation multiplied by $u(b)$. We get

$$b_1[u'(b)\cdot\overline{v(b)}-u(b)\cdot\overline{v'(b)}]=0.$$

Since $b_1\neq 0$, we obtain $B=0$.

The equality $A = 0$ is proved similarly.

Thus, we have shown that $B + A = 0$. Therefore, it follows from (2.113) that $\langle Lu, v\rangle = \langle u, L^+ v\rangle$ holds for all $u \in D(L)$ and for all $v \in D(L^+)$. That is, the operator L^+ is formally adjoint to the operator L. Consequently, $L^* \supset L^+$. From this and (2.111) we obtain the equality $L^* = L^+$. This is what we wanted to prove.

So, we have shown that the linear operator given by the differential expression $L^* v(x) = -\frac{d^2}{dx^2} v(x)$ on the domain

$$D(L^*) = \left\{ v \in L_2^2(a,b) : \overline{a_1} v'(a) + \overline{a_2} v(a) = 0,\ \overline{b_1} v'(b) + \overline{b_2} v(b) = 0 \right\}$$

is adjoint to L.

Comparing now the boundary conditions in the domains of the adjoint operators, it is easy to see that the operator L will be *self-adjoint* if and only if the equalities

$$b_1 \overline{b_2} - b_2 \overline{b_1} = 0 \text{ and } a_1 \overline{a_2} - a_2 \overline{a_1} = 0$$

simultaneously hold. This alone ensures that the numbers $b_1 \overline{b_2}$ and $a_1 \overline{a_2}$ are real numbers.

Consequently, the coefficients a_1, a_2, b_1, and b_2 can always be chosen as real numbers. To do this, it suffices to multiply the first boundary condition by $\overline{a_1}$ or by $\overline{a_2}$, and multiply the second boundary condition by $\overline{b_1}$ or by $\overline{b_2}$.

This case is exactly the problem that was first considered by Sturm for the more general differential expression

$$Lu(x) = \frac{d}{dx}\left(p(x) \frac{d}{dx} u(x) \right) + q(x) u(x).$$

Namely, from the property of the self-adjointness, obtained by Sturm, many further spectral properties of the operator L follow. $\square$

The above example is an explicit demonstration of the recipe for constructing the adjoint operator. We have deliberately outlined this method in such detail to demonstrate all the steps necessary for the construction of the adjoint operator in a simple example. Exactly the same method does allow one to find the adjoint operators in more complicated cases, when even the action and the form of boundary conditions (and also the class of smoothness of the domain) of the adjoint operator are not clear.

2.20 Two examples of nonclassical problems

In the previous section we found the adjoint operators in problems that are, so to speak, "classical". We will now introduce two examples of operators corresponding to non-classical problems. And, as we see here, in these cases the adjoint operator may have quite a different structure than the original operator. We will proceed following the same scheme as above: first we construct the inverse operator, then we find its adjoint operator, and then we take the inverse again.

Example 2.93 Consider the operator corresponding to a problem with the so-called "interior" conditions for the second-order differential operator

$$Lu(x) = -\frac{d^2}{dx^2}u(x),\ D(L) = \left\{u \in L_2^2(0,1):\ u(0) = 0, u(1/2) = 0\right\}.$$

Boundary conditions of such form are called ***interior conditions***, since values of the function at the point $x = 1/2$ that is interior for the interval $(0,1)$, take part in one of them.

Our goal is to define the operator adjoint to the operator L. First, we immediately note that the adjoint operator exists, since the domain $D(L)$ is dense in $L^2(0,1)$.

Second, we show that the operator is well-posed. A general solution of the differential equation

$$-u''(x) = f(x)$$

has the form

$$u(x) = -\int_0^x (x-t)f(t)dt + C_1 x + C_2, \tag{2.114}$$

where C_1, C_2 are some arbitrary (so far) constants.

Combining the boundary conditions

$$u(0) = 0,\ \ u(1/2) = 0,$$

with the formula (2.114), for finding the unknown constants C_1, C_2, we obtain the system of linear equations

$$\begin{cases} C_2 = 0, \\ 1/2C_1 + C_2 = \int_0^{1/2}(1/2-t)f(t)dt. \end{cases}$$

This system has the unique solution

$$C_1 = \int_0^{1/2}(1-2t)f(t)dt,\ C_2 = 0.$$

Substituting now the obtained result in formula (2.114), we obtain the explicit form of the solution of our problem:

$$u(x) = -\int_0^x (x-t)f(t)dt + \int_0^{1/2} x(1-2t)f(t)dt.$$

Consequently, the inverse operator has the form

$$L^{-1}f(x) = \int_0^1 k(x,t)f(t)dt, \tag{2.115}$$

where the integral kernel of the integral operator is given by the formula

$$k(x,t) = -(x-t)\theta(x-t) + x(1-2t)\theta(1/2-t). \tag{2.116}$$

It is clear that the integral kernel (2.116) is a continuous function. Therefore, the operator L^{-1} exists, is given by formula (2.115), is defined on the whole space $L^2(0,1)$, and is bounded, that is, the operator L is *well-posed*.

Let us find the operator adjoint to the operator L^{-1}. The operator $(L^{-1})^*$ will also be the integral operator

$$(L^{-1})^* g(x) = \int_0^1 \overline{k(x,t)} g(t)dt,$$

and its integral kernel is the function

$$\overline{k(x,t)} = (x-t)\theta(t-x) + t(1-2x)\theta(1/2-x). \tag{2.117}$$

Therefore, the operator $(L^{-1})^*$ has the form

$$(L^{-1})^* g(x) = \int_x^1 (x-t)g(t)dt + (1-2x)\theta(1/2-x)\int_0^1 tg(t)dt.$$

By Part 5 of Theorem 2.84 the operators $(L^{-1})^*$ and $(L^*)^{-1}$ coincide. Consider now the function $v(x)$ being the image in $v(x) = (L^*)^{-1} g(x)$, that is,

$$v(x) = \int_x^1 (x-t)g(t)dt + (1-2x)\theta(1/2-x)\int_0^1 tg(t)dt =$$

$$= \begin{cases} \int_x^1 (x-t)g(t)dt + (1-2x)\int_0^1 tg(t)dt, & \text{for } x \le 1/2, \\ \int_x^1 (t-x)g(t)dt, & \text{for } x \ge 1/2. \end{cases} \tag{2.118}$$

Here, when the functions $g(x)$ vary along the whole space $L^2(0,1)$, the functions $v(x)$ correspondingly vary over the whole image of the operator $(L^*)^{-1}$, that is, the whole domain of the operator L^*. Thus, for describing the operator L^* it is necessary to investigate the whole set of the functions $v(x)$, when $g(x)$ varies along the whole space $L^2(0,1)$.

From the analysis of formula (2.118) it follows that although the functions $v(x)$ from the domain of the operator L^* are continuous for $x = 1/2$, generally speaking, they are not continuously differentiable at the point $x = 1/2$. Obviously, the continuity condition of the first derivative at the point $x = 1/2$ is $\int\limits_0^1 tg(t)dt = 0$, that is the orthogonality condition $g(x) \perp x$ or $\langle g(x), x\rangle = 0$. Thus, already at this stage we see *a significant difference with respect to the class of smoothness* between the domains of the operators L and L^* .

Further, it is easy to see that for $x \ne 1/2$ the function $v(x)$ satisfies the differential equation

$$-v''(x) = g(x), \; x \ne 1/2. \tag{2.119}$$

Therefore, the action of the operator L^* is given by the differential expression

$$L^* v(x) = \frac{d^2}{dx^2} v(x),\ x \neq 1/2.$$

Let us now describe the domain of the operator L^*. It is easy to see that $v \in L_2^2(0,1/2) \cap L_2^2(1/2,1)$ holds for all $g \in L^2(0,1)$. Therefore,

$$D(L^*) \subset L_2^2(0,1/2) \cap L_2^2(1/2,1).$$

Let us find boundary conditions satisfied by the functions $v(x)$. To do this, instead of the function $g(x)$ we substitute its value from (2.119) into expression (2.118). Then, by a direct calculation, for $x < 1/2$ we get (we should not forget here that Eq. (2.119) holds only for $x \neq 1/2$):

$$\begin{aligned}
v(x) &= \int_x^1 (t-x)v''(t)dt - (1-2x)\int_0^1 tv''(t)dt \\
&= \int_x^{1/2} (t-x)v''(t)dt - (1-2x)\int_0^{1/2} tv''(t)dt \\
&\quad + \int_{1/2}^1 (t-x)v''(t)dt - (1-2x)\int_{1/2}^1 tv''(t)dt \\
&= \int_x^{1/2} d\left[(t-x)v'(t) - v(t)\right] - (1-2x)\int_0^{1/2} d\left[tv'(t) - v(t)\right] dt \\
&\quad + \int_{1/2}^1 d\left[(t-x)v'(t) - v(t)\right] - (1-2x)\int_{1/2}^1 d\left[tv'(t) - v(t)\right] dt \\
&= (1/2-x)v'(1/2-0) - v(1/2) + v(x) \\
&\quad -(1-2x)\left\{1/2v'(1/2-0)v(1/2) - v(0)\right\} \\
&\quad +(1-x)v'(1) - v(1) - (1/2-x)v'(1/2+0) + v(1/2) \\
&\quad -(1-2x)\left\{v'(1) - v(1) - 1/2v'(1/2+0) + v(1/2)\right\} \\
&= v(x) - v(0) + x\cdot\{2v(0) - 2v(1) + v'(1)\}.
\end{aligned}$$

Note that in the obtained equality the function $v(x)$ cancels from both sides, and the rest is a first-order polynomial in the variable x. Its equality to zero means the equality to zero of its coefficients. Therefore, the obtained equality is equivalent to the following two equalities:

$$\begin{cases} v(0) = 0, \\ v'(1) - 2v(1) + 2v(0) = 0. \end{cases} \tag{2.120}$$

Similarly, for $x > 1/2$ we calculate

$$v(x) = \int_x^1 (t-x)v''(t)dt = \int_x^1 d[(t-x)v'(t) - v(t)] =$$
$$= (1-x)v'(1) - v(1) + v(x) = v(x) + \left\{v'(1) - v(1)\right\} - x\cdot v'(1).$$

As above, in the obtained equality the function $v(x)$ cancels, and the rest is a first-order polynomial, for which the equality to zero means the equality to zero of its coefficients. Therefore, the obtained equality is equivalent to the following two equalities:

$$\begin{cases} v'(1) - v(1) = 0, \\ v'(1) = 0. \end{cases} \tag{2.121}$$

Considering systems (2.120) and (2.121) together, and taking into account the continuity of the function $v(x)$ at the point $x = 1/2$, we get

$$v(0) = 0,\ v(1) = 0,\ v'(1) = 0,\ v(1/2-0) = v(1/2+0). \tag{2.122}$$

These are the boundary conditions for all functions from the domain of the operator L^*. Thus, we get that

$$D(L^*) \subset \left\{ \begin{array}{l} v \in L_2^2(0,1/2) \cap L_2^2(1/2,1): \\ v(0) = 0,\ v(1) = 0,\ v'(1) = 0,\ v(1/2-0) = v(1/2+0) \end{array} \right\}. \tag{2.123}$$

Let us show now that, in fact, we have the equality in (2.123). To do this, we show that the operator given on the domain, coinciding with the right-hand side of (2.123), is the required operator L^*.

Let us denote by L^+ the operator given by the differential expression $L^+ v(x) = -\frac{d^2}{dx^2} v(x)$ on the domain

$$D(L^+) = \left\{ \begin{array}{l} v \in L_2^2(0,1/2) \cap L_2^2(1/2,1): \\ v(0) = 0,\ v(1) = 0,\ v'(1) = 0,\ v(1/2-0) = v(1/2+0) \end{array} \right\}.$$

By the embedding (2.123) it is clear that

$$L^* \subset L^+. \tag{2.124}$$

That is, the operator L^* is a restriction of the operator L^+.

On the other hand, for all $u \in D(L)$ and for all $v \in D(L^+)$ we have

$$\begin{aligned}
\langle Lu, v\rangle &= \int_0^1 Lu(x) \cdot \overline{v(x)} dx = -\int_0^{1/2} u''(x) \cdot \overline{v(x)} dx - \int_{1/2}^1 u''(x) \cdot \overline{v(x)} dx \\
&= -\int_0^{1/2} d\left[u'(x) \cdot \overline{v(x)} - u(x) \cdot \overline{v'(x)}\right] - \int_0^{1/2} u(x) \cdot \overline{v''(x)} dx - \int_{1/2}^1 d[u'(x) \cdot \overline{v(x)} \\
&+ u(x) \cdot \overline{v'(x)}] - \int_{1/2}^1 u(x) \cdot \overline{v''(x)} dx = -\left[u'(1/2) \cdot \overline{v(1/2)} - u(1/2)\overline{v'(1/2-0)}\right] \\
&+ \left[u'(0) \cdot \overline{v(0)} - u(0)\overline{v'(0)}\right] - \left[u'(1) \cdot \overline{v(1)} - u(1)\overline{v'(1)}\right] + \left[u'(1/2) \cdot \overline{v(1/2)}\right. \\
&\left. + u(1/2)\overline{v'(1/2+0)}\right] + \langle u, L^+ v\rangle \stackrel{def}{=} A + B + C + D + \langle u, L^+ v\rangle.
\end{aligned} \tag{2.125}$$

Using the boundary conditions

$$u \in D(L): \ u(0) = 0, \ u(1/2) = 0,$$

$$v \in D(L^+): \ v(0) = 0, \ v(1) = 0, \ v'(1) = 0, \ v(1/2-0) = v(1/2+0),$$

it is easy to show that the terms $A+B+C+D$ in (2.125) are equal to zero.

Therefore, it follows from (2.125) that $\langle Lu, v\rangle = \langle u, L^+ v\rangle$ holds for all $u \in D(L)$ and for all $v \in D(L^+)$. That is, the operator L^+ is formally adjoint to the operator L. Consequently, $L^+ \subset L^*$. From this and (2.124), we obtain the equality $L^* = L^+$.

So, we have shown that the operator adjoint to L is the linear operator given by the differential expression

$$L^* v(x) = \frac{d^2}{dx^2} v(x), \ x \neq 1/2,$$

on the domain

$$D(L^*) = \left\{ \begin{array}{l} v \in L_2^2(0,1/2) \cap L_2^2(1/2,1): \\ v(0) = 0, \ v(1) = 0, \ v'(1) = 0, \ v(1/2-0) = v(1/2+0) \end{array} \right\}.$$

Comparing now the domains of the adjoint operators, it is easy to see that they significantly differ both in the smoothness of the involved functions and in the quantity of conditions which they satisfy. We must note that the problem adjoint to the problem with "interior" conditions is the problem for which the domain contains non-smooth functions (the function itself and its first derivative admit an interval break at an interior point). □

In the following example we will construct the adjoint problem to one more non-classical problem.

Example 2.94 Consider the operator corresponding to the problem with so-called integral conditions for the second-order differential operator

$$Lu(x) = \frac{d^2}{dx^2} u(x),$$

$$D(L) = \left\{ u \in L_2^2(0,1): \ u(0) = 0, \ u(1) = \int_0^1 p(x) u(x) dx \right\},$$

where $p \in L^2(0,1)$ is a given real-valued function.

The boundary conditions of such kind are called ***integral conditions***, since in one of them not only the values of the function at a point, but also the values of the integral of the unknown function on the whole interval $(0,1)$ appear.

Our goal is to define the operator adjoint to the operator L. First of all, we immediately note that the adjoint operator exists, since the domain is dense in the space $L^2(0,1)$ (this is a consequence of $p \in L^2(0,1)$).

Second, we show that the operator is well-posed. A general solution of the differential equation

$$-u''(x) = f(x),$$

has the form

$$u(x) = -\int_0^x (x-t)f(t)dt + C_1x + C_2, \tag{2.126}$$

where C_1, C_2 are some arbitrary (so far) constants.

Combining this formula with the boundary conditions

$$u(0) = 0,\ u(1) = \int_0^1 p(x)u(x)dx,$$

we obtain the system of equations for the unknown constants C_1, C_2:

$$\begin{cases} C_2 = 0, \\ C_1 \left[1 - \int\limits_0^1 sp(s)ds\right] + C_2 \left[1 - \int\limits_0^1 p(s)ds\right] = P, \end{cases}$$

where $P = \int\limits_0^1 \left[(1-t) + \int\limits_t^1 (t-s)p(s)ds\right] f(t)dt$. This system has a unique solution provided that

$$\int_0^1 sp(s)ds \neq 1. \tag{2.127}$$

Thus, everywhere in what follows we assume that condition (2.127) holds.

Then, the system for the unknown constants C_1, C_2 has the unique solution

$$C_1 = \frac{P}{1 - \int_0^1 sp(s)ds},\ C_2 = 0.$$

Now substituting this into formula (2.126), we get the explicit form of the solution of our problem:

$$u(x) = -\int_0^x (x-t)f(t)dt + \int_0^1 \frac{x\left[(1-t) + \int_t^1 (t-s)p(s)ds\right]}{1 - \int_0^1 sp(s)ds} f(t)dt.$$

Consequently, the inverse operator has the form

$$L^{-1}f(x) = \int_0^1 k(x,t)f(t)dt, \tag{2.128}$$

where the integral kernel of the integral operator is given by

$$k(x,t) = -(x-t)\theta(x-t) + \frac{x\left[(1-t) + \int_t^1 (t-s)p(s)ds\right]}{1 - \int_0^1 sp(s)ds}. \tag{2.129}$$

Obviously, the kernel (2.129) is a continuous function. Therefore, the operator (2.128) is defined on the whole space $L^2(0,1)$ and is bounded, that is, the operator L is *well-posed*.

Let us find the operator adjoint to the operator L^{-1}. The operator $(L^{-1})^*$ will be also the integral operator

$$(L^{-1})^* g(x) = \int_0^1 \overline{k(x,t)} g(t) dt,$$

and its kernel is the function

$$\overline{k(x,t)} = (x-t)\theta(t-x) + \frac{t\left[(1-x) + \int_x^1 (x-s)p(s)ds\right]}{1 - \int_0^1 sp(s)ds}. \tag{2.130}$$

Therefore, the operator $(L^{-1})^*$ has the form

$$(L^{-1})^* g(x) = \int_x^1 (x-t)g(t)dt + \frac{\left[(1-x) + \int_x^1 (x-s)p(s)ds\right]}{1 - \int_0^1 sp(s)ds} \int_0^1 tg(t)dt.$$

By Part 5 of Theorem 2.84 the operators $(L^{-1})^*$ and $(L^*)^{-1}$ coincide. Consider now the function $v(x)$ which is the image in $v(x) = (L^*)^{-1} g(x)$, that is,

$$v(x) = \int_x^1 (x-t)g(t)dt + \frac{\left[(1-x) + \int_x^1 (x-s)p(s)ds\right]}{1 - \int_0^1 sp(s)ds} \int_0^1 tg(t)dt. \tag{2.131}$$

Here, when the functions $g(x)$ vary along the whole space $L^2(0,1)$, the functions $v(x)$ correspondingly vary along the whole image of the operator $(L^*)^{-1}$, that is, the whole domain of the operator L^*. Thus, in order to describe the operator L^* it is necessary to investigate the whole set of the functions $v(x)$, when $g(x)$ varies along the whole space $L^2(0,1)$.

First we find an equation to which the function (2.131) satisfies. To do this, we calculate the first and second derivatives:

$$v'(x) = \int_x^1 g(t)dt - \frac{\left[1 - \int_x^1 p(s)ds\right]}{1 - \int_0^1 sp(s)ds} \int_0^1 tg(t)dt, \tag{2.132}$$

$$v''(x) = -g(x) + Ap(x), \text{ where } A = const = -\frac{\int_0^1 tg(t)dt}{1 - \int_0^1 sp(s)ds}. \tag{2.133}$$

From Eq. (2.132) it is easy to see that $A = v'(1)$. Therefore, Eq. (2.133) can be written in the form

$$-v''(x) + p(x)v'(1) = g(x). \tag{2.134}$$

Unlike in the previous examples, Eq. (2.134) is not an entirely differential equation, since it contains the term $v'(1)$. Such equations, which contain (except the function itself and its derivatives) also traces of the function and/or its derivatives at some point, are called ***loaded differential equations***.

Thus, we have shown that the action of the adjoint operator L^* is given by the loaded differential expression

$$L^* v(x) = -\frac{d^2}{dx^2} v(x) + p(x) v'(1). \tag{2.135}$$

Let us describe now the domain of L^*. It is easy to see that $v \in L_2^2(0,1)$ holds for all $g \in L^2(0,1)$. Therefore, $D(L^*) \subset L_2^2(0,1)$.

Let us find boundary conditions for the functions $v \in D(L^*)$. To do this, instead of the function $g(x)$ we substitute its value from (2.134) into the expression (2.131). Then, by a direct calculation we get

$$\begin{aligned}
v(x) &= \int_x^1 (t-x)[v''(t) - p(t)v'(1)]dt \\
&+ \frac{\left[(x-1) + \int_x^1 (s-x)p(s)ds\right]}{1 - \int_0^1 sp(s)ds} \int_0^1 t[v''(t) - p(t)v'(1)]dt \\
&= \int_x^1 d[(t-x)v'(t) - v(t)] - v'(1) \int_x^1 (t-x)p(t)dt \\
&+ \frac{\left[(x-1) + \int_x^1 (s-x)p(s)ds\right]}{1 - \int_0^1 sp(s)ds} \left\{ \int_0^1 d[tv'(t) - v(t)]dt - v'(1) \int_0^1 tp(t)dt \right\} \\
&= (1-x)v'(1) - v(1) + v(x) - v'(1) \int_x^1 (t-x)p(t)dt \\
&+ \frac{\left[(x-1) + \int_x^1 (t-x)p(t)dt\right]}{1 - \int_0^1 tp(t)dt} \left\{ v'(1) - v(1) + v(0) - v'(1) \int_0^1 tp(t)dt \right\} \\
&= v(x) - v(1) + \frac{\left[(x-1) + \int_x^1 (t-x)p(t)dt\right]}{1 - \int_0^1 tp(t)dt} \{v(0) - v(1)\}.
\end{aligned}$$

Note that in the obtained equality the function $v(x)$ cancels out. Then, assuming here first $x = 1$, and then $x = 0$, we get

$$\begin{cases} v(0) = 0, \\ v(1) = 0. \end{cases} \tag{2.136}$$

These are indeed the required boundary conditions for functions in the domain of the operator L^*. Thus, we get that

$$D(L^*) \subset \{v \in L_2^2(0,1) : \ v(0) = 0, \ v(1) = 0\}. \tag{2.137}$$

Let us show now that, in fact, we have the equality in (2.137). To do this, we show that the operator given on the domain, given by the right-hand side of (2.137), is the required operator L^*.

We denote by L^+ the operator given by the expression

$$L^+ v(x) = -\frac{d^2}{dx^2} v(x) + p(x) v'(1),$$

on the domain

$$D(L^+) = \{v \in L_2^2(0,1) : v(0) = 0, v(1) = 0\}.$$

From the inclusion (2.137), it is clear that

$$L^* \subset L^+. \tag{2.138}$$

That is, the operator L^* is a restriction of the operator L^+.

On the other hand, for all $u \in D(L)$ and for all $v \in D(L^+)$ we have

$$\begin{aligned}
\langle Lu, v\rangle - \langle u, L^+ v\rangle &= \int_0^1 Lu(x) \cdot \overline{v(x)} dx - \int_0^1 u(x) \cdot \overline{L^+ v(x)} dx \\
&= \int_0^1 \left[-u''(x) \cdot \overline{v(x)} + u(x) \cdot \overline{v''(x)}\right] dx - \int_0^1 u(x) \cdot \overline{p(x) v'(1)} dx \\
&= \int_0^1 d\left[-u'(x) \cdot \overline{v(x)} + u(x) \cdot \overline{v'(x)}\right] - \overline{v'(1)} \int_0^1 p(x) u(x) dx \\
&= \left[-u'(1) \cdot \overline{v(1)} + u(1) \cdot \overline{v'(1)}\right] - \left[-u'(0) \cdot \overline{v(0)} + u(0) \cdot \overline{v'(0)}\right] \\
&- \overline{v'(1)} \int_0^1 p(x) u(x) dx.
\end{aligned} \tag{2.139}$$

Taking into account the boundary conditions $u \in D(L)$ and $v \in D(L^+)$:

$$u \in D(L) : u(0) = 0,\ u(1) = \int_0^1 p(x) u(x) dx,$$

$$v \in D(L^+) : v(0) = 0,\ v(1) = 0,$$

from (2.137) it is easy to see that $\langle Lu, v\rangle = \langle u, L^+ v\rangle$ for all $u \in D(L)$ and for all $v \in D(L^+)$.

That is, the operator L^+ is formally adjoint to the operator L. Consequently, $L^* \supset L^+$. From this and (2.138), we obtain the equality $L^* = L^+$.

So, we have shown that the operator adjoint to L is the linear operator given by the loaded differential expression (2.135) on the domain

$$D(L^*) = \left\{v \in L_2^2(0,1) : v(0) = 0,\ v(1) = 0\right\}.$$

This example shows that in spite of the fact that the action of the initial operator L is given by a seemingly self-adjoint expression, the action of the adjoint operator is given by a different expression. The action is not even completely differential. □

The two considered examples illustrate the complexity of the question of finding the actual adjoint operator. This is not so apparent if one only considers the classical boundary value problems. However, when studying non-classical problems (as the last two examples show) one should be rather careful in dealing with the justification and the properties of the adjoint problems.

Chapter 3

Elements of the spectral theory of differential operators

The aim of this chapter is to make a quick introduction to the basics of the spectral theory of differential operators. We follow the informal style of the previous chapters, aiming at explaining the main ideas rather than presenting the detailed proofs that can be found in many excellent books on the subject.

Starting with the basic notions of spectra, we first present the basics of the general theory in the case of bounded and unbounded self-adjoint operators. However, the main further interest for us lies in the non-self-adjoint cases. Here, the notions or associated eigenfunctions and root spaces naturally appear, also leading to different notions of bases in Hilbert spaces, such as Riesz bases, unconditional bases, etc. Thus, we spend some time discussing general biorthogonal Riesz bases and the associated Fourier analysis, and the notion of a convolution in Hilbert spaces. We complement the abstract material by many examples showing a variety of interesting phenomena that may appear in the analysis of non-self-adjoint problems.

3.1 Spectrum of finite-dimensional operators

The concept of the spectrum of a linear operator A acting in a Banach space X is a generalisation of the concept of eigenvalues for ordinary matrices. As was shown in Example 2.5, all linear operators in the finite-dimensional spaces are bounded and can be represented in the form of a matrix. Therefore, *the spectral theory of the finite-dimensional operators is the spectral theory of matrices*.

Consider the operator given by the square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \tag{3.1}$$

acting on the space of n-dimensional vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

with complex elements $x_j \in \mathbb{C}$. The elements of the matrix (a_{ij}) are given complex numbers.

A nonzero vector $x \neq 0$ is called an ***eigenvector of the matrix*** A, if there exists a (complex) number λ such that

$$Ax = \lambda x. \tag{3.2}$$

Such λ is called an ***eigenvalue of the matrix*** A. It is evident that an eigenvector of the matrix is not uniquely determined: if x is an eigenvector of the matrix A corresponding to the eigenvalue λ, then for any constant a the vector ax is also an eigenvector of the matrix A corresponding to the same eigenvalue λ. Often one fixes the number a such that the obtained eigenvector has the norm equal to 1.

We rewrite Eq. (3.2) in the form $(A - \lambda I)x = 0$, where I is the ***identity matrix*** (sometimes also called the ***unit matrix***), that is the $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere. Then we have the system of linear equations

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}. \tag{3.3}$$

It is well-known that this system of linear equations (3.3) has a unique solution if and only if $\det(A - \lambda I) \neq 0$. If this condition is satisfied, the unique solution of system (3.3) will be the zero vector: $x = 0$.

Thus, for the nonzero solution of system (3.3) to exist, it is necessary and sufficient that the number λ satisfies the equation

$$\triangle(\lambda) := \det(A - \lambda I) = 0. \tag{3.4}$$

Eq. (3.4) is called *the characteristic equation for the matrix* A.

It is easy to see that the left-hand side of Eq. (3.4) is a polynomial of order n in the variable λ. It is called the ***characteristic polynomial of the matrix*** A. Therefore, the eigenvalues of the matrix A are the roots of the polynomial $\triangle(\lambda)$.

Moreover, the *Cayley-Hamilton theorem* asserts that every matrix A satisfies its characteristic equation: if $\triangle(\lambda)$ is the characteristic polynomial of that matrix A, then $\triangle(A) = 0$.

A root of the polynomial $\triangle(\lambda)$ is a solution of the equation $\triangle(\lambda) = 0$: that is, a complex number λ_j such that $\triangle(\lambda_j) = 0$. The fundamental theorem of algebra states that every non-constant polynomial of order n in one variable with complex coefficients has exactly n complex roots λ_j, taking into account the multiplicity. Therefore, *the matrix A has exactly n complex eigenvalues λ_j, taking into account the multiplicity.*

One says that the root λ_j has multiplicity m, if the polynomial $\triangle(\lambda)$ under consideration is divisible by $(\lambda-\lambda_j)^m$ and is not divisible by $(\lambda-\lambda_j)^{m+1}$. For example, the polynomial $\lambda^2-2\lambda+1$ has a unique root equal to 1, of multiplicity 2. If the multiplicity of the root is greater than 1, then one says that it is "a multiple root".

Analogous to this, one says that *the eigenvalue λ_j has multiplicity m* if λ_j is a root of the polynomial $\triangle(\lambda)$ of multiplicity m. If $m=1$, then one says that the eigenvalue is ***simple***; and for $m>1$ one says that the eigenvalue is ***multiple***. The number m is called ***the multiplicity of the eigenvalue*** λ_j.

Example 3.1 Let an operator A be given by the matrix

$$A=\begin{pmatrix}2 & 1 & 0\\ 0 & 1 & \alpha\\ 0 & 0 & 1\end{pmatrix},$$

where α is some number.

It is easy to calculate that the characteristic polynomial is

$$\triangle(\lambda)=(\lambda-2)(\lambda-1)^2=0.$$

Therefore the eigenvalues of the matrix A will be: a simple eigenvalue $\lambda_1=2$ and a multiple eigenvalue $\lambda_2=1$ of multiplicity 2.

To find an eigenvector corresponding to $\lambda_1=2$, we look for a solution of the system $(A-\lambda_1 I)x=0$. We have

$$\begin{pmatrix}0 & 1 & 0\\ 0 & -1 & \alpha\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\quad\text{or}\quad\left\{\begin{aligned}x_2&=0,\\ -x_2+\alpha x_3&=0,\\ x_3&=0.\end{aligned}\right.$$

All solutions of this system have the form $x=(x_1,\ 0,\ 0)$, where x_1 is an arbitrary number. Let us choose $x_1=1$. Then $x^{(1)}=(1,\ 0,\ 0)$ will be an eigenvector corresponding to the eigenvalue $\lambda_1=2$.

To construct an eigenvector corresponding to the second eigenvalue $\lambda_2=1$, we have

$$\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & \alpha\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\quad\text{or}\quad\left\{\begin{aligned}x_1+x_2&=0,\\ \alpha x_3&=0,\\ 0&=0.\end{aligned}\right.\tag{3.5}$$

First consider the case $\alpha=0$. Then system (3.5) has solutions of the form $x=(x_1,-x_1,x_3)$, where x_1 and x_3 are arbitrary numbers. We see that the solutions of the system (3.5) form a two-parametric family. Let us choose x_1 and x_3 so as to obtain two linearly independent vectors. Then $x^{(2)}=(1,-1,\ 0)$ and $x^{(3)}=(0,\ 0,\ 1)$ will be eigenvectors corresponding to the eigenvalue $\lambda_2=1$.

Thus, in the case $\alpha=0$, the matrix A has two eigenvalues: a simple eigenvalue $\lambda_1=2$ and a multiple eigenvalue $\lambda_2=1$ of multiplicity 2. The eigenvector $x^{(1)}=(1,\ 0,\ 0)$ corresponds to the eigenvalue $\lambda_1=2$. And two eigenvectors $x^{(2)}=(1,-1,\ 0)$ and $x^{(3)}=(0,\ 0,\ 1)$ correspond to the multiple eigenvalue $\lambda_2=1$.

Now let $\alpha \neq 0$. Then from the second equation (3.5) we have $x_3 = 0$. Therefore all the solutions of the system (3.5) have the form $x = (x_1, -x_1, 0)$, where x_1 is an arbitrary number. Let us choose $x_1 = 1$. Then $x^{(2)} = (1, -1,\ 0)$ will be the eigenvector corresponding to the eigenvalue $\lambda_2 = 1$.

Note that although the multiplicity of the eigenvalue $\lambda_2 = 1$ is equal to two, only one eigenvector corresponds to this eigenvalue. The second (linearly independent) eigenvector does not exist.

In such cases the system of eigenvectors is complemented by the so-called *associated vectors* (we will give the exact definition below). Let us find a solution of the equation

$$(A - \lambda_2 I)x = x^{(2)}. \tag{3.6}$$

We obtain the system

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{or} \quad \left\{ \begin{aligned} x_1 + x_2 &= 1, \\ \alpha x_3 &= -1, \\ 0 &= 0. \end{aligned} \right. \tag{3.7}$$

From the second equation, in view of $\alpha \neq 0$ we have $x_3 = -1/\alpha$. Therefore all the solutions of system (3.7) have the form $x = (x_1, 1 - x_1, -1/\alpha)$, where x_1 is an arbitrary number. It is easy to see that these solutions can be represented in the form

$$x = (0,\ 1, -1/\alpha) + x_1 \cdot x^{(2)}.$$

This vector is called an ***associated vector***. It is not unique, up to a summand consisting of an eigenvector multiplied by a constant.

Thus, in the case $\alpha \neq 0$, the matrix A has two eigenvalues: a simple eigenvalue $\lambda_1 = 2$ and a multiple eigenvalue $\lambda_2 = 1$ of multiplicity 2. The eigenvector $x^{(1)} = (1,\ 0,\ 0)$ corresponds to the eigenvalue $\lambda_1 = 2$. And one eigenvector $x^{(2)} = (1, -1,\ 0)$ and one associated vector $x^{(3)} = (0,\ 1, -1/\alpha)$ correspond to the multiple eigenvalue $\lambda_2 = 1$. □

In this example, the right-hand side of Eq. (3.6) is the eigenvector $x^{(2)}$. Therefore, acting on (3.6) by the operator $(A - \lambda_2 I)$, we obtain that the associated vector $x^{(3)}$ satisfies the equation

$$(A - \lambda_2 I)^2 x^{(3)} = 0. \tag{3.8}$$

Thus, we come to a general definition of eigen- and associated vectors of an operator A, as solutions of an equation of the form (3.8).

Thus, a nonzero vector $x^{(0)}$ satisfying the equation

$$Ax^{(0)} = \lambda x^{(0)}, \quad \text{or} \quad (A - \lambda I)x^{(0)} = 0, \tag{3.9}$$

is called an ***eigenvector of the operator*** A. The number λ, for which there exists a nonzero solution of Eq. (3.9), is called an ***eigenvalue of the operator*** A.

It is evident that an eigenvector of the operator is not unique: if x is an eigenvector of the operator A corresponding to the eigenvalue λ, then for any constant a the vector ax is also an eigenvector of the operator A corresponding to the same eigenvalue λ.

Usually, such number a is chosen in the most convenient way. For example, it is fixed so that the obtained eigenvector has the norm equal to 1.

The vector x is called an ***associated vector of the operator*** A corresponding to the eigenvalue λ if for some integer number $m > 0$ the following relations hold

$$(A - \lambda I)^m x \neq 0, \quad (A - \lambda I)^{m+1} x = 0. \tag{3.10}$$

The number m is called the ***order of the associated vector*** x. Thus, in Example 3.1 the vector $x^{(3)}$ is the associated vector of the first order. Sometimes an eigenvector is also called an associated vector of zero order.

It is evident that the associated vectors are defined not uniquely. Indeed, if $x^{(0)}$ is an eigenvector of the operator A corresponding to the eigenvalue λ and $x^{(1)}$ is an associated vector of the operator A, then the vector $x^{(1)} + Cx^{(0)}$ is also an associated vector of the operator A for any choice of the constant C.

The eigen- and associated vectors are called ***root vectors of the operator*** A. The linear space spanned by all eigenvectors corresponding to a given eigenvalue is called an ***eigenspace*** of the linear operator A. The linear space spanned by all eigen- and associated vectors of the operator A corresponding to the same eigenvalue is called a ***root space***.

In a root subspace of the operator A, the eigen- and associated vectors corresponding to one eigenvalue can be organised in a *chain* as follows. Let $x^{(0)}$ be an eigenvector of the operator A corresponding to the eigenvalue λ. By the associated vector of the first order we call a vector $x^{(1)}$ satisfying the equation

$$(A - \lambda I) x^{(1)} = x^{(0)}. \tag{3.11}$$

Analogously to (3.11), other associated vectors of higher order can be found:

$$\begin{gathered} (A - \lambda I) x^{(2)} = x^{(1)}, \\ \cdots\cdots\cdots \\ (A - \lambda I) x^{(m)} = x^{(m-1)}. \end{gathered} \tag{3.12}$$

The vectors $x^{(0)}, x^{(1)}, \cdots, x^{(m)}$ are called a ***chain of the eigen- and associated vectors*** of the operator A corresponding to the eigenvalue λ.

The concept of a Jordan canonical form matrix is closely related to the concept of the eigen- and associated vectors of the matrix.

Consider the $k \times k$ square matrix of the form

$$J_k(\lambda_0) = \begin{pmatrix} \lambda_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \lambda_0 & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_0 \end{pmatrix}.$$

On the main diagonal here we have the same number λ_0, above the main diagonal there are ones, and the remaining elements are all zeros.

The characteristic polynomial of this matrix is $(\lambda-\lambda_0)^k=0$. Therefore, λ_0 is an eigenvalue of the matrix $J_k(\lambda_0)$ of multiplicity k. To construct eigenvectors of the matrix $J_k(\lambda_0)$ it is necessary to solve a homogeneous system of the linear equations

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_{k-1} \\ x_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}. \tag{3.13}$$

It is easy to see that the rank of the matrix in (3.13) is equal to $k-1$. Therefore the system (3.13) has only a one-parametric family of solutions. Consequently, the matrix $J_k(\lambda_0)$ has only one eigenvector $x^{(0)}=(1,0,\dots,0,0)$, and the multiplicity of the eigenvalue λ_0 is equal to k.

The matrix $J_k(\lambda_0)$ has a chain of associated vectors of the form

$$\begin{array}{rccccc} x^{(1)}=(0 & 1 & 0 & \dots & 0 & 0), \\ x^{(2)}=(0 & 0 & 1 & \dots & 0 & 0), \\ \dots & \dots & \dots & \dots & \dots & \\ x^{(k-1)}=(0 & 0 & 0 & \dots & 0 & 1). \end{array}$$

The matrix $J_k(\lambda_0)$ is called a ***Jordan block*** of order k corresponding to the eigenvalue λ_0. As we have shown this matrix has one eigenvector and $k-1$ associated vectors.

From several Jordan blocks corresponding to one eigenvalue λ_j one can construct a matrix

$$A(\lambda_j)=\begin{pmatrix} \boxed{J_1(\lambda_j)} & & & & \\ & \boxed{J_2(\lambda_j)} & & & \\ & & \ddots & & \\ & & & \boxed{J_{k-1}(\lambda_j)} & \\ & & & & \boxed{J_k(\lambda_j)} \end{pmatrix},$$

which is also called the ***Jordan block.***

Theorem 3.2 *Let a linear operator A act on a finite-dimensional space H of dimension n. Let its characteristic polynomial have the form*

$$\triangle(\lambda)=\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j},$$

where $\lambda_j\neq\lambda_k$ for $j\neq k$ and $m_1+m_2+\ldots+m_N=n$. Then there exists a basis in the space H consisting of eigen- and associated vectors of the operator A, in which the *matrix of the operator has a block-diagonal form*

$$J = \begin{pmatrix} \boxed{A(\lambda_1)} & & & & \\ & \boxed{A(\lambda_2)} & & & \\ & & \ddots & & \\ & & & \boxed{A(\lambda_{N-1})} & \\ & & & & \boxed{A(\lambda_N)} \end{pmatrix}, \tag{3.14}$$

where $A(\lambda_j)$ is a Jordan block corresponding to the eigenvalue λ_j .

The matrix (3.14) is called the ***Jordan normal form of a matrix***. The basis indicated in Theorem 3.2 is called a ***Jordan basis***.

Corollary 3.3 *In order that a matrix could be represented in a diagonal form it is necessary and sufficient that all its root subspaces consist only of eigenvectors.*

In general, the construction of the Jordan normal form and the Jordan basis is a quite complicated procedure. But the advantage of Theorem 3.2 is the fact that besides the opportunity of representing a finite-dimensional operator in the Jordan form, it indicates the method for obtaining such a representation. To do this it is sufficient to construct eigen- and associated vectors of the matrix.

Besides the results of Theorem 3.2, Jordan showed that for an arbitrary square matrix A over the field of complex numbers there always exists a square non-degenerate (i.e. with the determinant different from zero) matrix P such that

$$A = PJP^{-1},$$

where J is the Jordan normal form matrix for A. That is, *any square matrix A can be reduced to its Jordan normal form by a non-degenerate transformation P by the formula*

$$J = P^{-1}AP.$$

The matrix P consists of the columns being the vectors from the chain of the eigen- and associated vectors of the matrix A.

3.2 The resolvent and spectrum of an operator

The spectral theory of operators in infinite-dimensional spaces is much richer: while the spectrum of finite-dimensional operators consists only of eigenvalues, in the infinite-dimensional spaces we have other variants of the spectrum.

Consider a linear operator $A: X \to X$ with the domain $D(A)$ in a Banach space X. For any fixed value of the complex parameter $\lambda \in \mathbb{C}$ four cases are possible:

- The operator $(A-\lambda I)^{-1}$ *exists, is defined on the whole X and is bounded*. In this case the value λ is called ***regular***, and the operator

$$R_\lambda := (A-\lambda I)^{-1}$$

is called a ***resolvent of the operator*** A. The set of all the regular values λ is called the ***resolvent set*** $\rho(A)$, and the complement

$$\sigma(A) := \mathbb{C}\backslash\rho(A)$$

of the resolvent set is called the ***spectrum*** of the operator A.

- The operator $(A-\lambda I)^{-1}$ *does not exist*, that is, there is no unique inverse solvability. It means that there exists a nontrivial solution $x \neq 0$ of the equation

$$(A-\lambda I)x = 0. \tag{3.15}$$

In this case λ is called an ***eigenvalue*** of the operator A, and the nontrivial solution x of Eq. (3.15) is called an ***eigenvector*** of the operator A. The set of all the eigenvalues forms the so-called ***point spectrum*** $\sigma_p(A)$ of the operator A.

- The operator $(A-\lambda I)^{-1}$ *exists, is defined on a set everywhere dense in X but is not bounded*. The set of such values λ forms the so-called ***continuous spectrum*** $\sigma_c(A)$ of the operator A.

- The operator $(A-\lambda I)^{-1}$ *exists but is defined on a set which is not dense in* X. That is, the image of the operator $A-\lambda I$ is not dense in X:

$$\overline{R(A-\lambda I)} \neq X.$$

The set of such values λ forms the so-called ***residual spectrum*** $\sigma_r(A)$ of the operator A.

According to this classification, only the point spectrum and the resolvent set are present in the finite-dimensional case. This follows from the fact that if an inverse operator $(A-\lambda I)^{-1}$ exists, then it is given by a matrix which is immediately defined on the whole space X. Therefore *in the finite-dimensional space the spectrum of an operator is reduced to the point spectrum*, that is, to the eigenvalues, that we have discussed in detail in Section 3.1.

Example 3.4 In the space $C[a,b]$, consider the operator defined on the whole space by the formula

$$Tu(x) = \alpha \cdot u(x),\ \forall u \in C[a,b],$$

where α is a given number. A more general case of such operators was considered in Example 2.6. From the results of Example 2.6 it follows that the operator T is defined on the whole space $C[a,b]$ and is bounded. As was shown in Example 2.27, the operator T is invertible if and only if $\alpha \neq 0$.

Let us consider the spectral properties of the operator T. For our case, Eq. (3.15) for finding eigenvalues and eigenvectors has the form

$$(T-\lambda I)u(x) \equiv (\alpha-\lambda)u(x) = 0.$$

It is easy to see that the value $\lambda = \alpha$ is an eigenvalue of the operator T. The corresponding eigenfunctions are all functions from the space $C[a,b]$.

For the other values $\lambda \neq \alpha$ we consider the resolvent. It can be found from a solution of the equation

$$(T-\lambda I)u(x) \equiv (\alpha-\lambda)u(x) = f(x),$$

and has the form

$$(T-\lambda I)^{-1}f(x) \equiv \frac{1}{\alpha-\lambda}f(x).$$

Consequently, in this case the operator $(T-\lambda I)^{-1}$ exists, is defined on the whole space $C[a,b]$ and is bounded. Therefore, all the values $\lambda \neq \alpha$ belong to the resolvent set. □

Example 3.5 In the space $C[a,b]$, consider the operator defined on the whole space by the formula

$$Tu(x) = x\cdot u(x),\ \forall u \in C[a,b].$$

From the results of Example 2.6 it follows that the operator T is defined on the whole space $C[a,b]$ and is bounded. From the results of Example 2.27 it follows that the operator T is invertible if and only if $0 \notin [a,b]$.

Let us consider the spectral properties of this operator T. Eq. (3.15) for finding eigenvalues and eigenvectors is

$$(T-\lambda I)u(x) \equiv (x-\lambda)u(x) = 0.$$

It is easy to see that this implies $u(x) = 0$. Therefore, any value $\lambda \in \mathbb{C}$ is not an eigenvalue of the operator T, that is, *the operator has no point spectrum.*

The resolvent is constructed as a solution of the equation

$$(T-\lambda I)u(x) \equiv (x-\lambda)u(x) = f(x).$$

So, we have

$$u(x) = \frac{1}{x-\lambda}f(x).$$

Therefore, if $\lambda \notin [a,b]$, then the resolvent $(T-\lambda I)^{-1}$ exists, is defined on the whole space $C[a,b]$ and is bounded. Hence, the values $\lambda \notin [a,b]$ belong to the *resolvent set* $\rho(T)$ of the operator T.

For the values $\lambda_0 \in [a,b]$ the resolvent $(T-\lambda_0 I)^{-1}$ exists, but is defined only for those functions for which

$$f(\lambda_0) = 0. \tag{3.16}$$

Therefore the values $\lambda \in [a,b]$ belong to the spectrum of the operator.

In Example 2.8 we have shown that the linear functional given by the formula $Pf(x) = f(x_0)$ is bounded in the space $C[a,b]$. Therefore, by Theorem 2.19 the linear space of functions satisfying condition (3.16) is not dense in $C[a,b]$. It means that all the values $\lambda \in [a,b]$ belong to the *residual spectrum* $\sigma_r(T)$. □

Example 3.6 Consider the operator from Example 3.5 but now acting in the Hilbert space $L^2(a,b)$:

$$Tu(x) = x \cdot u(x),\ \forall u \in L^2(a,b).$$

As in Example 3.5 it is easy to show that for $\lambda \notin [a,b]$ a resolvent $(T-\lambda I)^{-1}$ exists, is defined on the whole space $L^2(a,b)$ by the formula

$$(T-\lambda I)^{-1} f(x) = \frac{1}{x-\lambda} f(x),$$

and is bounded. Consequently, the values $\lambda \notin [a,b]$ belong to the *resolvent set* $\rho(T)$ of the operator T.

For the values $\lambda \in [a,b]$ the resolvent $(T-\lambda I)^{-1}$ exists, but is defined not on all functions. For example, it is defined on all the functions $f \in C[a,b]$, for which (3.16) holds.

As shown in Example 2.8 the linear functional given by the formula $Pf(x) = f(x_0)$ is unbounded in the space $L^2(a,b)$. Therefore, by Theorem 2.19 the linear space of functions $f \in C[a,b]$ satisfying condition (3.16) is dense in $L^2(a,b)$. It means that all the values $\lambda \in [a,b]$ belong to the *continuous spectrum* $\sigma_c(T)$. □

The considered Examples 3.5 and 3.6 demonstrate that operators given by the same formula can have different spectral properties depending on spaces in which they are considered.

Example 3.7 In the space ℓ^2 of infinite square summable sequences, consider the linear operator A given by the formula

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

As is easily seen, the number $\lambda = 0$ is not an eigenvalue of the operator A.

The set of values $R(A)$ is not everywhere dense in ℓ^2: $R(A)$ is orthogonal to the nonzero element $(1,0,0,\ldots)$. Therefore, the number $\lambda = 0$ is a point of the *residual spectrum* $\sigma_r(A)$ of the operator A. □

Example 3.8 In the space of continuous functions $C[a,b]$, consider the operator acting by the formula

$$L_1 u(x) = \frac{d}{dx} u(x),\ a < x < b,$$

defined on the domain

$$D(L_1) = C^1[a,b] \subset C[a,b].$$

We have considered this operator in Examples 2.9 and 2.28. Let us show that any value λ belongs to the point spectrum of the operator L_1.

Indeed, from the equation of finding the eigenfunctions

$$(L_1 - \lambda I)\, u(x) \equiv \frac{d}{dx} u(x) - \lambda u(x) = 0, \; a < x < b,$$

we have $u(x) = e^{\lambda x}$ for all values $\lambda \in \mathbb{C}$. Therefore, a resolvent does not exist, and any value λ belongs to the point spectrum of the operator L_1. □

Example 3.9 In the space of continuous functions $C[a,b]$, consider the operator acting by the formula

$$L_2 u(x) = \frac{d}{dx} u(x), \; a < x < b,$$

given on the domain

$$D(L_2) = \left\{ u \in C^1[a,b] : u(a) = 0 \right\}.$$

This operator in contrast to the operator considered in Example 2.28 (or in the previous example) has a “smaller” domain.

We now consider the problem of eigenvalues for the operator L_2. Eigenvectors of the operator L_2 are all functions $u \in D(L_2)$, for which $L_2 u(x) - \lambda u(x) = 0$, that is, all continuously differentiable solutions of the differential equation

$$u'(x) - \lambda u(x) = 0, \;\; a < x < b,$$

for which $u(a) = 0$. All solutions of this equation have the form $u(x) = Ce^{\lambda x}$, $a \leq x \leq b$, where C is an arbitrary number. Since $u(a) = 0$, then, consequently, $u(x) \equiv 0$. And according to the corollary from Theorem 2.22 it means that the inverse operator $(L_2 - \lambda I)^{-1}$ to the operator L_2 exists. That is, for all the values $\lambda \in \mathbb{C}$ a resolvent exists.

This resolvent can be constructed in the explicit form

$$(L_2 - \lambda I)^{-1} f(x) = \int_a^x e^{\lambda (x-t)} f(t)dt, \; a \leq x \leq b,$$

for all $f \in C[a,b]$. Earlier, in Example 2.11, we have shown that such operator is defined on the whole space $L^2(a,b)$ and is bounded. Consequently, each value $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(L_2)$ of the operator L_2. □

Example 3.10 In the space of square integrable functions, consider the operator $L : L^2(a,b) \to L^2(a,b)$ given by the differential expression

$$L_3 u(x) = \frac{d}{dx} u(x), \; a < x < b,$$

on the domain

$$D(L_3) = \left\{ u \in L_1^2(a,b) : u(a) = u(b) = 0 \right\}.$$

We have considered the similar operator in Example 2.36. There, we have shown that the operator L_3 is not well-posed, since it is not everywhere solvable, though the inverse operator exists and is bounded on $R(L_3)$.

Let us consider the spectral properties of this operator L_3. As in Example 3.9, it is easy to show that any solution $u \in D(L_3)$ of the equation $L_3u(x) - \lambda u(x) = 0$ will be equal to zero. That is, any value $\lambda \in \mathbb{C}$ is not an eigenvalue of the operator L_3. Consequently, the operator $(L_3 - \lambda I)^{-1}$ exists. It can be constructed as a solution of the differential problem

$$u'(x) - \lambda u(x) = f(x), \;\; u(a) = u(b) = 0.$$

The solution of this problem has the form

$$(L_3 - \lambda I)^{-1} f(x) = \int_a^x e^{\lambda(x-t)} f(t)dt, \; a \leq x \leq b,$$

and exists only for those functions f, for which

$$\int_a^b e^{\lambda(b-t)} f(t)dt = 0.$$

That is, it exists only for the functions from the subspace of $L^2(a,b)$ that is orthogonal to the function $e^{\lambda(b-t)}$. Therefore, the image of the operator $(L_3 - \lambda I)$ is not dense in $L^2(a,b)$. It means that any value $\lambda \in \mathbb{C}$ belongs to the *residual spectrum* $\sigma_r(L_3)$. $\square$

Example 3.11 In $L^2(a,b)$, consider the operator given by the differential expression

$$L_4u(x) = \frac{d}{dx}u(x), \; a < x < b,$$

on the domain

$$D(L_4) = \left\{u \in L_1^2(a,b) : u(a) = u(b)\right\}.$$

Note that unlike the previous example the domain of the operator is given with the help of the nonlocal (and periodic) boundary condition $u(a) = u(b)$.

Let us consider the problem of eigenvalues for the operator L_4. Eigenvectors of the operator L_4 are all nonzero functions $u \in D(L_4)$, for which $L_4u(x) - \lambda u(x) = 0$, that is, all continuously differentiable solutions of the differential equation $u'(x) = \lambda u(x)$, $a < x < b$, for which $u(a) = u(b)$.

All solutions of this differential equation have the form $u(x) = Ce^{\lambda x}$, $a \leq x \leq b$, where C is an arbitrary number. Since $u(a) = u(b)$, we get the equation

$$C\left(e^{\lambda(b-a)} - 1\right) = 0.$$

The eigenfunction is not an identical zero, so that we must have $C \neq 0$. Consequently, we obtain the characteristic determinant

$$\triangle(\lambda) \equiv e^{\lambda(b-a)} - 1 = 0.$$

Solutions of this equation are numbers

$$\lambda_k = \frac{2}{b-a} k\pi i, \;\; k = 0, \pm 1, \pm 2, \ldots,$$

involving the imaginary i with $i^2 = -1$. Thus, the values λ_k are eigenvalues of the operator L_4, and

$$u_k(x) = e^{\lambda_k x}, \;\; k = 0, \pm 1, \pm 2, \ldots$$

will be the eigenfunctions corresponding to them.

For the other values $\lambda \neq \lambda_k$ a resolvent exists and can be constructed in the explicit form:

$$(L_4 - \lambda I)^{-1} f(x) = -\int_x^b e^{\lambda(x-t)} f(t) dt - \frac{1}{\triangle(\lambda)} \int_a^b e^{\lambda(x-t)} f(t) dt, \; a \leq x \leq b.$$

Since this operator for $\triangle(\lambda) \neq 0$ is defined on the whole space $L^2(a,b)$ and is bounded, all the values $\lambda \neq \lambda_k$ belong to the resolvent set $\rho(L_4)$ of the operator L_4. □

The considered Examples 3.8 - 3.11 demonstrate that for the operators given by the same formula (in our case, by the same differential expressions) the spectral properties can be absolutely different depending on the given domains of operators.

3.3 Spectral properties of bounded operators

In this section we briefly discuss the simplest properties of the spectrum for linear bounded operators. Let us formulate them in the form of theorems with brief proofs. We start with statements for general operators.

Theorem 3.12 *The spectrum of an operator A is the union of the point spectrum $\sigma_p(A)$, the continuous spectrum $\sigma_c(A)$ and the residual spectrum $\sigma_r(A)$:*

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

These spectra do not intersect:

$$\sigma_p(A) \cap \sigma_c(A) = \sigma_c(A) \cap \sigma_r(A) = \sigma_r(A) \cap \sigma_p(A) = \varnothing.$$

The proof of this theorem follows from definitions of different parts of the spectrum. □

The maximum of modules of elements of the spectrum of the operator A is called the ***spectral radius of*** A and is denoted by $r(A)$, so that

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

Theorem 3.13 *The spectrum of an operator A is contained in the closed ball in the complex plane of radius $\|A\|$ and centre at zero. More precisely, we have that the spectral radius of A satisfies the equality*

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

To prove this, we consider the equation

$$(A-\lambda I)u = f, \tag{3.17}$$

and show that the values λ, for which $|\lambda| > r(A)$, belong to the resolvent set. Indeed, by the properties of limits of sequences of numbers, there exists a number n_0 such that $|\lambda| > \|A^{n_0}\|^{1/n_0}$. Hence

$$\left\| \left(\frac{1}{\lambda}A\right)^{n_0} \right\| < 1. \tag{3.18}$$

We rewrite Eq. (3.17) in the form

$$u - \left(\frac{1}{\lambda}A\right)^{n_0} u = \frac{1}{\lambda}\sum_{j=0}^{n_0-1}\left(\frac{1}{\lambda}A\right)^{j} f.$$

We can apply Theorem 2.32 (the contraction mapping principle) to this equation using Eq. (3.18). Thus, the operator $(A-\lambda I)$ is invertible and, moreover, the inverse operator is bounded and defined on the whole space. Consequently, all values λ for which $|\lambda| > r(A)$ belong to the resolvent set. It means that the spectrum of the operator A is contained in the closed ball in the complex plane of the radius $r(A)$ with centre at zero. The theorem is proved. □

Corollary 3.14 *The resolvent set of a bounded operator is nonempty.*

The proof easily follows from the boundedness of the spectrum of the bounded operator. □

In Section 2.16 we have considered the Volterra operators. Taking into account the definition of the spectral radius, *we can give the definition of a Volterra operator as an operator whose spectral radius is equal to zero*.

Theorem 3.15 *The spectrum of an operator A is a closed set in the complex plane. The resolvent set is open.*

To prove this, we note that the first statement of the theorem follows from the second one. Therefore, let us show the second statement. Let λ_0 be a fixed point from the resolvent set of the operator A. Then the operator $(A-\lambda_0 I)^{-1}$ exists, is defined on the whole space, and is bounded. Let us choose a small number $\varepsilon > 0$ so that the inequality $\varepsilon\|(A-\lambda_0 I)^{-1}\| < 1$ holds. We rewrite the operator $A-\lambda I$ in the form of the product of two operators

$$A-\lambda I = (A-\lambda_0 I)\left[I - (\lambda-\lambda_0)(A-\lambda_0 I)^{-1}\right].$$

Here the first term is invertible since λ_0 is from the resolvent set. For any complex number λ satisfying the inequality $|\lambda - \lambda_0| < \varepsilon$, the second term is also invertible by Theorem 2.32 (the contracting mappings principle).

Consequently, the operator $(A - \lambda I)^{-1}$ exists, is defined on the whole space, and is bounded. That is, any point λ satisfying the inequality $|\lambda - \lambda_0| < \varepsilon$ belongs to the resolvent set of the operator A. The theorem is proved. □

Note that from the proof of Theorem 3.15 it follows that *the resolvent set of an operator is always an open set regardless of whether the operator itself is bounded or not*.

Theorem 3.16 *The spectrum of a bounded operator is nonempty.*

To prove this, we suppose the opposite, that $\sigma(A) = \varnothing$. It means that the spectral radius of the operator is equal to zero: $r(A) = 0$. That is,

$$\lim_{n\to\infty} \|A^n\|^{1/n} = 0.$$

Then the resolvent function $(A - \lambda I)^{-1}$ is well-defined for all $\lambda \neq 0$ and can be written in the form of the von Neumann series

$$(A - \lambda I)^{-1} = -\frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n.$$

As can be seen from here, $\left\|(A - \lambda I)^{-1}\right\| \to 0$, as $\lambda \to \infty$. If $\lambda = 0$ is also not in the spectrum, then A is invertible. Thus for any vectors $x, y \in H$ the function $f(\lambda) = \left\langle (A - \lambda I)^{-1} x, y \right\rangle$ is an analytic function tending to zero at infinity. Then by the Liouville theorem from the complex analysis $(A - \lambda I)^{-1} = 0$, which is impossible. Thus the spectrum is not empty. The theorem is proved. □

3.4 Spectrum of compact operators

The so-called Fredholm alternative can be considered as a central result of the theory of compact operators. The spectral theory of compact operators plays an important role in the spectral theory of differential operators. This is due to the fact that the operator inverses to the differential operators are compact in many cases. Therefore, in this section we consider the spectral properties of compact operators.

Since in a finite-dimensional space all operators are compact (see Theorem 2.61), the spectral properties of the compact operators in the finite-dimensional spaces have been essentially presented in Section 3.1. Therefore, in this section we consider the operators in infinite-dimensional spaces.

Theorem 3.17 *The point $\lambda = 0$ belongs to the spectrum of a compact operator A in an infinite-dimensional space X. The range $R(A)$ is not a closed set in X.*

Indeed, if one assumes that $\lambda = 0$ belongs to the resolvent set of a compact operator A, then the operator A^{-1} exists, is defined on the whole space, and is bounded. Then by Theorem 2.61 their composition $AA^{-1} = I$ would be compact. But the identity operator I in an infinite-dimensional space is a bounded, but not compact operator. Therefore, $\lambda = 0$ must belong to the spectrum of a compact operator. □

Corollary 3.18 *Let $k = k(x,y) \in L^2(\Omega \times \Omega)$. Then the integral equation*

$$\int_{\Omega} k(x,y)\varphi(y)dt = f(x), \quad x \in \Omega,$$

is solvable not for all $f \in L^2(\Omega)$.

If $k = k(x,y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$, then this integral equation is solvable not for all $f \in C(\overline{\Omega})$.

The equations of such type are called ***Fredholm equations of the first kind***, and by Corollary 3.18 such equations are ill-posed.

Historically, in studying integral equations, E. Fredholm first obtained results that did not hold for general operators. Later F. Riesz and J. Schauder showed that his results were a consequence of the compactness of such integral operators, and they developed the general theory of equations involving compact operators. One of the main results in this theory is the theorem on the spectral properties of a compact operator.

Theorem 3.19 (Riesz–Schauder) *The spectrum of a compact operator consists of zero and of a finite or countable set of eigenvalues of finite multiplicity. Moreover, only zero can be a limiting point of the spectrum. This number zero can be an eigenvalue of finite or infinite multiplicity.*

One can show that this theorem quickly implies the following well-known result:

Theorem 3.20 (Fredholm Alternative) *Let A be a compact operator in a Banach space X. Then there are only two alternatives:*

- *either the inhomogeneous equation*

$$Au - \lambda u = f$$

  has a unique solution for an arbitrary $f \in X$;

- *or the homogeneous equation*

$$Au - \lambda u = 0$$

  has a nonzero solution.

This theorem has a very fruitful extension for the case of operators in Hilbert spaces.

Theorem 3.21 (Fredholm Alternative) *Let A be a compact operator in a Hilbert space H with inner product $\langle u, v\rangle_H$. Then,*

- *either the equations*
$$(A - \lambda I)u = f, \tag{3.19}$$
$$(A^* - \overline{\lambda} E)v = g, \tag{3.20}$$
are solvable for any right-hand sides $f, g \in H$, and in this case the corresponding homogeneous equations
$$(A - \lambda I)u = 0, \tag{3.21}$$
$$(A^* - \overline{\lambda} E)v = 0 \tag{3.22}$$
have only zero solutions;
- *or these homogeneous equations* (3.21) *and* (3.22) *have the same (finite) number of linear independent solutions*
$$u_1, u_2, \ldots, u_n; \;\; v_1, v_2, \ldots, v_n.$$
In this case, for Eq. (3.19) *to have a solution it is necessary and sufficient that*
$$\langle f, v_j\rangle_H = 0, \; j = 1, 2, \ldots, n;$$
and for Eq. (3.20) *to have a solution it is necessary and sufficient that*
$$\langle g, u_j\rangle_H = 0, \; j = 1, 2, \ldots, n.$$

The Fredholm Alternative has very effective applications in the solvability theory of boundary value problems for differential equations. It makes it possible to conclude that there exists a solution of a boundary value problem only on the basis of the proven fact of the uniqueness of its solution. The methods used in the next example are quite universal and can be applied to a wide range of problems. Therefore, the following analysis will be carried out in more detail.

Example 3.22 Consider the problem of finding a solution of the ordinary differential equation
$$-u''(x) + q(x)u(x) = f(x), \; 0 < x < 1, \tag{3.23}$$
satisfying the boundary conditions
$$u(0) = 0, \;\; u(1) = 0. \tag{3.24}$$

Here $q(x) \geq 0$ is a given continuous function, and $f \in L^2(0,1)$ is a given function. A function $u \in L_2^2(0,1)$ satisfying Eq. (3.23) (in the sense of almost everywhere) and the boundary conditions (3.24) is called the *solution of problem* (3.23)-(3.24).

Let us multiply Eq. (3.23) by $u(x)$ and integrate over the interval $(0,1)$. Then, integrating by parts, we get

$$u'(x)u(x)\big|_0^1 + \int_0^1 |u'(x)|^2 dx + \int_0^1 q(x)|u(x)|^2 dx = \int_0^1 f(x)u(x)dx. \tag{3.25}$$

Here the first term is equal to zero by the boundary conditions (3.24). Also, taking these boundary conditions into account, we have

$$|u(x)| = \left| \int_0^x u'(t)dt \right| \leq \int_0^1 \left|u'(t)\right| dt \leq \left(\int_0^1 \left|u'(t)\right|^2 dt \right)^{1/2}.$$

In the last inequality we have used the integral Hölder inequality (2.20). Hence we have

$$\int_0^1 |u(x)|^2 dx \leq \int_0^1 \left|u'(t)\right|^2 dt.$$

Since $q(x) \geq 0$, from this and from (3.25) we get

$$\int_0^1 |u(x)|^2 dx + \int_0^1 \left|u'(t)\right|^2 dt \leq 2\int_0^1 \left|u'(t)\right|^2 dt \leq 2\int_0^1 |f(x)u(x)|dx. \tag{3.26}$$

For any number $\varepsilon \in (0,1)$ we have an elementary inequality

$$2|f(x)u(x)| \leq \frac{1}{\varepsilon}|f(x)|^2 + \varepsilon|u(x)|^2.$$

Then from (3.26) we have

$$\int_0^1 |u(x)|^2 dx + \int_0^1 \left|u'(t)\right|^2 dt \leq \frac{1}{\varepsilon(1-\varepsilon)} \int_0^1 |f(x)|^2 dx,$$

or, in terms of norms of the Sobolev spaces,

$$\|u\|_{L_1^2(0,1)} \leq \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \|f\|_{L^2(0,1)}. \tag{3.27}$$

Inequalities of the form (3.27) are called ***a priori estimates*** since they are obtained a priori without having an explicitly constructed solution of the original differential problem. A priori is Latin for “from before” and refers to the fact that the estimate for the solution is derived before the solution is known to exist.

Let us show now that problem (3.23)-(3.24) cannot have more than one solution. Suppose that there are two solutions of the problem (3.23)-(3.24): $u_1(x)$ and $u_2(x)$. Denote

$$u(x) := u_1(x) - u_2(x).$$

Then the function $u(x)$ satisfies the homogeneous equation

$$-u''(x) + q(x)u(x) = 0, \; 0 < x < 1,$$

and the boundary conditions (3.24). Then, by the a priori estimate (3.27) we have $u(x) \equiv 0$, that is, $u_1(x) = u_2(x)$.

Thus, inequality (3.27) provides the uniqueness of the solution of the problem (3.23)-(3.24). Let us prove the existence of the solution.

Let us twice integrate Eq. (3.23) over the interval $(0,x)$. Taking into account $u(0) = 0$, we get

$$-u(x) + Cx + \int_0^x (x-t)q(t)u(t)dt = \int_0^x (x-t)f(t)dt, \tag{3.28}$$

where C is an arbitrary constant. Let us find this constant satisfying the boundary condition $u(1) = 0$:

$$C = -\int_0^1 (1-t)q(t)u(t)dt - \int_0^1 (1-t)f(t)dt.$$

Substituting the obtained result into (3.28), after elementary transformations, we obtain

$$u(x) + \int_0^x (1-x)tq(t)u(t)dt + \int_x^1 x(1-t)q(t)u(t)dt = F(x), \tag{3.29}$$

where

$$F(x) = \int_0^x (1-x)tf(t)dt + \int_x^1 x(1-t)f(t)dt.$$

It is obvious that $F \in L^2(0,1)$ for $f \in L^2(0,1)$. Denote

$$k(x,t) := \theta(x-t)(1-x)tq(t) + \theta(t-x)x(1-t)q(t),$$

and by A we denote an integral operator with the kernel $k(x,t)$:

$$Au(x) := \int_0^1 k(x,t)u(t)dt.$$

Since the function $k(x,t)$ is continuous in the square $0 \le x,t \le 1$, the operator A is a compact operator in $L^2(0,1)$ (see Example 2.68).

Hence, Eq. (3.29) can be represented as an equation with the compact operator

$$(A+I)u = F, \tag{3.30}$$

to which one can apply the Fredholm Alternative. Therefore, from the uniqueness of the solution one gets its existence.

Thus, for all $F \in L^2(0,1)$ there exists the unique solution of Eq. (3.30). Let us show that it is the solution of the problem (3.23)-(3.24). From (3.29) it follows that $u(x)$ satisfies the boundary conditions (3.24).

By a direct calculation it is easy to make sure that $F \in L_2^2(0,1)$ for $f \in L^2(0,1)$ and $A\varphi \in L_2^2(0,1)$ for $\varphi \in L^2(0,1)$. Then from (3.30) we obtain that $u \in L_2^2(0,1)$. Therefore $u''(x)$ exists almost everywhere and any solution of Eq. (3.30) is the solution of the problem (3.23)-(3.24). Thus, we have proved

Lemma 3.23 *Let $q(x) \geq 0$ be a given continuous function. Then for any $f \in L^2(0,1)$ there exists a unique solution of the problem* (3.23)-(3.24).

The considered example shows how the application of the Fredholm Alternative allows one to prove the existence of the solution to the differential problem by only proving the uniqueness. □

Considering Example 3.22, we have applied the Fredholm Alternative to the particular integral equation (3.29). The following example demonstrates that the Fredholm Alternative can be applied to a wide class of the integral equations.

Example 3.24 *(Fredholm integral equations)* In $L^2(\Omega)$, consider the following equations for the unknown functions φ and ψ:

$$\varphi(x) - \int_\Omega k(x,t)\varphi(t)dt = f(x), \quad x \in \Omega; \tag{3.31}$$

$$\varphi(x) - \int_\Omega k(x,t)\varphi(t)dt = 0, \quad x \in \Omega; \tag{3.32}$$

$$\psi(x) - \int_\Omega k(t,x)\psi(t)dt = 0, \quad x \in \Omega. \tag{3.33}$$

Here $k = k(x,t) \in L^2(\Omega \times \Omega)$ is a given integral kernel of the integral equation, and $f \in L^2(\Omega)$ is a given function.

Equations of the form (3.31), unlike the equation from Corollary 3.18, are called ***Fredholm integral equations of the second kind***. Since the integral operator participating in these equations is compact in $L^2(\Omega)$, for these equations Theorem 3.21 with the Fredholm Alternative holds. Let us formulate this particular result in the form of a lemma.

Lemma 3.25 (Fredholm Alternative for integral equations). *The following statements are true:*

I. If Eq. (3.32) *has only a trivial solution, then Eq.* (3.31) *has a unique solution for any $f \in L^2(\Omega)$. But if Eq.* (3.32) *has a non-trivial solution, then the solution of Eq.* (3.31) *is not necessarily unique, and it is solvable not for all $f \in L^2(\Omega)$.*

II. Equations (3.32) *and* (3.33) *have the same finite number of linearly independent solutions.*

III. Eq. (3.31) *is solvable if and only if for any solution ψ of Eq.* (3.33) *the condition*

$$\int_\Omega f(x)\psi(x)dx = 0 \tag{3.34}$$

holds.

The consequence of this lemma is a classical Fredholm result on the solvability of integral equations in the space of continuous functions $C(\overline{\Omega})$.

Lemma 3.26 *Let the integral kernel $k = k(x,t)$ of the integral equation* (3.31) *be continuous: $k \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. The following statements are true:*

I. If Eq. (3.32) *has only a trivial solution, then Eq.* (3.31) *has a unique solution for any $f \in C(\overline{\Omega})$. But if Eq.* (3.32) *has a non-trivial solution, then the solution of Eq.* (3.31) *is not necessarily unique and it is solvable not for all $f \in C(\overline{\Omega})$.*

II. Equations (3.32) *and* (3.33) *have the same finite number of linearly independent solutions.*

III. Eq. (3.31) *is solvable if and only if for any solution ψ of Eq.* (3.33) *condition* (3.34) *holds.*

In Section 2.16 we have considered the Volterra operators, which are a particular case of the integral operators, whose spectral radius is equal to zero. Therefore, for such operators the corresponding homogeneous equation (3.32) has only the zero solution. Then from Lemmas 3.25 and 3.26 we get

Lemma 3.27 *Let $D = \left\{(x,t) \in \mathbb{R}^2 : a \leq t \leq x \leq b\right\}$. Then*

I. If $k = k(x,t) \in L^2(D)$, then the integral equation

$$\varphi(x) - \int_a^x k(x,t)\varphi(t)dt = f(x), \;\; a < x < b, \tag{3.35}$$

has the unique solution $\varphi \in L^2(a,b)$ for any $f \in L^2(a,b)$.

II. If $k = k(x,t) \in C(\overline{D})$, then the integral equation (3.35) *has the unique solution $\varphi \in C[a,b]$ for any $f \in C[a,b]$.*

Equations of the form (3.35) are called ***Volterra integral equations of the second kind***. By Lemma 3.27 such equations are always uniquely solvable. □

Let us consider one simple concrete example of an integral equation similar to that considered in Example 2.31.

Example 3.28 In the space of continuous functions $C[0,1]$, consider the integral equation

$$\varphi(x) - \int_0^1 3xt\varphi(t)dt = f(x), \; 0 < x < 1. \tag{3.36}$$

Since the integral kernel of this integral operator given by $k(x,t) = 3xt$ satisfies the condition $k \in C([0,1] \times [0,1])$, one can apply Lemma 3.26 to it.

Consider the corresponding homogeneous integral equation

$$\varphi(x) - 3\int_0^1 xt\varphi(t)dt = 0, \; 0 < x < 1. \tag{3.37}$$

Analysing Eq. (3.37), it is easy to see that all of its solutions have the form

$$\varphi(x) = cx, \text{ where } \; c = 3\int_0^1 t\varphi(t)dt.$$

Multiplying this equality by $3x$ and integrating the obtained result over the interval $[0,1]$, we find the identity

$$c = 3\int_0^1 x\varphi(x)dx = 3c\int_0^1 x^2 dx = c,$$

which is true for any c. Thus the homogeneous integral equation (3.37) has the nonzero solution $\varphi(x) = x$. Eq. (3.37) has no other linearly independent solutions.

Therefore, by Lemma 3.26 the integral equation (3.36) is solvable not for all $f \in C[0,1]$. For the existence of a solution of Eq. (3.36) it is necessary and sufficient that the condition

$$\int_0^1 xf(x)dx = 0$$

is satisfied. If this condition holds, then the solution of Eq. (3.36) is not unique and has the form

$$\varphi(x) = f(x) + Cx,$$

where C is an arbitrary constant. $\square$

The Fredholm integral equations are particular cases from the general theory of Noetherian operators.

In a Banach space X, a bounded operator T is called a ***Noetherian operator***, if $\dim\ker T < \infty$ and $\dim\operatorname{coker} T < \infty$. In particular, by the Riesz–Schauder theorem 3.19, all operators of the form $A - I$ with a compact operator A are Noetherian. In finite-dimensional spaces any linear operator is Noetherian.

For the Noetherian operators one can introduce the concept of the ***index of the operator***:

$$\operatorname{ind} T := \dim\ker T - \dim\operatorname{coker} T.$$

The Noetherian operators have the following properties:

- The adjoint operator to a Noetherian one is also Noetherian. Moreover,

$$\operatorname{ind} T^* = -\operatorname{ind} T.$$

- If the operators T and T_1 are Noetherian, then the operator TT_1 is also Noetherian. Moreover,

$$\operatorname{ind} TT_1 = \operatorname{ind} T + \operatorname{ind} T_1.$$

- If the operator T is Noetherian and A is compact, then the operator $T + A$ is also Noetherian. Moreover, the index of the operator does not change:

$$\operatorname{ind}(T + A) = \operatorname{ind} T.$$

A particular case of the Noetherian operator is the Fredholm operator. An operator T is called a ***Fredholm operator*** if $\operatorname{ind} T = 0$. The Fredholm operators have the following properties:

- The adjoint operator to the Fredholm one is also Fredholm.
- If the operators T and T_1 are Fredholm, then the operator TT_1 is also Fredholm.
- If the operator T is Fredholm and A is compact, then the operator $T+A$ is also Fredholm.
- The criterion of Nikolskii: an operator T is Fredholm if and only if it can be represented as the sum of two operators $T = S+A$, where the operator S is invertible and the operator A is compact.

An important subclass of the compact operators are self-adjoint compact operators. Let H be a separable Hilbert space and let A be an operator in H.

The linear bounded operator A is called a ***positive operator*** if $\langle Ax,x\rangle \geq 0$ for all $x \in H$. If A is a compact self-adjoint operator, then its spectrum consists of nonnegative real numbers of finite multiplicity.

Theorem 3.29 *Let A be a compact self-adjoint operator in a separable Hilbert space H and assume that $A \neq 0$. Then there exists at least one nonzero eigenvalue $\lambda \neq 0$ of the operator A.*

A compact self-adjoint operator in an infinite-dimensional separable Hilbert space has an infinite countable number of eigenvalues. To each eigenvalue $\lambda \neq 0$ there corresponds only a finite number of linearly independent eigenvectors. Such operator has no associated vectors. The normed system of its eigenvectors forms a complete orthonormal system in H.

Without dwelling on the proof of this theorem, we mention only a variational method for constructing the eigenvalues. For simplicity we assume that $\lambda = 0$ is not an eigenvalue of a compact self-adjoint operator A. By Theorem 3.29 there exists at least one nonzero eigenvalue $\lambda \neq 0$.

Consider the linear functional $Qx = \langle Ax,x\rangle$ on elements of the unit sphere $S \subset H$: $S = \{x \in H : \|x\| = 1\}$. Then

$$|Qx| = |\langle Ax,x\rangle| \leq \|Ax\|\|x\| \leq \|A\|, \ \forall x \in S.$$

Since $A \neq 0$, there exists $x \in S$ such that $\langle Ax,x\rangle > 0$.

By the *variational method* one proves that the largest eigenvalue of the positive compact self-adjoint operator A is given by the formula

$$\lambda_1 = \sup_{x\in S}\langle Ax,x\rangle. \tag{3.38}$$

Here the sup is achieved on an eigenvector x_1 corresponding to λ_1. Formula (3.38) and its generalisations will be of great importance for the analysis in the following chapters.

In the same way, one can consistently search for all the other eigenvalues. Let us denote by S_1 the subset of S orthogonal to the element x_1. By variational discussions one can show that there exists an element $x_2 \in S_1$ such that

$$\langle Ax_2,x_2\rangle = \lambda_2 = \sup_{x\in S_1}\langle Ax,x\rangle, \ \ Ax_2 = \lambda_2 x_2.$$

Here $x_2 \perp x_1$ and $\lambda_2 \leq \lambda_1$, since $S_1 \subset S$ and sup on S_1 cannot be less than sup on S.

Continuing this process, we obtain a nonincreasing sequence of eigenvalues of the operator A: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots > 0$ and a corresponding sequence of eigenvectors of the operator A: $x_1, x_2, x_3, \ldots$, which are pairwise orthogonal to each other. We can summarise it as follows.

Theorem 3.30 *Let A be a compact self-adjoint operator in an infinite-dimensional separable Hilbert space H and assume that $A \neq 0$. Then A has an infinite countable number of eigenvalues forming a nonincreasing sequence*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots > 0, \quad \lim_{n\to\infty} \lambda_n = 0.$$

The sequence of the corresponding normalised eigenvectors

$$x_1, x_2, x_3, \ldots$$

forms a complete orthonormal system in H.

For the first eigenvalue λ_1 the formula

$$\lambda_1 = \max_{x\in H:\|x\|=1} \langle Ax, x\rangle = \langle Ax_1, x_1\rangle,$$

is valid, and also

$$\lambda_1 = \|A\|;$$

for the n-th eigenvalue λ_n we have the equality

$$\lambda_n = \max_{x\in H:\|x\|=1, \langle x, x_j\rangle = 0, \forall j=1,\ldots,n-1} \langle Ax, x\rangle = \langle Ax_n, x_n\rangle.$$

Note that according to Theorem 3.30 the functional $\langle Ax, x\rangle$ reaches its maximum value on the unit sphere in the space H. This maximum is achieved on the first eigenvector and is equal to the first eigenvalue of the operator A.

If one considers the set S_1 of elements of the unit sphere orthogonal to the first eigenvector, then this functional also reaches its maximum value on S_1. This maximum is achieved on the second eigenvector and is equal to the second eigenvalue of the operator A. Continuing this procedure, one can construct all eigenvalues and eigenvectors of the operator A.

However there is no need to act so consistently. The following theorem gives the description for obtaining the n-th eigenvalue at once.

Theorem 3.31 (Courant minimax principle). *Let A be a compact self-adjoint operator in an infinite-dimensional separable Hilbert space H and assume that $A \neq 0$. Let x_1, x_2, ..., x_{n-1} be an arbitrary system of $n-1$ linearly independent elements in H. Denote by $m(x_1, x_2, \ldots, x_{n-1})$ the following maximum*

$$m(x_1, x_2, \ldots, x_{n-1}) := \sup_{x\in H:\langle x, x_j\rangle = 0, \forall j=1,\ldots,n-1} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

Then the n-th eigenvalue of the operator A is equal to the minimum among all these maxima when this minimum is taken with respect to all linearly independent elements x_1, x_2, ..., x_{n-1} *of the space H:*

$$\lambda_n = \inf_{x_j \in H, \forall j=1,\dots,n-1} m(x_1, x_2, \dots, x_{n-1}).$$

3.5 Hilbert-Schmidt theorem and its application

As we have shown in Section 3.1, for any finite-dimensional self-adjoint operator there exists a basis (consisting of eigenvectors) in which the Hermitian matrix of this self-adjoint operator has a diagonal form (see Theorem 3.2 and its Corollary). The Hilbert-Schmidt theorem generalises this result to the case of infinite-dimensional spaces.

Here and in the sequel we will say that the vector x in the Hilbert space is *normalised* if its length is one: $\|x\| = 1$.

Theorem 3.32 (Hilbert-Schmidt). *Let A be a compact self-adjoint operator on a Hilbert space H, and let* φ *be an arbitrary element of H. Then the element* $A\varphi \in H$ *decomposes as a converging Fourier series with respect to the system* x_j *of normalised eigenvectors of the operator A.*

By the decomposition into a Fourier series we mean that we have

$$A\varphi = \sum_{j=1}^{\infty} \lambda_j \langle \varphi, x_j \rangle x_j, \tag{3.39}$$

with the series convergent in H, and with λ_j *being the eigenvalues corresponding to the eigenvectors* x_j.

