Masami Isoda · Raimundo Olfos *Editors*

Teaching Multiplication with Lesson Study

Japanese and Ibero-American Theories for International Mathematics Education

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Foreword by Abraham Arcavi

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Foreword

Teacher: "Why are you doing your multiplication on the floor?"
Student: "You told me not to use tables."

https://www.rd.com/jokes/math/

Numerically literate adults know how to multiply numbers, either in their heads, with paper and pencil using standard or alternative algorithms, by estimating and adjusting, or, if all else fails, by resorting to a calculator. Many remember the distress (or for others the joy) caused by memorizing the multiplication table. Teachers and researchers know the range of difficulties students may face while learning how to multiply numbers. Much has been said, written, and researched about multiplication, its learning, and its teaching—what else is there to add? Well . . . a lot! And this book shows just how much.

This book contributes to the integration of existing studies while providing new insights by weaving together many important themes in contemporary mathematics education, as follows:

1. In the spirit of any serious design experiment (or, as in some traditions, didactic engineering), the book provides a thorough epistemological analysis of the common yet different meanings of the idea of multiplication, its many entailments, and its connections to its "predecessor ideas" (counting and addition) and to its "successors or extensions" (such as combinatory, ratio and proportion, division, fractions, decimals, and more).
2. Embedded in the theory and practice of the powerful Japanese tradition of lesson study, the book illustrates how epistemological analyses of the idea(s) of multiplication, coupled with awareness of the complexities of student cognition, can be translated into careful and detailed lesson plans, their practical implementation in authentic classrooms, and a posteriori analyses and refinements.

3. Faithful to the current wide recognition of the crucial role played by culture in mathematics education, this book travels to different countries to report, analyze, and learn how different educational traditions regard multiplication and consequently how to adopt different approaches and perspectives.

As an eager and ongoing learner of lesson study myself—having roots in Latin America, and in my past and present roles as a teacher, a grandparent, a curriculum developer, a mathematics teacher educator, and a researcher of mathematics education—I have found myself gaining new insights while reading the different sections of this book and enjoying them very much. Likewise, I am confident that readers—be they teachers, parents, teacher educators, curriculum developers, policy makers, or mathematics education researchers—will find themselves inspired by the different chapters of this book and relishing enriching educational practices.

Rehovot, Israel Abraham Arcavi

Preface

“When American teachers retire, almost all the lesson plans and practices they developed also retire. When Japanese teachers retire, they leave a legacy.” This is quoted from the journal *The Economist* (2007) regarding the difference of cultural practice in teaching of Japanese teachers in comparison with American teachers. However, what is that legacy? It is the product of lesson study.

Lesson study (*jugyou kenkyu* in Japanese) has influenced the world with the Japanese approach to developing students who learn mathematics by and for themselves. In Japan, the innovative challenges by teachers in lesson study have contributed not only to developing students and young teachers but also to research and the development of mathematics education in relation to reforms of the national curriculum standards and revision of textbooks. Despite the many contributions of lesson study mentioned in relation to teachers’ professional development—such as pedagogical content knowledge, the curriculum, and content knowledge for teaching[1]—leading Japanese teachers continuously revise their school curriculum and textbooks, and influence the revision of the national curriculum standards with innovative ideas based on their lesson study experiences.

The international leading project for lesson study under the Asia–Pacific Economic Cooperation (APEC) has been engaged in establishing the APEC lesson study community to improve and develop innovative mathematics education in the APEC region (2006–2018). In its third year of implementation, the delegates from the APEC economies were asked to evaluate the following aspects of the project, with the following results: useful for the improvement of the quality of mathematics education: 100% positive; influential for other subjects: 93% positive; useful for developing innovative teaching approach: 93% positive; useful for curriculum improvement: 80% positive; useful for sharing model teaching approaches: 80% positive; useful for developing teachers: 80% positive; useful for developing students: 80% positive; and useful for developing practical/local theories of mathematics

[1] For example, see Lewis and Pettry (2006); Lewis, Perry, and Murata (2006) and Figure 2 of http://www.criced.tsukuba.ac.jp/math/apec/.

education: 53% positive. All those issues are related to the content of the legacy, especially the lesser-known aspect of it, which is the practical/local theories of mathematics education as a product of lesson study. What are these theories?

Part I of this book answers the last question on the design science of mathematics education by illustrating the Japanese approach in teaching multiplication. It includes the theory of *Kyozaikenkyu* (the study of teaching materials), which spans over 100 years (see Isoda, 2007) and is embedded into the Japanese curriculum, textbooks, and teaching guidebooks. Its relation to a variety of proposals discussed in this book is shown in Part II by four leading Ibero-American researchers in primary mathematics education.

The Japanese engage in lesson study for developing students who learn mathematics by and for themselves. In other words, students are asked to use their previous knowledge for their new learning. This is the design principle of the Japanese curriculum, which corresponds to mathematization under the reorganization of past experience, as described by Freudenthal (1973). The Japanese had already established most of the ideas for their curricula and teaching through lesson study by the 1960s.

This book has two parts. In Part I, the Japanese approach to multiplication is explained in relation to lesson study. It is a coherent approach applied to curriculum standards, textbooks, and teaching practices for developing students who learn mathematics by and for themselves. It is the long-term product of lesson study under the design theories in mathematics education. The theoretical ideas used to produce lessons are also discussed. The nature of these Japanese theories is the design and reproducible science that enables teachers to predict the ideas of students before the class, based on what they have already learned under the curriculum sequence (not the same as the learning trajectory in some countries) and to produce new ideas which must be used in the future class through communication in class under the specific objective on the sequence of mathematics curriculum.

Part II—contributed by Ibero-American researchers—proposes, describes, and analyzes the teaching of multiplication in various contexts based on different theories such as French didactics, the learning trajectory, and ethnomathematics. In addition, it illustrates how the theories of mathematics education function.

This book identifies problems in the teaching of multiplication in elementary school mathematics beyond borders and language, and provides theoretical and practical ideas for possible solutions. It poses questions about the theories of mathematics education and what they are for. This book gives us opportunities to know how lesson study answers challenges from the perspective of design science for students and teachers.

Tsukuba, Japan — Masami Isoda
Valparaíso, Chile — Raimundo Olfos

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Contents

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Chapter 1
Introduction: Japanese Theories and Overview of the Chapters in This Book

Masami Isoda and Raimundo Olfos

This introductory chapter explains the origin of this book and provides overviews of every chapter in Parts I and II of the book. Part I of the book is aimed at explaining what multiplication and lesson study are in relation to the Japanese approach. It provides an overview of Japanese theories on mathematics education for developing students who learn mathematics by and for themselves and it provides necessary ideas to understand the Japanese approach and lesson study. Part II consists of contributions from leading researchers in Ibero-America. Through their contributions, this book provides various perspectives based on different theories of mathematics education which provide the opportunity to reconsider the teaching of multiplication and theories.

1.1 Origin of This Book

This book originated from collaborative research done by the editors since 2008. When Olfos studied Japanese lesson study with Isoda at the University of Tsukuba, Olfos was amazed by how Japanese students mathematically communicate the curriculum content and subject matter by themselves in their classroom under the problem-solving approach. They reorganize new mathematical knowledge by themselves based on what they have already learned under the learning trajectory in their curriculum sequence. Before he arrived in Japan, his image of problem solving was to recall and use learned content to solve new content. However, the Japanese problem-solving approach is done under the task sequence planned by the teachers

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and textbooks for students to reorganize mathematical knowledge by using what they have already learned. At the same time, he recognized that there were big differences in the curricular content, textbooks, tasks, and teaching content in schools. In Chile, most children attempt (but do not memorize at all) the multiplication table in the earlier grades. However, the Japanese teach the multiplication table for enabling students to learn how to extend the multiplication table by themselves. Through discussion about the Japanese approach from the perspective of Chile, the editors recognized that the teaching of multiplication is an exemplar for sharing Japanese practical theories in mathematics education to establish coherent and consistent alignment of the curriculum, teaching practice, and assessment.

To this end, the editors published lesson study books in Spanish (*La Enseñanza de la Multiplicación: El Estudio de Clases y las Demandas curriculares* [in English: *Teaching of Multiplication: Lesson Study for Curricular Demands*] (Isoda and Olfos, 2009a) and *El Enfoque de Resolución de Problemas: en la Enseñanza de la Matemática: a Partir del Estudio de Clases* [in English: *Problem Solving Approach: Mathematics Teaching on Lesson Study*] (Isoda and Olfos, 2009b) to explain the Japanese approach. For comparison of the first book with the Ibero-Americans' proposals, *Enseñanza de la Multiplicación: Desde el Estudio de Clases Japonés a las Propuestas iberoamericanas* [in English: *Teaching Multiplication: Japanese Lesson Study and Ibero-American Contributions*] was published in Isoda and Olfos, 2011 with leading researchers from Ibero-America.

Part I of this English-language book is a revision of Part I and Annex of the 2011 Spanish-language book on multiplication under the current international curriculum reform movement. It aims to develop twenty-first-century skills and competencies including the human character, values, attitudes, and way of thinking (Isoda and Katagiri, 2012; Mangao, Ahmad, and Isoda, 2017). Part II of this book comprises excerpts from Part II of the 2011 Spanish-language book. This is a new book on multiplication in relation to Japanese lesson study.

This chapter briefly explains the Japanese theories that are used by teachers for designing and implementing lessons, and gives an overview of the subsequent chapters to provides the perspectives of this book in teaching multiplication.

The Organisation for Economic Co-operation and Development (OECD) (2005) has defined competencies for curriculum reform for students to be able to succeed in this changing and refunctioning society by using the words "successful life" and "well functioning society". The United Nations (2015) seeks the establishment of high-quality education on SDG4. To address these issues, curriculum reforms are under way. This book provides ideas for high-quality education and theoretical overviews for better teaching of multiplication in a competency-based curriculum based on the experiences in Japan and Ibero-America.