To prove this theorem we take an eigenvalue of the operator A, $\lambda_1 = \|A\|$, which is the greatest by module, and the corresponding normalised eigenvector x_1. We denote by L_1 the one-dimensional subspace spanned by the eigenvector x_1:

$$L_1 := \{x \in H : x = Cx_1, \text{ where } C \text{ is a constant}\}.$$

We denote by H_1 the subspace orthogonal to x_1: $H_1 = \{x \in H : \langle x, x_1 \rangle = 0\}$. Thus, $H = L_1 \oplus H_1$. It is obvious that H_1 is invariant with respect to A, that is, $Ax \in H_1$ holds for all $x \in H_1$.

Therefore, one can consider A as an operator acting on H_1. And as in the previous case, one can define subspaces

$$L_2 = \{x \in H_1 : x = Cx_2, \text{ where } C \text{ is a constant}\}$$

and $H_2 = \{x \in H_1 : \langle x, x_2 \rangle = 0\}$ such that $H_1 = L_2 \oplus H_2$. Thus, $H = L_1 \oplus L_2 \oplus H_2$.

Continuing this process, at the n-th step we construct a subspace L_n generated by the eigenvector x_n and a subspace $H_n = \{x \in H : \langle x, x_j \rangle = 0,\ j = 1, \ldots, n-1\}$. We also have $H = L_1 \oplus L_2 \oplus \ldots \oplus L_n \oplus H_n$.

Let us now first consider the case when $Ax = 0$, for all $x \in H_n$ for some n. It means that then

$$H = L_1 \oplus L_2 \oplus \ldots \oplus L_n \oplus \ker A,$$

where $\ker A$ is the kernel of the operator A. In this case $\ker A$ is the set of all eigenvectors corresponding to the eigenvalue $\lambda_n = 0$.

Hence, any element $\varphi \in H$ can be represented in the form

$$\varphi = x_0 + \sum_{j=1}^{n} \langle \varphi, x_j \rangle x_j, \quad x_0 \in \ker A.$$

Hence

$$A\varphi = \sum_{j=1}^{n} \langle \varphi, x_j \rangle \lambda_j x_j, \quad x_0 \in \ker A,$$

and in this case the theorem and formula (3.39) are proved.

It is easy to see that in this case the operator A is actually finite-dimensional: its image is contained in the finite-dimensional subspace $L_1 \oplus L_2 \oplus \ldots \oplus L_n$.

Now consider the other case. Here the process of constructing eigenvectors can continue indefinitely. We have the estimate

$$\left\| A\left(\varphi - \sum_{j=1}^{n} \langle \varphi, x_j \rangle x_j\right) \right\|^2 \leq \|A\|_{H_n} \left\| \varphi - \sum_{j=1}^{n} \langle \varphi, x_j \rangle x_j \right\|^2$$

$$\leq \lambda_{n+1}^2 \left(\|\varphi\|^2 - \sum_{j=1}^{n} \left| \langle \varphi, x_j \rangle \right|^2 \right) \leq \lambda_{n+1}^2 \|\varphi\|^2.$$

Then, since $\lim\limits_{n\to\infty} \lambda_n = 0$ by Theorem 3.30, we have

$$A\varphi = \lim_{n\to\infty} A\left(\sum_{j=1}^{n} \langle \varphi, x_j \rangle x_j \right) = \sum_{j=1}^{\infty} \langle \varphi, x_j \rangle \lambda_j x_j,$$

which proves the theorem. □

The second case considered in the proof of the theorem is possible, for example, in the case when $\lambda = 0$ is not an eigenvalue of the operator A. From this we get the following important corollary.

Corollary 3.33 *If a compact self-adjoint operator A in a Hilbert space H is invertible, then the system of its eigenvectors forms a basis of the space H. That is, any element $\varphi \in H$ decomposes into a converging Fourier series with respect to the system x_j of normalised eigenvectors of the operator A:*

$$\varphi = \sum_{j=1}^{\infty} \langle \varphi, x_j \rangle x_j, \quad \|\varphi\|^2 = \sum_{j=1}^{\infty} \left| \langle \varphi, x_j \rangle \right|^2.$$

Here the second equality is called the ***Parseval's identity***. This is directly analogous to the Pythagorean theorem, which asserts that the sum of the squares of the components of a vector in an orthonormal basis is equal to the squared length of the vector.

Example 3.34 (*Integral equations with symmetric kernel*)
In the space $L^2(a,b)$, consider the Fredholm integral equation of the second kind

$$\varphi(x) - \mu \int_a^b k(x,t)\varphi(t)dt = f(x), \quad a < x < b, \tag{3.40}$$

with the symmetric kernel $k(x,t) = k(t,x)$ satisfying the condition $k \in L^2((a,b) \times (a,b))$. Here μ is a complex spectral parameter.

The integral operator

$$A\varphi(x) = \int_a^b k(x,t)\varphi(t)dt$$

in this equation has been already considered in Examples 2.11 and 2.68. As we have shown in Example 2.81, this operator is a self-adjoint compact operator in $L^2(a,b)$. Therefore, one can apply the Hilbert-Schmidt theorem 3.32 to Eq. (3.40).

As follows from the already discussed results, the integral operator A has at least one nonzero eigenvalue; all its eigenvalues λ_j are real; the eigenfunctions $\varphi_j(x)$ corresponding to different eigenvalues λ_j, are orthogonal to each other; to each eigenvalue there can correspond only a finite number of linearly independent eigenfunctions.

Applying the Hilbert-Schmidt theorem 3.32 and formula (3.39), we decompose the action of the operator A in a series with respect to an orthonormal basis of its eigenfunctions

$$A\varphi(x) = \sum_{j=1}^{\infty} \lambda_j \langle \varphi, \varphi_j \rangle \varphi_j(x). \tag{3.41}$$

Substituting this expression in Eq. (3.40), we get

$$\varphi(x) - \mu \sum_{j=1}^{\infty} \lambda_j \langle \varphi, \varphi_j \rangle \varphi_j(x) = f(x). \tag{3.42}$$

Taking the inner product with φ_i, we get

$$\langle \varphi, \varphi_i \rangle - \mu \sum_{j=1}^{\infty} \lambda_j \langle \varphi, \varphi_j \rangle \langle \varphi_j, \varphi_i \rangle = \langle f, \varphi_i \rangle.$$

Then from the orthogonality of eigenfunctions we get

$$\langle \varphi, \varphi_i \rangle - \mu \lambda_i \langle \varphi, \varphi_i \rangle = \langle f, \varphi_i \rangle. \tag{3.43}$$

Let first the number $\lambda = 1/\mu$ be not an eigenvalue of the operator A. Then from (3.43) we find for all $i \in \mathbb{N}$ that

$$\langle \varphi, \varphi_i \rangle = \frac{1}{1 - \lambda_i \mu} \langle f, \varphi_i \rangle.$$

Substituting the obtained values of Fourier coefficients in (3.42), we obtain a solution of the original equation (3.40)

$$\varphi(x) = f(x) + \mu \sum_{j=1}^{\infty} \frac{\lambda_j}{1-\lambda_j\mu} \langle f, \varphi_j \rangle \varphi_j(x). \tag{3.44}$$

Thus, in the case when the number $\lambda = 1/\mu$ is not the eigenvalue of the operator A, the integral equation (3.40) with a symmetric kernel for any right-hand side $f \in L^2(a,b)$ has the unique solution (3.44).

Consider now the case when the number $\lambda = 1/\mu$ is an eigenvalue of the operator A. Let this eigenvalue have multiplicity $m+1$. This means that for some k the eigenvalues $\lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+m}$ are the same.

Then from (3.43) one can define all coefficients except for $m+1$ of them, by

$$\langle \varphi, \varphi_i \rangle = \frac{1}{1-\lambda_i\mu} \langle f, \varphi_i \rangle, \; i \neq k, k+1, \ldots, k+m.$$

And for the remaining indices i from (3.43), the coefficients $\langle \varphi, \varphi_i \rangle$ cannot be defined. Instead, we obtain the condition

$$\langle f, \varphi_i \rangle = 0, \; i = k, k+1, \ldots, k+m. \tag{3.45}$$

Thus, in the case when the number $\lambda = 1/\mu$ is an eigenvalue of the operator A of multiplicity $m+1$, the integral equation (3.40) with the symmetric kernel has the solution only for those right-hand sides $f \in L^2(a,b)$, for which (3.45) hold. If this condition holds, then the solution of Eq. (3.40) exists. In this case, it is not unique and has the form

$$\varphi(x) = f(x) + \mu \sum_{j=1, j\neq k, k+1, \ldots, k+m}^{\infty} \frac{\lambda_j}{1-\lambda_j\mu} \langle f, \varphi_j \rangle \varphi_j(x) + \sum_{j=k}^{k+m} C_j \varphi_j(x),$$

where C_j are arbitrary constants. $\square$

3.6 Spectral properties of unbounded operators

In the previous sections we have considered the bounded operators. Such operators are defined on all elements of the space under consideration. Now let us consider unbounded linear operators. Such operators are defined not only by their action but also by their domain. We will consider only closed linear operators (see the definition in Section 2.12).

For the unbounded operators all definitions of the spectrum and the resolvent set (see Section 3.2) are preserved. But for completeness of the exposition some concepts will be explained again in this new context.

Consider a linear and, generally speaking, unbounded operator A acting in a Banach space X over the field of complex numbers. As we have shown in Section 3.1, the necessity of involving the complex numbers is caused by the fact that any square $n \times n$ matrix (even with real entries) always has n eigenvalues, but in general these eigenvalues will be complex.

As usual, by $D(A)$ we denote the domain of the operator A. Then on this domain $D(A)$ one can define the family of operators

$$(A - \lambda I) : D(A) \to X,$$

where I is the identity operator, and λ is a complex parameter called a ***spectral parameter***.

The value λ is called ***regular*** *if the operator* $(A - \lambda I)^{-1}$ *exists, is defined on the whole X, and is bounded.* In this case the operator

$$R_\lambda := (A - \lambda I)^{-1}$$

is called a ***resolvent of the operator*** A. The set of all regular values λ is called the ***resolvent set*** $\rho(A)$, and the complement of the resolvent set (refers to values not in $\rho(A)$) is called the ***spectrum*** of the operator $\sigma(A)$.

If the operator $(A - \lambda I)^{-1}$ *does not exist*, that is, there exists a nontrivial solution $x \neq 0$ of the equation

$$(A - \lambda I)x = 0, \tag{3.46}$$

then the number λ is called an ***eigenvalue of the operator*** A. For this value λ the nontrivial solution x of Eq. (3.46) is called an ***eigenvector of the operator*** A. The set of all eigenvalues forms the so-called ***point spectrum*** $\sigma_p(A)$ of the operator A.

If the operator $(A - \lambda I)^{-1}$ *exists, is defined on a set everywhere dense in X but is not bounded*, then such values λ are said to belong to the so-called ***continuous spectrum*** $\sigma_c(A)$ of the operator A.

And *if the operator* $(A - \lambda I)^{-1}$ *exists but is defined on a set which is not dense in* X, then such λ are said to belong to the ***residual spectrum*** $\sigma_r(A)$ of the operator A.

Due to this, note that if $\lambda \in \rho(A)$, then the operator $(A - \lambda I)^{-1}$ exists, is defined on the whole space X, and is bounded. Therefore (see Theorem 2.44) the operator $(A - \lambda I)^{-1}$ is closed. Then (see Theorem 2.45) the operator $A - \lambda I$ is also closed. Namely, this fact requires *considering the spectral properties only of the closed operators A*, otherwise each value λ will belong to the continuous spectrum $\sigma_c(A)$.

In the spectral theory of differential operators, in most cases one studies operators that have a compact resolvent for some λ. Then the results of the previous Sections 3.4-3.5 can be applied to the resolvent.

In this direction, ***Hilbert's identity*** (also called the resolvent identity) has an important value.

Lemma 3.35 (Hilbert's identity for the resolvent) *Let* $\lambda, \mu \in \rho(A)$. *Then for the resolvent* $R_\lambda = (A - \lambda I)^{-1}$ *of the operator A we have the identity*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu. \tag{3.47}$$

To prove this, for an arbitrary $f \in X$ we write two obvious identities

$$R_\lambda f = R_\mu (A - \mu E) R_\lambda f;$$

$$R_\mu f = R_\mu (A - \lambda I) R_\lambda f.$$

Now subtracting the second equality from the first one, by the arbitrariness of $f \in X$, we obtain (3.47). □

Corollary 3.36 *Resolvents commute with each other*

$$R_\lambda R_\mu = R_\mu R_\lambda, \ \forall \lambda, \mu \in \rho(A).$$

Corollary 3.37 *The resolvent R_λ is a differentiable operator-valued function of the parameter λ. We have*

$$R'(\lambda) = R^2(\lambda). \tag{3.48}$$

To prove this, we use the definition of a derivative and the Hilbert's identity (3.47), yielding

$$R'(\lambda) = \lim_{h \to 0} \frac{R(\lambda + h) - R(\lambda)}{h} = \lim_{h \to 0} \frac{(\lambda + h - \lambda) R(\lambda + h) R(\lambda)}{h} = R^2(\lambda).$$

Corollary 3.38 *The resolvent R_λ is an infinitely differentiable operator-valued function which is analytic on the set of regular points $\lambda \in \rho(A)$. That is, for all points $\lambda \in \rho(A)$ the operator R_λ is bounded and in the neighborhood of each point $\lambda_0 \in \rho(A)$ the function R_λ admits the expansion into a power series (converging with respect to the operator norm)*

$$R(\lambda) = R(\lambda_0) + \sum_{k=1}^{\infty} (\lambda - \lambda_0)^k C_k,$$

where C_k are bounded operators independent of λ.

The proof follows from the iteration of formula (3.48):

$$\left(\frac{d}{d\lambda} \right)^n R(\lambda) = n! R^{n+1}(\lambda),$$

and the continuity of the resolvent R_λ with respect to the parameter λ in the neighborhood of the point λ. □

Corollary 3.39 *If the resolvent R_λ is compact for one value λ, then it is compact for all other values $\lambda \in \rho(A)$.*

Theorem 3.40 *The resolvent set is open.*

The proof is the same as the proof of Theorem 3.15 on the openness of the resolvent set for bounded operators. □

As we have shown in Theorem 3.16, the spectrum of a bounded operator is nonempty. For the case of unbounded operators this is not so. In Example 3.9 we have shown that there exists an unbounded operator (an ordinary differential operator of the first order) for which any value λ belongs to the resolvent set. Let us now construct a more complicated example for the case of an ordinary differential operator of the second order.

Example 3.41 In $L^2(0,1)$, consider the operator acting by the formula

$$L_\alpha u(x) = -\frac{d^2}{dx^2}u(x),\ 0 < x < 1,$$

given on the domain

$$D(L_\alpha) = \left\{u \in L_2^2(0,1):\ u(0) = \alpha u(1),\ u'(0) = -\alpha u'(1)\right\},$$

where α is a fixed number.

Let us first consider the case when $\alpha^2 \neq 1$. By a direct calculation it is easy to show that the operator L_α is invertible and its inverse operator can be written in the form

$$u(x) = L_\alpha^{-1} f(x) = \int_0^x \frac{(1+\alpha)t-(1-\alpha)x-\alpha}{1-\alpha^2} f(t)dt + \int_x^1 \frac{\alpha(1+\alpha)t+\alpha(1-\alpha)x-\alpha}{1-\alpha^2} f(t)dt.$$

From this representation it is easy to see that L_α^{-1} is a compact operator in $L^2(0,1)$. By the Riesz–Schauder Theorem 3.19 the spectrum of a compact operator consists of zero and eigenvalues of finite multiplicity. Therefore the spectrum of the operator L_α can consist only of the eigenvalues.

Consider the problem of eigenvalues for the operator L_α. Eigenvectors of the operator L^2 are all functions $u \in D(L_\alpha)$, for which

$$L_\alpha u(x) \equiv -u''(x) = \lambda u(x).$$

All solutions of this differential equation have the form

$$u(x) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \frac{1}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda}x\right),$$

where C_1 and C_2 are arbitrary constants. Note that such representation of a solution includes also the case $\lambda = 0$. Passing to the limit for $\lambda \to 0$, we obtain $u(x) = C_1 + C_2 x$.

Satisfying the boundary conditions for the domain $D(A)$ for finding C_1 and C_2, we obtain the system of linear equations

$$\begin{cases} C_1\left(1-\alpha\cos\sqrt{\lambda}\right) - C_2\alpha\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda} = 0, \\ C_1\sqrt{\lambda}\alpha\sin\sqrt{\lambda} - C_2\left(1+\alpha\cos\sqrt{\lambda}\right) = 0. \end{cases} \tag{3.49}$$

The characteristic determinant of the spectral problem will be the determinant of this system. Calculating it, we have

$$\triangle(\lambda) = \alpha^2 - 1.$$

Then, since $\alpha^2 \neq 1$, we must have $C_1 = 0$ and $C_2 = 0$, that is, the operator L_α has no eigenvalues. That is, for all values $\lambda \in \mathbb{C}$ the resolvent exists.

Since L_α^{-1} is a compact operator, then by Corollary 3.45 the resolvent is compact (and, consequently, is bounded) for all values $\lambda \in \mathbb{C}$. Hence, for $\alpha^2 \neq 1$ each value $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(L_\alpha)$ of the operator L_α, and *the spectrum of the operator is empty*.

Consider now the case $\alpha^2 = 1$. Then $\triangle(\lambda) = 0$ for all values $\lambda \in \mathbb{C}$ and the system (3.49) has the nonzero solution. Therefore, each value $\lambda \in \mathbb{C}$ is an eigenvalue of the operator L_α, and the *resolvent set is empty*. □

Let us consider now the relation between the spectra of the adjoint operators. Let A be a linear (unbounded) operator with the domain $D(A) \subset H$ in a Hilbert space H. Assume that the domain of the operator is dense in the space H, that is, $\overline{D(A)} = H$. Then, as we have shown in Section 2.19, the adjoint operator A^* exists.

Lemma 3.42 *If the adjoint operator A^* exists, then the spectrum $\sigma(A)$ of the operator A and the spectrum $\sigma(A^*)$ of the operator A^* are symmetric with respect to the real axis, that is, if $\lambda \in \sigma(A)$ then $\overline{\lambda} \in \sigma(A^*)$.*

To prove this, it is sufficient to show that the resolvent sets have the same property. Indeed, if $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(A)$, then there exists the resolvent $R_\lambda = (A - \lambda I)^{-1}$ of the operator A which is defined and bounded on the whole space. But then there exists the adjoint operator $(R_\lambda)^*$, and applying Part 5 of Theorem 2.84, we have

$$\left((A - \lambda I)^{-1}\right)^* = (A^* - \overline{\lambda} I)^{-1}.$$

Hence, the point $\overline{\lambda}$ belongs to the resolvent set $\rho(A^*)$. □

From the proof of this lemma we obtain the following important property.

Corollary 3.43 *Let A be a closed linear operator in a Hilbert space H, with $\overline{D(A)} = H$. Then the resolvent set and the spectrum of the adjoint operator A^* are given by*

$$\rho(A^*) = \left\{\lambda : \overline{\lambda} \in \rho(A)\right\}, \quad \sigma(A^*) = \left\{\lambda : \overline{\lambda} \in \sigma(A)\right\}.$$

Moreover, we have $(R_\lambda(A))^* = R_{\overline{\lambda}}(A^*)$, that is,

$$\left((A - \lambda I)^{-1}\right)^* = (A^* - \overline{\lambda} I)^{-1}.$$

Lemma 3.44 *Let the point $\lambda = 0$ belong to the resolvent set of the operator A. If the number λ_0 is an eigenvalue of the operator A, then $1/\lambda_0$ is an eigenvalue of the operator A^{-1}. Moreover, the respective eigenvectors of the operators A and A^{-1} coincide.*

Indeed, let λ_0 be an eigenvalue of the operator A, and let x_0 be the corresponding eigenvector:

$$Ax_0 = \lambda_0 x_0, \; x_0 \in D(A), \; x_0 \neq 0.$$

In particular, we note that $\lambda_0 \neq 0$ since 0 is in the resolvent set. Since the operator A^{-1} is defined on all elements of the space, then we can apply A^{-1} to this equality. We obtain

$$x_0 = \lambda_0 A^{-1} x_0, \; x_0 \neq 0,$$

so that $1/\lambda_0$ is an eigenvalue of the operator A^{-1}, and x_0 is the corresponding eigenvector. □

From this lemma we obtain the following important property.

Corollary 3.45 *Let A be a closed linear operator in H, and assume that there exists a point $\lambda_0 \in \rho(A)$ such that the resolvent $R_{\lambda_0} = (A - \lambda_0 I)^{-1}$ is compact. Then the spectrum $\sigma(A)$ consists entirely of eigenvalues of A and is a countable set not having finite limiting points. In addition, the resolvent $R_\mu = (A - \mu E)^{-1}$ is compact for each $\mu \in \rho(A)$.*

The important special case of unbounded operators is the self-adjoint operator. Recall that the operator A is called self-adjoint if $A = A^*$ (with the equality of the domains). Such operators have plenty of specific spectral properties.

Theorem 3.46 *If a self-adjoint operator A has the inverse operator A^{-1}, then the range $R(A)$ of the operator A is dense in H and A^{-1} is a self-adjoint operator on H.*

Indeed, if $R(A)$ was not dense in H, then there would exist $z \neq 0$, for which $\langle Ax, z \rangle = 0$ for all $x \in D(A)$. That is, $\langle Ax, z \rangle = \langle x, 0 \rangle$ for all $x \in D(A)$. This means that $z \in D(A^*)$ and $A^* z = 0$. But the operator A is self-adjoint. This means that we have found $z \neq 0$, for which $Az = 0$. But this contradicts the existence of the inverse operator A^{-1}. □

As usual, we denote by $R(A)$ the range of the operator A.

Corollary 3.47 *If λ is not an eigenvalue of a self-adjoint operator A, then the range of $A - \lambda I$ is dense in H.*

Corollary 3.48 *A number λ is an eigenvalue of a self-adjoint operator A if and only if the range of $A - \lambda I$ is not dense in H: $\overline{R(A - \lambda I)} \neq H$.*

Corollary 3.49 *For an arbitrary self-adjoint operator on H, the Hilbert space H can be decomposed into the direct sum of the subspaces:*

$$H = R(A - \lambda I) \oplus \ker(A - \overline{\lambda} E).$$

Theorem 3.50 *Let $\lambda = \xi + i\eta$, where $\xi = \mathrm{Re}(\lambda)$ and $\eta = \mathrm{Im}(\lambda)$. If $\eta \neq 0$, then λ belongs to the resolvent set $\rho(A)$ of the self-adjoint operator A.*

Indeed, for arbitrary $x \in D(A)$ we have

$$\langle (A-\lambda I)x, (A-\lambda I)x\rangle = \langle Ax, Ax\rangle - 2\xi\langle Ax, a\rangle + |\lambda|^2\langle x, x\rangle$$

$$= \langle Ax, Ax\rangle - 2\xi\langle Ax, x\rangle + \xi^2\langle x, x\rangle + \eta^2\langle x, x\rangle = \langle (A-\xi I)x, (A-\xi I)x\rangle + \eta^2\langle x, x\rangle.$$

Hence we have

$$\|(A-\lambda I)x\|^2 \geq \eta^2\|x\|^2. \tag{3.50}$$

If we now assume that $(A-\lambda I)x = 0$, then from (3.50) we get that $x = 0$, that is, the number $\lambda = \xi + i\eta$ for $\eta \neq 0$ is not an eigenvalue.

Therefore, the operator $(A-\lambda I)^{-1}$ exists. By inequality (3.50) we have

$$\|(A-\lambda I)^{-1}x\| \leq \frac{1}{|\eta|}\|x\|$$

for all vectors $x \in D(A)$. Since the domain $D(A)$ is dense in H, then by Theorem 2.12 the operator $(A-\lambda I)^{-1}$ can be extended onto the whole space with preservation of the value of its norm. But, since the operator A is closed, the operator $(A-\lambda I)^{-1}$ is also closed. Hence, the range of the operator $(A-\lambda I)^{-1}$ is closed and coincides with the whole space H. That is, the number $\lambda = \xi + i\eta$ for $\eta \neq 0$ belongs to the resolvent set. $\square$

Corollary 3.51 *All eigenvalues of a self-adjoint operator are real.*

An operator A is called a ***positive operator*** if

$$\langle Ax, x\rangle \geq 0 \text{ for all } x \in D(A).$$

The operator A is called ***positive definite*** if there exists a number $\alpha > 0$ such that

$$\langle Ax, x\rangle \geq \alpha\langle x, x\rangle \text{ for all } x \in D(A).$$

Corollary 3.52 *If A is a positive definite self-adjoint operator, then A^{-1} exists, is defined on the whole space, and is bounded. That is, the number $\lambda = 0$ belongs to the resolvent set of the operator A.*

Indeed, from the definition of the positive definiteness by the Cauchy-Schwartz inequality it follows that

$$\|x\| \leq \frac{1}{\alpha}\|Ax\| \tag{3.51}$$

for all $x \in D(A)$. Hence, the number $\lambda = 0$ is not an eigenvalue of the operator A and A^{-1} exists. But, then by Corollary 3.49, $H = R(A)$, so that A^{-1} is defined on the whole space H. The boundedness A^{-1} follows from (3.51). $\square$

One of the important theorems of the spectral theory of self-adjoint operators is the analogue of the Hilbert-Schmidt theorem 3.32 for unbounded self-adjoint operators.

Theorem 3.53 (Hilbert-Schmidt theorem for unbounded self-adjoint operators). *Let A be an unbounded self-adjoint operator in a Hilbert space H with a compact resolvent. Then the system of its normalised eigenvectors forms an orthonormal basis of the space H. That is, any element $\varphi \in H$ can be decomposed into a converging Fourier series with respect to the system x_j of the normalised eigenvectors of the operator A:*

$$\varphi = \sum_{j=1}^{\infty} \langle \varphi, x_j \rangle x_j, \tag{3.52}$$

and the Parseval's identity

$$\|\varphi\|^2 = \sum_{j=1}^{\infty} \left| \langle \varphi, x_j \rangle \right|^2$$

holds.

The proof of this theorem follows from Corollary 3.33 and from Lemma 3.44. □

This theorem allows one to construct by the spectral method a solution of the operator equation

$$A\varphi = f, \ \varphi \in D(A), \ f \in H, \tag{3.53}$$

for a self-adjoint unbounded operator A.

Indeed, by Theorem 3.53 the solution φ and the right-hand side f of Eq. (3.53) can be written in the form of the converging series

$$\varphi = \sum_{j=1}^{\infty} \langle \varphi, x_j \rangle x_j, \quad f = \sum_{j=1}^{\infty} \langle f, x_j \rangle x_j.$$

Therefore, φ is a limit of the sequence $\varphi_n = \sum_{j=1}^{n} \langle \varphi, x_j \rangle x_j$ as $n \to \infty$. Here, φ_n is a finite sum. Hence,

$$A\varphi_n = \sum_{j=1}^{n} \lambda_j \langle \varphi, x_j \rangle x_j.$$

Since the operator A is closed, for any vector $\varphi \in D(A)$ the sequence $A\varphi_n$ converges to f, that is,

$$A\varphi = \sum_{j=1}^{\infty} \lambda_j \langle \varphi, x_j \rangle x_j = \sum_{j=1}^{\infty} \langle f, x_j \rangle x_j.$$

Since by Theorem 3.53 the system of the eigenfunctions x_j is a basis in H, from this we obtain

$$\langle \varphi, x_j \rangle = \frac{\langle f, x_j \rangle}{\lambda_j}.$$

Therefore, for any $f \in H$ the solution of Eq. (3.53) has the form

$$\varphi = \sum_{j=1}^{\infty} \frac{\langle f, x_j \rangle}{\lambda_j} x_j. \ \square$$

Simultaneously with solving Eq. (3.53) we have justified the following criterion.

Theorem 3.54 *Let $\varphi \in H$ be an arbitrary vector, and let $\langle \varphi, x_j \rangle$ be its Fourier coefficient in the expansion* (3.52) *with respect to the system of eigenvectors x_j of a self-adjoint operator A with a compact resolvent. Then, in order that $\varphi \in D(A)$ it is necessary and sufficient that the following series converges*

$$\sum_{j=1}^{\infty} |\lambda_j|^2 \, |\langle \varphi, x_j \rangle|^2 < \infty.$$

If this condition holds, then

$$A\varphi = \sum_{j=1}^{\infty} \lambda_j \langle \varphi, x_j \rangle x_j.$$

3.7 Some ordinary differential operators and their spectrum

As in the finite-dimensional case (see Section 3.1), in addition to eigenvectors the operators in infinite-dimensional spaces can also have associated vectors.

A vector $x \in H$ is called a ***root vector*** of an operator $A : H \to H$ corresponding to an eigenvalue λ_0 if $x \in D(A)$ and

$$(A - \lambda I)^{m+1} x = 0 \tag{3.54}$$

for some integer number $m > 0$. It is obvious that *any eigenvector of the operator is its root vector*. In addition to the eigenvectors, the associated vectors are also the root vectors.

Here, a vector x is called an ***associated vector*** of the operator A corresponding to the eigenvalue λ if for each integer number $m > 0$ the following relation holds

$$(A - \lambda I)^m x \neq 0, \;\; (A - \lambda I)^{m+1} x = 0. \tag{3.55}$$

The number m is called the ***order of the associated vector*** x.

It is evident that the associated vectors are defined not uniquely. Indeed, if $x^{(0)}$ is an eigenvector of the operator A corresponding to an eigenvalue λ, and $x^{(1)}$ is a corresponding associated vector of the operator A, then the vector $x^{(1)} + Cx^{(0)}$ is also an associated vector of the operator A for any choice of the constant C.

The eigen- and associated vectors are called root vectors of the operator A. The linear space spanned by all eigenvectors corresponding to a given eigenvalue is called an ***eigenspace*** of the linear operator A. The linear space spanned by all eigen- and associated vectors of the operator A corresponding to the same eigenvalue is called a ***root space***.

In the root space of the operator A corresponding to the same eigenvalue, the eigen- and associated vectors can be organised in a ***chain***. Let $x^{(0)}$ be an eigenvector of the operator A corresponding to the eigenvalue λ. A vector $x^{(1)}$ is called an ***associated vector of the first order*** it it satisfies the equation

$$(A - \lambda I) x^{(1)} = x^{(0)}. \tag{3.56}$$

Analogously to (3.56), other ***associated vectors of higher order*** are defined by

$$\begin{gathered}(A-\lambda I)x^{(2)} = x^{(1)},\\ \cdots\cdots\cdots\\ (A-\lambda I)x^{(m)} = x^{(m-1)}.\end{gathered} \tag{3.57}$$

Lemma 3.55 *Elements of a chain of eigen- and associated vectors are linearly independent.*

Indeed, let a chain of eigen- and associated vectors be defined by formulae (3.56)-(3.57). Consider their linear combination

$$C_0x^{(0)}+C_1x^{(1)}+\ldots+C_mx^{(m)}=0.$$

Applying the operator $(A-\lambda I)^m$ to this equality, we obtain

$$C_m(A-\lambda I)^m x^{(m)} = C_m x^{(0)} = 0.$$

Then, since $x^{(0)} \neq 0$, we have that $C_m = 0$. Continuing this procedure further, we obtain that $C_k = 0$ for all $k = 1,2,\ldots,m$, which proves the linear independence of the elements of the chain of the eigen- and associated vectors. $\square$.

We now consider an example of an ordinary differential operator of the first order having an associated vector.

Example 3.56 In $L^2(0,1)$, consider the operator given by the differential expression

$$L_\alpha u(x) = i\frac{d}{dx}u(x),\ 0<x<1,$$

on the domain

$$D(L_\alpha) = \left\{ u \in L_1^2(0,1) : u(0)-u(1) = \alpha \int\limits_0^1 u(t)dt \right\}.$$

Here $\alpha \in \mathbb{C}$ is a fixed number. Note that unlike in the previous examples, the domain of the operator is given by the nonlocal condition

$$u(0)-u(1) = \alpha \int_0^1 u(t)dt, \tag{3.58}$$

in which values of the function u at the interior points of the interval $(0,1)$ take part. Conditions of such type are *no longer the classical boundary conditions*.

In the special case of $\alpha = 0$ this operator has been considered in Example 2.91. There, we have shown that this operator L_0 is self-adjoint. For $\alpha \neq 0$ the structure of the adjoint operator L_α^* is more complicated. Such an example with the presence of integral conditions in the domain of a differential operator of the second order has been considered in Example 2.94.

Not dwelling on detail, we note that the operator L_α has a compact resolvent. Therefore, according to Corollary 3.45, the spectrum of the operator L_α can consist only of eigenvalues.

We now look at the eigenvalues of the operator L_α. Eigenvectors of the operator L_α are all nonzero vectors $u \in D(L_\alpha)$, for which $L_\alpha u - \lambda u = 0$, that is, all nonzero solutions of the differential equation

$$iu(x) - \lambda u'(x) = 0,\ 0 < x < 1,$$

for which (3.58) holds.

All solutions of this differential equation have the form $u(x) = Ce^{-i\lambda x}$, $0 \le x \le 1$, where C is an arbitrary constant. Using condition (3.58), we get the equation

$$C\left(e^{i\lambda} - 1\right)\left(\frac{i\alpha}{\lambda} + 1\right) = 0.$$

The eigenfunction is not an identical zero. Therefore, $C \neq 0$. Then we obtain the characteristic determinant

$$\triangle(\lambda) \equiv \frac{e^{i\lambda} - 1}{\lambda}(i\alpha + \lambda) = 0.$$

Solutions of this equation are numbers

$$\lambda_0 = -i\alpha \ \text{ and } \lambda_k = 2k\pi,\ \ k = \pm 1, \pm 2, \ldots.$$

Thus, if $\alpha \neq \pm 2k\pi i$ for all positive integers k, the values λ_k are eigenvalues of the operator L_α, and

$$u_k(x) = e^{-i\lambda_k x},\ \ k = 0, \pm 1, \pm 2, \ldots$$

will be the eigenfunctions corresponding to them.

And if $\alpha = 2m\pi i$ for some integer m, then the eigenvalues λ_0 and λ_m coincide. This is caused by the fact that $\lambda = \lambda_0$ is a multiple root of the characteristic determinant

$$\triangle(\lambda) \equiv \left(1 - \frac{\lambda}{\lambda_0}\right)^2 \prod_{k \in \mathbb{Z},\ k \neq m} \left(1 - \frac{\lambda}{\lambda_k}\right).$$

To this double eigenvalue $\lambda_0 = -i\alpha$ there corresponds only one eigenfunction $u_0(x) = e^{-\alpha x}$. Let us construct an associated function as a solution of the operator equation $L_\alpha u - \lambda_0 u = u_0$, that is, as a solution of the differential equation

$$iu'(x) + i\alpha u(x) = \ e^{-\alpha x},\ 0 < x < 1, \tag{3.59}$$

satisfying condition (3.58).

A general solution of Eq. (3.59) has the form

$$u(x) = -ixe^{-\alpha x} + Ce^{-\alpha x},$$

where C is an arbitrary constant. Then, as is easy to verify, conditions (3.58) hold for any C.

Thus, in the case when $\alpha = 2m\pi i$ for some integer m, the operator L_α has the eigenvalues and eigenvectors

$$\lambda_k = 2k\pi,\ u_k(x) = e^{-i\lambda_k x},\ \ k = \pm 1, \pm 2, \ldots.$$

Here all the eigenvalues, except λ_m, are simple. And the eigenvalue λ_m is double, with one eigenvector $u_m(x) = e^{-\alpha x}$ and one associated vector

$$u_{m1}(x) = -ixe^{-\alpha x} + Ce^{-\alpha x}$$

corresponding to it.

As we have noted earlier, an associated vector is defined not uniquely. The constant C can be chosen in a way most convenient for further applications. □

As we have indicated earlier, the associated vector can arise in cases in which the operator has multiple eigenvalues. However, the associated vectors for the multiple eigenvalues do not necessarily arise.

Example 3.57 In $L^2(0,1)$, consider the operator

$$L_\pi u(x) = -\frac{d^2}{dx^2}u(x),\ 0 < x < 1,$$

given on the domain

$$D(L_\pi) = \left\{u \in L_2^2(0,1):\ u'(0) = u'(1),\ u(0) = u(1)\right\}.$$

The boundary conditions

$$u'(0) = u'(1),\ u(0) = u(1) \tag{3.60}$$

defining the domain of the operator L_π are called *periodic boundary conditions*.

It is easy to see that $\lambda_0 = 0$ is an eigenvalue of the operator L_π, and $u_0(x) = 1$ is a corresponding eigenvector. Therefore, the operator L_π^{-1} does not exist. This means that to show the discreteness of the spectrum of the operator L_π it is necessary to show the compactness of its resolvent for another $\lambda \neq \lambda_0$. Not going into too-detailed calculations, we give the explicit form of the resolvent for $\lambda = -1$:

$$(L_\pi + I)^{-1} f(x) = \int_0^x \frac{e^{x-t} + e^{1+t-x}}{2(e-1)} f(t)dt + \int_x^1 \frac{e^{1+x-t} + e^{t-x}}{2(e-1)} f(t)dt. \tag{3.61}$$

From this explicit representation of the resolvent it is easy to see its compactness in $L^2(0,1)$. Therefore, according to Corollary 3.45, the spectrum of the operator L_π can consist only of the eigenvalues.

Let us now look for the eigenvalues of the operator L_π. Eigenvectors of the operator L_π are all nonzero vectors $u \in D(L_\pi)$, for which $L_\pi u - \lambda u = 0$, that is, all nonzero solutions of the differential equation $-u''(x) - \lambda u(x) = 0$, $0 < x < 1$, for which (3.60) holds.

For $\lambda \neq 0$ all solutions of this differential equation have the form

$$u(x) = C_1 \cos\left(\sqrt{\lambda} x\right) + C_2 \frac{1}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda} x\right),$$

where C_1 and C_2 are arbitrary constants.

Using the boundary conditions (3.60) for defining C_1 and C_2, we obtain the system of linear equations

$$\begin{cases} C_1\left(1 - \cos\sqrt{\lambda}\right) - C_2 \frac{1}{\sqrt{\lambda}} \sin\sqrt{\lambda} = 0, \\ C_1 \sqrt{\lambda} \sin\sqrt{\lambda} + C_2\left(1 - \cos\sqrt{\lambda}\right) = 0. \end{cases} \tag{3.62}$$

The characteristic determinant of the spectral problem will be the determinant of this system. Calculating it, we have

$$\triangle(\lambda) = 1 - \cos\sqrt{\lambda}.$$

Applying the trigonometric formulae, we rewrite this characteristic determinant in the form

$$\triangle(\lambda) = 2\left(\sin\frac{\sqrt{\lambda}}{2}\right)^2.$$

From this representation it is obvious that all roots $\lambda_k = (2k\pi)^2$, $k \in \mathbb{N}$, of the characteristic determinant will be double.

For these values $\lambda = \lambda_k$ the system (3.62) becomes an identity. Therefore, the coefficients C_1 and C_2 are not defined and can be chosen arbitrarily. Thus all functions of the form

$$C_1 \cos\left(\sqrt{\lambda_k} x\right) + C_2 \frac{1}{\sqrt{\lambda_k}} \sin\left(\sqrt{\lambda_k} x\right)$$

form an eigenspace corresponding to the eigenvalue λ_k. Since this family is two-parametric, one can choose two linearly independent elements in it.

Summarising, the operator L_π has one simple eigenvalue $\lambda_0 = 0$, to which there corresponds one eigenvector, and double eigenvalues $\lambda_k = (2k\pi)^2$, $k \in \mathbb{N}$, to each of which there correspond two eigenvectors

$$u_0(x) = 1; \;\; u_{k1}(x) = \cos(2k\pi x), \;\; u_{k2}(x) = \sin(2k\pi x). \tag{3.63}$$

As can be easily seen from the explicit representation (3.61) of the resolvent $(L_\pi + I)^{-1}$, it is a self-adjoint operator. Therefore, the operator $L_\pi + I$ is also self-adjoint. Consequently, the operator L_π is also self-adjoint. By Theorem 3.53 (the Hilbert-Schmidt theorem for unbounded self-adjoint operators) the system (3.63) of the eigenfunctions of the operator L_π forms a basis in $L^2(0,1)$. We also know this fact from the general university course of Analysis: this system is a classical trigonometric system, and the expansion into the series with respect to this system is the usual trigonometric Fourier series.

Thus, we have shown that the operator L_π has only double eigenvalues (except zero), to each of which there correspond two eigenvectors, and the corresponding system of the eigenfunctions is the classical trigonometric system. $\square$

The fact that the operator L_α has an associated vector (see Example 3.56) is caused not by the fact that one eigenvalue is multiple but is connected with the fact that this operator is not self-adjoint.

Lemma 3.58 *A self-adjoint operator cannot have associated vectors.*

Indeed, assume that $(A-\lambda_0 I)^2 x = 0$ holds for some $x \in D(A)$. Denote $x_0 := (A-\lambda_0 I)x$. Then $x_0 \in D(A)$ and $x_0 \in R(A-\lambda_0 I)$. Here, $(A-\lambda_0 I)x_0 = 0$, that is, x_0 is an eigenvector of the self-adjoint operator A, and λ_0 is its eigenvalue.

Therefore, by Corollary 3.49, $x_0 \perp R(A-\overline{\lambda_0} I)$. Since the eigenvalues of a self-adjoint operator are real numbers, we have $\lambda_0 = \overline{\lambda_0}$. Therefore, $x_0 \perp R(A-\lambda_0 I)$. But initially we have had that $x_0 \in R(A-\lambda_0 I)$. This is possible only for $x_0 = 0$.

Hence, we have $(A-\lambda_0 I)x = 0$, that is, x is an eigenvector of the operator A. $\square$

Let us give one more example of a differential operator having associated vectors.

Example 3.59 In $L^2(0,1)$, consider the operator

$$L_{SI}u(x) = -\frac{d^2}{dx^2}u(x),\ 0 < x < 1,$$

with the domain

$$D(L_{SI}) = \left\{u \in L_2^2(0,1):\ u'(0) = u'(1),\ u(0) = 0\right\}.$$

The boundary conditions

$$u'(0) = u'(1), \quad u(0) = 0, \tag{3.64}$$

defining the domain of the operator L_{SI} are called the ***Samarskii-Ionkin boundary conditions***.

The problem with boundary conditions of such type was first proposed by A. A. Samarskii in the 1970s in connection with the study of processes occurring in a parametrically unstable plasma. That problem was mathematically treated by N. I. Ionkin [54]. Therefore, the boundary conditions of the type (3.64) are called the conditions of the Samarskii-Ionkin type. The peculiarity of this problem is that it has *an infinite number of associated vectors*.

It is easy to see that $\lambda_0 = 0$ is an eigenvalue of the operator L_{SI}, and $u_0(x) = x$ is a corresponding eigenvector. Therefore, the operator L_{SI}^{-1} does not exist. This means that to show the discreteness of the spectrum of the operator L_{SI}, one could show the compactness of its resolvent for another $\lambda \neq \lambda_0$. Not dwelling on too-detailed calculations, we give an explicit form of the resolvent for $\lambda = -1$:

$$(L_{SI}+I)^{-1} f(x) = \int_0^x \frac{e^{x-t}+e^{1+t-x}}{2(e-1)} f(t)dt + \int_x^1 \frac{e^{1+x-t}+e^{t-x}}{2(e-1)} f(t)dt$$

$$+ \int_0^1 \frac{e^{1+x-t}+e^{x+t}}{2(e-1)^2} f(t)dt - \int_0^1 \frac{e^{2-x-t}+e^{1+t-x}}{2(e-1)^2} f(t)dt.$$

From this explicit representation of the resolvent it is easy to see its compactness in $L^2(0,1)$. Therefore, according to Corollary 3.45, the spectrum of the operator L_{SI} can consist only of the eigenvalues.

Let us look for the eigenvalues of the operator L_{SI}. Eigenvectors of the operator L_{SI} are all nonzero vectors $u \in D(L_{SI})$, for which $L_{SI}u - \lambda u = 0$, that is, all nonzero solutions of the differential equation $-u''(x) - \lambda u(x) = 0,\ 0 < x < 1$, for which (3.64) holds.

For $\lambda \neq 0$ all solutions of this differential equation have the form

$$u(x) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \frac{1}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda}x\right),$$

where C_1 and C_2 are arbitrary constants.

Using the boundary conditions (3.64) for defining C_1 and C_2 we obtain the system of linear equations

$$\begin{cases} C_1 = 0, \\ C_1\sqrt{\lambda}\sin\sqrt{\lambda} + C_2\left(1 - \cos\sqrt{\lambda}\right) = 0. \end{cases} \tag{3.65}$$

The characteristic determinant of the spectral problem will be the determinant of this system: $\triangle(\lambda) = 1 - \cos\sqrt{\lambda}$. Transforming it as in Example 3.57, we get

$$\triangle(\lambda) = 2\left(\sin\frac{\sqrt{\lambda}}{2}\right)^2.$$

All roots of this determinant, $\lambda_k = (2k\pi)^2$, $k \in \mathbb{N}$, are double. But from the system (3.65) we have $C_1 = 0$. Therefore, the operator L_{SI} cannot have two linearly independent eigenfunctions.

To each eigenvalue $\lambda_k = (2k\pi)^2$, $k \in \mathbb{N}$, there corresponds one eigenfunction

$$u_{k0}(x) = \sin(2k\pi x).$$

We now look for associated functions as solutions of the operator equations $L_{SI}u - \lambda_k u = u_{k0}$, that is, the solutions of the differential equations

$$-u''(x) - (2k\pi)^2 u(x) = \sin(2k\pi x),\ 0 < x < 1, \tag{3.66}$$

satisfying condition (3.64).

A general solution of Eq. (3.66) has the form

$$u(x) = \frac{x}{4k\pi}\cos(2k\pi x) + C_1\cos(2k\pi x) + C_2\sin(2k\pi x),$$

where C_1 and C_2 are arbitrary constants. Using the boundary conditions (3.64) for defining C_1 and C_2, we obtain the system of linear equations

$$\begin{cases} C_1 = 0, \\ C_2 - C_2 = 0. \end{cases} \tag{3.67}$$

Hence $C_1 = 0$, and C_2 is an arbitrary constant. Therefore,

$$u_{k1}(x) = \frac{x}{4k\pi}\cos(2k\pi x) + C_k \sin(2k\pi x),$$

where C_k are arbitrary constants, will be the associated functions of the operator L_{SI} corresponding to the eigenvalues $\lambda_k = (2k\pi)^2$ and to the eigenfunctions $u_{k0}(x) = \sin(2k\pi x)$. $\square$

The considered Examples 3.56 - 3.59 *demonstrate that the spectral properties of the differential operators can be rather complicated: some or all eigenvalues can be multiple; the root subspaces corresponding to the multiple eigenvalues can consist only of the eigenvectors or the associated vectors can participate; the general number of the associated vectors can be finite or infinite.* Such a variety does not give an opportunity to make any conclusions and indicate the spectral properties for the case of general operators. Therefore, there is a necessity of choosing a narrower class of the differential operators. One of such classes is formed by the so-called Sturm-Liouville operators.

3.8 Spectral theory of the Sturm-Liouville operator

One of the simplest ordinary differential operators which is frequently encountered in applications to many areas of different sciences is the ***Sturm-Liouville operator***. The Sturm-Liouville operator is an operator corresponding to the boundary value problem for an ordinary differential equations of the second order

$$Lu \equiv -u''(x) + q(x)u(x) = f(x),\ 0 < x < 1, \tag{3.68}$$

with the boundary conditions

$$\begin{cases} a_1 u'(0) + a_0 u(0) = 0, \\ d_1 u'(1) + d_0 u(1) = 0, \end{cases} \tag{3.69}$$

where the coefficients a_1, a_0, d_1, d_0 of the boundary condition are fixed real numbers ($a_1^2 + a_0^2 \neq 0$, $d_1^2 + d_0^2 \neq 0$); the coefficient $q \in C[0,1]$ in the equation (sometimes it is called a *potential*) is a real-valued function. As we will show later, this Sturm-Liouville problem is a self-adjoint problem. Since all the coefficients (in the equation and in the boundary conditions) are real, *in this section all the considerations are carried out over the field of real numbers*. In particular, we will also use the real inner product.

A principal difference between this boundary value problem and more general problems is that in the boundary condition (3.69) *one condition is set at the left endpoint of the interval, and the second condition is set at the right endpoint*. The conditions of such kind are called ***separated boundary conditions***. For comparison, the periodic boundary conditions (3.60) of the operator L_π, considered in Example 3.57 are nonlocal (in one boundary condition the values of the solution simultaneously enter both at the left endpoint and at the right endpoint of the interval) and do not belong to the class of separated boundary conditions.

Note that the more general equations of the second order

$$Lu \equiv -U''(t) + P(t)U'(t) + Q(t)U(t) = F(t), \quad a \le t \le b,$$

can be reduced to (3.60) with the help of the so-called *Liouville transformation*:

$$x = \frac{t-a}{b-a}, \; U(t) = u(x)\exp\left\{\frac{1}{2}\int_a^t P(s)ds\right\}.$$

Let us denote by L the closure in $L^2(0,1)$ of the linear operator

$$Ly = -u''(x) + q(x)u(x)$$

given on the linear space of functions $u \in C^2[0,1]$ satisfying the boundary conditions (3.69). *The spectrum and eigenfunctions of the boundary Sturm-Liouville problem* (3.68)-(3.69) *will be the spectrum and the eigenvectors of the operator L.*

To justify the discreteness of the spectrum of the Sturm-Liouville operator L we will show that it has a compact resolvent. We will also show that the solution of the problem can be constructed by means of Green's function.

First we give a formal definition and its corollaries, and then we show how Green's function is constructed.

The ***Green's function*** of the Sturm-Liouville boundary value problem (3.68)-(3.69) is a function $G = G(x,t)$ having the following properties:

1. The function $G(x,t)$ is continuous with respect to x and to t for all $x,t \in [0,1]$;

2. The first derivative with respect to x has a discontinuity at one point: for $x = t$. There, the value of the jump is equal to

$$G'_x(t-0;t) - G'_x(t+0;t) = 1;$$

3. For $x \neq t$ the function $G(x,t)$ is twice continuously differentiable with respect to the variable x, and satisfies the equation

$$L_x G(x,t) \equiv -G''_{xx}(x,t) + q(x)G(x,t) = 0, \; x \neq t;$$

4. For all $t \in (0,1)$ the function $G(x,t)$ with respect to the variable x satisfies the boundary conditions (3.69):

$$a_1 G'_x(0,t) + a_0 G(0,t) = 0, \;\; d_1 G'_x(1,t) + d_0 G(1,t) = 0.$$

Let us show that if such function $G(x,t)$ exists, then the solution of the Sturm-Liouville boundary value problem (3.68)-(3.69) is represented in the integral form

$$u(x) = \int_0^1 G(x,t)f(t)dt. \tag{3.70}$$

Indeed, the fulfilment of the boundary conditions (3.69) follows from the fourth property of Green's function. Further, by differentiating formula (3.69), we get

$$u'(x) = \int_0^x G'_x(x,t)f(t)dt + \int_x^1 G'_x(x,t)f(t)dt,$$

$$u''(x) = \int_0^x G''_{xx}(x,t)f(t)dt + G'_x(x,x-0)f(x)$$
$$+\int_x^1 G''_{xx}(x,t)f(t)dt - G'_x(x,x+0)f(x).$$

Now using the second and third properties of Green's function, it is easy to make sure that function (3.70) satisfies Eq. (3.68). Thus, (3.70) is indeed the solution of the boundary value Sturm-Liouville problem (3.68)-(3.69).

Furthermore, by the first property of Green's function, the function (3.70) will be continuous for any $f \in L^2(0,1)$. Therefore, from Eq. (3.68) we obtain that $u'' \in L^2(0,1)$, that is, the constructed solution belongs to the Sobolev space $L_2^2(0,1)$.

Assume that $\lambda = 0$ is not an eigenvalue of the operator. That is, the problem for the *homogeneous equation*

$$Ly = -u''(x) + q(x)u(x) = 0 \tag{3.71}$$

with the boundary conditions (3.69) has only a zero solution. Then Green's function is unique.

Indeed, if we suppose that there exist two different Green's functions $G_1(x,t)$ and $G_2(x,t)$, then we have two solutions to the boundary value problem (3.68)-(3.69):

$$u_1(x) = \int_0^1 G_1(x,t)f(t)dt, \;\; u_2(x) = \int_0^1 G_2(x,t)f(t)dt.$$

Then their difference

$$u_1(x) - u_2(x) = \int_0^1 \Big\{G_1(x,t) - G_2(x,t)\Big\}f(t)dt$$

is the solution of the homogeneous problem (3.69), (3.71). Therefore, $u_1(x) - u_2(x) = 0$. Consequently,

$$\int_0^1 \Big\{G_1(x,t) - G_2(x,t)\Big\}f(t)dt = 0$$

for all $f \in L^2(0,1)$. Hence $G_1(x,t) = G_2(x,t)$, which proves the uniqueness of Green's function.

Let us now show a method for constructing Green's function.

Denote by $u_0(x)$ and $u_1(x)$ the solutions of the homogeneous equation (3.69) satisfying, respectively, the initial conditions

$$u_0(0) = a_1, \; u_0'(0) = -a_0;$$

$$u_1(1) = d_1,\ u_1'(1) = -d_0.$$

The existence and uniqueness of such functions follow from the well-posedness of the Cauchy problem for a linear differential equation. It is obvious that the function $u_0(x)$ satisfies the first boundary condition in (3.69) (the condition at the point $x = 0$), while the function $u_1(x)$ satisfies the second boundary condition in (3.69) (the condition at the point $x = 1$).

Moreover, these functions are linearly independent. Indeed, if there was linear dependence, that is, $u_0(x) = Cu_1(x)$, then $u_0(x)$ would satisfy the boundary condition from (3.69). That is, the homogeneous problem (3.69), (3.71) would have the zero solution, which contradicts the assumption.

Therefore, the ***Wronskian*** of these two functions is different from zero:

$$W(x) = u_0(x)u_1'(x) - u_0'(x)u_1(x) \neq 0.$$

Moreover, since these functions are the solutions of the homogeneous equation (3.71), we have

$$W'(x) = u_0(x)u_1''(x) - u_0''(x)u_1(x) = u_0(x)q(x)u_1(x) - q(x)u_0(x)u_1(x) = 0.$$

That is, *the Wronskian does not depend on x*: $W(x) = W$.

Let us introduce the function

$$G(x,t) = \begin{cases} -\frac{1}{W}u_0(t)u_1(x),\ t \leq x, \\ \\ -\frac{1}{W}u_0(x)u_1(t),\ x \leq t. \end{cases} \tag{3.72}$$

It can be readily verified that the function (3.72) satisfies all the properties of Green's function. Thus, the following theorem is proved.

Theorem 3.60 *If the homogeneous problem* (3.69), (3.71) *has only a zero solution, then the inhomogeneous Sturm-Liouville boundary value problem* (3.68)-(3.69) *is uniquely solvable for any right-hand side* $f \in L^2(0,1)$. *The solution of the problem belongs to the class* $u \in L_2^2(0,1)$ *and is represented by Green's function by formula* (3.70). *Moreover, Green's function has the form* (3.72).

From the representation of solution by formula (3.70) it follows that if $\lambda = 0$ is not an eigenvalue of the Sturm-Liouville operator L, then the inverse operator L^{-1} exists, is defined on the whole space $L^2(0,1)$, and is bounded. This inverse operator is an integral operator in $L^2(0,1)$,

$$L^{-1}f(x) = \int_0^1 G(x,t)f(t)dt,$$

with the continuous kernel $G(x,t)$. Therefore, it is a Hilbert-Schmidt operator and, as a consequence, it is a compact operator in $L^2(0,1)$ (see Section 2.16).

As it is easy to see from (3.72), Green's function is a symmetric function: $G(x,t) = G(t,x)$. Consequently, the operator L^{-1} is self-adjoint in $L^2(0,1)$.

Therefore, the Sturm-Liouville operator L is also self-adjoint and all of its spectral properties are described by the Hilbert-Schmidt theorem 3.53 for unbounded self-adjoint operators, and also by Corollary 3.45 and Corollary 3.51. Let us formulate all these properties in the form of one theorem.

Theorem 3.61 *The Sturm-Liouville operator L is a self-adjoint linear operator in* $L^2(0,1)$. *The spectrum* $\sigma(L)$ *consists entirely of eigenvalues of L and is a countable set not having any finite limiting points. All eigenvalues of L are real. The eigenspaces corresponding to these eigenvalues consist only of eigenvectors (no associated vectors). The system of normalised eigenvectors forms an orthonormal basis of* $L^2(0,1)$. *That is, any element* $\varphi \in L^2(0,1)$ *can be decomposed into the converging Fourier series with respect to the system* $u_j(x)$ *of the normalised eigenvectors of the operator L:*

$$\varphi(x) = \sum_{j=1}^{\infty} \langle \varphi, u_j \rangle u_j(x).$$

Historically, in 1836, C. Sturm first formulated and investigated the spectral problem

$$-(p(x)u'(x))' + q(x)u(x) = \lambda u(x); \;\; u(0) = u(\ell) = 0,$$

which occurs when investigating the heat diffusion in an inhomogeneous rod. He constructed an analogue of the behavior of nonzero solutions to this problem (eigenfunctions) with classical trigonometric functions. The main attention was paid to the description of oscillatory properties of the eigenfunctions (number of zeros, their intermittency) and to the investigation of influence of coefficients of the equation and constants from boundary conditions on the placement of the eigenvalues. Almost immediately, J. Liouville, who was an expert in the theory expansions of functions into trigonometric series, joined these investigations. He considered questions of the expansion with respect to systems of the form $\{\cos(\rho_n x)\}$, $\{\sin(\rho_n x)\}$, where ρ_n are roots of some transcendent equation.

The intersection of their scientific interests led to the development of the spectral theory of the differential operator L, nowadays called the Sturm-Liouville operator. Significant results in this direction were obtained in works of H. Schwarz, E. Picard and H. Poincare. Apparently, the spectral theory of the Sturm-Liouville operator acquired a final form after the works of V. A. Steklov during 1896–1912.

Although we have formulated the main spectral properties of the Sturm-Liouville operator in Theorem 3.61 being a consequence of abstract results, let us also give brief proofs using the concrete form of the differential operator. In this way, also some of the spectral properties will be refined.

The ***spectral Sturm-Liouville problem*** is the problem of eigenfunctions and eigenvalues for the equation

$$Lu \equiv -u''(x) + q(x)u(x) = \lambda u(x), \; 0 < x < 1, \tag{3.73}$$

with the boundary conditions (3.69), that is,

$$\begin{cases} a_1 u'(0) + a_0 u(0) = 0, \\ d_1 u'(1) + d_0 u(1) = 0, \end{cases} \tag{3.69}$$

where the coefficients a_1, a_0, d_1, d_0 of the boundary condition (3.69) are fixed real numbers ($a_1^2 + a_0^2 \neq 0$, $d_1^2 + d_0^2 \neq 0$); the coefficient $q \in C[0,1]$ is a real-valued function. Solutions of Eq. (3.73) depend on a spectral parameter λ. Therefore, it is convenient to denote these solutions by $u(x,\lambda)$.

If for some λ_k the boundary value problem (3.73), (3.69) has a nontrivial solution $u(x,\lambda_k) \not\equiv 0$, then the number λ_k is called an ***eigenvalue***, and the corresponding function $u(x,\lambda_k)$ is called an ***eigenfunction*** of the boundary value problem (3.73), (3.69).

Lemma 3.62 *The eigenfunctions $u(x,\lambda_1)$ and $u(x,\lambda_2)$ that corresponds to two different eigenvalues $\lambda_1 \neq \lambda_2$ of the boundary value problem* (3.73), (3.69) *are orthogonal:*

$$\int_0^1 u(x,\lambda_1) u(x,\lambda_2) dx = 0.$$

Indeed, by integrating by parts, we easily get

$$\int_0^1 \{Lu(x,\lambda_1) \cdot u(x,\lambda_2) - u(x,\lambda_1) \cdot Lu(x,\lambda_2)\} dx = \int_0^1 \{-u'(x,\lambda_1) u(x,\lambda_2)$$

$$+ u(x,\lambda_1) u'(x,\lambda_2)\}' dx = W\{u(x,\lambda_1), u(x,\lambda_2)\}(x)\Big|_0^1, \tag{3.74}$$

where $W\{u(x,\lambda_1), u(x,\lambda_2)\}(x)$ is the Wronskian of two functions $u(x,\lambda_1)$ and $u(x,\lambda_2)$.

From the boundary conditions (3.69) we obtain that $W\{u(x,\lambda_1), u(x,\lambda_2)\}(0) = 0$ and $W\{u(x,\lambda_1), u(x,\lambda_2)\}(1) = 0$. Therefore, using $Lu(x,\lambda_j) = \lambda_j u(x,\lambda_j)$ in (3.74), we get

$$(\lambda_1 - \lambda_2) \int_0^1 u(x,\lambda_1) u(x,\lambda_2) dx = 0.$$

Thence, since $\lambda_1 \neq \lambda_2$, we complete the proof of the lemma. □

Lemma 3.63 *The eigenvalues of the boundary value problem* (3.73), (3.69) *are real.*

Indeed, since $q(x)$ and the coefficients of the boundary conditions are real, then if λ_1 is an eigenvalue of the boundary value problem (3.73), (3.69), and $u(x,\lambda_1)$ is a corresponding eigenfunction, then the number $\overline{\lambda_1}$ will be also the eigenvalue, and the function $u(x,\overline{\lambda_1}) = \overline{u(x,\lambda_1)}$ will be an eigenfunction.

Then, as in the proof of Lemma 3.62, we get

$$\left(\lambda_1 - \overline{\lambda_1}\right) \int_0^1 |u(x,\lambda_1)|^2 dx = \left(\lambda_1 - \overline{\lambda_1}\right) \int_0^1 u(x,\lambda_1) \overline{u(x,\lambda_1)} dx = 0.$$

So, if the number λ_1 is not real, then $u(x,\lambda_1) \equiv 0$. □

Since the function $q(x)$, generally speaking, is not a constant, then solutions of Eq. (3.73) cannot be written in an explicit form. The following well-known theorem is important for further investigations of the spectral properties of the Sturm-Liouville problem. We state it without proof.

Theorem 3.64 *Let $q \in C[0,1]$. Then for any parameters a_0 and a_1 there exists a unique solution $u(x,\lambda)$ of Eq.* (3.73) *satisfying the conditions*

$$u'(0) = a_0, \ u(0) = -a_1.$$

This solution $u(x,\lambda)$ for each fixed λ belongs to the class $u \in C^2[0,1]$, and for each fixed $x \in [0,1]$ it is an entire function of λ.

Although the result of the theorem on the existence of a solution is obvious enough (since this is the Cauchy problem with initial data for a linear equation), the important point is the fact that this solution is an entire function of λ, that is, a function that is holomorphic on the whole complex plane.

The next important fact of the spectral theory of the Sturm-Liouville problem is the asymptotics of eigenvalues λ_k and eigenfunctions $u(x,\lambda_k)$ as $k \to \infty$. It turns out that their asymptotics coincide with the asymptotics of eigenvalues and eigenfunctions of the problem with $q \equiv 0$.

For simplicity of the exposition we consider only the case $a_1 = d_1 = 1$. And, as is customary in this theory we denote $a_0 = -h$ and $d_0 = H$. Then the boundary conditions (3.69) can be rewritten in the form

$$u'(0) - hu(0) = 0, \ \ u'(1) + Hu(1) = 0; \ \ h, H \in \mathbb{R}. \tag{3.75}$$

We denote by $s(x,\lambda)$ the solution of the differential equation (3.73) satisfying the initial conditions

$$s'(0,\lambda) = 1, \ s(0,\lambda) = 0, \tag{3.76}$$

and by $c(x,\lambda)$ the solution of the differential equation (3.73) satisfying the initial conditions

$$c'(0,\lambda) = h, \ c(0,\lambda) = 1. \tag{3.77}$$

The function $s(x,\lambda)$ is called a ***solution of sine type***, and the function $c(x,\lambda)$ is called a ***solution of cosine type***.

Lemma 3.65 *Denote $\lambda = \mu^2$. Then the following equalities are true:*

$$s(x,\lambda) = \frac{\sin(\mu x)}{\mu} + \frac{1}{\mu}\int_0^x \sin(\mu(x-t))q(t)s(t,\lambda)dt, \tag{3.78}$$

$$c(x,\lambda) = \cos(\mu x) + h\frac{\sin(\mu x)}{\mu} + \frac{1}{\mu}\int_0^x \sin(\mu(x-t))q(t)c(t,\lambda)dt. \tag{3.79}$$

Let us prove only (3.78). Since $s(x,\lambda)$ satisfies Eq. (3.73), then

$$\int_0^x \sin(\mu(x-t))q(t)s(t,\lambda)dt = \int_0^x \sin(\mu(x-t))s''(t,\lambda)dt + \mu^2 \int_0^x \sin(\mu(x-t))s(t,\lambda)dt.$$

We apply twice integration by parts in the first integral on the right-hand side, and use the initial conditions (3.76). Then we get

$$\int_0^x \sin(\mu(x-t))q(t)s(t,\lambda)dt = -\sin(\mu x) + \mu s(x,\lambda),$$

that is, the formula (3.78). The equality (3.79) is proved in the same way. □

Note that equalities (3.78) and (3.79) are integral Volterra equalities of the second kind. Therefore, the existence of such functions $s(x,\lambda)$ and $c(x,\lambda)$ follows from the unique solvability of these equations. Moreover, by Theorem 3.64 these functions for each fixed $x \in [0,1]$ are the entire functions of λ.

Using these integral equations we get the following asymptotics of the functions $s(x,\lambda)$ and $c(x,\lambda)$ for large values of $|\lambda|$ which will be given without the proof. As before, we denote $\lambda = \mu^2$.

Lemma 3.66 *Denote $\mu = \delta + it$, where $\delta, t \in \mathbb{R}$. Then there exists a number $\mu_0 > 0$ such that for $|\mu| > \mu_0$ we have*

$$s(x,\lambda) = O\left(\frac{1}{|\mu|}e^{|t|x}\right), \quad c(x,\lambda) = O\left(e^{|t|x}\right). \tag{3.80}$$

More precisely, we have

$$s(x,\lambda) = \frac{\sin(\mu x)}{\mu} + O\left(\frac{1}{|\mu|^2}e^{|t|x}\right), \quad c(x,\lambda) = \cos(\mu x) + O\left(\frac{1}{|\mu|}e^{|t|x}\right). \tag{3.81}$$

These estimates are satisfied uniformly with respect to $x \in [0,1]$.

According to Lemma 3.63, the eigenvalues of the boundary value problem (3.73), (3.69) are real: $\mathrm{Im}(\mu) = t = 0$. Therefore, we obtain the following asymptotics for the eigenfunctions of the Sturm-Liouville problem:

$$s(x,\lambda) = \frac{\sin(\mu x)}{\mu} + O\left(\frac{1}{|\mu|^2}\right), \quad c(x,\lambda) = \cos(\mu x) + O\left(\frac{1}{|\mu|}\right). \tag{3.82}$$

It is easy to see that the function $c(x,\lambda)$ satisfies the boundary condition on the left boundary $u'(0) - hu(0) = 0$ for any λ. Therefore, the eigenvalues of the problem will be determined if we substitute the function $c(x,\lambda)$ into the boundary condition at the right boundary $u'(1) + Hu(1) = 0$. Then we obtain the equation

$$\triangle(\lambda) = c'(1,\lambda) + Hc(1,\lambda) = 0. \tag{3.83}$$

This equation is the equation for determining eigenvalues of the Sturm-Liouville problem. By analogy with the previous sections, we denote it by $\triangle(\lambda) = 0$ and call it the ***characteristic determinant***.

By differentiating equality (3.79) and using the second equality in (3.82), we get

$$c_x'(x,\lambda) = -\mu\sin(\mu x) + h\cos(\mu x) + O\left(\frac{1}{|\mu|}\right).$$

Substituting it and (3.82) into (3.83), we obtain the following equation

$$\triangle(\lambda) = -\mu\sin(\mu) + (h+H)\cos(\mu) + O\left(\frac{1}{|\mu|}\right) = 0. \tag{3.84}$$

For large values of μ this equation has solutions, which lie near solutions of the equation $\sin(\widehat{\mu}) = 0$, that is, near the points $\widehat{\mu}_k = k\pi$. This immediately implies the existence of an infinite set of eigenvalues of the Sturm-Liouville boundary value problem (3.73), (3.69).

By Theorem 3.64 the determinant $\triangle(\lambda)$ is an entire function of λ. Therefore, in (3.84) the term $O\left(\frac{1}{|\mu|}\right)$ is also an analytic function of λ. Let us differentiate $\triangle(\lambda)$:

$$\triangle'(\lambda) = -\mu\cos(\mu) - \sin(\mu) - (h+H)\sin(\mu) + O(1).$$

It is obvious that for sufficiently large k the term $\triangle'(\lambda)$ cannot vanish in the neighborhood of the points $\widehat{\mu}_k = k\pi$. This means that near each point $\widehat{\mu}_k = k\pi$ there is only one root of Eq. (3.84). Consequently, the *Sturm-Liouville boundary value problem has no multiple eigenvalues*.

For a more precise analysis we need the following well-known result.

Theorem 3.67 (Rouche theorem). *For any two complex-valued functions f and g, holomorphic inside some domain Ω with closed boundary $\partial\Omega$, if $|g(z)| < |f(z)|$ on $\partial\Omega$, then f and $f+g$ have the same number of zeros inside Ω, where each zero is counted as many times as its multiplicity.*

This theorem is usually used to simplify the problem of locating zeros, as follows. Given an analytic function, we write it as the sum of two parts, one of which is simpler and grows faster than (thus dominates) the other part. In this situation one usually says that f is the dominating part. We can then locate the zeros by looking only at the dominating part f.

In Eq. (3.84) the dominating part is $f(z) = -z\sin(z)$, and as we choose $g(z)$ as the remaining terms. It is obvious that conditions of the Rouche theorem hold if Ω is chosen as a disk of sufficiently large radius. Therefore, all roots of Eq. (3.84) starting from some number n_0 will lie in the neighborhood of the roots of the equation $f(z) = -z\sin(z) = 0$, that is, in the neighborhood of the points $\widehat{\mu}_k = k\pi$. Since all the roots of the equation $\sin(z) = 0$ are simple, all the roots of Eq. (3.84) starting from some number n_0 are simple.

Let us find asymptotics of the eigenvalues for $k \to \infty$. For this we denote $\mu_k := \widehat{\mu}_k + \delta_k \equiv k\pi + \delta_k$. Then Eq. (3.84) will have the form

$$-(k\pi + \delta_k)\sin(\delta_k) + (h+H)\cos(\delta_k) + O\left(\frac{1}{k}\right) = 0.$$

Then $\sin(\delta_k) = O\left(\frac{1}{k}\right)$, that is, $\delta_k = O\left(\frac{1}{k}\right)$. Thus, for sufficiently large k the eigenvalues of the Sturm-Liouville problem have the form

$$\lambda_k = \left(k\pi + O\left(\frac{1}{k}\right)\right)^2. \tag{3.85}$$

These asymptotics can be essentially refined under an assumption of higher smoothness of the function $q(x)$. For example, for $q \in C^1[0,1]$ the following formula for asymptotics is known:

$$\lambda_k = \left(k\pi + \frac{\alpha_1}{k} + O\left(\frac{1}{k^3}\right)\right)^2, \text{ where } \alpha_1 = h + H + \int_0^1 q(t)dt.$$

Asymptotic formulae for the eigenvalues also hold in the case when $a_1 = 0$ and/or $d_1 = 0$ in the boundary condition (3.69).

If $a_1 = 0$, $d_1 = 1$ and $d_0 = H$ or $a_1 = 1$, $a_0 = H$ and $d_1 = 0$, then instead of (3.85) we have

$$\lambda_k = \left(k\pi + \frac{\pi}{2} + O\left(\frac{1}{k}\right)\right)^2. \tag{3.86}$$

We separately write out the case when $a_1 = 0$ and $d_1 = 0$. Here the boundary conditions (3.69) will take the form

$$u(0) = u(1) = 0. \tag{3.87}$$

Lemma 3.68 *The asymptotics of eigenvalues of the Sturm-Liouville problem for Eq.* (3.73) *with the boundary conditions* (3.87) *have the form*

$$\lambda_k = \left(k\pi + O\left(\frac{1}{k}\right)\right)^2. \tag{3.88}$$

The asymptotics of normalised eigenfunctions have the form

$$u_k(x) = \sqrt{2}\sin(k\pi x) + O\left(\frac{1}{k}\right).$$

Under an additional condition $q \in C^1[0,1]$, the asymptotics (3.88) *can be refined as*

$$\lambda_k = \left(k\pi + \frac{\alpha_2}{k} + O\left(\frac{1}{k^3}\right)\right)^2, \textit{ where } \alpha_2 = \frac{1}{2}\int_0^1 q(t)dt.$$

From (3.85), (3.86) and (3.88) it follows that the Sturm-Lioville problem cannot have multiple eigenvalues starting from some number n_0. In fact, no eigenvalues can be multiple:

Lemma 3.69 *Each eigenvalue of the Sturm-Lioville problem has the multiplicity* 1.

Indeed, assume that two linearly independent eigenfunctions u_1 and u_2 correspond to some eigenvalue λ. These two functions are solutions of one differential equation

$$-u''(x)+q(x)u(x)=\lambda u(x),\ 0<x<1. \tag{3.89}$$

Therefore, their Wronskian differs from zero and does not depend on x: $W\{u_1,u_2\}(x)=W(0)\neq 0$.

Let us write the boundary conditions (3.69) for these two functions in the form

$$\begin{cases} a_1u_1'(0)+a_0u_1(0) & & =0\\ a_1u_2'(0)+a_0u_2(0) & & =0\\ & d_1u_1'(1)+d_0u_1(1) & =0\\ & d_1u_2'(1)+d_0u_2(1) & =0. \end{cases} \tag{3.90}$$

These equations can be considered as a system of homogeneous linear equations with respect to unknowns a_1, a_0, d_1, d_0.

The determinant of the system (3.90) differs from zero:

$$\triangle = W\{y_1,y_2\}(0)\cdot W\{y_1,y_2\}(1)=W^2(0)\neq 0.$$

Therefore, the system has only the zero solution $a_1=a_0=d_1=d_0=0$ that contradicts the condition $a_1^2+a_0^2\neq 0$, $d_1^2+d_0^2\neq 0$. This contradiction proves the impossibility of having two eigenfunctions corresponding to one eigenvalue. Since a self-adjoint operator cannot have associated vectors, it follows that all the eigenvalues of the Sturm-Liouville problem are simple. □

For completeness of the exposition we give one more fundamental result of Sturm on intermittency of solutions of various Sturm-Liouville equations.

Theorem 3.70 *Let two equations be given:*

$$-u''(x)+g(x)u(x)=0,\ 0<x<1,$$

$$-v''(x)+h(x)v(x)=0,\ 0<x<1.$$

If $g(x)>h(x)$ on the whole interval $[0,1]$, then between every two zeros of any non-trivial solution $u(x)$ of the first equation there is at least one zero of each solution $v(x)$ of the second equation.

From this theorem we obtain an interesting particular case.

Corollary 3.71 *Any solution of the equation*

$$-u''(x)+q(x)u(x)=0,\ 0<x<1,$$

for $q(x)>m^2>0$, for some m, can not have more than one zero.

Indeed, the equation $-v''(x)+m^2v(x)=0$ has the solution $v(x)=e^{mx}$ which does not vanish anywhere. Therefore, on the basis of Theorem 3.70 the solution $u(x)$ can have no more than one zero. □

Let us give without proof the most well-known Sturm theorem on the number of zeros of eigenfunctions on an interval. Its proof is based on the result of Theorem 3.70.

Theorem 3.72 (Sturm theorem on oscillation). *There exists an unboundedly increasing sequence of eigenvalues $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_2 \leq \ldots$ of the Sturm-Liouville boundary value problem* (3.73), (3.69). *Moreover, an eigenfunction corresponding to the eigenvalue λ_k has exactly k zeros inside the interval* $0 < x < 1$.

Note that, originally, it is based on this theorem that C. Sturm drew conclusions on the existence of an infinite number of eigenvalues of the Sturm-Liouville problem.

3.9 Spectral trace and Hilbert-Schmidt operators

To introduce the concept of the trace of an operator we return for a while to considering finite-dimensional operators and their spectral properties, which have been already partially described in Section 3.1.

Let X be an n-dimensional vector space and let an operator $A : X \to X$ in some basis of X be given by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}. \tag{3.91}$$

The sum of all diagonal elements of the matrix,

$$\operatorname{Tr} A = \sum_{i=1}^{n} a_{ii},$$

is called the ***matrix trace***.

It is well known that the multiplication of matrices, generally speaking, is not commutative: $AB \neq BA$. But the matrix traces of these matrices coincide:

Lemma 3.73 *Let A and B be $n \times n$ square matrices. Then*

$$\operatorname{Tr} AB = \operatorname{Tr} BA.$$

Indeed, if the matrices A and B are given by the coefficients a_{ij} and b_{ij}, then

$$\operatorname{Tr} AB = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}, \ \operatorname{Tr} AB = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_{ji},$$

which are equal. $\square$

From this lemma one obtains a remarkable property of the invariance of $\operatorname{Tr} A$ with respect to a chosen basis.

Theorem 3.74 *The matrix trace of a finite-dimensional operator does not depend on a basis in which the operator A is represented in the form of its matrix* (3.91).

Indeed, let P be the transformation matrix to a new basis. Then in the new basis the operator A is represented by the matrix $P^{-1}AP$. Applying Lemma 3.73 to the matrices $P^{-1}AP$ and P, we get

$$\mathrm{Tr}\,\left(P^{-1}AP\right) = \mathrm{Tr}\,\left(PP^{-1}A\right) = \mathrm{Tr}\,A. \square$$

The ***spectral trace*** of a finite-dimensional operator A is the sum of all of its eigenvalues.

Theorem 3.75 *The spectral trace of a finite-dimensional operator A coincides with its matrix trace:*

$$\mathrm{Tr}\,A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i. \tag{3.92}$$

To prove this fact we first show that the spectrum of a finite-dimensional operator A does not depend on a basis in which the operator A is represented as matrix (3.91). Indeed, let P be the transformation matrix to a new basis. Then in the new basis the operator A is represented by the matrix $P^{-1}AP$. The roots of the characteristic polynomial $\triangle(\lambda) = \det\left(P^{-1}AP - \lambda I\right)$ give its spectrum. Since $\det\left(P^{-1}\right) = (\det P)^{-1}$, we have

$$\det\left(P^{-1}AP - \lambda I\right) = \det\left(P^{-1}AP - \lambda P^{-1}IP\right) = \det\left(P^{-1}(A-\lambda I)P\right)$$

$$= \det\left(P^{-1}\right)\det(A-\lambda I)\det(P) = \det(A-\lambda I).$$

That is, the characteristic determinant of the operator A does not depend on a basis in which the operator A is represented as matrix (3.91). Consequently, the spectrum also does not depend on the choice of this basis.

Since the matrix trace also does not depend on the choice of this basis, then without loss of generality one can assume that the matrix A is represented in the Jordan normal form (3.14). Since in the Jordan normal form the eigenvalues of the matrix are diagonal elements: $a_{ii} = \lambda_i$, the equality (3.92) is proved. $\square$

The following question arises naturally: is the fact of the coincidence of the matrix and spectral traces true in other (not finite-dimensional) operators in Hilbert spaces?

Consider a compact operator A acting in a (infinite-dimensional) Hilbert space H. It is said that the ***operator*** A ***has a matrix trace*** if for any orthonormal basis φ_k the series

$$\mathrm{Tr}\,A = \sum_{k=1}^{\infty} \langle A\varphi_k, \varphi_k\rangle \tag{3.93}$$

converges absolutely. In this case the sum of the series (3.93) is called the ***matrix trace of the operator*** A. For self-adjoint operators having a matrix trace one has a formula analogous to formula (3.92):

Lemma 3.76 *Let A be a self-adjoint compact operator in a Hilbert space H, and let λ_k be its eigenvalues. If the series* (3.93) *converges absolutely, then its sum does not* *depend on the choice of an orthonormal basis φ_k. Moreover, the matrix trace of the operator coincides with its spectral trace:*

$$\operatorname{Tr} A = \sum_{k=1}^{\infty} \langle A\varphi_k, \varphi_k\rangle = \sum_{k=1}^{\infty} \lambda_k. \tag{3.94}$$

Indeed, let $\{\psi_j\}$ be an orthonormal basis consisting of eigenvectors of the self-adjoint compact operator A corresponding to the eigenvalues λ_j. Then, according to Corollary 3.33, from the Hilbert-Schmidt theorem 3.32 we have the expansion

$$\varphi_k = \sum_{j=1}^{\infty} \langle \varphi_k, \psi_j\rangle \psi_j, \quad \psi_j = \sum_{k=1}^{\infty} \langle \psi_j, \varphi_k\rangle \varphi_k. \tag{3.95}$$

Moreover, since $\|\varphi_k\| = \|\psi_j\| = 1$, by the Parseval's identity we have

$$\sum_{j=1}^{\infty} |\langle \varphi_k, \psi_j\rangle|^2 = 1, \quad \sum_{k=1}^{\infty} |\langle \varphi_k, \psi_j\rangle|^2 = 1.$$

Applying the operator A to the expansion (3.95), we get

$$A\varphi_k = \sum_{j=1}^{\infty} \langle \varphi_k, \psi_j\rangle \lambda_j \psi_j, \quad \langle A\varphi_k, \varphi_k\rangle = \sum_{j=1}^{\infty} \left|\langle \varphi_k, \psi_j\rangle\right|^2 \lambda_j.$$

Therefore, for the matrix trace we have

$$\sum_{k=1}^{\infty} \langle A\varphi_k, \varphi_k\rangle = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\langle \varphi_k, \psi_j\rangle|^2 \lambda_j = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\infty} \left|\langle \varphi_k, \psi_j\rangle\right|^2 = \sum_{j=1}^{\infty} \lambda_j.$$

This proves the independence of the matrix trace of a self-adjoint operator on the choice of a basis and gives the formula (3.94) for its equality with the spectral trace. □

For arbitrary (not necessarily self-adjoint) operators having the trace one also has the independence of the matrix trace on the choice of a basis.

Lemma 3.77 *If the series* (3.93) *converges absolutely, then its sum does not depend on the choice of an orthonormal basis φ_k.*

Indeed, let us represent the operator A in the form of the sum of its real and imaginary components:

$$A = A_R + A_I, \text{ where } A_R = \frac{1}{2}(A + A^*), \, A_I = \frac{1}{2i}(A - A^*).$$

It is obvious that the operators A_R and A_I are self-adjoint. We also have $\operatorname{Re}\langle A\varphi_k, \varphi_k\rangle = \langle A_R\varphi_k, \varphi_k\rangle$ and $\operatorname{Im}\langle A\varphi_k, \varphi_k\rangle = \langle A_I\varphi_k, \varphi_k\rangle$.

Therefore, by Lemma 3.76 for the self-adjoint operators the matrix traces of the operators A_R and A_I do not depend on the choice of a basis. Since

$$\sum_{k=1}^{\infty} \langle A\varphi_k, \varphi_k\rangle = \sum_{k=1}^{\infty} \langle A_R\varphi_k, \varphi_k\rangle + i\sum_{k=1}^{\infty} \langle A_I\varphi_k, \varphi_k\rangle,$$

the matrix trace of the operator A also does not depend on the choice of a basis. □

One of the remarkable results for operators having a trace is the theorem of V. B. Lidskii [74] from 1959, on the equality of matrix and spectral traces for arbitrary (not necessarily self-adjoint) operators (having a trace). The operators having the trace are called ***nuclear operators***.

Let us consider the *Schatten-von Neumann classes*, whose definition has been introduced in Section 2.15 and which we now briefly recall. In the class of compact operators in a Hilbert space H we can introduce the following parametrisation. One says that an ***operator*** $A : H \to H$ ***belongs to the class*** $S_p(H)$, if the following value is finite:

$$\|A\|_p = \left(\sum_{k=1}^{\infty} \|A\varphi_k\|_H^p \right)^{1/p} < \infty, \tag{3.96}$$

where $\varphi_k (k \in \mathbb{N})$ is an orthonormal basis in H. The value $\|A\|_p$ does not depend on the choice of the orthonormal basis φ_k. The classes $S_p(H)$ are called the ***Schatten-von Neumann classes***. They are nested: $S_p(H) \subset S_q(H)$ for $p \leq q$.

Theorem 3.78 (V. B. Lidskii, [74]) *Let $A \in S_1(H)$, that is, A is a nuclear operator. Then its matrix trace coincides with its spectral trace:*

$$\operatorname{Tr} A = \sum_{j=1}^{\infty} \lambda_j.$$

Let us describe the Schatten-von Neumann classes in terms of the eigenvalues. For example, if A is a compact self-adjoint operator, then as the basis φ_k we can choose the basis of its normalised eigenvectors. Then formula (3.96) has the form

$$\|A\|_p = \left(\sum_{k=1}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where λ_k are the eigenvalues of the operator A.

However, in the general case such substitution of definition (3.96) turns out to be impossible when considering general compact (not self-adjoint) operators which cannot have eigenvalues or can have associated vectors, or its system of eigenvectors does not form an orthonormal basis. To do this we need to introduce the concept of s-numbers of an operator.

Let A be an arbitrary compact operator in a Hilbert space H. Consider the operator $B = A^*A$. This operator is defined on the whole space H and is compact. For arbitrary vectors $\varphi, \psi \in H$ we have

$$\langle B\varphi, \psi \rangle = \langle A^*A\varphi, \psi \rangle = \langle A\varphi, A\psi \rangle = \langle \varphi, A^*A\psi \rangle = \langle \varphi, B\psi \rangle.$$

Hence the operator B is a self-adjoint operator.

We denote by μ_k the sequence of its nonzero eigenvalues, and let φ_k be the corresponding normalised eigenfunctions. Then

$$\mu_k = \mu_k \langle \varphi_k, \varphi_k \rangle = \langle B\varphi_k, \varphi_k \rangle = \langle A^*A\varphi_k, \varphi_k \rangle = \langle A\varphi_k, A\varphi_k \rangle = \|A\varphi_k\|^2 \geq 0.$$

Thus, *the operator $B = A^*A$ is a compact self-adjoint positive operator*. We denote its eigenvalues (numbered in ascending order) by s_k^2 ($s_k^2 = \mu_k$). The numbers s_k (we choose them as ≥ 0) are an important characteristic of the operator. They are called ***s*-numbers of the operator** or the ***singular values of the operator***.

Note that although a compact operator A itself may not have eigenvalues, its s-numbers always exist. Let us give an example.

Example 3.79 In $L^2(0,1)$, consider the integral operator

$$Vf(x) = \int_0^x f(t)dt.$$

We have considered this operator in Example 2.70, where we have shown that it is a Volterra operator. That is, it is a compact operator not having eigenvalues.

Let us show that s-numbers of the operator V exist and let us calculate them. The operator adjoint to the operator V is easily constructed:

$$V^*g(x) = \int_x^1 g(\tau)d\tau.$$

The corresponding operator $B = V^*V$ has the form

$$Bf(x) = V^*Vf(x) = \int_x^1 d\tau \int_0^\tau f(t)dt = \int_0^x (1-x)f(t)dt + \int_x^1 (1-t)f(t)dt.$$

This operator is self-adjoint as an integral operator with the symmetric kernel $k(x,t) = \theta(x-t)(1-x) + \theta(t-x)(1-t)$. This operator is compact since the kernel $k(x,t)$ is continuous. Let us find the eigenvalues of the operator B.

Denote

$$u(x) = Bf(x) = \int_0^x (1-x)f(t)dt + \int_x^1 (1-t)f(t)dt. \tag{3.97}$$

It is easy to see that $u \in L_2^2(0,1)$ and

$$-u''(x) = f(x), \tag{3.98}$$

that is, the operator B is inverse to an operator of some boundary value problem for the equation of the second order (3.98). Let us find boundary conditions of this problem. To do this, as in Example 2.93, instead of the function $f(x)$ we substitute its value from (3.98) into the expression (3.97). Then, by a direct calculation we get

$$u(x) = -(1-x)\int_0^x u''(t)dt - \int_x^1 (1-t)u''(t)dt = (1-x)u'(0) - u(1) + u(x)$$

for all elements from the range of the operator B. Hence we obtain the boundary conditions

$$u'(0) = 0, \ u(1) = 0. \tag{3.99}$$

Let us find the eigenvalues and eigenfunctions of this boundary value problem (3.98)-(3.99), that is, all nonzero solutions of the equation $-u''(x) = \lambda u(x)$, $0 < x < 1$, for which (3.99) holds.

It is easy to see that the number $\lambda = 0$ is not an eigenvalue. For $\lambda \neq 0$ all solutions of this differential equation have the form

$$u(x) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \frac{1}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda}x\right),$$

where C_1 and C_2 are arbitrary constants. Using the boundary conditions (3.99) for defining C_1 and C_2, we obtain

$$\begin{cases} C_2 = 0, \\ C_1 \cos\sqrt{\lambda} + C_2 \frac{1}{\sqrt{\lambda}} \sin\sqrt{\lambda} = 0. \end{cases} \tag{3.100}$$

Therefore, the eigenvalues of the problem are found from the equation

$$\triangle(\lambda) = \cos\sqrt{\lambda} = 0,$$

and can be calculated in the explicit form $\lambda_k = \left(k\pi - \frac{\pi}{2}\right)^2$, $k = 1,2,\ldots$. The eigenfunctions corresponding to them are $u_k(x) = \cos\left(\left(k\pi - \frac{\pi}{2}\right)x\right)$.

The discovered eigenvalues λ_k are the eigenvalues of the operator B^{-1}. Then by Lemma 3.44 the operator B has the eigenvalues λ_k^{-1}.

Thus, the integral operator V does not have eigenvalues, but its s-numbers are

$$s_k = \left(k\pi - \frac{\pi}{2}\right)^{-1}, \; k = 1,2,\ldots. \; \square$$

Thus, we have shown an example of a compact operator in a Hilbert space that has s-numbers while not having any eigenvalues. The s-numbers allow one to introduce the Schatten-von Neumann classes in terms of these s-numbers.

We say that an ***operator*** $A : H \to H$ ***belongs to the class*** $S_p(H)$, if

$$\|A\|_p := \left(\sum_{k=1}^{\infty} s_k^p\right)^{1/p} < \infty, \tag{3.101}$$

where $s_k (k \in \mathbb{N})$ is a descending sequence of s-numbers of the operator A. The value $\|A\|_p$ is called the ***Schatten-von Neumann norm of the operator*** A. They are nested: $S_p(H) \subset S_q(H)$ for $p \leq q$.

The operators from the class $S_1(H)$ are called ***nuclear operators (or trace-class operators)***, and the operators from the class $S_2(H)$ are called ***Hilbert-Schmidt operators***.

For any compact positive self-adjoint operator A one can introduce operators A^{α} called the ***power of an operator of order*** α. By x_k we denote an orthonormal basis of eigenvectors of the operator A corresponding to the eigenvalues $\lambda_k \geq 0$. Then by the Hilbert-Schmidt theorem 3.32 we have the spectral expansion

$$A\varphi = \sum_{k=1}^{\infty} \lambda_k \varphi_k x_k, \text{ where } \varphi_k = \langle \varphi, x_k \rangle. \tag{3.102}$$

The operator having the spectral expansion

$$A^{\alpha}\varphi = \sum_{k=1}^{\infty} \lambda_k^{\alpha} \varphi_k x_k, \text{ where } \varphi_k = \langle \varphi, x_k \rangle.$$

is called the ***power of the compact positive self-adjoint operator*** A of order $\alpha \geq 0$.

Since $\alpha \geq 0$, the convergence of this series follows.

In particular, one can define in this way the operator

$$\sqrt{A} := (A)^{1/2}$$

called the ***square root of the compact positive self-adjoint operator*** A. The square root is defined uniformly. It is obvious that $\left(\sqrt{A}\right)^2 = A$.

In the same way we define the operator

$$|A| := \sqrt{A^*A} = (A^*A)^{1/2}$$

called the ***modulus of the compact operator*** A. We note that the operator A^*A is automatically positive and self-adjoint. It is easy to see that *nonzero eigenvalues of the operator* $|A|$ *are the s-numbers of the operator* A.

Lemma 3.80 *For any compact operator* A *the operator* $|A|$ *is a compact positive self-adjoint operator.*

Indeed, let us first show the compactness of the operator $|A|$. Let $\{f_k\}_{k=1}^{\infty}$ be an arbitrary bounded ($\|f_k\| \leq M$) sequence of vectors from H. Let us show that one can choose a convergent subsequence from the sequence of their images $\{|A|f_k\}_{k=1}^{\infty}$. Since the operator A is compact, then A^*A is also compact. Therefore, one can choose a convergent subsequence from the sequence $\{A^*Af_k\}_{k=1}^{\infty}$. Without loss of generality, one can assume that the sequence $\{A^*Af_k\}_{k=1}^{\infty}$ already is convergent and, consequently, a Cauchy sequence.

By the boundedness of the sequence $\{f_k\}_{k=1}^{\infty}$ we have

$$\left\| |A|f_k - |A|f_j \right\|^2 = \langle |A|(f_k - f_j), |A|(f_k - f_j) \rangle = \langle A^*A(f_k - f_j), (f_k - f_j) \rangle$$
$$\leq \left\| A^*Af_k - A^*Af_j \right\| \|f_k - f_j\| \leq 2M \left\| A^*Af_k - A^*Af_j \right\|.$$

Therefore, the sequence $\{|A|f_k\}_{k=1}^{\infty}$ is also a Cauchy sequence and, therefore, converges.

Hence the operator $|A|$ maps an arbitrary bounded sequence into a convergent one and, therefore, is compact. □

A ***unitary operator*** on a Hilbert space H is a bounded linear operator $U : H \to H$ preserving the norm of the vectors:

$$\|Ux\| = \|x\|, \ \forall x \in H.$$

The unitary operators are boundedly invertible and $U^{-1} = U^*$. Moreover, the operator U is unitary if and only if $U^{-1} = U^*$.

Lemma 3.81 (Polar decomposition for operators) *For any compact positive operator A one has a polar decomposition of the form*

$$A = U|A|, \tag{3.103}$$

where U is a unitary operator.

Indeed, let μ_k be the eigenvalues and let x_k be the corresponding orthonormal system of eigenvectors of the compact positive self-adjoint operator $|A|$. All nonzero eigenvalues are s-numbers of the operator A. There can be zero eigenvalues among μ_k. It is obvious that

$$A^*Ax_k = \mu_k^2 x_k.$$

Denote by z_k the result of the action of the operator A on the basis vector x_k:

$$Ax_k = \mu_k z_k.$$

For those cases when $\mu_k = 0$ we choose z_k arbitrary but so that all such z_k form a basis in $\ker|A|$. Thus, the system $\{z_k\}_{k=1}^{\infty}$ is an orthonormal basis in H.

Consider the operator U acting on the basis vectors x_k by the formula

$$Ux_k = z_k.$$

Let us show that this operator is unitary. We represent the action of this operator on an arbitrary vector $f \in H$ in the form of an expansion with respect to the basis x_k:

$$Uf = U\sum_{k=1}^{\infty}\langle f, x_k\rangle x_k = \sum_{k=1}^{\infty}\langle f, x_k\rangle z_k, \text{ where } f = \sum_{k=1}^{\infty}\langle f, x_k\rangle x_k.$$

Therefore, the adjoint operator U^* has the representation

$$U^*g = \sum_{j=1}^{\infty}\langle g, z_j\rangle x_j, \text{ where } g = \sum_{j=1}^{\infty}\langle g, z_j\rangle z_j.$$

Then we calculate

$$UU^*g = \sum_{k=1}^{\infty}\left\langle \sum_{j=1}^{\infty}\langle g, z_j\rangle x_j, x_k\right\rangle z_k = \sum_{k=1}^{\infty}\langle g, z_k\rangle z_k = g,$$

that is, $UU^* = I$. It means that the operator U is unitary.

Consider the action by the operator $U|A|$ on the basis vectors x_k:

$$U|A|x_k = U\mu_k x_k = \mu_k Ux_k = \mu_k z_k = Ax_k.$$

This proves representation (3.103). The lemma is proved. □

By this lemma, one can apply the Hilbert-Schmidt theorem 3.32 to the operator $|A|$, so that we have the spectral expansion

$$|A|\varphi = \sum_{k=1}^{\infty} s_k\varphi_k u_k, \text{ where } \varphi_k = \langle\varphi, u_k\rangle, \tag{3.104}$$

where s_k are the eigenvalues, and u_k are the normalised eigenvectors of the operator $|A|$.

Applying the unitary operator U to the equality (3.104) from Lemma 3.81, we get

$$A\varphi = \sum_{k=1}^{\infty} s_k \langle \varphi, u_k \rangle U u_k.$$

Note that the vectors $v_k = U u_k$ belong to the range of the unitary operator U. Therefore, the system v_k is also an orthonormal basis in H. Thus, we have proved the following theorem which is an analogue of the Hilbert-Schmidt theorem for the case of non-self-adjoint operators.

Theorem 3.82 (Schmidt representation). *Let A be a compact operator in a Hilbert space H. Then one can find orthonormal systems of the vectors u_k and v_k, and also a non-increasing sequence of nonnegative numbers $s_k \geq 0$ such that*

$$A\varphi = \sum_{k=1}^{\infty} s_k \langle \varphi, u_k \rangle v_k, \ \forall \varphi \in H. \tag{3.105}$$

From the representation (3.105), since the sequence s_k is non-increasing, we obtain the estimate

$$\left\| A\varphi - \sum_{k=1}^{n} s_k \langle \varphi, u_k \rangle v_k \right\| = \left\| \sum_{k=n+1}^{\infty} s_k \langle \varphi, u_k \rangle v_k \right\| \leq s_{n+1} \|\varphi\|.$$

This fact demonstrates the direct connection between the decay order of the sequence of s-numbers of the operator A and the *approximation order of the operator A by finite-dimensional operators.* $\Box$

The following properties of the trace of a nuclear operator also follow from the representation (3.105). We give them without proof.

Theorem 3.83 *The following properties of the nuclear operators hold:*

1. *if A and B are nuclear operators, then operators $A+B$ and aA (where $a \in \mathbb{C}$ is a constant) are also nuclear. Moreover,*

$$\mathrm{Tr}\,(A+B) = \mathrm{Tr}\,A + \mathrm{Tr}\,B;$$

2. *if A is a nuclear operator, then the adjoint operator A^* is also nuclear and*

$$\mathrm{Tr}\,A = \mathrm{Tr}\,A^*;$$

3. *if A is a nuclear operator, then for any bounded operator B operators AB and BA are nuclear. Moreover,*

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA);$$

4. *if A and B are Hilbert-Schmidt operators, then operators AB and BA are nuclear.*

Let us demonstrate an application of formula (3.105) in a particular case of $L^2(\Omega)$.

Theorem 3.84 *Let T be a compact operator in the Hilbert space $L^2(\Omega)$. The operator T is a Hilbert-Schmidt operator if and only if it is an integral Hilbert-Schmidt operator. That is, if and only if there exists a function $k = k(x,y) \in L^2(\Omega \times \Omega)$ such that*

$$T\varphi(x) = \int_\Omega k(x,y)\varphi(y)dy, \ \forall \varphi \in L^2(\Omega). \tag{3.106}$$

Indeed, by the representation (3.105), in the case of the space $L^2(\Omega)$ we get

$$T\varphi(x) = \sum_{k=1}^{\infty} s_k \left(\int_\Omega \varphi(y) u_k(y) dy \right) \overline{v_k(x)} = \int_\Omega \left(\sum_{k=1}^{\infty} s_k u_k(y) \overline{v_k(x)} \right) \varphi(y) dy.$$

Denote

$$k(x,y) := \sum_{k=1}^{\infty} s_k u_k(y) \overline{v_k(x)}.$$

Then, by the orthonormality of bases u_k and v_k, we obtain

$$\|k\|^2_{L^2(\Omega\times\Omega)} = \sum_{k=1}^{\infty} |s_k|^2 .$$

The series in the right-hand part converges if and only if the operator T is a Hilbert-Schmidt operator. The theorem is proved. □

3.10 Schatten-von Neumann classes

Recall that an ***operator*** $A : H \to H$ on a Hilbert space H ***belongs to the class*** $S_p(H)$, if

$$\|A\|_p := \left(\sum_{k=1}^{\infty} s_k^p \right)^{1/p} < \infty, \tag{3.107}$$

where $s_k (k \in \mathbb{N})$ is a descending sequence of s-numbers of the operator A. The value $\|A\|_p$ is called the ***Schatten-von Neumann norm of the operator*** A.

The operators from the class $S_1(H)$ are called ***nuclear operators***, and the operators from the class $S_2(H)$ are called ***Hilbert-Schmidt operators***; these have been considered in Section 3.9. Here we briefly discuss the other values of p for the classes of integral operators, while referring to the abstract general theory to e.g. [49], [115],

[44]. For the basic theory of Schatten classes we refer the reader to [93], [115], [113]. For the trace class, see also [72].

Note that from the definition of the Schatten classes it follows that they are nested:

$$S_p(H) \subset S_q(H) \ \text{ for } \ p \leq q.$$

Moreover, they satisfy the important multiplication property

$$S_p S_q \subset S_r, \ \text{ where } \ \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \qquad 0 < p < q \leq \infty. \tag{3.108}$$

There arises an interesting question: under what additional conditions will an operator of the form (3.106), that is, the operator

$$T\varphi(x) = \int_\Omega k(x,y)\varphi(y)dy,$$

be nuclear or belong to $S_p(H)$? As T. Carleman showed in [28] in 1916, even in the one-dimensional case just the continuity of the kernel $k(x,t)$ on the whole square $[a,b] \times [a,b]$ does not guarantee that the corresponding operator T is nuclear or that T belongs to at least one of the classes $S_p(H)$ $(p < 2)$.

Example 3.85 (Carleman's example, 1916) In [28], T. Carleman constructed a periodic *continuous* function $\varkappa(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$ for which the Fourier coefficients c_k satisfy

$$\sum_{k=-\infty}^{\infty} |c_k|^p = \infty \qquad \text{for any } p < 2. \tag{3.109}$$

Now, consider the normal operator

$$Tf = f * \varkappa \tag{3.110}$$

acting on $L^2(\mathbb{T})$ (where $\mathbb{T}$ is the torus). The sequence $(c_k)_k$ gives a complete system of eigenvalues of this operator corresponding to the complete orthonormal system

$$\phi_k(x) = e^{2\pi ikx}, \ \ T\phi_k = c_k\phi_k.$$

The system ϕ_k is also complete for T^*, $T^*\phi_k = \overline{c_k}\phi_k$, so that the singular values of T are given by $s_k(T) = |c_k|$, and hence by (3.109) we have

$$\sum_{k=-\infty}^{\infty} s_k(T)^p = \infty \qquad \text{for any } p < 2.$$

A wide class of sufficient conditions for an integral operator T to belong to one of the classes $S_p(H)$ was obtained by P. E. Sobolevskii [120] in 1967.

Theorem 3.86 (P. E. Sobolevskii, [120], 1967). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega$, and let*

$$T\varphi(x) = \int_\Omega k(x,y)\varphi(y)dy.$$

If for some positive $\alpha > 0$ and $\beta > 0$ the integral kernel has the smoothness

$$k = k(x,y) \in W_{xy}^{\alpha\beta}(\Omega \times \Omega),$$

then the operator T belongs to the class $S_p\left(L^2(\Omega)\right)$ for

$$p > \frac{2n}{n+2(\alpha+\beta)}.$$

In this theorem $W_{xy}^{\alpha\beta}(\Omega \times \Omega)$ is the *Sobolev-Slobodetskii space* obtained by the closure with respect to the norm

$$\|k\|^2_{W_{xy}^{\alpha\beta}} = \int_\Omega \int_\Omega |k(x,y)|^2 dxdy$$

$$+ \sum_{[\alpha],[\beta]} \int_\Omega \int_\Omega \int_\Omega \int_\Omega \frac{\left|D_x^{[\alpha]} D_y^{[\beta]} \left(k(x_1,y_1) - k(x_2,y_1) - k(x_1,y_2) + k(x_2,y_2)\right)\right|^2}{|x_1 - x_2|^{n+2\{\alpha\}} |t_1 - y_2|^{n+2\{\beta\}}} dx_1 dx_2 dy_1 dy_2$$

of all functions $k = k(x,y)$ which are smooth in $\overline{\Omega} \times \overline{\Omega}$. Here $[\alpha]$ is the integer part of the number α and $\{\alpha\}$ is its fractional part. The summation is taken with respect to all partial derivatives of order $[\alpha]$ and $[\beta]$.

Theorem 3.86 clearly demonstrates that the increase in the smoothness of the kernel $k(x,y)$ of the integral operator (3.106) (with respect to even one of the variables x or y) leads to the inclusion of this operator into a smaller class $S_p(H)$.

The following condition was obtained in [34] for operators on *closed manifolds*, that is, on compact manifolds without boundary.

Theorem 3.87 ([34]) *Let M be a closed manifold of dimension n. Assume that k is in the Sobolev space, $k \in H^\mu(M \times M)$, for some $\mu > 0$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf)(x) = \int_M k(x,y) f(y) dy,$$

is in the Schatten classes $S_p(L^2(M))$ for

$$p > \frac{2n}{n+2\mu}.$$

In particular, if $\mu > \frac{n}{2}$, then T is trace class.

We refer to [34] for several more refined criteria, for example for the integral kernels in the mixed Sobolev spaces $H^{\mu_1,\mu_2}_{x,y}(M\times M)$.

We will also briefly review it here, but first, let us mention one corollary of such mixed Sobolev spaces criteria. We denote by $C^\alpha_x C^\beta_y(M\times M)$ the space of functions of class C^β with respect to y and C^α with respect to x.

Corollary 3.88 ([34]) *Let M be a closed manifold of dimension n. Let $k\in C^{\ell_1}_x C^{\ell_2}_y(M\times M)$ for some even integers $\ell_1,\ell_2\in 2\mathbb{N}_0$ such that $\ell_1+\ell_2>\frac{n}{2}$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf)(x)=\int_M k(x,y)f(y)dy,$$

is in $S_1(L^2(M))$, and its trace is given by

$$\operatorname{Tr}(T)=\int_M k(x,x)dx. \tag{3.111}$$

Let M be a closed manifold. In what follows, we will assume from the reader the basic knowledge of the theory of pseudo-differential operators, referring e.g. to [106] for the details.

Let P be an invertible first-order positive self-adjoint pseudo-differential operator on M. For example, if Δ_M denotes the positive Laplace operator on M, let us, for simplicity, fix

$$P:=(I+\Delta_M)^{1/2}.$$

For a function (or distribution) on $M\times M$, we will use the notation $P_y k(x,y)$ to indicate that the operator P is acting on the y-variable, the second factor of the product $M\times M$, and similarly for $P_x k(x,y)$ when it is acting on the x-variable.

We now define ***mixed Sobolev spaces*** $H^{\mu_1,\mu_2}_{x,y}(M\times M)$, of mixed regularity $\mu_1,\mu_2\geq 0$. To motivate the definition, we observe that for $k\in L^2(M\times M)$, we have

$$\|k\|^2_{L^2(M\times M)}=\int_{M\times M}|k(x,y)|^2dxdy=\int_M\left(\int_M |k(x,y)|^2dy\right)dx,$$

or we can also write this as

$$k\in L^2(M\times M)\Longleftrightarrow k\in L^2_x(M,L^2_y(M)). \tag{3.112}$$

In particular, this means that k_x defined by $k_x(y)=k(x,y)$ is well-defined for almost every $x\in M$ as a function in $L^2_y(M)$.

Let now $k\in L^2(M\times M)$ and let $\mu_1,\mu_2\geq 0$. We say that $K\in H^{\mu_1,\mu_2}_{x,y}(M\times M)$ if $k_x\in H^{\mu_2}(M)$ for almost all $x\in M$, and if $(I+P_x)^{\mu_1}k_x\in L^2_x(M,H^{\mu_2}_y(M))$. We set

$$\|k\|_{H^{\mu_1,\mu_2}_{x,y}(M\times M)}:=\left(\int_M \|(I+P_x)^{\mu_1}k_x\|^2_{H^{\mu_2}(M)}dx\right)^{1/2}.$$

In analogy to (3.112), we can also write this as

$$k\in H^{\mu_1,\mu_2}_{x,y}(M\times M)\Longleftrightarrow P^{\mu_1}_x P^{\mu_2}_y k\in L^2(M\times M). \tag{3.113}$$

We recall that by the elliptic regularity, the usual Sobolev space $H^{\mu}(M)$ can be characterised as the space of all distributions f such that $(I+P)^{\mu} f \in L^2(M)$, and this characterisation is independent of the choice of (an invertible first-order positive self-adjoint pseudo-differential operator) P. Similarly, the space $H_{x,y}^{\mu_1,\mu_2}(M \times M)$ is independent of the choice of a particular operator P as above.

One can show the following inclusions between the mixed and the standard Sobolev spaces:

$$H^{\mu_1+\mu_2}(M \times M) \subset H_{x,y}^{\mu_1,\mu_2}(M \times M) \subset H^{\min(\mu_1,\mu_2)}(M \times M), \tag{3.114}$$

for any $\mu_1, \mu_2 \geq 0$.

We will now give a condition for the integral operators to belong to the Schatten classes.

Theorem 3.89 ([34]) *Let M be a closed manifold of dimension n and let $\mu_1, \mu_2 \geq 0$. Let $k \in L^2(M \times M)$ be such that $k \in H_{x,y}^{\mu_1,\mu_2}(M \times M)$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf)(x) = \int_M k(x,y) f(y) dy,$$

is in the Schatten classes $S_r(L^2(M))$ for

$$r > \frac{2n}{n+2(\mu_1+\mu_2)}.$$

For $\mu_1, \mu_2 = 0$ the conclusion is trivial and can be sharpened to include $r = 2$.

We also note that combining Theorem 3.89 with the inclusions (3.114), one immediately obtains Theorem 3.87.

Corollary 3.90 *We have the following two special cases of Theorem 3.89 when no regularity is required in one of the variables. For example, for $\mu_1 = 0$, the condition $k \in L^2(M, H^{\mu}(M))$ implies that the corresponding operator T satisfies $T \in S_r$ for*

$$r > \frac{2n}{n+2\mu}.$$

In this case no regularity in the x-variable is imposed on the kernel.

The case $\mu_2 = 0$, imposing no regularity of K with respect to y, is dual to it. It also follows directly from the first one by considering the adjoint operator T^* and using the equality $\|T^*\|_{S_r} = \|T\|_{S_r}$.

In fact, more abstract results have been obtained in [36], which we will now briefly describe. They yield the aforementioned conclusions as special cases of the general statement.

Let $(X_j, \mathscr{M}_j, \mu_j)$ $(j = 1, 2)$ be measure spaces respectively endowed with a σ-finite measure μ_j on a σ-algebra $\mathscr{M}_j$ of subsets of X_j. We denote

$L^2(\mu_j) := L^2(X_j, \mu_j)$ the complex Hilbert space of square integrable functions on X_j. Let us consider the integral operators

$$Tf(x) = \int_{X_1} k(x,y) f(y) d\mu_1(y) \tag{3.115}$$

from $L^2(\mu_1)$ into $L^2(\mu_2)$. In analogy to (3.112) we observe that for $k \in L^2(\mu_2 \otimes \mu_1)$, we have

$$\|k\|^2_{L^2(\mu_2 \otimes \mu_1)} = \int_{X_2 \times X_1} |k(x,y)|^2 d\mu_2(x)\mu_1(y) = \int_{X_1} \left(\int_{X_2} |k(x,y)|^2 d\mu_2(x) \right) d\mu_1(y),$$

or we can also write

$$k \in L^2(\mu_2 \otimes \mu_1) \iff k \in L^2_y(\mu_1, L^2_x(\mu_2)).$$

In particular, this also means that k_y defined by $k_y(x) = k(x,y)$ is well-defined for almost every $y \in X_1$ as a function in $L^2_x(\mu_2)$.

For an operator E from $L^2(\mu_1)$ into $L^2(\mu_1)$ we will use the notation $E_x k(x,y)$ to emphasise that the operator E is acting on the x-variable. Analogously, we will also use the notation $E_y k(x,y)$ for an operator E from $L^2(\mu_2)$ into $L^2(\mu_2)$ acting on the y-variable.

The following statement asserts that if we know how some operators E_1, E_2 act on the integral kernel $k(x,y)$ of an operator T, and we know their spectral properties, we can obtain the spectral properties of the operator T.

Theorem 3.91 ([36]) *Let $(X_j, \mathscr{M}_j, \mu_j)$ $(j = 1,2)$ be σ-finite measure spaces. Let E_j $(j = 1,2)$ be linear invertible operators on $L^2(\mu_j)$ such that $E_j^{-1} \in S_{p_j}(L^2(\mu_j))$ for some $p_j > 0$. Let $k \in L^2(\mu_2 \otimes \mu_1)$ and let T be the integral operator from $L^2(\mu_1)$ to $L^2(\mu_2)$ defined by*

$$(Tf)(x) = \int_{X_1} k(x,y) f(y) d\mu_1(y).$$

Then the following holds:

(i) If $(E_2)_x (E_1)_y k \in L^2(\mu_2 \otimes \mu_1)$, then T belongs to the Schatten-von Neumann classes $S_r(L^2(\mu_1), L^2(\mu_2))$ for all $0 < r < \infty$ such that

$$\frac{1}{r} \le \frac{1}{2} + \frac{1}{p_1} + \frac{1}{p_2}.$$

Moreover,

$$\|T\|_{S_r} \le 2^{1 + \frac{2}{p_1} + \frac{1}{p_2}} \|E_1^{-1}\|_{S_{p_1}} \|E_2^{-1}\|_{S_{p_2}} \|(E_2)_x (E_1)_y k\|_{L^2(\mu_2 \otimes \mu_1)}. \tag{3.116}$$

(ii) If $(E_2)_x k \in L^2(\mu_2 \otimes \mu_1)$, then T belongs to the Schatten-von Neumann classes $S_r(L^2(\mu_1), L^2(\mu_2))$ for all $0 < r < \infty$ such that

$$\frac{1}{r} \le \frac{1}{2} + \frac{1}{p_2}.$$

Moreover,

$$\|T\|_{S_r} \leq 2^{\frac{1}{2}+\frac{1}{p_2}} \|E_2^{-1}\|_{S_{p_2}} \|(E_2)_x k\|_{L^2(\mu_2\otimes\mu_1)}. \tag{3.117}$$

(iii) If $(E_1)_y k \in L^2(\mu_2 \otimes \mu_1)$, then T belongs to the Schatten-von Neumann classes $S_r(L^2(\mu_1), L^2(\mu_2))$ for all $0 < r < \infty$ such that

$$\frac{1}{r} \leq \frac{1}{2} + \frac{1}{p_1}.$$

Moreover,

$$\|T\|_{S_r} \leq 2^{\frac{1}{2}+\frac{1}{p_1}} |E_1^{-1}\|_{S_{p_1}} \|(E_1)_y k\|_{L^2(\mu_2\otimes\mu_1)}. \tag{3.118}$$

The condition that $K \in L^2(\mu_2 \otimes \mu_1)$ in Theorem 3.91 is not restrictive. Indeed, conditions for $T \in S_r(L^2(\mu_1), L^2(\mu_2))$ for $r > 2$ do not require regularity of K and are given, for example, in Theorem 3.94. The case $0 < r < 2$ is much more subtle (as the classes become smaller), but if $T \in S_r(L^2(\mu_1), L^2(\mu_2))$ for $0 < r < 2$ then, in particular, T is a Hilbert-Schmidt operator, and hence the condition $k \in L^2(\mu_2 \otimes \mu_1)$ is also necessary.

The statement of Theorem 3.91 covers precisely the case $0 < r < 2$. Indeed, for example in Part (i), we have $r = \frac{2p_1p_2}{p_1p_2+2(p_1+p_2)}$ and hence we have $0 < r < 2$ since in general $0 < p_1, p_2 < \infty$. Thus, Theorem 3.91 provides a sufficient condition for Schatten classes S_r for $0 < r < 2$.

One can provide an alternative formulation in terms of the behaviour of the eigenvalue counting function of the operators E_1, E_2, also improving somewhat the decay rate of the singular numbers of the operator T.

We recall that for a self-adjoint operator E with discrete spectrum $\{\lambda_j\}_j$ its ***eigenvalue counting function*** is defined by

$$N(\lambda) := \#\{j : \lambda_j \leq \lambda\},$$

where λ_j's are counted with their respective multiplicities.

Theorem 3.92 ([36]) *Let $(X_i, \mathscr{M}_i, \mu_i)$ $(i = 1,2)$ be σ-finite measure spaces. For each $i = 1,2$, let E_i be an (essentially) self-adjoint operator on $L^2(\mu_i)$ such that the spectrum of its closure consists of a sequence of discrete and strictly positive eigenvalues $0 < \lambda_{1,i} \leq \lambda_{2,i} \leq \cdots$, whose eigenvectors are a basis of $L^2(\mu_i)$. Assume that for the eigenvalue counting function $N_i(\lambda)$ of E_i $(i = 1,2)$ there exist constants $C_i, p_i > 0$ such that*

$$N_i(\lambda) \leq C_i(1+\lambda)^{p_i} \text{ for all } \lambda > 0. \tag{3.119}$$

Let $k \in L^2(\mu_2 \otimes \mu_1)$ and let T be the integral operator from $L^2(\mu_1)$ to $L^2(\mu_2)$ defined by

$$(Tf)(x) = \int_{X_1} k(x,y) f(y) d\mu_1(y).$$

Then the following holds:

(i) *If* $(E_2)_x(E_1)_y k \in L^2(\mu_2 \otimes \mu_1)$, *then* T *belongs to the Schatten-von Neumann class* $S_r(L^2(\mu_1), L^2(\mu_2))$ *for all* $0 < r < \infty$ *such that*

$$\frac{1}{r} < \frac{1}{2} + \frac{1}{p_1} + \frac{1}{p_2},$$

and (3.116) *holds.*

Moreover, the sequence of singular values $(s_j(T))_j$ *satisfies the following estimate for the rate of decay:*

$$s_j(T) = o(j^{-\left(\frac{1}{2}+\frac{1}{p_1}+\frac{1}{p_2}\right)}).$$

(ii) *Let* E *be a linear invertible operator on* L^2 *as above such that its spectrum satisfies* (3.119) *for some* $p > 0$. *If either* $E_y k \in L^2(\mu_2 \otimes \mu_1)$ *or* $E_x k \in L^2(\mu_2 \otimes \mu_1)$, *then* T *belongs to the Schatten-von Neumann class* $S_r(L^2(\mu_1), L^2(\mu_2))$ *for all* $0 < r < \infty$ *such that*

$$\frac{1}{r} < \frac{1}{2} + \frac{1}{p},$$

and respectively (3.117) *or* (3.118) *holds.*

Moreover, the sequence of singular values $(s_j(T))_j$ *satisfies the following estimate for the rate of decay:*

$$s_j(T) = o(j^{-\left(\frac{1}{2}+\frac{1}{p}\right)}).$$

We note that the situation for Schatten classes S_p for $p > 2$ is simpler and, in fact, similar to that of $p = 2$. For the inclusion of some more recent results, let us briefly review a few other properties, for example, for convolution operators in the setting of compact Lie groups.

Theorem 3.93 ([35]) *Let* G *be a compact Lie group, and let* T *be a convolution operator,*

$$Tf(x) = f * \varkappa(x).$$

Then

$$\varkappa \in L^{p'}(G),\ 1 \leq p' \leq 2 \Longrightarrow T \in S_p(L^2(G)),\ \frac{1}{p'} + \frac{1}{p} = 1.$$

The converse of this is also true but for interchanged indices, i.e.

$$T \in S_p(L^2(G)),\ 1 \leq p \leq 2 \Longrightarrow \varkappa \in L^{p'}(G).$$

We refer to [35] for this as well as for the symbolic characterisation of Schatten classes in the setting of compact Lie groups.

There are several other conditions for the membership in the Schatten-von Neumann classes S_p for $p > 2$, for the case of integral operators on σ-finite measure spaces. The following condition was found independently by several mathematicians, we give here a version by [98]. Let X be a σ-finite measure space, and we define the class $L^{p,q}$ as consisting of (kernel) functions $k(x,y)$, $x,y \in X$, such that

$$\|k\|_{L^{p,q}} = \left(\int_X \left(\int_X |k(x,y)|^p dx \right)^{q/p} dy \right)^{1/q} < \infty.$$

Theorem 3.94 ([98]) *Let $p > 2$, $p' = p/(p-1)$, and let the integral kernel k of the operator*

$$(Tf)(x) = \int_X k(x,y) f(y) dy,$$

belong to $L^2(X \times X)$. Suppose that k and the adjoint kernel $k^*(x,y) = \overline{k(y,x)}$ belong to $L^{p',p}$. Then the integral operator T belongs to the Schatten-von Neumann class $S_p(L^2(X))$. Moreover, we have

$$\|T\|_{S_p} \leq (\|k\|_{L^{p',p}} \|k^*\|_{L^{p',p}})^{1/2}.$$

In fact, it was also shown in [45], quite elementarily, that the condition $k \in L^2(X \times X)$ is excessive and may be removed.

Under the above conditions, one can write the formula for the powers of the operator T. Thus, the following was shown in [45, Theorem 2.4]:

Theorem 3.95 *Let the integral kernel $k(x,y)$ of an operator T satisfy the conditions of Theorem 3.94 for some $p > 2$. Then for the operator T^m, which belongs to the trace class by the above theorem for any integer $m > p$, the following formula holds*

$$\mathrm{Tr}\,(T^m) = \int_{X^s} \left(\prod_{k=1}^{s} k(x_k, x_{k+1}) \right) dx_1 dx_2 \ldots dx_m, \tag{3.120}$$

where one identifies x_{m+1} with x_1.

Combining Theorem 3.91 with Theorem 3.94, we obtain the following extension.

Corollary 3.96 ([36]) *Let $(X, \mathscr{M}, \mu)$ be a measure space endowed with a σ-finite measure μ. Let E_j $(j = 1,2)$ be linear invertible operators on $L^2(X)$ such that $E_j^{-1} \in S_{p_j}(L^2(X))$ for some $p_j > 0$. Let $k \in L^2(X \times X)$ and let T be the integral operator from $L^2(X)$ to $L^2(X)$ defined by*

$$(Tf)(x) = \int_X k(x,y) f(y) d\mu(y).$$

Let $1 < q \leq 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

(i) If $(E_2)_x (E_1)_y k$ and $((E_2)_x (E_1)_y k)^* \in L^{q'}(X, L^q(X))$, then T belongs to the trace class $S_1(L^2(\mu))$ provided that

$$1 \leq \frac{1}{q'} + \frac{1}{p_1} + \frac{1}{p_2}.$$

Moreover, we have

$$\|T\|_{S_1} \leq 2^{1+\frac{1}{q'}+\frac{1}{p_1}} \|E_1^{-1}\|_{S_{p_1}} \|E_2^{-1}\|_{S_{p_2}} \times \\ \times \left(\|(E_2)_x (E_1)_y k\|_{L^{q'}(X,L^q(X))} \|((E_2)_x (E_1)_y k)^*\|_{L^{q'}(X,L^q(X))} \right)^{\frac{1}{2}}. \quad (3.121)$$

In particular, if $(E_2)_x (E_1)_y k \in L^2(X \times X)$, then T belongs to the trace class $S_1(L^2(X))$ provided that $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$.

(ii) Let E be a linear invertible operator on $L^2(X)$ such that $E^{-1} \in S_p(L^2(X))$ for some $p > 0$. If $E_x k, (E_x k)^* \in L^{q'}(X, L^q(X))$ or $E_y k, (E_y k)^* \in L^{q'}(X, L^q(X))$, then T belongs to the trace class $S_1(L^2(\mu))$ provided that

$$1 \leq \frac{1}{q'} + \frac{1}{p}.$$

Moreover, respectively one has

$$\|T\|_{S_1} \leq 2\|E^{-1}\|_{S_p} \left(\|E_x k\|_{L^{q'}(\Omega,L^q(\Omega))} \|(E_x k)^*\|_{L^{q'}(\Omega,L^q(\Omega))} \right)^{\frac{1}{2}}, \quad (3.122)$$

or

$$\|T\|_{S_1} \leq 2\|E^{-1}\|_{S_p} \left(\|E_y k\|_{L^{q'}(\Omega,L^q(\Omega))} \|(E_y k)^*\|_{L^{q'}(\Omega,L^q(\Omega))} \right)^{\frac{1}{2}}. \quad (3.123)$$

particular, if E is a linear invertible operator on $L^2(X)$ such that $E^{-1} \in S_2(L^2(X))$ and either $E_y k \in L^2(X \times X)$ or $E_x k \in L^2(X \times X)$, then T belongs to the trace class $S_1(L^2(X))$.

(iii) Moreover, assume additionally that X is a second countable topological space and $(X, \mathscr{M}, \mu)$ is a measure space endowed with a σ-finite Borel measure μ. Then under any of the assumptions (i) or (ii), the operator T is trace class on $L^2(\mu)$ and its trace is given by

$$\operatorname{Tr}(T) = \int_X \widetilde{k}(x,x)\, d\mu(x). \quad (3.124)$$

Here $\widetilde{k}(x,x)$ in (3.124) is defined using the averaging with respect to the martingale maximal function, see [36].

Consequently, Theorem 3.92 can be extended by using Corollary 3.96.

Corollary 3.97 ([36]) *Assume that E_1 and E_2 satisfy the assumptions of Theorem 3.92. Let $1 < q \leq 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then the following holds:*

(i) *If* $(E_2)_x(E_1)_y k$ *and* $((E_2)_x(E_1)_y k)^* \in L^{q'}(X, L^q(X))$*, then* T *belongs to the Schatten-von Neumann class* $S_r(L^2(X))$ *for all* $0 < r < \infty$ *such that*

$$\frac{1}{r} < \frac{1}{q'} + \frac{1}{p_1} + \frac{1}{p_2},$$

and (3.121) *holds.*

Moreover, the sequence of singular values $(s_j(T))_j$ *satisfies the following estimate for the rate of decay:*

$$s_j(T) = o(j^{-\left(\frac{1}{q'} + \frac{1}{p_1} + \frac{1}{p_2}\right)}).$$

(ii) *Let* E *be a linear invertible operator on* L^2 *as above such that its spectrum satisfies* (3.119) *for some* $p > 0$*. If* $E_x k, (E_x k)^* \in L^{q'}(X, L^q(X))$ *or* $E_y k, (E_y k)^* \in L^{q'}(X, L^q(X))$*, then* T *belongs to the Schatten-von Neumann class* $S_r(L^2(X))$ *for all* $0 < r < \infty$ *such that*

$$\frac{1}{r} < \frac{1}{q'} + \frac{1}{p},$$

and respectively (3.122) *or* (3.123) *holds.*

Moreover, the sequence of singular values $(s_j(T))_j$ *satisfies the following estimate for the rate of decay:*

$$s_j(T) = o(j^{-\left(\frac{1}{q'} + \frac{1}{p}\right)}).$$

The above criteria for the membership in the Schatten-von Neumann classes have a wide applicability; we refer to [36] for the detailed analysis of many examples in various settings.

3.11 Regularised trace for a differential operator

As we have seen in Section 3.9, the classical result of the matrix theory on the equality of matrix and spectral traces has been extended to the infinite-dimensional case. If an operator in the infinite-dimensional case is nuclear, then its matrix and spectral traces have a finite value and are equal to each other.

There arises a question of an analogue of these results for the case of unbounded operators. In this case the sequence of eigenvalues of a self-adjoint unbounded operator unboundedly increases. Therefore, the matrix and spectral traces of the operator do not exist. The further development of the theory has led to the formulation and investigation of the question on extending the notion of the trace invariance to operators not having a trace. The concept of the so-called *regularised traces* naturally arises.

If we consider the classical Sturm-Liouville problem

$$-u''(x)+q(x)u(x)=\lambda u(x);\ u'(0)-hu(0)=0,\ u'(\pi)+Hu(\pi)=0, \tag{3.125}$$

then, as has been shown in Section 3.8, for $q\in C^1[0,\pi]$ the eigenvalues of this problem have the asymptotics

$$\lambda_k=k^2+\alpha_1+O\left(\frac{1}{k^2}\right),\ \text{where}\ \alpha_1=\frac{2}{\pi}\left(h+H+\frac{1}{2}\int_0^\pi q(t)dt\right).$$

Therefore,

$$\sum_{k=1}^{\infty}\left(\lambda_k-k^2-\alpha_1\right)=\sum_{k=1}^{\infty}O\left(\frac{1}{k^2}\right)<\infty,$$

that is, we have obtained a convergent series.

Series of such type are called the ***regularised traces*** of the corresponding operators. As it has turned out (for the Sturm-Liouville problem), *although the eigenvalues of the Sturm-Liouville problem cannot be calculated in an explicit form, the sum of its regularised trace can be found exactly.*

In a more general form, we consider the problem of calculating the regularised traces of operators of the form

$$A=A_0+B,$$

where A_0 is an unbounded self-adjoint operator in a separable Hilbert space H with a compact resolvent, and B is an operator which is in some sense "subordinate" to the operator A_0.

The ***formula for the regularised traces*** for the operator A is a formula of the form

$$\sum_{k=1}^{\infty}\left(\lambda_k^p-\lambda_{k0}^p-c_k(p)\right)=F(p),$$

where λ_k and λ_{k0} are the eigenvalues of the operators A and A_0, respectively, p is an integer parameter called the ***order of the regularised trace***, and $c_k(p)$ and $F(p)$ are expressions in some explicit form.

For a wide class of abstract operators the formula for the regularised trace is obtained in the form

$$\sum_{k=1}^{\infty}(\lambda_k-\lambda_{k0})=\operatorname{Tr}B,$$

where $\operatorname{Tr}B$ is the spectral trace of some nuclear operator B. However, such class of operators of the form A_0+B, where B is nuclear, is not often used. In fact, more often, for the operator B one takes the operator of the multiplication by a function, $Bu(x)=q(x)u(x)$, which is not even a compact operator. Therefore, for such operators the additional investigation is needed.

The formula for the regularised trace of the first order was first obtained by I. M. Gelfand and B. M. Levitan in [43] (1953) for one concrete Sturm-Liouville problem:

$$-u''(x)+q(x)u(x)=\lambda u(x),\ x\in(0,\pi);\ \ u'(0)=0,\ u'(\pi)=0. \tag{3.126}$$

Under conditions $q \in C[0,\pi]$ and $\int_0^\pi q(x)dx = 0$, they obtained the formula

$$\sum_{k=0}^{\infty} \left(\lambda_k - k^2\right) = \frac{q(0)+q(\pi)}{4}.$$

Here λ_k are the eigenvalues of the problem (3.126). Noting that the second term in this sum, $\lambda_{k0} = k^2$, is the eigenvalue of the problem (3.126) for $q \equiv 0$. We obtain that formula (3.126) is a particular case of the general formula of calculating the regularised trace.

In further investigations the formulae for regularised traces were obtained for a wide class of differential operators including partial differential operators. It is known that the derivation of formulae for the regularised trace for a wide class of boundary value problems generated by ordinary differential expressions on a finite segment with complicated participation of a spectral parameter, reduces to studying regularised sums of roots of entire functions with a certain asymptotic structure. One can find a rather complete review of the modern state of this theory in the survey of V. A. Sadovnichii and V. E. Podol'skii [107]. Final formulations of the results have been obtained for most non-singular cases.

For the spectral problem (3.125) the formula of the regularised trace was obtained by B. M. Levitan [73] (1964) in the form

$$\sum_{k=1}^{\infty} \left(\lambda_k - k^2 - \alpha_1\right) = \frac{q(0)+q(\pi)}{4} - \frac{h+H}{\pi} - \frac{1}{2\pi}\int_0^\pi q(t)dt - \frac{h^2+H^2}{2}.$$

The further development of this theory consists in extending the formulae for the regularised trace to singular cases. These include problems in unbounded domains with non-smooth coefficients (which can be even generalised functions of Dirac delta-function type and/or its derivatives), problems in multiply-connected domains, etc.

Example 3.98 As a demonstration, we consider the following eigenvalue problem:

$$-u''(x) + q(x)u(x) = \lambda u(x),\ \ \frac{\pi}{n}(k-1) < x < \frac{\pi}{n}k,\ k = 1,\dots,n;\ n \geq 2; \tag{3.127}$$

$$u(0) = 0,\ u(\pi) = 0, \tag{3.128}$$

$$u\left(\frac{\pi k}{n} - 0\right) = u\left(\frac{\pi k}{n} + 0\right), \tag{3.129}$$

$$u'\left(\frac{\pi k}{n} - 0\right) = u'\left(\frac{\pi k}{n} + 0\right) - \beta_k \int_0^\pi u(t)dt,\ k = 1,\dots,n-1. \tag{3.130}$$

Here $q(x)$ is a sufficiently differentiable real-valued function, β_k are real constants, and λ is a spectral parameter.

One of the peculiarities of this problem is that the differential equation (3.127) is not satisfied at the interior points $\pi k/n$, $k = 1,\dots,n-1$, of the interval $(0,\pi)$.

At these points the solution is continuous according to the conditions (3.129), but the first derivative has jumps at these points according to the condition (3.130). The problems of such type are called *problems in punctured domains*.

Another peculiarity of this problem is that the used jump conditions (3.130) of the first derivative are sufficiently nonlocal: they involve an integral of the solution with respect to the whole interval $(0,\pi)$.

However, even for such complicated problems the first regularised trace can be written out in an explicit form. The problem is reduced to calculating a regularised sum of roots of quasi-polynomials.

Note that in [81] the formulae of the first regularised trace for the equation (3.127) with additional terms of the form $\sum_{k=1}^{n-1}\alpha_k u\left(\frac{\pi k}{n}\right)$ for $\beta_k = 0$, $k = 1,\dots,n-1$, were obtained. We will demonstrate the method of development of this result in the case of problems with integral conditions in a complex multiply-connected domain.

Theorem 3.99 *For a regularised trace of the problem* (3.127)-(3.130) *the following formula is valid:*

$$\sum_{m=0}^{\infty}\sum_{j=1}^{2n}\left(\lambda_{m,j}-(2nm+j)^2-\left(1+\frac{1}{2nm+j}\right)\frac{2}{\pi}\int_0^{\pi}q(t)dt\right) \tag{3.131}$$

$$=-\frac{q(0)+q(\pi)}{4}+\frac{1}{2\pi}\int_0^{\pi}q(t)dt,$$

where $\lambda_{m,j}$ are the eigenvalues of the problem (3.127)-(3.130). *Moreover, the eigenvalues have the asymptotics*

$$\lambda_{m,j}=s_{m,j}^2,\quad j=1,\dots,2n,\ m=m_0,m_0+1,\dots,$$

where

$$s_{m,j}=(2nm+j)+\frac{c_{1,j}}{2nm+j}+\frac{c_{2,j}}{(2nm+j)^2}+O\left(\frac{1}{(2nm+j)^3}\right); \tag{3.132}$$

$$c_{1,j}=\frac{1}{2\pi}\int_0^{\pi}q(t)dt,\quad c_{2,j}=\frac{1-(-1)^j}{2}\frac{i}{\pi}\sum_{k=1}^{n-1}(\beta_k+\beta_{n-k})e^{\frac{i\pi jk}{n}}. \tag{3.133}$$

Let us give a brief indication of the proof of this theorem. By standard calculations for Eq. (3.127) on each interval $I_k : \frac{\pi}{n}(k-1) < x < \frac{\pi}{n}k$, one can write out the asymptotics (for $|s|\to\infty$) for two linearly independent solutions of this equation:

$$u_{1,k}(x,s)\sim e^{isx}\sum_{\nu=0}^{\infty}\frac{a_{\nu,k}(x)}{s^{\nu}},\quad u_{2,k}(x,s)\sim e^{-isx}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{a_{\nu,k}(x)}{s^{\nu}},$$

where

$$a_{0,k}(x)\equiv 1,$$

$$a_{\nu,k}(x)=\frac{i}{2}\left\{a'_{\nu-1,k}(x)-a'_{\nu-1,k}\left(\frac{\pi}{n}(k-1)\right)-\int_{\frac{\pi}{n}(k-1)}^{x}q(t)a_{\nu-1,k}(t)dt\right\}. \tag{3.134}$$

Here it is assumed that the complex plane ($\lambda = s^2, s = \sqrt{\lambda}$) is divided into four sections by angles $\arg s = 0$ and $\arg s = \frac{\pi}{2}$ and the asymptotics exist in each of the four sections.

From the recurrent formula (3.134) we get

$$a_{1,k}(x) = -\frac{i}{2}\int_{\frac{\pi}{n}(k-1)}^{x} q(t)dt,$$

$$a_{2,k}(x) = \frac{1}{4}\left\{q(x) - q\left(\frac{\pi}{n}(k-1)\right) - \frac{1}{2}\left(\int_{\frac{\pi}{n}(k-1)}^{x} q(t)dt\right)^2\right\},$$

$$a_{\nu,k}\left(\frac{\pi}{n}(k-1)\right) = 0, \nu = 1,2;$$

$$u_{1,k}\left(\frac{\pi}{n}(k-1), s\right) = e^{i\frac{\pi(k-1)s}{n}}, \; u_{2,k}\left(\frac{\pi}{n}(k-1), s\right) = e^{-i\frac{\pi(k-1)s}{n}}.$$

On each of the intervals I_k a general solution of Eq. (3.127) is represented in the form

$$u(x) = A_k u_{1,k}(x,s) + B_k u_{2,k}(x,s).$$

Satisfying the boundary conditions (3.128) and the generalised "gluing" conditions (3.129), (3.130), we obtain a linear system of $2n$ equations for the constants A_k, B_k, whose determinant $\Delta(s)$ will be the characteristic determinant of the spectral problem (3.127)-(3.130). This function $\Delta(s)$ is determined by the following asymptotic expression:

$$\Delta(s) = e^{i\pi s}\left\{1 + \frac{a_1}{s} + \frac{a_2 - (\beta_1 + \ldots + \beta_{n-1})}{s^2} + O\left(\frac{1}{s^3}\right)\right\}$$

$$+ e^{-i\pi s}\left\{-1 + \frac{a_1}{s} - \frac{a_2 - (\beta_1 + \ldots + \beta_{n-1})}{s^2} + O\left(\frac{1}{s^3}\right)\right\}$$

$$+ \sum_{k=1}^{n-1} e^{i\frac{\pi k}{n}s}\left\{\frac{\beta_k + \beta_{n-k}}{s^2} + O\left(\frac{1}{s^3}\right)\right\} - \sum_{k=1}^{n-1} e^{-i\frac{\pi k}{n}s}\left\{\frac{\beta_k + \beta_{n-k}}{s^2} + O\left(\frac{1}{s^3}\right)\right\} + O\left(\frac{1}{s^4}\right).$$

Here

$$a_1 = -\frac{i}{2}\int_0^{\pi} q(t)dt, \; a_2 = \frac{1}{4}\left\{q(\pi) - q(0) - \frac{1}{2}\left(\int_0^{\pi} q(t)dt\right)^2\right\}.$$

Analysing the equation $\Delta(s) = 0$, we obtain that the problem (3.127)-(3.130) has $4n$ series of eigenvalues with the asymptotics (3.132), (3.133). Given that the function $\Delta(s)$ is odd, we obtain that along with the eigenvalues $s_{m,j}$ from formula (3.132), the numbers $-s_{m,j}$ are also the roots of the characteristic polynomial. We denote them by $s_{m,j+2n}, j = 1, \ldots, 2n$. Then we obtain the vanishing of the coefficients $c_{2,j}$ from (3.133) with the even numbers j. Thus, in terms of the spectral parameter λ we get $2n$ series of the eigenvalues.

The number of series of the eigenvalues in this problem is not classic. It is caused by the fact that the whole interval $[0,\pi]$ is divided in n subintervals.

Here the function $\Delta(s)$ belongs to the class K of entire functions of the first order. Therefore, for its analysis one can apply the method of calculating the regularised sum of roots of quasi-polynomials developed in the theory of regularised traces [107]. In view of the cumbersomeness of these calculations we will not give them here.

Remark 3.100 In a particular case when $\beta_k = 0$, $k = 1,\dots,n-1$, the problem (3.127)-(3.130) coincides with the Cauchy problem and formula (3.131) of Theorem 3.99 coincides with the classical result:

$$\sum_{m=0}^{\infty}\left(\lambda_m - m^2 - \frac{1}{\pi}\int_0^{\pi} q(t)dt\right) = -\frac{1}{4}\big(q(0)+q(\pi)\big) + \frac{1}{2\pi}\int_0^{\pi} q(t)dt.$$

The considered example once again demonstrates the possibilities of the methods of the theory of regularised traces. Although for concrete problems the eigenvalues themselves cannot be calculated in an explicit form, the regularised trace can often still be calculated in the form of an explicit formula. □

3.12 Eigenvalues of non-self-adjoint ordinary differential operators of the second order

In this section we consider basic properties of eigenvalues of non-selfadjoint boundary value problems of the general form for the second-order ordinary differential equations.

In $L^2(0,1)$, consider the operator L given by

$$Lu \equiv -u''(x) + q(x)u(x) = \lambda u(x),\ 0 < x < 1, \tag{3.135}$$

and two-point boundary conditions of the general form

$$\begin{cases} U_1(u) = a_{11}u'(0) + a_{12}u'(1) + a_{13}u(0) + a_{14}u(1) = 0, \\ U_2(u) = a_{21}u'(0) + a_{22}u'(1) + a_{23}u(0) + a_{24}u(1) = 0, \end{cases} \tag{3.136}$$

where $U_1(u)$ and $U_2(u)$ are linearly independent forms with arbitrary complex-valued coefficients and $q \in C[0,1]$ is an arbitrary complex-valued function.

We denote by L the closure in $L^2(0,1)$ of the operator given by the differential expression (3.135) on the linear space of functions $y \in C^2[0,1]$ satisfying the boundary conditions (3.136).

It is easy to justify that the operator L is a linear operator on $L^2(0,1)$ defined by (3.135) with the domain

$$D(L) = \left\{u \in L_2^2(0,1) : U_1(u) = U_2(u) = 0\right\}.$$

For the elements $u \in D(L)$ we understand the action of the operator $Lu = u''(x) + q(x)u(x)$ in the sense of almost everywhere on $(0,1)$.

By an ***eigenvector of the operator*** L corresponding to an eigenvalue $\lambda_0 \in \mathbb{C}$, we mean any nonzero vector $u_0 \in D(L)$ which satisfies the equation

$$Lu_0 = \lambda_0 u_0. \tag{3.137}$$

By an ***associated vector of the operator*** L of order m $(m = 1,2,\ldots)$ corresponding to the same eigenvalue λ_0 and the eigenvector u_0, we mean any function $u_m \in D(L)$ which satisfies the equation

$$Lu_m = \lambda_0 u_m + u_{m-1}. \tag{3.138}$$

The vectors $\{u_0, u_1, \ldots\}$ are called a ***chain of the eigen- and associated vectors*** of the operator L corresponding to the eigenvalue λ_0.

The eigenvalues of the operator L will be called the eigenvalues of the problem (3.135)-(3.136). *The eigen- and associated vectors of the operator L will be called eigen- and associated functions of the problem* (3.135)-(3.136). One can also say that the eigenfunction u_0 is a zero order associated function. The set of all eigen- and associated functions (they are collectively called root functions) corresponding to the same eigenvalue λ_0 forms a linear root space. This space is called a ***root space***.

The Sturm-Liouville problem considered in Section 3.8 is a particular case of problem (3.135)-(3.136). The peculiarity of the general case (3.135)-(3.136) is that in this case the operator L is not necessarily self-adjoint. Therefore, its spectral properties cannot be obtained from the results for the general self-adjoint operators.

We can form the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

from the coefficients of the boundary conditions (3.136) We denote by $A(ij)$ the matrix composed of the i-th and j-th columns of A, and denote

$$A_{ij} := \det A(ij), \ (1 \le i < j \le 4).$$

We denote by $c(x,\lambda)$ and $s(x,\lambda)$ the fundamental system of solutions of Eq. (3.135) (functions of cosine type and sine type) with the initial conditions

$$c(0,\lambda) = 1, \ c'(0,\lambda) = 0, \ s(0,\lambda) = 0, \ s'(0,\lambda) = 1. \tag{3.139}$$

Then the general solution of Eq. (3.135) is a linear combination of these functions:

$$u(x,\lambda) = C_1 c(x,\lambda) + C_2 s(x,\lambda),$$

where C_1 and C_2 are arbitrary constants. Substituting the general solution into the boundary conditions (3.136) for finding C_1 and C_2, we obtain the system of equations

$$\begin{cases} C_1 U_1(c(x,\lambda)) + C_2 U_1(s(x,\lambda)) = 0, \\ C_1 U_2(c(x,\lambda)) + C_2 U_2(s(x,\lambda)) = 0. \end{cases} \tag{3.140}$$

Hence, the boundary value problem (3.135)-(3.136) has a nonzero solution if and only if the system (3.140) has a nonzero solution. Therefore, the eigenvalues of this boundary value problem are given by the roots of its characteristic determinant

$$\triangle(\lambda)=\begin{vmatrix}U_1(c(x,\lambda)) & U_1(s(x,\lambda))\\ U_2(c(x,\lambda)) & U_2(s(x,\lambda))\end{vmatrix}.$$

Taking into account the initial condition (3.136), we can rewrite the characteristic determinant in the form

$$\triangle(\lambda)=\begin{vmatrix}a_{12}c'(1,\lambda)+a_{13}+a_{14}c(1,\lambda) & a_{11}+a_{12}s'(1,\lambda)+a_{14}s(1,\lambda)\\ a_{22}c'(1,\lambda)+a_{23}+a_{24}c(1,\lambda) & a_{21}+a_{22}s'(1,\lambda)+a_{24}s(1,\lambda)\end{vmatrix}. \tag{3.141}$$

Consider the ***Wronskian*** of the fundamental system of solutions $c(x,\lambda)$ and $s(x,\lambda)$,

$$W(x,\lambda)=c(x,\lambda)s'(x,\lambda)-c'(x,\lambda)s(x,\lambda).$$

Since these solutions are linearly independent, we have $W(x,\lambda)\neq 0$. Moreover, since these functions are the solutions of Eq. (3.135), then (we do not temporarily write the dependence on λ)

$$W'(x)=c(x)s''(x)-c''(x)s(x)=c(x)[q(x)-\lambda]s(x)-[q(x)-\lambda]c(x)s(x)=0.$$

That is, the Wronskian does not depend on x: $W(x,\lambda)=W(0,\lambda)$. Given the initial conditions (3.136) we easily obtain that $W(x,\lambda)=1$.

Using this fact, we calculate the determinant (3.141):

$$\triangle(\lambda)=-A_{12}c'(1,\lambda)-A_{23}s'(1,\lambda)-A_{14}c(1,\lambda)+A_{34}s(1,\lambda)-A_{13}-A_{24}. \tag{3.142}$$

In the particular case when $q(x)\equiv 0$, the fundamental system of solutions is written in the explicit form $c(x,\lambda)=\cos\left(\sqrt{\lambda}x\right)$ and $s(x,\lambda)=\frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}}$. Therefore, the characteristic determinant (3.142) takes the form

$$\triangle_0(\lambda)=A_{12}\sqrt{\lambda}\sin\sqrt{\lambda}-(A_{14}+A_{23})\cos\sqrt{\lambda}+A_{34}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}-A_{13}-A_{24}. \tag{3.143}$$

It is clear that it makes sense to consider the question of the eigenvalues of the problem (3.135)-(3.136) for $q(x)\equiv 0$ only for those boundary conditions, under which $\triangle_0(\lambda)$ differs from a constant. So we come to the concept of non-degenerate boundary conditions.

The boundary conditions (3.136), under one of three conditions

$$\begin{aligned}&(1)\ A_{12}\neq 0,\\ &(2)\ A_{12}=0,\ A_{14}+A_{23}\neq 0,\\ &(3)\ A_{12}=A_{14}+A_{23}=0,\ A_{34}\neq 0,\end{aligned} \tag{3.144}$$

are called the ***non-degenerate boundary conditions***. Accordingly, if

$$A_{12}=A_{14}+A_{23}=A_{34}=0, \tag{3.145}$$

then the boundary conditions (3.136) are called the ***degenerate boundary conditions***.

Although conditions (3.144) are allocated for the particular case $q(x) \equiv 0$, the non-degeneracy of the boundary conditions (3.136) guarantees the existence of the eigenvalues of the problem also in the case when $q(x) \not\equiv 0$. Then we have

Lemma 3.101 *Let the boundary conditions* (3.136) *be non-degenerate, that is, one of the three conditions* (3.144) *holds. Then for any* $q \in C[0,1]$*, the problem* (3.135)-(3.136) *has an infinite countable number of eigenvalues.*

This lemma is a consequence of a more general result which we introduce in a following section in Theorem 3.147. We have formulated this lemma here in order to emphasize the importance of assigning the class of non-degenerate conditions among all general boundary conditions.

Here it is necessary to note that *the boundary value problems with degenerate boundary conditions generally can have no eigenvalues or can have a countable number of eigenvalues, and there may be cases when any complex number is an eigenvalue of the problem.*

Example 3.102 To demonstrate some degenerate boundary conditions we consider the spectral problem

$$-u''(x) = \lambda u(x),\ u'(0) + \alpha u'(1) = 0,\ u(0) - \alpha u(1) = 0, \tag{3.146}$$

where $\alpha \in \mathbb{C}$ is a fixed number. It is easy to see that (3.145) holds for all α. Therefore, the boundary conditions in the problem (3.146) are degenerate.

For our problem from (3.143) we have

$$\triangle_0(\lambda) = -A_{13} - A_{24} = -1 + \alpha^2.$$

Then we obtain that for $\alpha^2 \neq 1$ the problem (3.146) does not have eigenvalues, and for $\alpha^2 = 1$ each number $\lambda \in \mathbb{C}$ is an eigenvalue of this problem.

Moreover, in [19] a general case was considered: the spectral problem

$$-u''(x) + q(x)u(x) = \lambda u(x),\ u'(0) + \alpha u'(1) = 0,\ u(0) - \alpha u(1) = 0,$$

with a continuous coefficient $q(x)$. It is obvious that for $\alpha = 0$ this problem is the Cauchy problem and does not have eigenvalues. It was shown in [19] that for $\alpha^2 \neq 1$ this problem does not have eigenvalues if and only if the coefficient $q(x)$ is symmetric:

$$q(x) = q(1-x), \ \ \forall x \in [0,1]. \quad \square$$

For the second-order equations there are no degenerate boundary conditions of the other kind than these conditions (see, for example, [19]). The concept of the degenerate boundary conditions can be similarly introduced also for higher-order differential equations. Then, as is shown in [1], the degenerate boundary conditions of the other kind exist. The final description of all the degenerate boundary conditions is not complete even for equations of the third and fourth orders.

The following example demonstrates that even in the case $q(x) \equiv 0$ the eigenvalues cannot always be found in an explicit form.

Example 3.103 Consider the spectral problem

$$-u''(x) = \lambda u(x),\ u'(1) - \alpha u(1) = 0,\ u(0) = 0, \tag{3.147}$$

where $\alpha \in \mathbb{C}$ is a fixed number. It is easy to see that for all α the case (2) from (3.144) holds: $A_{12} = 0$, $A_{14} + A_{23} = 1 \neq 0$. Therefore, the boundary conditions of the problem (3.147) are non-degenerate. If α is a real number, then the problem (3.147) is the self-adjoint Sturm-Liouville problem, whose spectral properties were considered in Section 3.8.

For our problem, from (3.143) we have

$$\triangle_0(\lambda) = -\cos\sqrt{\lambda} + \alpha\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}.$$

Therefore, we obtain that the eigenvalues of the problem will be squares of positive roots of the equation

$$\cot\mu = \frac{\alpha}{\mu},\ \mu = \sqrt{\lambda}. \tag{3.148}$$

Solutions of this equation for $\alpha \neq 0$ cannot be found in an explicit form. But the Rouche theorem 3.67 can be applied to this equation. According to this theorem, the roots of Eq. (3.148) are asymptotically close to the roots $\cot\mu_0 = 0$. Since $\mu_0 = k\pi - \pi/2$, $(k = 1, 2, \dots)$, we will look for the roots of Eq. (3.148) in the form

$$\mu_k = k\pi - \frac{\pi}{2} + \delta_k.$$

Substituting this into Eq. (3.148), we get

$$\cot\left(\delta_k - \frac{\pi}{2}\right) = \frac{\alpha}{k\pi - \frac{\pi}{2} + \delta_k}.$$

Therefore, $\cot\left(\delta_k - \frac{\pi}{2}\right) = O\left(\frac{1}{k}\right)$, that is, $\delta_k = O\left(\frac{1}{k}\right)$. Thus, for sufficiently large k the eigenvalues of the spectral problem (3.147) have the form

$$\lambda_k = \left(k\pi - \frac{\pi}{2} + O\left(\frac{1}{k}\right)\right)^2.$$

The eigenfunctions of the spectral problem (3.147) have the representation

$$u_k(x) = \sin\left(k\pi - \frac{\pi}{2} + O\left(\frac{1}{k}\right)\right)x,$$

and also cannot be written out in an explicit form.

This example demonstrates that even for equations of a simple form ($q(x) \equiv 0$) it is necessary to involve general methods for investigating the spectral properties. $\square$

Regardless of the concept of the non-degeneracy of the boundary conditions we now introduce the concept of *regular boundary conditions*. This concept was first introduced by G. D. Birkhoff in his works in 1908 in [14], [15], for n-th order general ordinary differential operators

$$u^{(n)}(x)+p_2(x)u^{(n-2)}(x)+\ldots+p_{n-1}(x)u'(x)+p_n(x)u(x)=\lambda u(x), \qquad (3.149)$$

with n linearly independent boundary conditions of the general form

$$U_j(u)\equiv\sum_{s=0}^{n-1}\left(a_{js}u^{(s)}(0)+b_{js}u^{(s)}(1)\right)=0,\ j=1,\ldots,n.$$

Replacing, if it is necessary, the boundary forms $U_j(u)$ by their linear combinations, one can always achieve that the boundary conditions have the form

$$a_ju^{(k_j)}(0)+b_ju^{(k_j)}(1)+\sum_{s=0}^{k_j-1}\left(a_{js}u^{(s)}(0)+b_{js}u^{(s)}(1)\right)=0,\ j=1,\ldots,n, \qquad (3.150)$$

where $|a_j|+|b_j|>0$, $n-1\geq k_1\geq k_2\geq\ldots\geq k_n\geq 0$, $k_j>k_{j+2}$.

Let us give the *definition of Birkhoff regular boundary conditions*. We denote by $\varepsilon_j=\exp\left(i\frac{2\pi j}{n}\right)$, $j=1,\ldots,n$, the roots of order n from 1.

In the odd case $n=2m-1$ the "normed" boundary conditions (3.150) are called the ***regular boundary conditions*** if the numbers θ_0 and θ_1 defined by the equality

$$\theta_0+\theta_1 s=\begin{vmatrix} a_1\varepsilon_1^{k_1} & \ldots & a_1\varepsilon_{m-1}^{k_1} & (a_1+sb_1)\varepsilon_m^{k_1} & b_1\varepsilon_{m+1}^{k_1} & \ldots & b_1\varepsilon_n^{k_1}\\ a_2\varepsilon_1^{k_2} & \ldots & a_2\varepsilon_{m-1}^{k_2} & (a_2+sb_2)\varepsilon_m^{k_2} & b_2\varepsilon_{m+1}^{k_2} & \ldots & b_2\varepsilon_n^{k_2}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ a_n\varepsilon_1^{k_n} & \ldots & a_n\varepsilon_{m-1}^{k_n} & (a_n+sb_n)\varepsilon_m^{k_n} & b_n\varepsilon_{m+1}^{k_n} & \ldots & b_n\varepsilon_n^{k_n}\end{vmatrix}$$

are different from zero.

In the even case $n=2m$, the "normed" boundary conditions (3.150) are called the ***regular boundary conditions*** if the numbers θ_{-1} and θ_1 defined by the equality

$$\frac{\theta_{-1}}{s}+\theta_0+\theta_1 s$$

$$=\begin{vmatrix} a_1\varepsilon_1^{k_1} & \ldots & a_1\varepsilon_{m-1}^{k_1} & (a_1+sb_1)\varepsilon_m^{k_1} & (a_1+\frac{1}{s}b_1)\varepsilon_{m+1}^{k_1} & b_1\varepsilon_{m+2}^{k_1} & \ldots & b_1\varepsilon_n^{k_1}\\ a_2\varepsilon_1^{k_2} & \ldots & a_2\varepsilon_{m-1}^{k_2} & (a_2+sb_2)\varepsilon_m^{k_2} & (a_2+\frac{1}{s}b_2)\varepsilon_{m+1}^{k_2} & b_2\varepsilon_{m+2}^{k_2} & \ldots & b_2\varepsilon_n^{k_2}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ a_n\varepsilon_1^{k_n} & \ldots & a_n\varepsilon_{m-1}^{k_n} & (a_n+sb_n)\varepsilon_m^{k_n} & (a_n+\frac{1}{s}b_n)\varepsilon_{m+1}^{k_n} & b_n\varepsilon_{m+2}^{k_n} & \ldots & b_n\varepsilon_n^{k_n}\end{vmatrix}$$

are different from zero.

The regular boundary conditions are called ***Birkhoff regular***. Note that in defining the regularity there appear only the coefficients from terms with the highest derivatives in the normed boundary conditions $a_ju^{(k_j)}(0)+b_ju^{(k_j)}(1)$, and the coefficients $p_s(x)$ of the differential expression (3.149) do not appear.

Later, the regular boundary conditions were studied by many authors. An important subclass of the regular boundary conditions, the so-called ***strengthened regular boundary conditions*** was defined. In the case of the odd order $n = 2m-1$ of Eq. (3.149) all the regular boundary conditions are strengthened regular. And in the case of the even order $n = 2m$ of Eq. (3.149) the regular boundary conditions, for which $\theta_0^2 - 4\theta_{-1}\theta_1 \neq 0$, are called ***strengthened regular***.

The important result established by Birkhoff consisted in estimating the resolvent of a regular differential operator and in establishing its spectrum asymptotics. We will give this result in the formulation from the monograph of M. A. Naimark [84] (1967) and only for the case of the even order $n = 2m$ in Eq. (3.149). Namely, this case will be considered below.

Theorem 3.104 (see [84]) *The eigenvalues of the n-th order differential operator* (3.149) *generated by the Birkhoff regular boundary conditions* (3.150) *form two infinite sequences λ_k' and λ_k'' ($k = N, N+1, N+2, \dots$), where N is some integer. Let the order of Eq.* (3.149) *be an even number $n = 2m$. Then*

1. *For $\theta_0^2 - 4\theta_{-1}\theta_1 \neq 0$, that is, for strengthened regular boundary conditions we have that all eigenvalues starting with some eigenvalue are simple and have the asymptotics*

$$\lambda_k' = (2k\pi)^m \left\{ (-1)^m - \frac{m \ln \xi'}{k\pi i} + O\left(\frac{1}{k^2}\right) \right\}, \tag{3.151}$$

$$\lambda_k'' = (2k\pi)^m \left\{ (-1)^m - \frac{m \ln \xi''}{k\pi i} + O\left(\frac{1}{k^2}\right) \right\}, \tag{3.152}$$

where ξ' and ξ'' are roots of the equation

$$\theta_1 \xi^2 + \theta_0 \xi + \theta_{-1} = 0, \tag{3.153}$$

and $\ln \xi$ *denotes some fixed value of the natural logarithm;*

2. *For $\theta_0^2 - 4\theta_{-1}\theta_1 = 0$, that is, for not strengthened regular boundary conditions we have that all eigenvalues starting with some eigenvalue can be simple or twofold, and have the asymptotics*

$$\lambda_k' = (2k\pi)^m \left\{ (-1)^m - \frac{m \ln \xi}{k\pi i} + O\left(\frac{1}{k^{3/2}}\right) \right\}, \tag{3.154}$$

$$\lambda_k'' = (2k\pi)^m \left\{ (-1)^m - \frac{m \ln \xi}{k\pi i} + O\left(\frac{1}{k^{3/2}}\right) \right\}, \tag{3.155}$$

where ξ is a (double) root of Eq. (3.153).

We see from this theorem the difference in asymptotics of the eigenvalues of the strengthened regular and not strengthened regular boundary value problems. As is clarified in subsequent investigations, namely the presence of multiple eigenvalues, or series of eigenvalues infinitely close to each other, causes the main difficulties in investigating the basis property of the system of root functions.

The general description of the regular and strengthened regular boundary conditions is cumbersome. Of course, for a concrete type of the boundary value problems the checking of regularity conditions may be a feasible task. However, the complete description of all classes of regular and strengthened regular boundary conditions is not complete even for fourth-order ordinary differential operators.

For the case of general boundary value problems for the second-order equation (3.135)-(3.136) such a classification does not cause difficulties. Let us single out second-order conditions. We have $n = 2$, $m = 1$, $\varepsilon_1 = -1$, $\varepsilon_2 = 1$.

Let first $A_{12} \neq 0$. In this case the boundary conditions (3.136) have the normed form. We have $a_1 = a_{11}$, $b_1 = a_{12}$, $a_2 = a_{21}$, $b_2 = a_{22}$, $k_1 = k_2 = 1$. We calculate the determinant

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} (a_1 + sb_1)\varepsilon_1^{k_1} & (a_1 + \frac{1}{s}b_1)\varepsilon_2^{k_1} \\ (a_2 + sb_2)\varepsilon_1^{k_2} & (a_2 + \frac{1}{s}b_2)\varepsilon_2^{k_2} \end{vmatrix}$$

$$= \begin{vmatrix} -(a_{11} + sa_{12}) & (a_{11} + \frac{1}{s}a_{12}) \\ -(a_{21} + sa_{22}) & (a_{21} + \frac{1}{s}a_{22}) \end{vmatrix}$$

$$= \left(s - \frac{1}{s}\right) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = A_{12}\left(s - \frac{1}{s}\right).$$

Then $\theta_{-1} = -1$, $\theta_1 = 1$, $\theta_0 = 0$. In this case the boundary conditions (3.136) are regular. Since $\theta_0^2 - 4\theta_{-1}\theta_1 = 4 \neq 0$, in this case the conditions are strengthened regular.

Let now $A_{12} = 0$, and $|a_{11}| + |a_{12}| > 0$. Then the boundary conditions (3.136) can be reduced to the normed form (we do not change the notations of the coefficients)

$$\left\{ \begin{aligned} a_{11}u'(0) + a_{12}u'(1) + a_{13}u(0) + a_{14}u(1) &= 0, \\ a_{23}u(0) + a_{24}u(1) &= 0. \end{aligned} \right. \tag{3.156}$$

We have $a_1 = a_{11}$, $b_1 = a_{12}$, $a_2 = a_{23}$, $b_2 = a_{24}$, $k_1 = 1$, $k_2 = 0$. We calculate the determinant

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} -(a_{11} + sa_{12}) & a_{11} + \frac{1}{s}a_{12} \\ a_{23} + sa_{24} & a_{23} + \frac{1}{s}a_{24} \end{vmatrix} = -\left(s + \frac{1}{s}\right)(A_{14} + A_{23}) - 2(A_{13} + A_{24}).$$

Then, $\theta_{-1} = \theta_1 = -(A_{14} + A_{23})$, $\theta_0 = -2(A_{13} + A_{24})$. That is, in this case the boundary conditions (3.156) are regular under the additional condition $A_{14} + A_{23} \neq 0$. The same condition provides the validity of the assumption $|a_{11}| + |a_{12}| > 0$. The condition of the strengthened regularity will be written in the form $\theta_0^2 - 4\theta_{-1}\theta_1 = (A_{13} + A_{24})^2 - (A_{14} + A_{23})^2 \neq 0$.

Consider the remaining case $A_{12}=0$, with $a_{11}=a_{12}=0$. Then the boundary conditions (3.136) can be reduced to the normed form

$$\begin{cases} a_{13}u(0)+a_{14}u(1)=0, \\ a_{23}u(0)+a_{24}u(1)=0. \end{cases} \tag{3.157}$$

We have $a_1=a_{13}$, $b_1=a_{14}$, $a_2=a_{23}$, $b_2=a_{24}$, $k_1=k_2=0$. We calculate the determinant

$$\frac{\theta_{-1}}{s}+\theta_0+\theta_1 s=\begin{vmatrix} a_{13}+sa_{14} & a_{13}+\frac{1}{s}a_{14} \\ a_{23}+sa_{24} & a_{23}+\frac{1}{s}a_{24} \end{vmatrix}=A_{34}\left(\frac{1}{s}-s\right).$$

Then, $\theta_{-1}=-\theta_1=A_{34}$, $\theta_0=0$. The inequality $A_{34}\neq 0$ is satisfied in view of the linear independence of the boundary conditions (3.157). Hence, in this case the boundary conditions (3.157) are regular. Since $\theta_0^2-4\theta_{-1}\theta_1=4A_{34}^2\neq 0$, these boundary conditions are strengthened regular.

Since in this case $a_{11}=a_{12}=a_{21}=a_{22}=0$, all determinants $A_{12}=A_{13}=A_{14}=A_{23}=A_{24}=0$ are equal to zero, except that $A_{34}\neq 0$.

Let us formulate the obtained result in the form of a theorem.

Theorem 3.105 *The boundary conditions* (3.136) *are regular in the following three cases:*

$$\begin{aligned} &(1)\ A_{12}\neq 0, \\ &(2)\ A_{12}=0,\ A_{14}+A_{23}\neq 0, \\ &(3)\ A_{12}=A_{13}=A_{14}=A_{23}=A_{24}=0,\ A_{34}\neq 0. \end{aligned} \tag{3.158}$$

Here, the boundary conditions will be strengthened regular in the cases (1) *and* (3)*, and in the case* (2) *under the additional condition*

$$A_{13}+A_{24}\neq\pm\left(A_{14}+A_{23}\right). \tag{3.159}$$

Corollary 3.106 *For the case of the second-order equation* (3.135) *all the regular boundary conditions* (3.136) *are non-degenerate.*

Indeed, it is easy to verify the statement of the corollary by comparing conditions (3.144) and (3.158). Here, the boundary conditions can be non-degenerate and simultaneously irregular in the case when $A_{12}=0$, $A_{14}+A_{23}=0$, $A_{34}\neq 0$, and one of the determinants $|A_{13}|+|A_{14}|+|A_{23}|+|A_{24}|>0$ is not equal to zero.