1.2 Overview of Japanese Theories for Designing Lessons

Through Japan's remarkable economic growth up until the early 1990s and its highest achievements in several international surveys in mathematics (such as the International Association for the Evaluation of Educational Achievement (IEA)

Trends in International Mathematics and Science Study (TIMSS), and the Program for International Student Assessment (PISA)) since 1964, Japanese education and its system have become internationally influential. Especially in the 1980s, the system of the Japanese national curriculum standards influenced England and the USA.[1] The Japanese ways of teaching began to spread to developing countries through JICA in the 1990s.[2] Japanese teaching approaches and lesson study have been learned internationally since around 2000 in relation to Stigler and Hiebert (1999) and NCMS (2000). Especially, NCMS praised Japanese lesson study from the aspect of teachers' collaboration for long term development. Then, collaboration of teachers modeled by lesson study became one of research trends. Robutti et al. (2016) mentioned that it is not always successful because Japanese lesson study is a kind of cultural practice and there are missing informations. Indeed, Japanese lesson study has a long tradition dating back to 1873 (Isoda, 2007, 2020; Makinae, 2010, 2016; Baba, Ueda, Ninomiya, and Hino, 2018). The first guidebook for lesson study was written by Wakabayashi and Shirai (1883) and was aimed at improvement of teaching and learning by adoption of the Pestalozzi method and Zen/Confucian–style dialectical questioning. The book explained the principles of teaching under the Pestalozzi method, dialectical questioning and tasks for inquiry, objective-based lesson planning, and ways of critical discussion after observation of the class (such as preferred teaching materials and methods, and observed activities

[1] In Japan, schools that follow the national curriculum standards are recognized as schools that are supported by the government. The national standards are the bases for textbook authorization and national assessments. Authorized textbooks follow the standards, 90% of curriculum standards content on compulsory education and 80% on senior secondary education. After authorization, they can be called textbooks and freely selected by the district. Every school is supposed to manage its own curriculum under these conditions. Most schools' curricula follow the textbooks' recommended curriculum; however, lessons are planned beyond these limitations, depending on the teachers. It looks like a top-down system; however, it includes lesson study, which involves a bottom-up system. For example, well-recognized approaches and teaching materials will be embedded into the new edition of textbooks. For revision of the curriculum, a laboratory school usually proposes new approaches and teaching materials. In Japan, there are no private educational consultants who provide schools with their own/original curriculum, lesson plans, worksheets, tools, and methods of teaching. However, lesson study produces learning communities for innovation and sharing of ideas on curriculum development and implementation in every classroom. In Japan, results of lesson studies will be embedded into curriculum and textbooks. In countries which do not have the consistent curriculum alignment such as the countries do not have national standards, or countries that teachers don't have custom to follow the standards even they have, teachers usually use worksheets copied from various different resources. In this type of worksheet culture, it is not easy to establish a coherent system under the curriculum alignment (see such as Squires, 2012) like the system in Japan. Indeed, on the worksheet culture, the teaching time distribution to the contents are not the same and if teachers use different worksheets, they find it difficult to estimate and utilize what students have learned in the past. Then, the teachers have to try to make sense of the teaching content at every class. In Japan, teachers are able to engage in sense making (McCallum, 2018) for future learning to be able students to learn by and for themselves. See Chap. 5.

[2] Singapore had the opportunity to study the Japanese system and the Japanese approach at the end of the 1970s (in a 5-year project with Japanese overseer development assistance). Since 1982, the Ministry of Education, Japan, has provided an 18-month program at teacher education universities. Each year, more than 150 teachers from Southeast Asia and other region study the Japanese approach, which includes learning mathematics education, in the program.

of the teacher and students), with exemplary protocols for every subject in the curriculum. This was the beginning of Japanese general theories for designing lessons, which are not known internationally, as we discuss in the preface to this book.

Japanese theories for mathematics education as for the school subject specific theories (Herbst & Chazan, 2016) are based on the didactics[3] of lesson study involving math educators, which can be seen from four perspectives (Isoda, 2020). The first perspective is the theories that clarify the aims and objectives in every class. The national curriculum standards constitute an authorized document that explains the objectives. To clarify the objective of teaching, math educators have prepared related theories such as mathematical thinking. The second perspective is the terminologies used to distinguish conceptual differences in teaching content. The third perspective is the theory used to establish the curriculum sequence and task sequence. The fourth perspective is the theory used to manage lessons. These theories have been prepared by math educators through lesson study.[4]

[3] To date, a limited number of research articles on lesson study have focused on Japanese cases, such as Miyakawa and Winsløw (2013), analyzing lesson study by using French didactics. In Part I of this book, Japanese didactics mean design theories of practice to develop students' competency to learn mathematics by and for themselves under the curriculum. This means that Japanese didactics is oriented toward realizing the aims of mathematics education. For teachers, it is not necessary to mean the theoretical frameworks for social scientific analysis on empirical studies even it can be used for (see such as Huang and Shimizu, 2016).

[4] In the community of math educators, when we say "theory," most math educators might imagine theoretical frameworks, such as French didactics, which are used for observation, analysis, and description of the research object. It is necessary to contribute to the research community for educators. On the other hand, the Japanese mathematics education theories which used by teachers and educators orient the design science and are necessary to develop and explain better teaching practices for students as for reproducible science (Isoda, 2015a). Thus, the bases are the aims and objectives, followed by the terminology to distinguish the teaching content, and then the sequence of teaching and the method of teaching. For example, French Didactics does not include aims and objectives in its theories although it includes anthropological approaches in mathematics education and design-based research (see the *Encyclopedia of Mathematics Education (Lerman,* 2014)). In French didactics, the aims and objectives are analyzed under the terminology/framework on didactics: see such as Rasmussen & Isoda (2018) in the case of mathematical thinking. Conversely, when Japanese teachers refer to the aims and objectives in curriculum documents as terminology for their lesson plan, they continuously use the same terminology. For example, developing students who learn mathematics by and for themselves is written into the curriculum document (see Ministry of Education, 1998). The teachers try to prepare teaching sequences, materials, and methods to develop students toward this shared aim which functions likely an axiom. On this meaning, Japanese Theories are the aims and objectives based, normative, theories for educators and teachers. Lesson study has been functioning for their theorizations. Japanese theories are referred and functioning in various lesson study community though the national level publication for designing, observing and explaining the classes and students' developments. Japanese math educators also use social scientific theories however it is not the major scope in Part I for illustrating the theories on lesson study.

1.2.1 Mathematical Thinking and Activity: Aims and Objectives

The Japanese aims of education have been described as three pillars: human character formation (such as values and attitudes), general thinking skills (such as mathematical thinking and ideas), and specific knowledge and skills (such as mathematical knowledge and skills). If we change the terminology, the principle aims are common not only for Japan but also for other countries such as the Southeast Asian countries (Mangao et al., 2017).

The first two pillars are usually explained as higher-order thinking skills in many countries and also as the learning content for learning how to learn. It is usual for teachers to write or share these objectives through the lesson plan. According to the Japanese principle of the national curriculum, these aims are symbolized by a single concept: "Developing students who learn mathematics by and for themselves" (Shimizu, 1984). In Japanese mathematics education, this has been recognized in relation to mathematical activities as for reorganization of living and life (Ministry of Education, 1947). The activity has been re-explained as mathematical thinking and attitude (Ministry of Education, 1956) by Japanese math educators, who have tried to explain it further. Shigeo Katagiri (see Katagiri, Sakurai, and Takahasi, 1969 and Katagiri, Sakurai, Takahasi, and Oshima, 1971), who was a curriculum specialist in primary school mathematics in the Ministry of Education, established the framework for mathematical thinking with teachers (Isoda and Katagiri, 2012, 2016).[5]

In Japanese lesson study, Table 1.1 is used for clarifying the curriculum, task sequence, teaching materials,[6] and methods of teaching. It is not a list of hints such as the strategies for solving problems adapted from Pólya (1945) but is used for precise descriptions of objectives for every teaching material in the lesson and for considering its processes as for preparation of future learning. It is also used to write the lesson plan for clarifying the objectives of teaching, which explains why it is necessary to practice like that. Table 1.1 is used for writing these objectives more concretely and clearly with teaching materials.[7] This framework also provides the general study theme of lesson study beyond every objective of the teaching content.[8]

Katagiri also developed the list for questioning in the classroom in relation to teaching phases for Table 1.1.

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[5] The Japanese teachers' manner of preparation will be illustrated in Chap. 7.

[6] Teaching materials mean the content or the task of mathematics embedded objectives in the curriculum. In Japanese mathematics education, development of mathematical thinking is a part of the aims of the national curriculum standards.

[7] Katagiri's framework is historically known. There are several projects for further revision of his framework on the context of 21st century skills (Mangao et al., 2017) in ASEAN region and computational thinking for 4th Industrial revolution in APEC region.

[8] The recursive process of lesson study can be continued according to the study theme even though the teaching content changes every time.

Table 1.1 Types of mathematical thinking according to Katagiri (published in English in 2012)

I. Mathematical attitudes: Mindset
1. Attempting to grasp one's own problems, objectives, or entities clearly by oneself
(a) Attempting to have questions
(b) Attempting to be aware problematic
(c) Attempting to find further problems from situation
2. Attempting to take logical-reasonable actions (reasonableness)
(a) Attempting to take actions that match the objectives
(b) Attempting to establish a perspective
(c) Attempting to think based on the data that can be used, previously learned items, and assumptions
3. Attempting to represent matters clearly and simply: Clarity
(a) Attempting to record and communicate problems and results clearly and simply
(b) Attempting to sort and organize objects when representing them
4. Attempting to seek better ways and ideas
(a) Attempting to raise thinking from the objects to operations
(b) Attempting to evaluate thinking both objectively and subjectively, and to refine thinking
(c) Attempting to economize thought and effort
II. Mathematical thinking related to mathematical methods: Mathematical Ways of Thinking
1. Inductive thinking
2. Analogical thinking
3. Deductive thinking
4. Integrative thinking (including extension)
5. Developmental thinking
6. Abstract thinking (thinking that abstracts, concretizes, and idealizes, and thinking that clarifies conditions)
7. Thinking that simplifies
8. Thinking that generalizes
9. Thinking that specializes
10. Thinking that symbolizes
11. Thinking that represents by numbers, quantities, figures and diagrams
III. Mathematical thinking related to mathematical contents: Mathematical Ideas
1. Clarifying sets of objects for consideration and objects excluded from sets, and clarifying conditions for inclusion (the idea of sets)
2. Focusing on constituent elements (units) and their sizes and relationships (the idea of units)
3. Attempting to think based on the fundamental principles of expressions and the permanence of form (the idea of expression)
4. Clarifying and extending the meaning of things and operations, and attempting to think based on this (the idea of operation)
5. Attempting to formalize operation methods (the idea of algorithms)
6. Attempting to grasp the big picture of objects and operations, and using the result of this understanding (the idea of approximation)
7. Focusing on basic rules and properties (the idea of fundamental properties)
8. Attempting to focus on what is determined by one's decisions, finding rules of relationships between variables, and using relationship (functional thinking)
9. Attempting to express propositions and relationships as formulas, and to read their meaning (the idea of formulas)

1.2.2 Terminology and Sequences: Extension and Integration

The terminology distinguish conceptual differences and its development in curriculum content. It includes the technical terms to distinguish conjectural differences such as different meaning of multiplication and the representations such as Tape Diagram and Proportional Number Lines for overcoming such differences. It is necessary to explain the process of reorganization of mathematical concepts in the curriculum sequence. The Japanese established most of it between 1900 and the 1960s (see the special issues of the *Journal of Mathematics Education* published by the Japan Society of Mathematical Education in 2010).[9] The related terminology for multiplication will be explained in Chaps. 3 and 4 in Part I of this book in relation to the historical development of school mathematics and current research perspectives. Japanese teachers need to learn the terminology of school mathematics for developing students who learn mathematics by and for themselves because the school curriculum sequence cannot exist as a system deduced from the set and axioms such as pure mathematics (see Freudenthal (1973)).

The sequence in the Japanese curriculum standards has been explained by the principle of "extension and integration" since 1968, which is oriented toward enhancing mathematical activities and developing mathematical thinking. It corresponds to the principle of reinvention by Freudenthal (1973) who proposed mathematization as the reorganization of mathematical experience (see Isoda, 2018).

Under this principle, the school mathematics curriculum can be seen as a set of partially ordered local mathematics theories, like a net that is consistent within every local theory like a knot; however, on extending and integrating local theories, the net has some inconsistencies in connecting the local theories, like entangled strings among knots. Japanese textbooks are written for students to be able to extend and integrate mathematics by and for themselves (see Chaps. 4 and 7).[10] The questions mentioned in item 1(a) of the "I. Mathematical attitudes" section of Table 1.1 (by Katagiri) are written for producing such mathematical problematic situations, not only for problems posed in real-world situations such as mathematical modeling.

Such inconsistencies through the extension and integration of local theories in relation to multiplication are explained by adaptation of the conceptual and procedural knowledge to meaning and procedure in Fig. 1.1 (Isoda, 1992, 1996, 2009).[11]

[9] In relation to multiplication in this book, Izsák and Beckmann (2019) discussed the same idea, such as the definition of multiplication by measurements and proportional number lines; however, the Japanese established it in 1960s. See Chapter 3.

[10] It means that Japanese textbook has the task sequence for Zone of Proximal Development (ZPD; Vygotski, 1978) by using what they already learned and preparing for future. Murata (2008) illustrated the function of tape diagram as a model for ZPD.

[11] In 1992, Isoda proposed the design theory of the task sequence with adaptation of conceptual and procedural knowledge from Hiebert (1986) and published eight lesson study books in Japanese as the product of lesson studies; more than 200 lessons, ranging from the first grade to the tenth grade, were produced using this theory. Later, similar theories were also proposed by Hiroshi Tanaka and Kei Ohono, teachers of the Elementary School at the University of Tsukuba.

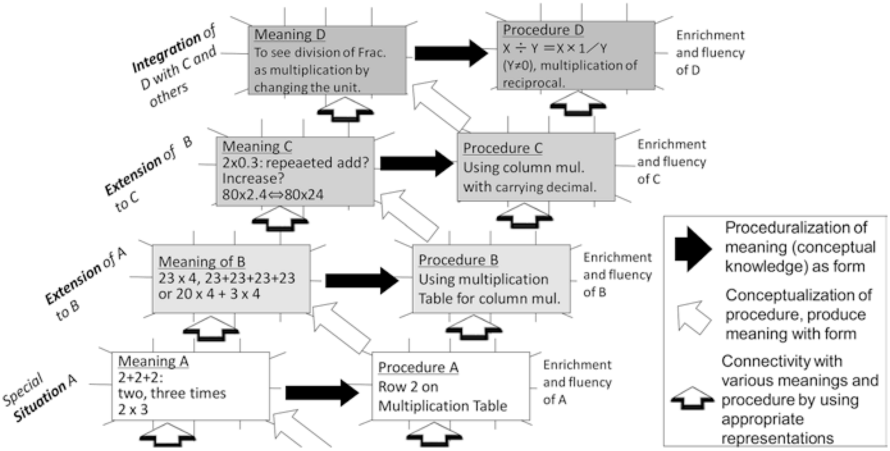

Fig. 1.1 Simplified extension and integration process of multiplication (*mul.*) in the task sequence detailed in the textbooks, which is explained by conceptual and procedural knowledge (Isoda, 2009)

How do you explain the sequence of conceptual development in Fig. 1.1? Conceptual and procedural knowledge are used to explain the development of personal knowledge; however, in Fig. 1.1, we use them to design and explain task sequences in the curriculum. In the curriculum sequence, as in textbooks, these are not discussed at the same time. Conceptual knowledge is usually taught for meaning; however, it needs to use some known form of procedure. After introducing the meaning of multiplication as a binary operation (expression), the multiplication table is proceduralized from repeated addition; otherwise, students cannot distinguish it from addition as a new operation. In the process of extension and integration, inconsistencies usually appear. For example, for doing multidigit multiplication, students need to see the multidigit numbers under the base ten system for applying the multiplication table instead of just repeated addition. For the extension of multiplication to multidigit numbers with column methods, multiplication as repeated addition should be integrated with the base ten system by using the rule of distribution. If we extend multiplication from whole numbers to decimals, the product of multiplication becomes small in case. It cannot be explained well as repeated addition. In the Japanese textbooks and Japanese teachers' lesson design (as shown in later chapters) these processes are discussed more precisely in relation to the task sequence.

In the terminology of the "learning trajectory", progressive relationship of conceptual and procedural knowledge in Fig. 1.1 are not easily seen as two different sides of the same coin.[12] On the task sequence in Japanese textbooks, as in the curriculum, it might be clearly distinguished. One of the reasons is that it is possible for students to learn the procedure without knowing when the procedure should be used. This may seem like a strange statement; however, the textbook provides the opportunity to exercise a set of similar tasks for getting fluency of the procedure. “If *A*, then *B*” is the format of the procedure. In the exercise in the chapter, students do only exercise *B* for solving given tasks. The condition, a part of *A*, is not necessary to consider in the exercise for practicing the same tasks.

Before the extension of multiplication to decimal numbers, the product of multiplication only increases: “If it is multiplication of whole numbers, then the products become large.” However, until extension of whole number to decimals, whole numbers are numbers, so it looks correct to say, “If it is multiplication of numbers, then the products become large.” This is possible learning content for students through the exercise in the textbook chapter. The necessity for all students to think about conditions in relation to *A* will be provided when students learn multiplication of decimals. Actually, when students learn whole numbers, they do not know about decimals. Students are able to learn *A* when they encounter multiplication of decimal numbers.[13] Another reason is related to the shortage of the capacity of working memory. If we limit working memory, procedures are very convenient and firster for doing multiplication. Students do not need to consider the meaning of *A*, because the numbers given in the exercise are not decimals. They have already established a convenient procedure that can be used without considering the original meaning of *A*. After students attain fluency in the procedure, many students do not feel the necessity to go back to and interpret the original meaning of the situations. Many of them lose/compartmentalize it because they do not need to think about the condition of *A* as long as they are applying it to learned situations. The opportunity for extension and integration is a chance to reorganize their mathematics by comparing what they already knew and their developed mathematical ideas. At the moment of extension and integration on the task sequence, students are able to establish the significant meaning.

[12] This metaphor was popularized by Sfard (1991). She illustrated by using the history of mathematics. In the case of Japan, it can be illustrated by using the textbooks under the curriculum standards. Isoda (1992) established his theory on his lesson study groups in Japan such as province at Sapporo, Ibaraki, Tokyo, Toyama, Fukuoka and Okinawa by using the theories of Hiebert (1986). Simon (1995) characterized ‘hypothetical learning trajectory’ on teachers’ instructional design. Clements & Sarama (2004) characterized ‘learning trajectory’ by a learning goal, developmental progressions of thinking and learning, and a sequence of instructional tasks. Japanese textbooks under the national curriculum standards are well established the task sequence which enable to develop mathematical thinking by using already learned and for preparing the future learning. Japanese textbooks are products of the huge experience and challenges of lesson study in whole Japan. On the consequence of their recursive revisions, all six textbook-companies’ series under the national curriculum standards become similar. Japanese teachers are able to produce their learning trajectry based on the experience of lesson study as the design and reproducible science. Thus, it is not just a personal hypotheths.

[13] Tall (2013) explained this with his terminology “met before.”

1.2.3 Problem-Solving Approach: Not Only a Teaching Method

If you have a chance to observe a lesson in a Japanese elementary school, the Japanese problem-solving approach looks the same as an open-ended approach,[14] which involves posing an unknown task, solving the task in various ways, comparing solutions with the whole class, and summarize. However, the Japanese problem-solving approach is prepared in the following ideas: aims and objectives for developing students who learn mathematics by and for themselves, terminologies to explain the learning content, the curriculum and task sequence which connect past, current and future learning, and the teaching materials. On the other hand, an open-ended approach is characterized by an open-ended task. Consequently, the teaching materials used in the Japanese problem-solving approach are not the same as open-ended approach for an independent task, topic, or content of mathematics because problem-solving approach is explained under the aims, objectives, task sequences and preparation of future learning.

In the Japanese problem-solving approach,[15] the task given by the teacher to the class means that the teacher prepare the teaching material which embeds the objectives in the task sequence. Thus, when you read a Japanese textbooks without considering the context and objective embed in the task sequence, it is just reading content but not regarding it as teaching materials. On the Japanese problem-solving approach, students reinvent the objective of the class from the given task as problematic. It was planned by the teacher to encourage them to think mathematically. The contradictions in the planned task sequence are necessary in this context. Given this limitation, the following exemplar on how the Japanese use the board in the lesson is meaningful (Fig. 1.2).

In this book, the Japanese approach means all those consequences and does not imply just a method of teaching like the scaffolding used to construct a building. Every component is explained by the theories and used for designing the classroom.

These theories are the models that will be illustrated in Part I of this book. Please note that there are several other theories in Japan, and many of them have been proposed through critical discussions such as curriculum sequences. For example, the extension and integration principle provides task sequences that go against the general-to-specific principle proposed by the mathematicians' group of Hiraku Toyama since the 1950s with the name of the water supply method (a metaphor from general-to-specific, see Kobayasi, 1989). Against the general-to-specific approach, several counter theories were proposed to support extension and integra-

[14] In Japan, open-ended tasks appeared before and in the middle of World War II. This idea was proposed by Shimada in 1977 (published in English by Becker and Shimada (1997) and theoretically elaborated by Nohda (1983, 2000)). In Japan, the problem-solving approach was named arround 1950 in the context of progressivism and then was renamed in the 1980s in the context of the Agenda for Action (NCTM, 1980). The teaching style itself could be seen in the format of the lesson plan before World War II.

[15] Tall (2013) explained it as "lesson study" because "problem solving" in English merely implies solving an unknown task or exercise.

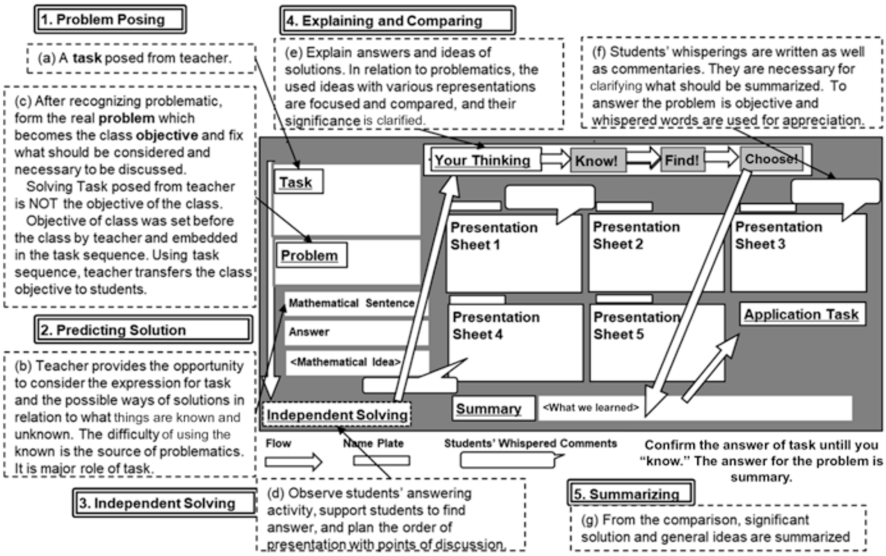

Fig. 1.2 A lesson plan format by using the board in the problem-solving approach (Isoda, 2012)

tion, some called it mathematization or discovery, such as Ito's theory as a representative of 1961, 1962a, 1962b, 1962c, 1963a, 1963b, 1963c. Ito's theory to mediate ideas by models (representations) such as proportional number lines (Ito, 1968) was named "discovery methods" by Ito (in English, 1971).[16] Toshio Odaka established schema theories (1975, 1979, 1980) for a problem-solving approach (see School Mathematics Study Society at the Junior Secondary School of the Tokyo University of Education, 1969, 1970, 1971, 1972), inspired by the idea of Piaget for supporting the extension and integration principle from the tradition of mathematization in the 1943 national textbook. Odaka produced a counter theory to explain an appropriate curriculum and task sequence—called the "exemplar approach"—against Toyama's general-to specific sequence and schema theory, and completed as his task sequence for problem solving approach (Odaka & Okamoto, 1982)

[16] Ito established the Japanese theory of proportional number lines which also included the idea of the definition of multiplication by measurements and tape diagram, wrote textbooks, and published seven guidebooks for teachers. He proposed proportional number lines for overcoming the inconsistency between local theories in the process of extension and integration. His theory is an integration of existed theories as discovery methods. Currently, his discovery method by consistent using of diagram can be seen from the perspective of the representation theory for Zone of Proximal Development. Isoda learned the theory for proportional number line from Prof. Tatsuro Miwa at the University of Tsukuba in 1981 on his undergraduate class for the elementary school mathematics curriculum. The same idea can be seen in Izak and Beckman (2019). Ito proposed his theory against the definition by attributes in relation to the Toyama group (Kobayasi, 1989). The theory by Toyama group will be discussed in Chap. 3.