Example 3.107 Consider the spectral problem

$$-u''(x)=\lambda u(x),\ u'(0)-\alpha u(1)=0,\ u(0)=0, \tag{3.160}$$

where $\alpha\in\mathbb{C}$ is a fixed number. It is easy to see that for all $\alpha\neq 0$ item (3) from (3.144) holds: $A_{12}=0$, $A_{14}+A_{23}=0$, $A_{34}=\alpha\neq 0$. Therefore, the boundary conditions of the problem (3.160) are non-degenerate.

Here, the determinant $A_{13} = 1$ is not equal to zero since condition (3) from (3.158) does not hold. Hence, the boundary conditions of the problem (3.160) are not regular.

For our problem from (3.143) we have

$$\triangle_0(\lambda) = A_{34}\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - A_{13} = \alpha\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - 1.$$

From this we obtain that the eigenvalues are the roots of this equation in the form

$$\sin\sqrt{\lambda} = \frac{\sqrt{\lambda}}{\alpha},$$

and they have asymptotics which do not coincide at all with the asymptotics (3.151)-(3.155) in Theorem 3.104. □

For convenience of use we reformulate Theorem 3.105 in terms of coefficients of the boundary conditions (3.136).

Theorem 3.108 *The boundary conditions* (3.136) *are regular, if one of the following three conditions holds:*

$$\begin{aligned} &(1)\ a_{11}a_{22} - a_{12}a_{21} \neq 0;\\ &(2)\ a_{11}a_{22} - a_{12}a_{21} = 0,\ |a_{11}| + |a_{12}| > 0,\ a_{11}a_{24} + a_{12}a_{23} \neq 0;\\ &(3)\ a_{11} = a_{12} = a_{21} = a_{22} = 0,\ a_{13}a_{24} - a_{14}a_{23} \neq 0. \end{aligned} \tag{3.161}$$

The regular boundary conditions are strengthened regular in the first and third cases, and in the second case under the additional condition

$$a_{11}a_{23} + a_{12}a_{24} \neq (a_{11}a_{24} + a_{12}a_{23}). \tag{3.162}$$

In the case (1) from (3.161), the regular boundary conditions can be reduced to the form

$$\begin{cases} u'(0) + a_{13}u(0) + a_{14}u(1) = 0,\\ u'(1) + a_{23}u(0) + a_{24}u(1) = 0. \end{cases}$$

The boundary conditions of such type are called ***boundary conditions solvable with respect to the highest derivative***.

For example, the *Sturm type boundary conditions*

$$u'(0) - hu(0) = 0,\ \ u'(1) + Hu(1) = 0,$$

are strengthened regular. Therefore the Sturm-Liouville problem is a strengthened regular boundary value problem. The particular case of this problem occurring when $h = H = 0$:

$$u'(0) = 0,\ \ u'(1) = 0,$$

is called the *Neumann (or second-type) boundary conditions*, and the corresponding boundary value problem is called the Neumann problem or the second boundary value problem.

In case (3) from (3.161), it is easy to see that it is a first-type boundary condition

$$u(0)=0, \;\; u(1)=0.$$

They are also called the *Dirichlet boundary condition*, and the corresponding boundary value problem is called the Dirichlet problem.

Let us consider now regular but not strengthened regular boundary conditions for the second-order equation (3.135). As follows from Theorem 3.105, such boundary conditions can occur only in case (2) from (3.158) when conditions (3.159) do not simultaneously hold:

$$A_{12}=0, \; A_{14}+A_{23}\neq 0, \; A_{13}+A_{24}=\pm\left(A_{14}+A_{23}\right).$$

Let us represent these conditions in a more convenient form in terms of the coefficients of the boundary condition (3.136). As follows from Theorem 3.108, the regular but not strengthened regular boundary conditions can be written in the form

$$\begin{cases} a_{11}u'(0)+a_{12}u'(1)+a_{13}u(0)+a_{14}u(1)=0, \\ \qquad\qquad\qquad\qquad a_{23}u(0)+a_{24}u(1)=0, \end{cases} \tag{3.163}$$

when $|a_{11}|+|a_{12}|>0$ and two conditions

$$a_{11}a_{24}+a_{12}a_{23}\neq 0, \tag{3.164}$$

$$a_{11}a_{23}+a_{12}a_{24}=\left(a_{11}a_{24}+a_{12}a_{23}\right) \tag{3.165}$$

simultaneously hold.

Theorem 3.109 ([86], [110]) *If the boundary conditions* (3.136) *are regular but not strengthened regular, they can be always reduced to the form* (3.163) *(with* $|a_{11}|+|a_{12}|>0$*) of one of the following four types:*

$$\begin{array}{lll} I. & a_{11}=a_{12}, & a_{23}\neq -a_{24}; \\ II. & a_{11}=-a_{12}, & a_{23}\neq a_{24}; \\ III. & a_{23}=a_{24}, & a_{11}\neq -a_{12}; \\ IV. & a_{23}=-a_{24}, & a_{11}\neq a_{12}. \end{array} \tag{3.166}$$

Indeed, condition (3.165) can be written in the form

$$\left(a_{11}\pm a_{12}\right)\left(a_{23}\pm a_{24}\right)=0,$$

that is, even one of the equalities of condition (3.166) holds. If one of these equalities holds, condition (3.164) provides the validity of the corresponding inequality from (3.166). The theorem is proved. □

Corollary 3.110 *All regular, but not strengthened regular boundary conditions can be reduced to one of the four forms:*

$$\begin{cases} u'(0)-u'(1)+au(0)+bu(1)=0, \\ \qquad\qquad\qquad u(0)+\alpha u(1)=0, \end{cases} \quad \begin{cases} u'(0)+u'(1)+au(0)+bu(1)=0, \\ \qquad\qquad\qquad u(0)-\alpha u(1)=0, \end{cases}$$

$$\begin{cases} u'(0) + \alpha u'(1) + au(0) + bu(1) = 0, \\ u(0) - u(1) = 0, \end{cases} \quad \begin{cases} u'(0) - \alpha u'(1) + au(0) + bu(1) = 0, \\ u(0) + u(1) = 0, \end{cases}$$

where $\alpha \neq 1$, and the coefficients a and b can be arbitrary.

For $\alpha = 1$ these boundary conditions are degenerate (and consequently, are not regular).

3.13 Biorthogonal systems in Hilbert spaces

As we have demonstrated in Sections 3.5 and 3.8, the eigenvectors of self-adjoint operators are important for the expansion of an arbitrary vector in the form of a series. According to the Hilbert-Schmidt theorem 3.32 for a compact self-adjoint operator A in a Hilbert space H, any element $A\varphi \in H$ decomposes into a convergent Fourier series with respect to the system x_k of normalised eigenvectors of the operator A.

According to Theorem 3.61, the Sturm-Liouville operator L is a self-adjoint linear operator in $L^2(0,1)$ and any element $f \in L^2(0,1)$ is decomposed into converging Fourier series with respect to the system $u_k(x)$ of the normed eigenvectors of the operator L:

$$f(x) = \sum_{k=1}^{\infty} \langle f, u_k \rangle u_k(x).$$

For the case of non-selfadjoint operators the situation is much more complicated. As follows from Example 3.102, the non-self-adjoint operator may have no eigenvalues and eigenfunctions. The operator of the Samarskii-Ionkin problem, as shown in Example 3.59, has an infinite number of associated vectors. Such a variety does not allow us to apply the general methods simultaneously to all non-self-adjoint operators.

Therefore, the results for ordinary differential operators with general order boundary conditions cannot be obtained as a consequence of the spectral theory of self-adjoint operators.

To consider a possibility of representing an arbitrary element in the form of an expansion into a series with respect to root vectors of a differential operator, it is necessary that there is a sufficient number of them. So, we naturally come to the concept of the completeness of a system of elements.

A system of elements of a Hilbert space is said to be a ***complete system*** if any vector orthogonal to all vectors of this system is equal to zero. In a Hilbert space, the properties of completeness and closeness of the system are equivalent. The system of elements $\{x_k\}$ is said to be a ***closed system*** in the Hilbert space H if the linear span of this system is everywhere dense in H. That is, any element of the space H can be approximated by a linear combination of elements of this system $\{x_k\}$ with any accuracy in the norm of the space H.

An important analogue of the linear independence of elements in an infinite-dimensional space is the concept of the minimality of a system. The system of elements $\{x_k\}$ is said to be a ***minimal system*** in H if none of its elements belongs to the closure of the linear span of the other elements of this system.

Two systems of elements $\{x_k\}$ and $\{z_k\}$ are said to be ***biorthogonal systems*** in H if the relation

$$\langle x_k, z_j \rangle = \delta_{kj} \equiv \begin{cases} 1, & k = j, \\ 0, & k \neq j; \end{cases} \tag{3.167}$$

holds for all values of the indices k and j. Here δ_{kj} is the Kronecker delta.

In particular, any orthonormal system $\{x_k\}$ is biorthogonal to itself: $\langle x_k, x_j \rangle = \delta_{kj}$.

Example 3.111 In $L^2(0,1)$, consider the system of functions $u_k(x) = \alpha^{-x} e^{2k\pi ix}$, $k \in \mathbb{Z}$, where $\alpha \in \mathbb{C}$ is a fixed number. It is easy to calculate that

$$\langle u_k, u_j \rangle = \int_0^1 \alpha^{-x} e^{2k\pi ix} \overline{\alpha}^{-x} e^{-2j\pi ix} dx = \int_0^1 |\alpha|^{-2x} e^{2(k-j)\pi ix} dx \neq \delta_{kj}$$

for $|\alpha| \neq 1$. Therefore, the system $\{u_k\}$ is not orthogonal and cannot be the system biorthogonal to itself.

It is easy to make sure that $v_k(x) = \overline{\alpha}^x e^{2k\pi ix}$, $k \in \mathbb{Z}$, is a biorthogonal system. Indeed, we have

$$\langle u_k, v_j \rangle = \int_0^1 \alpha^{-x} e^{2k\pi ix} \alpha^x e^{-2j\pi ix} dx = \int_0^1 e^{2(k-j)\pi ix} dx = \delta_{kj}.$$

Note that the system $\{u_k\}$ is a system of eigenfunctions of the boundary value problem for the first-order ordinary differential operator

$$u'(x) = \lambda u(x), \; u(0) = \alpha u(1),$$

corresponding to the eigenvalues $\lambda_k = -\ln\alpha + 2k\pi i$. Consequently, the biorthogonal system $\{v_k\}$ is the system of eigenfunctions of the adjoint boundary value problem:

$$-v'(x) = \overline{\lambda} v(x), \; \overline{\alpha} v(0) = v(1). \quad \square$$

We will continue the discussion of this setting in Example 3.143. □

The closeness and minimality of biorthogonal systems are closely related to each other and are the dual concepts.

Theorem 3.112 (see [68]) *A system $\{x_k\}$ is minimal in H if and only if there exists a biorthogonal system dual to it, that is, a system $\{z_k\}$, such that* (3.167) *holds. Moreover, if the original system $\{x_k\}$ is at the same time closed and minimal in H, then its (biorthogonally) dual system $\{z_k\}$ is uniquely defined.*

Example 3.113 Let a closed and minimal system in H have the form $\{x_{0k}, x_{1k}\}_{k\in\mathbb{N}}$. Then by Theorem 3.112 there exists a biorthogonal system dual to it, that is, a system $\{z_{0k}, z_{1k}\}_{k\in\mathbb{N}}$, such that

$$\langle x_{ik}, z_{jn}\rangle = \delta_{ij}\delta_{kn}, \text{ where } i, j = 0, 1 \text{ and } k, n \in \mathbb{N}$$

holds.

Let $\{C_k\}_{k\in\mathbb{N}}$ be a scalar sequence. We now consider the system

$$\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}},$$

and show that this system is a closed and minimal system in H and construct a biorthogonal system for it.

The minimality of this system is evident and it follows from the minimality of the initial system $\{x_{0k}, x_{1k}\}_{k\in\mathbb{N}}$. Suppose that the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}}$ is not closed. Then there exists a vector $f \in H$ that is orthogonal to all vectors of this system:

$$\langle x_{0k}, f\rangle = 0, \ \langle x_{1k} + C_k x_{0k}, f\rangle = 0, \ \forall k \in \mathbb{N}.$$

Then we have

$$\langle x_{0k}, f\rangle = 0, \ \langle x_{1k}, f\rangle = 0, \ \forall k \in \mathbb{N},$$

that is, this vector f is orthogonal to all vectors of the system $\{x_{0k}, x_{1k}\}_{k\in\mathbb{N}}$. Since this system is closed in H, we have $f = 0$, which proves the closeness of the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}}$.

Let us now show that the system $\left\{z_{0k} - \overline{C_k} z_{1k}, z_{1k}\right\}_{k\in\mathbb{N}}$ is a biorthogonal system. Indeed, firstly, equalities

$$\begin{aligned}
&\langle x_{0k}, z_{0n} - \overline{C_k} z_{1n}\rangle = \langle x_{0k}, z_{0n}\rangle = \delta_{kn};\\
&\langle x_{0k}, z_{1n}\rangle = 0;\\
&\langle x_{1k} + C_k x_{0k}, z_{0n} - \overline{C_k} z_{1n}\rangle = C_k\langle x_{0k}, z_{0n}\rangle - C_k\langle x_{1k}, z_{1n}\rangle = C_k\delta_{kn} - C_k\delta_{kn} = 0;\\
&\langle x_{1k} + C_k x_{0k}, z_{1n}\rangle = \langle x_{1k}, z_{1n}\rangle = \delta_{kn};
\end{aligned}$$

hold for all values of the indices $k, n \in \mathbb{N}$ regardless of the choice of the numeric sequence $\{C_k\}_{k\in\mathbb{N}}$.

Therefore, for the systems $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}}$ and $\left\{z_{0k} - \overline{C_k} z_{1k}, z_{1k}\right\}_{k\in\mathbb{N}}$, the biorthogonality conditions hold.

Secondly, as we have proved, the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}}$ is closed in H. Therefore, according to Theorem 3.112, the biorthogonal system is uniquely defined. □

In Example 3.111 for the system consisting of eigenfunctions of the boundary value problem for the differential equation, the biorthogonal system turned out to be the system consisting of eigenfunctions of the adjoint boundary value problem. The following general result holds.

Theorem 3.114 *Let A be a densely defined operator on a Hilbert space H with a compact resolvent. Suppose that its system of root vectors is a closed and minimal system in H. Then the system biorthogonal to it consists of the root vectors of the operator A^*.*

To prove this, we denote by $\{x_k\}_{k\in\mathbb{N}}$ the system of root vectors of the operator A, and let $\{z_k\}_{k\in\mathbb{N}}$ be the biorthogonally adjoint system. We write the equation for eigen- and associated vectors in a single form

$$Ax_k - \lambda_k x_k = \theta_k x_{k-1},$$

where $\theta_k = 0$ if x_k is an eigenvector, and $\theta_k = 1$ if x_k is an associated vector (in this case we additionally require that $\lambda_k = \lambda_{k-1}$).

For all values of the indices $k, j \in \mathbb{N}$ we consider the inner product

$$0 = \left\langle Ax_k - \lambda_k x_k - \theta_k x_{k-1}, z_j \right\rangle = \left\langle x_k, A^* z_j \right\rangle - \left\langle x_k, \overline{\lambda}_k z_j \right\rangle - \theta_k \left\langle x_{k-1}, z_j \right\rangle$$

$$= \left\langle x_k, A^* z_j - \overline{\lambda}_j z_j - \theta_{j+1} z_{j+1} \right\rangle + (\lambda_j - \lambda_k) \left\langle x_k, z_j \right\rangle + \theta_{j+1} \left\langle x_k, z_{j+1} \right\rangle - \theta_k \left\langle x_{k-1}, z_j \right\rangle.$$

In view of the biorthogonality conditions, the second term here is equal to zero for all $k, j \in \mathbb{N}$, and the third and fourth terms are equal to zero for $k \neq j+1$. But if $k = j+1$, then the third and fourth terms are equal to zero in the sum. Thus, for all $k, j \in \mathbb{N}$ we get

$$\left\langle x_k, A^* z_j - \overline{\lambda}_j z_j - \theta_{j+1} z_{j+1} \right\rangle = 0.$$

Since the system $\{x_k\}_{k\in\mathbb{N}}$ is closed in H, we have

$$A^* z_j - \overline{\lambda}_j z_j = \theta_{j+1} z_{j+1}.$$

Let $x_n, x_{n+1}, \ldots, x_{n+m}$ be a chain of the eigenvector x_n and all the associated vectors $x_{n+1}, \ldots, x_{n+m}$ of an operator A corresponding to one eigenvalue $\lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+m}$. Then, $\theta_n = 0$ and $\theta_{n+1} = \ldots = \theta_{n+m} = 1$. Also, $\theta_{n+m+1} = 0$.

Hence,

$$A^* z_{n+m} - \overline{\lambda}_n z_{n+m} = 0,$$

$$A^* z_{n+m-1} - \overline{\lambda}_n z_{n+m-1} = z_{n+m}, \ \ldots, \ A^* z_n - \overline{\lambda}_n z_n = z_{n+1}.$$

Thus, $z_n, z_{n+1}, \ldots, z_{n+m}$ is a chain of the eigenvector z_{n+m} and of all the associated vectors $z_{n+m-1}, \ldots, z_n$ of the operator A^* corresponding to one eigenvalue $\overline{\lambda}_n$. Note that unlike the initial system, the numbering inside the chains for the adjoint system goes in the inverse direction: from the most recent associated vector to the first one.

Theorem 3.114 is proved. □

As we have noted earlier, the associated vectors of an operator are constructed not uniquely, but up to a linear combination of the eigenvector and the associated vectors of the lower order. Modulo this, the biorthogonal system is constructed uniquely. Hence, for a fixed system of the eigen- and associated vectors of an operator A, one needs to choose as the biorthogonal system an arbitrary but well-defined system of eigen- and associated vectors of the operator A^*. The following example demonstrates this fact.

Example 3.115 Let a closed and minimal system in H have the form $\{x_{0k}, x_{1k}\}_{k\in\mathbb{N}}$ and consist of eigenvectors x_{0k} and associated vectors x_{1k} of the operator A. That is, all eigenvalues of the operator A are twofold and

$$Ax_{0k} - \lambda_k x_{0k} = 0, \;\; Ax_{1k} - \lambda_k x_{1k} = x_{0k}.$$

By Theorem 3.112 there exists a biorthogonal system $\{z_{0k}, z_{1k}\}_{k\in\mathbb{N}}$, dual to it. According to Theorem 3.114, this biorthogonal system consists of eigen- and associated vectors of the adjoint operator. Here, z_{1k} is the eigenvector, and z_{0k} is the associated vector:

$$A^* z_{1k} - \overline{\lambda}_k z_{1k} = 0, \;\; A^* z_{0k} - \overline{\lambda}_k z_{0k} = z_{1k}.$$

Let us choose another chain of eigen- and associated vectors of the operator A: $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}}$, where C_k are some constants. Then, as follows from Example 3.113, this system is the closed and minimal system in H, and the biorthogonal system will be the system $\left\{z_{0k} - \overline{C}_k z_{1k}, z_{1k}\right\}_{k\in\mathbb{N}}$.

As is easy to see, in this new system the vector z_{1k} is an eigenvector, and $z_{0k} - \overline{C}_k z_{1k}$ is the associated vector. □

3.14 Biorthogonal expansions and Riesz bases

Let $\{x_k\}$ be a closed and minimal system in a Hilbert space H, and let $\{z_k\}$ be a system biorthogonal to it, that is, we have

$$\langle x_k, z_j \rangle = \delta_{kj}$$

is the Kronecker's delta, which is equal to 1 for $j = k$ and 0 otherwise, see (3.167). Assume that the equality

$$f = \sum_{k=1}^{\infty} f_k x_k$$

holds with some numbers f_k. If we form the inner product of this with $\{z_j\}$, by the biorthogonality condition (3.167) we formally obtain the equalities

$$f_k = \langle f, z_k \rangle.$$

The numbers f_k are said to be the ***Fourier coefficients*** with respect to the biorthogonal system.

The ***biorthogonal expansion*** of the vector $f \in H$ with respect to $\{x_k\}$ is the series

$$f \sim \sum_{k=1}^{\infty} \langle f, z_k \rangle x_k. \tag{3.168}$$

Here the sign $\sim$ means the correspondence of this series to this function. We cannot put the sign of equality until we justify the convergence of the series in the right-hand side of (3.168).

The system $\{x_k\}$ is said to form a ***basis of the space*** H if, for any element $f \in H$, there exists a unique expansion of f with respect to the elements of this system:

$$f = \sum_{k=1}^{\infty} f_k x_k. \tag{3.169}$$

That is, the series (3.169) is convergent to f in the norm of the space H. The same definition of a basis also makes sense for *Banach spaces.*

It is obvious that any basis $\{x_k\}$ is a closed and minimal system in H, and, therefore by Theorem 3.112, we can uniquely find its biorthogonal dual system $\{z_k\}$, and hence the expansion of any element of f with respect to the basis $\{x_k\}$ coincides with its biorthogonal expansion (3.168). Thus, *the basis property of a closed and minimal system is equivalent to the convergence of the biorthogonal expansions*.

A basis $\{x_k\}$ in the space H is said to be an ***unconditional basis***, if it remains a basis for any permutation for its elements.

If the system $\{x_k\}$ is a basis (unconditional basis) of the space H, then the biorthogonally adjoint system $\{z_k\}$ is also a basis (unconditional basis) of the space H.

Theorem 3.116 ([46]) *In order for a system $\{x_k\}$ to be a basis in a Banach space X it is necessary and sufficient that there exists a positive constant $\alpha > 0$ such that for all indices $k = 1, 2, \ldots$, we have*

$$\rho\left(S_k, L^k\right) := \inf_{x' \in S_k,\ x'' \in L^k} \|x' - x''\| \geq \alpha. \tag{3.170}$$

Here

$$S_k = \{x \in L_k : \|x\| = 1\}$$

is the unit sphere in L_k, where

$$L_k = \text{Span } \{x_1, \ldots, x_k\}$$

is the linear space spanned by the vectors $\{x_1, \ldots, x_k\}$, and

$$L^k = \overline{\text{Span } \{x_{k+1}, x_{k+2}, \ldots\}}.$$

Of course, the checking of condition (3.170) for all indices $k = 1, 2, \ldots$ is not an easy task even for systems given in an explicit form. Therefore, there has been intensive research for justifying various kinds of necessary and sufficient conditions.

A harder requirement than the minimality is the concept of a uniform minimal system. We say that the system $\{x_k\}$ is ***uniformly minimal*** in H, if there exists a constant $\alpha > 0$ such that for all values of the indices k, we have

$$\rho(x_k, E_k) := \inf_{x \in E_k} \|x_k - x\| \geq \alpha,$$

where E_k is the closure of the linear span of all elements $\{x_j\}$ with indices $j \neq k$.

Theorem 3.117 (see [68]) *A closed and minimal system* $\{x_k\}$ *is uniformly minimal in a Hilbert space* H *if and only if there exists a constant* $C > 0$ *such that for all values of the indices* k *we have*

$$\|x_k\| \cdot \|z_k\| \leq C < \infty, \tag{3.171}$$

where $\{z_k\}$ *is the biorthogonal system to* $\{x_k\}$*.*

Indeed, let $\{x_k\}$ be a basis in H. Then the biorthogonal expansions $\sum_{k=1}^{\infty} \langle f, z_k \rangle x_k$ converge in H. Therefore, $\|\langle f, z_k \rangle x_k\| \to 0$ as $k \to \infty$. That is, $\langle f, z_k \|x_k\| \rangle \to 0$ as $k \to \infty$ for all $f \in H$. Hence, the norms of the elements $z_k \|x_k\|$ are totally bounded, that is, the condition (3.171) holds. □

Corollary 3.118 *Any basis in* H *is a uniformly minimal system, and, therefore,* (3.171) *holds.*

The condition (3.171) is said to be a ***condition for the uniform minimality*** of the system $\{x_k\}$. Indeed, there exist examples of closed, minimal and uniformly minimal systems which are not a basis. Therefore, (3.171) is a necessary condition for being a basis.

Example 3.119 Consider a particular case of the system from Example 3.113. Let an orthonormal system in H have the form $\{x_{0k}, x_{1k}\}_{k \in \mathbb{N}}$. Then

$$\langle x_{ik}, x_{jn} \rangle = \delta_{ij} \delta_{kn};\ i, j = 0, 1;\ k, n \in \mathbb{N}$$

holds.

Let $\{C_k\}_{k \in N}$ be some scalar sequence. Consider the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k \in \mathbb{N}}$. As follows from Example 3.113, this system is a closed and minimal system in H and the biorthogonal system for it will be the system $\left\{x_{0k} - \overline{C_k} x_{1k}, x_{1k}\right\}_{k \in \mathbb{N}}$.

Let us now verify the condition for the uniform minimality (3.171). We have

$$\|x_{0k}\| \cdot \|x_{0k} - \overline{C_k} x_{1k}\| = \sqrt{1 + |C_k|^2},$$

$$\|x_{1k} + C_k x_{0k}\| \cdot \|x_{1k}\| = \sqrt{1 + |C_k|^2}.$$

Therefore, the condition for the uniform minimality is a uniform boundedness of the sequence $\{C_k\}_{k \in \mathbb{N}}$. If this sequence is not bounded, then the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k \in \mathbb{N}}$ does not form a basis in H.

Let us show now that if the sequence $\{C_k\}_{k \in \mathbb{N}}$ is uniformly bounded, then the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k \in \mathbb{N}}$ forms a basis in H. In this case there exists a constant $\widehat{C}$ such that $|C_k| \leq \widehat{C}$ for all $k \in \mathbb{N}$.

For an arbitrary vector $f \in H$, consider a partial sum of the biorthogonal series

$$S_n = S_{0n} + S_{1n}, \text{ where } S_{0n} = \sum_{k=1}^{n} \langle f, x_{0k} - \overline{C}_k x_{1k} \rangle x_{0k},\ S_{1n} = \sum_{k=1}^{n} \langle f, x_{1k} \rangle \left(x_{1k} + C_k x_{0k}\right).$$

It is easy to see that for each fixed n the partial sum S_n coincides with the partial sum of the expansion of the vector f with respect to the orthonormal basis $\{x_{0k}, x_{1k}\}_{k\in\mathbb{N}}$:

$$\widehat{S}_{0n} + \widehat{S}_{1n} = \sum_{k=1}^{n} \langle f, x_{0k}\rangle x_{0k} + \sum_{k=1}^{n} \langle f, x_{1k}\rangle x_{1k}.$$

Hence, if the partial sum S_n converges, then it converges to f.

To prove the convergence of S_n it is sufficient to show the convergence of the partial sums S_{0n} and S_{1n}. Let us show that these sequences are Cauchy sequences. We have

$$\|S_{0n} - S_{0(n+p)}\|^2 = \left\| \sum_{k=n+1}^{n+p} \left\langle f, x_{0k} - \overline{C}_k x_{1k} \right\rangle x_{0k} \right\|^2 = \sum_{k=n+1}^{n+p} \left| \left\langle f, x_{0k} - \overline{C}_k x_{1k} \right\rangle \right|^2$$

$$\leq 2 \sum_{k=n+1}^{n+p} |\langle f, x_{0k}\rangle|^2 + 2 \sum_{k=n+1}^{n+p} |\langle f, x_{1k}\rangle|^2 \, |C_k|^2 .$$

Hence, since the sequence $\{C_k\}_{k\in\mathbb{N}}$ is uniformly bounded, we obtain

$$\|S_{0n} - S_{0(n+p)}\|^2 \leq 2\|\widehat{S}_{0n} - \widehat{S}_{0(n+p)}\|^2 + 2|\widehat{C}|^2 \|\widehat{S}_{1n} - \widehat{S}_{1(n+p)}\|^2.$$

Since $\widehat{S}_{0n}$ and $\widehat{S}_{1n}$ are Cauchy sequences, then S_{0n} is also a Cauchy sequence. The proof for S_{1n} is similar.

Hence, the sequences of the partial sums S_{0n} and S_{1n} converge in H. That is, for an arbitrary vector $f \in H$ its biorthogonal series converges to f:

$$f = \sum_{k=1}^{\infty} \langle f, x_{0k} - \overline{C}_k x_{1k}\rangle x_{0k} + \sum_{k=1}^{\infty} \langle f, x_{1k}\rangle \left(x_{1k} + C_k x_{0k}\right).$$

This means that the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}}$ is a basis in H. □

Let us formulate a final result of Example 3.115 in the form of a lemma.

Lemma 3.120 *Let an orthonormal system in H have the form $\{x_{0k}, x_{1k}\}_{k\in\mathbb{N}}$. Let $\{C_k\}_{k\in\mathbb{N}}$ be some scalar sequence. Then the system $\{x_{0k}, x_{1k} + C_k x_{0k}\}_{k\in\mathbb{N}}$ is a closed and minimal system in H. This system forms a basis of the space H if and only if the sequence $\{C_k\}_{k\in\mathbb{N}}$ is bounded.*

In her fundamental paper, N.K. Bari [9] (1951) introduced new definitions which turned out to be widely used in the future.

A complete and minimal system $\{x_k\}$ is said to be a ***Bessel system*** in H if for any vector $f \in H$ the series of squares of coefficients of its biorthogonal expansion converges:

$$\sum_{k=1}^{\infty} |\langle f, z_k\rangle|^2 < \infty,$$

where $\{z_k\}$ is a biorthogonal system to $\{x_k\}$. In other words, the system $\{x_k\}$ is said to be Bessel in H if there exists a constant $M > 0$ such that for any $f \in H$, the following ***Bessel-type inequality*** holds:

$$\sum_{k=1}^{\infty} |\langle f, z_k \rangle|^2 \leq M \|f\|^2.$$

A complete and minimal system $\{x_k\}$ is said to be a ***Hilbert system*** in H if for any sequence of numbers ξ_k such that $\sum_{k=1}^{\infty} |\xi_k|^2 < \infty$ there exists a unique vector f, for which these ξ_k are the coefficients of its biorthogonal expansion: $\xi_k = \langle f, z_k \rangle$. In other words, the system $\{x_k\}$ is said to be Hilbert in H if there exists a constant $m > 0$ such that for any $f \in H$, the following ***Hilbert-type inequality*** holds:

$$m\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, z_k \rangle|^2.$$

The Bessel and Hilbert properties for systems from a biorthogonal pair $\{x_k\}$ and $\{z_k\}$ are dual to one another: if one of the systems is a Bessel system in H, then the other is a Hilbert system in H, and vice versa ([9]).

A complete and minimal system $\{x_k\}$ is said to be a ***Riesz basis*** in H, if it is simultaneously a Bessel and a Hilbert system in H.

Theorem 3.121 ([9]) *A closed and minimal system $\{x_k\}$ in H is a Riesz basis in H if and only if there exists a bounded invertible operator S such that the system $\{Sx_k\}$ is an orthonormal basis in H.*

If the system $\{x_k\}$ is a Riesz basis in H, then its biorthogonal system $\{z_k\}$ is also a Riesz basis in H.

Corollary 3.122 *If a system $\{x_k\}$ is a Riesz basis in H, then it is also a basis in H.*

Indeed, let the system $\{x_k\}$ be a Riesz basis in H. Then by Theorem 3.121 there exists a bounded invertible operator S such that the system $\{Sx_k\}$ is an orthonormal basis in H. For an arbitrary vector $f \in H$ we write an expansion of the vector Sf with respect to this orthonormal basis $\{Sx_k\}$:

$$Sf = \sum_{k=1}^{\infty} \langle Sf, Sx_k \rangle Sx_k.$$

Since the operator S is bounded and defined on the whole H, then the adjoint operator S^* exists. Therefore,

$$Sf = \sum_{k=1}^{\infty} \langle f, S^* Sx_k \rangle Sx_k.$$

Acting on this equality by the operator S^{-1}, we will have

$$f = \sum_{k=1}^{\infty} \langle f, S^* Sx_k \rangle x_k.$$

Thus, for the arbitrary vector $f \in H$ we obtain its expansion into a series with respect to elements of the system $\{x_k\}$. Therefore, the system $\{x_k\}$ is a basis in H.

Note that in the process of proving the lemma we also have found that the biorthogonally adjoint system is $\{S^*Sx_k\}$. □

Thus, the Riesz bases are bases equivalent to the orthonormal bases. Other significant results have been obtained earlier for such bases. The following theorem gives the connection between the concepts of the Riesz basis and an unconditional basis.

Theorem 3.123 (E. R. Lorch [76], I. M. Gelfand [42]) *A closed and minimal system $\{x_k\}$ in H is a Riesz basis in H if and only if it is an unconditional basis **almost normalised** in H, i.e. there exist constants $m > 0$ and $M > 0$ such that for all values of the indices k we have*

$$0 < m \leq \|x_k\| \leq M < \infty.$$

Note that if a basis is almost normalised, then the biorthogonal basis to it is also almost normalised.

Let us give two more important facts indicating another similarity of the Riesz bases with the orthonormal ones.

Theorem 3.124 ([9]) *Let a system $\{x_k\}$ be a Riesz basis in H. If the series*

$$\sum_{k=1}^{\infty} |f_k|^2 < \infty,$$

converges, then also the series

$$\sum_{k=1}^{\infty} f_k x_k < \infty$$

is convergent with respect to the norm of the space H, and the converse is also true.

Theorem 3.125 ([9]) *Let a system $\{x_k\}$ be a Riesz basis in H. Then for any vector $f \in H$ we have the two-sided inequality*

$$m\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, x_k \rangle|^2 \leq M\|f\|^2.$$

The inequality in Theorem 3.125 can be improved to an equality if we choose the equivalent norms for the appearing expressions appropriately. Let us explain this following [105].

Let us take biorthogonal systems

$$\mathscr{U} := \{u_k |\ u_k \in H\}_{k\in\mathbb{N}}$$

and

$$\mathscr{V} := \{v_k |\ v_k \in H\}_{k\in\mathbb{N}}$$

in a separable Hilbert space H. We assume that $\mathscr{U}$ (and hence also $\mathscr{V}$) is a Riesz basis in H.

It is convenient to introduce the $\mathscr{U}$- and $\mathscr{V}$-Fourier transforms (coefficients) by formulae

$$\mathscr{F}_{\mathscr{U}}(f)(k) := \langle f, v_k \rangle =: \widehat{f}(k) \tag{3.172}$$

and

$$\mathscr{F}_{\mathscr{V}}(g)(k) := \langle g, u_k \rangle =: \widehat{g}_*(k), \tag{3.173}$$

respectively, for all $f, g \in H$ and for each $k \in \mathbb{N}$. Here $\widehat{g}_*$ stands for the $\mathscr{V}$-Fourier transform of the function g. Indeed, in general $\widehat{g}_* \neq \widehat{g}$. Their inverses are given by

$$(\mathscr{F}_{\mathscr{U}}^{-1} a)(x) := \sum_{k \in \mathbb{N}} a(k) u_k \tag{3.174}$$

and

$$(\mathscr{F}_{\mathscr{V}}^{-1} a)(x) := \sum_{k \in \mathbb{N}} a(k) v_k. \tag{3.175}$$

Let us denote by

$$l^2_{\mathscr{U}} = l^2(\mathscr{U})$$

the linear space of complex-valued functions a on $\mathbb{N}$ such that $\mathscr{F}_{\mathscr{U}}^{-1} a \in H$, i.e. if there exists $f \in H$ such that $\mathscr{F}_{\mathscr{U}} f = a$. Then the space of sequences $l^2_{\mathscr{U}}$ is a Hilbert space with the inner product

$$\langle a, \, b \rangle_{l^2_{\mathscr{U}}} := \sum_{k \in \mathbb{N}} a(k) \, \overline{(\mathscr{F}_{\mathscr{V}} \circ \mathscr{F}_{\mathscr{U}}^{-1} b)(k)}, \tag{3.176}$$

for arbitrary $a, b \in l^2_{\mathscr{U}}$. The reason for this choice of the definition is the following formal calculation:

$$\begin{aligned}
\langle a, \, b \rangle_{l^2_{\mathscr{U}}} &= \sum_{\xi \in \mathbb{N}} a(k) \, \overline{(\mathscr{F}_{\mathscr{V}} \circ \mathscr{F}_{\mathscr{U}}^{-1} b)(k)} \\
&= \sum_{k \in \mathbb{N}} a(k) \overline{\left(\mathscr{F}_{\mathscr{U}}^{-1} b, u_k \right)} \\
&= \left(\left[\sum_{k \in \mathbb{N}} a(k) u_k \right], \mathscr{F}_{\mathscr{U}}^{-1} b \right) \\
&= \langle \mathscr{F}_{\mathscr{U}}^{-1} a, \, \mathscr{F}_{\mathscr{U}}^{-1} b \rangle,
\end{aligned}$$

which implies the Hilbert space properties of the space of sequences $l^2_{\mathscr{U}}$. The norm of $l^2_{\mathscr{U}}$ is then given by the formula

$$\|a\|_{l^2_{\mathscr{U}}} = \left(\sum_{k \in \mathbb{N}} a(k) \, \overline{(\mathscr{F}_{\mathscr{V}} \circ \mathscr{F}_{\mathscr{U}}^{-1} a)(k)} \right)^{1/2}, \quad \text{for all } a \in l^2_{\mathscr{U}}. \tag{3.177}$$

We note that individual terms in this sum may be complex-valued but the formula implies that the whole sum is real and nonnegative.

Analogously, we introduce the Hilbert space

$$l^2_{\mathscr{V}} = l^2(\mathscr{V})$$

as the space of sequences a on $\mathbb{N}$ such that $\mathscr{F}_{\mathscr{V}}^{-1}a \in H$, with the inner product

$$\langle a,\, b\rangle_{l^2_{\mathscr{V}}} := \sum_{k\in\mathbb{N}} a(k)\, \overline{(\mathscr{F}_{\mathscr{U}} \circ \mathscr{F}_{\mathscr{V}}^{-1} b)(k)}, \tag{3.178}$$

for arbitrary $a, b \in l^2_{\mathscr{V}}$. The norm of $l^2_{\mathscr{V}}$ is given by the formula

$$\|a\|_{l^2_{\mathscr{V}}} = \left(\sum_{k\in\mathbb{N}} a(k)\, \overline{(\mathscr{F}_{\mathscr{U}} \circ \mathscr{F}_{\mathscr{V}}^{-1} a)(k)} \right)^{1/2}$$

for all $a \in l^2_{\mathscr{V}}$. The spaces of sequences $l^2_{\mathscr{U}}$ and $l^2_{\mathscr{V}}$ are thus generated by biorthogonal systems $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$.

Since by Theorem 3.121 the Riesz bases are equivalent to an orthonormal basis by an invertible linear transformation, we have the equality between the spaces $l^2_{\mathscr{U}} = l^2_{\mathscr{V}} = \ell^2(\mathbb{N})$ as sets.

With the definitions as above, we have the following Plancherel identity essentially established in [103], and finally in [105]:

Theorem 3.126 (Biorthogonal Plancherel identity, [105]) *If $f, g \in H$, then $\widehat{f}, \widehat{g} \in l^2_{\mathscr{U}}$, $\widehat{f}_*, \widehat{g}_* \in l^2_{\mathscr{V}}$, and the inner products* (3.176), (3.178) *take the form*

$$(\widehat{f},\, \widehat{g})_{l^2_{\mathscr{U}}} = \sum_{k\in\mathbb{N}} \widehat{f}(k)\, \overline{\widehat{g}_*(k)}$$

and

$$(\widehat{f}_*,\, \widehat{g}_*)_{l^2_{\mathscr{V}}} = \sum_{k\in\mathbb{N}} \widehat{f}_*(k)\, \overline{\widehat{g}(k)},$$

respectively. In particular, we have

$$\overline{(\widehat{f},\, \widehat{g})_{l^2_{\mathscr{U}}}} = (\widehat{g}_*,\, \widehat{f}_*)_{l^2_{\mathscr{V}}}.$$

The Parseval identity takes the form

$$\langle f, g\rangle_H = (\widehat{f}, \widehat{g})_{l^2_{\mathscr{U}}} = \sum_{k\in\mathbb{N}} \widehat{f}(k)\, \overline{\widehat{g}_*(k)}. \tag{3.179}$$

Furthermore, for any $f \in H$, we have $\widehat{f} \in l^2_{\mathscr{U}}$, $\widehat{f}_* \in l^2_{\mathscr{V}}$, and

$$\|f\|_H = \|\widehat{f}\|_{l^2_{\mathscr{U}}} = \|\widehat{f}_*\|_{l^2_{\mathscr{V}}}. \tag{3.180}$$

Now, let us briefly discuss a few other features of the biorthogonal Fourier analysis, such as the ℓ^p-types spaces, and their interpolation properties and duality. For this, we introduce a scale of Banach spaces $\{H^p\}_{1\le p\le\infty}$ with the norms $\|\cdot\|_p$ such that

$$H^p \subseteq H$$

and with the property

$$|\langle x, y\rangle_H| \le \|x\|_{H^p} \|y\|_{H^q} \tag{3.181}$$

for all $1 \leq p \leq \infty$, where $\frac{1}{p}+\frac{1}{q}=1$. We assume that $H^2 = H$, and that H^p are real interpolation properties in the following sense:

$$(H^1, H^2)_{\theta,p} = H^p,\ 0<\theta<1,\ \frac{1}{p} = 1-\frac{\theta}{2},$$

and

$$(H^2, H^\infty)_{\theta,p} = H^p,\ 0<\theta<1,\ \frac{1}{p} = \frac{1-\theta}{2}.$$

We also assume that $\mathscr{U} \subset H^p$ and $\mathscr{V} \subset H^p$ for all $p \in [1,\infty]$.

Here, we will not discuss the basics of the real interpolation, referring the reader, for example, to [13]. However, let us give some examples. If $H = L^2(\Omega)$ for some Ω, we could take $H^p = L^2(\Omega)\cap L^p(\Omega)$. If $H = S_2(\mathscr{K})$ is the Hilbert space of the Hilbert-Schmidt operators on a Hilbert space $\mathscr{K}$, then we can take $H^p = S_2(\mathscr{K})\cap S_p(\mathscr{K})$, where $S_p(\mathscr{K})$ stands for the space of p-Schatten-von Neumann operators on $\mathscr{K}$.

We now introduce the p-Lebesgue versions of the spaces of Fourier coefficients. We define spaces $l^p_{\mathscr{U}} = l^p(\mathscr{U})$ as the spaces of all $a : \mathbb{N} \to \mathbb{C}$ such that

$$\|a\|_{l^p(\mathscr{U})} := \left(\sum_{k\in\mathbb{N}} |a(k)|^p \|u_k\|^{2-p}_{H^\infty}\right)^{1/p} < \infty, \quad \text{for } 1 \leq p \leq 2, \tag{3.182}$$

and

$$\|a\|_{l^p(\mathscr{U})} := \left(\sum_{k\in\mathbb{N}} |a(k)|^p \|v_k\|^{2-p}_{H^\infty}\right)^{1/p} < \infty, \quad \text{for } 2 \leq p < \infty, \tag{3.183}$$

and, for $p=\infty$, we define

$$\|a\|_{l^\infty(\mathscr{U})} := \sup_{k\in\mathbb{N}} \left(|a(k)| \cdot \|v_k\|^{-1}_{H^\infty}\right) < \infty.$$

Here, without loss of generality, we can assume that $u_k \neq 0$ and $v_k \neq 0$ for all $k \in \mathbb{N}$, so that the above spaces are well-defined.

Analogously, we introduce spaces $l^p_{\mathscr{V}} = l^p(\mathscr{V})$ as the spaces of all $b : \mathbb{N} \to \mathbb{C}$ such that

$$\|b\|_{l^p(\mathscr{V})} = \left(\sum_{k\in\mathbb{N}} |b(k)|^p \|v_k\|^{2-p}_{H^\infty}\right)^{1/p} < \infty, \quad \text{for } 1 \leq p \leq 2,$$

$$\|b\|_{l^p(\mathscr{V})} = \left(\sum_{k\in\mathbb{N}} |b(k)|^p \|u_k\|^{2-p}_{H^\infty}\right)^{1/p} < \infty, \quad \text{for } 2 \leq p < \infty,$$

$$\|b\|_{l^\infty(\mathscr{V})} = \sup_{k\in\mathbb{N}} \left(|b(k)| \cdot \|u_k\|^{-1}_{H^\infty}\right).$$

The introduced spaces have the expected interpolation properties.

Theorem 3.127 ([105]) *For* $1 \leq p \leq 2$, *we have*

$$(l^1(\mathscr{U}), l^2(\mathscr{U}))_{\theta,p} = l^p(\mathscr{U}),$$

$$(l^1(\mathscr{V}), l^2(\mathscr{V}))_{\theta,p} = l^p(\mathscr{V}),$$

where $0 < \theta < 1$ *and* $p = \frac{2}{2-\theta}$.

In a standard way, the Plancherel identity in Theorem 3.126, the obvious $H^1 - \ell^\infty$ and $\ell^1 - H^\infty$ estimates following from (3.181), and the interpolation in Theorem 3.127 imply the following ***Hausdorff-Young inequality***.

Theorem 3.128 ([105]) *Assume that* $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then there exists a constant* $C_p \geq 1$ *such that*

$$\|\widehat{f}\|_{l^{p'}(\mathscr{U})} \leq C_p \|f\|_{H^p} \quad \text{and} \quad \|\mathscr{F}_{\mathscr{U}}^{-1} a\|_{H^{p'}} \leq C_p \|a\|_{l^p(\mathscr{U})} \tag{3.184}$$

for all $f \in H^p$ *and* $a \in l^p(\mathscr{U})$. *Similarly, for all* $b \in l^p(\mathscr{V})$ *we obtain*

$$\|\widehat{f}_*\|_{l^{p'}(\mathscr{V})} \leq C_p \|f\|_{H^p} \quad \text{and} \quad \|\mathscr{F}_{\mathscr{V}}^{-1} b\|_{H^{p'}} \leq C_p \|b\|_{l^p(\mathscr{V})}. \tag{3.185}$$

Finally, we record the duality between spaces $l^p(\mathscr{U})$ and $l^q(\mathscr{V})$:

Theorem 3.129 ([105]) *Let* $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$(l^p(\mathscr{U}))' = l^q(\mathscr{V}) \quad \text{and} \quad (l^p(\mathscr{V}))' = l^q(\mathscr{U}).$$

We now go back to discussing the Riesz bases. It is necessary to pay attention to the fact that even a simplest transformation among the basis elements can lead to the violation of the basis property. The following example demonstrates this.

Example 3.130 Let $\{x_k\}_{k\in\mathbb{N}}$ be a Riesz basis in H. Consider the sequence

$$y_k = x_1 + x_{k+1}, \ k \in \mathbb{N}.$$

Let us show that the system $\{y_k\}_{k\in\mathbb{N}}$, although closed and almost normalised in H, does not give a basis in H.

Let us check first of all that this system $\{y_k\}_{k\in\mathbb{N}}$ is closed. Suppose that it is not closed. Then there exists an element $f_0 \in H$, $f_0 \neq 0$, orthogonal to all elements of the system $\{y_k\}_{k\in\mathbb{N}}$. Then we have

$$\langle f_0, y_k \rangle = \langle f_0, x_1 \rangle + \langle f_0, x_{k+1} \rangle = 0, \ k \in \mathbb{N}.$$

Therefore, $\langle f_0, x_k \rangle = -\langle f_0, x_1 \rangle$ for all $k > 1$. But by Theorem 3.124 we have $\langle f_0, x_k \rangle \to 0$ as $k \to \infty$. Hence, $\langle f_0, x_k \rangle = 0$ for all $k \in \mathbb{N}$. Then $f_0 = 0$, which proves the closedness of the system $\{y_k\}_{k\in\mathbb{N}}$.

Let us show that the system $\{y_k\}_{k\in\mathbb{N}}$ does not give a basis in H. Indeed, suppose that this system is a basis in H. Then the element x_1 must be also represented in the form of the expansion with respect to the basis:

$$x_1 = \sum_{k=1}^{\infty} \xi_k y_k.$$

Then,

$$0 = \left(\sum_{k=1}^{\infty} \xi_k - 1\right) x_1 + \sum_{k=2}^{\infty} \xi_{k-1} x_k.$$

Since $\{x_k\}_{k\in\mathbb{N}}$ is a Riesz basis in H, we get

$$\sum_{k=1}^{\infty} \xi_k = 1; \;\; \xi_k = 0, \; k \geq 1.$$

It is obvious that this system of equations has no solutions. Hence, the system $\{y_k\}_{k\in\mathbb{N}}$ is not a basis in H. $\square$

The transformations among the elements of the basis, considered in this example, although looking simple and innocent, have turned out to be quite significant.

A remarkable property of the Riesz bases is their stability with respect to small (in some sense) perturbations. The system of the functions $\{y_k\}$ is said to be ***quadratically close*** to $\{x_k\}$ if the scalar series

$$\sum_{k=1}^{\infty} \|y_k - x_k\|^2 < \infty$$

converges.

Theorem 3.131 ([9]) *Any minimal system that is quadratically close to a Riesz basis in H, is also a Riesz basis in H.*

Example 3.132 In $L^2(0,1)$, consider a system of functions $\{u_k(x)\}_{k=0}^{\infty}$, where

$$u_k(x) = \cos(\mu_k x),$$

and μ_k are the solutions of the equation $\tan\mu = \frac{\alpha}{\mu}$, and α is an arbitrary complex number. This system is the system of eigenfunctions of the boundary value problem (3.147) considered in Example 3.103. In that example we have shown that the sequence μ_k has the asymptotics

$$\mu_k = k\pi + \delta_k, \;\; \delta_k = O(1/k). \tag{3.186}$$

Using Theorem 3.131, we can show that this system is a Riesz basis in $L^2(0,1)$. Let us choose a system $v_0(x) = 1$, $v_k(x) = \cos(k\pi x)$, $k = 1,2,\ldots$. It is well known that the system of functions $\left\{1, \sqrt{2}\cos(k\pi x)\right\}$ is an orthonormal basis in $L^2(0,1)$. The expansion into a series with respect to this basis is the classical trigonometric Fourier series.

Our system $\{v_k(x)\}_{k=0}^{\infty}$ is obtained from this orthonormal basis by multiplying by the constant $\sqrt{2}$. Such action is a bounded and invertible operator in $L^2(0,1)$. Therefore, according to Theorem 3.121, the system $\{v_k\}_{k=0}^{\infty}$ is a Riesz basis in $L^2(0,1)$.

Let us now show that the systems $\{u_k(x)\}_{k=0}^{\infty}$ and $\{v_k(x)\}_{k=0}^{\infty}$ are quadratically close. Indeed, since

$$u_k(x) - v_k(x) = \cos(\mu_k x) - \cos(k\pi x) = 2\sin\left(k\pi + \frac{1}{2}\delta_k\right)x \cdot \sin\left(\frac{1}{2}\delta_k\right)x,$$

in view of the asymptotics (3.186), we get that $u_k(x) - v_k(x) = O\left(\frac{1}{k}\right)$. Hence, the series $\sum\limits_{k=1}^{\infty} \|u_k - v_k\|^2 < \infty$ is convergent.

Thus, the systems $\{u_k(x)\}_{k=0}^{\infty}$ and $\{v_k(x)\}_{k=0}^{\infty}$ are quadratically close. By Theorem 3.131 this implies that the system $\{u_k\}_{k=0}^{\infty}$ of the eigenfunctions of the boundary value problem (3.147) is a Riesz basis in $L^2(0,1)$. $\square$

The property (3.171) is a necessary condition for a system $\{x_k\}$ to be a basis. However, an essential inconvenience for this condition is the need to use the biorthogonal system $\{z_k\}$, whose construction may not be an easy task.

Let us give one more necessary condition for unconditional bases in Hilbert spaces.

Theorem 3.133 *Let a system $\{x_k\}$ be closed, minimal and almost normalised in H. Assume that there are infinitely increasing sequences of integers k_i and n_j such that the following inner products are different from zero*

$$|\langle x_{k_i}, x_{n_j}\rangle| \geq \alpha > 0 \tag{3.187}$$

as $i, j \to \infty$. Then the system $\{x_k\}$ is not an unconditional basis in H.

To prove this, we suppose the opposite, namely, that the system $\{x_k\}$ is an unconditional basis in H. Since this system is almost normalised in H, then by Theorem 3.123 it is a Riesz basis in H. Hence (see Theorem 3.121), a biorthogonally adjoint system $\{z_k\}$ is also a Riesz basis in H.

For all indices k_i we represent vectors x_{k_i} in the form of a biorthogonal expansion with respect to the basis $\{z_k\}$:

$$x_{k_i} = \sum_{k=1}^{\infty} \langle x_{k_i}, x_k\rangle z_k.$$

Since $\{z_k\}$ is a Riesz basis in H, then by Theorem 3.124 the series of squares of the Fourier coefficients converges:

$$\sum_{k=1}^{\infty} |\langle x_{k_i}, x_k\rangle|^2 < \infty.$$

This implies that the sequence $\langle x_{k_i}, x_k\rangle$ goes to zero as $k \to \infty$, which contradicts condition (3.187). The theorem is proved. $\square$.

Example 3.134 Let us demonstrate an application of Theorem 3.133 using the example of the system from Example 3.130. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Riesz basis in H. Consider the sequence

$$y_k = x_1 + x_{k+1},\ k \in \mathbb{N}.$$

In Example 3.130 we have shown that the system $\{y_k\}_{k\in\mathbb{N}}$ is closed and almost normalised in H. For all indices $k \neq n$ we calculate the inner products

$$\langle y_k, y_n\rangle = \langle x_1 + x_{k+1}, x_1 + x_{n+1}\rangle = \|x_1\|^2 + \langle x_1, x_{n+1}\rangle + \langle x_{k+1}, x_1\rangle + \langle x_{k+1}, x_{n+1}\rangle.$$

Since $\{x_k\}_{k\in\mathbb{N}}$ is a Riesz basis in H, here the second, third and fourth terms vanish as $k \to \infty$ and $n \to \infty$. However, the first term remains fixed. Hence, the system $\{u_k\}_{k\in\mathbb{N}}$ satisfies condition (3.187) of Theorem 3.133 and, therefore, is not a basis. □

Let us give another necessary condition for the unconditional bases in a Hilbert space.

Theorem 3.135 *Let $\{x_k\}$ be a closed and minimal system in H. If the system $\{x_k\}$ is an unconditional basis in H, then*

$$\lim_{k\to\infty} \left| \left\langle \frac{x_k}{\|x_k\|}, \frac{x_{k+1}}{\|x_{k+1}\|} \right\rangle \right| < 1. \tag{3.188}$$

Indeed, let us denote $y_k := \frac{x_k}{\|x_k\|}$. Then $\|y_k\| = 1$ and, therefore, in view of Theorem 3.123, the system $\{y_k\}$ is a Riesz basis in H. Hence, by Theorem 3.116 there exists a constant $\alpha > 0$ such that for all constants C we have

$$\|y_{k+1} - Cy_k\| \geq \alpha.$$

Therefore, we have

$$\alpha^2 \leq \langle y_{k+1} - Cy_k, y_{k+1} - Cy_k\rangle = 1 + |C|^2 - \overline{C}\langle y_{k+1}, y_k\rangle - C\overline{\langle y_{k+1}, y_k\rangle}.$$

Let us represent the coefficient C in the form of $C = t\langle y_{k+1}, y_k\rangle$, where $t \in \mathbb{R}$ is an arbitrary real coefficient. Then for any t we have

$$0 \leq \left(1 - \alpha^2\right) - 2t\,|\langle y_{k+1}, y_k\rangle|^2 + t^2\,|\langle y_{k+1}, y_k\rangle|^2.$$

This is a quadratic polynomial of a single variable t. Since this inequality must hold for all $t \in \mathbb{R}$, the discriminant of the quadratic equation is non-positive, that is,

$$\triangle = |\langle y_{k+1}, y_k\rangle|^4 - \left(1 - \alpha^2\right)|\langle y_{k+1}, y_k\rangle|^2 \leq 0.$$

Therefore,

$$|\langle y_{k+1}, y_k\rangle|^2 \leq 1 - \alpha^2.$$

Since $\alpha > 0$, this proves inequality (3.188). □

Corollary 3.136 *For a system $\{x_k\}$ to be an unconditional basis in H, it is necessary that the inequality* (3.188) *holds.*

Let us show an application of this necessary condition in an example of a system of eigenfunctions of the boundary value problem.

Example 3.137 Consider the spectral problem

$$-u''(x) = \lambda u(x),\ u'(0) - u'(1) - \alpha u(1) = 0,\ u(0) = 0, \tag{3.189}$$

where $\alpha \in \mathbb{C}$ is a fixed number.

For all α, the part (2) from (3.144) holds: $A_{12} = 0$, $A_{23} + A_{14} = -1 \neq 0$. Therefore, the boundary conditions of problem (3.189) are non-degenerate.

If $\alpha = 0$, then this problem is said to be the Samarskii-Ionkin problem. We have considered it in Example 3.59, where we have shown that all eigenvalues of the problem (except the zero value) are twofold, and the root subspaces consist of one eigenfunction and one associated function.

Let us consider the problem (3.189) for $\alpha \neq 0$. For this case, from (3.143) we have

$$\triangle_0(\lambda) = \cos\sqrt{\lambda} + \alpha\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - 1 = 2\sin\frac{\sqrt{\lambda}}{2}\left(\alpha\frac{\cos\frac{\sqrt{\lambda}}{2}}{\sqrt{\lambda}} - \sin\frac{\sqrt{\lambda}}{2}\right).$$

Therefore, we obtain that the eigenvalues of the problem are squares of roots of two equations

$$\sin\mu = 0,\ \ \tan\mu = \frac{\alpha}{2\mu},\quad \text{where } \mu = \frac{\sqrt{\lambda}}{2}. \tag{3.190}$$

Solutions of the first equation can be found explicitly to be $\mu_{1k} = k\pi$. And for the solutions of the second equation in (3.190), we can write out an asymptotic formula using the Rouche theorem 3.67: $\mu_{2k} = k\pi + \delta_k$, where $\delta_k = O\left(\frac{1}{k}\right)$. From this it is obvious that all the eigenvalues are simple.

From the second condition for the problem (3.189) we obtain that all eigenfunctions have the form $u(x) = \sin\sqrt{\lambda}x$.

Thus, the spectral problem (3.189) has two series of simple eigenvalues

$$\lambda_{1k} = (k\pi)^2,\ k = 1,2,\ldots;\ \ \lambda_{2k} = (k\pi + \delta_k)^2,\ k = 0,1,2,\ldots,$$

with the corresponding eigenfunctions

$$u_{1k}(x) = \sin(k\pi x),\ k = 1,2,\ldots;\ \ u_{2k}(x) = \sin\left((k\pi + \delta_k)x\right),\ k = 0,1,2,\ldots,$$

The problem has no associated functions.

Let us calculate the inner product

$$\langle u_{1k}, u_{2k}\rangle = \int_0^1 \sin(k\pi x)\sin\left(\left(k\pi + \overline{\delta_k}\right)x\right)dx = \frac{\sin\left(\overline{\delta_k}\right)}{2\overline{\delta_k}}\frac{4k\pi}{4k\pi + \overline{\delta_k}}.$$

Since $\delta_k = O\left(\frac{1}{k}\right)$, we obtain that

$$\lim_{k\to\infty} \langle u_{1k}, u_{2k}\rangle = \frac{1}{2}.$$

It is easy to calculate norms of the eigenfunctions:

$$\|u_{1k}\| = \frac{1}{\sqrt{2}}; \quad \lim_{k\to\infty} \|u_{2k}\| = \frac{1}{\sqrt{2}}.$$

As a result, we finally have

$$\lim_{k\to\infty} \left| \left\langle \frac{u_{1k}}{\|u_{1k}\|}, \frac{u_{2k}}{\|u_{2k}\|} \right\rangle \right| = 1.$$

Hence, the necessary condition for the unconditional basis property (3.188) is not satisfied. Therefore, the system $\{\sin(k\pi x), \sin((k\pi+\delta_k)x)\}$ of all eigenfunctions of the problem (3.189) is not an unconditional basis in $L^2(0,1)$. □

3.15 Convolutions in Hilbert spaces

Let us now discuss convolutions in Hilbert spaces corresponding to biorthogonal systems. We continue with the setting of the previous section, so we fix the biorthogonal systems

$$\mathscr{U} := \{u_k |\ u_k \in H\}_{k\in\mathbb{N}}$$

and

$$\mathscr{V} := \{v_k |\ v_k \in H\}_{k\in\mathbb{N}}$$

in a separable Hilbert space H. We assume that $\mathscr{U}$ (and hence also $\mathscr{V}$) is a Riesz basis in H, i.e. any element of H has a unique decomposition with respect to the elements of H.

We will be also using the Fourier analysis notation as set in (3.172)-(3.175).

We now define the ***$\mathscr{U}$– and $\mathscr{V}$–convolutions*** in the following form:

$$f \star_{\mathscr{U}} g := \sum_{k\in\mathbb{N}} \langle f, v_k\rangle \langle g, v_k\rangle u_k \tag{3.191}$$

and

$$h \star_{\mathscr{V}} j := \sum_{k\in\mathbb{N}} \langle h, u_k\rangle \langle j, u_k\rangle v_k \tag{3.192}$$

for appropriate elements $f, g, h, j \in H$. We can call them ***biorthogonal convolutions.*** These convolutions are clearly commutative and associative, and have a number of properties expected from convolutions, which we will briefly discuss here. First we note that these convolutions are well-defined.

Theorem 3.138 *Let $f \star_{\mathscr{U}} g$ and $h \star_{\mathscr{V}} j$ be defined by* (3.191) *and* (3.192)*, respectively. Then there exists a constant $M > 0$ such that we have*

$$\|f \star_{\mathscr{U}} g\|_H \leq M \sup_{k\in\mathbb{N}} \|u_k\|_H \|f\|_H \|g\|_H, \quad \|h \star_{\mathscr{V}} j\|_H \leq M \sup_{k\in\mathbb{N}} \|v_k\|_H \|h\|_H \|j\|_H, \tag{3.193}$$

for all $f, g, h, j \in H$.

The statement follows from the Cauchy-Schwarz inequality, from Theorem 3.124, and from the uniform boundedness

$$\sup_{k\in\mathbb{N}} \|u_k\|_H + \sup_{k\in\mathbb{N}} \|v_k\|_H < \infty,$$

assured by Theorem 3.123.

The above convolutions have been introduced in [105] where also their main properties have been analysed. Some properties have been also established in [61]. In the rest of this section we follow the analysis of [105].

There is a straightforward relation between $\mathscr{U}$- and $\mathscr{V}$-convolutions, and the Fourier coefficients defined in (3.172)-(3.175).

Theorem 3.139 *For arbitrary $f, g, h, j \in H$ we have*

$$\widehat{f \star_{\mathscr{U}} g} = \widehat{f}\,\widehat{g}, \quad \widehat{h \star_{\mathscr{V}} j}_* = \widehat{h}_* \,\widehat{j}_*.$$

Therefore, the convolutions are commutative and associative.

Let $K : H \times H \to H$ be a bilinear mapping. If for all $f, g \in H$, the form $K(f, g)$ satisfies the property

$$\widehat{K(f,g)} = \widehat{f}\,\widehat{g} \tag{3.194}$$

then K is the $\mathscr{U}$–convolution, i.e. $K(f, g) = f *_{\mathscr{U}} g$.

Similarly, if $K(f, g)$ satisfies the property

$$\widehat{K(f,g)}_* = \widehat{f}_*\,\widehat{g}_* \tag{3.195}$$

then K is the $\mathscr{V}$–convolution, i.e. $K(f, g) = f *_{\mathscr{V}} g$.

Indeed, by direct calculations we have

$$\begin{aligned}
\mathscr{F}_{\mathscr{U}}(f \star_{\mathscr{U}} g)(k) &= \langle \sum_{l\in\mathbb{N}} \widehat{f}(l)\widehat{g}(l) u_l, v_k \rangle \\
&= \sum_{l\in\mathbb{N}} \widehat{f}(l)\widehat{g}(l) \langle u_l, v_k \rangle \\
&= \widehat{f}(k)\widehat{g}(k).
\end{aligned}$$

The commutativity follows from the bijectivity of the $\mathscr{U}$–Fourier transform, also implying the associativity. This can be also seen from the definition:

$$\begin{aligned}
((f \star_{\mathscr{U}} g) \star_{\mathscr{U}} h) &= \sum_{k\in\mathbb{N}} \langle \sum_{l\in\mathbb{N}} \widehat{f}(l)\widehat{g}(l)u_l, v_k\rangle \widehat{h}(k)u_k \\
&= \sum_{k\in\mathbb{N}} \widehat{f}(k)\widehat{g}(k)\widehat{h}(k)u_k \\
&= \sum_{k\in\mathbb{N}} \widehat{f}(k) \left[\sum_{l\in\mathbb{N}} \widehat{g}(l)\widehat{h}(l)\langle u_l, v_k\rangle\right] u_k \\
&= \sum_{k\in\mathbb{N}} \widehat{f}(k)\langle \sum_{l\in\mathbb{N}} \widehat{g}(l)\widehat{h}(l)u_l, v_k\rangle u_k \\
&= (f \star_{\mathscr{U}} (g \star_{\mathscr{U}} h)).
\end{aligned}$$

Next, let us show that K is the $\mathscr{U}$-convolution under the assumption (3.194). The similar property for $\mathscr{V}$-convolutions under assumption (3.195) follows by simply replacing $\mathscr{U}$ by $\mathscr{V}$ in the part concerning $\mathscr{U}$-convolutions.

Since for arbitrary $f,g \in H$ and for $K(f,g) \in H$ the property (3.194) holds, we can obtain $K(f,g)$ from the inverse $\mathscr{U}$–Fourier transform formula:

$$K(f,g) = \sum_{k\in\mathbb{N}} \widehat{K(f,g)}(k)u_k = \sum_{k\in\mathbb{N}} \widehat{f}(k)\,\widehat{g}(k)u_k.$$

The last expression defines the $\mathscr{U}$–convolution. $\square$

Very much as for the classical distributions, we can define "distribution-type" behaviour in Hilbert spaces and to link it to properties of convolutions.

In contrast to previous sections where we have analysed the spectrum of operators, we now look at operators corresponding to given collections of eigenvalues. So, we fix some sequence $\Lambda := \{\lambda_k\}_{k\in\mathbb{N}}$ of complex numbers such that the series

$$\sum_{k\in\mathbb{N}} (1+|\lambda_k|)^{-s_0} < \infty, \tag{3.196}$$

converges for some $s_0 > 0$.

We then associate to the pair $(\mathscr{U}, \Lambda)$ a linear operator $L : H \to H$ by the formula

$$Lf := \sum_{k\in\mathbb{N}} \lambda_k \langle f, v_k\rangle u_k, \tag{3.197}$$

for those $f \in H$ for which this series converges in H. The operator L is densely defined in H since $Lu_k = \lambda_k u_k$ for all $k \in \mathbb{N}$, and $\mathscr{U}$ is a Riesz basis in H. In particular, we observe that $\operatorname{Span}(\mathscr{U}) \subset D(L) \subset H$. We will say that *$L$ is the operator associated to the pair $(\mathscr{U}, \Lambda)$.*

In a similar way to the above, we define the operator $L^* : H \to H$ by

$$L^* g := \sum_{k\in\mathbb{N}} \overline{\lambda_k}\langle g, u_k\rangle v_k,$$

for those $g \in H$ for which it makes sense. Then L^* is densely defined since $L^* v_k = \overline{\lambda_k} v_k$ and $\mathscr{V}$ is a basis in H, $\operatorname{Span}(\mathscr{V}) \subset D(L^*) \subset H$. One can also readily check that the equality

$$\langle Lf, g\rangle = \langle f, L^* g\rangle = \sum_{k\in\mathbb{N}} \lambda_k \langle f, v_k\rangle\langle g, u_k\rangle$$

holds on their domains, so that the notation L^* is justified.

Let us describe the space of rapidly decreasing sequences on $\mathbb{N}$, in analogy to the Schwartz rapidly decaying functions on $\mathbb{R}^n$. We will write $\varphi \in \mathscr{S}(\mathbb{N})$ if for any $M < \infty$ there exists a constant $C_{\varphi,M}$ such that

$$|\varphi(k)| \leq C_{\varphi,M}(1+|\lambda_k|)^{-M} \tag{3.198}$$

holds for all $k \in \mathbb{N}$. The topology on $\mathscr{S}(\mathbb{N})$ is given by the seminorms p_k, $k \in \mathbb{N}_0$, defined by

$$p_k(\varphi) := \sup_{k\in\mathbb{N}} (1+|\lambda_k|)^k |\varphi(k)|.$$

Continuous anti-linear functionals on $\mathscr{S}(\mathbb{N})$ are of the form

$$\varphi \mapsto \langle w, \varphi\rangle := \sum_{k\in\mathbb{N}} w(k)\overline{\varphi(k)},$$

where functions $w : \mathbb{N} \to \mathbb{C}$ grow at most polynomially at infinity, i.e. there exist constants $M < \infty$ and $C_{w,M}$ such that

$$|w(k)| \leq C_{w,M}(1+|\lambda_k|)^M$$

holds for all $k \in \mathbb{N}$. Such distributions $w : \mathbb{N} \to \mathbb{C}$ form the space of distributions which we denote by $\mathscr{S}'(\mathbb{N})$, with the distributional duality extending the inner product on $\ell^2(\mathbb{N})$.

Let us now define spaces of test functions and distribution associated to the given biorthogonal systems and the set Λ. The following constructions can be also expressed in terms of the so-called rigged Hilbert spaces, but we will avoid such language for the simplicity of the exposition:

(i) the spaces of $(\mathscr{U}, \Lambda)$- and $(\mathscr{V}, \Lambda)$-test functions are defined by

$$\mathscr{C}^\infty_{\mathscr{U},\Lambda} := \bigcap_{j=0}^{\infty} \mathscr{C}^j_{\mathscr{U},\Lambda},$$

where

$$\mathscr{C}^j_{\mathscr{U},\Lambda} := \{\phi \in H : \ |\langle \phi, v_k\rangle| \leq C(1+|\lambda_k|)^{-j} \text{ for some constant } C \text{ for all } k \in \mathbb{N}\},$$

and

$$\mathscr{C}^\infty_{\mathscr{V},\Lambda} := \bigcap_{j=0}^{\infty} \mathscr{C}^j_{\mathscr{V},\Lambda},$$

where

$$\mathscr{C}^j_{\mathscr{V},\Lambda} := \{\psi \in H : \ |\langle \psi, u_\xi\rangle| \leq C(1+|\lambda_k|)^{-j} \text{ for some constant } C \text{ for all } k \in \mathbb{N}\}.$$

The topology of these spaces can be defined by a natural choice of seminorms.

(ii) We can define spaces of $(\mathscr{U},\Lambda)$– and $(\mathscr{V},\Lambda)$–distributions by

$$\mathscr{D}'_{\mathscr{U},\Lambda} := (\mathscr{C}^\infty_{\mathscr{V},\Lambda})' \text{ and } \mathscr{D}'_{\mathscr{V},\Lambda} := (\mathscr{C}^\infty_{\mathscr{U},\Lambda})',$$

as spaces of linear continuous functionals on $\mathscr{C}^\infty_{\mathscr{V},\Lambda}$ and $\mathscr{C}^\infty_{\mathscr{U},\Lambda}$, respectively. We can extend the inner product on H to cover the above duality by

$$\langle u,\phi\rangle_{\mathscr{D}'_{\mathscr{U},\Lambda},\mathscr{C}^\infty_{\mathscr{V},\Lambda}} := \langle u,\phi\rangle_H, \tag{3.199}$$

extending the inner product on H for $u,\phi\in H$, and similarly for the pair $\mathscr{D}'_{\mathscr{V},\Lambda} := (\mathscr{C}^\infty_{\mathscr{U},\Lambda})'$.

(iii) the $\mathscr{U}$– and $\mathscr{V}$–Fourier transforms

$$\mathscr{F}_{\mathscr{U}}(\phi)(k) := \langle\phi,v_k\rangle =: \widehat{\phi}(k)$$

and

$$\mathscr{F}_{\mathscr{V}}(\psi)(k) := \langle\psi,u_k\rangle =: \widehat{\psi}_*(k),$$

respectively, for arbitrary $\phi\in\mathscr{C}^\infty_{\mathscr{U},\Lambda}$, $\psi\in\mathscr{C}^\infty_{\mathscr{V},\Lambda}$ and for all $k\in\mathbb{N}$. By duality these Fourier transforms extend to $\mathscr{D}'_{\mathscr{U},\Lambda}$ and $\mathscr{D}'_{\mathscr{V},\Lambda}$, respectively. Here we have

$$\langle\mathscr{F}_{\mathscr{U}}(w),a\rangle = \langle w,\mathscr{F}_{\mathscr{V}}^{-1}(a)\rangle, \quad w\in\mathscr{D}'_{\mathscr{U},\Lambda},\ a\in\mathscr{S}(\mathbb{N}). \tag{3.200}$$

Indeed, for $w\in H$ we can calculate

$$\begin{aligned}\langle\mathscr{F}_{\mathscr{U}}(w),a\rangle = \langle\widehat{w},a\rangle_{\ell^2(\mathbb{N})} &= \sum_{k\in\mathbb{N}}\langle w,v_k\rangle\overline{a(k)}\\ &= \left\langle w,\sum_{k\in\mathbb{N}}a(k)v_k\right\rangle = \langle w,\mathscr{F}_{\mathscr{V}}^{-1}a\rangle,\end{aligned}$$

justifying definition (3.200). Similarly, we define

$$\langle\mathscr{F}_{\mathscr{V}}(w),a\rangle = \langle w,\mathscr{F}_{\mathscr{U}}^{-1}(a)\rangle, \quad w\in\mathscr{D}'_{\mathscr{V},\Lambda},\ a\in\mathscr{S}(\mathbb{N}). \tag{3.201}$$

The Fourier transforms of elements of $\mathscr{D}'_{\mathscr{U},\Lambda},\mathscr{D}'_{\mathscr{V},\Lambda}$ can be characterised by the property that, for example, for $w\in\mathscr{D}'_{\mathscr{U},\Lambda}$, there is $N>0$ and $C>0$ such that

$$|\mathscr{F}_{\mathscr{U}}w(k)| \le C(1+|\lambda_k|)^N, \quad \text{for all } k\in\mathbb{N}.$$

(iv) $\mathscr{U}$- and $\mathscr{V}$-convolutions can be extended by the same formula:

$$f\star_{\mathscr{U}} g := \sum_{k\in\mathbb{N}}\widehat{f}(k)\widehat{g}(k)u_k = \sum_{k\in\mathbb{N}}\langle f,v_k\rangle\langle g,v_k\rangle u_k$$

for example, for all $f\in\mathscr{D}'_{\mathscr{U},\Lambda}$ and $g\in\mathscr{C}^\infty_{\mathscr{U},\Lambda}$. It is well-defined since the series converges in view of properties from (i) above and assumption (3.196). By the commutativity we can also take the convolution in the other order. Similarly,

$$h\star_{\mathscr{V}} j := \sum_{k\in\mathbb{N}}\widehat{h}_*(k)\widehat{j}_*(k)v_k = \sum_{k\in\mathbb{N}}\langle h,u_k\rangle\langle j,u_k\rangle v_k$$

for all $h\in\mathscr{D}'_{\mathscr{V},\Lambda}$, $j\in\mathscr{C}^\infty_{\mathscr{V},\Lambda}$.

The space $\mathbb{C}^{\infty}_{\mathscr{U},\Lambda}$ can be also described in terms of the domain of the operator L in (3.197) and of its iterated powers. Namely, we have the equality

$$\mathscr{C}^{\infty}_{\mathscr{U},\Lambda} = \bigcap_{k\in\mathbb{N}} D(L^k), \tag{3.202}$$

where

$$D(L^k) := \{f \in H : \ L^i f \in H, i = 2, ..., k-1\},$$

and similarly

$$\mathscr{C}^{\infty}_{\mathscr{V},\Lambda} = \bigcap_{k\in\mathbb{N}} D((L^*)^k),$$

where

$$D((L^*)^k) := \{g \in H : \ (L^*)^i g \in H, i = 2, ..., k-1\}.$$

Summarising the above definitions and observations, we note the basic properties of the described extensions of the Fourier transforms:

Theorem 3.140 *The $\mathscr{U}$-Fourier transform $\mathscr{F}_{\mathscr{U}}$ is a bijective homeomorphism from $\mathscr{C}^{\infty}_{\mathscr{U},\Lambda}$ to $\mathscr{S}(\mathbb{N})$. Its inverse*

$$\mathscr{F}^{-1}_{\mathscr{U}} : \mathscr{S}(\mathbb{N}) \to \mathscr{C}^{\infty}_{\mathscr{U},\Lambda}$$

is given by

$$\mathscr{F}^{-1}_{\mathscr{U}} h = \sum_{k\in\mathbb{N}} h(k) u_k, \quad h \in \mathscr{S}(\mathbb{N}), \tag{3.203}$$

so that the Fourier inversion formula becomes

$$f = \sum_{k\in\mathbb{N}} \widehat{f}(k) u_k \quad \textit{for all } f \in \mathscr{C}^{\infty}_{\mathscr{U},\Lambda}. \tag{3.204}$$

Similarly, $\mathscr{F}_{\mathscr{V}} : \mathscr{C}^{\infty}_{\mathscr{V},\Lambda} \to \mathscr{S}(\mathbb{N})$ is a bijective homeomorphism and its inverse

$$\mathscr{F}^{-1}_{\mathscr{V}} : \mathscr{S}(\mathbb{N}) \to \mathscr{C}^{\infty}_{\mathscr{V},\Lambda}$$

is given by

$$\mathscr{F}^{-1}_{\mathscr{V}} h := \sum_{k\in\mathbb{N}} h(k) v_k, \quad h \in \mathscr{S}(\mathbb{N}), \tag{3.205}$$

so that the conjugate Fourier inversion formula becomes

$$f = \sum_{k\in\mathbb{N}} \widehat{f}_*(k) v_k \quad \textit{for all } f \in \mathscr{C}^{\infty}_{\mathscr{V},\Lambda}. \tag{3.206}$$

By (3.200) *the Fourier transforms extend to linear continuous mappings $\mathscr{F}_{\mathscr{U}} : \mathscr{D}'_{\mathscr{U},\Lambda} \to \mathscr{S}'(\mathbb{N})$ and $\mathscr{F}_{\mathscr{V}} : \mathscr{D}'_{\mathscr{V},\Lambda} \to \mathscr{S}'(\mathbb{N})$.*

Given the above properties, we can also extend the corresponding properties of $\mathscr{U}$- and $\mathscr{V}$-convolutions from those in Theorem 3.139:

Theorem 3.141 *For all $f \in \mathscr{D}'_{\mathscr{U},\Lambda}, g \in \mathscr{C}^{\infty}_{\mathscr{U},\Lambda}$, $h \in \mathscr{D}'_{\mathscr{V},\Lambda}$, $j \in \mathscr{C}^{\infty}_{\mathscr{V},\Lambda}$ we have*

$$\widehat{f \star_{\mathscr{U}} g} = \widehat{f}\,\widehat{g}, \quad \widehat{h \star_{\mathscr{V}} j_*} = \widehat{h}_*\,\widehat{j}_*.$$

The convolutions are commutative and associative. If $g \in \mathscr{C}^{\infty}_{\mathscr{U},\Lambda}$ then for all $f \in \mathscr{D}'_{\mathscr{U},\Lambda}$ we have

$$f \star_{\mathscr{U}} g \in \mathscr{C}^{\infty}_{\mathscr{U},\Lambda}. \tag{3.207}$$

The proof is analogous to the same verification as in Theorem 3.139. The property (3.207) follows from the fact that $\widehat{g} \in \mathscr{S}(\mathbb{N})$ implies that the series

$$\sum_{k\in\mathbb{N}} \lambda_k^j \widehat{f}(k)\widehat{g}(k)u_k$$

converges for any $j \in \mathbb{N}$. □

The operator L associated to the pair $(\mathscr{U},\Lambda)$, as defined in (3.197) by

$$Lf := \sum_{k\in\mathbb{N}} \lambda_k \langle f, v_k\rangle u_k,$$

is bi-invariant in the following sense, and, moreover, its resolvent can be described in terms of the convolution:

Theorem 3.142 *If $L : H \to H$ is associated to the pair $(\mathscr{U},\Lambda)$ then we have*

$$L(f \star_{\mathscr{U}} g) = (Lf) \star_{\mathscr{U}} g = f \star_{\mathscr{U}} (Lg)$$

for all $f, g \in \mathscr{C}^{\infty}_{\mathscr{U},\Lambda}$. Moreover, the resolvent of the operator L is given by the formula

$$\mathscr{R}_\lambda f := (L-\lambda I)^{-1} f = g_\lambda \star_{\mathscr{U}} f, \quad \lambda \notin \Lambda,$$

where I is the identity operator in H and

$$g_\lambda = \sum_{k\in\mathbb{N}} \frac{1}{\lambda_k - \lambda} u_k.$$

The first statement follows immediately from the definition of L and the calculations:

$$\mathscr{F}_{\mathscr{U}}(L(f \star_{\mathscr{U}} g))(k) = \lambda_k \widehat{f}(k)\widehat{g}(k)$$

and

$$\mathscr{F}_{\mathscr{U}}((Lf) \star_{\mathscr{U}} g)(k) = \mathscr{F}_{\mathscr{U}}(Lf)(k)\widehat{g}(k) = \lambda_k \widehat{f}(k)\widehat{g}(k),$$

for all $k \in \mathbb{N}$.

The second statement follows from the following calculation:

$$\begin{aligned} g_\lambda \star_{\mathscr{U}} f &= \sum_{k\in\mathbb{N}} \frac{1}{\lambda_k-\lambda}\widehat{f}(k)u_k \\ &= \sum_{k\in\mathbb{N}} \widehat{f}(k)(L-\lambda I)^{-1}u_k \\ &= (L-\lambda I)^{-1}\left(\sum_{k\in\mathbb{N}} \widehat{f}(k)u_k\right) \\ &= (L-\lambda I)^{-1} f \\ &= \mathscr{R}_\lambda f, \end{aligned}$$

where we used the continuity of the resolvent. □

Sometimes the abstract convolution can be worked out explicitly, as we will show in the following example.

Example 3.143 Consider the operator $\mathrm{O}_h^{(1)} : L^2(0,1) \to L^2(0,1)$ defined

$$\mathrm{O}_h^{(1)} := -i\frac{d}{dx},$$

where $h > 0$, on the domain $(0,1)$ with the boundary condition

$$hy(0) = y(1).$$

We have briefly discussed such operators in Example 3.111. In the case $h = 1$ we have $\mathrm{O}_1^{(1)}$ with periodic boundary conditions, and the systems $\mathscr{U}$ and $\mathscr{V}$ of eigenfunctions of $\mathrm{O}_1^{(1)}$ and its adjoint $\mathrm{O}_1^{(1)*}$ coincide, and are given by

$$\mathscr{U} = \mathscr{V} = \{u_k(x) = e^{2\pi ixk},\ k \in \mathbb{Z}\}.$$

This leads to the setting of the classical Fourier analysis on the circle which can be viewed as the interval $(0,1)$ with periodic boundary conditions.

For $h \neq 1$, the operator $\mathrm{O}_h^{(1)}$ is not self-adjoint. The spectral properties of $\mathrm{O}_h^{(1)}$ are well-known: the spectrum of $\mathrm{O}_h^{(1)}$ is discrete and is given by

$$\lambda_k = -i\ln h + 2k\pi,\ k \in \mathbb{Z}.$$

The corresponding bi-orthogonal families of eigenfunctions of $\mathrm{O}_h^{(1)}$ and its adjoint are given by

$$\mathscr{U} = \{u_k(x) = h^x e^{2\pi ixk},\ k \in \mathbb{Z}\}$$

and

$$\mathscr{V} = \{v_k(x) = h^{-x} e^{2\pi ixk},\ k \in \mathbb{Z}\},$$

respectively. They form Riesz bases, and $\mathrm{O}_h^{(1)}$ is the operator associated to the pair $\mathscr{U}$ and $\Lambda = \{\lambda_k = -i\ln h + 2k\pi\}_{k\in\mathbb{Z}}$.

Clearly, all the previous constructions work with the index set $\mathbb{Z}$ instead of $\mathbb{N}$.

It was shown in [105] that one can work out the abstract convolution expressions for the $\mathscr{U}$–convolution in some special cases. Let us give an example.

We consider the special case

$$H = L^2(0,1),\ \mathscr{U} = \{u_k(x) = h^x e^{2\pi ixk},\ k \in \mathbb{Z}\},$$

tand $\Lambda = \{\lambda_k = -i\ln h + 2k\pi\}_{k\in\mathbb{Z}}$. Then the operator $L : L^2(0,1) \to L^2(0,1)$ associated to the pair $(\mathscr{U},\Lambda)$ by formula (3.197) coincides with $\mathrm{O}_h^{(1)}$. The corresponding $\mathscr{U}$-convolution can be written in the integral form

$$(f \star_{\mathscr{U}} g)(x) = \int_0^x f(x-t)g(t)dt + \frac{1}{h}\int_x^1 f(1+x-t)g(t)dt, \tag{3.208}$$

which is the so-called ***Kanguzhin's convolution*** that was studied in [62]. In particular, when $h = 1$, this gives

$$(f \star_{\mathscr{U}} g)(x) = \int_0^1 f(x-t)g(t)dt,$$

the usual convolution on the circle.

Let us prove the formula (3.208). Let us denote

$$K(f,g)(x) := \int_0^x f(x-t)g(t)dt + \frac{1}{h}\int_x^1 f(1+x-t)g(t)dt.$$

Then we can calculate

$$\begin{aligned}
\mathscr{F}_{\mathscr{U}}(K(f,g))(k) &= \int_0^1\int_0^x f(x-t)g(t)h^{-x}e^{-2\pi ixk}dtdx \\
&+ \frac{1}{h}\int_0^1\int_x^1 f(1+x-t)g(t)h^{-x}e^{-2\pi ixk}dtdx \\
&= \int_0^1\left[\int_t^1 f(x-t)h^{-x}e^{-2\pi ixk}dx\right]g(t)dt \\
&+ \int_0^1\left[\int_0^t f(1+x-t)h^{-(1+x)}e^{-2\pi i(1+x)k}dx\right]g(t)dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1\left[\int_t^1 f(x-t)h^{-(x-t)}e^{-2\pi i(x-t)k}dx\right]g(t)h^{-t}e^{-2\pi itk}dt \\
&+ \int_0^1\left[\int_0^t f(1+x-t)h^{-(1+x-t)}e^{-2\pi i(1+x-t)k}dx\right]g(t)h^{-t}e^{-2\pi itk}dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1\left[\int_0^{1-t} f(z)h^{-z}e^{-2\pi izk}dz\right]g(t)h^{-t}e^{-2\pi itk}dt \\
&+ \int_0^1\left[\int_1^{1-t} f(z)h^{-z}e^{-2\pi izk}dx\right]g(t)h^{-t}e^{-2\pi itk}dt
\end{aligned}$$

$$= \int_0^1 \left[\int_0^1 f(z) h^{-z} e^{-2\pi i z k} dz \right] g(t) h^{-t} e^{-2\pi i t k} dt$$
$$= \widehat{f}(k)\, \widehat{g}(k).$$

Consequently, by Theorem 3.139, we must have $K(f,g) = f *_{\mathscr{U}} g$. $\square$

In the above constructions we had an operator L associated to the pair $(\mathscr{U}, \Lambda)$ by the formula (3.197), where $\mathscr{U}$ was a Riesz basis of eigenfunctions corresponding to the discrete spectrum Λ of the operator L, with $Lu_k = \lambda_k u_k$. However, it may happen that we want to analyse an operator which does not have eigenfunctions forming a basis. In this case, as was discussed in the preceding sections, one looks at the associated functions.

In this case one can work with another, more general, version of the convolution.

This, let $L : H \to H$ be a linear densely defined operator in H. We say that a bilinear associative and commutative operation $*_L$ is an ***L-convolution*** if for any $f, g \in D(L^\infty)$ we have

$$L(f \star_L g) = (Lf) \star_L g = f \star_L (Lg). \tag{3.209}$$

Here we have denoted $D(L^\infty) := \bigcap_{k\in\mathbb{N}} D(L^k)$ with $D(L^k) := \{f : L^i f \in H, i = 2, ..., k-1\}$.

Theorem 3.142 implies that the $\mathscr{U}$-convolution is a special case of L-convolutions.

However, the converse is not true, that is, an L-convolution does not have to be a $\mathscr{U}$-convolution for any choice of the set Λ.

Example 3.144 Let us consider an L-convolution associated to the so-called Ionkin operator considered in [54], see Example 3.59 (and also Examples 3.148 and 3.152, and Lemma 3.153). The ***Ionkin operator*** $L : H \to H$ is the operator in $L^2(0,1)$ given by

$$L = -\frac{d^2}{dx^2},\ x \in (0,1),$$

with the boundary conditions

$$u(0) = 0,\ u'(0) = u'(1).$$

It has eigenvalues

$$\lambda_k = (2\pi k)^2,\ k \in \mathbb{Z}_+,$$

and an extended set of root functions

$$u_0(x) = x,\ u_{2k-1}(x) = \sin(2\pi k x),\ u_{2k}(x) = x\cos(2\pi k x),\ k \in \mathbb{N},$$

which give a basis in $L^2(0,1)$. We denote this basis by $\mathscr{U}$. The corresponding biorthogonal basis is given by

$$v_0(x) = 2,\ v_{2k-1}(x) = 4(1-x)\sin(2\pi k x),\ v_{2k}(x) = 4\cos(2\pi k x),\ k \in \mathbb{N}.$$

The derivation of these formulae go back to Ionkin [54], see also Lemma 3.153. The so-called ***Ionkin-Kanguzhin convolution*** appearing in the analysis related to the Ionkin problem is given by the formula

$$\begin{aligned} f\star_L g(x) := \frac{1}{2}\int_x^1 & f(1+x-t)g(t)dt \\ & +\frac{1}{2}\int_{1-x}^1 f(x-1+t)g(t)dt + \int_0^x f(x-t)g(t)dt \\ & -\frac{1}{2}\int_0^{1-x} f(1-x-t)g(t)dt + \frac{1}{2}\int_0^x f(1+t-x)g(t)dt, \end{aligned} \tag{3.210}$$

where we have indicated that it is, indeed, an L-convolution in the sense of (3.209), that is, it satisfies the relation

$$L(f\star_L g) = (Lf)\star_L g = f\star_L (Lg),$$

which follows from the analysis in [63] and [105]. For the collection

$$\mathcal{U} := \{u_\xi :\ u_0(x) = x, u_{2\xi-1}(x) = \sin(2\pi\xi x), u_{2\xi}(x) = x\cos(2\pi\xi x), \xi \in \mathbb{N}\},$$

it can be readily checked that the corresponding $\mathcal{U}$-Fourier transform satisfies

$$\widehat{f\star_L g}(0) = \widehat{f}(0)\widehat{g}(0),$$

$$\widehat{f\star_L g}(2k) = \widehat{f}(2k)\widehat{g}(2k),$$

$$\widehat{f\star_L g}(2k-1) = \widehat{f}(2k-1)\widehat{g}(2k) + \widehat{f}(2k)\widehat{g}(2k) + \widehat{f}(2k)\widehat{g}(2k-1),\ k \in \mathbb{N}.$$

Therefore, by Theorem 3.139, the L-convolution does not coincide with the $\mathcal{U}$–convolution for any choice of numbers Λ.

Remark 3.145 Based on the biorthogonal Fourier analysis as set in (3.172)-(3.175) and on the notion of the convolution as described in Section 3.15, one can develop the full global theory of pseudo-differential operators and the corresponding symbolic calculus based on the biorthogonal systems and expansions. This becomes very useful in many problems since such an approach incorporates the boundary conditions in the general form into the symbolic calculus. We refer the reader to [103] for the development of this theory, as well as to [104] for its more general version.

3.16 Root functions of second-order non-self-adjoint ordinary differential operators

Compared to the self-adjoint case, the spectral theory of non-self-adjoint differential operators has some significant differences. For example, while the eigenvalues of any self-adjoint operator always exist and are always real, then eigenvalues of a non-self-adjoint operator may not exist, and, if they exist, they can be complex. Moreover, a self-adjoint operator has the system of eigenvectors forming an orthonormal basis, while the system of eigenvectors of a non-self-adjoint operator can be non-closed in H. In some cases a non-self-adjoint operator, in addition to eigenvectors, may also have associated vectors; the system of root (that is, eigen- and associated vectors) vectors can be non-closed in H or, if it is closed, it may not form a basis in H.

These and other peculiarities of the non-self-adjoint operators make it difficult to construct a unified spectral theory for this case. Therefore, we may single out different classes of boundary value problems for which it is still possible to study properties of eigenvalues and root vectors (including questions on expansions in biorthogonal series with respect to the system of root functions).

Since the volume of results available in the literature is quite large, we will dwell in this section only on a few main (from our point of view) results, giving the reader an idea of the appearing complications. And we will consider only the case of second-order differential operators.

In this section we consider main properties of eigen- and associated functions of the non-self-adjoint boundary value problems of the general form for a second-order ordinary differential operator.

In $L^2(0,1)$, consider the operator L given by a differential expression

$$Ly \equiv -u''(x)+q(x)u(x)=\lambda u(x),\ 0<x<1, \tag{3.211}$$

and two-point boundary conditions of the general form

$$\begin{cases} U_1(u)=a_{11}u'(0)+a_{12}u'(1)+a_{13}u(0)+a_{14}u(1)=0, \\ U_2(u)=a_{21}u'(0)+a_{22}u'(1)+a_{23}u(0)+a_{24}u(1)=0, \end{cases} \tag{3.212}$$

where $U_1(u)$ and $U_2(u)$ are linearly independent forms with arbitrary complex-valued coefficients, and $q\in C[0,1]$ is an arbitrary complex-valued function.

We denote by L the closure in $L^2(0,1)$ of the operator given by the differential expression (3.211) on the linear space of functions $u\in C^2[0,1]$ satisfying the boundary conditions (3.212).

It is easy to justify that the operator L is a linear operator on $L^2(0,1)$ defined by (3.211) with the domain

$$D(L)=\left\{u\in L_2^2(0,1): U_1(u)=0,\ U_2(u)=0\right\}.$$

For $u\in D(L)$ we understand the action of the operator $Lu=-u''(x)+q(x)u(x)$ in the sense of almost everywhere on $(0,1)$.

By an ***eigenvector of the operator*** L corresponding to an ***eigenvalue*** $\lambda_0\in\mathbb{C}$, we mean any nonzero vector $u_0\in D(L)$ which satisfies the equation

$$Lu_0=\lambda_0 u_0. \tag{3.213}$$

By an ***associated vector of the operator*** L of order m $(m = 1, 2, \ldots)$ corresponding to the same eigenvalue λ_0 and the eigenvector u_0, we mean any function $u_m \in D(L)$ which satisfies the equation:

$$Lu_m = \lambda_0 u_m + u_{m-1}. \tag{3.214}$$

The vectors $\{u_0, u_1, \cdots\}$ are called a ***chain of the eigen- and associated vectors of the operator*** L corresponding to the eigenvalue λ_0.

The eigenvalues of the operator L will be called the ***eigenvalues of problem*** (3.211)-(3.212). The eigen- and associated vectors of the operator L will be called ***eigen- and associated functions of problem*** (3.211)-(3.212). One can also say that the eigenfunction u_0 is a zero order associated function. The set of all eigen- and associated functions (they are collectively called ***root functions***) corresponding to the same eigenvalue λ_0 forms a root linear space. This space is called a ***root space***.

Avoiding repetition, in this section we use the definitions and terminology from Sections 3.12-3.14.

In Example 3.102 we have demonstrated the example of a boundary value problem with degenerate boundary conditions, which does not have eigenvalues. We now consider an example of a problem with degenerate boundary conditions, which has eigenvalues, but its system of eigenfunctions is not closed.

Example 3.146 Consider the spectral problem

$$-u''(x) + q(x)u(x) = \lambda u(x),\ u'(0) + \alpha u'(1) = 0,\ u(0) - \alpha u(1) = 0, \tag{3.215}$$

with a continuous coefficient q, where $\alpha \in \mathbb{C}$ is a fixed number. It is easy to see that for all α the condition (3.145) holds: $A_{12} = 0, A_{14} + A_{23} = 0$. Therefore, *the boundary conditions of the problem* (3.215) *are degenerate*.

It was shown in [19] that this problem for $\alpha^2 \neq 1$ and $\alpha \neq 0$ does not have eigenvalues if and only if the coefficient q is symmetric:

$$q(x) = q(1-x) \tag{3.216}$$

for all $x \in [0, 1]$.

Assume that the condition (3.216) holds not on the whole interval $[0, 1]$ but only on some part of it. That is, suppose that there exists a positive number $\delta < 1/2$ such that condition (3.216) holds for $x \in [0, \delta]$. Then this condition also holds for $x \in [1-\delta, \delta]$. Assume that this condition does not hold for $x \in (\delta, 1-\delta)$. In this case, as was shown in [19], the problem (3.215) has eigenvalues. Let us show that the system of the eigen- and associated functions of the problem is not closed in $L^2(0, 1)$.

Let $u(x)$ be an eigenfunction of the problem (3.215) corresponding to an eigenvalue λ.

Denote $U(x) := u(x) - \alpha u(1-x)$. It is easy to see that $U(0) = 0$ and $U'(0) = 0$. For $x \in [0, \delta]$ by condition (3.216) we have

$$\begin{aligned} &-u''(x) + q(x)u(x) + \alpha\left\{-u''(1-x) + q(x)u(1-x)\right\} \\ &\qquad = -u''(x) + q(x)u(x) + \alpha\left\{-u''(1-x) + q(1-x)u(1-x)\right\} \\ &\qquad\qquad = \lambda u(x) + \alpha\lambda u(1-x). \end{aligned}$$

Therefore, $U(x)$ on the interval $x \in [0,\delta]$ is a solution of the Cauchy problem

$$-U''(x) + q(x)U(x) = \lambda U(x), \;\; U(0) = 0, \; U'(0) = 0.$$

In view of the uniqueness of solutions to the Cauchy problems we have $U(x) \equiv 0$ for all $x \in [0,\delta]$. Therefore,

$$u(x) \equiv \alpha u(1-x), \; \forall x \in [0,\delta]. \tag{3.217}$$

Similarly we can prove that the first-order associated function also satisfies (3.216). Continuing, we obtain that all root functions of the problem (3.215) satisfy the condition (3.216). Let us show that due to this reason the system of root functions is not closed in $L^2(0,1)$.

Consider a function $v(x)$ that is not identically zero, satisfying the conditions

$$\overline{\alpha} v(x) + v(1-x) \equiv 0, \; \forall x \in [0,\delta]; \;\; v \equiv 0, \; \forall x \in [\delta, 1-\delta]. \tag{3.218}$$

For an arbitrary root function $u(x)$ of the problem (3.215), using the second condition from (3.218), we calculate the inner product

$$\begin{aligned} \langle u, v \rangle = \int_0^\delta u(x)\overline{v(x)}dx + \int_{1-\delta}^1 u(x)\overline{v(x)}dx \\ = \int_0^\delta u(x)\overline{v(x)}dx + \int_0^\delta u(1-x)\overline{v(1-x)}dx \\ = \int_0^\delta u(1-x)\overline{\{\overline{\alpha}v(x) + v(1-x)\}}dx. \end{aligned}$$

Therefore, by the first condition from conditions (3.218), we have $\langle u, v \rangle = 0$. That is, each root function of the problem (3.215) is orthogonal to $v(x)$. Since the function $v(x)$ is not zero, this proves that the system of root functions of the problem (3.215) is not closed in $L^2(0,1)$.

Note that in [20] one also proved an inverse result: for $\alpha^2 \neq 1$, the system of root functions of the boundary value problem (3.215) is closed in $L^2(0,1)$ if and only if there is no $\delta > 0$ such that condition (3.216) holds for $x \in [0,\delta)$. □

To consider problems whose system of root functions gives a basis, we need to avoid boundary value problems whose system of root functions is not closed. As Example 3.146 shows, such cases can be among problems with degenerate boundary conditions. For the boundary value problems of the general form (3.211)-(3.212) the following important result holds, which we record without proof.

Theorem 3.147 (see [80]) *The system of eigen- and associated functions of the boundary value problem* (3.211)-(3.212) *with nondegenerate boundary conditions is closed in* $L^2(0,1)$. *For any nondegenerate conditions, the spectrum of the problem* (3.211)-(3.212) *consists of a countable set* $\{\lambda_n\}$ *of eigenvalues with only one limit point* ∞, *and the dimensions of the corresponding root subspaces are uniformly bounded by the same constant.*

For an arbitrary system of elements of a Hilbert space, generally speaking, the completeness of this system does not yet entail the basis property (for example see Lemma 3.120, Example 3.130, Example 3.134). The same fact holds also for nondegenerate problems of the general form: *the completeness of the system of eigen- and associated functions of the problem with nondegenerate conditions does not yet guarantee the basis properties of this system of eigen- and associated functions*. We have considered one such problem in Example 3.137.

Consider an example of a problem with the nondegenerate boundary conditions whose system of eigenfunctions is complete but does not form a basis.

Example 3.148 Consider the spectral problem

$$-u''(x) = \lambda u(x), \; u'(0) = u'(1), \; u(0) = 0. \tag{3.219}$$

We have considered this problem in Example 3.59. There, we have shown that the problem (3.219) has the eigenvalues: $\lambda_0 = 0$ (simple) and $\lambda_k = (2k\pi)^2$, $k \in \mathbb{N}$ (double). The simple eigenvalue λ_0 has the eigenfunction $u_0(x) = x$. For the double eigenvalues λ_k one has one eigenfunction $u_{k0}(x) = \sin(2k\pi x)$ and one associated function

$$u_{k1}(x) = \frac{x}{4k\pi}\cos(2k\pi x) + C_k \sin(2k\pi x),$$

where C_k are arbitrary constants. The arbitrariness of this constant is caused by the fact that the associated functions of the problem are not uniquely defined.

Let us check the necessary basis condition (3.188). We calculate the scalar product

$$\langle u_{k0}, u_{k1}\rangle = \int_0^1 \sin(2k\pi x)\left(\frac{x}{4k\pi}\cos(2k\pi x) + C_k \sin(2k\pi x)\right) dx = \frac{\overline{C}_k}{2} - \frac{1}{(4k\pi)^2}.$$

We also calculate the norms of the root functions:

$$\|y_{k0}\| = \frac{1}{\sqrt{2}}; \;\; \|y_{k1}\|^2 = \left|\frac{C_k}{2}\right|^2 - \frac{C_k + \overline{C}_k}{(4k\pi)^2} + \frac{1}{96k\pi} + \frac{1}{(8k\pi)^2}.$$

If the sequence $C_k\sqrt{k}$ is unbounded, then there exists a subsequence k_j such that $C_{k_j}\sqrt{k_j} \to \infty$. Then

$$\overline{\lim_{k\to\infty}} \left|\left\langle \frac{y_{1k}}{\|y_{1k}\|}, \frac{y_{2k}}{\|y_{2k}\|}\right\rangle\right| = 1.$$

Hence, in this case the necessary condition for an unconditional basis (3.188) does not hold. Therefore, such system of root functions of the problem (3.219) does not give an unconditional basis in $L^2(0,1)$.

In the case when the sequence $C_k\sqrt{k}$ is bounded, we get

$$\lim_{k\to\infty} \left|\left\langle \frac{y_{1k}}{\|y_{1k}\|}, \frac{y_{2k}}{\|y_{2k}\|}\right\rangle\right| = \lim_{k\to\infty} \frac{\left|C_k\sqrt{k}\right|}{\sqrt{\left|C_k\sqrt{k}\right|^2 + \frac{1}{24\pi}}} < 1.$$

This means that for such C_k the necessary condition (3.188) for an unconditional basis holds. To make a conclusion whether such a system of root functions of the problem (3.219) is a basis or not, we need an additional investigation. □

The considered Examples 3.137 and 3.148 show that the completeness of a system of root functions (which is guaranteed by Theorem 3.147 for all problems with nondegenerate boundary conditions) does not guarantee that the system is a basis. Therefore, we need to single out a new class among problems with the nondegenerate boundary conditions for which the basis property of the system of root functions can be guaranteed.

One of these classes, for example, is the class of Sturm-Liouville problems which we have considered in Section 3.8. But this class consists only of self-adjoint boundary value problems. The coefficients of boundary conditions and of a differential equation are real. The root subspaces consist only of eigenfunctions. The system of all eigenfunctions forms an orthonormal basis. This class of boundary value problems has turned out to be too narrow for modelling arising new physical phenomena. It was necessary to construct a wider class of problems.

An important step was made at the beginning of the 1960s by G. M. Kesel'man [67] and V. P. Mihailov [82]. They singled out a narrower class among regular boundary conditions providing the Riesz basis of the system of root functions. Boundary conditions of such type were called strengthened regular (the exact definition and asymptotics of eigenvalues are given in Theorem 3.104). They showed that eigenvalues of problems with strengthened regular conditions starting from some moment become simple (onefold), and this means that they have only eigenfunctions (without associated functions). Thus, *the problem with strengthened regular boundary conditions can have no more than a finite number of associated functions*. In this sense the problem with strengthened regular conditions "is similar" to a self-adjoint one.

Let us formulate this result for a case of the problem (3.211)-(3.212) for a second-order equation.

From the coefficients of the boundary conditions (3.212) we form the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

We denote by $A(ij)$ the matrix composed of the i-th and j-th columns of A, and denote $A_{ij} := \det A(ij)$ $(1 \leq i < j \leq 4)$. From Theorem 3.105 we have that the boundary conditions (3.212) are regular in the following three cases:

$$\begin{aligned} &(1)\, A_{12} \neq 0, \\ &(2)\, A_{12} = 0,\; A_{14} + A_{23} \neq 0, \\ &(3)\, A_{12} = A_{13} = A_{14} = A_{23} = A_{24} = 0,\; A_{34} \neq 0. \end{aligned} \tag{3.220}$$

Here, the boundary conditions will be strengthened regular in the cases (1) and (3), and in the case (2) under the additional condition

$$A_{13} + A_{24} \neq \pm (A_{14} + A_{23}). \tag{3.221}$$

Theorem 3.149 ([67], [82]). *Let the boundary conditions* (3.212) *be regular and strengthened regular, that is, let one of conditions* (3.220) *and* (3.221) *hold. Then the system of root functions of the boundary value problem* (3.211)-(3.212) *is a Riesz basis in* $L^2(0,1)$.

Note that the result of this theorem does not depend on the coefficient $q(x)$ of Eq. (3.211), but depends only on coefficients of the boundary condition. Thus, *the basis property of the system of root functions of the problem with the strengthened regular boundary conditions is stable with respect to the coefficients (the potential) of the equation*.

A particular case of the strengthened regular boundary conditions are Sturm type conditions. It follows from Theorem 3.149 that the system of root functions of the problem with the Sturm type boundary conditions (see Example 2.92) is a Riesz basis.

As the further investigations show, the class of the strengthened regular boundary conditions turns out to be a single class in which the basis property of root functions is completely defined by the coefficients of the boundary condition and does not depend on the behavior of the coefficients. The theory of the basis property for the boundary value problems with not strengthened regular boundary conditions is far from completion.

Relatively recently in 2006, A. S. Makin [78] considered one subclass of the not strengthened regular boundary conditions and showed that the system of root functions of problems with such conditions gives a Riesz basis regardless of the behavior of the coefficient $q(x)$.

Theorem 3.150 (A. S. Makin [78]) *If* $A_{14}+A_{23}=0$ *and* $A_{34}\neq 0$*, then the system of eigen- and associated functions is a Riesz basis in* $L^2(0,1)$*, and the spectrum is asymptotically simple.*

The boundary conditions satisfying the conditions of Theorem 3.150 can be reduced to one of two forms:

$$\begin{cases} u'(0)-u'(1)+au(0)=0, \\ \qquad\qquad u(0)-u(1)=0, \end{cases} \quad \text{or} \quad \begin{cases} u'(0)+u'(1)+au(0)=0, \\ \qquad\qquad u(0)+u(1)=0, \end{cases}$$

where the coefficient a is different from zero: $a\neq 0$.

This subclass (singled out by A. S. Makin) of the not strengthened regular boundary conditions is the only boundary condition, in addition to strengthened regular ones, that ensure the Riesz basis property of the system of root functions for any potential $q(x)$. For the other cases of the not strengthened regular boundary conditions it is shown (see [79]) that the set of coefficients $q(x)$, for which the system of root functions of the problem (3.211)-(3.212) is a Riesz basis in $L^2(0,1)$, is dense in $L^1(0,1)$. Here, the set of coefficients $q(x)$, for which the system of root functions of (3.211)-(3.212) does not give a Riesz basis in $L^2(0,1)$, is also dense in $L^1(0,1)$. This fact demonstrates the instability of the basis property of the root functions with respect to small changes of the coefficient $q(x)$.

The question of such instability of the basis property of not strengthened regular problems was studied by V. A. Ilin. He showed that the basis conditions of the system of root functions for a wide class of non-self-adjoint differential operators cannot in principle be expressed in terms of boundary conditions. In particular he constructed an example for such a phenomenon.

Consider the boundary value problem

$$-u''(x)+p(x)u'(x)+q(x)=\lambda u(x), \;\; u'(0)=u'(1), \; u(0)=0. \tag{3.222}$$

We have considered this problem for $p(x)=q(x)=0$ in Examples 3.59 and 3.148. It was shown in [53] that in this case the system of eigen- and associated functions of problem (3.222) is a Riesz basis in $L^2(0,1)$. And if $p(x)=\varepsilon\left(x-\frac{1}{2}\right)$ and $q(x)=\frac{\varepsilon^2}{4}\left(x-\frac{1}{2}\right)^2+\frac{\varepsilon}{2}$, then for any small $\varepsilon>0$ the system of root functions of the problem (3.222) consists only of the eigenfunctions and does not give an unconditional basis in $L^2(0,1)$.

Thus, two operators with the same boundary conditions and arbitrarily close (in any metric) infinitely differentiable coefficients have fundamentally different properties: one has a basis from root functions, and the other does not. This example clearly shows that the study of the basis property of a system of root functions of non-self-adjoint operators in terms of the boundary conditions and the smoothness of the coefficients is generally rather impossible.

Based on this thesis, V. A. Ilin proposed in [51] a new interpretation of the root functions, abandoning the specific form of the boundary conditions. Since for the case of the strengthened regular boundary conditions all problems have the system of root functions forming a basis, then the theory of V. A. Ilin actually deals with problems with conditions that are not strengthened regular. Since the explicit form of the boundary conditions is not used in the considerations, then by the root functions one means the system of arbitrary complex-valued functions $\{u_k(x)\}_{k=1}^{\infty}$ that are different from identical zero satisfying almost everywhere in $[0,1]$ the equation

$$Lu_k-\lambda_k u_k=\theta_k u_{k-1}, \tag{3.223}$$

where $\theta_k=0$ if u_k is an eigenfunction, and $\theta_k=1$ if u_k is an associated function (in this case we additionally require that $\lambda_k=\lambda_{k-1}$). We assume that all chains of root functions entering the system are finite, i.e. $m(\lambda)<\infty$. Denote by $\{u_k\}$ the system of root functions enumerated in a certain way.

The theory of V. A. Ilin received further developments for several classes of differential equations and problems. Let us give without proof the simplest version of the theorem on necessary and sufficient conditions for the basis property of a system of root functions.

Let $\{u_k\}$ be a system of root functions (understood in the sense of equality (3.223)) of an operator given by the differential expression (3.211). We require the following Condition A [51]:

(A1): the system of root functions $\{u_k\}$ of an operator L is closed and minimal in $L^2(0,1)$;

(A2): the ranks of eigenfunctions are uniformly bounded:

$$\sup_{\lambda\in\Lambda} m(\lambda)<\infty;$$

(A3): the following estimate holds uniformly in $t > 0$:

$$\sum_{\lambda\in\Lambda:\ \left|\mathrm{Re}\sqrt{\lambda}-t\right|\leq 1} 1 \leq B_1,$$

which will be called a "sum of units" in what follows;

(A4): the set of eigenvalues Λ lies inside a certain parabola (which is called the Carleman parabola), i.e. the following estimate holds uniformly in $\lambda \in \Lambda$:

$$\left|\mathrm{Im}\sqrt{\lambda}\right| \leq B_2.$$

The latter condition on the spectrum of the operator will be called the Carleman condition;

By Theorem 3.112, condition (A1) ensures the existence of a unique system $\{v_k\}$ biorthogonally dual to $\{u_k\}$. Also, we assume that

(A5): the biorthogonal dual system $\{v_k\}$ consists of root functions of the formally adjoint operator L^* given by the expression

$$L^* v(x) = -v''(x) + \overline{q(x)}v(x).$$

Theorem 3.151 (V.A. Ilin [51]) *Let the potential $q(x)$ of the operator L be Lebesgue integrable on $(0,1)$, and let conditions (A1)-(A5) hold. Then each of the systems $\{u_k\}$ and $\{v_k\}$ is an unconditional basis in $L^2(0,1)$ if and only if the following estimate of the product of norms holds uniformly in $k \in \mathbb{N}$:*

$$\|u_k\| \cdot \|v_k\| \leq C. \tag{3.224}$$

Thus, under the assumptions (A1)-(A5), the condition (3.171) of the uniform minimality of the system $\{u_k\}$, which is the necessary condition for the basis property, also becomes a sufficient condition for the system of root functions to be an unconditional basis.

The condition "sum of units" (A3) expresses a serious requirement for the uniform distribution of eigenvalues: in each vertical strip of the width 1 the quantity of the numbers $\sqrt{\lambda}$ is bounded by the same uniform constant.

We note that for the system of root functions of a concrete boundary-value problem, all the conditions of Theorem 3.151 are easily verified. Indeed, the completeness of the system $\{u_k\}$ is proved by using the well-known abstract theorems. For example, the closedness of the system $\{u_k\}$ can be a consequence of the non-degeneracy of the boundary conditions. The minimality is implied by the completeness in $L^2(0,1)$ of the biorthogonally dual system $\{v_k\}$. Conditions (A2), (A3), and (A4) are verified by using the leading term of the asymptotics of eigenvalues, and estimate (3.224) is verified by using the leading term of the asymptotics of root functions.

Example 3.152 Let us continue the consideration of the problem (3.219) from Example 3.148. Consider a system of root functions consisting of one eigenfunction $u_0(x)$ and a pair of eigen- and associated functions

$$u_0(x) = x,\ u_{k0}(x) = \sin(2k\pi x),\ u_{k1}(x) = \frac{x}{4k\pi}\cos(2k\pi x) + C_k \sin(2k\pi x). \tag{3.225}$$

These root functions form a closed and minimal system in $L^2(0,1)$. This is a consequence of the non-degeneracy of the boundary conditions of problem (3.219). Therefore condition (A1) is satisfied.

Condition (A2) is satisfied since all eigenvalues (except the one-fold eigenvalue λ_0) are double (two-fold). The eigenvalues of problem (3.219) are known in the explicit form : $\lambda_k = (2k\pi)^2$, $k \in \mathbb{N}$. Therefore conditions (A3) and (A4) are also satisfied.

The biorthogonal system $\{v_k\}$ is constructed of the root functions of the problem adjoint to (3.219):

$$-v''(x) = \lambda v(x),\ v'(0) = 0,\ v(0) = v(1). \tag{3.226}$$

The root functions of the problem (3.226) can be constructed in the explicit form: $v_0(x) = A_0$ is an eigenfunction; $v_{k1}(x) = A_k \cos(2k\pi x)$ are eigenfunctions; $v_{k0}(x) = A_k \frac{(1-x)}{4\pi k} \sin(2k\pi x) + B_k \cos(2k\pi x)$ are associated functions. Here A_0, A_k and B_k are arbitrary constants.

To construct a biorthogonal system, we need to choose these coefficients A_0, A_k and B_k according the biorthogonality condition:

$$\langle u_0, v_0\rangle = 1,\ \ \langle u_{k0}, v_{k0}\rangle = 1,\ \ \langle u_{k1}, v_{k1}\rangle = 1,\ \ \langle u_{k1}, v_{k0}\rangle = 0. \tag{3.227}$$

From the first condition we get

$$1 = \langle u_0, v_0\rangle = \overline{A}_0 \int_0^1 x dx = \frac{1}{2}\overline{A}_0.$$

Then, $A_0 = 2$. We use the third condition from (3.227):

$$1 = \langle u_{k1}, v_{k1}\rangle = \overline{A}_k \int_0^1 \left\{ \frac{x}{4k\pi} \cos(2k\pi x) + C_k \sin(2k\pi x) \right\} \cos(2k\pi x)\, dx = \frac{\overline{A}_k}{16k\pi}.$$

Then, $A_k = 16k\pi$. Therefore the eigen- and associated functions have the form $v_{k1}(x) = 16k\pi \cos(2k\pi x)$, $v_{k0}(x) = 4(1-x)\sin(2k\pi x) + B_k \cos(2k\pi x)$

Given the obtained result, we have the second condition in (3.227):

$$\langle u_{k0}, v_{k0}\rangle = \int_0^1 \sin(2k\pi x) \left\{ 4(1-x)\sin(2k\pi x) + \overline{B}_k \cos(2k\pi x) \right\} dx = 1.$$

To choose B_k, we use the fourth condition in (3.227):

$$0 = \int_0^1 \left\{ \frac{x}{4k\pi} \cos(2k\pi x) + C_k \sin(2k\pi x) \right\} \left\{ 4(1-x)\sin(2k\pi x) + \overline{B}_k \cos(2k\pi x) \right\} dx$$

$$= C_k + \frac{\overline{B}_k}{16k\pi}.$$

Then, $B_k = -16k\pi C_k$.

Thus, the system

$$\begin{gathered} v_0(x) = 2,\ v_{k1}(x) = 16k\pi \cos(2k\pi x), \\ v_{k0}(x) = 4(1-x)\sin(2k\pi x) - 16k\pi C_k \cos(2k\pi x) \end{gathered} \tag{3.228}$$

will be biorthogonal to the system of root functions (3.225). Under such choice of the root functions of the problem (3.226) condition (A5) will hold. Thus, all conditions (A1)-(A5) from Theorem 3.151 hold.

Let us check for what constants C_k the necessary and sufficient basis condition (3.224) from Theorem 3.151 holds. We calculate the norms of elements of the biorthogonal system:

$$\|u_{k0}\| = \frac{1}{\sqrt{2}}; \;\; \|u_{k1}\|^2 = \left|\frac{C_k}{2}\right|^2 - \frac{C_k+\overline{C}_k}{(4k\pi)^2} + \frac{1}{96k\pi} + \frac{1}{(8k\pi)^2},$$

$$\|v_{k0}\|^2 = 2(8k\pi)^2\,|C_k|^2 + 8\left(C_k+\overline{C}_k\right) + \frac{8}{3} - \frac{1}{(k\pi)^2}; \;\; \|v_{k1}\| = \frac{16k\pi}{\sqrt{2}};$$

If the sequence $C_k\sqrt{k}$ is unbounded, then there exists a subsequence k_j such that $C_{k_j}\sqrt{k_j} \to \infty$. Then

$$\overline{\lim_{k\to\infty}}\|u_{k0}\|\|v_{k0}\| = \infty, \text{ and } \overline{\lim_{k\to\infty}}\|u_{k1}\|\|v_{k1}\| = \infty.$$

Hence, in this case the requirement of the criterion (3.224) for an unconditional basis does not hold. Therefore, the system (3.225) of root functions of the problem (3.219) does not give an unconditional basis in $L^2(0,1)$.

This result (on absence of the basis property) has been previously obtained in Example 3.148 by using the necessary condition (3.188) for an unconditional basis. But in that example we could not substantiate the basis of the system of root functions for the case of other sequences C_k. For such sequences C_k we use the criterion (3.224) for unconditional bases from Theorem 3.151.

In the case when the sequence $C_k\sqrt{k}$ is bounded, we get

$$\lim_{k\to\infty}\|u_{k0}\|\|v_{k0}\| < \infty, \text{ and } \lim_{k\to\infty}\|u_{k1}\|\|v_{k1}\| < \infty.$$

This means that for such C_k the condition (3.224) for the unconditional bases holds. Hence, under the choice of such sequences C_k the system (3.225) of root functions of problem (3.219) forms an unconditional basis in $L^2(0,1)$. □

The considered Example 3.152 demonstrates the essence of one problem of the spectral theory of non-self-adjoint operators, the so-called "*problem of choosing associated functions*". It consists in the fact that for one choice of the associated functions the system of root functions of the operator is an unconditional basis, while for another choice of the associated functions the system of root functions cannot give a basis.

Usually, when one considers the systems of eigen- and associated functions, the constants C_k are chosen independently of the index: $C_k = C$. Then the system of eigen- and associated functions of the Samarskii-Ionkin problem (3.219) has the form

$$u_0(x) = x, \; u_{k0}(x) = \sin(2k\pi x), \; u_{k1}(x) = \frac{x}{4k\pi}\cos(2k\pi x) + C\sin(2k\pi x). \quad (3.229)$$

From the results of Example 3.152, we obtain the following result for the system (3.229).

Lemma 3.153 *The system of eigen- and associated functions of the Samarskii-Ionkin problem* (3.219) *is closed and minimal in* $L^2(0,1)$. *It has the form* (3.229), *where C is an arbitrary constant. This system for a special choice of the constant* $C=0$ *is an unconditional basis in* $L^2(0,1)$, *and ceases to be a basis for any other choice* $C \neq 0$.

In accordance with the demonstrated problem the term "for a special choice of associated functions" is used in the scientific literature. It is used in cases when one considers spectral properties of problems having an infinite number of associated functions. This term demonstrates that the basis property depends not only on properties of the problem under consideration, but also on the choice of the system of associated functions.

Studying this problem, Ilin [50] in 1976 constructed the theory of a ***reduced system of eigen- and associated functions*** for ordinary differential operators. Specifically, the reduced system always has the basis property if there is at least one choice of eigen- and associated functions possessing this property. However, the elements of the reduced system do not satisfy the equation for determining associated functions.

In the context of choosing associated functions, the question arises as to what relations the eigen- and associated functions must satisfy to avoid this effect?

To solve this problem, it was suggested in [108] and [109] to use a modified definition of the associated functions. An important result for justifying such new definition of the associated functions is the following theorem:

Theorem 3.154 ([108], [109]) *Let the system* $\{u_{k0}(x); u_{k1}(x)\}_{k\in\mathbb{N}}$ *consisting of eigen- and associated functions of the operator L be an unconditional basis of the space* $L^2(0,1)$. *Then the system*

$$\{u_{k0}(x); u_{k1}(x) + Cu_{k0}(x)\}_{k\in\mathbb{N}} \tag{3.230}$$

is an unconditional basis in $L^2(0,1)$ *if and only if the uniform estimate*

$$\|u_{k0}\| \leq C_0 \|u_{k1}\| \tag{3.231}$$

is valid for all indices k.

Let us give a brief proof. Denote by $\{v_{k0}(x); v_{k1}(x)\}$ the system biorthogonal to the initial system $\{u_{k0}(x); u_{k1}(x)\}$. As follows from Example 3.113, the system (3.230) is a closed and minimal system in $L^2(0,1)$, and the system $\left\{v_{k0} - \overline{C}v_{k1}, v_{k1}\right\}_{k\in\mathbb{N}}$ will be its biorthogonal system.

Let the system (3.230) form an unconditional basis in $L^2(0,1)$. Let us show that the inequality (3.227) must be valid. But first we note that the basis property of the system (3.230) leads to the validity of the following two conditions as a necessary condition for the basis property (condition (3.171) for the uniform minimality of the system):

$$\begin{aligned} &\|u_{k0}\| \cdot \|v_{k0} - \overline{C}v_{k1}\| \leq C_1, \\ &\|u_{k1} + Cu_{k0}\| \cdot \|v_{k1}\| \leq C_2. \end{aligned} \tag{3.232}$$

In addition, we have the estimates

$$\|u_{k0}\| \cdot \|v_{k0}\| \leq C_3,$$
$$\|u_{k1}\| \cdot \|v_{k1}\| \leq C_4, \tag{3.233}$$

which follow from the unconditional basis property of the system $\{u_{k0}(x); u_{k1}(x)\}$ as a necessary condition for the basis property.

By using the estimates (3.232) and (3.233), one can estimate the quantity

$$\|u_{k0}\| \cdot \|v_{k1}\|.$$

This estimate follows from the inequalities

$$|C|\|u_{k0}\| \cdot \|v_{k1}\| = \|u_{k0}\| \cdot \|\overline{C}v_{k1} - v_{k0} + v_{k0}\| \leq \|u_{k0}\| \left\{ \|v_{k0} - \overline{C}v_{k1}\| + \|v_{k0}\| \right\}$$
$$\leq \|u_{k0}\| \cdot \|v_{k0} - \overline{C}v_{k1}\| + \|u_{k0}\| \cdot \|v_{k0}\| \leq C_1 + C_3,$$

where C_1 and C_3 are the constants occurring in (3.232) and (3.233), respectively, so that

$$\|u_{k0}\| \cdot \|v_{k1}\| \leq C_5 = (C_1 + C_3)|C|^{-1}. \tag{3.234}$$

By the biorthogonality condition we have $\langle u_{k1}, v_{k1} \rangle = 1$. Therefore, the second of inequalities (3.233) can be rewritten in the form

$$1 = \langle u_{k1}, v_{k1} \rangle \leq \|u_{k1}\| \cdot \|v_{k1}\| \leq C_4,$$

which implies the inequalities

$$\frac{1}{\|u_{k1}\|} \leq \|v_{k1}\| \leq \frac{C_4}{\|u_{k1}\|}.$$

Substituting the obtained result into (3.234), we obtain $\|u_{k0}\| \left(\|u_{k1}\|\right)^{-1} \leq C_5$, which proves inequality (3.231).

Let us now prove the converse statement. To prove that the system (3.230) is a basis, we show the criterion (3.224) from Theorem 3.151. Since the system of root functions $\{u_{k0}(x); u_{k1}(x)\}$ is an unconditional basis, conditions (A1)-(A5) of Theorem 3.151 hold. To prove the sufficiency of condition (3.231), we show that relation (3.231) implies conditions (3.232). First inequalities in (3.232) are valid, since

$$\|u_{k0}\| \cdot \|v_{k0} - \overline{C}v_{k1}\| \leq \|u_{k0}\| \cdot \|v_{k0}\| + |C|\|u_{k0}\| \cdot \|v_{k1}\|$$
$$\leq \|u_{k0}\| \cdot \|v_{k0}\| + |C|C_0\|u_{k1}\| \cdot \|v_{k1}\| \leq C_3 + |C|C_0C_4.$$

Here we have used relations (3.233), which are valid by the basis property of the system $\{u_{k0}(x); u_{k1}(x)\}$.

In a similar way, one can prove the validity of the second inequalities in (3.232). We have thus justified (3.224).

Since all assumptions of Theorem 3.151 are satisfied, it follows that the system (3.230) is an unconditional basis of the space $L^2(0,1)$. The proof of Theorem 3.154 is complete. □

In connection with the assertion of the above theorem, we especially note that estimates of the form (3.231) (for the case in which the right-hand side does not contain the spectral parameter in a positive power) are not natural at least in the case of second-order differential operators if there exist infinitely many associated functions, and they are given by the formula

$$Lu_{k1} = \lambda_k u_{k1} + u_{k0}. \tag{3.235}$$

Moreover, uniform estimates with respect to the index k (see [53, p. 11])

$$\|u_{k0}\| = O(1)\left(1 + \left|\operatorname{Im}\sqrt{\lambda_k}\right|\right)\|u_{k1}\|$$

hold for eigen- and associated functions of the second-order differential operator given by the differential expression (3.211) for all $q \in L^1(0,1)$.

This estimate completely contradicts inequality (3.231). The obtained contradiction together with Theorem 3.154 lead to the conclusion that if some system of eigen- and associated functions $\{u_{k0}(x); u_{k1}(x)\}$ is an unconditional basis, then no system of eigen- and associated functions of the form $\{u_{k0}(x); u_{k1}(x) + Cu_{k0}(x)\}$ for $C \neq 0$ can be an unconditional basis.

It should be noted that to correct such situation in some special cases one sometimes uses another formula for constructing associated functions, which differs from (3.235).

In [108] and [109] one suggested the following formula for constructing the associated functions:

$$Lu_{k1} = \lambda_k \left(u_{k1} + u_{k0}\right). \tag{3.236}$$

If the associated functions of the operator L given by the differential expression (3.211) are defined with the use of relation (3.236), then one can show that anti-a priori estimates of the form (3.231) (whose right-hand side does not contain a positive power of the spectral parameter) are valid, and, consequently, the investigation of the basis property of eigen- and associated functions of this operator is not related to any difficulty in the choice of associated functions.

By the above theorems, if there is at least one choice of eigen- and associated functions given by (3.236) which is an unconditional basis in $L^2(0,1)$, then any other system of eigen- and associated functions given by (3.236) also is an unconditional basis, since estimates of the form (3.231) are always valid in this case.

Thus, the use of formula (3.236) for constructing the system of eigen- and associated functions makes it possible to completely avoid the problem of choosing associated functions for the case of the second-order operator given by the differential expression (3.211). Some of these results have been extended to the case of higher-order differential operators and for an arbitrary (finite) number of associated functions in each root subspace.

Although in the general case the theory of bases of the systems of root functions of ordinary differential operators is far from completion, we note two works in which significant results for a wide class of problems were obtained.

For the case of the second-order operator with $q \equiv 0$, that is, for the operator

$$Lu = -u''(x),$$

the complete spectral characteristic of all boundary value problems on the interval $(0,1)$ was given in [70]. In this paper the authors obtained the results on the spectrum $\sigma(L)$ of L, on the algebraic multiplicities of the eigenvalues $\lambda \in \sigma(L)$ and the ascents of the operators $L - \lambda I$, on the boundedness of the family of all finite sums of the projections associated with L, on the density of the generalised eigenfunctions, and on the existence of bases consisting of generalised eigenfunctions.

In particular, they obtained either explicit or the asymptotic formulae for the eigenvalues of L, explicit algebraic multiplicities and ascents, explicit or asymptotic formulae for the projections; furthermore, they either obtained explicit bounds for the family of all finite sums of these projections, or they showed that such families are unbounded.

A survey of results on the spectral theory of differential operators generated by ordinary differential expressions and also by partial differential expressions of the elliptic type obtained with the help of the theory of V. A. Ilin, as well as their development and subsequent application, was given in [53]. The main focus there is on the non-self-adjoint case.

A. A. Skalikov in [118] considered the general case of the problem for the n-th order equation (3.149) with regular boundary conditions of the general form (3.150). For all cases of the regular boundary conditions he proved that *the system of eigen- and associated functions forms an unconditional basis with brackets (that is, a basis of subspaces) in L^2 and, moreover, no more than two eigenfunctions are combined into the brackets*.

Let us recall that *a system $\{H_k\}_{k=1}^{\infty}$ of subspaces* is called a basis in a Hilbert space H if any vector $f \in H$ is uniquely expanded into a series

$$f = \sum_{k=1}^{\infty} f_k, \quad \text{where } f_k \in H_k.$$

A basis $\{H_k\}_{k=1}^{\infty}$ of subspaces is orthogonal if $H_k \perp H_j$ for $k \neq j$. A system $\{H_k\}_{k=1}^{\infty}$ is called a ***Riesz basis of subspaces*** if there exists a bounded and boundedly invertible operator S such that the system $\{S(H_k)\}_{k=1}^{\infty}$ is an orthogonal basis of subspaces in H. If the boundary conditions are regular, but not strengthened regular, then it is not guaranteed that the system $\{u_k\}_{k=1}^{\infty}$ of eigenfunctions and associated functions of the operator L forms a Riesz basis. However, in this case, the Riesz basis property holds and the subspaces are two-dimensional.

This result was widely used in further works on the spectral theory and was developed for several other classes of operators. For example, in a recent work [112] the concepts of regularity and strengthened regularity for a matrix two-dimensional Dirac operator were introduced. If the boundary conditions are strengthened regular, then the spectrum of the Dirac operator is asymptotically simple and the system of eigen- and associate functions is a Riesz basis. If the boundary conditions are regular, but not strengthened regular, then all eigenvalues of the Dirac operator are asymptotically double, and the system formed by the corresponding two-dimensional root subspaces of the Dirac operator is a Riesz basis of subspaces (Riesz basis with brackets).

Problems closely related to the results of this section were considered by V. A. Ilin in [52]. He studied the uniform and absolute convergence and the convergence in the norm of L^2 for biorthogonal expansions of functions in the Sobolev class L^2_{2p} with respect to a system of eigen- and associated functions of the non-self-adjoint second-order elliptic operator

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + c(x)u(x), \tag{3.237}$$

which is defined in an arbitrary domain $\Omega \in \mathbb{R}^n$ and has smooth coefficients; moreover, the coefficients $a_{ij}(x)$ are assumed to be real-valued, and all the remaining coefficients are complex-valued.

The eigen- and associated functions are understood in the generalised sense, which permits considering arbitrary boundary conditions. Namely, an eigenfunction of the operator (3.237) corresponding to an eigenvalue λ_k is defined as an arbitrary complex-valued function $u_{k0} \in L^2(\Omega)$ that is not identically zero, belongs to the class $C^2(\Omega)$, and is a solution of the equation $Lu_{k0} = \lambda_k u_{k0}$. Similarly, an associated function of order $j = 1, 2, \ldots$ corresponding to the same eigenvalue λ_k and eigenfunction u_{k0} is defined as an arbitrary complex-valued function $u_{kj} \in L^2(\Omega)$ that belongs to the class $C^2(\Omega)$ and is a solution of the equation $Lu_{kj} = \lambda_k u_{kj} + u_{kj-1}$.

Theorem 3.155 (V. A. Ilin [52]) *Let $\{u_k(x)\}$ be an arbitrary closed and minimal system in $L^2(\Omega)$ consisting of eigen- and associated functions of operator* (3.237) *(understood in the above-mentioned generalised sense), and let the following conditions be satisfied:*
1. The system $\{v_k(x)\}$ biorthogonal in $L^2(\Omega)$ to the system $\{u_k(x)\}$ consists of eigen- and associated functions (understood in the above-mentioned generalised sense) of the differential operator

$$L^*v = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\overline{b_i(x)}v\right) + \overline{c(x)}v(x), \tag{3.238}$$

which is adjoint to operator (3.237)*; more precisely, each function $v_k(x)$ belongs to the class $C^2(\Omega)$ and is a solution of the equation $L^*v_k + \overline{\lambda}_k v_k = \widehat{\theta}_k v_{k+1}$ inside the domain Ω, where $\overline{\lambda}_k$ is the complex conjugate of λ_k and the number $\widehat{\theta}_k$ is equal either to zero [in this case, $v_k(x)$ is referred to as an eigenfunction of operator* (3.238)*] or to unity [in this case, we require that $\lambda_k = \lambda_{k+1}$, and $v_k(x)$ is referred to as an associated function of operator* (3.238)*].*
2. For any compact set K in the domain Ω, there exists a constant $C(K)$ such that the inequality $\|u_k\|_{L^2 z(\Omega)} \cdot \|v_k\|_{L^2(K)} \leq C(K)$ holds for all indices k.
3. The number of associated functions corresponding to each eigenfunction of the system $\{u_k(x)\}$ is uniformly bounded.

4. The eigenvalues λ_k satisfy two inequalities:
(a) $|Im\lambda_k| \le M\sqrt{|Re\lambda_k|}$;
(b) $N(\lambda) \equiv \sum_{|\lambda_k|\le\lambda} 1 \le M_1\lambda^{\alpha}$, valid for some $\alpha > 0$ and for an arbitrary $\lambda > 0$;
5. The function $f(x)$ is compactly supported in the domain Ω and belongs to the Sobolev class L^2_{2p} for some index p. Then the spectral expansion

$$S_\lambda(x,f) = \sum_{|\lambda_k|\le\lambda} \langle f, v_k\rangle u_k(x)$$

converges to $f(x)$ in the metric of $L^2(\Omega)$ as $\lambda\to\infty$ for $p > \alpha$ and, in addition, is convergent absolutely and uniformly on any compact set K in the domain Ω for $p > \alpha + \frac{n-1}{4}$. As usual, here $\langle f,g\rangle = \int\limits_\Omega f(x)\overline{g(x)}dx$.

The ideas of this theorem were developed further in [58], where biorthogonal decompositions for an arbitrary (not necessarily elliptic) operator were considered.

Let $\Omega \in \mathbb{R}^n$, and let L be a linear operator with dense domain $D(L)$ in $L^2(\Omega)$. We assume that the operator L is invertible and that L^{-1} is a compact operator. Then its spectrum can consist only of eigenvalues λ_k of finite multiplicity with the only limiting point at infinity. Most linear differential operators have such properties.

We use the new definition of associated functions suggested in [108] and [109] (see also (3.236)). The eigenfunctions are defined as solutions of the equation $Lu_{k0} = \lambda_k u_{k0}$, $u_{k0} \in D(L)$, and chains u_{kj} of associated functions of u_{k0} are defined as solutions of the equations $Lu_{kj} = \lambda_k\big(u_{kj} + u_{kj-1}\big)$, $u_{kj} \in D(L)$, $j = 1,2,\dots,m$. Obviously, the number m of associated functions depends on the index k: $m = m_k$, but to simplify the notation, we omit this index. We denote by $\{v_k(x)\}$ the system that is biorthogonal to $\{u_k(x)\}$ and consists of root functions of the adjoint operator L^*.

Theorem 3.156 ([58]) *Let $\{u_k\}$ be an arbitrary closed and minimal system in $L^2(\Omega)$ that consists of root functions of the operator L understood in the sense of the new definition of associated functions, let $\{v_k\}$ be its biorthogonal system, and let the following conditions be satisfied:*
1. The number of associated functions u_{kj} corresponding to each eigenfunction u_{k0} of the system $\{u_k(x)\}$ is uniformly bounded;
2. The inequalities

$$\|u_{kj}\| \le C\|u_{kj+1}\|,\ \|v_{km-j}\| \le C\|v_{km-j-1}\|,\ j = 0,1,\dots,m, \tag{3.239}$$

hold inside chains of eigenfunctions u_{k0} and v_{km} and associated functions u_{kj} and v_{km-j}, $j = 1,2,\dots,m$, of the operators L and L^*;
3. There exists a positive integer p such that

$$\sum_{k=1}^{\infty} |\lambda_k|^{-p} < \infty,$$

where λ_k are eigenvalues of the operator L;
4. For any index k, there exists a constant $C > 0$ such that the estimate

$$\|u_k\|\cdot\|v_k\| \le C$$

holds and, in terms of eigen- and associated functions, this estimate can be rewritten in the form

$$\|u_{kj}\| \cdot \|v_{kj}\| \leq C, \quad j = 0, 1, \ldots, m.$$

Then the relation

$$f(x) = \sum_{k=1}^{\infty} \langle f, v_k \rangle u_k(x)$$

is satisfied for any function $f \in D(L^p)$ in the sense of the norm of the space $L^2(\Omega)$.

Note that condition (3.239) is not a restrictive requirement. As is shown in Theorem 3.154, such conditions between eigenfunctions and associated functions are natural for the basis property of systems of root functions.

The assertions of Theorems 3.155 and 3.156 naturally lead to the introduction of the following notion.

An arbitrary closed and minimal in $L^2(\Omega)$ system of root functions $\{u_k(x)\}$ of the operator L is referred to as an ***almost basis*** in $L^2(\Omega)$ if there exists a set $W \subset L^2(\Omega)$, everywhere dense in $L^2(\Omega)$, such that the biorthogonal expansion $\sum_{k=1}^{\infty} \langle f, v_k \rangle u_k(x)$ of an arbitrary function $f \in W$ (where $\{v_k(x)\}$ is the biorthogonal system of $\{u_k(x)\}$), converges to the function f in the sense of the metric of the space $L^2(\Omega)$.

For example, the system of root functions satisfying the assumptions of Theorem 3.155 is an almost basis in the space $L^2(\Omega)$.

Corollary 3.157 *Let the assumptions of Theorem 3.156 hold. Then root functions of the operator L are an almost basis in the space $L^2(\Omega)$.*

Indeed, the statement follows from the density of the domain $D(L^p)$ in $L^2(\Omega)$. $\square$

In conclusion, we note that studies of the questions about bases and convergence of biorthogonal (spectral) expansions for various classes of differential operators are actively continuing.

Chapter 4

Symmetric decreasing rearrangements and applications

Historically, in the celebrated book *Inequalities* by Hardy, Littlewood and Pólya [47], the systematic analysis of symmetric decreasing rearrangements of nonnegative measurable functions was given. Since this foundational book, there have been many developments dedicated to symmetrizations of functions and their applications. Nowadays, the methods and techniques of symmetric decreasing rearrangements of functions and other related symmetrizations have become one of the important tools in analysis and its applications.

Compared to previous chapters, in this chapter we present the material at an advanced level, in the spirit of [75], [26] and [122]. We review backgrounds on symmetric decreasing rearrangements of functions and then give some of their applications. We have also reviewed a part of the work [27], which has important applications in astrophysics.

The structure of the present chapter is as follows. Roughly speaking, the symmetric decreasing rearrangement u^* of a positive measurable function u is a lower semicontinuous function such that u^* is equimeasurable with u. To present a formal definition of u^* we follow the layer-cake decomposition approach. The definition will be given with concrete examples of elementary functions and some of their main properties, for instance, the L^p-norm preserving and L^p-distance decreasing properties are discussed.

It should be noted that we have chosen a simpler definition of the symmetric decreasing rearrangement which, although sometimes less general, is better suited for our further considerations and is useful in various applications. Then we present and prove several fundamental rearrangement inequalities, namely the Hardy-Littlewood inequality, the Riesz inequality, the Pólya-Szegő inequality and the Talenti inequalities. We first prove the Hardy-Littlewood inequality by using the layer-cake decomposition and Fubini's theorem. Then we give the proof of the Pólya-Szegő theorem by using the co-area formula and Jensen's inequality. As usual, the Riesz inequality is proved by induction over the dimension. Finally, we prove the Talenti inequalities which are also called Talenti's comparison principles for the Laplacian and the p-Laplacian.

Consequently, we discuss the compactness properties of symmetric decreasing rearrangements of minimising sequences arising in various problems in mathematical physics. We state and give the complete proof of the Burchard-Guo theorem on compactness via symmetric decreasing rearrangements, which has interesting applications in the dynamical stability analysis of gaseous stars and stability of symmetric steady states in galactic dynamics. Then we list some applications of symmetric decreasing rearrangement in mathematical physics. First, we give an example of applications to the Brownian motion, then an application to the theory of sound.

It is proved that the deepest bass note is produced by the circular drum among all drums of the same area (as the circular drum). Moreover, we show that among all bodies of a given volume in the three-dimensional space with constant density, the ball has a gravitational field of the highest energy. We also give an application of the symmetric decreasing rearrangement in dynamical stability problems of gaseous stars. Finally, we discuss very briefly the stability of a symmetric steady state in galactic dynamics via the symmetric decreasing rearrangement of functions.

4.1 Symmetric decreasing rearrangements

In this section the symmetric rearrangement of sets and the symmetric decreasing rearrangement of functions are defined and some properties are noted. We present some examples and applications of symmetric decreasing rearrangements to illustrate the definitions and techniques.

4.1.1 Definitions and examples

Let Ω be a measurable (bounded) set in $\mathbb{R}^d$. An open ball (centred at 0) Ω^* is called a ***symmetric rearrangement (or just symmetrization) of the set*** Ω if

$$|\Omega^*| = |\Omega|$$

and

$$\Omega^* = \left\{ x \in \mathbb{R}^d : \, \sigma_d |x|^d < |\Omega| \right\},$$

where $\sigma_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the surface area of the unit ball of the d-dimensional Euclidean space. Here and in the sequel we denote by $|\Omega|$ the volume (the Lebesgue measure) of the set Ω.

Let u be a nonnegative measurable function vanishing at infinity in the sense that all of its positive level sets have a finite measure, that is,

$$\mathrm{Vol}\left(\{x : u(x) > t\}\right) < \infty, \quad (\forall t > 0).$$

It is easy to see that any nonnegative function $u \geq 0$ can be represented in terms of its level sets in the following way

$$u(x) = \int_0^\infty \chi_{\{u(x)>t\}} dt, \tag{4.1}$$

where χ is the characteristic function. This representation is also called the ***layer-cake decomposition.***

We now give the central definition for this chapter.

Definition 4.1 *Let $u \geq 0$ be a nonnegative measurable function vanishing at infinity. Then the function*

$$u^*(x) := \int_0^\infty \chi_{\{u(x)>t\}^*} dt$$

is called the ***symmetric decreasing rearrangement of*** u*.*

We note that u^* is a lower semicontinuous function since its level sets are open. Moreover, it is uniquely determined by the *distribution function*

$$\mu_u(t) := \mathrm{Vol}\{x : u(x) > t\}.$$

By Definition 4.1, the function u^* is nonnegative and *equimeasurable* with u, that is, the corresponding level sets of these functions have the same volume:

$$\mu_u(t) = \mu_{u^*}(t), \ \ (\forall t > 0). \tag{4.2}$$

From this and (4.1) we also have

$$M(u(x)) = M(u^*(x)), \tag{4.3}$$

for any non-decreasing function $M : \mathbb{R}^+ \to \mathbb{R}^+$.

Example 4.2 Let $\Omega = [-1,3]$ and let

$$u(x) = \begin{cases} x+1, & x \in [-1,2], \\ 9-3x, & x \in [2,3]. \end{cases}$$

Then the symmetric rearrangement of the set $[-1,3]$ is $[-2,2]$ and the symmetric decreasing rearrangement of u is

$$u^*(x) = 3 - \frac{3}{2}|x|, \quad \text{in } [-2,2].$$

Example 4.3 Let $\Omega = [-3,4]$ and let

$$u(x) = \begin{cases} -x-3, & x \in [-3,-1], \\ -x+1, & x \in [-1,0], \\ 1, & x \in [0,3], \\ -x+4, & x \in [3,4]. \end{cases}$$

Then the symmetric rearrangement of the set $[-3,4]$ is $[-3.5,3.5]$ and the symmetric decreasing rearrangement of u is

$$u^*(x) = \begin{cases} x+3.5, & x \in [-3.5,-2.5], \\ 1, & x \in [-2.5,-1], \\ x+2, & x \in [-1,0], \\ -x+2, & x \in [0,1], \\ 1, & x \in [1,2.5], \\ -x+3.5, & x \in [2.5,3.5]. \end{cases}$$

Example 4.4 Let $\Omega = [-2,2]$ and consider the function

$$u(x) = \exp^{-|x|}, \quad x \in [-2,2].$$

Then the symmetric decreasing rearrangement of u is

$$u^*(x) = \exp^{-|x|}, \quad x \in [-2,2].$$

In this case the function u is equal to its symmetric decreasing rearrangement, because this function is itself a symmetric decreasing function on a symmetric domain.

4.1.2 Some properties

Let us discuss some important well-known properties of the symmetric decreasing rearrangements.

Lemma 4.5 (Order-preserving) *Let u and v be nonnegative measurable functions. If*

$$u(x) \leq v(x), \quad \forall x \in \mathbb{R}^d,$$

then

$$u^*(x) \leq v^*(x), \quad \forall x \in \mathbb{R}^d.$$

To show this, by using Definition 4.1 and $u(x) \leq v(x)$, $\forall x \in \mathbb{R}^d$, we simply obtain

$$u^*(x) = \int_0^\infty \chi_{\{u(x)>t\}^*} dt \leq \int_0^\infty \chi_{\{v(x)>t\}^*} dt = v^*(x), \quad \forall x \in \mathbb{R}^d,$$

which proves Lemma 4.5. □

Lemma 4.6 (L^p-norm preserving) *For any* $1 \leq p \leq \infty$ *we have*

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(\Omega^*)}.$$

for any nonnegative measurable function u in $L^p(\Omega)$.

To prove this (let us say, in the case $p = 2$), by using the layer-cake decomposition (4.1), Fubini's theorem, and (4.2), we calculate

$$\int_\Omega |u(x)|^2 dx = \int_\Omega \int_0^\infty \chi_{\{u^2(x)>t\}} dt dx$$

$$= \int_0^\infty \mathrm{Vol}\left(\left\{u^2(x) > t\right\}\right) dt = \int_0^\infty \mathrm{Vol}\left(\{u(x) > s\}\right) 2sds$$

$$= \int_0^\infty \mu_u(s) 2sds = \int_0^\infty \mu_{u^*}(s) 2sds$$

$$= \int_0^\infty \mathrm{Vol}(\{u^*(x) > s\}) 2sds = \int_0^\infty \mathrm{Vol}(\{u^*(x)^2 > t\}) dt$$

$$= \int_\Omega \int_0^\infty \chi_{\{u^{*2}(x) > t\}} dtdx = \int_\Omega |u^*(x)|^2 dx.$$

Note that following exactly the same proof (using p instead of 2), we prove the same preserving statement for the L^p-norm with $1 \le p \le \infty$. Lemma 4.6 is proved. □

Lemma 4.7 (Non-expansivity) *Let $H : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function with $H(0) = 0$. Then for any nonnegative functions u and v on $\mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} H(u^*(x) - v^*(x)) dx \le \int_{\mathbb{R}^d} H(u(x) - v(x)) dx. \tag{4.4}$$

To show the statement, we will use the following consequence of the Hardy-Littlewood inequality (see Theorem 4.9):

$$\int_{\mathbb{R}^d} u(x) \chi_{\{v(x) \le t\}} dx \le \int_{\mathbb{R}^d} u^*(x) \chi_{\{v^*(x) \le t\}} dx. \tag{4.5}$$

Let us write

$$H = H_- + H_+,$$

where

$$H_-(x) = 0 \quad \text{for} \quad x \ge 0 \quad \text{and} \quad H_-(x) = H(x) \quad \text{for} \quad x \le 0,$$

$$H_+(x) = 0 \quad \text{for} \quad x \le 0 \quad \text{and} \quad H_+(x) = H(x) \quad \text{for} \quad x \ge 0.$$

As H_- and H_+ are convex functions, we can prove Lemma 4.7 for H_- and H_+ separately.

Since H_+ is convex, by the fundamental lemma of calculus, one has

$$H_+(x) = \int_0^x H_+'(t) dt,$$

and $H_+'(x)$ is a nondecreasing function of x. Thus, we get

$$-H_+(u(x) - v(x)) = \int_{v(x)}^{u(x)} H_+'(u(x) - t) dt = \int_0^\infty H_+'(u(x) - t) \chi_{\{v \le t\}}(x) dt. \tag{4.6}$$

Integrating over the whole $\mathbb{R}^d$ we get

$$\int_{\mathbb{R}^d} H_+(u(x)-v(x))dx = -\int_{\mathbb{R}^d}\int_0^\infty H_+^{'}(u(x)-t)\chi_{\{v\le t\}}(x)dtdx.$$

By using Fubini's theorem we can change the order of integration for s and x in the right-hand side, i.e.

$$\int_{\mathbb{R}^d} H_+(u(x)-v(x))dx = -\int_0^\infty\int_{\mathbb{R}^d} H_+^{'}(u(x)-t)\chi_{\{v\le t\}}(x)dxdt.$$

Now combining inequality (4.5) and property (4.3) we obtain

$$\int_{\mathbb{R}^d} H_+(u(x)-v(x))dx = -\int_0^\infty\int_{\mathbb{R}^d} H_+^{'}(u(x)-t)\chi_{\{v\le t\}}(x)dxdt$$
$$\ge -\int_0^\infty\int_{\mathbb{R}^d} H_+^{'}(u^*(x)-t)\chi_{\{v^*\le t\}}(x)dxdt = \int_{\mathbb{R}^d} H_+(u^*(x)-v^*(x))dx,$$

that is,

$$\int_{\mathbb{R}^d} H_+(u(x)-v(x))dx \ge \int_{\mathbb{R}^d} H_+(u^*(x)-v^*(x))dx. \tag{4.7}$$

A similar calculation for H_- gives

$$\int_{\mathbb{R}^d} H_-(u(x)-v(x))dx \ge \int_{\mathbb{R}^d} H_-(u^*(x)-v^*(x))dx. \tag{4.8}$$

These inequalities yield (4.4), so that Lemma 4.7 is proved. □

Corollary 4.8 (L^p-distance decreasing) *Let* $1\le p\le\infty$. *Then for any two nonnegative measurable functions u and v on $\mathbb{R}^d$ we have*

$$\|u-v\|_{L^p(\mathbb{R}^d)} \ge \|u^*-v^*\|_{L^p(\mathbb{R}^d)}. \tag{4.9}$$

To see this for $p<\infty$ (the case $p=\infty$ is a modification of it), we take the nonnegative convex function $H(t)=|t|^p$ with $H(0)=0$, and apply Lemma 4.7. This gives

$$\int_{\mathbb{R}^d} |u^*(x)-v^*(x)|^p dx \le \int_{\mathbb{R}^d} |u(x)-v(x)|^p dx, \tag{4.10}$$

proving Corollary 4.8. □

4.2 Basic inequalities

The goal of this section is to discuss fundamental inequalities in the symmetric decreasing rearrangement theory. This section is based on the online lecture note of Burchard [26]. We believe that the shortest proofs of these inequalities are given in that note and, for the sake of completeness, below we recapture some of these short proofs of the following well-known results.

4.2.1 Hardy-Littlewood inequality

The following is a fundamental result showing how the symmetric decreasing rearrangements behave with respect to the inner product.

Theorem 4.9 (Hardy-Littlewood inequality) *Let u and v be nonnegative measurable functions, which vanish as $|x| \to \infty$, and let u^* and v^* be their symmetric decreasing rearrangements. If $\int_{\mathbb{R}^d} u^* v^* < \infty$ is finite, then*

$$\int_{\mathbb{R}^d} u(x)v(x)dx \leq \int_{\mathbb{R}^d} u^*(x)v^*(x)dx. \tag{4.11}$$

To show this, we observe that for any two measurable sets X and Y of finite volume we have the inequality

$$\mathrm{Vol}(X^* \cap Y^*) = \min\{\mathrm{Vol}(X^*), \mathrm{Vol}(Y^*)\} \geq \mathrm{Vol}(X \cap Y), \tag{4.12}$$

where X^* and Y^* are the balls (centred at 0) which are the rearrangements of X and Y, respectively. By the layer-cake decomposition (4.1), the Fubini theorem and (4.12), we get

$$\begin{aligned}
\int_{\mathbb{R}^d} u(x)v(x)dx &= \int_{\mathbb{R}^d} u(x) \int_0^\infty \chi_{\{v(x)>t\}} dt dx \\
&= \int_{\mathbb{R}^d} \int_0^\infty \chi_{\{u(x)>s\}} \int_0^\infty \chi_{\{v(x)>t\}} ds dt dx \\
&= \int_0^\infty \int_0^\infty \mathrm{Vol}(\{u > s\} \cap \{v > t\}) ds dt \\
&\leq \int_0^\infty \int_0^\infty \mathrm{Vol}(\{u^* > s\} \cap \{v^* > t\}) ds dt \\
&= \int_{\mathbb{R}^d} \int_0^\infty \chi_{\{u^*(x)>s\}} \int_0^\infty \chi_{\{v^*(x)>t\}} ds dt dx \\
&= \int_{\mathbb{R}^d} u^*(x)v^*(x)dx,
\end{aligned}$$

which proves the Hardy-Littlewood inequality. $\square$

4.2.2 Riesz inequality

The Riesz inequality is another important rearrangement inequality with a wide range of applications; for example, the Riesz inequality plays a decisive role in proofs of embedding theorems. For the first time in the one-dimensional case the inequality was proved by F. Riesz in [96]. Later in [119], S. L. Sobolev generalised this inequality to the multidimensional case by using induction over the dimension. Note that Hardy, Littlewood and Pólya gave a different proof in their book [47].

Theorem 4.10 (Riesz inequality) *For any nonnegative measurable functions $u, v, \varepsilon : \mathbb{R}^d \to \mathbb{R}^+$ vanishing as $|x| \to \infty$, we have*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x)\varepsilon(x-y)v(y)dxdy \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u^*(x)\varepsilon^*(x-y)v^*(y)dxdy. \tag{4.13}$$

The inequality (4.13) is understood to hold for those u, v and ε for which the right-hand side in (4.13) is finite.

Let us now prove the Riesz inequality (4.13). Let us first prove that

$$\int_X \chi_Y * \chi_Z \leq \int_{X^*} \chi_{Y^*} * \chi_{Z^*}, \tag{4.14}$$

where $X, Y, Z \subset \mathbb{R}^d$ are measurable sets of finite volume and χ is the characteristic function of a corresponding set. After using the layer cake representation (4.1) for nonnegative measurable functions u, v and ε one can easily see that showing (4.14) is sufficient to prove inequality (4.13). Let us introduce the following notation for the left-hand side of (4.14):

$$L(X,Y,Z) := \int_X \chi_Y * \chi_Z.$$

To prove inequality (4.14) we use induction over the Euclidian dimensions.

The case $d = 1$. To proceed in the one-dimensional case we will use the Brascamp-Lieb-Luttinger sliding method. Let X, Y and Z be intervals such that

$$X = x_0 + X^*, \quad Y = y_0 + Y^* \quad \text{and} \quad Z = z_0 + Z^*.$$

We will employ the following family of the symmetric rearrangements:

$$X(t) = x_0 e^{-t} + X^*, \quad Y(t) = y_0 e^{-t} + Y^* \quad \text{and} \quad Z(t) = z_0 e^{-t} + Z^*,$$

so that $X(0) = X$ and $X(t) \to X^*$ as $t \to \infty$. For these families we have

$$L(X(t), Y(t), Z(t)) = \int_{(x_0 - y_0 - z_0)e^{-t} + X^*} \chi_{Y^*} * \chi_{Z^*}.$$

This integral can be computed in an explicit form and it is a symmetric decreasing function. Therefore, the value $L(X(t), Y(t), Z(t))$ is nondecreasing in t, proving the case of the sets being the intervals.

Now consider the case

$$X = \bigcup_{k=1}^{m} X_k,$$

that is, when X is the union of m (a finite number) intervals X_k. Then we replace any two intervals such that their closures intersect by their union, so that the distance between any two intervals can be assumed to be positive. Let us denote by t_1 the point when the sub-interval closures for the family $X_k(t)$ first intersect, and set

$$X(t) = \bigcup_{k=1}^{m} X_k(t), \quad t \leq t_1.$$

At t_1, we combine the collided intervals, and we keep going with the same procedure for the new set of intervals. We follow the same scheme for Y and Z. It can be shown that the functional L increases with t. Thus, this implies the desired inequality for unions of intervals. For general sets, the inequality is proved by approximation, using that L is continuous with respect to symmetric differences, and that the symmetric decreasing rearrangement can only decrease the symmetric difference.

Induction. Let us assume that the Riesz inequality holds in dimensions from 1 to $d-1$. Let $X,Y,Z \subset \mathbb{R}^d$. For any $\hat{x} \in \mathbb{R}^{d-1}$, let us denote by $X_{\hat{x}}$, $Y_{\hat{y}}$ and $Z_{\hat{z}}$ the intersection of X, Y and Z with the line through $(\hat{x},0)$, $(\hat{y},0)$ and $(\hat{z},0)$, respectively. By the Fubini theorem we obtain

$$L(X,Y,Z) = \int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}^{d-1}} L(X_{\hat{x}}, Y_{\hat{y}}, Z_{\hat{x}-\hat{y}}) d\hat{x} d\hat{y}.$$

We know from the case $d=1$ of the Riesz inequality that L increases under the symmetric decreasing rearrangement. Moreover, by the induction hypothesis the $d-1$-dimensional case of the inequality implies that L increases under the symmetric decreasing rearrangement. It is clear that L is invariant under simultaneous rotations of X,Y and Z. Thus, its continuity implies

$$L(X,Y,Z) \le L(X^*,Y^*,Z^*),$$

proving the general case of the Riesz inequality. □

4.2.3 Pólya-Szegő inequality

The following inequality comes from a more general inequality for functions of the Sobolev class $L_1^p(\mathbb{R}^d)$ with $1 \le p \le \infty$, from [90]. We are mostly interested in the case $p=2$ which we will use later in the next chapter.

Theorem 4.11 (Pólya-Szegő inequality) *Let $1 \le p \le \infty$. Then for any $u \in L_1^p(\mathbb{R}^d)$ we have*

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \ge \|\nabla u^*\|_{L^p(\mathbb{R}^d)}.$$

Let us outline the proof of this theorem in the case $p=2$ as it can be modified to also cover the whole range of $1 \le p \le \infty$. We will rely on the following *co-area formula* (see e.g. [75]):

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx = \int\limits_0^\infty \int\limits_{u^{-1}(t)} |\nabla u| d\sigma dt,$$

where $d\sigma$ is the integration over the $d-1$-dimensional level sets.

Let us first consider the integrand in the right-hand side. Since $s \mapsto s^{-1}$ is convex, by the Jensen inequality we obtain that

$$\int\limits_{u^{-1}(t)} |\nabla u| \frac{d\sigma}{\mathrm{Per}(\{u>t\})} \ge \left(\int\limits_{u^{-1}(t)} |\nabla u|^{-1} \frac{d\sigma}{\mathrm{Per}(\{u>t\})} \right)^{-1}, \tag{4.15}$$

where Per denotes the surface area of a set. Moreover, we have

$$\int_{t_1}^{t_2} \int_{u^{-1}(t)} |\nabla u|^{-1} d\sigma dt = \mathrm{Vol}(\{x : t_1 < u(x) < t_2, |\nabla u(x)| \neq 0\}), \tag{4.16}$$

for every semi-interval $(t_1, t_2]$. The volume of the set of critical points decreases under the symmetric-decreasing rearrangement, so the right-hand side of (4.16) increases. From the left-hand side of (4.16) for u and u^* we get

$$\int\limits_{u^{-1}(t)} |\nabla u|^{-1} d\sigma \leq \int\limits_{(u^*)^{-1}(t)} |\nabla u^*|^{-1} d\sigma,$$

for almost every $t > 0$. By using this and replacing u with u^*in (4.15) we arrive at

$$\int\limits_{u^{-1}(t)} |\nabla u| d\sigma \geq \mathrm{Per}(\{u > t\})^2 \left(\int\limits_{u^{-1}(t)} |\nabla u|^{-1} d\sigma \right)^{-1}$$

$$\geq \mathrm{Per}(\{u^* > t\})^2 \left(\int\limits_{(u^*)^{-1}(t)} |\nabla u^*|^{-1} d\sigma \right)^{-1} = \int\limits_{(u^*)^{-1}(t)} |\nabla u^*| d\sigma,$$

where we have used the following isoperimetric inequality (see, e.g. [2]):

$$\mathrm{Per}(\{u > t\}) \geq \mathrm{Per}(\{u^* > t\}). \tag{4.17}$$

Integrating over t we obtain the desired result, finishing the proof of the Pólya-Szegő inequality. □

4.2.4 Talenti's comparison principles

In this section we prove the celebrated Talenti comparison principle ([125]), which states that the symmetric decreasing rearrangement (Schwarz rearrangement) of the Newtonian potential of a charge distribution is pointwise smaller than the potential resulting from symmetrizing the charge distribution itself. Talenti's comparison principle can be also extended to the Dirichlet p-Laplacian and the Dirichlet uniformly elliptic boundary value problems.

Theorem 4.12 (Talenti's inequality) *Consider a smooth nonnegative function f with* $\mathrm{supp} f \subset \Omega \subset \mathbb{R}^d$*, $d \geq 3$, for a bounded set Ω, and its symmetric decreasing rearrangement f^*. If solutions u and v of*

$$-\Delta u = f, \quad -\Delta v = f^*,$$

vanish as $|x| \to \infty$, then

$$u^*(x) \leq v(x), \quad \forall x \in \mathbb{R}^d.$$

Note that u and v exist, and are uniquely determined by the equation, i.e.

$$u(x) = \int_\Omega \varepsilon_d(x-y) f(y)\,dy$$

and

$$v(x) = \int_B \varepsilon_d(x-y) f^*(y)\,dy,$$

where $\varepsilon_d(\cdot)$ is the fundamental solution of the Laplacian, that is,

$$\varepsilon_d(x-y) = \frac{1}{(d-2)\sigma_d|x-y|^{d-2}}, \quad d \geq 3,$$

σ_d is the surface area of the d-dimensional unit ball, and B is the ball centred at the origin with $|B| = |\Omega|$, where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^d$. They are nonnegative since the fundamental solution is nonnegative. The inequality also holds for nonnegative measurable functions f vanishing as $|x| \to \infty$.

Let us outline the proof of this theorem. Consider the distribution function of u given by

$$\mu(t) = \mathrm{Vol}\{x:\ u(x) > t\}.$$

First we will estimate this distribution function in terms of f^*. It is clear that μ is differentiable at $t > 0$ almost everywhere. Equality (4.16) yields

$$-\mu'(t) \geq \int_{\{u=t\}} |\nabla u|^{-1} d\sigma,$$

for almost every $t > 0$. Restating the integrand on the right-hand side as

$$|\nabla u(x)|^{-1} = \sup_{\tau>0}\{2\tau - \tau^2|\nabla u(x)|\}, \tag{4.18}$$

we have

$$-\mu'(t) \geq \sup_{\tau>0}\{2\tau \int_{\{u=t\}} 1 d\sigma - \tau^2 \int_{\{u=t\}} |\nabla u| d\sigma\}. \tag{4.19}$$

The first term on the right-hand side satisfies

$$\int_{\{u=t\}} 1 d\sigma = \mathrm{Per}(\{u > t\}) \geq \mathrm{Per}(\{u^* > t\}),$$

where we have used (4.17).