Tadao Kaneko, written by Sakai and Hasegawa (1989), also theorized a task sequence for specific-to-general and an exercise sequence for general-to-special (1987). Shigeru Shimada proposed the open-ended approach (1977; originally he began the study in the 1960s) based on his experience of the 1943 textbook under the mathematization principle, and Nobuhiko Nohda retheorized it as the open approach (1983). There were discussions about embedding open-ended tasks into textbooks in the 1980s.[17] Their theory for open-ended tasks itself did not indicate the manner to establish the task sequence for conceptual development in curriculum, directly. Odaka's, Kaneko's, and Isoda's theories were proposed for the task sequence as for conceptual development on the curriculum under the principle.

1.2.4 Change Approaches for Developing Students and Teachers

Lesson study around the world is usually focusing on the open class such as that shown in Fig. 1.3. The most necessary activity for any teacher is the preparation of the lesson for setting the teaching materials which embed the objectives into the teaching content and process. In the process of planning the board writing (Fig. 1.3), teachers usually prepare various types of questioning for inquiry (known in Japanese as *hatsumon*).

There are three types of questioning by teachers and students from the viewpoints of the objectives:[18] The first type is questions of mathematical interest such as the task given by the teacher (see box (a) in Fig. 1.2), and the problematic posed by the students (see box (c) in Fig. 1.2). The second type is questions on teaching and learning in the teaching phases (see boxes 1–5 in Fig. 1.2) to provide the opportunity for reflections on what the students have learned that day in the summary (see box 5 in Fig. 1.2). The third type is meta-questions, which enable students to provide questions (like teachers) by and for themselves internally, such as "What do you want to do next?"[19]

In the lesson study process, after the open class, in the postclass discussions, the quality of the classroom communication by students is usually a subject for critique. If the students did the third type of questions by themselves well, a major point of discussion after the class observation is usually how the teacher developed the students to think mathematically. The teacher usually explains his or her everyday efforts to prepare future learning with deep understanding of the teaching materials and sequences. To develop values, attitudes, and mathematical thinking, the second and third types of questions are necessary; however, the questions do not exist without the

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[17] Their discussion about questioning in mathematics was not so far to the questioning the world (Chevallard, 2015), and the Study and Research Path (SRP) (Winsløw, Matheron, and Mercier, 2013), which is related to the open approach by Nohda (2000), is a good framework to illustrate the open inquiry process. However, the Japanese problem-solving approach is more oriented to the task sequence to achieve the objectives and aims of the curriculum.

[18] See the introductory chapter and pp. 127–128 in Isoda and Katagiri (2012).

[19] See the Introductory Chapter of Isoda and Katagiri (2012). Hideyo Emori (2013) also mentioned similar ideas in classroom communication.

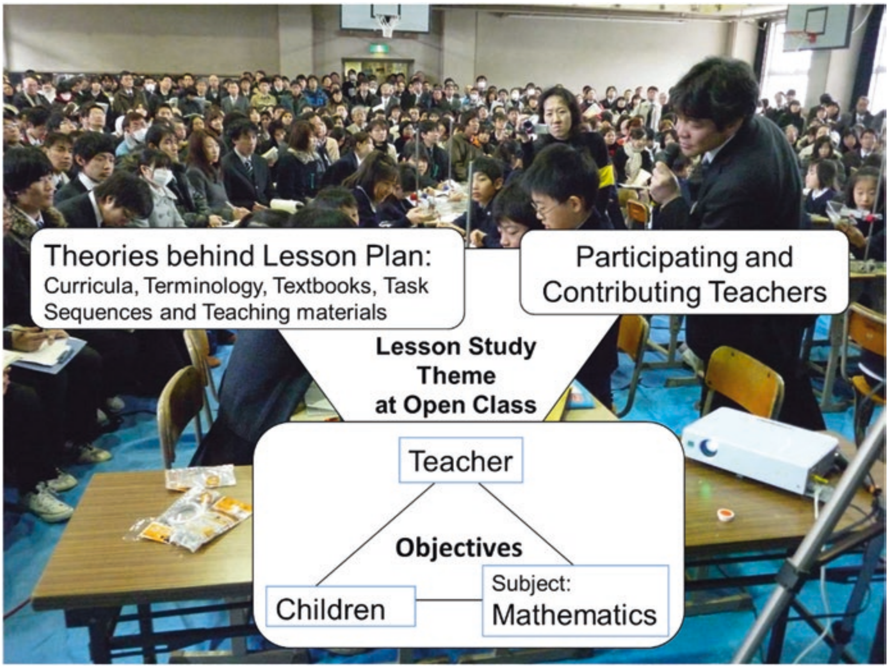

Fig. 1.3 What shall we discuss and observe?

first type in mathematics class. Thus, all questions are not just for the method of teaching but are also associated with the teaching materials and task sequences. In this context, Japanese teachers try to develop teaching materials by clearly embedding the objectives into the content under the task sequence in their preparation of the class.

In the postclass discussion, student-centered approaches are usually recommended instead of teacher-centered approach, and sometimes the qualities of the subject, mathematics, are not focused on, even they observed students activity. In Japanese lesson study, teachers discuss the achievement of the objectives and the study theme in the lesson plan. The study theme, innovative proposals, and challenges, are explained by the teacher with the teaching materials, as well as the specific objective of the task under the sequence. Then, alternative possibilities in relation to the objective and study theme are discussed based on their observation of class. All of them are related to the teaching materials.

Depending on the objective, a teacher-centered approach may be preferable. For developing students to learn mathematics by and for themselves, the view of mathematics is not the same. It depends on the teaching material and the objective prepared by the teacher (see Fig. 1.4). Exercise is necessary for acquisition of fluency in knowledge and skills. If the objectives are focused just on acquisition, mathematics can be seen as a set of knowledge and skills. An open approach is possible if teachers change the ordinal task to be open ended. Here, mathematics has various answers, which are the subject of communications. By using open-ended tasks,

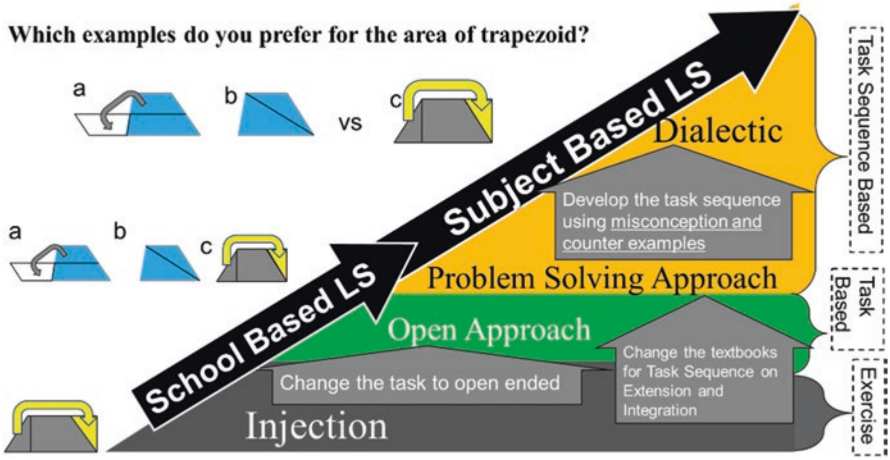

Fig. 1.4 Various approaches for developing students on lesson study (LS)

students and teachers usually learn ways of learning/teaching in the classroom. If they can use a textbook that has an appropriate task sequence for students based on the extension and integration principle, students can learn mathematics as a subject of extension and integration. This is a way to change the approach from being teacher centered to student centered. It is usually one of target of school based lesson study which is done within every school.

If teachers can use student's misconceptions and counter examples, they are able to develop students who have minds for proof and refutation (Lakatos, 1976; Isoda, 2015b).[20] However, dialectical discussions are not easy for students who are only learning mathematics through teachers' explanations and exercises. It is also difficult to plan dialectical discussions for most teachers, because they cannot imagine the process for proof and refutation in school mathematics. Thus, in school-based lesson study, primary schools usually choose a problem-solving approach as the goal of training teachers by using appropriate textbooks.

On the subject of mathematics-based lesson study, many teachers are already able to teach with an open and problem-solving approach in their classes, and then they try to establish their own original task sequences for their classroom students. Such challenges are usually seen in the mathematics lesson study group at the Elementary School attached to the University of Tsukuba and the Sapporo mathematics lesson study group. These teachers produce the textbooks and try to make it possible to practice in school-based lesson study. The examples of lessons shown in Part I of this book were produced by these teachers.

In school-based lesson study at elementary schools, it is not easy for every teacher to focus on mathematics as dialectics. In this sense, it is oriented toward a student-centered approach. Subject-based lesson study is done by math major teach-

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[20] Lakatos was Hegelian in the context of Karl Popper; proof and refutation are a kind of dialectic. It is the bases for to develop critical thinkers in mathematics class. However, dialectic discussion in the classroom is not popular in the world even though it can be seen from first grade of elementary school.

ers and is oriented toward a subject-centered approach. In Part I of this book, we focus on Japanese multiplication, which is shared by the Japanese mathematics lesson study groups and textbooks as a result of long-term development of a subject-centered approach as well as a student-centered approach.

1.3 Overview of Chapters in Part I: The Japanese Approach

Part I of this book illustrates the Japanese approach to multiplication through comparison of other perspectives. The theories used to design Japanese lesson study will be also illustrated.

In Chap. 2, the national curricula of seven countries are compared for confirmation of their differences and diversities, and for posing questions regarding teaching of multiplication, which will be answered in Chaps. 3, 4, 5, 6, and 7 in relation to the Japanese approach. The questions posed in Chap. 2 are related to the meanings and definition of multiplication, the necessity of appropriate selection of meanings for teaching other content such as division and extension of numbers (illustrated in Chap. 4), ways of teaching the meaning of multiplication and the multiplication table in relation to memorization, ways of teaching algorithms (the column method) for multiplication of multidigit numbers, and appropriate grades for introducing multiplication.