From the Gauss divergence theorem we obtain the following equality for the second term

$$\int_{\{u=t\}} |\nabla u| d\sigma = -\int_{\{u=t\}} \nabla u \cdot N d\sigma = \int_{\{u>t\}} -\Delta u(y)\,dy,$$

where N is the exterior normal to the level set $\{u=t\}$. From the Poisson equation $-\Delta u = f$, we get

$$\int_{\{u=t\}} |\nabla u| d\sigma = \int_{\{u>t\}} f(y)dy = \int_{\{u^*>t\}} f^* dy.$$

By using these inequalities and minimising over τ, from (4.19) we obtain

$$-\mu'(t) \geq \mathrm{Per}(\{u^*>t\})^2 \left(\int_{\sigma_d |y|^d < \mu(t)} f^*(y)dy \right)^{-1}.$$

We also have

$$-\mu'(t) \int_{\sigma_d |y|^d < \mu(t)} f^*(y)dy \geq (d\sigma_d)^2 \left(\frac{\mu(t)}{\sigma_d} \right)^{2-\frac{2}{d}}.$$

To represent the above inequality in terms of u^*, we set $u^*(x) = \eta(|x|)$ for some non-increasing function η, and use a new variable $t = \eta(r)$. By definition we have $\mu \circ \eta(r) = \sigma_d r^d$. It implies that $\mu'(t)\eta'(r) = d\sigma_d r^{d-1}$. Therefore, we get

$$-\eta'(r) \leq (d\sigma_d r^{d-1})^{-1} \int_{|y|<r} f^*(y)dy. \tag{4.20}$$

Now integrating over r and using that η vanishes at infinity, we obtain

$$\eta'(r) \leq (d\sigma_d)^{-1} \int_r^\infty \int_{|y|<s} f^*(y) s^{-d+1} dyds.$$

All the above inequalities will be equalities if the function f is a symmetric decreasing function. By using the Fubini theorem and computing the integral over s in an explicit form, we arrive at

$$u^*(x) \leq v(x) = \frac{1}{d(d-2)\sigma_d} \int_{R^d} f^*(y)(\max\{|x|,|y|\})^{-d+2} dy, \ \forall x \in \mathbb{R}^d.$$

This expression of v is equivalent to the ordinary representation of v in terms of the Newton potential, finishing the proof of Talenti's inequality. □

Talenti's comparison principle can be extended to the Dirichlet boundary value problems for uniformly elliptic second-order differential operators and for the p-Laplacian (see, e.g. of [26, Section 4.3] as well as [125] and [126]).

Theorem 4.13 (Talenti's comparison principle for the Dirichlet p-Laplacian) *Let $1 < p < \infty$. Consider a (smooth) nonnegative function f in a smooth bounded domain*

$\Omega \subset \mathbb{R}^d$, $d \geq 1$, and its symmetric decreasing rearrangement f^*. Then solutions u and v of

$$-\Delta_p u = f \text{ in } \Omega,\ u|_{\partial\Omega} = 0,$$

and

$$-\Delta_p v = f^* \text{ in } B,\ v|_{\partial B} = 0,$$

satisfy

$$u^*(x) \leq v(x), \quad \forall x \in B.$$

Here B is the ball centred at the origin with $|B| = |\Omega|$, and $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^d$.

4.3 Properties of symmetric rearrangement sequences

In this section we study the compactness properties of minimising sequences of symmetric decreasing rearrangements. We formulate the Burchard-Guo theorem [27] on compactness via symmetric decreasing rearrangements. Some of its interesting applications will be considered in the following sections.

4.3.1 Burchard-Guo theorem

Let us consider the following energy integral

$$I(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x)K(x-y)u(y)dxdy, \tag{4.21}$$

where the positive kernel $K \in L^1_{loc}(\mathbb{R}^d)$ is a symmetric-decreasing function in $\mathbb{R}^d$. Using the Riesz inequality we have

$$I(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x)K(x-y)u(y)dxdy$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u^*(x)K^*(x-y)u^*(y)dxdy = I(u^*). \tag{4.22}$$

As in the previous sections u is a nonnegative measurable function which vanishes as $|x| \to \infty$, and its symmetric-decreasing rearrangement is denoted by u^*. Let F : $\mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function satisfying $F(0) = 0$ and let

$$J(u) := \int_{\mathbb{R}^d} F(|\nabla u|)dx. \tag{4.23}$$

From Lemma 4.7 and the Pólya-Szegő inequality we obtain

$$J(u) = \int_{\mathbb{R}^d} F(|\nabla u|)dx \geq \int_{\mathbb{R}^d} F(|\nabla u^*|)dx = J(u^*), \tag{4.24}$$

for all nonnegative measurable function u on $\mathbb{R}^d$ vanishing at infinity.

Let us assume that the kernel K is a positive strictly symmetric decreasing function and that F is a nonnegative strictly convex function which satisfies $F(0) = 0$.

By the continuity of I with respect to the norm defined by the positive definite quadratic form I, we obtain that

$$\lim_{n\to\infty} I(u_n^* - v) = 0 \tag{4.25}$$

implies

$$\overline{\lim_{n\to\infty}} I(u_n) \le \lim_{n\to\infty} I(u_n^*) = I(v).$$

In this section we denote by u_n the sequence of nonnegative functions which vanishes as $|x| \to \infty$, and v is a symmetric decreasing function.

Similarly, by using (4.24) and the Fatou lemma, from

$$\lim_{n\to\infty} J(u_n^* - v) = 0, \tag{4.26}$$

it follows that

$$\underline{\lim}_{n\to\infty} J(u_n) \ge \underline{\lim}_{n\to\infty} J(u_n^*) \ge J(v).$$

Now setting $u_n \equiv v$ and $v = u^*$, inequalities (4.22) and (4.24) are verified. Thus, both (4.22) and (4.24) are preserved under taking limits.

The following theorem shows that from (4.25) one obtains that the sequence u_n converges to v modulo translations.

Theorem 4.14 (Burchard-Guo theorem) *Let the sequence of rearrangements u_n^* converge to v in the sense that*

$$\lim_{n\to\infty} I(u_n^* - v) = 0. \tag{4.27}$$

If

$$\lim_{n\to\infty} I(u_n) = I(v), \tag{4.28}$$

then there exists a sequence of translations T_n in $\mathbb{R}^d$ such that

$$\lim_{n\to\infty} I(T_n u_n - v) = 0. \tag{4.29}$$

In the proof of the Burchard-Guo theorem it is useful to represent a function u as a sum of *levels*, $u = u^b + u^a$, where

$$u^b = [\min\{u, u^*(R^{-1})\} - u^*(R)]_+ \tag{4.30}$$

is finite, and

$$u^a = u - u^b = [\min\{u, u^*(R)\} + [u - u^*(R^{-1})]_+ \tag{4.31}$$

is zero for large enough R. Here, as usual, the lower index $+$ means that we get only its positive value, that is, if the value in the bracket is negative then we get zero as a result. Clearly, in the case of equimeasurability of u and v it follows that u^b and

u^a are equimeasurable to v^b and v^a, respectively. It is easy to see that the expansion commutes with translations and symmetric decreasing rearrangements.

We will prove Theorem 4.14 in the next section, and now we collect a number of technical lemmas that will be useful in its proof.

Lemma 4.15 *For fixed $R > 1$ and $I_0, J_0 > 0$ as in* (4.21) *and* (4.23)*, respectively, define the middle level u^b by* (4.30) *, i.e.*

$$u^b = [\min\{u, u^*(R^{-1})\} - u^*(R)]_+,$$

where u is a nonnegative measurable function which vanishes as $|x| \to \infty$.

Then we can always find constants $C_1(R, I_0)$ and $C_2(R, J_0)$ such that

$$\|u^b\|_\infty \leq C_1(R, I_0), \tag{4.32}$$

for any u such that $I(u^*) \leq I_0$, and

$$\|u^b\|_\infty \leq C_2(R, J_0), \tag{4.33}$$

for any u such that $J(u^*) \leq J_0$.

Let us prove this lemma. As $\|u^b\|_\infty$ grows with R, it is sufficient to show Lemma 4.15 for large R. First, using the fact that K and u^* are symmetric decreasing, we get

$$I(u^*) \geq \int \int_{|x|,|y|<R^{-1}} u^*(x)K(x-y)u^*(y)dxdy$$

$$\geq K(2R^{-1})(d\sigma_d R^{-d} u^*(R^{-1}))^2 \geq K(2R^{-1})(d\sigma_d R^{-d}\|u^b\|_\infty)^2,$$

where $\sigma_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the surface area of the unit ball in $\mathbb{R}^d$ and $\|u^b\|_\infty \leq u(R^{-1})$ has been used. We get (4.32) as $K(2R^{-1}) > 0$ for large enough R by assumption. To prove (4.33), we define a function Q on $\mathbb{R}^+$ such that $|\nabla u^*(x)| = Q(|x|)$, and by using polar coordinates we have

$$J(u^*) \geq \int_{R^{-1}}^{R} F(Q(r)) d\sigma_d r^{d-1} dr$$

$$\geq d\sigma_d R^{1-d}(R - R^{-1}) F\left(\int_{R^{-1}}^{R} Q(r)\frac{dr}{R - R^{-1}}\right) \geq d\sigma_d R^{2-d} F\left(\frac{\|u^b\|_\infty}{R}\right).$$

In the second step, we bound r^{d-1} from below by R^{1-d} and then use the Jensen inequality. Since $tF(x/t)$ is non-increasing in t, $R - R^{-1}$ can be replaced by R in the last step. Now (4.33) follows since F is strictly increasing, so that Lemma 4.15 is proved. □

Lemma 4.16 *Let us fix* $R>1$ *and expand* u_n, u_n^* *and* v *into levels as in* (4.30)-(4.31). *If*

$$\lim_{n\to\infty} J(u_n^* - v) = 0, \tag{4.34}$$

then

$$\lim_{n\to\infty} J(u_n^{*b} - v^b) = 0 \text{ and } \lim_{n\to\infty} J(u_n^{*a} - v^a) = 0. \tag{4.35}$$

If, additionally,

$$\lim_{n\to\infty} J(u_n) = J(v), \tag{4.36}$$

then

$$\lim_{n\to\infty} J(u_n^b) = J(v^b) \text{ and } \lim_{n\to\infty} J(u_n^a) = J(v^a). \tag{4.37}$$

Let us prove this lemma. By

$$\nabla u^{*b}(x) = \nabla u^*(x)\mathbf{1}_{R^{-1}\le|x|\le R},$$

the condition (4.34) can be rewritten as

$$\lim_{n\to\infty}\{J(u_n^{*b} - v^b) + J(u_n^{*a} - v^a)\} = 0,$$

which implies that both terms approach zero as in (4.35). To prove (4.37), by using

$$\nabla u^b(x) = \nabla u(x)\mathbf{1}_{u^*(R)\le u(x)\le u^*(R^{-1})},$$

the condition (4.36) can be rewritten as

$$\lim_{n\to\infty}\{J(u_n^b - J(v^b)) + J(u_n^a - J(v^a))\} = 0.$$

The (4.37) follows since the limit of each term is nonnegative by (4.26). The similar claim holds for the convolution functional I, so that Lemma 4.16 is proved. □

Lemma 4.17 *Let us fix* $R>1$, *and expand* u_n, u_n^* *and* v *into levels as in* (4.30)-(4.31). *If*

$$\lim_{n\to\infty} (u_n^* - v) = 0, \tag{4.38}$$

then

$$\lim_{n\to\infty} I(u_n^{*b} - v^b) = 0 \text{ and } \lim_{n\to\infty} I(u_n^{*a} - v^a) = 0. \tag{4.39}$$

If, additionally,

$$\lim_{n\to\infty} I(u_n) = I(v), \tag{4.40}$$

then

$$\lim_{n\to\infty} I(u_n^b) = I(v^b) \text{ and } \lim_{n\to\infty} I(u_n^a) = I(v^a). \tag{4.41}$$

To prove Lemma 4.17 we need some preliminary estimates:

Lemma 4.18 *We have*

$$I(v) \geq K(2R)\left(\int\limits_{|x|\leq R} v(x)dx\right)^2 \tag{4.42}$$

and

$$I(v) \geq \left(\int\limits_{|x|\geq 2R} v(x)K(|x|+R)dx\right)\left(\int\limits_{|x|<R} v(x)dx\right) \tag{4.43}$$

for any $R > 0$. In addition, for every $h \in L^1(\mathbb{R}^d)$ supported in $\{x : |x| \leq R_0\}$, and each $\varepsilon > 0$ there exists $R > 0$ depending only on K, R_0, and ε such that

$$\int\limits_{|x|\geq R} v(x)K*h(x)dx \leq \varepsilon\|h\|_1 I(v)^{1/2}. \tag{4.44}$$

Let us prove Lemma 4.18. It is clear that (4.42)-(4.43) are valid since K and v are nonnegative and symmetric decreasing functions. To establish the weak remainder estimate in (4.25), we consider two cases. If

$$\|v\|_{L^1} \leq \frac{(3/2)^{d-1}}{\varepsilon} I(v)^{1/2},$$

then we obtain for $R > R_0$

$$\int\limits_{|x|\geq R} v(x)|K*h(x)|dx \leq \|h\|_{L^1}\|v\|_{L^1}K(R-R_0)$$

$$\leq \|h\|_{L^1}\frac{(3/2)^{d-1}K(R-R_0)}{\varepsilon} I(v)^{1/2},$$

and (4.25) follows by taking R sufficiently large such that $K(R-R_0)(3/2)^{d-1} \leq \varepsilon^2$ If

$$\int\limits_{|x|\geq R_1} v(x)dx > \frac{(3/2)^{d-1}}{\varepsilon} I(v)^{1/2},$$

for $R_1 \geq R_0$, then we have for $R \geq 4R_1$, that

$$\int\limits_{|x|\geq R} v(x)|K*h(x)|dx \leq \|h\|_{L^1}\int\limits_{|x|\geq 4R_1} v(x)K(|x|-R_1)dx.$$

On the other hand, we have

$$\int\limits_{|x|\geq 4R_1} v(x)K(|x|-R_1)dx \leq \int\limits_{|x|\geq 2R_1} v(x)K(|x|+R_1)\left(1+\frac{2R_1}{|x|}\right)^{d-1} dx$$

$$\leq (3/2)^{d-1} \frac{I(v)}{\int_{|x|\leq R_1} v(x)dx} \leq \varepsilon I(v)^{1/2}.$$

In the first step, $v(x) \leq v(|x|-2R_1)$ is estimated using the polar coordinates. In the second step, $|x|+2R_1 \leq (3/2)|x|$ is used and (4.43). Plugging the last inequality into the preceding equation again gives (4.44). Lemma 4.18 is proved. □

Lemma 4.19 *For some $R>1$ and a sequence of nonnegative symmetric decreasing functions v_n on $\mathbb{R}^d$ vanishing at infinity we use the decompositions* (4.30)-(4.31). *If*

$$\lim_{n\to\infty} I(v_n - v) = 0, \tag{4.45}$$

then

$$\lim_{n\to\infty} I(v_n^b - v^b) = 0 \;\; and \;\; \lim_{n\to\infty} I(v_n^a - v^a) = 0. \tag{4.46}$$

Let us prove Lemma 4.19. First we need to show that a subsequence of v_n approaches v pointwise a. e.; then (4.46) is established by applying the Fatou lemma to

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \{[v_n(x)+v(x)][v_n(y)+v(y)]$$
$$-[v_n^\sharp(x)-v^\sharp(x)][v_n^\sharp(y)-v^\sharp(y)]\}K(x-y)dxdy,$$

for $\sharp = b, a$. Now let us show pointwise convergence. It is clear that

$$\lim_{n\to\infty} I(v_n) = I(v)$$

by (4.45). By the Cauchy-Schwarz inequality, the assumption (4.45) implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_n(x)K(x-y)h(y)dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(x)K(x-y)h(y)dxdy,$$

for all h with $I(h)<\infty$. It is the same as saying that $K*v_n$ converges to $K*v$ in the distribution sense. By (4.42) we see that the sequence v_n is uniformly bounded in L^1_{loc}. Since all functions v_n are symmetric decreasing, a subsequence (still denoted by v_n) can be taken such that

$$v_n \rightharpoonup g\delta_0 + v_0$$

in the distribution sense, and

$$v_n \to v_0$$

pointwise a. e. Here $g \geq 0$, δ_0 is the Dirac distribution at 0, and $v_0 \geq 0$ is a symmetric decreasing function satisfying $I(v_0)<\infty$. We need to prove that $g=0$. To do it, let us fix a function $h \in C_0^\infty$. As $\sup_{n\geq 1} I(v_n) < \infty$, estimate (4.44) in Lemma 4.18 implies that for each $\varepsilon > 0$ there exists an $R>0$ such that

$$\sup_{n\geq 1}\int_{|x|\geq R} v_n(x)|K*h(x)|dx \leq \varepsilon.$$

This yields

$$\int_{\mathbb{R}^d} (K * v)h = \lim_{n\to\infty} \int_{\mathbb{R}^d} v_n(x) K * h(x) dx$$
$$= \lim_{R\to\infty} \lim_{n\to\infty} \int\limits_{|x|\le R} v_n(x) K * h(x) dx = \int_{\mathbb{R}^d} K * (g\delta_0 + v_0)(x) h(x) dx,$$

where we have used that $K * v_n$ and v_n converge in the sense of distributions. As h is arbitrary, finally, we arrive at

$$K * (g\delta_0 + v_0) = K * v,$$

which implies that $g\delta_0 + v_0 = g$ by the positivity of K, and the pointwise convergence also follows. Lemma 4.19 is now proved. □

Proof of Lemma 4.17. Applying Lemma 4.19 to the sequence u_n^* of symmetric decreasing rearrangements, we see that (4.38) implies (4.39). On the other hand, we have

$$\lim_{n\to\infty} I(u_n^b) \le I(v^b) \quad \textit{and} \quad \lim_{n\to\infty} I(u_n^a) \le I(v^a),$$

by (4.38). Similarly, using the Riesz inequality and then the continuity with respect to the norm defined by the positivity of the quadratic form I, we obtain

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_n^b K(x-y) u_n^a(y) dxdy \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v^b K(x-y) v^a(y) dxdy.$$

Combining these inequalities we verify (4.41) of Lemma 4.17. □

Lemma 4.20 *Let u^* be a symmetric decreasing rearrangement of u^* supported on a ball of radius R_0 and let $I(u^*) < \infty$. Then there exists a translation T such that*

$$I(u^* - u) \ge (K(2R_0) - K(R_1)) \left(\int\limits_{|x|>R_1} Tu(x) dx \right)^2,$$

for any $R_1 > 2R_0$.

To prove Lemma 4.20, let us represent K as

$$K = [K - K(2R_0)]_+ + \min[K, K(2R_0)].$$

Since both terms on the right-hand side are nonnegative and symmetric decreasing, by the Riesz inequality we obtain

$$I(u^*) - I(u) \ge \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u^*(x) u^*(y) \min[K(x-y), K(2R_0)] dxdy$$
$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x) u(y) \min[K(x-y), K(2R_0)] dxdy \ge 0.$$

We rewrite the first integral on the right-hand side in the form

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u^*(x)u^*(y)\min[K(x-y),K(2R_0)]$$

$$=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u^*(x)u^*(y)K(2R_0)dxdy=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)u(y)K(2R_0)dxdy,$$

where in the first line we have used that u is supported in the ball with radius R_0, and we use that u is equimeasurable with its u^*. We have

$$I(u^*)-I(u)\geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)u(y)(K(2R_0)-\min[K(x-y),K(2R_0)])dxdy$$

$$+(K(2R_0)-K(R_1))\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)u(y)\mathbf{1}_{|x-y|\geq R_1}dxdy,$$

for any $R_1>2R_0$.

Setting $h(y)=\int u(x)\mathbf{1}_{|x-y|\geq R_1}dx$, by using the mean-value theorem there exists a point x_0 such that

$$\int_{\mathbb{R}^d} u(y)h(y)dy\geq h(x_0)\int_{\mathbb{R}^d} u(y)dy.$$

We have shown that

$$I(u^*)-I(u)\geq (K(2R_0)-K(R_1))\int_{\mathbb{R}^d} u(y)\int_{\mathbb{R}^d} u(x)\mathbf{1}_{|x-y|\geq R_1}dxdy$$

$$\geq (K(2R_0)-K(R_1))\int_{\mathbb{R}^d} u(x)\mathbf{1}_{|x-x_0|\geq R_1}dx\int_{\mathbb{R}^d} u(y)dy$$

$$\geq (K(2R_0)-K(R_1))\left(\int_{\mathbb{R}^d} u(x)\mathbf{1}_{|x-x_0|\geq R_1}dx\right)^2.$$

Putting $Tu(x)=u(x+x_0)$ ends the proof of Lemma 4.20. □

We denote by u_n a sequence of nonnegative functions in L^2, and, as in the whole section, I is defined by (4.21), with the nonnegative symmetric decreasing kernel K. Then the following lemma is valid.

Lemma 4.21 *If $u_n \rightharpoonup u$ and $u_n^* \rightharpoonup v$ converge weakly in L^2 for some functions u and v, then*

$$I(u)\leq I(v).$$

If K is strictly symmetric decreasing and $I(v)<\infty$, then from the equality it follows that there exists a translation T with $Tu=v$.

Let us prove Lemma 4.21. For an arbitrary nonnegative function $h\in L^2$ we have

$$\int u(x)h(x)dx=\lim_{n\to\infty}\int u_n(x)h(x)dx$$

$$\leq \lim_{n\to\infty}\int v_n^*(x)h^*(x)dx=\int v(x)h^*(x)dx.$$

By equimeasurability of u and u^* and the bathtub principle ([75]), for every $R > 0$ we have

$$\int\limits_{|x|<R} u^*(x)dx = \sup_{A:Vol(A)=\sigma_d R^d} \int\limits_A u^*(x)dx$$

$$= \sup_{A:Vol(A)=\sigma_d R^d} \int\limits_A u(x)dx \le \int\limits_{|x|<R} v(x)dx,$$

where σ_d is the surface area of the unit ball of the d-dimensional Euclidian space. Using the layer-cake decomposition we arrive at

$$\int u^*(x)h(x)dx \le \int v(x)h(x)dx, \tag{4.47}$$

for any symmetric decreasing h. If h is strictly symmetric decreasing and the integrals are not infinite, then the equality in (4.47) is valid only for $u^* = v$. Then the Riesz inequality gives

$$I(u) \le I(u^*) \le \int u^*(x)K * v(x)dx \le I(v), \tag{4.48}$$

where (4.47) has been used with $h = K * u^*$ and then with $h = K * v$. If K is strictly symmetric decreasing, then the equality in the Riesz inequality implies that there exists a translation T with $Tu = u^*$. In addition, since obviously both $K * u^*$ and $K * v$ are strictly symmetric decreasing, the equality in (3.12) implies $u^* = v$. Lemma 4.21 is proved. □

4.3.2 Proof of Burchard-Guo theorem

Suppose that u_n are uniformly bounded and their symmetric decreasing rearrangements u_n^* are supported in a ball with radius R. By Lemma 4.20, there exists a sequence of translations T_n such that

$$\int\limits_{|x|\ge 3R} T_n u_n(x)dx \le \left(\frac{I(u^*) - I(u)}{K(2R) - K(3R)}\right)^{\frac{1}{2}} \to 0, \;\; n \to \infty. \tag{4.49}$$

Since $\|T_n u_n\|_{L^2}^2 = \|u_n^*\|_{L^2}^2$ is uniformly bounded, the sequence $T_n u_n$ is weakly compact in L^2, that is, there exists a subsequence (again denoted by u_n) and a function u such that

$$T_n u_n \rightharpoonup u, \;\; n \to \infty, \tag{4.50}$$

converges weakly in L^2. From Lemma 4.21 we see that $I(u)$ is finite. We need to prove $I(T_n u_n - u) \to 0$ when $n \to \infty$. To do it, first let us fix $\varepsilon > 0$ and set

$$K = K\mathbf{1}_{|x|<\varepsilon} + K\mathbf{1}_{|x|\ge\varepsilon} := K^s + K^c.$$

Thus, we have

$$\begin{aligned} I(T_n u_n - u) = &\int_{|x|<3R} (T_n u_n - u) K^c * (T_n u_n - u) dx \\ &+ \int_{|x|\geq 3R} (T_n u_n - u) K^c * (T_n u_n - u) dx \\ &+ \int_{\mathbb{R}^d} (T_n u_n - u) K^s * (T_n u_n - u) dx. \end{aligned}$$

Here the first integral on the right-hand side disappears because of the compactness (by the Hilbert-Schmidt theorem) of $\{(K^c * T_n u_n)\mathbf{1}_{|x|<3R}\}_{n\geq 1}$ in L^2, and

$$(K^c * T_n u_n)\mathbf{1}_{|x|<3R} \to (K^c * u)\mathbf{1}_{|x|<3R}, \;\; n \to \infty.$$

By using (4.49) we get

$$\begin{aligned} &\int T_n u_n (K^c * T_n u_n)\mathbf{1}_{|x|\geq 3R} dx \\ &\leq \|K^c\|_{L^\infty} \|u\|_{L^1} \int_{|x|\geq 3R} u_n(x) dx \to 0, \;\; n \to \infty, \end{aligned}$$

for the second integral. The last integral is bounded by

$$\int_{\mathbb{R}^d} T_n u_n(x) K^s * u_n(x) dx \leq \|u_n\|_{L^\infty} \|u_n\|_{L^1} \int_{|x|\leq \varepsilon} K(x) dx,$$

which can be chosen small by taking ε sufficiently small.

We obtain $I(T_n u_n - u) \to 0$ for the positive kernel $K \in L^1_{loc}(\mathbb{R}^d)$.

Since we have $I(u) = I(v^*)$, Lemma 4.21 implies that $T_0 u = v$ for some translation T_0. Thus, this implies that

$$\inf_T I(T_0 u_n - v) \leq I(T_0 T_n u_n - v) \to 0, \;\; n \to \infty,$$

for a suitable sub-sequence. This proves (4.29) when u_n^* are uniformly bounded and supported on the same ball as the limit is independent of the subsequence.

Consider a sequence u_n, which satisfies the convergence assumptions of the Burchard-Guo theorem.

In cases when u_n and v have large level sets or/and they are not uniformly bounded, we represent them in the form

$$u_n = u_n^b + u_n^a$$

and

$$v = v^b + v^a,$$

using (4.30)-(4.31), where $R > 1$ is a sufficiently large number which will be chosen below.

By the Cauchy-Schwarz inequality and the fact that Tu_n is equimeasurable with u_n, we obtain

$$\inf_T I(Tu_n - v) \leq 3\{\inf_T I(Tu_n^b - v^b) + I(u_n^a) + I(v^a)\}. \tag{4.51}$$

Lemma 4.15 says that the sequence u_n^b is uniformly bounded, and by definition their symmetric decreasing rearrangements are supported on a ball of radius R.

According to Lemma 4.17, we see that the assumptions of the Burchard-Guo theorem are also satisfied by the functions u_n^b, with v changed by v^b.

In the first part of the proof it is shown that

$$\lim_{n\to\infty} \inf_T I(Tu_n^b - v^b) = 0.$$

Moreover, by Lemma 4.17, we have

$$\lim_{n\to\infty} I(Tu_n^a) = I(v^a).$$

Letting $n \to \infty$ in (3.46), we arrive at

$$\overline{\lim_{n\to\infty}} \inf_T I(Tu_n - v) \leq 6I(v^a).$$

Since the right-hand side can be chosen arbitrarily small by taking R sufficiently large, the Burchard-Guo theorem follows. $\square$

4.4 Applications in mathematical physics

In this section we give a number of applications of the technique of the symmetric decreasing rearrangements to several problems of mathematical physics.

4.4.1 Brownian motion in a ball

Let $z(t) = (z_1(t), z_2(t), z_3(t))$ be the three-dimensional Brownian motion, i.e. $z_1(t), z_2(t)$ and $z_3(t)$ are independent Wiener processes (see e.g. [55]). By definition

$$\text{Prob}\{y + z(\tau) \in \Omega\} = \int_\Omega (2\pi\tau)^{-\frac{3}{2}} \exp(-\frac{1}{2}\tau^{-1}|x-y|^2)dx, \tag{4.52}$$

where $\text{Prob}\{y + z(\tau) \in \Omega\}$ means the probability of $y + z(\tau)$ being in the set $\Omega \subset \mathbb{R}^3$ at time τ.

Lemma 4.22 *For any $\tau > 0$, we have*

$$\int_\Omega \text{Prob}\{y + z(\tau) \in \Omega\}dy \leq \int_{\Omega^*} \text{Prob}\{y + z(\tau) \in \Omega^*\}dy. \tag{4.53}$$

Note that geometrically (4.53) means that *Brownian particles starting from Ω are more likely to leave Ω than those starting from the ball Ω^* with the same volume as Ω to leave that ball.*

Let us now prove (4.53). Recall the Riesz inequality from Theorem 4.10 stating that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(y)g(z-y)h(y)dydz \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^*(y)g^*(z-y)h^*(y)dydz, \tag{4.54}$$

where f^*, g^* and h^* are the symmetric decreasing rearrangement of positive measurable functions f, g and h, respectively. Now that for $\tau > 0$ the function $\exp(-\frac{1}{2}\tau^{-1}|z|^2)$ is itself a strictly symmetrically decreasing function, so it does not change its formula under the rearrangement. By using the Riesz inequality (4.54) we have

$$\int_{\Omega} \mathrm{Prob}\{y+z(\tau)\in\Omega\}dy = \int_{\Omega}\int_{\Omega}(2\pi\tau)^{-\frac{3}{2}}\exp(-\frac{1}{2}\tau^{-1}|z-y|^2)dzdy$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\chi_{\Omega}(z)(2\pi\tau)^{-\frac{3}{2}}\exp(-\frac{1}{2}\tau^{-1}|z-y|^2)\chi_{\Omega}(y)dzdy$$

$$\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\chi_{\Omega^*}(z)(2\pi\tau)^{-\frac{3}{2}}\exp(-\frac{1}{2}\tau^{-1}|z-y|^2)\chi_{\Omega^*}(y)dzdy$$

$$= \int_{\Omega^*}\int_{\Omega^*}(2\pi\tau)^{-\frac{3}{2}}\exp(-\frac{1}{2}\tau^{-1}|z-y|^2)dzdy = \int_{\Omega^*}\mathrm{Prob}\{y+z(\tau)\in\Omega^*\}dy,$$

where $\chi_{\Omega}(z)$ is the characteristic function of the domain Ω, i.e. $\chi_{\Omega}(z)=1$ if $z\in\Omega$, $\chi_{\Omega}(z)=0$ if $z\in\mathbb{R}^3\setminus\Omega$. This completes the proof. $\Box$

4.4.2 Optimal membrane shape for the deepest bass note

Let $\Omega\subset\mathbb{R}^d$, $d\geq 2$, be a bounded domain with a smooth boundary $\partial\Omega$. Consider the following spectral problem for the Dirichlet Laplacian

$$-\Delta u = \lambda u, \quad x\in\Omega, \tag{4.55}$$

$$u = 0, \quad x\in\partial\Omega. \tag{4.56}$$

It is known from the spectral theory by arguments like those in Chapter 3 that the operator (4.55)-(4.56) is positive defined, self-adjoint and compact. Therefore, we know from Section 3.4 that it has a countable set of positive real eigenvalues that we can enumerate in increasing order (each time repeated according to multiplicity). Let us denote the smallest eigenvalue by $\lambda_1(\Omega)$. It is also known that $\lambda_1(\Omega)$ is simple and the corresponding eigenfunction $u_1(x)$ can be chosen positive.

Theorem 4.23 (Rayleigh-Faber-Krahn inequality) *The ball is a minimiser of the first eigenvalue of the Dirichlet Laplacian among all domains of a given measure:*

$$\lambda_1(\Omega)\geq\lambda_1(\Omega^*),$$

where Ω^* is the ball with $|\Omega^*|=|\Omega|$.

When $d = 2$ this theorem is interpreted as showing that the deepest bass note is produced by the circular drum among all drums with the same area as the circular drum, which was stated for the first time by Lord Rayleigh in his work [92]. In general, beyond the intrinsic interest of geometric extremum problems, Rayleigh-Faber-Krahn type inequalities produce a priori bounds for spectral invariants of operators on arbitrary domains. For example, in other words, the Rayleigh-Faber-Krahn inequality says that the operator norm of the inverse operator to the Dirichlet Laplacian is maximised in the ball among all Euclidean domains of a given volume.

Let us now prove Theorem 4.23. Let Ω be an open bounded (smooth) domain with the same measure as the ball Ω^*. Let u_1 be the first eigenfunction, i.e. an eigenfunction associated to $\lambda_1(\Omega)$. As we mentioned one can use that u_1 is positive in Ω, so we can introduce its symmetric decreasing rearrangement u_1^*. From Lemma 4.6, we get

$$\int_\Omega |u_1(x)|^2 dx = \int_{\Omega^*} |u_1^*(x)|^2 dx. \tag{4.57}$$

From the Pólya-Szegő inequality in Theorem 4.11, we have

$$\int_{\Omega^*} |\nabla u_1^*(x)|^2 dx \leq \int_\Omega |\nabla u_1(x)|^2 dx. \tag{4.58}$$

Now using (4.57), (4.58) and the variational principle, we arrive at

$$\begin{aligned}\lambda_1(\Omega) &= \frac{\int_\Omega |\nabla u_1(x)|^2 dx}{\int_\Omega |u_1(x)|^2 dx} \\ &\geq \frac{\int_{\Omega^*} |\nabla u_1^*(x)|^2 dx}{\int_{\Omega^*} |u_1^*(x)|^2 dx} \geq \inf_{v \in \mathring{L}_1^2(\Omega^*)} \frac{\int_{\Omega^*} |\nabla v(x)|^2 dx}{\int_{\Omega^*} |v(x)|^2 dx} = \lambda_1(\Omega^*).\end{aligned}$$

This proves the Rayleigh-Faber-Krahn inequality. □

4.4.3 Maximiser body of the gravitational field energy

Let Ω be a 3-dimensional body with a fixed volume and constant density ρ. Assume that the body $\Omega \subset \mathbb{R}^3$ generates the gravitational field with energy E_Ω.

It is known from the classical theory of gravity that the gravitational (Newtonian) potential u is represented in the form

$$u(z) = \int_\Omega \frac{1}{4\pi |z-y|} dy,$$

and it satisfies the Laplace equation, that is,

$$\triangle u(z) = \begin{cases} 1, & \text{for } z \in B, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding energy of the gravitational field is given by

$$E_\Omega = \int_{\mathbb{R}^3} |\nabla u|^2 dz,$$

and using Green's first formula and the fact that the gravitational potential vanishes as $|z| \to \infty$, this can be rewritten as

$$E_\Omega = \int_\Omega \int_\Omega \frac{1}{4\pi|z-y|} dydz. \tag{4.59}$$

Using the Riesz inequality (Theorem 4.10) and the fact that $\frac{1}{|z|}$ is a symmetric decreasing function for all $z \in \mathbb{R}^3$, we obtain

$$\begin{aligned} E_\Omega = \int_\Omega \int_\Omega \frac{1}{4\pi|z-y|} dydz = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \chi_\Omega(z)\chi_\Omega(y) \frac{1}{4\pi|z-y|} dydz \\ \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\chi_\Omega(z))^*(\chi_\Omega(y))^* \left(\frac{1}{4\pi|z-y|}\right)^* dydz \\ = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \chi_{\Omega^*}(z)\chi_{\Omega^*}(y) \frac{1}{4\pi|z-y|} dydz = \int_{\Omega^*} \int_{\Omega^*} \frac{1}{4\pi|z-y|} dydz = E_{\Omega^*}, \end{aligned}$$

where, as usual, χ is the characteristic function of the corresponding domain.

Thus, we obtain that

$$E_\Omega \leq E_{\Omega^*},$$

for an arbitrary $\Omega \subset \mathbb{R}^3$ with $|\Omega| = |\Omega^*|$, where Ω^* is the ball. Therefore, we have shown that *the ball has the gravitational field with maximal energy among all bodies with the same volume as the ball.*

4.4.4 Dynamical stability problem of gaseous stars

In the present section we briefly discuss an application of the Burchard-Guo theorem to the dynamical stability of gaseous stars. Following [27] we give an alternative proof of G. Rein's result [95] on the stability of gaseous stars. It is known that a self-gravitating star is expressed by the following *Euler-Poisson system*:

$$\begin{cases} \partial_t \mu + \nabla \cdot (\mu u) = 0, \\ \mu \partial_t u + \mu (u \cdot \nabla) u = -\nabla P(\mu) - \mu \nabla V, \end{cases} \tag{4.60}$$

$$\Delta V = 4\pi\mu,$$

with the condition at infinity

$$\lim_{|x|\to\infty} V(t,x) = 0,$$

where $\mu \geq 0$ is the mass density, u is the velocity field of a gaseous star, and $V = V_\mu$ is the gravitational (Newtonian) potential which can be written in the form

$$V_\mu(t,x) = -\int_{\mathbb{R}^3} |x-y|^{-1} \mu(t,x) dy. \tag{4.61}$$

Let us consider the special case when $P(\mu) = \mu^\gamma$. Formally, from (4.60) one obtains the conservation of the energy functional

$$E(\mu, u) = \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 \mu dx + \frac{1}{\gamma - 1}\int_{\mathbb{R}^3} \mu^\gamma dx - \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mu(x)|x-y|^{-1}\mu(y)dxdy.$$

On the right-hand side, the first term gives the kinetic energy, the second term represents the pressure contribution, and the last term expresses the gravitational potential energy.

The steady states are obtained by minimising the functional

$$H = \frac{1}{\gamma - 1}\int_{\mathbb{R}^3} \mu^\gamma dx - \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mu(x)|x-y|^{-1}\mu(y)dxdy, \tag{4.62}$$

with the mass restriction $\int_{\mathbb{R}^3} \mu(x)dx = M$.

A symmetric minimiser is expressed by the formula

$$\mu_0(x) = c(\gamma)[E_0 - V_{\mu_0}(x)]_+^{\frac{1}{\gamma-1}}. \tag{4.63}$$

Here $E_0 \leq 0$ is the Lagrange multiplier corresponding to the mass constraint, and $V_{\mu_0}(x)$ is the potential obtained by using (4.61) with μ_0. This minimiser is always unique up to a translation.

Theorem 4.24 ([95]) *For some $\gamma > 4/3$, the symmetric steady-state solution $\mu_0(x)$ is dynamically stable up to translations, among all weak solutions which satisfy the mass constraint and whose energy is not greater than the initial energy.*

The “distance” from μ_0 to μ is given by

$$d(\mu, \mu_0) = \frac{1}{\gamma - 1}\int_{\mathbb{R}^3} \mu^\gamma - \mu_0^\gamma + (V_{\mu_0} - E_0)(\mu - \mu_0)dx.$$

It is clear that the integrand is nonnegative since $\gamma > 4/3$ (one can see it expanding into the Taylor series around μ_0 in (4.63)). It is important to obtain for any minimising sequence μ_n that there exists a translation sequence T_n on $\mathbb{R}^3$ with

$$\|\nabla V_{T_n\mu_n} - \nabla V_{\mu_0}\|_{L^2(\mathbb{R}^3)} \to 0. \tag{4.64}$$

Let us now prove Theorem 4.24. Let us introduce the notation

$$I(\mu) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mu(x)|x-y|^{-1}\mu(y)dxdy = \|\nabla V_\mu\|^2_{L^2(\mathbb{R}^3)},$$

which expresses the gravitational potential energy corresponding to μ.

It follows directly from [95, Lemma 4.1] that the symmetric minimising sequences are compact.

It was also proved in [95] that (4.64) is valid without any translation, that is,

$$\lim_{n\to\infty} I(\mu_n - \mu_0) = 0.$$

Now we take a general minimising sequence μ_n such that $\lim_{n\to\infty}\int\mu_n = M$. By using the fact that μ_n is equimeasurable with μ_n^* and the Riesz inequality, we obtain that the symmetrization sequence μ_n^* by itself is a minimising sequence, so that

$$\lim_{n\to\infty} I(\mu_n) = \lim_{n\to\infty} I(\mu_n^*) = I(\mu_0).$$

By the first step,

$$\lim_{n\to\infty} I(\mu_n^* - \mu_0) = 0.$$

By using the fact that the kernel $K(x-y) = |x-y|^{-1}$ is a positive strictly symmetric decreasing function and belongs to L^1_{loc}, the convergence (4.64) follows directly from the Burchard-Guo Theorem 4.14. □

4.4.5 Stability of symmetric steady states in galactic dynamics

In this section we briefly discuss another application of the symmetric decreasing rearrangements to the stability in a large ensemble of stars (for instance, a galaxy). This section is based on [27] and we refer to it for further discussions.

Let us consider a galactic system of stars (a large ensemble of stars, for example, a galaxy) under the gravitational field which is created by themselves.

It is almost impossible to study the dynamics of every single star separately. For this reason, the most basic models of theoretical physics to study galaxy dynamics are based on the kinetic theory, in which the star system is described by a density $q(t,x,u)$ of the phase space instead of a density $\rho(t,x)$ and a velocity field $u(t,x)$.

Let us introduce the notations for the position and the momentum variables by $(x,u) \in \mathbb{R}^3 \times \mathbb{R}^3$, respectively.

In astrophysics, the galaxy dynamics or the global cluster dynamics are described by the following *Vlasov-Poisson equations*:

$$\begin{cases} \partial_t q + u\cdot\nabla_x q - \nabla_x u\cdot\nabla_u q = 0, \\ \Delta V = 4\pi\rho, \end{cases} \tag{4.65}$$

where

$$\rho(t,x) = \int q(t,x,u)du \tag{4.66}$$

is the particle density associated with q, and the gravitational potential V again solves (4.61).

The Hamiltonian energy (the sum of the kinetic and potential energies)

$$E(q) = E_{kinetic}(q) + E_{potential}(\rho)$$
$$= \frac{1}{2}\int\int |u|^2 q(x,u)dudx - \frac{1}{2}\int\int \rho(x)|x-y|^{-1}\rho(y)dxdy$$

is conserved by the equations (4.65).

Note that the rare collisions among stars are neglected in this kind of model. In fact, the Vlasov-Poisson system has more properties, in particular, a scaling symmetry and a continuum of conserved quantities are given by the following Casimir functionals

$$C(q) =: \int \int Q(q(x,u))dudx,$$

where Q is any measurable function which satisfies suitable growth conditions. In the present section $Q(q) = q^{1+\frac{1}{k}}$ with $0 < k < \frac{3}{2}$ will play a special role.

One obtains the steady states by minimising the functionals

$$C(q) + E(q) \tag{4.67}$$

under the essential restriction that the total mass $\int \int q dudx = M$ should be a positive constant.

We consider the minimisation problem for (4.67),

$$\inf_{q:\int\int q(x,u)dudx=M} \{C(q) + E(q)\} =$$

$$= \inf_{\rho:\int \rho(x)dx=M} \{ \inf_{q:\int q(\cdot,u)du=\rho} \{C(q) + E_{kinetic}(q) + E_{potential}(\rho)\}\},$$

in two steps.

Step I: The inner minimisation is equivalent to calculating for a fixed particle density ρ the quantity $G \circ \rho$, where

$$G(r) = \inf\{\int Q(g(u)) + \frac{1}{2}|u|^2 g(u)du \mid 0 \le g \in L^1(\mathbb{R}^3), \int g(u)du = r\}, \tag{4.68}$$

for $r \geq 0$. Since Q is a strictly convex function, the minimiser in (4.68) is expressed in terms of r in a unique form. Thus, any minimising phase space density for (4.67) is uniquely expressed by the associated particle density.

Using their Legendre transforms $\hat{Q}$ of Q and $\hat{G}$ of G, it can be written as the following relationship

$$\hat{G}(t) = \int \hat{Q}(t - \frac{|u|^2}{2})du.$$

From $Q(q) = q^{1+\frac{1}{k}}$ we obtain (up to a multiplicative constant) $G(\rho) = \rho^\gamma$ with $\gamma = 1 + \frac{1}{k+\frac{3}{2}} \in (\frac{4}{3}, \frac{5}{3})$.

Step II: Thus, the outer minimisation problem is simplified to minimising

$$H = \int G(\rho(x))dx - \int \int \rho(x)|x-y|^{-1}\rho(y)dxdy \tag{4.69}$$

over particle densities ρ with the mass restriction $\int \rho(x)dx = M$. This problem has the same form as (4.62), studied in the previous section. In particular, the existence of symmetric steady states with the particle density is given by (4.63). The associated symmetric minimising phase space density is expressed by

$$q_0(x,u) = \left[E_0 - \frac{|u|^2}{2} - V_{\rho_0}(x)\right]_+^k, 0 < k < \frac{3}{2}.$$

From the stability point of view for the Vlasov-Poisson equations all the needed knowledge for the variational problem in (4.67) can be taken from its simplified form in (4.69).

To see it, let us consider a minimising sequence q_n for (4.67) and the relevant sequence of particle densities ρ_n which is determined by (4.66).

According to the fact that ρ_n is a minimising sequence for the simplified problem in (4.69), we obtain from the previous section that ρ_n approaches (up to suitable translations T_n) some particle density ρ_0 and $\nabla V_{T_n\rho_n} \to \nabla V_{\rho_0}$ in L^2.

Using a corresponding subsequence and applying the special form of Q, it can be assumed that the sequence of phase space densities T_nq_n converges weakly in $L^{1+\frac{1}{k}}$ to some function u_0. From the compactness of $\nabla V_{T_nq_n}$ in L^2, the Casimir energy functional $E+C$ is lower semi-continuous, and its value must converge along the sequence, and we establish that $T_nq_n \to q_0$ strongly in $L^{1+\frac{1}{k}}$. This means that q_0 is the unique minimiser for the given problem in (4.67) expressed by $\rho_0(x)$. Therefore, there exists a sequence of translations T_n such that $T_nq_n \to q_0$. $\square$

Chapter 5

Inequalities of spectral geometry

In this chapter we will discuss the isoperimetric inequalities and other related inequalities of the spectral geometry for integral operators of several types, appearing as solutions to different boundary value problems for elliptic and parabolic partial differential equations.

We start by discussing the logarithmic potential operators in 2-dimensions, subsequently moving to the Riesz and Bessel potential operators. Consequently, we show how the analysis can be extended to the Riesz potential operators also in the spherical and hyperbolic geometries.

Next, we concentrate on several cases of non-self-adjoint operators. This is a case much more rarely encountered in the literature. Here we discuss different versions of the isoperimetric inequalities for the singular numbers, for the heat operators of different types: higher-order heat operators, as well as the heat operators with the Cauchy-Dirichlet, Cauchy-Robin, Cauchy-Neumann and Cauchy-Dirichlet-Neumann boundary conditions.

Most of the results in this chapter are based on the papers by the authors and we follow the presentation therein for our exposition here.

5.1 Introduction

In Rayleigh's famous book *Theory of Sound* (first published in 1877, [92]), by using some explicit computation and physical interpretations, he stated that the disc minimises (among all domains of the same area) the first eigenvalue of the Dirichlet Laplacian. The proof of this conjecture was obtained about 50 years later, simultaneously (and independently) by G. Faber and E. Krahn. Nowadays, the Rayleigh-Faber-Krahn inequality has been established for many other operators; see e.g. [48] for further references (see also [8] and [90], as well as the already described Theorem 4.23). Among other things, in this chapter we prove the Rayleigh-Faber-Krahn inequality for the integral operators of convolution type, i.e. it is proved that the ball is a minimiser of the first eigenvalue of the convolution type integral operator among all domains of a given measure.

By using the Feynman-Kac formula and spherical rearrangements, Luttinger proved in [77] that the disc D is a maximiser of the partition function of the Dirichlet

Laplacian among all domains of the same area as D for all positive values of time, i.e.

$$\sum_{i=1}^{\infty} \exp(-t\mu_i^{\mathscr{D}}(\Omega)) \leq \sum_{i=1}^{\infty} \exp(-t\mu_i^{\mathscr{D}}(D)), \quad \forall t > 0,\ |\Omega| = |D|,$$

where $\mu_i^{\mathscr{D}}, i = 1, 2, \ldots,$ are the characteristic numbers of the Dirichlet Laplacian. *The characteristic numbers are defined as the inverses of the eigenvalues, and there is a natural correspondence between the eigenvalues of a problem and the characteristic numbers of its solution operator.*

From here by using the Mellin transform one obtains

$$\sum_{i=1}^{\infty} \frac{1}{[\mu_i^{\mathscr{D}}(\Omega)]^p} \leq \sum_{i=1}^{\infty} \frac{1}{[\mu_i^{\mathscr{D}}(D)]^p}, \quad |\Omega| = |D|, \tag{5.1}$$

when $p > 1$, $\Omega \subset \mathbb{R}^2$. In this chapter we discuss an analogue of this Luttinger's inequality for the integral operator of the convolution type. For example, we consider the logarithmic potential operator: in this setting the main difficulty arises from the fact that the logarithmic potential is not positive and that we cannot use the Brascamp-Lieb-Luttinger type rearrangement inequalities directly.

Thus, we are mainly interested in questions of spectral geometry. The main reason why the results are useful, beyond the intrinsic interest of geometric extremum problems, is that they produce *a priori* bounds for spectral invariants of operators on arbitrary domains. For a good general review of isoperimetric inequalities for the Dirichlet, Neumann and other Laplacians we can refer to [12].

We also show that under certain restrictions for indices, the Schatten norms of the Riesz potentials $\mathscr{R}_{\alpha,\Omega}$ over sets of a given measure are maximised on balls. More precisely, we can summarise this type of discussion as follows:

- Let $0 < \alpha < d$ and let Ω^* be a ball in $\mathbb{R}^d$; we set $p_0 := d/\alpha$. Then for any integer p with $p_0 < p \leq \infty$ we have

  $$\|\mathscr{R}_{\alpha,\Omega}\|_p \leq \|\mathscr{R}_{\alpha,\Omega^*}\|_p, \tag{5.2}$$

  for any domain Ω with $|\Omega| = |\Omega^*|$. Here $\|\cdot\|_p$ stands for the Schatten p-norm, $|\cdot|$ for the Lebesgue measure. The proof is based on the application of a suitably adapted Brascamp-Lieb-Luttinger inequality. Note that for $p = \infty$ this result gives a variant of the famous Rayleigh-Faber-Krahn inequality for the Riesz potentials (and hence also for the Newton potential).

- We also establish the Hong-Krahn-Szegő inequality: the maximum of the second eigenvalue of $\mathscr{R}_{\alpha,\Omega}$ among bounded open sets with a given measure is approached by the union of two identical balls with mutual distance going to infinity.

There is a vast number of papers dedicated to the above type of results for Dirichlet, Neumann and other Laplacians, see, for example, [23], [48] and references therein. For instance, the questions of Rayleigh-Faber-Krahn type are still open for boundary value problems of the bi-Laplacian (see [48, Chapter 11]). The main difficulty arises because the resulting operators of these boundary value problems are not positive for higher powers of the Laplacian. The same is the situation for the Schatten p-norm inequalities: the result for the Dirichlet Laplacian can be obtained from Luttinger's inequality [77] but very little is known for other Laplacians (see [38]). The Hong-Krahn-Szegő inequality for the Robin Laplacian was proved recently [66] (see [24] for further discussions). So, in general, until now there were no examples of a boundary value problem for the poly-Laplacian ($m > 1$) for which all the above results had been proved. It seems that there are also no isoperimetric results for the fractional order Riesz or/and Bessel potentials either.

We believe that Kac's boundary value problem (5.3) with (5.5) serves as the first example of such a boundary value problem, for which all the above results are true. This problem describes the nonlocal boundary conditions for the poly-Laplacian corresponding to the polyharmonic Newton potential operator.

In a bounded connected domain $\Omega \subset \mathbb{R}^d$ with a piecewise C^1 boundary $\partial\Omega$, we consider the polyharmonic equation

$$(-\Delta_x)^m u(x) = f(x), \quad x \in \Omega, \quad m \in \mathbb{N}. \tag{5.3}$$

To relate the polyharmonic Newton potential

$$u(x) = \int_\Omega \varepsilon_{2m,d}(|x-y|) f(y) dy, \, f \in L^2(\Omega), \tag{5.4}$$

to the boundary value problem (5.3) in Ω, we can use the result of [60] asserting that for each function $f \in L^2(\Omega)$, $\operatorname{supp} f \subset \Omega$, the polyharmonic Newton potential (5.4) belongs to the class $H^{2m}(\Omega)$ and satisfies, for $i = 0, 1, \ldots, m-1$, the nonlocal boundary conditions

$$\begin{aligned} &-\frac{1}{2}(-\Delta_x)^i u(x) + \\ &\qquad + \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y} (-\Delta_y)^{m-i-1-j} \varepsilon_{2(m-i),d}(|x-y|)(-\Delta_y)^j (-\Delta_y)^i u(y) dS_y \\ &- \sum_{j=0}^{m-i-1} \int_{\partial\Omega} (-\Delta_y)^{m-i-1-j} \varepsilon_{2(m-i),d}(|x-y|) \frac{\partial}{\partial n_y} (-\Delta_y)^j (-\Delta_y)^i u(y) dS_y = 0, \, x \in \partial\Omega. \end{aligned} \tag{5.5}$$

Conversely, if a function $u \in H^{2m}(\Omega)$ satisfies (5.3) and the boundary conditions (5.5) for $i = 0, 1, \ldots, m-1$, then it defines the polyharmonic Newton potential by the formulae (5.4).

Therefore, our analysis (of the special case of the Riesz (or Bessel) potential) of the polyharmonic Newton potential (5.4) implies the corresponding result for the boundary value problem (5.3) with (5.5). Note that the analogue of the problem (5.3) with (5.5) for the Kohn Laplacian and its powers on the Heisenberg group have been recently investigated in [101]. We note that there are certain interesting questions concerning such operators, lying beyond Schatten classes properties, see e.g. [39] for different regularised trace formulae.

On the other hand, in G. Pólya's work [89] he proves that the first eigenvalue of the Dirichlet Laplacian is minimised in the equilateral triangle among all triangles of given area.

In this chapter our other aim is to extend some of the known results for the self-adjoint operators to non-self-adjoint operators. Thus, for example, we prove a Pólya type inequality for the Cauchy-Dirichlet heat operator, that is, that the first s-number of the Cauchy-Dirichlet heat operator is minimised in the equilateral triangular cylinder among all triangular cylinders of given volume. We also discuss a number of other inequalities of the spectral geometry for the heat operators with different boundary conditions.

5.2 Logarithmic potential operator

Let us consider the ***logarithmic potential operator*** on $L^2(\Omega)$ defined by

$$\mathscr{L}_\Omega f(x) := \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y) dy, \quad f \in L^2(\Omega), \tag{5.6}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded set and

$$|x-y| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$

Since the logarithmic potential operator $\mathscr{L}_\Omega$ is a compact and self-adjoint operator, all of its eigenvalues are discrete and real. The characteristic numbers (the inverses of the eigenvalues) of $\mathscr{L}_\Omega$ may be enumerated in ascending order of their modulus,

$$|\mu_1(\Omega)| \leq |\mu_2(\Omega)| \leq ...,$$

where $\mu_i(\Omega)$ is repeated in this series according to its multiplicity. We denote the corresponding eigenfunctions by $u_1, u_2, ...$, so that for each characteristic number μ_i there is a unique corresponding normalised eigenfunction u_i,

$$u_i = \mu_i(\Omega) \mathscr{L}_\Omega u_i, \ \ i = 1, 2, ...$$

It is known, see e.g. [59], that the equation

$$u(x) = \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y) dy$$

is equivalent to the equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{5.7}$$

with the nonlocal integral boundary condition

$$-\pi u(x)+\int_{\partial\Omega}\frac{\partial}{\partial n_y}\ln\frac{1}{|x-y|}u(y)dS_y-\int_{\partial\Omega}\ln\frac{1}{|x-y|}\frac{\partial u(y)}{\partial n_y}dS_y=0, \tag{5.8}$$

for all $x\in\partial\Omega$, where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at a point y on the boundary $\partial\Omega$, which is assumed piecewise C^1 here.

Spectral analysis on the logarithmic potential have been investigated in many papers (see, e.g. [3], [5], [18], [40], [56], [59], [111], [128], [129]). Here we discuss some spectral geometric inequalities of the logarithmic potential $\mathscr{L}_\Omega$, that is also, some spectral geometric inequalities of the nonlocal Laplacian (5.7)-(5.8). For a general review of spectral geometric inequalities for the Dirichlet, Neumann and other Laplacians we refer to Benguria, Linde and Loewe in [12].

Thus, in this section we show that the disc is a maximiser of the Schatten p-norm of the logarithmic potential operator among all domains of a given measure in $\mathbb{R}^2$, for all even integers $2\le p<\infty$. We also discuss polygonal versions of this result; in particular, we show that the equilateral triangle has the largest Schatten p-norm among all triangles of a given area. For the logarithmic potential operator on bounded open or triangular domains, we also present analogies of the Rayleigh-Faber-Krahn or Pólya inequalities, respectively. This section is completely based on our open access paper [99].

5.2.1 Spectral geometric inequalities and examples

We assume that $\Omega\subset\mathbb{R}^2$ is an open bounded set and we consider the logarithmic potential operator on $L^2(\Omega)$ of the form

$$\mathscr{L}_\Omega f(x)=\int_\Omega\frac{1}{2\pi}\ln\frac{1}{|x-y|}f(y)dy,\quad f\in L^2(\Omega). \tag{5.9}$$

We also assume that the operator $\mathscr{L}_\Omega$ is positive. In Landkof [69, Theorem 1.16, p. 80] the positivity of the operator $\mathscr{L}_\Omega$ is proved in domains $\overline{\Omega}\subset U$, where U is the unit disc. In general, $\mathscr{L}_\Omega$ is not a positive operator. For any bounded open domain Ω the logarithmic potential operator $\mathscr{L}_\Omega$ can have at most one negative eigenvalue, see Troutman [128] (see also Kac [56]). Note that for positive self-adjoint operators the singular values equal the eigenvalues.

By $\|\mathscr{L}_\Omega\|_p$ we denote the Schatten p-norm of the logarithmic potential operators. It is known that $\mathscr{L}_\Omega$ is a Hilbert-Schmidt operator.

Theorem 5.1 *Let D be a disc. Let Ω be a bounded open domain with the same Lebesque measure as the disc D. Assume that the logarithmic potential operator is positive on both sets Ω and D. Then*

$$\|\mathscr{L}_\Omega\|_p\le\|\mathscr{L}_D\|_p \tag{5.10}$$

for any integer $2\le p<\infty$.

Note that we will see from the proof that for even integers p we do not need to assume the positivity of the logarithmic potential operator for the above result to be true:

Theorem 5.2 *We have*

$$\|\mathscr{L}_\Omega\|_p \leq \|\mathscr{L}_D\|_p \tag{5.11}$$

for all even integer $2 \leq p < \infty$ and any bounded open domain Ω with $|\Omega| = |D|$. Here D is a disc and $|\cdot|$ is the Lebesque measure of a set.

The right-hand side of the formula (5.11) can be calculated explicitly by a direct calculation of the logarithmic potential eigenvalues in the unit disc, see [3, Theorem 3.1]. For example, we have

$$\|\mathscr{L}_\Omega\|_p \leq \|\mathscr{L}_U\|_p = \left(\sum_{m=1}^{\infty} \frac{3}{j_{0,m}^{2p}} + \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \frac{2}{j_{l,m}^{2p}}\right)^{\frac{1}{p}}, \tag{5.12}$$

for any even $2 \leq p < \infty$ and any bounded open domain Ω with $|\Omega| = |U|$. Here U is the unit disc and j_{km} denotes the m^{th} positive zero of the Bessel function J_k of the first kind of order k.

We also have the following Rayleigh-Faber-Krahn inequality for the logarithmic potential when $p = \infty$:

Theorem 5.3 (Rayleigh-Faber-Krahn inequality) *The disc D is a minimiser of the characteristic number of the logarithmic potential $\mathscr{L}_\Omega$ with the smallest modulus among all domains of a given measure, that is,*

$$\|\mathscr{L}_\Omega\|_\infty \leq \|\mathscr{L}_D\|_\infty \tag{5.13}$$

for an arbitrary bounded open domain $\Omega \subset \mathbb{R}^2$ with $|\Omega| = |D|$.

From [3, Corollary 3.2] we calculate explicitly the operator norm in the right-hand side of (5.13). Let $D \equiv U$ be the unit disc. Then by Theorem 5.3 we have

$$\|\mathscr{L}_\Omega\|_\infty \leq \|\mathscr{L}_U\|_\infty = \frac{1}{j_{01}^2}$$

for any bounded open domain Ω with $|\Omega| = |D|$. Here $\|\cdot\|_\infty$ is actually the operator norm of the logarithmic potential on the space L^2.

As discussed before, the logarithmic potential operator can be related to a non-local boundary value problem for the Laplacian, so above theorems are valid for the eigenvalues (5.7)-(5.8) as well.

To prove Theorem 5.3 first we need some preliminary discussions: The eigenfunctions of the logarithmic potential $\mathscr{L}_\Omega$ may be chosen to be real as its kernel is real. First let us prove that u_1 cannot change sign in the domain Ω, that is,

$$u_1(x)u_1(y) = |u_1(x)u_1(y)|,\ x, y \in \Omega.$$

Indeed, in the opposite case, by the continuity of the function $u_1(x)$, there would be neighborhoods $U(x_0,r) \subset \Omega$ such that

$$|u_1(x)u_1(y)| > u_1(x)u_1(y), \quad x,y \in U(x_0,r) \subset \Omega.$$

On the other hand we have

$$\int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x_0 - z|} \frac{1}{2\pi} \ln \frac{1}{|z - x_0|} dz > 0, \quad x_0 \in \Omega. \tag{5.14}$$

From here by continuity it is simple to check that there exists $\rho > 0$ such that

$$\int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x - z|} \frac{1}{2\pi} \ln \frac{1}{|z - y|} dz > 0, \quad x,y \in U(x_0,\rho) \subset U(x_0,r). \tag{5.15}$$

Now let us introduce a new function

$$\widetilde{u}_1(x) := \begin{cases} |u_1(x)|, \ x \in U(x_0,\rho), \\ u_1(x), \ x \in \Omega \backslash U(x_0,\rho). \end{cases} \tag{5.16}$$

Then we obtain

$$\begin{aligned} \frac{(\mathscr{L}_\Omega^2 \widetilde{u}_1, \widetilde{u}_1)}{\|\widetilde{u}_1\|^2} &= \frac{1}{\|\widetilde{u}_1\|^2} \int_\Omega \int_\Omega \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x - z|} \frac{1}{2\pi} \ln \frac{1}{|z - y|} dz \widetilde{u}_1(x) \widetilde{u}_1(y) dx dy \\ &> \frac{1}{\|u_1\|^2} \int_\Omega \int_\Omega \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x - z|} \frac{1}{2\pi} \ln \frac{1}{|z - y|} dz u_1(x) u_1(y) dx dy = \frac{1}{\mu_1^2}, \end{aligned} \tag{5.17}$$

where μ_1^2 is the smallest characteristic number of $\mathscr{L}_\Omega^2$ and u_1 is the eigenfunction corresponding to μ_1^2, i.e.

$$u_1 = \mu_1^2 \mathscr{L}_\Omega^2 u_1.$$

Therefore, by the variational principle we also have

$$\frac{1}{\mu_1^2} = \sup_{f \in L^2(\Omega), f \neq 0} \frac{\langle \mathscr{L}_\Omega^2 f, f \rangle}{\|f\|^2}. \tag{5.18}$$

This means that the strong inequality (5.17) contradicts the variational principle (5.18) because $\|\widetilde{u}_1\|_{L^2} = \|u_1\|_{L^2} < \infty$.

Since u_1 is nonnegative it follows that μ_1 is simple. Indeed, if there were an eigenfunction v_1 linearly independent of u_1 and corresponding to μ_1, then for all real c the linear combination $u_1 + cv_1$ also would be an eigenfunction corresponding to μ_1 and therefore, by what has been proved, it could not become negative in Ω. As c is arbitrary, this is impossible. Thus, we have proved the following fact.

Lemma 5.4 *The characteristic number μ_1 of the logarithmic potential $\mathscr{L}_\Omega$ with the smallest modulus is simple, and the corresponding eigenfunction u_1 can be chosen nonnegative.*

Let us now prove Theorem 5.3. As discussed above, $\mu_1^2(\Omega)$ is the smallest characteristic number of $\mathscr{L}_\Omega^2$ and u_1 is the eigenfunction corresponding to μ_1^2, i.e.

$$u_1 = \mu_1^2(\Omega)\mathscr{L}_\Omega^2 u_1.$$

By Lemma 5.4 the first characteristic number μ_1 of the operator $\mathscr{L}_\Omega$ is simple; the corresponding eigenfunction u_1 can be chosen positive in Ω, and in view of Lemma 5.4 we can apply the above construction to the first eigenfunction u_1. The rearrangement inequality for the logarithmic kernel (cf. [29, Lemma 2]) gives

$$\int_\Omega \int_\Omega \int_\Omega u_1(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1(x) dz dy dx \leq \int_D \int_D \int_D u_1^*(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1^*(x) dz dy dx. \quad (5.19)$$

where u_1^* is the symmetric decreasing rearrangement of u_1.

For the proof of the rearrangement inequality (5.19) for the logarithmic kernel, see Lemma 5.8. The proof is almost the same with the slight difference that in this case the symmetric decreasing rearrangement is used instead of the Steiner symmetrization.

In addition, for each nonnegative function $u \in L^2(\Omega)$ we have

$$\|u\|_{L^2(\Omega)} = \|u^*\|_{L^2(D)}. \quad (5.20)$$

Therefore, from (5.64), (5.20) and the variational principle for the positive operator $\mathscr{L}_D^2$, we get

$$\mu_1^2(\Omega) = \frac{\int_\Omega |u_1(x)|^2 dx}{\int_\Omega \int_\Omega \int_\Omega u_1(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1(x) dz dy dx} \geq$$
$$\frac{\int_D |u_1^*(x)|^2 dx}{\int_D \int_D \int_D u_1^*(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1^*(x) dz dy dx} \geq$$
$$\inf_{v \in L^2(D), v \neq 0} \frac{\int_D |v(x)|^2 dx}{\int_D \int_D \int_D v(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} v(x) dz dy dx} = \mu_1^2(D).$$

Finally, note that 0 is not a characteristic number of $\mathscr{L}_D$ (see [128, Corollary 1]), that is, $0 < |\mu_1(D)|$. □

Throughout this chapter, the Brascamp-Lieb-Luttinger inequality [22] will be used often. Therefore, let us state its suitable version explicitly. For the proof we refer to [116, Theorem 14.8]:

Theorem 5.5 (Brascamp-Lieb-Luttinger inequality) *Let $f_1, \ldots, f_n$ be nonnegative functions in $\mathbb{R}^d$ and let $f_1^*, \ldots, f_n^*$ be their symmetric decreasing rearrangements. Fix an integer m. Let $\{a_{jk}\}_{1 \leq j \leq n; 1 \leq k \leq m}$ be an $n \times m$ real matrix. Then*

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^n f_j \left(\sum_{k=1}^m a_{jk} y_k \right) dy_1 \ldots dy_m \leq \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^n f_j^* \left(\sum_{k=1}^m a_{jk} y_k \right) dy_1 \ldots dy_m. \quad (5.21)$$

Note that taking $m = n$, $a_{jk} = 1$ if $j = k$ and $a_{jk} = -1$ if $k = j+1$ with $a_{nn+1} = a_{n1}$, otherwise $a_{jk} = 0$, Theorem (5.5) becomes

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f_1(y_1 - y_2) f_2(y_2 - y_3) \ldots f_n(y_n - y_1)\, dy_1 \ldots dy_n \leq \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f_1^*(y_1 - y_2) f_2^*(y_2 - y_3) \ldots f_n^*(y_n - y_1)\, dy_1 \ldots dy_n. \quad (5.22)$$

Let us now prove Theorem 5.2. We first prove the Brascamp-Lieb-Luttinger type rearrangement inequality for the logarithmic kernel (which is not nonnegative function).

Let D be a disc centred at the origin. Then

$$\int_{\Omega} \cdots \int_{\Omega} \frac{1}{2\pi} \ln \frac{1}{|y_1 - y_2|} \cdots \frac{1}{2\pi} \ln \frac{1}{|y_p - y_1|} dy_1 \ldots dy_p \leq \int_{D} \cdots \int_{D} \frac{1}{2\pi} \ln \frac{1}{|y_1 - y_2|} \cdots \frac{1}{2\pi} \ln \frac{1}{|y_p - y_1|} dy_1 \ldots dy_p, \quad (5.23)$$

for any $p = 2, 3, \ldots$, and for any bounded open set Ω with $|\Omega| = |D|$. Here we prove it for $p = 2$ and the proof is based on the proof of [29, Lemma 2]. The proof for arbitrary p is essentially the same as the case $p = 2$. Let us fix $r_0 > 0$ and consider the function

$$f(r) := \begin{cases} \frac{1}{2\pi} \ln \frac{1}{r}, \ r \leq r_0, \\ \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi} \int_{r_0}^{r} s^{-1} \frac{1+r_0^2}{1+s^2} ds, \ r > r_0. \end{cases} \quad (5.24)$$

Let us show that the function $f(r)$ is strictly decreasing and has a limit of $r \to \infty$. If $r \leq r_0$ then

$$f(r_1) = \frac{1}{2\pi} \ln \frac{1}{r_1} > \frac{1}{2\pi} \ln \frac{1}{r_2} = f(r_2)$$

for $r_1 < r_2$. If $r > r_0$ then

$$f(r) = \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi} \int_{r_0}^{r} s^{-1} \frac{1+r_0^2}{1+s^2} ds = \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi}(1+r_0^2)[\ln r - \frac{1}{2}\ln(1+r^2) - \ln r_0 + \frac{1}{2}\ln(1+r_0^2)]. \quad (5.25)$$

Thus $f(r_1) > f(r_2)$ for $r_1 < r_2$, that is, $f(r)$ is strictly decreasing. From (5.25) it is easy to see that

$$\lim_{r\to\infty} f(r) = \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi}(1+r_0^2)[-\ln r_0 + \frac{1}{2}\ln(1+r_0^2)]. \quad (5.26)$$

We use the notation

$$f_\infty := \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi}(1+r_0^2)[-\ln r_0 + \frac{1}{2}\ln(1+r_0^2)].$$

By construction $\frac{1}{2\pi}\ln\frac{1}{r}-f(r)$ is decreasing. Thus, if we define

$$h_1(r)=f(r)-f_\infty$$

we have the decomposition

$$\frac{1}{2\pi}\ln\frac{1}{r}=h_1(r)+h_2(r)$$

where h_1 is positive strictly decreasing function and h_2 is decreasing. Hence by the Brascamp-Lieb-Luttinger rearrangement inequality we have

$$\int_\Omega\int_\Omega h_1(|y_1-y_2|)h_1(|y_2-y_1|)dy_1dy_2\leq \int_D\int_D h_1(|y_1-y_2|)h_1(|y_2-y_1|)dy_1dy_2 \quad (5.27)$$

and

$$\int_\Omega\int_\Omega h_2(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2\leq \int_D\int_D h_2(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2. \quad (5.28)$$

Thus it remains to show that

$$\int_\Omega\int_\Omega h_1(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2\leq \int_D\int_D h_1(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2, \quad (5.29)$$

which does not follow directly from the Brascamp-Lieb-Luttinger rearrangement inequality since h_2 is not positive. Define for $R>0$ the function

$$q_R(r):=\begin{cases} h_2(r)-h_2(R),\ r\leq R,\\ 0,\ r>R,\end{cases} \quad (5.30)$$

and note that by the monotone convergence we have

$$I_\Omega(h_1,h_2)=\lim_{R\to\infty}[I_\Omega(h_1,q_R)+h_2(R)\int_\Omega\int_\Omega h_1(|y_1-y_2|)dy_1dy_2], \quad (5.31)$$

with the notation

$$I_\Omega(f,g)=\int_\Omega\int_\Omega f(|y_1-y_2|)g(|y_2-y_1|)dy_1dy_2. \quad (5.32)$$

Since h_1 and q_R are positive and nonincreasing, we have

$$I_\Omega(h_1,q_R)\leq I_D(h_1,q_R)$$

by the Brascamp-Lieb-Luttinger rearrangement inequality. Noting that

$$\int_\Omega\int_\Omega h_1(|y_1-y_2|)dy_1dy_2 \leq \int_D\int_D h_1(|y_1-y_2|)dy_1dy_2,$$

we obtain

$$I_\Omega(h_1,h_2) = \lim_{R\to\infty}[I_\Omega(h_1,q_R)+h_2(R)\int_\Omega\int_\Omega h_1(|y_1-y_2|)dy_1dy_2] \leq$$
$$\lim_{R\to\infty}[I_D(h_1,q_R)+h_2(R)\int_D\int_D h_1(|y_1-y_2|)dy_1dy_2] = I_D(h_1,h_2), \quad (5.33)$$

completing the proof of (5.23). Since the logarithmic potential operator is a Hilbert-Schmidt operator, by using bilinear expansion of its iterated kernels (see, for example, [130]) we obtain for $p\geq 2$, $p\in\mathbb{N}$,

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j^p(\Omega)} = \int_\Omega\cdots\int_\Omega\frac{1}{2\pi}\ln\frac{1}{|y_1-y_2|}\cdots\frac{1}{2\pi}\ln\frac{1}{|y_p-y_1|}dy_1...dy_p. \quad (5.34)$$

Recalling the inequality (5.23) stating that

$$\int_\Omega\cdots\int_\Omega\frac{1}{2\pi}\ln\frac{1}{|y_1-y_2|}\cdots\frac{1}{2\pi}\ln\frac{1}{|y_p-y_1|}dy_1...dy_p \leq$$

$$\int_D\cdots\int_D\frac{1}{2\pi}\ln\frac{1}{|y_1-y_2|}\cdots\frac{1}{2\pi}\ln\frac{1}{|y_p-y_1|}dy_1...dy_p, \quad (5.35)$$

we obtain

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j^p(\Omega)} \leq \sum_{j=1}^{\infty}\frac{1}{\mu_j^p(D)}, \quad p\geq 2,\ p\in\mathbb{N}, \quad (5.36)$$

for any bounded open domain $\Omega\subset\mathbb{R}^2$ with $|\Omega|=|D|$. Taking even p in (5.36) we complete the proof of Theorem 5.2. □

Finally, we note that the inequality (5.36) also proves Theorem 5.1 when the logarithmic potential operator is positive. □

5.2.2 Isoperimetric inequalities over polygons

The techniques of the previous section do not allow us to prove Theorem 5.2 for all $p>1$. In view of the Dirichlet Laplacian case, it seems reasonable to conjecture that the Schatten p-norm is still maximised on the disc also for all $p>1$.

On the other hand, we can ask the same question of maximising the Schatten p-norms in the class of polygons with a given number n of sides. We denote by $\mathscr{P}_n$ the class of plane polygons with n edges. We now aim at describing the maximiser for Schatten p-norms of the logarithmic potential $\mathscr{L}_\Omega$ in $\mathscr{P}_n$. According to the previous section, it is natural to conjecture that it is the n-regular polygon.

Let us prove this for $n=3$:

Theorem 5.6 *The equilateral triangle is a maximiser of Schatten p-norms of $\mathscr{L}_\Omega$ for any even integer $2 \le p < \infty$ among all triangles of a given area, that is, if Δ is the equilateral triangle, then we have*

$$\|\mathscr{L}_\Omega\|_p \le \|\mathscr{L}_\Delta\|_p \tag{5.37}$$

for any even integer $2 \le p \le \infty$ and any bounded open triangle Ω with $|\Omega| = |\Delta|$.

Similarly, we have the following $\mathscr{P}_3$ analogue of Theorem 5.1:

Theorem 5.7 *Let Δ be an equilateral triangle and let Ω be a bounded open triangle with $|\Omega| = |\Delta|$. Assume that the logarithmic potential operator is positive for Ω and Δ. Then*

$$\|\mathscr{L}_\Omega\|_p \le \|\mathscr{L}_\Delta\|_p \tag{5.38}$$

for any integer $2 \le p < \infty$.

Let u be a nonnegative, measurable function on $\mathbb{R}^2$, and let x^2 be a line through the origin of $\mathbb{R}^2$. Choose an orthogonal coordinate system in $\mathbb{R}^2$ such that the x^1-axis is perpendicular to x^2. A nonnegative, measurable function $u^\star(x|x^2)$ on $\mathbb{R}^2$ is called the ***Steiner symmetrization*** with respect to x^2 of the function $u(x)$, if $u^\star(x^1, x^2)$ is a symmetric decreasing rearrangement with respect to x^1 of $u(x^1, x^2)$ for each fixed x^2. The Steiner symmetrization (with respect to the x^1-axis) $\Omega^\star$ of a measurable set Ω is defined in the following way: if we write (x^1, z) with $z \in \mathbb{R}$, and let

$$\Omega_z = \{x^1 : (x^1, z) \in \Omega\},$$

then

$$\Omega^\star = \{(x^1, z) \in \mathbb{R} \times \mathbb{R} : x^1 \in \Omega_z^*\},$$

where Ω^* is a symmetric rearrangement of Ω (see the proof of Theorem 5.3). We obtain:

Lemma 5.8 *For a positive function u and a measurable $\Omega \subset \mathbb{R}^2$ we have*

$$\int_\Omega \int_\Omega \int_\Omega u(y) \ln \frac{1}{|y-z|} \ln \frac{1}{|z-x|} u(x) dz dy dx \le \int_{\Omega^\star} \int_{\Omega^\star} \int_{\Omega^\star} u^\star(y) \ln \frac{1}{|y-z|} \ln \frac{1}{|z-x|} u^\star(x) dz dy dx, \tag{5.39}$$

where $\Omega^\star$ and $u^\star$ are Steiner symmetrizations of Ω and u, respectively.

Let us prove Lemma 5.8. The proof is based on the proof of [29, Lemma 2]. Let us fix $r_0 > 0$ and consider the function

$$f(r) := \begin{cases} \ln \frac{1}{r}, \ r \le r_0, \\ \ln \frac{1}{r_0} - \int_{r_0}^r s^{-1} \frac{1+r_0^2}{1+s^2} ds, \ r > r_0. \end{cases} \tag{5.40}$$

The function $f(r)$ is strictly decreasing and has a limit of $r \to \infty$,

$$\lim_{r\to\infty} f(r) =: f_\infty.$$

Since $f(r)$ is strictly decreasing, $\ln \frac{1}{r} - f(r)$ is decreasing. Thus if we define

$$h_1(r) = f(r) - f_\infty$$

we have the decomposition

$$\ln \frac{1}{r} = h_1(r) + h_2(r)$$

where h_1 is a positive strictly decreasing function and h_2 is decreasing. Hence by the Brascamp-Lieb-Luttinger rearrangement inequality for the Steiner symmetrization (see [22, Lemma 3.2]) we have

$$\int_\Omega \int_\Omega \int_\Omega u(y)h_1(|y-z|)h_1(|z-x|)u(x)dzdydx \leq \int_{\Omega^\star} \int_{\Omega^\star} \int_{\Omega^\star} u^\star(y)h_1(|y-z|)h_1(|z-x|)u^\star(x)dzdydx. \quad (5.41)$$

Thus, it remains to show that

$$\int_\Omega \int_\Omega \int_\Omega u(y)h_2(|y-z|)h_2(|z-x|)u(x)dzdydx \leq \int_{\Omega^\star} \int_{\Omega^\star} \int_{\Omega^\star} u^\star(y)h_2(|y-z|)h_2(|z-x|)u^\star(x)dzdydx \quad (5.42)$$

and

$$\int_\Omega \int_\Omega \int_\Omega u(y)h_1(|y-z|)h_2(|z-x|)u(x)dzdydx \leq \int_{\Omega^\star} \int_{\Omega^\star} \int_{\Omega^\star} u^\star(y)h_1(|y-z|)h_2(|z-x|)u^\star(x)dzdydx, \quad (5.43)$$

which does not follow directly from the Brascamp-Lieb-Luttinger rearrangement inequality since h_2 is not positive. Define for $R > 0$ the function

$$q_R(r) := \begin{cases} h_2(r) - h_2(R), \ r \leq R, \\ 0, \ r > R, \end{cases} \quad (5.44)$$

and note that by the monotone convergence we have

$$I_\Omega(u,h_2) = \lim_{R\to\infty} \left[I_\Omega(u,q_R) + 2h_2(R)J_\Omega(u,q_R) + h_2^2(R)\left(\int_\Omega u(x)dx\right)^2 \right], \quad (5.45)$$

with the notations

$$I_\Omega(u,g) = \int_\Omega \int_\Omega \int_\Omega u(y)g(|y-z|)g(|z-x|)u(x)dzdydx \quad (5.46)$$

and

$$J_\Omega(u,g)=\int_\Omega\int_\Omega\int_\Omega u(y)g(|z-x|)u(x)dzdydx. \tag{5.47}$$

Since q_R is positive and nonincreasing, and noting that

$$\int_\Omega u(x)dx=\int_{\Omega^\star}u^\star(x)dx,$$

we obtain

$$I_\Omega(u,q_R)\le I_{\Omega^\star}(u^\star,q_R),$$

and

$$J_\Omega(u,q_R)\le J_{\Omega^\star}(u^\star,q_R),$$

by the Brascamp-Lieb-Luttinger rearrangement inequality. Therefore,

$$\begin{aligned}I_\Omega(u,h_2)&\\ &=\lim_{R\to\infty}\left[I_\Omega(u,q_R)+2h_2(R)J_\Omega(u,q_R)+h_2^2(R)\left(\int_\Omega u(x)dx\right)^2\right]\le\\ &\lim_{R\to\infty}\left[I_{\Omega^\star}(u^\star,q_R)+2h_2(R)J_{\Omega^\star}(u^\star,q_R)+h_2^2(R)\left(\int_{\Omega^\star}u^\star(x)dx\right)^2\right]\\ &=I_{\Omega^\star}(u^\star,h_2).\end{aligned} \tag{5.48}$$

This proves the inequality (5.42). Similarly, now let us show that the inequality (5.43) is valid. We have

$$\widetilde{I}_\Omega(u,h_2)=\lim_{R\to\infty}[\widetilde{I}_\Omega(u,q_R)+h_2(R)\widetilde{J}_\Omega(u,h_1)], \tag{5.49}$$

with the notations

$$\widetilde{I}_\Omega(u,g)=\int_\Omega\int_\Omega\int_\Omega u(y)h_1(|y-z|)g(|z-x|)u(x)dzdydx \tag{5.50}$$

and

$$\widetilde{J}_\Omega(u,h_1)=\int_\Omega\int_\Omega\int_\Omega u(y)h_1(|y-x|)u(x)dzdydx. \tag{5.51}$$

Since both q_R and h_1 are positive and nonincreasing,

$$\widetilde{I}_\Omega(u,q_R)\le\widetilde{I}_{\Omega^\star}(u^\star,q_R), \tag{5.52}$$

and

$$\widetilde{J}_\Omega(u,h_1)\le\widetilde{J}_{\Omega^\star}(u^\star,h_1), \tag{5.53}$$

by the Brascamp-Lieb-Luttinger rearrangement inequality. Therefore, we obtain

$$\widetilde{I}_\Omega(u,h_2)=\lim_{R\to\infty}[\widetilde{I}_\Omega(u,q_R)+h_2(R)\widetilde{J}_\Omega(u,h_1)]\le$$

$$\lim_{R\to\infty}[\widetilde{I}_{\Omega^\star}(u^\star,q_R)+h_2(R)\widetilde{J}_{\Omega^\star}(u^\star,h_1)]=\widetilde{I}_{\Omega^\star}(u^\star,h_2).$$

This proves the inequality (5.43). $\square$

Lemma 5.8 implies the following analogy of the Pólya theorem [89] for the operator $\mathscr{L}_\Omega$.

Theorem 5.9 *The equilateral triangle Δ centred at the origin is a minimiser of the first characteristic number of the logarithmic potential $\mathscr{L}_\Omega$ among all triangles of a given area, i.e.*

$$\frac{1}{|\mu_1(\Omega)|} \leq \frac{1}{|\mu_1(\Delta)|}$$

for any triangle $\Omega \subset \mathbb{R}^2$ with $|\Omega| = |\Delta|$.

In other words, Theorem 5.9 says that the operator norm of $\mathscr{L}_\Omega$ is maximised in an equilateral triangle among all triangles of a given area.

Let us prove Theorem 5.9. According to Lemma 5.4, the first characteristic number μ_1 of the operator $\mathscr{L}_\Omega$ is simple; the corresponding eigenfunction u_1 can be chosen positive in Ω. Using the fact that by applying a sequence of the Steiner symmetrizations with respect to the mediator of each side, a given triangle converges to an equilateral one (see e.g. [48, Figure 3.2]), from (5.39) we have

$$\begin{aligned}\int_\Omega \int_\Omega \int_\Omega u_1(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1(x) dz dy dx \leq \\ \int_\Delta \int_\Delta \int_\Delta u_1^\star(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1^\star(x) dz dy dx. \quad (5.54)\end{aligned}$$

Thus, from (5.54) and the variational principle for the positive operator $\mathscr{L}_\Delta^2$, we get

$$\begin{aligned}\mu_1^2(\Omega) = \frac{\int_\Omega |u_1(x)|^2 dx}{\int_\Omega \int_\Omega \int_\Omega u_1(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1(x) dz dy dx} \geq \\ \frac{\int_\Delta |u_1^\star(x)|^2 dx}{\int_\Delta \int_\Delta \int_\Delta u_1^\star(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1^\star(x) dz dy dx} \geq \\ \inf_{v \in L^2(\Delta)} \frac{\int_\Delta |v(x)|^2 dx}{\int_\Delta \int_\Delta \int_\Delta v(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} v(x) dz dy dx} = \mu_1^2(\Delta).\end{aligned}$$

Note that here we have used the fact that the Steiner symmetrization preserves the L^2-norm. □

Finally, let us prove Theorem 5.6 and Theorem 5.7. Actually, the proofs of Theorem 5.6 and Theorem 5.7 follow by the same steps as the proofs of Theorem 5.2 and Theorem 5.1, with the difference that now the Steiner symmetrization is used. According to the property of the Steiner symmetrization (cf. [22, Lemma 3.2]), in the same way as for the symmetric-decreasing rearrangement (5.35), it is clear that any Steiner symmetrization increases (or at least does not decrease) the Schatten p-norms for even integers $p \geq 2$. Thus, for the proofs we only need to recall the fact that a sequence of Steiner symmetrizations with respect to the mediator of each side, a given triangle converges to an equilateral one. □

Any quadrilateral can be transformed into a rectangle by using a sequence of three Steiner symmetrizations (see [48, Figure 3.3]). That is, it is sufficient to seek the maximisation problem among rectangles for $\mathscr{P}_4$. However, for $\mathscr{P}_5$ (pentagons and others), the Steiner symmetrization increases, in general, the number of sides. This prevents us from using the same technique for general polygons $\mathscr{P}_n$ with $n \geq 5$.

5.3 Riesz potential operators

In this section we discuss the spectral properties and several spectral geometric inequalities for the Riesz potential operators. Our exposition here of the Riesz and Bessel operators follows our open access paper [97] with G. Rozenblum.

Thus, let us consider the ***Riesz potential operators***

$$(\mathscr{R}_{\alpha,\Omega} f)(x) := \int_\Omega \varepsilon_{\alpha,d}(|x-y|) f(y) dy, \quad f \in L^2(\Omega), \quad 0 < \alpha < d, \tag{5.55}$$

where $\Omega \subset \mathbb{R}^d$ is a set with finite Lebesgue measure,

$$\varepsilon_{\alpha,d}(|x-y|) = c_{\alpha,d} |x-y|^{\alpha-d} \tag{5.56}$$

and $c_{\alpha,d}$ is a positive constant,

$$c_{\alpha,d} = 2^{\alpha-d} \pi^{-d/2} \frac{\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)}.$$

The Riesz potential operators generalise the Riemann-Liouville operators to the settings of several variables, and the Newton potential operators to fractional orders. The fact that $c_{\alpha,d}|x|^{\alpha-d}$ is the Fourier transform of the function $|\xi|^{-\alpha}$ in $\mathbb{R}^d$ says that the kernel is the fundamental solution of $(-\Delta)^{\alpha/2}$, i.e.

$$(-\Delta_y)^{\alpha/2} \varepsilon_{\alpha,d}(x-y) = \delta_x,$$

where δ_x is the Dirac distribution. In particular, for an even integer $\alpha = 2m$ with $0 < m < d/2$, the function

$$\varepsilon_{2m,d}(|x|) = c_{2m,d} |x|^{2m-d}, \tag{5.57}$$

is the fundamental solution to the polyharmonic equation of order $2m$ in $\mathbb{R}^d$.

Thus, the polyharmonic (Newton) potential

$$(\mathscr{L}^{-1}_{2m,\Omega} f)(x) := \int_\Omega \varepsilon_{2m,d}(|x-y|) f(y) dy, \, f \in L^2(\Omega), \tag{5.58}$$

is a particular case of the Riesz potential,

$$\mathscr{L}^{-1}_{2m,\Omega} = \mathscr{R}_{2m,\Omega}. \tag{5.59}$$

In the case $m = 1$, i.e. for the Laplacian, under the assumption of a sufficient regularity of the boundary of Ω (for example, piecewise C^1), it is known, see e.g. [59], that the equation

$$u(x) = (\mathscr{L}_{2,\Omega}^{-1} f)(x) = \int_\Omega \varepsilon_{2,d}(|x-y|) f(y) dy \tag{5.60}$$

is equivalent to the equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{5.61}$$

with the following nonlocal integral boundary condition

$$-\frac{1}{2}u(x) + \int_{\partial\Omega} \frac{\partial \varepsilon_{2,d}(|x-y|)}{\partial n_y} u(y) dS_y - \int_{\partial\Omega} \varepsilon_{2,d}(|x-y|) \frac{\partial u(y)}{\partial n_y} dS_y = 0, \ x \in \partial\Omega, \tag{5.62}$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at the point $y \in \partial\Omega$. This approach was further expanded in [60] to polyharmonic operators.

Discussions in this section, as it concerns integer values $m \geq 1$, take care of a generalisation of the boundary value problem (5.61)-(5.62). Moreover, for non-integer values of m (i.e. $\alpha \notin 2\mathbb{Z}$), the operator (5.55) acts as the interior term in the resolvent for boundary problems for the fractional power of the Laplacian, see, e.g. [114].

In the present section we show that the ball is a maximiser of some Schatten p-norms of the Riesz potential operators among all domains of a given measure in $\mathbb{R}^d$. In particular, the result is valid for the polyharmonic Newton potential operator, which is related to a nonlocal boundary value problem for the poly-Laplacian, so we also show isoperimetric inequalities for its eigenvalues as well, namely, analogues of Rayleigh-Faber-Krahn and Hong-Krahn-Szegö inequalities. Before we present these results we give some preliminaries on basic spectral properties of the Riesz potential operators.

5.3.1 Spectral properties of $\mathscr{R}_{\alpha,\Omega}$

Consider the spectral problem of the Riesz potential operators

$$\mathscr{R}_{\alpha,\Omega} u = \int_\Omega \varepsilon_\alpha(|x-y|) u(y) dy = \lambda u, \ \ u \in L^2(\Omega), \tag{5.63}$$

where the kernel is

$$\varepsilon_{\alpha,d}(|x-y|) := c_{\alpha,d}|x-y|^{\alpha-d},$$

and $0 < \alpha < d$. We may sometimes drop the subscripts α, d and Ω in the notation of the operator and the kernel, provided this does not cause confusion. Recall that in the case of the Newton potential operator it is the same as considering the spectrum of the operator corresponding to the boundary value problem (5.3)-(5.5), which we call $\mathscr{L} = \mathscr{L}_{2m,\Omega}$, in a bounded connected domain $\Omega \subset \mathbb{R}^d$ with a piecewise C^1 continuous boundary $\partial\Omega$, that is,

$$(-\Delta_x)^m u(x) = \lambda^{-1} u(x), \ x \in \Omega, \ m \in \mathbb{N}, \tag{5.64}$$

with the nonlocal boundary conditions (5.5).

Let $\Omega \subset \mathbb{R}^d$ be a set with finite Lebesgue measure. Then the well-known Schur test shows immediately that $\mathscr{R}_{\alpha,\Omega}$ is bounded in $L^2(\Omega)$. Also, it will be shown soon that this operator is compact in $L^2(\Omega)$ as well and belonging to certain Schatten-von Neumann classes S_p. Since the Riesz kernel is symmetric, the operator $\mathscr{R}_{\alpha,\Omega}$ is self-adjoint. For compact self-adjoint operators the singular values are equal to the moduli of (nonzero) eigenvalues, and the corresponding eigenfunctions form a complete orthogonal basis on L^2. In addition, if the operator is nonnegative, the words 'moduli of' in the previous sentence can be omitted.

Thus, the eigenvalues of the Riesz potential operator $\mathscr{R}_{\alpha,\Omega}$ can be enumerated in the descending order of their moduli,

$$|\lambda_1| \geq |\lambda_2| \geq ...$$

where λ_j is repeated in this series according to its multiplicity. The corresponding eigenfunctions can be denoted by $u_1, u_2, \ldots$, so that for each eigenvalue λ_j one can fix a unique normalised eigenfunction u_j:

$$\mathscr{R}_{\alpha,\Omega} u_j = \lambda_j u_j, \quad j = 1, 2, \ldots.$$

The following lemma asserts that the operator $\mathscr{R}_{\alpha,\Omega}$ is compact and evaluates the decay rate of its singular numbers.

Lemma 5.10 *Let $\Omega \subset \mathbb{R}^d$ be a measurable set with finite Lebesgue measure, $0 < \alpha < d$. Then*

(a) The Riesz potential operator $\mathscr{R}_{\alpha,\Omega}$ is nonnegative; this means, in particular, that all eigenvalues are nonnegative,

$$\lambda_j \equiv \lambda_j(\mathscr{R}_{\alpha,\Omega}) = |\lambda_j(\mathscr{R}_{\alpha,\Omega})| = s_j.$$

(b) For the eigenvalues λ_j of $\mathscr{R}_{\alpha,\Omega}$ the following estimate holds:

$$\lambda_j \leq C|\Omega|^{\vartheta} j^{-\vartheta},$$

where $\vartheta = \alpha/d$. In particular, this implies the compactness of the operator $\mathscr{R}_{\alpha,\Omega}$.

In fact, for the Riesz operator $\mathscr{R}_{\alpha,\Omega}$, the inequality

$$\lambda_j \leq C|\Omega|^{\vartheta} j^{-\vartheta},$$

is actually accompanied by an asymptotic formula for its eigenvalues. Indeed, as we will see from the proof below, we can extend the operator $\mathscr{R}_{\alpha,\Omega}$ to an operator $\tilde{\mathscr{R}}_{\alpha,\Omega}$ on the whole space $\mathbb{R}^d$ without changing its nonzero singular numbers: in fact they are related by

$$\tilde{\mathscr{R}}_{\alpha,\Omega} = \mathscr{R}_{\alpha,\Omega} \oplus \mathbf{0},$$

in the direct sum decomposition $L^2(\mathbb{R}^d) = L^2(\Omega) \oplus L^2(\mathbb{R}^d \setminus \Omega)$. Consequently, we can write $\tilde{\mathscr{R}}_{\alpha,\Omega} = (2\pi)^d T^*T$ for some explicitly given operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$,

$$Tf(x) = \int_\Omega \varepsilon_{\alpha/2,d}(|x-y|)f(y)dy,$$

for which we have the asymptotics

$$s_j(T) \sim Cj^{-\alpha/(2d)}|\Omega|^{\alpha/(2d)}.$$

In fact, such asymptotics are known for more general convolution type operators, also with an explicitly given constant C. For a bounded set Ω, this asymptotics is a particular case of general results of M. Birman and M. Solomyak [16] concerning integral operators with weak polarity in the kernel. One can easily dispose of this boundedness condition using the estimate and the asymptotic approximation procedure as has been done many times since the early 1970s (see, for instance, in [17] and [94]).

Let us prove Lemma 5.10. Using the formula for the fundamental solutions, it is easy to see that

$$\varepsilon_{\alpha',d} * \varepsilon_{\alpha'',d}(|x-y|) \equiv \int_{\mathbb{R}^d} \varepsilon_{\alpha',d}(|x-z|)\varepsilon_{\alpha'',d}(|z-y|)dz$$
$$= (2\pi)^d \varepsilon_{\alpha'+\alpha'',d}(|x-y|), \quad (5.65)$$

for $0 < \alpha', \alpha'' < \alpha' + \alpha'' < d$. Since $|\xi|^{-(\alpha'+\alpha'')} = |\xi|^{-\alpha'}|\xi|^{-\alpha''}$, this well-known relation follows, for instance, from the fact, already mentioned, that $\varepsilon_{\alpha,d}$ is the Fourier transform of $|\xi|^{-\alpha}$, and from the relation between the Fourier transform of a product and the convolution of the Fourier transforms. We consider the operator

$$\tilde{\mathscr{R}}_{\alpha,\Omega} : \quad L^2(\mathbb{R}^d) \ni f \mapsto \chi_\Omega(x) \int_{\mathbb{R}^d} \varepsilon_{\alpha,d}(|x-y|)\chi_\Omega(x)(y)f(y)dy \in L^2(\mathbb{R}^d), \quad (5.66)$$

where χ_Ω is the characteristic function of Ω. In the direct sum decomposition $L^2(\mathbb{R}^d) = L^2(\Omega) \oplus L^2(\mathbb{R}^d \setminus \Omega)$ the operator $\tilde{\mathscr{R}}_{\alpha,\Omega}$ is represented as $\mathscr{R}_{\alpha,\Omega} \oplus \mathbf{0}$, so the nonzero singular numbers of operators $\tilde{\mathscr{R}}_{\alpha,\Omega}$ and $\mathscr{R}_{\alpha,\Omega}$ coincide. According to the convolution property (5.65), the operator $\tilde{\mathscr{R}}_{\alpha,\Omega}$ can be represented as $(2\pi)^d T^*T$, where $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$,

$$Tf(x) = \int \varepsilon_{\alpha/2,d}(|x-y|)\chi_\Omega(y)f(y)dy. \quad (5.67)$$

In fact, the above relations show that the operator $\tilde{\mathscr{R}}_{\alpha,\Omega} = T^*T$ and, further on, the operator $\mathscr{R}_{\alpha,\Omega}$ are nonnegative. This proves the case (a) in the lemma.

Moreover, the eigenvalues of $\mathscr{R}_{\alpha,\Omega}$ equal the squares of the singular numbers of T. Thus, we can apply the Cwikel estimate, see [30], concerning the singular numbers estimates for integral operators with a kernel of the form $h(x-y)g(y)$. In our case, $h = \varepsilon_{\alpha/2,d}$, $g = \chi_\Omega$, and thus the Cwikel's [30, estimate (1)], with $p = 2d/\alpha$ gives

$$s_j(T) \le Cj^{-1/p}\|\chi_\Omega\|_{L^p} = Cj^{-\alpha/(2d)}|\Omega|^{\alpha/(2d)}, \quad (5.68)$$

with certain constant $C = C(\alpha, d)$ and, finally,

$$s_j(\mathscr{R}_{\alpha,\Omega}) \leq C j^{-\theta} |\Omega|^{\theta}, \quad \theta = \alpha/d. \tag{5.69}$$

This completes the proof of Lemma 5.10. □

It follows from Lemma 5.10 that the Riesz potential operator $\mathscr{R}_{\alpha,\Omega}$ belongs to each Schatten class S_p with $p > p_0 = \alpha/d$ and

$$\|\mathscr{R}_{\alpha,\Omega}\|_p = \left(\sum_{j=1}^{\infty} \lambda_j(\Omega)^p \right)^{\frac{1}{p}}, \ 1 \leq p < \infty. \tag{5.70}$$

In some further discussions, we need to calculate the trace of certain trace class integral operators. For a positive trace class integral operator $\mathbf{K}$ with continuous kernel $K(x,y)$ on a (nice) set Ω, it is well known that

$$\operatorname{Tr}(\mathbf{K}) = \int_{\Omega} K(x,x) dx.$$

This fact cannot be used directly in our more general setting, since our set Ω is not supposed to be nice and we cannot grant the continuity of the kernels in question. So we need to do some additional work.

In general, for an integral operator $\mathbf{K}$ with kernel $K(x,y)$ an exact criterion for membership in the Schatten-von Neumann classes S_p in terms of the kernel exists only for $p = 2$, that is, for Hilbert-Schmidt operators (but see also conditions for Schatten classes in terms of the regularity of the kernel in Section 3.10). Namely for $\mathbf{K}$ to belong to S_2, it is necessary and sufficient that $\iint_{\Omega\times\Omega} |K(x,y)|^2 dxdy < \infty$ (see Theorem 3.84), moreover, $\|\mathbf{K}\|_2^2$ equals exactly the above integral and for the trace class operator $\mathbf{K}^*\mathbf{K}$ the same integral equals its trace.

Now let us apply the trace formula in Theorem 3.95 to our kernel

$$K(x,y) = \varepsilon_{\alpha,d}(|x-y|), \ \ x,y \in \Omega.$$

For its definition it follows that the kernel $K(x,y)$ belongs to $L^{p',p}(\Omega \times \Omega)$ for any $p > \frac{d}{\alpha}$. That is, for the trace of $\mathbf{K}^s$ formula (3.120) is valid, and thus, for $s > p_0 = \frac{d}{\alpha}$ we have

$$\begin{aligned} \sum \lambda_j(\mathscr{R}_{\alpha,\Omega})^s &= \operatorname{Tr}(\mathscr{R}^s_{\alpha,\Omega}) \\ &= \int\limits_{\Omega^s} \left(\prod_{k=1}^{s} K(x_k, x_{k+1}) \right) dx_1 dx_2 \ldots dx_s, \ \ x_{s+1} \equiv x_1. \end{aligned} \tag{5.71}$$

Note that for the membership in the Schatten classes S_p with $p < 2$ usually a certain regularity of the kernel is required. We can recall from Theorem 3.87 (or from Theorem 3.91) that if the integral kernel K of an operator $\mathbf{K}f(x) = \int_{\Omega} K(x,y) f(y) dy$ satisfies

$$K \in H^{\mu}(\Omega \times \Omega)$$

for a space Ω of dimension d then

$$\mathbf{K} \in S_p(L^2(\Omega)) \text{ for } p > \frac{2d}{d+2\mu}.$$

In the case of the Riesz potential with $K(x,y) = \varepsilon_{\alpha,d}(|x-y|)$ it implies that

$$\mathscr{R}_{\alpha,\Omega} \in S_p(L^2(\Omega)) \text{ for } p > \frac{d}{\alpha}.$$

As was already mentioned, if the integral kernel $K^{(s)}$ of the operator $\mathbf{K}^s$ is not continuous, the formula $\mathrm{Tr}\,(\mathbf{K}^s) = \int_\Omega K^{(s)}(x,x)dx$ may not hold but it can be replaced by the formula (3.120) (and hence also (5.71)). However, there is also another expression for the trace: if $\widetilde{K^{(s)}}$ denotes the averaging of $K^{(s)}$ with respect to the martingale maximal function, we have

$$\mathrm{Tr}\,(\mathbf{K}^s) = \int_\Omega \widetilde{K^{(s)}}(x,x)dx,$$

as in (3.124). For the description of $\widetilde{K^{(s)}}$, its properties and further references we refer to [34, Section 4].

5.3.2 Spectral geometric inequalities for $\mathscr{R}_{\alpha,\Omega}$

In this section we present some spectral geometric inequalities for the Riesz potential operator $\mathscr{R}_{\alpha,\Omega}$. As usual, here $|\Omega|$ denotes the Lebesgue measure of Ω.

Theorem 5.11 *Let Ω^* be a ball in $\mathbb{R}^d$. For any integer p with $p_0 := \frac{d}{\alpha} < p \leq \infty$, we have*

$$\|\mathscr{R}_{\alpha,\Omega}\|_p \leq \|\mathscr{R}_{\alpha,\Omega^*}\|_p, \tag{5.72}$$

for all Ω with $|\Omega| = |\Omega^*|$.

Since the integral kernel of $\mathscr{R}_{\alpha,\Omega}$ is positive, the statement, sometimes called Jentsch's theorem, applies, see, e.g. [94]. We give it here without proof since it is very similar to the proofs of Lemma 5.4 or Lemma 5.18.

Lemma 5.12 *The eigenvalue λ_1 of $\mathscr{R}_{\alpha,\Omega}$ with the largest modulus is positive and simple; the corresponding eigenfunction u_1 is positive, and any other eigenfunction u_j, $j > 1$, is sign changing in Ω.*

Note that we have already established in Lemma 5.10 that all $\lambda_j(\Omega)$, $i = 1,2,...$, are positive for any domain Ω, so the positivity of λ_1 is already known, since the operator $\mathscr{R}_{\alpha,\Omega}$ is nonnegative; what is important is the positivity of u_1.

First we prove the following analogue of ***Rayleigh-Faber-Krahn theorem*** for the operator $\mathscr{R}_{\alpha,\Omega}$, that is, $p = \infty$ case in Theorem 5.11. We will use this fact further on.

Lemma 5.13 (Rayleigh-Faber-Krahn inequality) *The ball Ω^* is a maximiser of the first eigenvalue of the operator $\mathscr{R}_{\alpha,\Omega}$ among all domains of a given volume, i.e.*

$$0 < \lambda_1(\Omega) \leq \lambda_1(\Omega^*)$$

for an arbitrary domain $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$.

In other words Lemma 5.13 says that the operator norm of $\mathscr{R}_{\alpha,\Omega}$ is maximised in the ball among all Euclidean domains of a given volume.

Let us prove Lemma 5.13. According to the Riesz inequality in Theorem 4.10 and the fact that $\varepsilon_\alpha(|x-y|)$ is a symmetric-decreasing function, we obtain

$$\int_\Omega \int_\Omega u_1(y)\varepsilon_\alpha(|y-x|)u_1(x)dydx \leq \int_{\Omega^*}\int_{\Omega^*} u_1^*(y)\varepsilon_\alpha(|y-x|)u_1^*(x)dydx, \tag{5.73}$$

where u_1^* is symmetric decreasing rearrangement of the function u_1. In addition, for each nonnegative function $u \in L^2(\Omega)$ we have

$$\|u\|_{L^2(\Omega)} = \|u^*\|_{L^2(\Omega^*)}. \tag{5.74}$$

Therefore, from (5.73), (5.74) and the variational principle for $\lambda_1(\Omega^*)$, we get

$$\lambda_1(\Omega) = \frac{\int_\Omega \int_\Omega u_1(y)\varepsilon_\alpha(|y-x|)u_1(x)dydx}{\int_\Omega |u_1(x)|^2 dx} \leq$$

$$\frac{\int_{\Omega^*}\int_{\Omega^*} u_1^*(y)\varepsilon_\alpha(|y-x|)u_1^*(x)dydx}{\int_{\Omega^*} |u_1^*(x)|^2 dx} \leq$$

$$\sup_{v\in L^2(\Omega^*), v\neq 0} \frac{\int_{\Omega^*}\int_{\Omega^*} v(y)\varepsilon_\alpha(|y-x|)v(x)dydx}{\int_{\Omega^*} |v(x)|^2 dx} = \lambda_1(\Omega^*),$$

completing the proof. □

Let us now prove Theorem 5.11. According to Theorem 3.95, we have

$$\sum_{j=1}^{\infty} \lambda_j^p(\Omega) = \int_\Omega \cdots \int_\Omega \varepsilon_\alpha(|y_1-y_2|)\ldots\varepsilon_\alpha(|y_p-y_1|)dy_1\ldots dy_p, \ p > p_0, \ p \in \mathbb{N}. \tag{5.75}$$

It follows from the Brascamp-Lieb-Luttinger inequality ([22]) that

$$\int_{\Omega^p} \varepsilon_\alpha(|y_1-y_2|)\ldots\varepsilon_\alpha(|y_p-y_1|)dy_1\ldots dy_p$$
$$\leq \int_{\Omega^*} \cdots \int_{\Omega^*} \varepsilon_\alpha(|y_1-y_2|)\ldots\varepsilon_\alpha(|y_p-y_1|)dy_1\ldots dy_p. \tag{5.76}$$

This implies

$$\sum_{j=1}^{\infty} \lambda_j^p(\Omega) \leq \sum_{j=1}^{\infty} \lambda_j^p(\Omega^*), \ p \in \mathbb{N}, \ p > p_0, \tag{5.77}$$

for $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$. Here the fact that the kernel ε_α is a symmetric-decreasing function in $\Omega^* \times \Omega^*$ is used, i.e.

$$\varepsilon_\alpha^*(|x-y|) = \varepsilon_\alpha(|x-y|), \;\; x, y \in \Omega^* \times \Omega^*. \;\; \Box$$

We also present the following analogue of the ***Hong-Krahn-Szegő inequality*** for the Riesz potential operators:

Theorem 5.14 (Hong-Krahn-Szegő inequality) *The maximum of the second eigenvalue $\lambda_2(\Omega)$ of $\mathscr{R}_{\alpha,\Omega}$ among all sets $\Omega \subset \mathbb{R}^d$ with a given measure is approached by the union of two identical balls with mutual distance going to infinity.*

In Theorem 5.14 we have $\lambda_2(\Omega) > 0$ since all the eigenvalues of $\mathscr{R}_{\alpha,\Omega}$ are non-negative (see Lemma 5.10). Note that a similar result for the Dirichlet Laplacian is called the Hong-Krahn-Szegő inequality. See, for instance, [23] and [66] for further reference. We also refer to [24] which deals with the second eigenvalue of a nonlocal and nonlinear p-Laplacian operator.

To prove Theorem 5.14, the classical two-ball trick, Lemma 5.12 and Lemma 5.13 can be used. Thus, in the remainder of this subsection we prove Theorem 5.14.

Introducing the following sets:

$$\Omega^+ := \{x : u_2(x) > 0\}, \; \Omega^- := \{x : u_2(x) < 0\},$$

we have,

$$u_2(x) > 0, \; \forall x \in \Omega^+ \subset \Omega, \; \Omega^+ \neq \{\emptyset\},$$

$$u_2(x) < 0, \; \forall x \in \Omega^- \subset \Omega, \; \Omega^- \neq \{\emptyset\},$$

and it follows from Lemma 5.12 that the sets Ω^- and Ω^+ both have positive Lebesgue measure. Denoting

$$u_2^+(x) := \begin{cases} u_2(x), \text{ in } \Omega^+, \\ 0, \text{ otherwise}, \end{cases} \tag{5.78}$$

and

$$u_2^-(x) := \begin{cases} u_2(x), \text{ in } \Omega^-, \\ 0, \text{ otherwise}, \end{cases}$$

we have

$$\lambda_2(\Omega) u_2(x) = \int_{\Omega^+} \varepsilon_\alpha(|x-y|) u_2^+(y) dy + \int_{\Omega^-} \varepsilon_\alpha(|x-y|) u_2^-(y) dy, \; x \in \Omega.$$

Multiplying both sides of this equality by $u_2^+(x)$ and integrating over Ω^+ we obtain

$$\begin{aligned} \lambda_2(\Omega) \int_{\Omega^+} |u_2^+(x)|^2 dx = \int_{\Omega^+} u_2^+(x) \int_{\Omega^+} \varepsilon_\alpha(|x-y|) u_2^+(y) dy dx \\ + \int_{\Omega^+} u_2^+(x) \int_{\Omega^-} \varepsilon_\alpha(|x-y|) u_2^-(y) dy dx, \; x \in \Omega. \end{aligned}$$

The second term on the right-hand side of this inequality is non-positive since the integrand is non-positive. Thus, we get

$$\lambda_2(\Omega)\int_{\Omega^+}|u_2^+(x)|^2dx \le \int_{\Omega^+}u_2^+(x)\int_{\Omega^+}\varepsilon_\alpha(|x-y|)u_2^+(y)dydx,$$

that is,

$$\frac{\int_{\Omega^+}u_2^+(x)\int_{\Omega^+}\varepsilon_\alpha(|x-y|)u_2^+(y)dydx}{\int_{\Omega^+}|u_2^+(x)|^2dx}\ge\lambda_2(\Omega).$$

By the variational principle, we obtain

$$\lambda_1(\Omega^+)=\sup_{v\in L^2(\Omega^+),v\not\equiv0}\frac{\int_{\Omega^+}v(x)\int_{\Omega^+}\varepsilon_\alpha(|x-y|)v(y)dydx}{\int_{\Omega^+}|v(x)|^2dx}$$

$$\ge\frac{\int_{\Omega^+}u_2^+(x)\int_{\Omega^+}\varepsilon_\alpha(|x-y|)u_2^+(y)dydx}{\int_{\Omega^+}|u_2^+(x)|^2dx}\ge\lambda_2(\Omega).$$

Similarly, we establish

$$\lambda_1(\Omega^-)\ge\lambda_2(\Omega).$$

Therefore, we have

$$\lambda_1(\Omega^+)\ge\lambda_2(\Omega),\ \lambda_1(\Omega^-)\ge\lambda_2(\Omega). \tag{5.79}$$

Now we introduce B^+ and B^-, the balls of the same volume as Ω^+ and Ω^-, respectively. According to Lemma 5.13, we obtain

$$\lambda_1(B^+)\ge\lambda_1(\Omega^+),\ \lambda_1(B^-)\ge\lambda_1(\Omega^-). \tag{5.80}$$

Comparing (5.79) and (5.80), we get

$$\min\{\lambda_1(B^+),\lambda_1(B^-)\}\ge\lambda_2(\Omega). \tag{5.81}$$

Now let us consider the set $B^+\cup B^-$, with the balls $B^\pm$ placed at distance l, that is,

$$l=\mathrm{dist}(B^+,B^-),$$

Let us denote by $u_1^\circledast$ the first normalised eigenfunction of $\mathscr{R}_{\alpha,B^+\cup B^-}$ and take u_+ and u_- being the first normalised eigenfunctions of each single ball, i.e. of operators $\mathscr{R}_{\alpha,B^\pm}$. Moreover, we introduce the function $v^\circledast\in L^2(B^+\cup B^-)$, which equals u_+ in B^+ and γu_- in B^-. Because the functions $u_+,u_-,u^\circledast$ are positive, it is possible to find a real number γ so that $v^\circledast$ is orthogonal to $u_1^\circledast$. We can write

$$\int_{B^+\cup B^-}\int_{B^+\cup B^-}v^\circledast(x)v^\circledast(y)\varepsilon_\alpha(|x-y|)dxdy=\sum_{i=1}^4\mathscr{I}_i, \tag{5.82}$$

where

$$\mathscr{I}_1:=\int_{B^+}\int_{B^+}u_+(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy,$$

$$\mathscr{I}_2 := \int_{B^+}\int_{B^-} u_+(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy,$$

$$\mathscr{I}_3 := \gamma\int_{B^-}\int_{B^+} u_-(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy,$$

$$\mathscr{I}_4 := \gamma^2\int_{B^-}\int_{B^-} u_-(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy.$$

By using the variational principle, we obtain

$$\lambda_2(B^+\cup B^-) = \sup_{v\in L^2(B^+\bigcup B^-),\, v\perp u_1,\, \|v\|=1}\int_{B^+\cup B^-}\int_{B^+\cup B^-} v(x)v(y)\varepsilon_\alpha(|x-y|)dxdy.$$

By construction $v^\circledast$ is orthogonal to u_1, so we get

$$\lambda_2(B^+\cup B^-) \geq \int_{B^+\cup B^-}\int_{B^+\cup B^-} v^\circledast(x)v^\circledast(y)\varepsilon_\alpha(|x-y|)dxdy = \sum_{i=1}^{4}\mathscr{I}_i.$$

Moreover, since u_+ and u_- are the first normalised eigenfunctions (by Lemma 5.4 both are positive everywhere) of each single ball B^+ and B^-, we have

$$\lambda_1(B^\pm) = \int_{B^\pm}\int_{B^\pm} u_\pm(x)u_\pm(y)\varepsilon_\alpha(|x-y|)dxdy$$

Summarising the above facts, we obtain

$$\lambda_2(B^+\cup B^-) \geq \frac{\begin{array}{c}\int_{B^+}\int_{B^+} u_+(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy\\ +\gamma^2\int_{B^-}\int_{B^-} u_-(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy + \mathscr{I}_2+\mathscr{I}_3\end{array}}{\begin{array}{c}\lambda_1(B^+)^{-1}\int_{B^+}\int_{B^+} u_+(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy\\ +\gamma^2\lambda_1(B^-)^{-1}\int_{B^-}\int_{B^-} u_-(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy\end{array}}. \quad (5.83)$$

Since the kernel $\varepsilon_\alpha(|x-y|)$ tends to zero as $x\in B^\pm,\ y\in B^\mp$ and $l\to\infty$, we observe that

$$\lim_{l\to\infty}\mathscr{I}_2 = \lim_{l\to\infty}\mathscr{I}_3 = 0,$$

thus

$$\lim_{l\to\infty}\lambda_2(B^+\bigcup B^-) \geq \max\{\lambda_1(B^+),\, \lambda_1(B^-)\}, \quad (5.84)$$

where $l = \mathrm{dist}(B^+,B^-)$. The inequalities (5.81) and (5.84) imply that the optimal set for λ_2 does not exist. Moreover, taking $\Omega\equiv B^+\bigcup B^-$ with $l = \mathrm{dist}(B^+,B^-)\to\infty$, and B^+ and B^- being identical, from the inequalities (5.81) and (5.84) we arrive at

$$\lim_{l\to\infty}\lambda_2(B^+\bigcup B^-) \geq \min\{\lambda_1(B^+),\, \lambda_1(B^-)\} = \lambda_1(B^+)$$
$$= \lambda_1(B^-) \geq \lim_{l\to\infty}\lambda_2(B^+\cup B^-),$$

and this implies that the maximising sequence for λ_2 is given by a disjoint union of two identical balls with mutual distance going to ∞. $\square$

5.4 Bessel potential operators

Following the analysis of Riesz potential operators in the previous section, we now briefly discuss some questions of the spectral geometry and boundary properties for Bessel potential operators in open bounded Euclidean domains. In particular, the results apply to differential operators related to a nonlocal boundary value problem for the Helmholtz equation, so we obtain isoperimetric inequalities for its eigenvalues as well.

In $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$, consider the ***Bessel potential operators***

$$(\mathscr{B}_{\alpha,\Omega} f)(x) := \int_\Omega \varepsilon_{\alpha,d}(|x-y|) f(y) dy, \quad f \in L^2(\Omega), \quad 0 < \alpha < d, \tag{5.85}$$

where

$$\varepsilon_{\alpha,d}(|x-y|) = c_{\alpha,d} \frac{K_{\frac{d-\alpha}{2}}(|x-y|)}{|x-y|^{\frac{d-\alpha}{2}}}, \tag{5.86}$$

with

$$c_{\alpha,d} = \frac{2^{\frac{2-m-\alpha}{2}}}{\pi^{d/2}\Gamma(\alpha/2)}.$$

Here K_ν is the modified Bessel function of the second kind (the McDonald function):

$$K_\nu(z) = \frac{\pi}{2\sin \upsilon\pi} \left(\mathscr{I}_{-\nu}(z) - \mathscr{I}_\nu(z)\right), \quad \nu \neq 0, \pm 1, \pm 2, \ldots,$$

$$K_n(z) = \lim_{\nu \to n} K_\nu(z), \quad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\mathscr{I}_\nu = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

For an even integer $\alpha = 2m$ with $0 < m < d/2$, the kernel $\varepsilon_{2m,d}(|x|)$ is the fundamental solution to the poly-Helmholtz equation of order $2m$ in $\mathbb{R}^d$:

$$(I - \Delta_x)^m \varepsilon_{\alpha,d}(|x-y|) = \delta_y, \; m = 1, 2, \ldots,$$

where I is the identity operator, Δ_x is the Laplacian with respect to the point $x \in \mathbb{R}^d$ and δ_y is the Dirac distribution at the point $y \in \mathbb{R}^d$.

In a bounded connected set (domain) $\Omega \subset \mathbb{R}^d$ with a piecewise C^1 boundary $\partial\Omega$, as an analogue to (5.62) we consider the poly-Helmholtz equation

$$\mathscr{L}u(x) := (I - \Delta_x)^m u(x) = f(x), \quad x \in \Omega,, \; m = 1, 2, \ldots. \tag{5.87}$$

To relate the Bessel potential (5.85) ($\alpha = 2m$ and $0 < m < d/2$) to the boundary value problem (5.87) in Ω, we will show that for any $f \in L^2(\Omega)$, the Bessel potential (5.85) belongs to the functional class $H^{2m}(\Omega)$ and satisfies the following nonlocal integral boundary conditions

$$-\frac{1}{2}\mathscr{L}^i u(x)+\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\frac{\partial}{\partial n_y}\mathscr{L}^{m-i-1-j}\varepsilon_{2(m-i),d}(|x-y|)\mathscr{L}^{j+i}u(y)dS_y$$
$$-\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathscr{L}^{m-i-1-j}\varepsilon_{2(m-i),d}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j+i}u(y)dS_y=0,\ x\in\partial\Omega, \quad (5.88)$$

for $i=0,1,\dots,m-1$. Conversely, if $u\in H^{2m}(\Omega)$ satisfies (5.87) and the boundary conditions (5.88) for $i=0,1,\dots,m-1$, then it coincides with the (polyharmonic) volume potential (defined by the formula (5.85)). On the other hand, this means that the analysis of the Bessel potential operators (5.85) yields corresponding result for the boundary value problem (5.87)–(5.88).

Thus, we can summarise the facts of this section as:

- Let Ω^* be a ball in $\mathbb{R}^d$. Then for any integer p with $d/\alpha<p\leq\infty$ we have

$$\|\mathscr{B}_{\alpha,\Omega}\|_p\leq\|\mathscr{B}_{\alpha,\Omega^*}\|_p,\ 0<\alpha<d, \quad (5.89)$$

  for any domain Ω with $|\Omega|=|\Omega^*|$. Here $\|\cdot\|_p$ is the Schatten p-norm and $|\cdot|$ is the Lebesgue measure. Note that for $p=\infty$ this result gives an analogue of the famous Rayleigh-Faber-Krahn inequality for the Bessel potentials.

- Also, we construct a well-posed boundary value problem (5.87)-(5.88) for the poly-Helmholtz equation, which is related to the Bessel potential.

5.4.1 Spectral properties of $\mathscr{B}_{\alpha,\Omega}$

According to the discussions of Reisz potential operators in Section 5.3, in the same way the eigenvalues of $\mathscr{B}_{\alpha,\Omega}$ may be enumerated in descending order,

$$\lambda_1\geq\lambda_2\geq\dots$$

where λ_j is repeated in this series with respect to its multiplicity. As usual we denote the corresponding eigenfunctions by $u_1,u_2,\dots,$ so that for each eigenvalue λ_j one and only one corresponding normalised eigenfunction u_j is fixed,

$$\mathscr{B}_{\alpha,\Omega}u_j=\lambda_j u_j,\ \ j=1,2,\dots.$$

Recall the Bessel operator $\mathscr{B}_{\alpha,\Omega}$ is nonnegative, means, in particular, that all eigenvalues are nonnegative and equal to its singular values

$$\lambda_j\equiv\lambda_j(\mathscr{B}_{\alpha,\Omega})=|\lambda_j(\mathscr{B}_{\alpha,\Omega})|=s_j.$$

In addition, for each eigenvalue λ_j one has the inequality

$$\lambda_j\leq C|\Omega|^{\vartheta}j^{-\vartheta}, \quad (5.90)$$

where $\vartheta = \alpha/d$. This implies the compactness of the operator in $\Omega \subset \mathbb{R}^d$, which is a measurable set with finite Lebesgue measure. This also means the Bessel operator $\mathscr{B}_{\alpha,\Omega}$ belongs to all Schatten class S_p with $p > \alpha/d$ and

$$\|\mathscr{B}_{\alpha,\Omega}\|_p = \left(\sum_{j=1}^{\infty} \lambda_j(\Omega)^p\right)^{\frac{1}{p}}, \ 1 \le p < \infty. \tag{5.91}$$

Furthermore, applying Theorem 3.95 to the Bessel operator kernel $K(x,y) = \varepsilon_{\alpha,d}(|x-y|)$, $x,y \in \Omega$, that is, since the measure of Ω is finite, the kernel $K(x,y)$ belongs to $L^{p',p}(\Omega \times \Omega)$ for any $p > \frac{d}{\alpha}$. Therefore, for $s > \frac{d}{\alpha}$ we have

$$\begin{aligned}\sum \lambda_j(\mathscr{B}_{\alpha,\Omega})^s &= \mathrm{Tr}\,(\mathscr{B}^s_{\alpha,\Omega}) \\ &= \int\limits_{\Omega^s} \left(\prod_{k=1}^{s} K(x_k, x_{k+1})\right) dx_1 dx_2 \dots dx_s, \ \ x_{s+1} \equiv x_1. \end{aligned} \tag{5.92}$$

Thus, we have:

Theorem 5.15 *For any integer p with $\frac{d}{\alpha} < p \le \infty$, we have*

$$\|\mathscr{B}_{\alpha,\Omega}\|_p \le \|\mathscr{B}_{\alpha,\Omega^*}\|_p, \quad 0 < \alpha < d, \tag{5.93}$$

for any domain Ω with $|\Omega| = |\Omega^*|$, where Ω^* is a ball in $\mathbb{R}^d$.

By Jentsch's theorem (as in Lemma 5.4 or Lemma 5.18), the eigenvalue λ_1 of $\mathscr{B}_{\alpha,\Omega}$ with the largest modulus is positive and simple; the corresponding eigenfunction u_1 is positive, and any other eigenfunction u_j, $j > 1$, is sign changing in Ω. Note that the positivity of λ_1 is already known, since the operator $\mathscr{B}_{\alpha,\Omega}$ is nonnegative; what is important is the positivity of u_1.

Proof of the analogue of the famous Rayleigh-Faber-Krahn theorem for the operator $\mathscr{B}_{\alpha,\Omega}$ is the same as the proof of Lemma 5.13. That is for $p = \infty$: The ball Ω^* is a maximiser of the first eigenvalue of the operator $\mathscr{B}_{\alpha,\Omega}$ among all domains of a given volume, i.e.

$$0 < \lambda_1(\Omega) \le \lambda_1(\Omega^*)$$

for an arbitrary domain $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$. In other words it means that the operator norm of $\mathscr{B}_{\alpha,\Omega}$ is maximised in the ball among all Euclidean domains of a given volume.

Theorem 5.15 can be proved in the same way as Theorem 5.11. Note that an analogue of Theorem 5.14 is also proved the same way for the Bessel operators.

5.4.2 Boundary properties of $\mathscr{B}_{\alpha,\Omega}$

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a piecewise smooth boundary $\partial\Omega \in C^1$. For $m \in \mathbb{N}$, we denote

$$\mathscr{L}^m := \mathscr{L}\mathscr{L}^{m-1}, \quad m = 2, 3, \dots,$$

with

$$\mathscr{L} = I - \Delta.$$

Then for $m = 1, 2, \ldots$, we consider the equation

$$\mathscr{L}^m u(x) = f(x),\ x \in \Omega, \tag{5.94}$$

for a given $f \in L^2(\Omega)$ and

$$u(x) = \int_\Omega f(y)\varepsilon_{2m}(|x-y|)dy \tag{5.95}$$

in $\Omega \subset \mathbb{R}^d$, where $\varepsilon_{2m}(|x-y|)$ is a fundamental solution of (5.94).

A simple calculation shows that the Bessel potential (5.95) is a solution of (5.94) in Ω. Also it is known that if $f \in L^2(\Omega)$, then $u \in H^{2m}(\Omega)$ (see, e.g. [6]).

We now describe a boundary condition on $\partial\Omega$ such that with this boundary condition equation (5.94) has a unique solution in $H^{2m}(\Omega)$, which coincides with (5.95).

Theorem 5.16 *For any $f \in L^2(\Omega)$, the Bessel potential* (5.95) *is a unique solution of equation* (5.94) *in $H^{2m}(\Omega) \cap H^{2m-1}(\overline{\Omega})$ with the m boundary conditions*

$$-\frac{\mathscr{L}^i u(x)}{2} + \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathscr{L}^{j+i} u(y) \frac{\partial}{\partial n_y} \mathscr{L}^{m-1-j} \varepsilon_{2m}(|x-y|) dS_y$$
$$- \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathscr{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathscr{L}^{j+i} u(y) dy = 0, \quad x \in \partial\Omega, \tag{5.96}$$

for all $i = 0, 1, \ldots, m-1$.

The remainder of this section is devoted to the proof of Theorem 5.16. Applying Green's second formula for each $x \in \Omega$, we calculate

$$\begin{aligned} u(x) &= \int_\Omega f(y)\varepsilon_{2m}(|x-y|)dy = \int_\Omega \mathscr{L}^m u(y)\varepsilon_{2m}(|x-y|)dy \\ &= \int_\Omega \mathscr{L}^{m-1} u(y) \mathscr{L} \varepsilon_{2m}(|x-y|)dy - \int_{\partial\Omega} \mathscr{L}^{m-1} u(y) \frac{\partial}{\partial n_y} \varepsilon_{2m}(|x-y|) dS_y \\ &+ \int_{\partial\Omega} \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathscr{L}^{m-1} u(y) dS_y = \int_\Omega \mathscr{L}^{m-2} u(y) \mathscr{L}^2 \varepsilon_{2m}(|x-y|)dy \end{aligned}$$

$$
\begin{aligned}
&-\int_{\partial\Omega} \mathscr{L}^{m-2}u(y)\frac{\partial}{\partial n_y}\mathscr{L}\varepsilon_{2m}(|x-y|)dS_y \\
&+\int_{\partial\Omega} \mathscr{L}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{m-2}u(y)dS_y \\
&-\int_{\partial\Omega} \mathscr{L}^{m-1}u(y)\frac{\partial}{\partial n_y}\varepsilon_{2m}(|x-y|)dS_y \\
&+\int_{\partial\Omega} \varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1}u(y)dS_y = \ldots \\
&= u(x) - \sum_{j=0}^{m-1}\int_{\partial\Omega} \mathscr{L}^{j}u(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)dS_y \\
&+\sum_{j=0}^{m-1}\int_{\partial\Omega} \mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j}u(y)dS_y, \quad x\in\Omega.
\end{aligned}
$$

where $\frac{\partial}{\partial n_y}$ is the unit outer normal at the point y on the boundary $\partial\Omega$.

This gives the identity

$$
\begin{aligned}
&\sum_{j=0}^{m-1}\int_{\partial\Omega} \mathscr{L}^{j}u(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)dS_y \\
&\qquad -\sum_{j=0}^{m-1}\int_{\partial\Omega} \mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j}u(y)dS_y = 0, \quad x\in\Omega. \qquad (5.97)
\end{aligned}
$$

Using the properties of the double- and single-layer potentials as x approaches the boundary $\partial\Omega$ from the interior, (5.97) gives

$$
\begin{aligned}
&-\frac{u(x)}{2} + \sum_{j=0}^{m-1}\int_{\Omega} \mathscr{L}^{j}u(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)dS_y \\
&\qquad -\sum_{j=0}^{m-1}\int_{\partial\Omega} \mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j}u(y)dS_y = 0, \quad x\in\partial\Omega.
\end{aligned}
$$

Thus, this relation is one of the boundary conditions of (5.95). Let us derive the remaining boundary conditions. To do it, we set

$$
\mathscr{L}^{m-i}\mathscr{L}^{i}u = f, \quad i = 0,1,\ldots,m-1, \quad m = 1,2,\ldots, \qquad (5.98)
$$

and carry out similar calculations just as above. This yields

$$\begin{aligned}
\mathscr{L}^i u(x) &= \int_\Omega f(y)\mathscr{L}^i \varepsilon_{2m}(|x-y|)dy = \int_\Omega \mathscr{L}^{m-i}\mathscr{L}^i u(y)\mathscr{L}^i \varepsilon_{2m}(|x-y|)dy \\
&= \int_\Omega \mathscr{L}^{m-i-1}\mathscr{L}^i u(y)\mathscr{L}\mathscr{L}^i \varepsilon_{2m}(|x-y|)dy \\
&- \int_{\partial\Omega} \mathscr{L}^{m-i-1}\mathscr{L}^i u(y)\frac{\partial}{\partial n_y}\mathscr{L}^i \varepsilon_{2m}(|x-y|)dS_y \\
&+ \int_{\partial\Omega} \mathscr{L}^i \varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{m-i-1}\mathscr{L}^i u(y)dS_y \\
&= \int_\Omega \mathscr{L}^{m-i-2}\mathscr{L}^i u(y)\mathscr{L}^2\mathscr{L}^i \varepsilon_{2m}(|x-y|)dy \\
&- \int_{\partial\Omega} \mathscr{L}^{m-i-2}\mathscr{L}^i u(y)\frac{\partial}{\partial n_y}\mathscr{L}\mathscr{L}^i \varepsilon_{2m}(|x-y|)dS_y \\
&+ \int_{\partial\Omega} \mathscr{L}\mathscr{L}^i \varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{m-i-2}\mathscr{L}^i u(y)dS_y \\
&- \int_{\partial\Omega} \mathscr{L}^{m-i-1}\mathscr{L}^i u(y)\frac{\partial}{\partial n_y}\mathscr{L}^i \varepsilon_{2m}(|x-y|)dS_y \\
&+ \int_{\partial\Omega} \mathscr{L}^i \varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{m-i-1}\mathscr{L}^i u(y)dS_y \\
&= \ldots = \int_\Omega \mathscr{L}^i u(y)\mathscr{L}^{m-i}\mathscr{L}^i \varepsilon_{2m}(|x-y|)dy \\
&- \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathscr{L}^j\mathscr{L}^i u(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-i-1-j}\mathscr{L}^i \varepsilon_{2m}(|x-y|)dS_y \\
&+ \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathscr{L}^{m-i-1-j}\mathscr{L}^i \varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^j\mathscr{L}^i u(y)dS_y \\
&= \mathscr{L}^i u(x) - \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathscr{L}^{j+i} u(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)dS_y \\
&+ \sum_{j=0}^{m-i-1}\int_{\partial\Omega} \mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j+i} u(y)dS_y, \quad x\in\Omega,
\end{aligned}$$

where, $\mathscr{L}^i\varepsilon_m$ is a fundamental solution of equation (5.98), that is,

$$\mathscr{L}^{m-i}\mathscr{L}^i\varepsilon_{2m} = \delta, \qquad i = 0,1,\ldots,m-1.$$

The previous relations imply

$$\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathscr{L}^{j+i}u(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)dS_y \\ -\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j+i}u(y)dS_y=0$$

for any $x\in\Omega$ and $i=0,1,\ldots,m-1$. According to the properties of the double- and single-layer potentials as x approaches the boundary $\partial\Omega$ from the interior of Ω, we obtain

$$-\frac{\mathscr{L}^i u(x)}{2}+\sum_{j=0}^{m-i-1}\int_{\Omega}\mathscr{L}^{j+i}u(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)dS_y \\ -\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j+i}u(y)dS_y=0,\quad x\in\partial\Omega,$$

are all boundary conditions of (5.95) for each $i=0,1,\ldots,m-1$.

To prove the uniqueness of the solution, let us show that if a function $w\in H^{2m}(\Omega)\cap H^{2m-1}(\overline{\Omega})$ satisfies the equation $\mathscr{L}^m w=f$ and the boundary conditions (5.96), then it coincides with the solution (5.95). Indeed, otherwise the function

$$v=u-w\in H^{2m}(\Omega)\cap H^{2m-1}(\overline{\Omega}),$$

where u is the generalised volume potential (5.95), satisfies the homogeneous equation

$$\mathscr{L}^m v=0 \tag{5.99}$$

and the boundary conditions (5.96), i.e.

$$I_i(v)(x):=-\frac{\mathscr{L}^i v(x)}{2}+\sum_{j=0}^{m-i-1}\int_{\Omega}\mathscr{L}^{j+i}v(y)\frac{\partial}{\partial n_y}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)dS_y \\ -\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathscr{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathscr{L}^{j+i}v(y)dS_y=0,\quad i=0,1,\ldots,m-1,$$

for $x \in \partial\Omega$. By applying the Green formula to the function $v \in H^{2m}(\Omega) \cap H^{2m-1}(\overline{\Omega})$ and by following the lines of the above argument, we obtain

$$\begin{aligned}
0 &= \int_{\Omega} \mathscr{L}^m v(x) \mathscr{L}^i \varepsilon_{2m}(|x-y|)dy \\
&= \int_{\Omega} \mathscr{L}^{m-i} \mathscr{L}^i v(x) \mathscr{L}^i \varepsilon_{2m}(|x-y|)dy \\
&= \int_{\Omega} \mathscr{L}^{m-1} v(x) \mathscr{L} \mathscr{L}^i \varepsilon_{2m}(|x-y|)dy \\
&- \int_{\partial\Omega} \mathscr{L}^{m-1} v(x) \frac{\partial}{\partial n_y} \mathscr{L}^i \varepsilon_{2m}(|x-y|)dS_y \\
&+ \int_{\partial\Omega} \mathscr{L}^i \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathscr{L}^{m-1} v(x)dS_y = \ldots \\
&= \mathscr{L}^i v(x) - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathscr{L}^{j+i} v(y) \frac{\partial}{\partial n_y} \mathscr{L}^{m-1-j} \varepsilon_{2m}(|x-y|)dS_y \\
&+ \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathscr{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathscr{L}^{j+i} v(y)dS_y, \ i = 0, 1, \ldots, m-1.
\end{aligned}$$

By passing to the limit as $x \to \partial\Omega$, we obtain the relations

$$\mathscr{L}^i v(x) \,|_{x\in\partial\Omega} = I_i(v)(x) \,|_{x\in\partial\Omega} = 0, \quad i = 0, 1, \ldots, m-1. \tag{5.100}$$

Assuming for the moment the uniqueness of the solution of the boundary value problem

$$\mathscr{L}^m v = 0, \tag{5.101}$$

$$\mathscr{L}^i v \,|_{\partial\Omega} = 0, \quad i = 0, 1, \ldots, m-1,$$

we get that $v = u - w \equiv 0$, for all $x \in \Omega$, i.e. w coincides with u in Ω. Thus (5.95) is the unique solution of the boundary value problem (5.94), (5.96) in Ω.

It remains to argue that the boundary value problem (5.101) has a unique solution in $H^{2m}(\Omega) \cap H^{2m-1}(\overline{\Omega})$. Denoting $\tilde{v} := \mathscr{L}^{m-1} v$, this follows by induction from the uniqueness in $C^2(\Omega) \cap C^1(\overline{\Omega})$ of the problem

$$\mathscr{L}\tilde{v} = 0, \quad \tilde{v} \,|_{\partial\Omega} = 0.$$

The proof of Theorem 5.16 is complete. □

5.5 Riesz transforms in spherical and hyperbolic geometries

We now discuss the spectral geometric properties of the Riesz transforms in non-flat geometries, namely, in the cases of the spheres and of the hyperbolic spaces.

Thus, let M be a complete, connected, simply connected Riemannian manifold of constant sectional curvature. Let dy be the Riemannian measure on M, and let $d(x,y)$ be the Riemannian geodesic distance. As is well-known, the three possibilities for M are the sphere $\mathbb{S}^n$, the Euclidean space $\mathbb{R}^n$ and the real hyperbolic space $\mathbb{H}^n$ for positive, zero, and negative curvature, respectively. As the Euclidean case has been already treated in Section 5.3, we now discuss the other two cases.

Let us consider the Riesz transform

$$\mathscr{R}_\alpha f(x) := \int_M \frac{1}{d(x,y)^\alpha} f(y)dy, \tag{5.102}$$

where f will be assumed to be compactly supported in an open bounded set $\Omega \subset M$. In fact, we can also consider the family of operators

$$\mathscr{R}_{\alpha,\Omega} f(x) := \int_\Omega \frac{1}{d(x,y)^\alpha} f(y)dy, \quad 0 < \alpha < n, \tag{5.103}$$

depending on Ω. In the present section we are interested in the behaviour of the first and second eigenvalues of operators $\mathscr{R}_{\alpha,\Omega}$.

Now we describe the results on $\mathbb{S}^n$ and $\mathbb{H}^n$, similar to the Euclidean results of Section 5.3. In the case of the real hyperbolic space $\mathbb{H}^n$ we also establish the analogue of the Hong-Krahn-Szegő inequality on $\mathbb{R}^n$, namely, the description of Ω for which the second eigenvalue is maximised. In fact, our results apply to a more general class of convolution type operators than the Riesz transforms (5.103) that we will now describe.

Let $\Omega \subset \mathbb{S}^n$ or $\Omega \subset \mathbb{H}^n$ be an open bounded set. We consider the following integral operator $\mathscr{K}_\Omega : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\mathscr{K}_\Omega f(x) := \int_\Omega K(d(x,y)) f(y)dy, \quad f \in L^2(\Omega), \tag{5.104}$$

which we assume to be compact. Here $d(x,y)$ is the distance between the points x and y in the space $\mathbb{S}^n$ or $\mathbb{H}^n$. Throughout this section we assume that the kernel $K(\cdot)$ is (say, a member of $L^1(\mathbb{S}^n)$ or $L^1_{loc}(\mathbb{H}^n)$) real, positive and non-increasing, that is, the kernel $K : [0,\infty) \to \mathbb{R}^+$ satisfies

$$K(\rho_1) \geq K(\rho_2) \quad \text{if} \quad \rho_1 \leq \rho_2. \tag{5.105}$$

Since the kernel K is a real and symmetric function, $\mathscr{K}_\Omega$ is a self-adjoint operator. So, all of its eigenvalues are real. Thus, the eigenvalues of $\mathscr{K}_\Omega$ can be enumerated in the descending order of their moduli,

$$|\lambda_1| \geq |\lambda_2| \geq ..., \tag{5.106}$$

where $\lambda_j = \lambda_j(\Omega)$ is repeated in this series according to its multiplicity. Further on, the corresponding eigenfunctions will be denoted by $u_1, u_2, ...$, so that for each eigenvalue λ_j there is a unique corresponding normalised eigenfunction u_j:

$$\mathscr{K}_\Omega u_j = \lambda_j(\Omega) u_j, \;\; j = 1, 2, \ldots .$$

As we mentioned above, in this section we are interested in spectral geometric inequalities of the convolution type operator $\mathscr{K}_\Omega$ for the first and the second eigenvalues.

Summarising the main results of this section for operators $\mathscr{K}_\Omega$ in both cases of $\mathbb{S}^n$ or $\mathbb{H}^n$, we present the following facts:

- Rayleigh-Faber-Krahn type inequality: the first eigenvalue of $\mathscr{K}_\Omega$ is maximised on the geodesic ball among all domains of a given measure in $\mathbb{S}^n$ or $\mathbb{H}^n$;
- Hong-Krahn-Szegő type inequality: the maximum of the second eigenvalue of (positive) $\mathscr{K}_\Omega$ among bounded open sets with a given measure in $\mathbb{H}^n$ is achieved by the union of two identical geodesic balls with mutual distance going to infinity.

The presentation of this section follows our open access paper [102].

5.5.1 Geometric inequalities for the first eigenvalue

Let M denote $\mathbb{S}^n$ or $\mathbb{H}^n$. We assume that $\Omega \subset M$ is an open bounded set, and consider compact integral operators on $L^2(\Omega)$ of the form

$$\mathscr{K}_\Omega f(x) = \int_\Omega K(d(x,y))f(y)dy, \quad f \in L^2(\Omega), \tag{5.107}$$

where the kernel K is real, positive and non-increasing, that is, K satisfies (5.105). By $|\Omega|$ we will denote the Riemannian measure of Ω. We prove the following analogue of the Rayleigh-Faber-Krahn inequality for the integral operator $\mathscr{K}_\Omega$.

Theorem 5.17 (Rayleigh-Faber-Krahn inequality) *The geodesic ball $\Omega^* \subset M$ is a maximiser of the first eigenvalue of the operator $\mathscr{K}_\Omega$ among all domains of a given measure in M, that is, more precisely we have*

$$\lambda_1(\Omega) \leq \lambda_1(\Omega^*) \tag{5.108}$$

for an arbitrary domain $\Omega \subset M$ with $|\Omega| = |\Omega^*|$.

Note that, in other words, Theorem 5.17 says that the operator norm $\|\mathscr{K}_\Omega\|$ is maximised in a geodesic ball among all domains of a given measure.

Since the integral kernel of $\mathscr{K}$ is positive, the following statement, sometimes called Jentsch's theorem, applies. However, for completeness we restate and give its proof on the symmetric space M (that is, $\mathbb{S}^n$ or $\mathbb{H}^n$). This also shows that (5.108) is the inequality between positive numbers, that is, $0 < \lambda_1(\Omega) \leq \lambda_1(\Omega^*)$.

Lemma 5.18 *The first eigenvalue λ_1 (with the largest modulus) of the convolution type compact operator $\mathscr{K}$ is positive and simple; the corresponding eigenfunction u_1 can be chosen positive.*

Let us prove Lemma 5.18.

Since as the kernel K is real the eigenfunctions of the convolution type compact operator $\mathscr{K} = \mathscr{K}_\Omega$ may be chosen to be real. Let us first show that u_1 cannot change its sign in $\Omega \subset M$, i.e.

$$u_1(x)u_1(y) = |u_1(x)u_1(y)|, \; x, y \in \Omega \subset M.$$

In the opposite case, in view of the continuity of the function $u_1(x)$, there would be neighbourhoods $U(x_0, r) \subset \Omega$ and $U(y_0, r) \subset \Omega$ such that

$$|u_1(x)u_1(y)| > u_1(x)u_1(y), \; x \in U(x_0, r) \subset \Omega, \; y \in U(y_0, r) \subset \Omega,$$

and so due to

$$\int_\Omega K(d(x,z))K(d(z,y))dz > 0 \tag{5.109}$$

we get

$$\begin{aligned}\frac{(\mathscr{K}^2|u_1|, |u_1|)}{\|u_1\|^2} &= \frac{1}{\|u_1\|^2}\int_\Omega\int_\Omega\int_\Omega K(d(x,z))K(d(z,y))dz|u_1(x)||u_1(y)|dxdy \\ &> \frac{1}{\|u_1\|^2}\int_\Omega\int_\Omega\int_\Omega K(d(x,z))K(d(z,y))dz u_1(x)u_1(y)dxdy = \lambda_1^2. \end{aligned} \tag{5.110}$$

Using the fact

$$\lambda_1^2 u_1 = \mathscr{K}^2 u_1,$$

and by the variational principle we establish

$$\lambda_1^2 = \sup_{f \in L^2(\Omega), f \not\equiv 0} \frac{\langle \mathscr{K}^2 f, f\rangle}{\|f\|^2}. \tag{5.111}$$

It follows that the strict inequality (5.110) contradicts the variational principle (5.111).

Now it remains to show the eigenfunction $u_1(x)$ cannot become zero in Ω and therefore can be chosen positive in Ω. If it is not so, then there would be a point $x_0 \in \Omega$ such that

$$0 = \lambda_1^2 u_1(x_0) = \int_\Omega\int_\Omega K(d(x_0,z))K(d(z,y))dz u_1(y)dy,$$

from which, due to condition (5.109), the contradiction follows: $u_1(y) = 0$ for almost all $y \in \Omega$.

From positivity of u_1 it follows that λ_1 is simple. Indeed, if λ_1 is not simple, that is, if there were an eigenfunction $\widetilde{u}_1$ linearly independent of u_1 and corresponding to λ_1, then for all real c every linear combination $u_1 + c\widetilde{u}_1$ would also be an eigenfunction corresponding to λ_1 and therefore, by what has been proved, it could not become zero in Ω. As c is arbitrary, this is impossible. Finally, it remains to show that λ_1 is positive. It is trivial since u_1 and the kernel are positive. □

We can now prove Theorem 5.17. Let Ω be a bounded measurable set in M (where, as above, M is $\mathbb{S}^n$ or $\mathbb{H}^n$). We denote its symmetric rearrangement by Ω^*, which is simply an open geodesic ball centred at 0 with the measure equal to the measure of Ω, i.e. $|\Omega^*| = |\Omega|$.

Let u be a nonnegative measurable function in Ω such that all its positive level sets have finite measure. The symmetric decreasing rearrangement of u (we refer to [7] and [10] for more detailed discussions on this subject) can be defined in the same way as in Section 4.1: Let u be a nonnegative measurable function in $\Omega \subset M$. The function

$$u^*(x) := \int_0^\infty \chi_{\{u(x)>t\}^*} dt \tag{5.112}$$

is called the (radially) symmetric decreasing rearrangement of a nonnegative measurable function u.

By Proposition 5.18 the first eigenvalue λ_1 of the operator $\mathscr{K}$ is simple; the corresponding eigenfunction u_1 can be chosen positive in $\Omega \subset M$. Recall the Riesz-Sobolev inequality (see e.g. Symmetrization Lemma in [11]):

$$\int_\Omega \int_\Omega u_1(y)K(d(y,x))u_1(x)dydx \leq \int_{\Omega^*} \int_{\Omega^*} u_1^*(y)K(d(y,x))u_1^*(x)dydx. \tag{5.113}$$

Moreover, we have

$$\|u\|_{L^2(\Omega)} = \|u^*\|_{L^2(\Omega^*)}, \tag{5.114}$$

for each nonnegative function $u \in L^2(\Omega)$ Thus, from (5.113), (5.114) and the variational principle for $\lambda_1(\Omega^*)$, we obtain

$$\lambda_1(\Omega) = \frac{\int_\Omega \int_\Omega u_1(y)K(d(y,x))u_1(x)dydx}{\int_\Omega |u_1(x)|^2 dx} \leq \frac{\int_{\Omega^*} \int_{\Omega^*} u_1^*(y)K(d(y,x))u_1^*(x)dydx}{\int_{\Omega^*} |u_1^*(x)|^2 dx}$$

$$\leq \sup_{v \in L^2(\Omega^*), v \neq 0} \frac{\int_{\Omega^*} \int_{\Omega^*} v(y)K(d(y,x))v(x)dydx}{\int_{\Omega^*} |v(x)|^2 dx} = \lambda_1(\Omega^*). \qquad \square$$

5.5.2 Geometric inequalities for the second eigenvalue

Let us consider the maximisation problem of the second eigenvalue for positive operators $\mathscr{K}_\Omega$ on $\mathbb{H}^n$ among open sets of a given measure. We keep all the notation of the previous section.

Theorem 5.19 *If the kernel K of the positive integral operator $\mathscr{K}_\Omega$ on $\mathbb{H}^n$ satisfies*

$$K(\rho) \to 0 \ \text{ as } \ \rho \to \infty, \tag{5.115}$$

then the maximum of the second eigenvalue λ_2 among bounded open sets in $\mathbb{H}^n$ of a given measure is achieved by the union of two identical geodesic balls with mutual distance going to infinity. Moreover, this maximum is equal to the first eigenvalue of one of the two geodesic balls.

A similar type of results on $\mathbb{R}^n$, called the Hong-Krahn-Szegő inequality, was given in Theorem 5.14. Note that in Theorem 5.19 we consider only domains $\Omega \subset \mathbb{H}^n$ for which $\mathscr{K}_\Omega$ are positive operators. However, this assumption can be relaxed. In fact, the statement of Theorem 5.19 and its proof remain valid if we only assume that the second eigenvalues $\lambda_2(\Omega)$ of considered operators $\mathscr{K}_\Omega$ are positive. In the case of $\mathbb{R}^n$ and Riesz transforms (5.103), Theorem 5.19 holds without the positivity assumption since the Riesz transforms $\mathscr{R}_{\alpha,\Omega}$ on $\mathbb{R}^n$ are positive, see Lemma 5.10. We do not have an analogue of Theorem 5.19 on $\mathbb{S}^n$ because the assumption (5.115) does not make sense due to compactness of the sphere.

Let us prove Theorem 5.19. By introducing the sets

$$\Omega^+ := \{x : u_2(x) > 0\},\ \Omega^- := \{x : u_2(x) < 0\},$$

we have

$$u_2(x) \gtrless 0,\ \forall x \in \Omega^\pm \subset \Omega \subset \mathbb{H}^n,\ \Omega^\pm \neq \{\emptyset\},$$

and it follows from Proposition 5.18 that both Ω^- and Ω^+ have a positive measure. Denoting

$$u_2^\pm(x) := \begin{cases} u_2(x), \text{ in } \Omega^\pm, \\ 0, \text{ otherwise,} \end{cases} \tag{5.116}$$

we have

$$\lambda_2(\Omega)u_2(x) = \int_{\Omega^+} K(d(x,y))u_2^+(y)dy + \int_{\Omega^-} K(d(x,y))u_2^-(y)dy,\ \ x \in \Omega.$$

This gives

$$\begin{aligned}\lambda_2(\Omega)\int_{\Omega^+}|u_2^+(x)|^2dx = \int_{\Omega^+} u_2^+(x)\int_{\Omega^+} K(d(x,y))u_2^+(y)dydx \\ + \int_{\Omega^+} u_2^+(x)\int_{\Omega^-} K(d(x,y))u_2^-(y)dydx,\ \ x \in \Omega.\end{aligned}$$

Since the last term (on the right-hand side) is non-positive we have

$$\lambda_2(\Omega)\int_{\Omega^+}|u_2^+(x)|^2dx \leq \int_{\Omega^+} u_2^+(x)\int_{\Omega^+} K(d(x,y))u_2^+(y)dydx,$$

that is,

$$\frac{\int_{\Omega^+} u_2^+(x)\int_{\Omega^+} K(d(x,y))u_2^+(y)dydx}{\int_{\Omega^+}|u_2^+(x)|^2dx} \geq \lambda_2(\Omega).$$

On the other hand, the variational principle yields

$$\begin{aligned}\lambda_1(\Omega^+) = \sup_{v\in L^2(\Omega^+),v\not\equiv 0} \frac{\int_{\Omega^+} v(x)\int_{\Omega^+} K(d(x,y))v(y)dydx}{\int_{\Omega^+}|v(x)|^2dx} \\ \geq \frac{\int_{\Omega^+} u_2^+(x)\int_{\Omega^+} K(d(x,y))u_2^+(y)dydx}{\int_{\Omega^+}|u_2^+(x)|^2dx} \geq \lambda_2(\Omega).\end{aligned}$$

By the same argument we obtain

$$\lambda_1(\Omega^-) \geq \lambda_2(\Omega).$$

Therefore, we showed that

$$\min\{\lambda_1(\Omega^+), \lambda_1(\Omega^-)\} \geq \lambda_2(\Omega). \tag{5.117}$$

Now let us introduce B^+ and B^-, the geodesic balls of the same measure as Ω^+ and Ω^-, respectively. According to Theorem 5.17, we have

$$\lambda_1(B^+) \geq \lambda_1(\Omega^+) \text{ and } \lambda_1(B^-) \geq \lambda_1(\Omega^-). \tag{5.118}$$

From (5.117) and (5.118), we get

$$\min\{\lambda_1(B^+),\, \lambda_1(B^-)\} \geq \lambda_2(\Omega). \tag{5.119}$$

We also consider the set $B^+ \cup B^-$, with the geodesic balls B^+ and B^- placed at distance l, that is,

$$l = \mathrm{dist}(B^+, B^-).$$

Let us denote by $u_1^{\circledast}$ the first normalised eigenfunction of $\mathscr{K}_{B^+\cup B^-}$ and take u_+ and u_- being the first normalised eigenfunctions of each single geodesic ball B^+ and B^-, respectivily, that is, of operators $\mathscr{K}_{B^\pm}$. Let us also introduce the function $v^{\circledast} \in L^2(B^+ \cup B^-)$, which equals u_+ in B^+ and γu_- in B^-. Since the functions u_+, u_-, and $u^{\circledast}$ are positive, it is possible to find a real number γ so that $v^{\circledast}$ is orthogonal to $u_1^{\circledast}$. We observe that

$$\int_{B^+\cup B^-}\int_{B^+\cup B^-} v^{\circledast}(x)v^{\circledast}(y)K(d(x,y))dxdy = \sum_{i=1}^{4}\mathscr{I}_i, \tag{5.120}$$

where

$$\mathscr{I}_1 := \int_{B^+}\int_{B^+} u_+(x)u_+(y)K(d(x,y))dxdy,$$

$$\mathscr{I}_2 := \gamma\int_{B^+}\int_{B^-} u_+(x)u_-(y)K(d(x,y))dxdy,$$

$$\mathscr{I}_3 := \gamma\int_{B^-}\int_{B^+} u_-(x)u_+(y)K(d(x,y))dxdy,$$

and

$$\mathscr{I}_4 := \gamma^2\int_{B^-}\int_{B^-} u_-(x)u_-(y)K(d(x,y))dxdy.$$

On the other hand, the variational principle implies

$$\lambda_2(B^+ \cup B^-) = \sup_{v\in L^2(B^+\bigcup B^-),\, v\perp u_1,\, \|v\|=1}\int_{B^+\cup B^-}\int_{B^+\cup B^-} v(x)v(y)K(d(x,y))dxdy.$$

Since by construction $v^{\circledast}$ is orthogonal to u_1, we have

$$\lambda_2(B^+ \cup B^-) \geq \int_{B^+\cup B^-}\int_{B^+\cup B^-} v^{\circledast}(x)v^{\circledast}(y)K(d(x,y))dxdy = \sum_{i=1}^{4}\mathscr{I}_i.$$

Moreover, since u_+ and u_- are the first normalised eigenfunctions (by Proposition 5.18 both are positive everywhere) of each single geodesic ball B^+ and B^-, we obtain

$$\lambda_1(B^\pm) = \int_{B^\pm}\int_{B^\pm} u_\pm(x)u_\pm(y)K(d(x,y))dxdy.$$

Summarising the above facts, we get

$$\lambda_2(B^+ \cup B^-) \geq \sum_{i=1}^{4} \mathscr{I}_i \geq \frac{\sum_{i=1}^4 \mathscr{I}_i}{1+\gamma^2} = \frac{\mathscr{I}_1 + \mathscr{I}_4 + \mathscr{I}_2 + \mathscr{I}_3}{\lambda_1(B^+)^{-1}\mathscr{I}_1 + \lambda_1(B^-)^{-1}\mathscr{I}_4}. \tag{5.121}$$

Since the function $K(d(x,y))$ tends to zero as $x \in B^\pm,\ y \in B^\mp$ and $l \to \infty$, we observe that

$$\lim_{l\to\infty} \mathscr{I}_2 = \lim_{l\to\infty} \mathscr{I}_3 = 0,$$

therefore,

$$\lim_{l\to\infty} \lambda_2(B^+ \bigcup B^-) \geq \max\{\lambda_1(B^+), \lambda_1(B^-)\}, \tag{5.122}$$

where $l = \operatorname{dist}(B^+, B^-)$. The inequalities (5.119) and (5.122) imply that the optimal set for λ_2 does not exist. However, taking $\Omega \equiv B^+ \bigcup B^-$ with $l = \operatorname{dist}(B^+, B^-) \to \infty$, and B^+ and B^- being identical, from the inequalities (5.119) and (5.122) we obtain

$$\begin{aligned}\lim_{l\to\infty} \lambda_2(B^+ \bigcup B^-) &\geq \min\{\lambda_1(B^+), \lambda_1(B^-)\} = \lambda_1(B^+) \\ &= \lambda_1(B^-) \geq \lim_{l\to\infty} \lambda_2(B^+ \cup B^-),\end{aligned} \tag{5.123}$$

and this implies that the maximising sequence for λ_2 is given by a disjoint union of two identical geodesic balls with mutual distance going to ∞. □

Finally, let us briefly mention the convolution type integral operator $\mathscr{K}_\Omega : L^2(\Omega) \to L^2(\Omega)$ in the form

$$\mathscr{K}_\Omega f(x) := \int_\Omega K(|x-y|)f(y)dy, \quad f \in L^2(\Omega), \tag{5.124}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded set, which we assume to be compact. Thus, we assume that the operator $\mathscr{K}_\Omega$ is compact and, also the kernel $K(|x|)$ is (a member of $L^1_{loc}(\mathbb{R}^d)$) real, positive and non-increasing, i.e. that the function $K : [0,\infty) \to \mathbb{R}$ satisfies

$$K(\rho) > 0 \quad \text{for any} \quad \rho \geq 0, \tag{5.125}$$

and

$$K(\rho_1) \geq K(\rho_2) \quad \text{if} \quad \rho_1 \leq \rho_2. \tag{5.126}$$

Since $\mathscr{K}_\Omega$ is a self-adjoint operator all of its eigenvalues and characteristic numbers are real (recall that the characteristic numbers are the inverses of the eigenvalues). Thus, the characteristic numbers of $\mathscr{K}_\Omega$ can be enumerated in ascending order of their modulus,

$$|\mu_1(\Omega)| \leq |\mu_2(\Omega)| \leq \ldots, \tag{5.127}$$

where $\mu_i(\Omega)$ is repeated in this series according to its multiplicity. As usual, the corresponding eigenfunctions can be denoted by $u_1, u_2, ...$, so that for each characteristic number μ_i there is a unique corresponding (normalised) eigenfunction u_i,

$$u_i = \mu_i(\Omega)\mathscr{K}_\Omega u_i, \ \ i = 1, 2, \ldots.$$

Note that examples of operators $\mathscr{K}_\Omega$ often appear as solutions to differential equations. For instance, the Peierls integral operator, that is,

$$\mathscr{P}_\Omega f(x) = \int_\Omega \frac{1}{4\pi} \frac{e^{-|x-y|}}{|x-y|^2} f(y)dy, \quad f \in L^2(\Omega), \ \Omega \subset \mathbb{R}^3,$$

appears as the inverse operator to the one-speed neutron transport operator in Ω. In this way, the eigenvalues of the differential operators correspond to characteristic numbers of operators $\mathscr{K}_\Omega$.

Note that some arguments in the previous sections also hold for more general convolution-type operators: the ball is a maximiser of the Schatten p-norm of some convolution type integral operators $\mathscr{K}_\Omega$ (with certain kernel conditions) among all domains of a given measure in $\mathbb{R}^d$. Moreover, the equilateral triangle has the largest Schatten p-norm among all triangles of a given area. However, this seems to still require certain additional assumptions on the kernel, which we will not dwell upon here.

5.6 Heat potential operators

In the following sections we discuss some inequalities of the spectral geometry of non-self-adjoint operators. In this case, as the eigenvalues may be complex, one cannot immediately talk about maximisation problems for them. However, one can still consider their moduli, or the corresponding singular s-numbers, provided that the appearing operators are compact. In the following sections we will demonstrate some of such properties in the example of the heat operators, first without, and then with boundary conditions of different types. In such cases, it is useful to find some ways to change the operator in a way that it becomes self-adjoint while keeping track of the spectral properties under such change. If the resulting (composed) operators are self-adjoint, the developed methods can be applied to it.

In this section we start with heat operators. We will show that the circular cylinder is a maximiser of the Schatten p-norm of the (generalised) heat potential operators among all cylindric domains of a given volume. We also show that the equilateral triangular prism has the largest Schatten p-norm among all triangular prisms of a given measure. Furthermore, we will discuss the cylindric analogues of the Rayleigh-Faber-Krahn and Hong-Krahn-Szegő inequalities.

5.6.1 Basic properties

Let $\Omega \subset \mathbb{R}^d$ be a simply-connected set with smooth boundary $\partial\Omega$, so $D = \Omega \times (0,T)$ is a cylindrical domain. Let us consider the following generalised ***heat potential operator*** $\mathscr{H}_\Omega : L^2(D) \to L^2(D)$:

$$\mathscr{H}_\Omega f(x,t) := \int_0^t \int_\Omega K_{m,d}(|x-\xi|, t-\tau) f(\xi,\tau) d\xi d\tau, \ \forall f \in L^2(D), \tag{5.128}$$

where $t \in (0,T)$, $m = 1,2,\ldots$, and

$$K_{m,d}(|x|,t) = \frac{\theta(t) t^{m-1}}{(2\sqrt{\pi t})^d} e^{\frac{-|x|^2}{4t}},$$

with the Heaviside function

$$\theta(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

This $K_{m,d}(|x|,t)$ is the fundamental solution of the Cauchy problem for the high-order heat equation, i.e.

$$\left(\frac{\partial}{\partial t} - \Delta_x\right)^m K_{m,d}(|x-\xi|, t-\tau) = 0,$$

also satisfying its adjoint equation

$$\left(-\frac{\partial}{\partial \tau} - \Delta_\xi\right)^m K_{m,d}(|x-\xi|, t-\tau) = 0,$$

and

$$\lim_{t\to\tau} K_{m,d}(|x-\xi|, t-\tau) = \delta(x-\xi),$$

for all $x, \xi \in \mathbb{R}^d$, where δ is the Dirac delta 'function'.

The operators of such type are higher-order analogues of the usual heat potential operators, and some boundary value problems for them have been considered, e.g. in [124].

The generalised heat potential operator $\mathscr{H}_\Omega$ is a non-self-adjoint operator in $L^2(D)$. We introduce an *involution operator* $P : L^2(D) \to L^2(D)$ in the form

$$Pu(x,t) := u(x, T-t), \quad t \in (0,T).$$

This operator has the following properties

$$P^2 = I, \ P = P^* \text{ and } P = P^{-1},$$

where I is the identity operator, P^* is the adjoint operator to the involution operator P, and P^{-1} is the inverse operator to P.

Thus, the involution operator P acts on the operator $\mathscr{H}_\Omega$ by the formula

$$P\mathscr{H}_\Omega u = \int_0^{T-t}\int_\Omega K_{m,d}(|x-\xi|, T-t-\tau)u(\xi,\tau)d\xi d\tau. \tag{5.129}$$

A direct computation shows that

$$\langle P\mathscr{H}_\Omega u, v\rangle_{L^2(D)} =$$

$$\int_0^T\int_\Omega\left(\int_0^{T-t}\int_\Omega K_{m,d}(|x-\xi|, T-t-\tau)u(\xi,\tau)d\xi d\tau\right)v(x,t)dxdt$$
$$= \int_0^T\int_0^{T-t}\int_\Omega\int_\Omega K_{m,d}(|x-\xi|, T-t-\tau)u(\xi,\tau)v(x,t)d\xi dx d\tau dt$$
$$= \int_0^T\int_0^{T-\tau}\int_\Omega\int_\Omega K_{m,d}(|x-\xi|, T-t-\tau)u(\xi,\tau)v(x,t)d\xi dx dt d\tau$$
$$= \int_0^T\int_\Omega u(\xi,\tau)\left(\int_0^{T-\tau}\int_\Omega K_{m,d}(|x-\xi|, T-t-\tau)v(x,t)dxdt\right)d\xi d\tau$$
$$= \int_0^T\int_\Omega u(\xi,\tau)\left(\int_0^{T-\tau}\int_\Omega K_{m,d}(|\xi-x|, T-\tau-t)v(x,t)dxdt\right)d\xi d\tau$$
$$= \langle u, P\mathscr{H}_\Omega v\rangle_{L^2(D)}.$$

Thus, we arrive at $P\mathscr{H}_\Omega = (P\mathscr{H}_\Omega)^*$ in $L^2(D)$, i.e. $P\mathscr{H}_\Omega$ is a self-adjoint operator. This proves

Lemma 5.20 *The operator $P\mathscr{H}_\Omega$ is a self-adjoint operator in $L^2(D)$.*

Recall that the s-numbers of a compact operator A are the eigenvalues of the operator $(A^*A)^{1/2}$, where A^* is the adjoint operator to A. By the properties of the operator P, we obtain

$$(P\mathscr{H}_\Omega)^2 = (P\mathscr{H}_\Omega)^*(P\mathscr{H}_\Omega) = \mathscr{H}_\Omega^* P^* P\mathscr{H}_\Omega = \mathscr{H}_\Omega^* P^2\mathscr{H}_\Omega = \mathscr{H}_\Omega^*\mathscr{H}_\Omega,$$

yielding

Lemma 5.21 *The s-numbers of the operator $\mathscr{H}_\Omega$ coincide with eigenvalues of the operator $P\mathscr{H}_\Omega$.*

As a consequence of Lemma 5.21 we obtain $\|\mathscr{H}_\Omega\|_p = \|P\mathscr{H}_\Omega\|_p$ for $\mathscr{H}_\Omega \in S_p$.