In Chap. 3, in relation to the question of defining the meaning of multiplication, the definition of multiplication in pure mathematics is confirmed first. Then, situations explaining the meaning of multiplication and the various types and uses of properties are discussed. Further, the Japanese definition of multiplication by measurement, which extends the group of groups, is introduced. This definition of multiplication becomes a keyword to explain the Japanese approach. The difference in language structure that produces inconsistency in teaching multiplication in Indo-European languages is also explained. The learning of children using their language and ways of thinking through the subject of elementary school mathematics are also presented. The terminologies explained in Chaps. 3 and 4 provide the bases of multiplication and lesson study in other chapters.

Chapter 4, in relation to the question of teaching of other content, illustrates the Japanese consistent curriculum sequence and terminologies to adapt the idea of multiplication to division and extend it to decimals and fractions. It further illustrates how the Japanese curriculum and textbooks are planned to develop and reorganize students' mathematical ideas for multiplication up to proportionality under the Japanese definition of multiplication by measurement. It also explains that the reason why many countries in Central America, the Pacific, Southeast Asia, and so on choose Japanese textbooks is this consistent sequence to extend students' ideas for future learning.

Chapter 5, in relation to the question of the definition and meaning of multiplication, illustrates how Japanese teachers introduce the meaning of multiplication under the definition of multiplication by measurement in two examples of lesson study. The capacity of the students to set the unit for measurement is enhanced in the Japanese classroom. Data on how Japanese students develop in the curriculum sequence are also provided. After these discussions, the Japanese and Chilean approaches for introducing the meaning of multiplication are compared. The way in which the Chilean curriculum explains how students make sense of multiplication is also shown. On the other hand, the Japanese engage in sense making for multiplication by enhancing the capacity of the students to set the unit for measurement.

Chapter 6, in relation to the question of the grade level at which the multiplication table should be taught, illustrates how the Japanese approach enables students to learn the skill to extend what they have learned and the significance of their learning. In Japan, the multiplication table is taught in the second grade, and this chapter explains three reasons for this. The first reason is the high achievement. The second reason is that students are able to extend the multiplication table by themselves in an appropriate teaching sequence. The third reason is that memorizing the table is an enjoyable activity for students as a cultural practice. In relation to the subtheme of this book, this chapter illustrates how students are able to extend the multiplication table by themselves in the task sequence.

Chapter 7, in relation to the question of multidigit multiplication, illustrates the extension of multiplication in vertical form and the column method, from single digit numbers to multidigit numbers, which includes the process of integration with addition in vertical form. Because multiplication of multidigit numbers by the column method is not repeated addition, students have to extend and integrate what they have already learned. In the Japanese approach, teachers prepare an appropriate task sequence that enables students to devise various approaches for vertical form and to choose the appropriate form in relation to the base ten place value system. Using the exemplar of lesson study and the teaching sequence in the textbook, three principles on how to design the task sequence using what the students have already learned are illustrated.

As explained in Section 1.1, to understand the Japanese approach to multiplication, readers have to note how the coherent alignment between the curriculum, textbooks, and teaching in Japan is planned through lesson study (see Isoda, Stephens, Ohara, and Miyakawa, 2007; Miyakawa and Winsløw, 2019). In Japan, the Ministry reforms the national curriculum every decade.[21] Textbooks, which condense teachers' experience of lesson study in relation to every subject, are revised every 4 years.[22] Teachers must use the textbooks approved by the government, although they can create their own school curriculum. Teachers must also follow the national curriculum sequence. For countries in which every teacher teaches mathematics using their own independent curriculum, it may look like Japanese teachers are restricted by the national curriculum; however, this is a misunderstanding because the curriculum is the product of lesson study by teachers.[23]

In Japan, half of the university math educators are well-experienced schoolteachers. Most of these math educators work with teachers in the schools and are challenged

[21] This has been a gradual transformation of the system over several decades. For example, the subject of English has been introduced in three steps: in the first decade, it was recommended as an activity; in the second decade, English activity was done every week; and in the third decade, English became a subject to be learned.

[22] Textbooks are usually written by leading teachers of lesson study, and math educators usually contribute editing. Teachers in experimental schools usually collaborate with math educators for innovation of mathematics teaching.

[23] There are various misunderstanding for Japanese lesson study (see such as Isoda, 2015a; Fujii, 2014).

to devise innovations in mathematics education. Members of curriculum reform committees are usually leading researchers and teachers in national-level lesson study. They are usually the authors of textbooks and members of national assessment committees. If they are engaged in lesson study, they try to embed their achievements in the national curriculum standards and textbooks that follow the national standards.

In this manner, lesson study has given way to harmonious progress between curriculum design and classroom management. It contributes to the aspiration of offering mathematics education centered on a problem-solving approach that is connected to the demands of addressing the content established in the curriculum, the teaching materials, and the task sequences, and developing students' interest/values and attitude in learning mathematics and mathematical thinking.

1.4 Overview of Chapters in Part II, Focusing on Ibero-American Countries

Part II of this English edition develops a proposal for teaching multiplication and offers reflections on it by leading researchers in Ibero-American countries, which provide diverse original views and deep critiques but are not necessarily representatives of the national approaches.

In Chap. 8, a contribution from Dr. Ubiratan D'Ambrosio and Dr. Claudia Sabba of Brazil is presented, which invites us to appreciate the development of their original ideas—an ethnomathematical perspective on the question of the idea of multiplication. The teaching approach is grounded on miniprojects that integrate diverse areas of knowledge in the Waldorf Schools tradition in Sao Paulo. There, the concept of multiplication is constructed together with the geometry of plane figures through the elaboration of mathematical thinking together with figures mounted on a circular wooden table. These ideas are connected to the use of photos taken using students' cellular phones to introduce the concept of proportionality. They take photos of their bodies and faces, and use them to study Leonardo da Vinci's *Vitruvian Man*.

Chapter 9 presents a contribution from Dr. David Block and Laura Resendiz of Mexico. They share a teaching sequence for addressing multiplication constructed and validated in the framework of French didactics engineering. The teaching proposal is made up of a sequence of didactic situations about a kind of proportionality relation in which each value of a set—the number of necklaces—is made to correspond, in another set, to pairs, threes, or *N*s of values (numbers of beads of different colors required for that number of necklaces). The sequence includes multiplication, division, and proportionality problems. Also, the results of application of the sequence in a group of fourth-grade students (9–10 years of age) are presented.

The contribution in Chap. 10 comes from Professors Fatima Mendes, Jouana Brocardo, and Helia Oliveira of Portugal. The authors, bearing in mind the pedagogical notion of the "path," show how a teacher, as a sailor, adjusts the sails to correct the path and reach port, taking responsibility for third-grade students' learning regarding multiplication. The paths are associated with potential levels of achievement, learning goals, and competencies to be reached. So, while the study of learning tasks is connected to microdidactics and the study of teaching sequences is connected to mesodidactics, the study of hypothetical learning paths lies in the macrodidactic context particular to longitudinal study, which addresses the evolution of students' understanding of a concept over years.

Chapter 11 provides the last contribution, from Dr. Maria del Carmen Chamorro of Spain, who reflects on why so many children fail in learning multiplication in elementary schools in Ibero-American countries. She describes four problems: students' lack of understanding, lack of development of skill in written calculation, inappropriateness of common teaching methods, and the presence of the algorithm without controlling how it is produced. Dr. Chamorro points out the virtues of Japanese teaching with respect to the importance of meaning or semantic dimension, the importance given in Japan to the use of manipulatives, and the relevance of the cultural dimension.

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Part I
Japanese Approach for Multiplication: Comparison with other Countries, and Theoretical, Historical, and Empirical Analysis for Lesson Study

Chapter 2
Multiplication of Whole Numbers in the Curriculum: Singapore, Japan, Portugal, the USA, Mexico, Brazil, and Chile

Raimundo Olfos, Masami Isoda, and Soledad Estrella

This chapter shows how the teaching of multiplication is structured in national curriculum standards (programs) around the world. (The documents are distributed by national governments via the web. Those documents are written in different formats and depths. For understanding the descriptions of the standards, we also refer to national authorized textbooks for confirmation of meanings). The countries chosen for comparison in this case are two countries in Asia, one in Europe, two in North America, and two in South America: Singapore, Japan, Portugal, the USA (where the Common Core State Standards (2010) are not national but are agreed on by most of the states), Mexico, Brazil, and Chile, from the viewpoint of their influences on Ibero-American countries. (The National Council of Teachers of Mathematics (NCTM) standards (published in 2000) and the Japanese and Singapore textbooks have been influential in Latin America. Additionally, Portugal was selected to be compared with Brazil). To distinguish between each country's standard and the general standards described here, the national curriculum standards are just called the "program." The comparison shows the differences in the programs for multiplication in these countries in relation to the sequence of the description and the way of explanation. The role of this chapter in Part I of this book is to provide the introductory questions that will be discussed in Chaps. 3, 4, 5, 6, and 7 to explain the features of the Japanese approach. (As is discussed in Chap. 1, the Japanese approach

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includes the Japanese curriculum, textbooks, and methods of teaching which can be used for designing classes, as has been explored in Chile (see (Estrella, Mena, Olfos, Lesson Study in Chile: a very promising but still uncertain path. In Quaresma, Winslow, Clivaz, da Ponte, Ni Shuilleabhain, Takahashi (eds), Mathematics lesson study around the world: Theoretical and methodological issues. Cham: Springer, pp. 105–22, 2018). The comparison focuses on multiplication of whole numbers. In multiplication, all of these countries seem to have similar goals—namely, for their students to grasp the meaning of multiplication and develop fluency in calculation. However, are they the same? By using the newest editions of each country's curriculum standards, comparisons are done on the basis of the manner of writing, with assigned grades for the range of numbers, meanings, expression, tables, and multi-digit multiplication. The relationship with other specific content such as division, the use of calculators, the treatment of multiples, and mixed arithmetic operations are beyond the scope of this comparison. Those are mentioned only if there is a need to show diversity.

2.1 Comparison of Curricular Standards' Descriptions for Introducing Multiplication in Different Countries

In the various programs (National Curriculum Standards and related documents), the meaning of multiplication is usually given using situations for multiplication. The way to find the answer (product) of multiplication is known as repeated addition. However, the sequence of descriptions and ways of explanation are very different in terms of their format and terminology. Thus, here, we would like to briefly illustrate the differences in format, terminology, and ways of explanation in their introduction of multiplication.

In Singapore (Ministry of Education, Singapore, 2012), the term "multiplication" appears with "division" as one joint category—"multiplication and division"—from the first grade until the third grade. In the first grade (Ministry of Education, Singapore, 2012, p. 35), multiplication and division are explained: concepts of multiplication and division, use of "×," multiplying within 40, dividing within 20, and solving one-step word problems involving multiplication and division with pictorial representations. The students should be given opportunities to experience the following:

- Making equal groups using concrete objects and counting the total number of objects in the groups by repeated addition using phrases such as "2 groups of 5" and "2 fives"
- Sharing a given number of concrete objects/picture cutouts and explaining how the sharing is done and whether the objects can be shared equally
- Dividing a set of concrete objects into equal groups and discussing the grouping and sharing concept of division

In Japan (Ministry of Education, Culture, Sports, Science and Technology; MEXT, 2017), the program is written in two categories: (A) knowledge and skills;

and (B) competencies for thinking, making decisions, and representing. Mathematical activity is autonomous and objective oriented, with the aims of self-directedness, interactivity, and deep learning. In this framework, multiplication is introduced in the second grade as described below.