5.6.2 Spectral geometric inequalities for the heat potential

In this section we discuss the basic spectral geometric inequalities for the heat potential operators based on the observation in Lemma 5.21.

Theorem 5.22 (Rayleigh-Faber-Krahn inequality) *The first eigenvalue of the operator $P\mathscr{H}_\Omega$ is maximised in the circular cylinder $C = B \times (0,T)$ with $B \subset \mathbb{R}^d$ being the open ball, that is,*

$$0 < \lambda_1(D) \leq \lambda_1(C),$$

for $|\Omega| = |B|$, where $D = \Omega \times (0,T)$ with Ω being a bounded simply-connected open set with smooth boundary $\partial\Omega$. Here $|\Omega|$ is the Lebesgue measure of the domain Ω.

Let us now prove this theorem.

A symmetric rearrangement of a bounded measurable set $D = \Omega \times (0,T)$ in $\mathbb{R}^{d+1}$ can be defined as the circular cylinder $C = B \times (0,T)$ with the measure equal to the measure of D, i.e. $|D| = |C|$. Let u be a nonnegative measurable function in D, such that all its positive level sets have finite measure. As before, to define a cylindrical symmetric-decreasing rearrangement of u we can use the layer-cake decomposition as in Section 4.1, expressing a nonnegative function u in terms of its level sets with respect to the variable x for each $t \in (0,T)$:

$$u(x,t) = \int_0^\infty \chi_{\{u(x,t)>z\}} dz, \quad \forall t \in (0,T), \tag{5.130}$$

where χ is the characteristic function of the domain. The function

$$u^*(x,t) = \int_0^\infty \chi_{\{u(x,t)>z\}^*} dz, \quad \forall t \in (0,T), \tag{5.131}$$

is called a ***cylindrical symmetric decreasing rearrangement*** of a nonnegative measurable function u.

Let us consider the eigenvalue problem

$$P\mathscr{H}_\Omega u = \lambda u.$$

By the variational principle for the self-adjoint operator $P\mathscr{H}_\Omega$, we get

$$\lambda_1(D) = \frac{\int_0^T \int_0^{T-t} \int_\Omega \int_\Omega K_{m,d}(|x-\xi|, T-t-\tau) u_1(\xi,\tau) u_1(x,t) d\xi dx d\tau dt}{\|u_1\|^2_{L^2(D)}},$$

where $u_1(x,t)$ is the first eigenfunction of $P\mathscr{H}_\Omega$.

By the Riesz inequality (for the cylindrical symmetric decreasing rearrangement, which can be shown in a similar way to Theorem 4.10), we have

$$\int_0^T \int_0^{T-t} \int_\Omega \int_\Omega K_{m,d}(|x-\xi|, T-t-\tau) u_1(\xi,\tau) u_1(x,t) d\xi dx d\tau dt$$

$$\leq \int_0^T \int_0^{T-t} \int_B \int_B K_{m,d}(|x-\xi|, T-t-\tau) u_1^*(\xi,\tau) u_1^*(x,t) d\xi dx d\tau dt. \tag{5.132}$$

We also have

$$\|u\|_{L^2(D)} = \|u^*\|_{L^2(C)}, \tag{5.133}$$

for each nonnegative function $u \in L^2(D)$. From (5.133) and (5.132), it follows that

$$\begin{aligned}
\lambda_1(D) &= \frac{\int_0^T \int_0^{T-t} \int_\Omega \int_\Omega K_{m,d}(|x-\xi|, T-t-\tau)u_1(\xi,\tau)u_1(x,t)d\xi dx d\tau dt}{\|u_1\|^2_{L^2(D)}} \\
&\le \frac{\int_0^T \int_0^{T-t} \int_B \int_B K_{m,d}(|x-\xi|, T-t-\tau)u_1^*(\xi,\tau)u_1^*(x,t)d\xi dx d\tau dt}{\|u_1\|^2_{L^2(C)}} \\
&\le \sup_{v\in L^2(C), v\neq 0} \frac{\int_0^T \int_0^{T-t} \int_B \int_B K_{m,d}(|x-\xi|, T-t-\tau)v(\xi,\tau)v(x,t)d\xi dx d\tau dt}{\|v\|^2_{L^2(C)}} \\
&= \lambda_1(C). \quad \square
\end{aligned}$$

Due to Lemma 5.21, the eigenvalues of the self-adjoint operator $P\mathscr{H}_\Omega$ coincide with the s-numbers of the operator $\mathscr{H}_\Omega$, in particular, $\lambda_1(P\mathscr{H}_\Omega) = s_1(\mathscr{H}_\Omega)$. Therefore, $\|\mathscr{H}_\Omega\| = \|P\mathscr{H}_\Omega\|$ for any Ω, i.e. the norm of the operator $\mathscr{H}_\Omega$ is maximised in the cylinder C, that is,

$$\|\mathscr{H}_\Omega\| \le \|\mathscr{H}_B\|.$$

Moreover, we can compare some of the Schatten norms.

Theorem 5.23 *Let $P\mathscr{H} \in S_{p_0}$. For each integer p we have*

$$\|P\mathscr{H}_\Omega\|_p \le \|P\mathscr{H}_B\|_p, \quad p_0 \le p < \infty,$$

for all Ω such that $|\Omega| = |B|$.

Let us prove this theorem. For all integer p with $p_0 \le p < \infty$, we have

$$\sum_{j=1}^{\infty} \lambda_j^p(P\mathscr{H}) =$$

$$\int_0^T \int_0^{T-\tau_1} \dots \int_0^{T-\tau_{p-1}} \int_\Omega \dots \int_\Omega \prod_{k=1}^{p} K_{m,d}(|\xi_k - \xi_{k+1}|, T-\tau_k-\tau_{k+1}) d\tau_1 \dots d\tau_p d\xi_1 \dots d\xi_p,$$

where $\xi_1 = \xi_{p+1}$ and $\tau_1 = \tau_{p+1}$. By the Brascamp-Lieb-Luttinger inequality, we get

$$\begin{aligned}
\sum_{i=1}^{\infty} \lambda_i^p(D) &= \int_0^T \int_0^{T-\tau_1} \dots \int_0^{T-\tau_{p-1}} \int_\Omega \dots \int_\Omega \prod_{k=1}^{p} K_{m,d}(|\xi_k - \xi_{k+1}|, T-\tau_k-\tau_{k+1}) dz \\
&\le \int_0^T \int_0^{T-\tau_1} \dots \int_0^{T-\tau_{p-1}} \int_B \dots \int_B \prod_{k=1}^{p} K^*_{m,d}(|\xi_k - \xi_{k+1}|, T-\tau_k-\tau_{k+1}) dz \\
&= \sum_{i=1}^{\infty} \lambda_i^p(C),
\end{aligned}$$

where $\xi_1 = \xi_{p+1}$, $\tau_1 = \tau_{p+1}$ and $dz = d\tau_1 ... d\tau_p d\xi_1 ... d\xi_p$. Here the following fact is used:

$$K_{m,d}(|x-y|, T-t-\tau) = K^*_{m,d}(|x-y|, T-t-\tau).$$

Therefore, we arrive at

$$\sum_{i=1}^{\infty} \lambda_i^p(D) \leq \sum_{i=1}^{\infty} \lambda_i^p(C), \ p_0 \leq p < \infty.$$

That is,

$$\|P\mathscr{H}_D\|_p \leq \|P\mathscr{H}_C\|_p, \ p_0 \leq p < \infty,$$

completing the proof. □

Theorem 5.24 *The supremum of the second eigenvalue λ_2 of the operator $P\mathscr{H}_\Omega$ among all cylindric domains D with a given measure is approached by a disjoint union of two identical cylinders with mutual distance going to ∞.*

Let us prove this theorem. Let $D^+ = \{(x,t) : u_2(x,t) > 0\}$, and $D^- = \{(x,t) : u_2(x,t) < 0\}$, so that

$$u_2(x,t) > 0, \ t \in (0,T), \forall x \in \Omega^+ \subset \Omega, \Omega^+ \neq \{\emptyset\},$$

$$u_2(x,t) < 0, \ t \in (0,T), \forall x \in \Omega^- \subset \Omega, \Omega^- \neq \{\emptyset\}.$$

We also introduce the notations

$$u_2^+(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^+, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$u_2^-(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^-, \\ 0, & \text{otherwise}. \end{cases}$$

On the other hand, we have

$$\begin{aligned} &\lambda_2(D)u_2(x,t) \\ &= \int_0^{T-t} \int_{\Omega^+} K_{m,d}(|x-\xi|, T-t-\tau)u_2(\xi,\tau)d\xi d\tau \\ &+ \int_0^{T-t} \int_{\Omega^-} K_{m,d}(|x-\xi|, T-t-\tau)u_2(\xi,\tau)d\xi d\tau. \end{aligned}$$

And multiplying by $u_2^+(x,t)$ as well as integrating over $\Omega^+ \times (0,T)$, we obtain

$$\begin{aligned} &\lambda_2(D)\|u_2^+\|^2_{L^2(D^+)} \\ &= \int_0^T \int_0^{T-t} \int_{\Omega^+} \int_{\Omega^+} K_{m,d}(|x-\xi|, T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz \\ &+ \int_0^T \int_0^{T-t} \int_{\Omega^+} \int_{\Omega^-} K_{m,d}(|x-\xi|, T-t-\tau)u_2^-(x,t)u_2^+(\xi,\tau)dz, \end{aligned} \tag{5.134}$$

where $dz = d\xi dx d\tau dt$.

Since
$$\int_0^{T-t}\int_{\Omega^-} K_{m,d}(|x-\xi|,T-t-\tau)u_2^-(\xi,\tau)d\xi d\tau<0,$$
we have
$$\begin{aligned}&\lambda_2(D)\|u_2^+\|^2_{L^2(D^+)}\\ &=\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K_{m,d}(|x-\xi|,T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz\\ &+\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^-} K_{m,d}(|x-\xi|,T-t-\tau)u_2^-(x,t)u_2^+(\xi,\tau)dz\\ &\leqslant\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K_{m,d}(|x-\xi|,T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz.\end{aligned}$$
This means
$$\lambda_2(D)\leqslant\frac{\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K_{m,d}(|x-\xi|,T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz}{\|u_2^+\|^2_{L^2(D^+)}}.$$

Furthermore, we have
$$\begin{aligned}\lambda_2(D)&\leqslant\frac{\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K_{m,d}(|x-\xi|,T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz}{\|u_2^+\|^2_{L^2(D^+)}}\\ &\leqslant\sup_{v\in L^2(D^+)}\frac{\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K_{m,d}(|x-\xi|,T-t-\tau)v^+(x,t)v^+(\xi,\tau)dz}{\|v\|^2_{L^2(D^+)}}\\ &=\lambda_1(D^+).\end{aligned}$$

Similarly, we see
$$\lambda_2(D)\leqslant\lambda_1(D^-).$$
Finally, we have
$$\lambda_2(D)\leqslant\lambda_1(D^-),\ \ \lambda_2(D)\leqslant\lambda_1(D^+). \tag{5.135}$$
By Theorem 5.22 we have
$$\lambda_1(D^+)<\lambda_1(C^+),\quad \lambda_1(D^-)<\lambda_1(C^-). \tag{5.136}$$
From (5.135) and (5.136), we obtain
$$\lambda_2(D)\leqslant\min\{\lambda_1(C^+),\lambda_1(C^-)\}.$$

Let l be the distance between C^+ and C^-, i.e. $l=dist(C^+,C^-)$. Here C^+ and C^- are the circular cylinders of the same measure as D^+ and D^-, respectively.

Let us denote by $u_1^{\circledast}(x,t)$ the first normalised eigenfunction of the operator $P\mathscr{H}_{C^+\cup C^-}$ and take u_+ and u_- being the first (normalised) eigenfunctions of each circular cylinder, i.e. of the corresponding operators $P\mathscr{H}_{C^\pm}$. Let $f^{\circledast} \in L^2(C^+ \cup C^-)$ be a function such that

$$f^{\circledast} = \begin{cases} u_+(x,t), & (x,t) \in C^+, \\ \gamma u_-(x,t), & (x,t) \in C^-, \end{cases}$$

where γ is a real number, such that $f^{\circledast}$ is orthogonal to $u_1^{\circledast}$, moreover, u_+, u_- and $u^{\circledast}$ are positive functions.

Let us denote

$$\sum_{i=1}^{4} U_i := \int_0^T \int_0^{T-t} \int_{B^+ \cup B^-} \int_{B^+ \cup B^-} K_{m,d}(|x-\xi|, T-t-\tau) f^{\circledast}(\xi,\tau) f^{\circledast}(x,t) d\xi dx d\tau dt,$$

where

$$U_1 := \int_0^T \int_0^{T-t} \int_{B^+} \int_{B^+} K_{m,d}(|x-\xi|, T-t-\tau) u_+(\xi,\tau) u_+(x,t) d\xi dx d\tau dt,$$

$$U_2 := \int_0^T \int_0^{T-t} \int_{B^+} \int_{B^-} K_{m,d}(|x-\xi|, T-t-\tau) u_-(\xi,\tau) u_+(x,t) d\xi dx d\tau dt,$$

$$U_3 := \gamma \int_0^T \int_0^{T-t} \int_{B^-} \int_{B^+} K_{m,d}(|x-\xi|, T-t-\tau) u_+(\xi,\tau) u_-(x,t) d\xi dx d\tau dt,$$

$$U_4 := \gamma^2 \int_0^T \int_0^{T-t} \int_{B^-} \int_{B^-} K_{m,d}(|x-\xi|, T-t-\tau) u_-(\xi,\tau) u_-(x,t) d\xi dx d\tau dt.$$

The variational principle implies that

$$\lambda_2(C^+ \cup C^-) = \sup_{v \in L^2(C^+ \cup C^-), v \perp u_1} \int_0^T \int_0^{T-t} \int_{B^+ \cup B^-} \int_{B^+ \cup B^-} K_{m,d}(|x-\xi|, T-t-\tau) v(\xi,\tau) v(x,t) d\xi dx d\tau dt$$
$$\geq \int_0^T \int_0^{T-t} \int_{B^+ \cup B^-} \int_{B^+ \cup B^-} K_{m,d}(|x-\xi|, T-t-\tau) f^{\circledast}(\xi,\tau) f^{\circledast}(x,t) d\xi dx d\tau dt = \sum_{i=1}^{4} U_i.$$

Moreover, u_+ and u_- are the first (normalised) eigenfunctions of each circular cylinder C^+ and C^-, that is, we have

$$\lambda_1(C^\pm) = \int_0^T \int_0^{T-t} \int_{B^\pm} \int_{B^\pm} u_\pm(x,t) u_\pm(\xi,\tau) K_{m,d}(|x-\xi|, T-t-\tau) d\xi dx d\tau dt.$$

Thus, we arrive at

$$\lambda_2(C^+ \cup C^-) \geq \frac{U_1 + U_2 + U_3 + U_4}{(\lambda_1(D^+))^{-1} U_1 + (\lambda_1(D^-))^{-1} U_2}.$$

Since for $x \in B^{\pm}$, $\xi \in B^{\mp}$ the kernel $K_{m,d}(|x-\xi|, T-t-\tau)$ tends to zero as $l \to \infty$, we notice that

$$\lim_{l \to 0} U_2 = \lim_{l \to 0} U_3 = 0.$$

That is, we get

$$\lambda_2(C^+ \bigcup C^-) \geqslant \max\{\lambda_1(C^+), \lambda_1(C^-)\}, \tag{5.137}$$

where $l = dist(C^+, C^-)$. On the other hand, from inequalities (5.135) and (5.137), we obtain

$$\begin{aligned}\lim_{l \to \infty} \lambda_2 \left(C^+ \bigcup C^-\right) &\geq \min\{\lambda_1(C^+), \lambda_1(C^-)\} = \lambda_1(C^+) = \lambda_1(C^-) \\ &\geq \lim_{l \to \infty} \lambda_2 \left(C^+ \cup C^-\right),\end{aligned}$$

and this yields that the maximising sequence for λ_2 corresponds to the union of two identical circular cylinders with mutual distance going to infinity. □

5.6.3 The case of triangular prisms

Let us now discuss the same question of maximising the Schatten p-norms in the class of triangular prisms with a given measure. That is, we look for the maximiser for Schatten p-norms of the heat potential operator $\mathscr{H}_\Omega$ in the class of triangular prisms with a given measure. According to the previous section, it is natural to conjecture that it is the equilateral triangular prism.

Theorem 5.25 *Let $\triangle$ be an equilateral triangle and assume that $P\mathscr{H}_\triangle \in S_q(L^2(\triangle))$ for some $q > 1$. Let Ω be any triangle with $|\Omega| = |\triangle|$. Then*

$$\|P\mathscr{H}_\Omega\|_p \leq \|P\mathscr{H}_\triangle\|_p \tag{5.138}$$

for any integer p such that $q \leq p < \infty$.

The proof of Theorem 5.25 is based on the same scheme as the proof of Theorem 5.22, with the only difference that now we use Steiner's symmetrization. Here we use the fact that by a sequence of Steiner symmetrizations with respect to the mediator of each side, a given triangle converges to an equilateral one. The rest of the proof is exactly the same as the proof of Theorem 5.22. □

We now present the following analogue of the ***Pólya theorem*** ([89]) for the integral operator $\mathscr{H}_\Omega$ for triangles Ω. This also says that the operator norm of $\mathscr{H}_\Omega$ is maximised on the equilateral triangular cylinder among all triangular cylinders of a given measure.

Theorem 5.26 *Let $\triangle$ be an equilateral triangle. The equilateral triangular cylinder $(0,T) \times \triangle$ is a maximiser of the first eigenvalue of the operator $P\mathscr{H}_\Omega$ among all prisms $(0,T) \times \Omega$ of a given measure, that is,*

$$0 < \lambda_1((0,T) \times \Omega) \leq \lambda_1((0,T) \times \triangle),$$

for any triangle $\Omega \subset \mathbb{R}^2$ with $|\Omega| = |\triangle|$.

Let us prove Theorem 5.26.

The first eigenvalue $\lambda_1(\Omega)$ of the operator $P\mathscr{H}_\Omega$ is positive and simple, moreover, the corresponding eigenfunction u_1 can be chosen positive in $(0,T)\times\Omega$. Since applying a sequence of Steiner symmetrizations with respect to the mediator of each side, a given triangle converges to an equilateral one, and $K_{m,d}(|x-y|,T-t-\tau)$ is a non-increasing function, we have

$$\int_0^T\int_0^{T-t}\int_0^{T-\tau}\int_E u_1(y,\tau_1)F(x,y,\xi;t,\tau,\tau_1)u_1(x,t)dz \\ \leq \int_0^T\int_0^{T-t}\int_0^{T-\tau}\int_{E_\triangle} u_1^\star(y,\tau_1)F(x,y,\xi;t,\tau,\tau_1)u_1^\star(x,t)dz, \quad (5.139)$$

where

$$F(x,y,\xi;t,\tau,\tau_1)=K_{m,d}(|y-\xi|,T-\tau-\tau_1)K_{m,d}(|\xi-x|,T-t-\tau)$$

and $dz=d\xi dydxd\tau_1 d\tau dt$. Thus, by (5.139) and the variational principle for the positive self-adjoint operator $(P\mathscr{H}_\triangle)^2$, we obtain

$$\begin{aligned}
&\lambda_1^2((0,T)\times\Omega)\\
&=\frac{\int_0^T\int_0^{T-t}\int_0^{T-\tau}\int_E u_1(y,\tau_1)F(x,y,\xi;t,\tau,\tau_1)u_1(x,t)dz}{\int_0^T\int_\Omega|u_1(x,t)|^2dxdt}\\
&\leq\frac{\int_0^T\int_0^{T-t}\int_0^{T-\tau}\int_{E_\triangle} u_1^\star(y,\tau_1)F(x,y,\xi;t,\tau,\tau_1)u_1^\star(x,t)dz}{\int_0^T\int_\triangle|u_1^\star(x,t)|^2dxdt}\\
&\leq\inf_{v\in L^2((0,T)\times\triangle)}\frac{\int_0^T\int_0^{T-t}\int_0^{T-\tau}\int_{E_\triangle} v(y,\tau_1)F(x,y,\xi;t,\tau,\tau_1)v(x,t)dz}{\int_0^T\int_\triangle|v(x,t)|^2dxdt}\\
&=\lambda_1^2((0,T)\times\triangle),
\end{aligned}$$

where

$$F(x,y,\xi;t,\tau,\tau_1)=K_{m,d}(|y-\xi|,T-\tau-\tau_1)K_{m,d}(|\xi-x|,T-t-\tau)$$

and $dz=d\xi dydxd\tau_1 d\tau dt$. Here we have used the fact that the Steiner symmetrization preserves the L^2-norm. Since $\lambda_1((0,T)\times\Omega)$ and $\lambda_1((0,T)\times\triangle)$ are positive we get

$$0<\lambda_1((0,T)\times\Omega)\leq\lambda_1((0,T)\times\triangle)$$

for any triangle $\Omega\subset\mathbb{R}^2$ with $|\Omega|=|\triangle|$. $\square$

5.7 Cauchy-Dirichlet heat operator

In this section we show that a circular cylinder is a minimiser of the first s-number of the Cauchy-Dirichlet heat operator among all cylindric domains of a given measure. It is a (non-self-adjoint) analogue of the Rayleigh-Faber-Krahn inequality for the Cauchy-Dirichlet heat operator. We also discuss analogues of the Hong-Krahn-Szegő and Payne-Pólya-Weinberger inequalities for the heat operator.

As mentioned in a previous section, the classical Rayleigh-Faber-Krahn inequality asserts that the first eigenvalue of the Laplacian with the Dirichlet boundary condition in $\mathbb{R}^d$, $d \geq 2$, is minimised in a ball among all domains of the same measure. However, the minimum of the second Dirichlet Laplacian eigenvalue is achieved by the union of two identical balls. This fact is called a Hong-Krahn-Szegő inequality. In the present section analogues of both inequalities are discussed for the heat operator.

Payne, Pólya and Weinberger ([87] and [88]) studied the ratio of $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}$ for the Dirichlet Laplacian and conjectured that the ratio of $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}$ is maximised in the disk among all domains of the same area. Later in [4], Ashbaugh and Benguria proved this conjecture for $\Omega \subset \mathbb{R}^d$. In the present section we also investigate the same ratio for s-numbers of the Cauchy-Dirichlet heat operator, and prove an analogue of a Payne-Pólya-Weinberger type inequality for the heat operator. The aim of this section is to extend these results for a non-self-adjoint operator. Thus, we discuss the following facts:

- Rayleigh-Faber-Krahn type inequality: the first s-number of the Cauchy-Dirichlet heat operator is minimised in the circular cylinder among all Euclidean cylindric domains of a given measure;
- Hong-Krahn-Szegő type inequality: the second s-number of the Cauchy-Dirichlet heat operator is minimized in the union of two identical circular cylinders among all Euclidean cylindric domains of a given measure;
- Payne-Pólya-Weinberger type inequality: the ratio $\frac{s_2}{s_1}$ is maximized in the circular cylinder;
- Ashbaugh-Benguria type inequality: the maximum of the ratio $\frac{s_4}{s_2}$ among cylindric bounded open sets with a given measure is achieved by the union of two identical cylinders.

5.7.1 Spectral geometric inequalities for the Cauchy-Dirichlet heat operator

Let $\Omega \subset \mathbb{R}^d$ be a simply-connected bounded set with smooth boundary $\partial\Omega$, and let $D = \Omega \times (0, T)$, $T > 0$, be a cylindrical domain. We consider the ***Cauchy-Dirichlet***

heat operator $\diamondsuit : L^2(D) \to L^2(D)$ in the form

$$\diamondsuit u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0,\ x \in \Omega, \\ u(x,t) = 0,\ x \in \partial\Omega,\ \forall t \in (0,T). \end{cases} \tag{5.140}$$

It is easy to check that the operator $\diamondsuit$ is a non-self-adjoint operator in $L^2(D)$ and the adjoint operator $\diamondsuit^*$ to the operator $\diamondsuit$ is

$$\diamondsuit^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,T) = 0,\ x \in \Omega, \\ v(x,t) = 0,\ x \in \partial\Omega,\ \forall t \in (0,T). \end{cases} \tag{5.141}$$

Moreover, a direct calculation yields that the operator $\diamondsuit^*\diamondsuit$ has the formula

$$\diamondsuit^*\diamondsuit u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0,\ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0,\ x \in \Omega, \\ u(x,t) = 0,\ x \in \partial\Omega,\ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0,\ x \in \partial\Omega,\ \forall t \in (0,T). \end{cases} \tag{5.142}$$

Let us introduce operators $M, L : L^2(\Omega) \to L^2(\Omega)$ by

$$Mz(x) = \begin{cases} -\Delta z(x), \\ z(x) = 0,\ x \in \partial\Omega. \end{cases} \tag{5.143}$$

and

$$Lz(x) = \begin{cases} \Delta^2 z(x), \\ z(x) = 0,\ x \in \partial\Omega, \\ \Delta z(x) = 0,\ x \in \partial\Omega. \end{cases} \tag{5.144}$$

Lemma 5.27 *The first eigenvalue of the operator L is minimised in the ball B among all domains Ω of the same measure with $|B| = |\Omega|$.*

Let us prove this lemma. Let μ be an eigenvalue of the operator M. A straightforward calculation from (5.143) gives that

$$\begin{cases} \Delta^2 z(x) = \mu^2 z(x), \\ z(x) = 0,\ x \in \partial\Omega, \\ \Delta z(x) = 0,\ x \in \partial\Omega. \end{cases} \tag{5.145}$$

Thus, $M^2 = L$ and $\mu^2 = \lambda$, where λ is an eigenvalue of the operator L. Now using the Rayleigh-Faber-Krahn inequality (Lemma 5.22) we have

$$\lambda_1(B) = \mu_1^2(B) \leq \mu_1^2(\Omega) = \lambda_1(\Omega),$$

i.e. $\lambda_1(B) \leq \lambda_1(\Omega)$. Here λ_1 and μ_1 are the first eigenvalues of L and M, respectively. $\Box$

Theorem 5.28 (Rayleigh-Faber-Krahn inequality for s-numbers) *Let $C = B \times (0,T)$ be a circular cylinder with $B \subset \mathbb{R}^d$ an open ball. The first s-number of the operator $\diamondsuit$ is minimised on the circular cylinder C among all cylindric domains of a given measure, that is,*

$$s_1(C) \leq s_1(D),$$

for all D with $|D| = |C|$, where $|D|$ is the Lebesgue measure of the domain D.

In other words, the statement of Theorem 5.28 says that the operator norm of the operator $\diamondsuit^{-1}$ is maximised on the circular cylinder C among all cylindric domains of a given measure, i.e.

$$\|\diamondsuit^{-1}\|_D \leq \|\diamondsuit^{-1}\|_C.$$

To prove this, recall that the symmetric rearrangement of $D = \Omega \times (0,T)$ is a circular cylinder $C = B \times (0,T)$ with the measure equal to the measure of D, i.e. $|D| = |C|$. Let u be a nonnegative measurable function in D. Then its cylindric symmetric decreasing rearrangement is

$$u^*(x,t) = \int_0^\infty \chi_{\{u(x,t)>z\}^*} dz, \ \forall t \in (0,T). \tag{5.146}$$

We consider the following spectral problem

$$\diamondsuit^*\diamondsuit u = su,$$

$$\diamondsuit^*\diamondsuit u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = su(x,t), \\ u(x,0) = 0, \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ x \in \Omega, \\ u(x,t) = 0, \ x \in \partial\Omega, \ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0, \ x \in \partial\Omega, \ \forall t \in (0,T). \end{cases} \tag{5.147}$$

Let us set $u(x,t) = X(x)\varphi(t)$, and take $u_1(x,t) = X_1(x)\varphi_1(t)$ as the first eigenfunction of the operator $\diamondsuit^*\diamondsuit$. Then we have

$$-\varphi_1''(t)X_1(x) + \varphi_1(t)\Delta^2 X_1(x) = s_1\varphi_1(t)X_1(x). \tag{5.148}$$

By using the variational principle for the self-adjoint compact positive operator $\diamondsuit^*\diamondsuit$, we obtain

$$s_1(D) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega (\Delta X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega (-\mu_1(\Omega)X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt\int_\Omega X_1^2(x)dx + \mu_1^2(\Omega)\int_0^T \varphi_1^2(t)dt\int_\Omega (X_1(x))^2dx}{\int_0^T \varphi_1^2(t)dt\int_\Omega X_1^2(x)dx},$$

where $\mu_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian.

For each nonnegative function $u \in L^2(D)$, we have

$$\int_\Omega |X_1(x)|^2dx = \int_B |X_1^*(x)|^2dx. \tag{5.149}$$

Combining Lemma 5.27 and (5.149), we calculate

$$\begin{aligned}
s_1(D) &= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt\int_\Omega (X_1(x))^2dx + \mu_1^2(\Omega)\int_0^T \varphi_1^2(t)dt\int_\Omega (X_1(x))^2dx}{\int_0^T \varphi_1^2(t)dt\int_\Omega (X_1(x))^2dx}\\
&\geq \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt\int_B (X_1^*(x))^2dx + \mu_1^2(B)\int_0^T \varphi_1^2(t)dt\int_B (X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt\int_B (X_1^*(x))^2dx}\\
&= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt\int_B (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt\int_B (-\mu_1(B)X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt\int_B (X_1^*(x))^2dx}\\
&= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt\int_B (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt\int_B (\Delta X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt\int_B (X_1^*(x))^2dx}\\
&= \frac{-\int_0^T\int_B u_1^*(x,t)\frac{\partial^2 u_1^*(x,t)}{\partial t^2}dxdt + \int_0^T\int_B u_1^*(x,t)\Delta_x^2 u_1^*(x,t)dxdt}{\int_0^T\int_B (u_1^*(x,t))^2dxdt}\\
&\geq \inf_{z(x,t)\neq 0} \frac{-\int_0^T\int_B z(x,t)\frac{\partial^2 z(x,t)}{\partial t^2}dxdt + \int_0^T\int_B z(x,t)\Delta_x^2 z(x,t)dxdt}{\int_0^T\int_B z^2(x,t)dxdt} = s_1(C).
\end{aligned}$$

The proof is complete. □

Theorem 5.29 (Hong-Krahn-Szegő inequality for s-numbers) *The second s-number of the operator $\diamondsuit$ is minimised on the union of two identical circular cylinders among all cylindric domains of the same measure.*

Let $D^+ = \{(x,t) : u(x,t) > 0\}$, $D^- = \{(x,t) : u(x,t) < 0\}$, and introduce the following notations

$$u_2^+(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^+,\\ 0, & \text{otherwise},\end{cases}$$

and

$$u_2^-(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^-,\\ 0, & \text{otherwise}.\end{cases}$$

We will need the following fact in the proof.

Lemma 5.30 *For the operator $\diamondsuit^*\diamondsuit$ we obtain the equalities*

$$s_1(D^+) = s_1(D^-) = s_2(D).$$

Let us prove this lemma. For the operator M we have the equality

$$\mu_1(\Omega^+) = \mu_1(\Omega^-) = \mu_2(\Omega). \tag{5.150}$$

Let us solve the spectral problem (5.147) by using Fourier's method in the domain $D^\pm$, so

$$\begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = s(D^\pm)u(x,t), \\ u(x,0) = 0,\ x \in \Omega^\pm, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0,\ x \in \Omega^\pm, \\ u(x,t) = 0,\ x \in \partial\Omega^\pm,\ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0,\ x \in \partial\Omega^\pm,\ \forall t \in (0,T). \end{cases} \tag{5.151}$$

Thus, we arrive at the spectral problems for $\varphi(t)$ and $X(x)$:

$$\begin{cases} \Delta^2 X(x) = \mu^2(\Omega^\pm)X(x),\ x \in \Omega^\pm, \\ X(x) = 0,\ x \in \partial\Omega^\pm, \\ \Delta X(x) = 0,\ x \in \partial\Omega^\pm, \end{cases} \tag{5.152}$$

and

$$\begin{cases} \varphi''(t) + (s(D^\pm) - \mu^2(\Omega^\pm))\varphi(t) = 0,\ t \in (0,T), \\ \varphi(0) = 0, \\ \varphi'(T) + \mu(\Omega^\pm)\varphi(T) = 0. \end{cases} \tag{5.153}$$

It also gives that

$$\tan\sqrt{s(D^\pm) - \mu^2(\Omega^\pm)}T = -\frac{\sqrt{s(D^\pm) - \mu^2(\Omega^\pm)}}{\mu(\Omega^\pm)}. \tag{5.154}$$

Now for the domains D and $D^\pm$ we have

$$\begin{cases} \tan\sqrt{s_1(D^+) - \mu_1^2(\Omega^+)}T = -\frac{\sqrt{s_1(D^+) - \mu_1^2(\Omega^+)}}{\mu_1(\Omega^+)}, \\ \tan\sqrt{s_1(D^-) - \mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_1(D^-) - \mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}, \\ \tan\sqrt{s_2(D) - \mu_2^2(\Omega)}T = -\frac{\sqrt{s_2(D) - \mu_2^2(\Omega)}}{\mu_2(\Omega)}. \end{cases}$$

By using (5.150) we establish that

$$\begin{cases} \tan\sqrt{s_1(D^+) - \mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_1(D^+) - \mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}, \\ \tan\sqrt{s_1(D^-) - \mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_1(D^-) - \mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}, \\ \tan\sqrt{s_2(D) - \mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_2(D) - \mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}. \end{cases}$$

Finally, we get

$$s_1(D^+) = s_1(D^-) = s_2(D). \qquad \square \tag{5.155}$$

Let us now prove Theorem 5.29.

We will use the notation for C and D from Theorem 5.28.

Let us state the spectral problem for the second s-number of the Cauchy-Dirichlet heat operator (that is, the second eigenvalue of (5.142)) in the circular cylinder C:

$$s_2(C)v_2(x,t) = -\frac{\partial^2 v_2(x,t)}{\partial t^2} + \Delta_x^2 v_2(x,t). \tag{5.156}$$

where $v_2(x,t)$ is the second eigenfunction of the operator $\diamondsuit^*\diamondsuit$ in the circular cylinder C.

Let $B = B^+ \cup B^-$. Then by multiplying (5.156) by $v_2^+(x,t)$ and integrating over $B^+ \times (0,T)$, we get

$$\begin{aligned}
s_2(C)\int_0^T\int_{B^+} v_2(x,t)v_2^+(x,t)dxdt &= s_2(C)\int_0^T\int_{B^+}(v_2^+(x,t))^2dxdt \\
&= -\int_0^T\int_{B^+} v_2^+(x,t)\frac{\partial^2 v_2(x,t)}{\partial t^2}dxdt + \int_0^T\int_{B^+} v_2^+(x,t)\Delta_x^2 v_2(x,t)dxdt \\
&= -\int_0^T\int_{B^+} v_2^+(x,t)\frac{\partial^2 v_2^+(x,t)}{\partial t^2}dxdt + \int_0^T\int_{B^+} v_2^+(x,t)\Delta_x^2 v_2^+(x,t)dxdt.
\end{aligned} \tag{5.157}$$

After this, we get

$$\begin{aligned}
s_2(C) &= \frac{-\int_0^T\int_{B^+} v_2^+(x,t)\frac{\partial^2 v_2^+(x,t)}{\partial t^2}dxdt + \int_0^T\int_{B^+} v_2^+(x,t)\Delta_x^2 v_2^+(x,t)dxdt}{\int_0^T\int_{B^+}(v_2^+(x,t))^2dxdt} \\
&\leq \sup_{z(x,t)\neq 0} \frac{-\int_0^T\int_{B^+} z(x,t)\frac{\partial^2 z(x,t)}{\partial t^2}dxdt + \int_0^T\int_{B^+} z(x,t)\Delta_x^2 z(x,t)dxdt}{\int_0^T\int_{B^+} z^2(x,t)dxdt} = s_1(C^+).
\end{aligned} \tag{5.158}$$

Similarly, if we multiply (5.156) by $v_2^-(x,t)$ and intergrate over $B^- \times (0,T)$, we have

$$\begin{cases} s_2(C) \leq s_1(C^+) \\ s_2(C) \leq s_1(C^-). \end{cases} \tag{5.159}$$

From the Rayleigh-Faber-Krahn inequality in Theorem 5.28, we obtain

$$\begin{cases} s_1(C^+) \leq s_1(D^+) \\ s_1(C^-) \leq s_1(D^-). \end{cases} \tag{5.160}$$

By using Lemma (5.30) we arrive at

$$s_2(C) \leq \min(s_1(C^+), s_1(C^-)) \leq s_1(D^+) = s_1(D^-) = s_2(D). \qquad \square$$

We now show an analogue of the Payne-Pólya-Weinberger inequality. We keep the notation for C and D from Theorem 5.28.

Theorem 5.31 *The ratio $\frac{s_2(D)}{s_1(D)}$ is maximised in the circular cylinder among all cylindric domains of the same measure, i.e.*

$$\frac{s_2(D)}{s_1(D)} \leq \frac{s_2(C)}{s_1(C)},$$

for all D with $|D| = |C|$.

To prove this theorem, let us restate the first and second s-numbers in the forms

$$\begin{aligned} s_1(D) &= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega \Delta^2 X_1(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx} \\ &= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \mu_1^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}. \end{aligned} \tag{5.161}$$

and

$$\begin{aligned} s_2(D) &= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega \Delta^2 X_2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx} \\ &= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \mu_2^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}, \end{aligned} \tag{5.162}$$

From [4] we have

$$\frac{\mu_2(\Omega)}{\mu_1(\Omega)} \leq \frac{\mu_2(B)}{\mu_1(B)}. \tag{5.163}$$

Using this and (5.133) we obtain

$$\begin{aligned} \frac{s_2(D)}{s_1(D)} &= \frac{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \mu_2^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}}{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \mu_1^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}} \\ &\leq \frac{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2dx + \mu_2^2(B)\int_0^T \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}}{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_1^*(x))^2dx + \mu_1^2(B)\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}} \end{aligned}$$

$$= \frac{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt\int_B (X_2^*(x))^2dx + \int_0^T \varphi_1^2(t)dt\int_B(\Delta X_2^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt\int_B (X_2^*(x))^2dx}}{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt\int_B (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt\int_B(\Delta X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt\int_B (X_1^*(x))^2dx}}$$

$$= \frac{\frac{-\int_0^T\int_B u_2^*(x,t)\frac{\partial^2 u_2^*(x,t)}{\partial t^2}dxdt + \int_0^T\int_B u_2^*(x,t)\Delta_x^2 u_2^*(x,t)dxdt}{\int_0^T\int_B (u_2^*(x,t))^2dxdt}}{\frac{-\int_0^T\int_B u_1^*(x,t)\frac{\partial^2 u_1^*(x,t)}{\partial t^2}dxdt + \int_0^T\int_B u_1^*(x,t)\Delta_x^2 u_1^*(x,t)dxdt}{\int_0^T\int_B (u_1^*(x,t))^2dxdt}} = \frac{s_2(C)}{s_1(C)}. \quad \square \tag{5.164}$$

Theorem 5.32 *The maximum of the ratio $\frac{s_4(D)}{s_2(D)}$ among cylindric bounded open sets with a given measure is achieved by the union of two identical circular cylinders.*

To see this, we recall the fact (see, e.g. [48]) that $\frac{\mu_4^2(\Omega)}{\mu_2^2(\Omega)} \leq \frac{\mu_4^2(B)}{\mu_2^2(B)}$. The rest of the proof is similar to the proof of Theorem 5.31. $\square$

5.7.2 The case of polygonal cylinders

The main motivation of the present section is the Polya inequality which asserts that the equilateral triangle is a minimiser of the first eigenvalue of the Dirichlet Laplacian among all triangles of a given area. In this section we prove Polya type inequalities for the Cauchy-Dirichlet heat operator in polygonal cylindric domains of a given measure. That is, in particular, we prove that the s_1-number of the Cauchy-Dirichlet heat operator is minimised on the equilateral triangular cylinder among all triangular cylinders of given measure, which means that the operator norm of the inverse operator is maximised on the equilateral triangular cylinder among all triangular cylinders of a given measure.

Let $D_1 = \Omega_1 \times (0,T)$ and $D_2 = \Omega_2 \times (0,T)$ be cylindrical domains, where $\Omega_1 \subset \mathbb{R}^2$ is a triangle and $\Omega_2 \subset \mathbb{R}^2$ is a quadrilateral. We denote an equilateral triangular cylinder by

$$C_\triangle := \triangle \times (0,T),$$

where $\triangle \subset \mathbb{R}^2$ is an equilateral triangle with $|\triangle| = |\Omega_1|$, and a quadratic cylinder

$$C_\square := \square \times (0,T),$$

where $\square \subset \mathbb{R}^2$ is a square with $|\square| = |\Omega_2|$. Here, as usual, $|\Omega|$ is the area of the domain Ω.

Let us introduce operators $T, L : L^2(\Omega) \to L^2(\Omega)$ by the formulae

$$Tz(x) := \begin{cases} -\Delta z(x) = \mu z(x), \\ z(x) = 0, \ x \in \partial\Omega. \end{cases} \tag{5.165}$$

and

$$Lz(x) := \begin{cases} \Delta^2 z(x) = \lambda z(x), \\ z(x) = 0, \ x \in \partial\Omega, \\ \Delta z(x) = 0, \ x \in \partial\Omega. \end{cases} \tag{5.166}$$

Lemma 5.33 *The first eigenvalue of the operator L is minimised on the equilateral triangle (square) among all triangles (quadrilaterals) with $|\triangle| = |\Omega|$ ($|\square| = |\Omega|$).*

Let us prove this lemma. The Pólya theorem ([89]) for the operator T says that the equilateral triangle (square) is a minimiser of the first Dirichlet Laplacian eigenvalue among all triangles (quadrilaterals) Ω_1 (Ω_2) of the same area with $|\triangle| = |\Omega_1|(|\square| = |\Omega_2|)$. Let us calculate T^2 from (5.165),

$$T^2 z(x) = \begin{cases} \Delta^2 z(x) = \mu^2 z(x), \\ z(x) = 0, \ x \in \partial\Omega, \\ \Delta z(x) = 0, \ x \in \partial\Omega. \end{cases} \tag{5.167}$$

That is, $T^2 = L$ and $\mu^2 = \lambda$. Thus, we obtain that

$$\lambda_1(\triangle) = \mu_1^2(\triangle) \le \mu_1^2(\Omega_1) = \lambda_1(\Omega_1)$$

and

$$\lambda_1(\square) = \mu_1^2(\square) \le \mu_1^2(\Omega_2) = \lambda_1(\Omega_2).$$

Then, we arrive at $\lambda_1(\triangle) \le \lambda_1(\Omega_1)$ and $\lambda_1(\square) \le \lambda_1(\Omega_2)$. $\square$

Theorem 5.34 *The s_1-number of the operator $\diamondsuit$ is minimised on the equilateral triangular cylinder among all triangular cylinders of given measure, that is,*

$$s_1(C_\triangle) \le s_1(D_1),$$

with $|D_1| = |C_\triangle|$.

To prove this theorem, consider the following spectral problem

$$\diamondsuit^* \diamondsuit u = su,$$

$$\diamondsuit^* \diamondsuit u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = su(x,t), \\ u(x,0) = 0, \ x \in \Omega_1, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ x \in \Omega_1, \\ u(x,t) = 0, \ x \in \partial\Omega_1, \ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0, \ x \in \partial\Omega_1, \ \forall t \in (0,T). \end{cases} \tag{5.168}$$

The domain $D_1 = \{(x,t) | x \in \Omega_1 \subset \mathbb{R}^2, \ t \in (0,T)\}$ is a cylindrical domain and we can have $u(x,t) = X(x)\varphi(t)$, so that $u_1(x,t) = X_1(x)\varphi_1(t)$ is the first eigenfunction of the operator $\diamondsuit^*\diamondsuit$. We can also restate this fact (5.168) as

$$-\varphi_1''(t) X_1(x) + \varphi_1(t) \Delta^2 X_1(x) = s_1 \varphi_1(t) X_1(x). \tag{5.169}$$

By the variational principle for the operator $\diamondsuit^*\diamondsuit$ and after a straightforward calculation, we obtain

$$s_1(D_1) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_{\Omega_1} (\Delta X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_{\Omega_1}(-\mu_1(\Omega_1)X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \mu_1^2(\Omega_1)\int_0^T \varphi_1^2(t)dt \int_{\Omega_1}(X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx},$$

where $\mu_1(\Omega_1)$ is the first eigenvalue of the operator Dirichlet Laplacian.

For each nonnegative function $X \in L^2(\Omega_1)$ we have

$$\int_{\Omega_1} |X_1(x)|^2 dx = \int_{\triangle} |X_1^\star(x)|^2 dx, \tag{5.170}$$

where $X^\star$ is the Steiner symmetrization of the function X. By Lemma 5.33 and (5.170) we obtain

$$s_1(D_1) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \mu_1^2(\Omega_1)\int_0^T \varphi_1^2(t)dt \int_{\Omega_1}(X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx}$$

$$\geq \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\triangle}(X_1^\star(x))^2 dx + \mu_1^2(\triangle)\int_0^T \varphi_1^2(t)dt \int_{\triangle}(X_1^\star(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\triangle}(X_1^\star(x))^2 dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\triangle}(X_1^\star(x))^2 dx + \int_0^T \varphi_1^2(t)dt \int_{\triangle}(-\mu_1(\triangle)X_1^\star(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\triangle}(X_1^\star(x))^2 dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\triangle}(X_1^\star(x))^2 dx + \int_0^T \varphi_1^2(t)dt \int_{\triangle}(\Delta X_1^\star(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\triangle}(X_1^\star(x))^2 dx}$$

$$= \frac{-\int_0^T \int_{\triangle} \frac{\partial^2 u_1^\star(x,t)}{\partial t^2} dxdt + \int_0^T \int_{\triangle}(\Delta_x^2 u_1^\star(x,t))^2 dxdt}{\int_0^T \int_{\triangle}(u_1^\star(x,t))^2 dxdt}$$

$$\geq \inf_{z(x,t)\neq 0} \frac{-\int_0^T \int_{\triangle} z_t(x,t)z(x,t)dxdt + \int_0^T \int_{\triangle}(\Delta z(x,t))^2 dxdt}{\int_0^T \int_{\triangle} z^2(x,t)dxdt} = s_1(C_\triangle).$$

The proof is complete. $\Box$

Theorem 5.35 *The s_1-number of the operator $\diamondsuit$ is minimised on the quadratic cylinder among all quadrangular cylinders of a given measure, i.e.*

$$s_1(C_\Box) \leq s_1(D_2),$$

with $|D_2| = |C_\Box|$.

The proof of this theorem is similar to the proof of Theorem 5.34.

5.8 Cauchy-Robin heat operator

In this section we consider the heat operator with the Robin boundary condition.

The techniques developed in the previous section allow us to show that the first s-number of the Cauchy-Robin heat operator is minimised on a circular cylinder among all cylindric (Lipschitz) domains of a given measure. In addition, following the same idea from the previous section we see that the second s-number is minimised on the disjoint union of two identical circular cylinders among all cylindric (Lipschitz) domains of the same measure.

The *Bossel-Daners inequality* (see [21] and [31]) shows that the first eigenvalue of the Laplacian with the Robin boundary condition is minimised on a ball among all Lipschitz domains (in $\mathbb{R}^d$, $d \geq 2$) of the given measure. However, as in the case of the Dirichlet Laplacian, among all domains of the given measure, the minimiser of the second eigenvalue of the Robin Laplacian on a bounded Lipschitz domain consists of the disjoint union of two balls. In the present section analogues of both inequalities are discussed for the heat operator. That is, we show that the first s-number of the Cauchy-Robin heat operator is minimised on the circular cylinder among all cylindric (Lipschitz) domains of a given measure, and its second s-number is minimised on the set consisting of the disjoint union of two identical circular cylinders among all cylindric (Lipschitz) domains of a given measure.

These isoperimetric inequalities have been mainly studied for the operators related to the Laplacian, for instance, for the p-Laplacians and bi-Laplacians. However, there are also many papers on this subject for other types of compact operators. For instance, at the beginning of this chapter we discussed results for self-adjoint operators. So the goal of these sections is to further extend those known isoperimetric inequalities to non-self-adjoint operators, namely in this case, for the heat operator (see, e.g. [64], which we follow in our exposition in this part).

Thus, in this section we discuss the following facts:

- The first s-number of the Cauchy-Robin heat operator is minimised on the circular cylinder among all cylindric (Lipschitz) domains of a given measure;
- The minimiser of the second s-number of the Cauchy-Robin heat operator among cylindric bounded open (Lipschitz) sets with a given measure consists of the disjoint union of two identical circular cylinders.

5.8.1 Isoperimetric inequalities for the first s-number

Let $\Omega \subset \mathbb{R}^d$ be a simply-connected set with (piecewise) smooth boundary $\partial\Omega$ and $D = \Omega \times (0,T)$ be a cylindrical domain. We consider the ***heat operator for the***

Cauchy-Robin problem $\diamondsuit_R : L^2(D) \to L^2(D)$ in the form

$$\diamondsuit_R u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \ (x,t) \in D, \\ u(x,0) = 0, \ x \in \Omega, \\ \alpha u(x,t) + \frac{\partial u(x,t)}{\partial n} = 0, \ x \in \partial\Omega, \ \forall t \in (0,T), \ \alpha > 0. \end{cases} \tag{5.171}$$

It is simple to check that the operator $\diamondsuit_R$ is a non-self-adjoint operator in $L^2(D)$, and the adjoint operator $\diamondsuit_R^*$ to the operator $\diamondsuit_R$ is

$$\diamondsuit_R^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \ (x,t) \in D, \\ v(x,T) = 0, \ x \in \Omega, \\ \alpha v(x,t) + \frac{\partial v(x,t)}{\partial n} = 0, \ x \in \partial\Omega, \ \forall t \in (0,T), \ \alpha > 0. \end{cases} \tag{5.172}$$

A direct calculation gives that the operator $\diamondsuit_R^* \diamondsuit_R$ has the formula

$$\diamondsuit_R^* \diamondsuit_R u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \ (x,t) \in D, \\ u(x,0) = 0, \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ x \in \Omega, \\ \alpha u(x,t) + \frac{\partial u(x,t)}{\partial n} = 0, \ x \in \partial\Omega, \ \forall t \in (0,T), \\ \alpha \Delta_x u(x,t) + \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ x \in \partial\Omega, \ \forall t \in (0,T), \ \alpha > 0. \end{cases}$$

We consider a (circular) cylinder $C = B \times (0,T)$, where $B \subset \mathbb{R}^d$ is an open ball. We will denote by $|\Omega|$ the Lebesgue measure of the set Ω.

Let us introduce operators T_R and L_R in $L^2(\Omega)$ by the formulae

$$T_R z(x) := \begin{cases} -\Delta z(x) = \mu z(x), \ x \in \Omega, \\ \alpha z(x) + \frac{\partial z(x)}{\partial n} = 0, \ x \in \partial\Omega, \ \alpha > 0, \end{cases} \tag{5.173}$$

and

$$L_R z(x) := \begin{cases} \Delta^2 z(x) = \lambda z(x), \ x \in \Omega, \\ \alpha z(x) + \frac{\partial z(x)}{\partial n} = 0, \ x \in \partial\Omega, \\ \alpha \Delta z(x) + \frac{\partial \Delta z(x)}{\partial n} = 0, \ x \in \partial\Omega, \ \alpha > 0. \end{cases} \tag{5.174}$$

Lemma 5.36 *The first eigenvalue of the operator L_R is minimised on the ball B among all Lipschitz domains Ω of the same measure with $|B| = |\Omega|$.*

Let us prove this statement. The Bossel-Daners (see [21] and [31]) inequality is valid for the Robin Laplacian, that is, the ball is a minimiser of the first eigenvalue of the operator T among all Lipschitz domains Ω with $|B| = |\Omega|$. A straightforward calculation from (5.173) gives that

$$T_R^2 z(x) = \begin{cases} \Delta^2 z(x) = \mu^2 z(x), \ x \in \Omega, \\ z(x) = 0, \ x \in \partial\Omega, \\ \alpha \Delta z(x) + \frac{\partial \Delta z(x)}{\partial n} = 0, \ x \in \partial\Omega, \ \alpha > 0. \end{cases} \tag{5.175}$$

Thus, $T_R^2 = L_R$ and $\mu^2 = \lambda$. Now using the Bossel-Daners inequality we have

$$\lambda_1(B) = \mu_1^2(B) \le \mu_1^2(\Omega) = \lambda_1(\Omega),$$

i.e. $\lambda_1(B) \le \lambda_1(\Omega)$. $\square$

We keep the notation for C and D from Theorem 5.28.

Theorem 5.37 *The first s-number of the operator $\diamondsuit_R$ is minimised on the circular cylinder C among all cylindric Lipschitz domains D of a given measure, that is,*

$$s_1^R(C) \le s_1^R(D),$$

for all D with $|D| = |C|$.

Let us prove this theorem. Recall that $D = \Omega \times (0,T)$ is a bounded measurable set in $\mathbb{R}^{d+1}$. Its symmetric rearrangement $C = B \times (0,T)$ is the circular cylinder with the measure equal to the measure of D, i.e. $|D| = |C|$. Consider the spectral problem

$$\diamondsuit_R^* \diamondsuit_R u = s^R u,$$

$$\diamondsuit_R^* \diamondsuit_R u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = s^R u(x,t), \ \ (x,t) \in D, \\ u(x,0) = 0, \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ x \in \Omega, \\ \alpha u(x,t) + \frac{\partial u(x,t)}{\partial n} = 0, \ x \in \partial\Omega, \ \forall t \in (0,T), \\ \alpha \Delta_x u(x,t) + \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ x \in \partial\Omega, \ \forall t \in (0,T), \ \alpha > 0. \end{cases}$$

Since the domain D is cylindrical we can seek $u(x,t) = X(x)\varphi(t)$ and $u_1(x,t) = X_1(x)\varphi_1(t)$ for the first eigenfunction of the operator $\diamondsuit_R^* \diamondsuit_R$. Thus, we have

$$-\varphi_1''(t) X_1(x) + \varphi_1(t) \Delta^2 X_1(x) = s_1^R \varphi_1(t) X_1(x).$$

By the variational principle for the self-adjoint compact positive operator $\diamondsuit^* \diamondsuit$, we get

$$s_1^R(D) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega (\Delta X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega (-\mu_1(\Omega) X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \mu_1^2(\Omega) \int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx},$$

where $\mu_1(\Omega)$ is the first eigenvalue of the Robin Laplacian.

For each nonnegative function $f \in L^2(D)$ we have

$$\int_\Omega |f(x,t)|^2 dx = \int_B |f^*(x,t)|^2 dx, \ \forall t \in (0,T), \tag{5.176}$$

where as usual f^* is the (cylindric) symmetric decreasing rearrangement of the function f. Combining Lemma 5.36 and (5.176), we obtain

$$s_1^R(D) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega (X_1(x))^2dx + \mu_1^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega (X_1(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega (X_1(x))^2dx}$$

$$\geq \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_1^*(x))^2dx + \mu_1^2(B)\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt \int_B X_1^*(x)(\mu_1^2(B)X_1^*(x))dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt \int_B X_1^*(x)\Delta^2 X_1^*(x)dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}$$

$$= \frac{-\int_0^T \int_B u_1^*(x,t)\frac{\partial^2 u_1^*(x,t)}{\partial t^2}dxdt + \int_0^T \int_B u_1^*(x,t)\Delta_x^2 u_1^*(x,t)dxdt}{\int_0^T \int_B (u_1^*(x,t))^2dxdt}$$

$$\geq \inf_{z(x,t)\neq 0} \frac{-\int_0^T \int_B z(x,t)\frac{\partial^2 z(x,t)}{\partial t^2}dxdt + \int_0^T \int_B z(x,t)\Delta_x^2 z(x,t)dxdt}{\int_0^T \int_B z^2(x,t)dxdt} = s_1^R(C).$$

The proof is complete. $\square$

5.8.2 Isoperimetric inequalities for the second s-number

A direct consequence of Theorem 5.37 is that the operator norm of the operator $\diamondsuit_R^{-1}$ is maximised in the circular cylinder C among all cylindric domains of a given measure, i.e.

$$\|\diamondsuit_R^{-1}\|_D \leq \|\diamondsuit_R^{-1}\|_C.$$

Theorem 5.38 *The second s-number of the operator $\diamondsuit_R$ is minimised on the disjoint union of two identical circular cylinders among all cylindric Lipschitz domains of the same measure.*

Let us prove this theorem. First we solve the following initial boundary value problem by the Fourier method:

$$\begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = s^R(D)u(x,t), \ (x,t)\in D, \\ u(x,0) = 0, \ x\in\Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ x\in\Omega, \\ \alpha u(x,t) + \frac{\partial u(x,t)}{\partial n} = 0, \ x\in\partial\Omega, \ \forall t\in(0,T), \\ \alpha\Delta_x u(x,t) + \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ x\in\partial\Omega, \ \forall t\in(0,T), \alpha>0. \end{cases}$$

So we arrive at the spectral problems for $\varphi(t)$ and $X(x)$, that is,

$$\begin{cases} \Delta^2 X(x) = \mu^2(\Omega) X(x), \ x \in \Omega, \\ \alpha X(x) + \frac{\partial X(x)}{\partial n} = 0, \ x \in \partial\Omega, \\ \alpha \Delta X(x) + \frac{\partial \Delta X(x)}{\partial n} = 0, \ x \in \partial\Omega, \ \alpha > 0, \end{cases}$$

and

$$\begin{cases} \varphi^{''}(t) + (s^R(D) - \mu^2(\Omega))\varphi(t) = 0, \, t \in (0,T), \\ \varphi(0) = 0, \\ \varphi^{'}(T) + \mu(\Omega)\varphi(T) = 0. \end{cases}$$

It also gives that

$$\tan\sqrt{s^R(D) - \mu^2(\Omega)}T = -\frac{\sqrt{s^R(D) - \mu^2(\Omega)}}{\mu(\Omega)}.$$

We have $s_2^R(D) = s_2^R(\mu)$, and for the second s-number we get

$$\tan\sqrt{s_2^R(\mu_i) - \mu_i^2}T = -\frac{\sqrt{s_2^R(\mu_i) - \mu_i^2}}{\mu_i}.$$

Thus, we have

$$(s_2^R(\mu_i))' = \frac{2s_2^R(\mu_i)\cos^2\sqrt{s_2^R - \mu_i^2}T + 2\mu_i^3 T}{\mu_i^2 T + \mu_i \cos^2\sqrt{s_2^R - \mu_i^2}T} \quad i = 1, 2, \dots .$$

These s-numbers and μ_i are positive, so

$$(s_2^R(\mu_i))' > 0 \ \ i = 1, 2, \dots .$$

It means that the function $s_2^R(\mu_i)$ is monotonically increasing. From [66] we have $\mu_2(\Omega) \geq \mu_2(B)$, and thus we arrive at

$$s_2^R(D) \geq s_2^R(C). \qquad \square$$

5.9 Cauchy-Neumann and Cauchy-Dirichlet-Neumann heat operators

In [117], Siudeja proved certain (isoperimetric) eigenvalue inequalities for the mixed Dirichlet-Neumann Laplacian operator in the right and equilateral triangles. As many other isoperimetric inequalities these inequalities have physical interpretation. In fact, the eigenvalues can be related to the (time dependent) survival probability of the Brownian motion on a triangle, dying on the Dirichlet boundary, and reflecting on the Neumann part. Thus, it is clear that enlarging the Dirichlet part leads to a shorter survival time. Moreover, having the Dirichlet condition on one long side gives a larger chance of dying, than having a shorter Dirichlet side. In the present section we extend such properties to the mixed Cauchy-Dirichlet-Neumann heat operator in the right and equilateral triangular cylinders (prisms).

In [71], Laugersen and Siudeja proved that the first nonzero Neumann Laplacian eigenvalue is maximised on the equilateral triangle among all triangles of given area. Below we also present a version of such inequality for the Cauchy-Neumann heat operator on the triangular cylinders.

One of classical inequalities in this direction is the Szegő-Weinberger inequality (see, e.g. [121], and [132]) which shows that the first nonzero eigenvalue of the Laplacian with the Neumann boundary condition is maximised in a ball among all Lipschitz domains in $\mathbb{R}^d$, $d \geq 2$, of the same measure. In this section an analogue of the Szegő-Weinberger inequality is also presented for the heat operator. That is, we prove that the second s-number of the Cauchy-Neumann heat operator is maximised on the circular cylinder among all (Euclidean) cylindric Lipschitz domains of a given volume.

Spectral isoperimetric inequalities have been mainly studied for the Laplacian related operators, for instance, for the p-Laplacians and bi-Laplacians. However, there are also many recent works on this subject for other types of compact operators. All these works were for self-adjoint operators. Here our main interest is to describe the extensions of the known isoperimetric inequalities to the case of non-self-adjoint operators. Summarising main results of the present section, we prove the following facts:

- The second s-number of the Cauchy-Neumann heat operator is maximised on the circular cylinder among all (Euclidean) cylindric Lipschitz domains of a given volume.

- We show the s-numbers inequalities for the (mixed) Cauchy-Dirichlet-Neumann heat operator in the right and equilateral triangular cylinders.

- The second s-number of the Cauchy-Neumann heat operator is maximised on the equilateral triangular cylinder among all triangular cylinders with a given volume.

5.9.1 Basic properties

Let $D = \Omega \times (0,1)$ be a triangular cylindrical domain, where $\Omega \subset \mathbb{R}^2$ is a triangle. We consider the ***Cauchy-Dirichlet and Cauchy-Neumann heat operators*** $\diamondsuit_D$, $\diamondsuit_N$: $L^2(D) \to L^2(D)$ respectively, given by the formulae

$$\diamondsuit_D u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0, \ x \in \Omega, \\ u(x,t) = 0, \ x \in \partial\Omega, \ \forall t \in (0,1), \end{cases} \tag{5.177}$$

and

$$\diamondsuit_N u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0,\ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ \forall t \in (0,1). \end{cases} \tag{5.178}$$

Here $\partial\Omega$ is the boundary of Ω and $\frac{\partial}{\partial n}$ is the normal derivative on the boundary. The operators $\diamondsuit_D$ and $\diamondsuit_N$ are compact, but these are non-self-adjoint operators in $L^2(D)$. The adjoint operators $\diamondsuit_D^*$ and $\diamondsuit_N^*$ to the operators $\diamondsuit_D$ and $\diamondsuit_N$ can be written as

$$\diamondsuit_D^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,1) = 0,\ x \in \Omega, \\ v(x,t) = 0,\ x \in \partial\Omega,\ \forall t \in (0,1), \end{cases} \tag{5.179}$$

and

$$\diamondsuit_N^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,1) = 0,\ x \in \Omega, \\ \frac{\partial \Delta_x v(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ \forall t \in (0,1). \end{cases} \tag{5.180}$$

A direct calculation gives that the operators $\diamondsuit_D^*\diamondsuit_D$ and $\diamondsuit_N^*\diamondsuit_N$ can be written in the forms

$$\diamondsuit_D^*\diamondsuit_D u(x,t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0,\ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0,\ x \in \Omega, \\ u(x,t) = 0,\ x \in \partial\Omega,\ \forall t \in (0,1), \\ \Delta_x u(x,t) = 0,\ x \in \partial\Omega,\ \forall t \in (0,1), \end{cases} \tag{5.181}$$

and

$$\diamondsuit_N^*\diamondsuit_N u(x,t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0,\ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0,\ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ \forall t \in (0,1). \end{cases} \tag{5.182}$$

Let $D_\triangle = \triangle \times (0,1)$ be a cylindrical domain, where $\triangle \subset \mathbb{R}^2$ is a right triangle with the sides of length $L \geq M \geq S$ (that is, with the boundary $\partial\triangle = \{L,M,S\}$). We also denote by L, M, S the sides of the right triangle with respect to their lengths (cf. [117]). We consider the Cauchy-Dirichlet-Neumann heat operator $\diamondsuit_\triangle : L^2(D_\triangle) \to L^2(D_\triangle)$ in the form

$$\diamondsuit_\triangle u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0,\ x \in \triangle, \\ u(x,t) = 0,\ x \in D \subset \{L,M,S\},\ \forall t \in (0,1) \\ \frac{\partial u(x,t)}{\partial n} = 0,\ x \in \partial\triangle \backslash D,\ \forall t \in (0,1). \end{cases} \tag{5.183}$$

Here $D \in \{L, M, S\}$ means that D is one of the sides where we set the Dirichlet condition. Its adjoint operator $\diamondsuit^*_\triangle$ can be written as

$$\diamondsuit^*_\triangle v(x,t) := \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ u(x,1) = 0, \ x \in \triangle, \\ \Delta v(x,t) = 0, \ x \in D \subset \{L,M,S\}, \ \forall t \in (0,1) \\ \frac{\partial \Delta v(x,t)}{\partial n} = 0, \ x \in \partial\triangle \backslash D, \ \forall t \in (0,1). \end{cases} \tag{5.184}$$

A direct calculation gives that the operator $\diamondsuit^*_\triangle \diamondsuit_\triangle$ has the following formula

$$\diamondsuit^*_\triangle \diamondsuit_\triangle u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \ x \in \triangle, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \ x \in \triangle, \\ u(x,t) = 0, \ x \in D \subset \{L,M,S\}, \ \forall t \in (0,1), \\ \frac{\partial u(x,t)}{\partial n} = 0, \ x \in \partial\triangle \backslash D, \ \forall t \in (0,1), \\ \Delta_x u(x,t) = 0, \ x \in D \subset \{L,M,S\}, \ \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ x \in \partial\triangle \backslash D, \ \forall t \in (0,1). \end{cases} \tag{5.185}$$

Let s_1^N and s_2^N be the first and second s-numbers of the Cauchy-Neumann problem, respectively. Let s_1^{side} be the first s-number of the spectral problem with the Dirichlet condition to this side. That is, s_1^{SL} would correspond to the Dirichlet conditions imposed on the shortest and longest sides. Let s_1^D be the first s-number of the Cauchy-Dirichlet heat operator. We will be using these notations in the subsections that follow.

5.9.2 On the Szegő-Weinberger type inequality

Let Ω be a simply-connected Lipschitz set with smooth boundary $\partial\Omega$ with $|B| = |\Omega|$, where $|\Omega|$ is the Lebesgue measure of the domain Ω.

Let us introduce the operators $T, L : L^2(\Omega) \to L^2(\Omega)$, respectively, by

$$Tz(x) := \begin{cases} -\Delta z(x) = \mu z(x), \\ \frac{\partial z(x)}{\partial n} = 0, \ x \in \partial\Omega, \end{cases} \tag{5.186}$$

and

$$Lz(x) := \begin{cases} \Delta^2 z(x) = \lambda z(x), \\ \frac{\partial z(x)}{\partial n} = 0, \ x \in \partial\Omega, \\ \frac{\partial \Delta z(x)}{\partial n} = 0, \ x \in \partial\Omega. \end{cases} \tag{5.187}$$

Lemma 5.39 *The second eigenvalue of the operator L is maximised on the ball B among all Lipschitz domains Ω of the same measure with $|\Omega| = |B|$.*

Let us prove this lemma. The Szegő-Weinberger inequality is valid for the Neumann Laplacian, that is, the ball is a maximiser of the second eigenvalue of the operator T among all Lipschitz domains Ω with $|B| = |\Omega|$. A straightforward calculation from (5.186) gives that

$$T^2 z(x) = \begin{cases} \Delta^2 z(x) = \mu^2 z(x), \\ z(x) = 0, \ x \in \partial\Omega, \\ \frac{\partial \Delta z(x)}{\partial n} = 0, \ x \in \partial\Omega. \end{cases}$$

Thus, $T^2 = L$ and $\mu^2 = \lambda$. Now using the Szegő-Weinberger inequality, we see that

$$\lambda_2(B) = \mu_2^2(B) \geq \mu_2^2(\Omega) = \lambda_2(\Omega),$$

that is, $\lambda_2(B) \geq \lambda_2(\Omega)$. $\square$

Let $D = \Omega \times (0,1)$ be a cylindrical domain, where $\Omega \subset \mathbb{R}^d$ is a simply-connected Lipschitz set with smooth boundary $\partial\Omega$. We consider the heat operator with the Cauchy-Neumann conditions, $\diamondsuit : L^2(D) \to L^2(D)$. We also denote by $C = B \times (0,1)$ a circular cylinder, where $B \subset \mathbb{R}^d$ is an open ball.

Theorem 5.40 *The second s-number of the operator $\diamondsuit$ is maximised on the circular cylinder C among all cylindric Lipschitz domains of a given measure, that is,*

$$s_2^N(C) \geq s_2^N(D),$$

for all D with $|D| = |C|$.

Let us prove this theorem. Consider the spectral problem

$$\diamondsuit_N^* \diamondsuit_N u = su,$$

$$\diamondsuit_N^* \diamondsuit_N u(x,t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = s^N u(x,t), \\ u(x,0) = 0, \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0, \ x \in \partial\Omega, \ \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ x \in \partial\Omega. \end{cases} \tag{5.188}$$

We can set $u(x,t) = X(x)\varphi(t)$, with $u_2(x,t) = X_2(x)\varphi_1(t)$ the second eigenfunction of the operator $\diamondsuit_N^* \diamondsuit_N$, where $\varphi_1(t)$ and $X_2(x)$ are the first and second eigenfunctions with respect to variables t and x. Consequently, we have

$$-\varphi_1^{''}(t) X_2(x) + \varphi_1(t) \Delta^2 X_2(x) = s_2^N \varphi_1(t) X_2(x). \tag{5.189}$$

Now by the variational principle for the self-adjoint compact positive operator $\diamondsuit_N^* \diamondsuit_N$, we get

$$s_2^N(D) = \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \int_0^1 \varphi_1^2(t)dt \int_\Omega X_2(x)\Delta^2 X_2(x)dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \int_0^1 \varphi_1^2(t)dt \int_\Omega \lambda_2(\Omega)(X_2(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}$$

$$= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \lambda_2(\Omega)\int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx},$$

where $\lambda_2(\Omega)$ is the second eigenvalue of the operator L. For each nonnegative function $X \in L^2(\Omega)$ we have

$$\int_\Omega |X_1(x)|^2 dx = \int_B |X_1^*(x)|^2 dx, \text{ with } |\Omega| = |B|, \tag{5.190}$$

where X^* is the symmetric decreasing rearrangement of X.

By applying Lemma 5.39 and (5.190), we get

$$s_2^N(D) = \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_\Omega (X_2(x))^2 dx + \lambda_2(\Omega)\int_0^1 \varphi_2^2(t)dt \int_\Omega (X_2(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega (X_2(x))^2 dx}$$

$$\leq \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2 dx + \lambda_2(B)\int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2 dx}$$

$$= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2 dx + \int_0^1 \varphi_1^2(t)dt \int_B X_2^*(x)(\lambda_2(B)X_2^*(x))dx}{\int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2 dx}$$

$$= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2 dx + \int_0^1 \varphi_1^2(t)dt \int_B X_2^*(x)\Delta^2 X_2^*(x)dx}{\int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2 dx}$$

$$= \frac{-\int_0^1 \int_B u_2^*(x,t)\frac{\partial^2 u_2^*(x,t)}{\partial t^2}dxdt + \int_0^1 \int_B u_2^*(x,t)\Delta_x^2 u_2^*(x,t)dxdt}{\int_0^1 \int_B (u_2^*(x,t))^2 dxdt}$$

$$\leq \sup_{\nu(x,t)\neq 0} \frac{-\int_0^1 \int_B \nu(x,t)\frac{\partial^2 \nu(x,t)}{\partial t^2}dxdt + \int_0^1 \int_B \nu(x,t)\Delta_x^2 \nu(x,t)dxdt}{\int_0^1 \int_B \nu^2(x,t)dxdt} = s_2^N(C).$$

The proof is complete. □

5.9.3 Inequalities for the Cauchy-Dirichlet-Neumann operator

Let us consider the eigenvalue problems for the operators $T_\triangle, L_\triangle : L^2(\triangle) \to L^2(\triangle)$, respectively, defined by

$$T_\triangle z(x) := \begin{cases} -\Delta z(x) = \beta z(x), \\ z(x) = 0, \ x \in D \subset \{L,M,S\}, \\ \frac{\partial z(x)}{\partial n} = 0, \ \partial\triangle \backslash D, \end{cases} \tag{5.191}$$

and

$$L_\triangle z(x) := \begin{cases} \Delta^2 z(x) = \eta z(x), \\ z(x) = 0, \ x \in D \subset \{L,M,S\}, \\ \frac{\partial z(x)}{\partial n} = 0, \ x \in \partial\triangle \backslash D, \\ \Delta z(x) = 0, \ x \in D \subset \{L,M,S\}, \\ \frac{\partial \Delta z(x)}{\partial n} = 0, \ x \in \partial\triangle \backslash D. \end{cases} \tag{5.192}$$

Lemma 5.41 *For the operator $L_\triangle$ and any right triangle $\triangle$ with the smallest angle α with $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$, we have the eigenvalue inequalities*

$$0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_2^N < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1^D. \tag{5.193}$$

When $\alpha = \frac{\pi}{6}$, $\eta_1^M = \eta_2^N$, and for $\alpha = \frac{\pi}{4}$ (right isosceles triangle) we have $S = M$ and $\eta_2^N < \eta_1^L$. All other inequalities stay sharp in these cases. For an arbitrary triangle we have

$$\min\{\eta_1^S, \eta_1^M, \eta_1^L\} < \eta_2^N \leq \eta_1^{SM} \leq \eta_1^{SL} \leq \eta_1^{ML}, \tag{5.194}$$

for any lengths of sides. However, it is possible that $\eta_2^N > \eta_1^L$ (for any small perturbation of the equilateral triangle) or $\eta_2^N < \eta_1^M$ (for the right triangle with $\alpha < \frac{\pi}{6}$).

Let us prove this lemma. It is easy to see that $L_\triangle = T_\triangle^2$. It means that $\eta = \beta^2$. From [117], for any right triangle with the smallest angle $\alpha \in (\frac{\pi}{6}, \frac{\pi}{4})$, we have the inequalities

$$0 = \beta_1^N < \beta_1^S < \beta_1^M < \beta_2^N < \beta_1^L < \beta_1^{SM} < \beta_1^{SL} < \beta_1^{ML} < \beta_1^D.$$

This yields

$$0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_2^N < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1^D.$$

For $\alpha = \frac{\pi}{6}$ we have $\beta_1^M = \beta_2^N$, so that $\eta_1^M = \eta_2^N$, and for $\alpha = \frac{\pi}{4}$ (right isosceles triangle) we have $S = M$ and $\beta_2^N < \beta_1^L$, so that $\eta_2^N < \eta_1^L$. For an arbitrary triangle we have

$$\min\{\beta_1^S, \beta_1^M, \beta_1^L\} < \beta_2^N \leq \beta_1^{SM} \leq \beta_1^{SL} \leq \beta_1^{ML},$$

for any lengths of sides. Moreover, we have

$$\min\{\eta_1^S, \eta_1^M, \eta_1^L\} < \eta_2^N \leq \eta_1^{SM} \leq \eta_1^{SL} \leq \eta_1^{ML}.$$

However, it is possible that $\beta_2^N > \beta_1^L$ in the case $\eta_2^N > \eta_1^L$ (for any small perturbation of the equilateral triangle) or $\beta_2^N < \beta_1^M$, after that $\eta_2^N < \eta_1^M$ (for right triangle with $\alpha < \frac{\pi}{6}$). This completes the proof. $\square$

Theorem 5.42 *For any right triangular cylinder $D_\triangle$, with the smallest angle α with $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$, we have the inequalities for s-numbers:*

$$\frac{\pi^2}{4} = s_1^N < s_1^S < s_1^M < s_2^N < s_1^L < s_1^{SM} < s_1^{SL} < s_1^{ML} < s_1. \tag{5.195}$$

For $\alpha = \frac{\pi}{6}$ we have $s_1^M = s_2^N$, and for $\alpha = \frac{\pi}{4}$ (right isosceles triangular cylinder) we have $S = M$ and $s_2^N < s_1^L$. All other inequalities stay sharp in these cases. For an arbitrary triangular cylinder we have

$$\min\{s_1^S, s_1^M, s_1^L\} < s_2^N \leq s_1^{SM} \leq s_1^{SL} \leq s_1^{ML}, \tag{5.196}$$

for any lengths of sides. However, it is possible that $s_2^N > s_1^L$ (for any small perturbation of the equilateral triangular cylinder) or $s_2^N < s_1^M$ (for right triangular cylinders with $\alpha < \frac{\pi}{6}$).

To prove this theorem, let us prove first the inequality in (5.195). To do it, we solve the following problem by the Fourier method:

$$\Diamond_{\triangle}^{*}\Diamond_{\triangle}u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0,\ x \in \triangle, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0,\ x \in \triangle, \\ u(x,t) = 0,\ x \in D \subset \{L,M,S\},\ \forall t \in (0,1), \\ \frac{\partial u(x,t)}{\partial n} = 0,\ x \in \partial\triangle \backslash D,\ \forall t \in (0,1), \\ \Delta_x u(x,t) = 0,\ x \in D \subset \{L,M,S\},\ \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0,\ x \in \partial\triangle \backslash D,\ \forall t \in (0,1). \end{cases} \tag{5.197}$$

Thus, we arrive at the spectral problems for $\varphi(t)$ and $X(x)$ separately, i.e

$$\begin{cases} \Delta^2 X(x) = \beta^2(\triangle)X(x),\ x \in \triangle, \\ X(x) = 0,\ x \in D \subset \{L,M,S\},\ \forall t \in (0,1), \\ \frac{\partial X(x)}{\partial n} = 0,\ x \in \partial\triangle \backslash D, \\ \Delta X(x) = 0,\ x \in D \subset \{L,M,S\}, \\ \frac{\partial \Delta X(x)}{\partial n} = 0,\ x \in \partial\triangle \backslash D, \end{cases} \tag{5.198}$$

and

$$\begin{cases} \varphi''(t) + (s-\beta^2)\varphi(t) = 0,\ t \in (0,1), \\ \varphi(0) = 0, \\ \varphi'(1) + \beta(\triangle)\varphi(1) = 0. \end{cases} \tag{5.199}$$

It also gives that

$$\tan\sqrt{s-\beta^2} = -\frac{\sqrt{s-\beta^2}}{\beta}. \tag{5.200}$$

We have (see, [117]) that $0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_2^N < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1^D$ and

$$\tan\sqrt{s(\beta)-\beta^2} = -\frac{\sqrt{s(\beta)-\beta^2}}{\beta}. \tag{5.201}$$

It is easy to see that

$$s'(\beta) = \frac{2s(\beta)\cos^2\sqrt{s-\beta^2}}{\beta^2 + \beta\cos^2\sqrt{s-\beta^2}}. \tag{5.202}$$

The s-numbers and β are positive, so that

$$s_1'(\beta)' > 0. \tag{5.203}$$

It means the function $s(\eta)$ is monotonically increasing. If $\beta_1^N = 0$ from (5.153) we take $s_1^N = \frac{\pi^2}{4}$ and [117] and from Lemma 5.41 we take $0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1$, and thus we get

$$\frac{\pi^2}{4} = s_1^N < s_1^S < s_1^M < s_1^L < s_1^{SM} < s_1^{SL} < s_1^{ML} < s_1.$$

Let's prove the second part of inequality (5.193), $s_1^M < s_2^N < s_1^L$, and from Lemma 5.41 we get $\eta_1^M < \eta_2^N < \eta_1^L$. The operator $\diamondsuit_\triangle^*\diamondsuit_\triangle$ is a self-adjoint and compact operator. Hence, we have a complete orthonormal system in $L^2(D_\triangle)$, and thus

$$\int_{D_\triangle} u_i u_j dxdt = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Now using Lemma 5.41 we obtain

$$\begin{aligned}
s_1^M &= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt\int_\triangle (X_1^M(x))^2dx + \int_0^1 \varphi_1^2(t)dt\int_\triangle X_1^M\Delta^2 X_1^M(x)dx}{\int_0^1\int_\triangle (u_1^M(x,t))^2dxdt} \\
&= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt + \int_\triangle X_1^M\Delta^2 X_1^M(x)dx}{\int_0^1\int_\triangle (u_1^M(x,t))^2dxdt} = \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt + \eta_1^M}{\int_0^1\int_\triangle (u_1^M(x,t))^2dxdt} \\
&< \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt\int_\triangle (X_2^N(x)dx)^2 + \int_0^1 \varphi_1(t)\varphi_1(t)dt\int_\triangle X_2^N(x)\Delta^2 X_2^N(x)dx}{\int_0^1\int_\triangle (u_2^N(x,t)^2dxdt} \\
&= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt + \int_\triangle \eta_2^N(X_2^N(x))^2dx}{\int_0^1\int_\triangle (u_2^N(x,t))^2dxdt} = \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt + \eta_2^N}{\int_0^1\int_\triangle (u_2^N(x,t))^2dxdt} \\
&= s_2^N < \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt + \eta_1^L}{\int_0^1\int_\triangle (u_2^L(x,t))^2dxdt} \\
&= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt\int_\triangle (X_1^L(x)dx)^2 + \int_0^1 \varphi_1(t)\varphi_1(t)dt\int_\triangle X_1^L(x)\Delta^2 X_1^L(x)dx}{\int_0^1\int_\triangle (u_2^L(x,t))^2dxdt} \\
&= s_1^L. \qquad (5.204)
\end{aligned}$$

The rest of the equalities and inequalities follow from the monotonicity property (5.203). $\square$

Theorem 5.43 *For all triangular cylinders, the second s-number of the Cauchy-Neumann heat operator* (5.112) *satisfies*

$$s_2^N(\Omega) \leq (2.78978609910027)^2 + \left(\frac{4\pi^2}{3\sqrt{3}}\right)^2,$$

and the equality holds if and only if the triangular cylinder coincides with the equilateral triangular cylinder $\Omega^* \times (0,1)$*, that is,* $|\Omega| = |\Omega^*|$.

Let us prove this theorem. By using the fact that s-numbers are monotonically increasing (see (5.203)) and the main result of [71] we obtain

$$s_2^N(\Omega) \leq s_2^N\left(\frac{4\pi^2}{3\sqrt{3}}\right).$$

A straightforward calculation in (5.201) gives

$$s_2^N(\Omega) \le \ s_2^N\left(\frac{4\pi^2}{3\sqrt{3}}\right) \cong (2.78978609910027)^2 + \left(\frac{4\pi^2}{3\sqrt{3}}\right)^2. \qquad \square$$

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