Students should be nurtured to be able to acquire the following through mathematics activities for multiplication (MEXT, 2017, p. 51):

- To acquire the following knowledge and skills:
  - Understand the meaning of multiplication and know situations where multiplication is used
  - Represent situations where multiplication is used with algebraic expressions and interpret these expressions
  - Understand simple properties that hold for multiplication
  - Learn the multiplication table up to 9×9 and multiply 1-digit numbers accurately
  - Know ways of multiplication of a 2-digit number by a 1-digit number in simple cases
- To acquire the following competency for thinking, making decisions, and representing:
  - Focusing on mathematical relations, thinking about the meaning and ways of calculation (operation), finding the properties of multiplication, and, by using these properties, utilizing calculation and confirming the result of calculation
  - Focusing on mathematical relations and utilizing multiplication in daily life

In Portugal (Ministério da Educação e Ciencia, 2013), the teaching of multiplication begins in the second grade. In the first grade, addition and subtraction are written in different categories but in the second grade, they are written in the same category. Multiplication and division appear in different categories in the second grade. This implies that depending on the grade level, the content is seen from different perspectives. The term "multiplication" is explained for the second grade as follows (however, this clearly implies that the content is short) (Ministério da Educação e Ciencia, Portugal, 2013, p. 7):

- Additive and combinatorial meaning
- The symbol "×" with the terms "factor" and "product"
- The product of 1 and 0
- Multiplication tables for 2, 3, 4, 5, 6, and 10
- The terms "double," "triple," "quadruple," and "quintuple"
- One- or two-step problems that involve multiplicative situations in the additive and combinatorics sense

In the USA, instead of the national standards, the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers (NGAC and CCSSO), 2010) have a hierarchy—grade, domain, cluster, and standards—and formal multiplication is specified in the third grade. The domain “Operations and Algebraic Thinking” has seven standards, categorized into the following three clusters for multiplication:

- Representing and solving problems involving multiplication and division
- Understanding the properties of multiplication and the relationship between multiplication and division
- Multiplying and dividing within 100

In the other domains, multiplication is also mentioned. Additionally, the memorization of multiplication tables is done in the third grade.

In Mexico (Secretaría de Educación Pública, 2017a, 2017b), the curriculum framework is written under the following hierarchy: domain, learning expectation, didactical orientation, and assessment. Multiplication is introduced in the second grade.

The learning expectation for multiplication is as follows (Secretaría de Educación Pública, México, 2017a):

- Solve multiplication problems with natural numbers less than 10

In didactic orientation, the following processes of teaching are described:

  - Problems regarding repeated quantities
  - Making multiplication explicit
  - Problems of counting in rectangular arrays
  - Mental calculation and application of the product of digits in the cases of 5 and 2

In Brazil (Ministério da Educação, 2017), a National Common Curricular Base is written under the following hierarchy: thematic units, object of knowledge, and abilities according to the competencies of mathematics.

For the thematic unit of numbers in mathematics in the second grade (Ministério da Educação, 2017, pp. 280–281), the object of knowledge is as follows:

- Problems that imply addition of equal groups (multiplication)
- Problems that imply meanings of double, half, triple, and third

The abilities are:

- To solve and elaborate problems of multiplication (by 2, 3, 4, and 5) with the idea of adding equal parcels by means of strategies and forms of personal registry with or without the support of images and/or manipulative materials
- To solve and elaborate problems involving double, half, triple, and third, with the support of images or manipulative materials, using personal strategies

In the Chilean Curricular Framework (MINEDUC, 2012, p. 104) for the second grade, the skills that students should be able to acquire to show that they understand multiplication are as follows:

- Using concrete and pictorial representations
- Expressing multiplication as the addition of equal summands

- Using the distributive property as a strategy for building the multiplication tables of 2, 5, and 10
- Solving problems that involve the multiplication tables of 2, 5, and 10

Based on the comparison of the programs above, there are differences in the format and terminology used to introduce multiplication.

- Some countries do not mention the meaning of multiplication. Combinatorics meaning is uniquely introduced in Portugal. In Japan, the word "meaning" is just said without referring to any idea. What is the meaning of multiplication?
- Some countries introduce complete multiplication tables in the same grade while others introduce 2 and 5 in the upper grades or in several grades. How they are different.
- Some countries introduce factors and others do not. Why?
- There is a variety in the ranges of numbers when introducing multiplication. In Brazil, multiple is mentioned as half and third. It already implies multiplication of fractions when it is introduced. In the case of other countries, how do they prepare to extend multiplication to larger whole numbers, decimals, and fractions?

Those differences provide some perspectives on what we shall compare.

With regard to differences in the format and terminology, as well as the grade level when multiplication is introduced, there are differences in the depth of writing. For example, some countries discuss the method of teaching and assessment of their programs, and others do not. For comparison of the depth of writing, their guidebooks and authorized textbooks should be referred to.

2.2 Comparison of the Assigned Grade Levels for Multiplication

How are the assigned grades and teaching sequence of multiplication different? The tables in this chapter—in relation to the range of numbers, meanings, tables, and multidigit multiplication—highlight the differences in specific aspects in dealing with multiplication according to the programs of the countries.

2.2.1 Range of Digits

When is it possible to discuss multiplication and multidigit multiplication? We can use the number of digits as an indicator (Table 2.1). Addition and subtraction are deeply dependent on the base ten place value system for making ten. On the other hand, the counting unit in multiplication is produced depending on the multiplicand and is not only limited to ten as a unit. The numbers up to 100 are necessary for

Table 2.1 School grades in which different countries introduce extensions of the range of digits

Country	School grade in which concept is introduced						
	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
Up to 100 or so	1	1	1	1	1	1	1
Up to 1000	2	2	2	2	2	–	2
Up to 10,000	4	3	4	–	3	2	–
Up to 100,000	–	4	5	–	4	3	–
Up to 1 million	–	5	–	3	5	–	4
Up to thousand, or millions	5	–	–	4	–	3	–

learning the multiplication table, and further extension of numbers produces the need for multidigit multiplication. For teaching multiplication of a 3-digit number by a 2-digit number, it is necessary to deal with numbers up to 100,000.

Based only on the viewpoint of the range of numbers, if the multiplication table is up to 10 by 10, it is not impossible to introduce the multiplication table from the first grade in all countries. When do they actually introduce multiplication?

2.2.2 *The Meaning of Multiplication*

For discussing the meaning of multiplication, every country introduces counting by 2s or by 5s in the first or second grade. Most countries, except Singapore and the USA, introduce multiplication from the second grade. The USA begins to introduce repeated addition in the second grade but the definition of multiplication is provided in the third grade. Singapore introduces multiplication within 40 in the first grade.

Multiplication as a group of groups and length based on unit length in a tape diagram are not mentioned in Mexico. An array diagram with a rectangular shape is not mentioned in the Portugal and Singapore programs. The area of a rectangle is introduced in the upper grades in every country. Combinatorics is discussed only in Portugal, and proportionality is mentioned only in Brazil and Japan (Table 2.2).

From Table 2.2, the following questions emerge: Is an array diagram an alternative for a group of groups? Is repeated addition the same as a group of groups? How do the different meanings contribute to understanding of multiplication? Is repeated addition the only way to obtain the result when multiplication is introduced? Why is combinatorics used when multiplication is still being introduced? Is the use of equal amounts in a group the way to explain the meaning of multiplication?

Table 2.2 School grades in which different countries introduce various meanings of multiplication

Country	School grade in which concept is introduced						
	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
Counting by 2s	1	1	1	2	1	1[a]	2
Situation for adding equal quantities	2	2	2	2	1	2	2
Repeated addition	2	2	2	2	1	2	2
Group as a unit or group of groups (without repeated addition)	2[a]	–	3	2	1	2	3
Length based on unit length (tape diagram)	2	–	3	2	1	2	3
Array diagram (or rectangular shape)	3	2	3	–	–	2	2
Area (rectangle)	4	4	5	3	3	4	3
Proportionality	–	–	4	–	–	3[b]	–
Combinatorics	–	–	–	2	–	–	–

[a]Includes interpretation of examples provided in the guidebook
[b]Means a proportional number line or tape diagram

Table 2.3 School grades in which different countries introduce mathematical expressions of multiplication

Country	School grade in which concept is introduced						
	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
Use of “×” symbol	2[a]	2	3[b]	2	1	2	3
Multiplication expression explained in Table 2.2	2?	2	3[b]	2	1?	2	3
Multiplier Multiplicand	–	–	4 5	–	–	2	–
Product	4	2	5	2	3	2[c]	3
Factor	4	3	–	2	4	–	3

[a]In the second grade, Chile uses the dot, for instance a Croix, as a multiplication symbol
[b]This is not explicit in the National Common Curricular Base
[c]In Japan, the term “product”, as well as sum, difference and quotient, is the content of teaching for students at the fourth grade. Until fifth grade, student call it as a value of multiplication. On the other hands, the term “product” itself is appeared from the second grade in the guidebook

2.2.3 The Definition of Multiplication

The “×” symbol is introduced in different grades in different countries (see Table 2.3). In particular, the USA introduces repeated addition in the second grade, and introduces multiplication and symbol “×” in the third grade. Brazil, on the other hand, introduces them in later grades. It is only in Japan that the multiplier and multiplicand are introduced at the beginning. In contrast, factors are not introduced in Brazil and Japan.

As a binary operation, multiplication is defined as $N \times N$ and provides the image/value for every pair of natural numbers. The numerical expression of multiplication can be introduced with various meanings as provided in Table 2.2. The meaning is not only limited to repeated addition. Some countries introduce different terms such as factors, multiplier, and product providing exemplary numerical expressions with the symbol "×" for introducing multiplication. Why do some counties use it? How do the terms, symbols, and explanations contribute to the definition of multiplication?

2.2.4 Multiplication Tables

Multiplication tables are completed up to the second or third grade (see Table 2.4). Singapore spends 3 years on them, while Portugal, the USA, and Japan spend only 1 year.

Why do some countries divide the multiplication table into several grades and others do not? What criteria are used to select the order for discussion of the table?

2.2.5 Use of Algorithm or Column Method for Multiplication

The teaching of techniques to calculate the product of multidigit numbers varies among countries (see Table 2.5). Japan starts to teach T0 × U in the second grade (the simplest case of TU × U); the others start in the third grade. Some countries require 3 years, such as Chile, and others only 1 year, such as Portugal.

Comparing the countries, questions emerge, like the following: Why is the product of a 3-digit number by a 2-digit number considered? Why is KHTU × TU not considered?

Table 2.4 School grades in which different countries introduce multiplication tables

	School grade in which concept is introduced						
Country	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
The rows of 2 and 5	2	2	2	2	1 (2)	2	3
The rows of 3 and 4	3	2 and 3	2	2	2	2	3
The rows of 6 and 8	3	2 and 3	–	2 and 3	3	2	3
The rows of 7 and 9	4	3	2	2	4	2	3
Multiplying by 10	2	3	3	3	2	2	3

Table 2.5 School grades in which multiplication of multi-digits of numbers and vertical forms are introduced

Country	School grade in which concept is introduced						
	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
TU × U	3	3	3	3	3	2	3
HTU × U	4	3[a]	4	3	3	3	–
KHTU × U	–	4[a]	4	3	4	4	4
TU × TU	5	3[a]	4	3	4	3	4
HTU × TU	–	4	4[b]	3	4	3	5
Algorithm in vertical form (column methods)	4	4	4	3	3	3	5

U units, *T* tens, *H* hundreds, *K* thousands

[a]Programs in Mexico demand that the product has at most three digits in the third grade and five digits in the fourth grade, so 20 × 30 is acceptable in the third grade but 30 × 40 is not

[b]In Brazil, the teaching of algorithms in the fourth grade is up to 5-digit numbers

2.2.6 Comparing the Results with Previous Research

The results of this comparison are not the same as the results of the comparisons made in the 2000s in our previous Spanish-language edition (Isoda and Olfos, 2009). Ten years ago, there was more diversity under the previous curricula. The teaching of multiplication at the various grade levels was very much different. For example, in the case of Chile, multiplication tables are presently taught in the second and third grades, but they were taught in the third and fourth grades 10 years ago, so now it is much easier to share teaching approaches beyond individual nations.

2.3 Questions for Later Chapters

Those comparisons in the tables show how the teaching of multiplication is assigned at each grade level. The sequences and content of teaching are different. From the comparisons, we raise several questions, which will be discussed in Chaps. 3, 4, 5, 6, and 7. Here the questions are summarized as follows:

- In relation to the meaning of multiplication: What is the meaning of multiplication? Repeated addition? A group of groups? What is combinatorics? How many meanings do we have for multiplication? (See Chaps. 3 and 5.)
- In relation to the expression of multiplication: What are the multiplier and multiplicand? Why do some countries teach them and others do not? Why do some countries introduce factors instead of the multiplier and multiplicand? (See Chaps. 3 and 7.)

- In relation to the multiplication table: Why do some countries teach it at different grade levels? What is the difference? (See Chap. 6.)
- Some countries introduce the complete multiplication table in the same grade level while others introduce 2 and 5 at the upper grade level or at several grade levels. How are they different? (See Chap. 6.)
- Why do some countries teach multiplication algorithms (the column method or vertical form) in later grades? Why do they not teach it at the same time as the teaching of multiplication of several digits? (See Chap. 7.)
- How is the teaching of multiplication of whole numbers related to other numbers such as fractions and decimals, and how is it related to other operations such as division? (See Chap. 4.)

These questions can be explained in every country's context. In Chaps. 3, 4, 5, 6, and 7, these questions will be answered in comparison with the Japanese approach by explaining the reasons why the Japanese prefer such a teaching sequence and ways of teaching. This provides alternative perspectives for the other countries, as has been explored in Chile (Estrella, Mena, & Olfos, 2018). Because of the difference of the format of the national curriculum documents, we did not analyze the philosophical, mathematical, educational reason of every curriculum sequence. Every curriculum for teaching multiplication exists under its school system, educational culture, historical background, and reform issues. This comparison is done for posing these basic questions which make clear the perspectives to explain Japanese Approach for multiplication.

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Chapter 3
Problematics for Conceptualization of Multiplication

Masami Isoda and Raimundo Olfos

This chapter addresses the problematics for the conceptualization of multiplication in school mathematics and fundamental difficulties, which include semantics for defining multiplication meaningfully, syntax in relation to languages, and difficulties that originate from historical transitions. The chapter discusses the contradictions or inconsistencies in the various meanings of multiplication in school mathematics situations. Many of these problems of multiplication are originated from European languages. This discussion of these problematics provides some answers to the questions posed in Chap. 2 and provides bases for the necessity to consider the Japanese approach described in Chaps. 4, 5, 6, and 7 of this book. The terminology of multiplication discussed here is related to mathematical usages of multiplication in relation to situations and models. Educational terminology used for multiplication to explain the curriculum and task sequences for designing lessons are discussed in Chap. 4 of this book.

3.1 Definitions of Multiplication and Their Meanings in Situations in School Mathematics

Mathematics curricula look well designed and consistent for learned adults; however, they usually have a number of inconsistencies for learners. Given this essential nature of mathematics curricula, the learning sequence used for mathematics, such as the curriculum and task sequence, can be explained by reorganization of

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mathematics, such as mathematization (Freudenthal, 1973; see Chap. 1 of this book and Isoda, 2018). Here, several inconsistencies in the definitions and meanings of multiplication are confirmed.

Multiplication as an operation can be explained in several ways, depending on the context (see Freudenthal (1983)). Here, some definitions and meanings which can be seen in curriculum documents, textbooks, and research articles will be illustrated in relation to problematics. These definitions and meanings will provide some answers to the questions posed in Chap. 2 and the necessary didactic questions for considering the Japanese challenges to established teaching sequences for developing the concept of multiplication in later chapters.

3.1.1 The Concept of Multiplication in Pure Mathematics in Relation to School Mathematics

In the formal context of pure mathematics, multiplication is defined by axioms such as the field theory of numbers.[1] Multiplication is defined as a binary operation and is distinguished from addition. In relation to abstract algebra, upper secondary school mathematics usually focuses on these two operations: division should be represented by multiplication of the dividend and the reciprocal (multiplicative inverse) of the divisor, and subtraction should be represented by addition of the minuend and the opposite (additive inverse) of the subtrahend. Multiplication and addition allow the rule of commutativity as a field axiom, such as $2 \times 3 = 3 \times 2$ and $2 + 3 = 3 + 2$. On the other hand, subtraction and division change their values if the order of numbers changes: $2 \div 3 \neq 3 \div 2$, and $3 - 2 \neq 2 - 3$. It provides one of the necessity in school mathematics to reorganize the four arithmetic operations at the elementary school level into the two major operations at the university level. In relation to Set theory, multiplication can be seen as Cartesian products. The value of multiplication can be seen as a cardinal number of the set of ordered pairs.

In elementary school, students learn all four arithmetic operations on their basis of life under their languages.[2] Depending on the learning trajectories under their own curriculum, students encounter contradictions (inconsistencies), which produce several gaps between arithmetic and the two operations in field theory.[3]

[1] The axioms for numbers are not only limited to the field theory. There are theories for the number system based on the algebraic extensions from the axiom of Peano. Further extension to real numbers is done by the Dedekind cut and hyperreal numbers (Tall, 2013). Complex numbers do not maintain the axiom of order. The R-module in relation to vector space can be another perspective for the number system. Vergnaud (1983) also discussed the "multiplicative structure" in relation to modern mathematics. This chapter is written from the Japanese and Chilean authors' perspective of the bases for the Japanese approach, which was established up to 1960s and is illustrated in Part I of this book.

[2] The matter of language will be discussed in Sect. 3.2.

[3] A simple example of miscalculation is $2 \div \frac{3}{5} \times 5 = \frac{2}{3}$, instead of $\frac{50}{3}$.

In formal algebra, natural numbers are introduced with Peano's axiom and the number systems are extended through progressive introduction of the four operations, magnitude[4] relations (the equivalence relation (=)), and order relations (greater or less than (> or <)) (see Michell and Ernst (1996)).[5] In elementary school, the equivalence of numbers can also be confirmed in every operation: $1 + 4 = 2 + 3 = 3 + 2 = 4 + 1$, $2 - 1 = 3 - 2 = 4 - 3 = \ldots$, $2 \times 3 = 3 \times 2$, $2 \div 1 = 4 \div 2 = 6 \div 3 = \ldots$ On natural number, the commutativity of multiplication also illustrates the equivalence of products. Within the natural numbers, the equivalence of values in addition and multiplication are finite but that in subtraction and division are infinite.

In school mathematics, the concept of multiplication is developed through reorganization of the process for multiplication (see Chap. 1, Fig. 1.1).[6] In elementary school, multiplication is usually introduced as repeated addition. Within a few years, children have to distinguish both addition and multiplication as independent operations. The elementary school curriculum usually treats the relationships between multiplication and division, and between addition and subtraction, as inverse operations, such as division of fractions is multiplication of reciprocal numbers. Teachers need to help students reorganize the four operations into two operations when the numbers are extended to positive and negative numbers. The rules of commutativity, associativity, and distributivity are usually introduced at the earlier stage of elementary school in preparation for future reorganizations.

For introducing multiplication of whole numbers, it can be defined as repeated addition, which is useful for getting the products of the multiplication table. In developing the multiplication table, the pattern "the product increases by the multiplier" for each row is used and, mathematically, it will be explained by the distributive law. For students, the row of 1—such as 1×1, 1×2, and 1×3—is not easy to explain by repetition because the row of 1 is the same as counting and not adding. Thus, we use the permanence of form (see Table 1.1 in Chap. 1). There is no counting - objects for the row of 0, thus the row of 0 is normally never discussed. Extension of the multiplication table from 9 by 9 to 10 by 10, or more, is easier for students if we use the pattern (permanence of form) supported by the distributive law. If the multiplication table is established at once, it will provide an alternative way to get the value of multiplication as the product.[7]

As mentioned in Chap. 1, Fig. 1.1, extending the numbers to multidigit multiplication is done by the column method which is a mixture of the unit (multiplier) in the

[4] Here, the magnitude is used for the size of the number such as larger or less in mathematics without indicating a concrete unit quantity on concrete situation such as just "3," not "3 marbles," which is called a denominate number (a number with "marbles" as the denomination for the unit of quantity).

[5] English translation of Otto Holder's German text (1901), Journal of Mathematical Psychology 40, 235–252 (1996).

[6] Tall (2013, 2019) sketched the process of reorganization on his terminology of three words of mathematics.

[7] In Japan, in the process of extension, the permanence of form has been enhanced in relation to mathematical thinking (see Chap. 1, Table 1.1) since 1956. It is used in the same way as the historical meaning of the extension of numbers, such as that described by George Peacock (for example, see Eves, 1997, p. 111).

multiplication table and addition in the base ten place value system. Extension to decimals and fractions causes overgeneralization of the definition of multiplication as repeated addition because addition of natural numbers always increases; however, it does not work with decimals and fractions. This is the problematic (which means "inconsistency" in elementary school mathematics, see Chap. 1) because multiplication of decimals (and fractions) does not always increase as repeated addition does on whole numbers. To overcome this, we need to follow the idea of the base ten system to find an alternative decimal unit such as 0.1 and $\frac{1}{10}$ to see it as a unit fraction, and the vertical form multiplication algorithm (the column method) using the multiplication table. In this process, for instance, if $9 \times 8 = 72$, then $90 \times 8 = 720$; associativity and commutativity can be used, such as $90 \times 8 = 9 \times 10 \times 8 = 9 \times 8 \times 10 = (9 \times 8) \times 10 = 720$. The distributive law is also necessary to introduce the multiplication algorithm (the column method), which will be explained in Chaps. 4 and 5.

3.1.2 Multiplicative Situations, Expression, and Translations

Formally, multiplication is a binary operation to get the product, just as addition is to get the sum. It is an expression in the world of mathematics without any concrete situation.[8] On the other hand, in applying multiplication in life, several meanings depending on the situation should be learned, particularly with regard to translations (interpretation) between the situation and the multiplication expression throughout the school curriculum. These meanings are usually expressed with everyday language to represent multiplication in situations and relations (mapping/arrow/correspondence) as a translation between situations and multiplication (expressions). Everyday language is necessary to represent reasoning in elementary school; it also brings limitations, such as the row of 1 in the multiplication table, which has already been mentioned. Here, we would like to consider several meanings of multiplication in relation to situations.

3.1.2.1 Origin of Written Situations

Multiplicative situations can be found in the ancient Babylonian language, Sumerian (Muroi, 2017), represented as *A a-rá B túm A*. Here, *túm* means "carry" and implies repeated addition. It means "A, B times" ($B \times A$), however, there were no expressions to represent it as a binary operation. Kazuo Muroi translated the following inheritance text for explaining the Sumerian sense of the base 60 system:

> There were 1,1,1,1 on base 60 system (= 219661 in base 10 system) rams and 13,13 on base 60 system (= 793) shepherd boys. How many rams did each boy receive? Each boy received 4,37 on base 60 system (= 277). There were 1,1,1,1 (= 219661) rams and 13 shepherd boys.

[8] Historically, the column method appeared much earlier than the expression.

> How many rams did each boy receive? Each boy received 4,41,37 (= 277 × 61 = 16897) There were 1,1,1 (= 3661) rams and 7 shepherd boys. How many rams did each boy receive? Each boy received 8,43 (= 523).

UET 5121 (from around the eighteen century BCE) was used in Muroi's Japanese translation; see also Figulla and Martin (1953) and Friberg (2007).

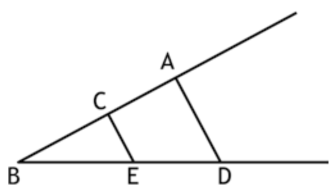
For finding the answers, the Sumerians used various tables on tablets; however, they did not write down the process of calculation. According to Muroi, the division of $a \div b$ is calculated as $a \times 1/b$ by using the reciprocal number table. For us, the quotation is a multiplicative situation; however, it is not the same as our multiplication as a binary operation. In division of the integers $a \div b$, a is not always divisible by b; it is a finite decimal or a recurring decimal. In the case of $1 \div 7$, this produces a recurring decimal. In the base 60 system, the numbers 2, 3, and 5 as factors of 60 are called *a-rá-gub-ba*, which means an ordinal factor. Seven in the base 60 system is the first number for which the reciprocal becomes a recurring decimal.[9] This implies that the number sense for multiplication in the base 60 system is not the same as that in the base ten system. For example, in the binary system, multiplication becomes addition. In this book, we focus on multiplication in the base ten system.

3.1.2.2 In Situations of Geometry with Proportionality

In Euclid's Elements, the idea of multiplication is discussed as "multiple/multiplicity" in the ancient Greek language in relation to ratio and proportion (Chemla, Chorlay, & Rabouin, 2016; Saito, 2008). It is *not* the same as the current meaning of multiplication in school, which is represented by expressions with "×" as the symbol of operation. During the era of Euclid, there was no algebraic expression. For example, a current expression such as $x^2 + a$ would have no meaning for Euclid because it would imply the addition of (a segment) to (a square). In the context of the Euclidian Elements, the product can be measured with a plane (two-dimensional) unit by associating the unit as measurable with multiplicity. For Euclid, measurable means the existence of the greatest common divisor.

To create algebraic representation as a universal language (mathematics), Descartes redefined the four operations as constructions with segments although he used "∝" instead of the current "=". Figure 3.1 was used for redefining multiplication in his book of geometry, published in 1637.

Fig. 3.1 Descartes (1637)



[9] Muroi mentioned that this is an origin of a myth which distinguishes 7 from other decimals.

In Fig. 3.1, $BE{:}BC = BD{:}BA$, then $BE \times BA = BC \times BD$. If BA is a unit, $BE = BC \times BD$. This is the definition of multiplication according to Descartes. This diagram was also used by Euclid. However, in the context of Euclid, $BE \times 1 = BC \times BD$ is acceptable because "an area = another area" but $BE = BC \times BD$ is not, because "a segment = an area" is inappropriate. Descartes established expressions beyond the limitations of dimension.

Descartes reorganized geometry as a part of his universal mathematics with algebraic expressions. His motivation was to shift mathematics from distinguished subjects such as geometry, arithmetic, astronomy, and music to algebra (universal mathematics). In his geometry, he needed to explain the appropriateness of using algebraic notation. In this context, the current meaning of multiplication, which is represented by expressions, becomes possible to use beyond Euclid.

We can extend Descartes's procedure of geometric construction to multiplication of negative numbers "$(-) \times (-) = (+)$" although the negative sign was not independently discussed during his time, unlike today.

3.1.2.3 In Situations with Quantities and Definition by Measurement

In the context of quantities, multiplication is the operation used to get the total quantity when the unit quantity and the number of units are known. This is the definition (explanation) in the Japanese curriculum documents, but it was not written in the textbook directory (Isoda, 2010). Here, we call it the definition of multiplication by measurement.[10] This definition degenerates to a group of groups or a set of groups, which was mentioned in Chap. 2, if it is limited to the natural numbers. It is consistent with Descartes's definition when we adapt it to geometric construction. If we apply this definition to measurement with geometric construction, it is to measure the length when the length of the unit and the number of units are known. Here, the length of the unit and the number of units can be real numbers if we extend the segments to lines (according to Euclid, the line can be extendable). On the other hand, a set of groups is usually imagined as whole numbers by students. Definition by measurement can be extended from natural numbers to real numbers. It does not contradict repeated addition such as a set of groups and can be applied to real numbers.

The Japanese textbooks from the third to the sixth grades use proportional number lines[11] (see Chap. 4 and Fig. 3.1) based on this definition (Isoda, Murata, & Yap, 2015, Grade 2, p. 9; Isoda & Murata, 2011, Grade 2, p. 9). Even in the second-grade textbooks, an approach to that meaning is provided by sentences such as "number of

[10] "Get the total quantity when the unit quantity and the number of units are known" is not actually a measuring activity; however, it is well connected with the proportional number line, which will be explained fully in Chap. 4. Definition by measurement is named by Shizumi Shimizu (Curriculum Specialist in the MEXT, personal communication). The definition was known in the 1960s at least (see Ito, 1968). Recently, Izak and Beckmann (2019) provided the same ideas for a world researchers.

[11] The proportional number line for elementary school mathematics was systematized by Ito (1972). By using the textbooks (Hitotsumatsu et al., 2005), Murata (2008) illustrated the tape diagram as the model for Zone of Proximal Development.

pieces of 3 cm tape and their lengths" (Isoda et al., 2015, Grade 2, p. 13; Isoda & Murata, 2011, Grade 2, p. 14). This definition is consistent with repeated addition when we limit the quantities to whole numbers or integers. If the measure and the value of the unit are natural numbers, the product can be seen as "repeated addition of the quantity corresponding to the unit" but when they are not, the definition applies to multiplication of decimals, fractions, and any measurement.[12] Both of these meanings have been written in the guidebook for the Japanese curriculum since the 1960s and can also be seen in Freudenthal (1983). For the extension of multiplication, this definition by measurement can also be applied to fractions and decimals with proportionality by using proportional number lines in Japan, serving as a mediational means (model/representation) for definition by measurement before formal definition of the proportion. Theoretically, the proportional number line is consistent with the Descartes[13] similarity in Fig. 3.1. Proportionality can be seen as the natural extension of multiplication in relation to definition by measurement.

Definition by measurement is not popular in the world. For example, in the Chilean curriculum (MINEDUC, 2013a, p. 152), repeated addition has been chosen as the definition. It looks like there is no inconsistency in interpreting the given example "In each of 6 boxes are 4 brushes, how many total brushes are there?" in the context of repeated addition rather than definition by measurement. However, the Chilean definition of repeated addition cannot be extended directly to decimals and fractions (see Chap. 5).

3.1.2.4 Contradictions between Repeated Addition and Situations with Quantities

In real-life situations, numbers usually appear with measurement units (quantities); these are called denominate numbers, such as "2 cups."[14] In this example, "2" is the number and "cups" is the denomination, with "a cup" as the unit of measurement to be counted. The "2" in "2 cups" can be seen as a mapping from the world of numbers in mathematics to the world of measurement in real life, setting the translation rule by seeing a cup as a counting unit. In this correspondence, the relationship of magnitude (greater than, less than, or equivalence to) is kept.

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[12] This works for real numbers. Multiplication of real numbers should be redefined for extension of real numbers to complex numbers.

[13] If the intersecting lines in Fig. 3.1 become parallel lines, they are proportional number lines. First Japanese translation of Descartes's Geometry was 1949 by Kouno.

[14] In mathematics (not in real life), quantity as magnitude is defined with the axiom of the magnitude relationship (the equivalence relationship and order relationship) without any physical unit quantity. In this section, quantity means the physical quantity and the quantities produced from physical quantities referring to a measurement quantity in real life where numbers are usually denominated with a measurement unit. In English, a denominate number such as "3 apples" refers to the measurement–quantity unit "apple," whereas in some other languages—such as Thai, Japanese, and so on—the measurement–quantity unit does not correspond to the denomination well. For example, "3 cups," "3 apples," "3 tomatoes," etc., in English are all said as *3 ko* ("3 pieces") in Japanese; *3 ko* is the denominate number. However, *ko* is not as clear as a measurement unit in English.

The Japanese definition of multiplication is introduced by situations with denominations by using measurement units such as the following: "If there are 3 apples for each dish (3 apples per dish) and 4 dishes, then the total number of apples is 12 apples." It is not the same as just saying "3 apples and 4 dishes." In the case of a number with a denomination, the situation can be represented by a physical expression such as "(dishes) × (apples/dish) = (apples)." Here, "per dish part" is canceled out by the quantity "dish" in the multiplication, and what remains is the measurement unit "apple."

Multiplication is an operation in the world of numbers. However, with regard to interpretation in situations, it includes a metaphysical interpretation among physical quantities (measurement units) used in real life. As for the scaffolding used to support the interpretation and translation between a situation with physical measurement units and the world of mathematics, mathematical sentences of quantities such as "(dishes) × (apples/dish) = (apples)" are used even though they are mathematical informal–physical representations, which are not formally allowed as mathematical expressions in the world of mathematics.

The interpretation of "physical expression" in the situation (see Kobayashi, 1986) "(dishes) × (apples/dish) = (apples)" is inconsistent with the repeated addition of "(apples/dish)" in mathematics, which can be discussed as follows:

4 (dishes) × 3 (apples/dish) = 12 (apples)

≠ 3 (apples/dish) + 3 (apples/dish) + 3 (apples/dish) + 3 (apples/dish)[15]

≠ 12 (apples/dish), or ≠ (12 apples)/(4 dishes) =3 (apples/dish)

However, in mathematics textbooks, it will be as follows.

$$\underbrace{3\text{ (apples)} + 3\text{ (apples)} + 3\text{ (apples)} + 3\text{ (apples)}}_{4\text{ (dishes)}} = 12\text{ (apples)}$$

This inconsistency is related to embedding the ways of explanation in the quantities (several measurements) in the situation into the world of number operations without quantity. In general, the quantity for a denomination such as apples can be added because the quantity implies the measurement unit for counting, which is an apple. However, the measurement unit (quantity) produced by the rate of different units such as "apples/dish" cannot be added. To avoid such inconsistencies, when repeatedly adding (((3 + 3) +3) +3), we should see only the part of apples by disregarding the part of the "every (or per) dish" in each term and counting "4 dishes" repeatedly. Thus, we can say that repeated addition is the way to find the product by regarding 3 "apples" and 4 "dishes" instead of regarding 3 "apples/dishes" in the situation even if it is hiding the idea to see "3 apples" as one set for the dish. The translation between situations and multiplication is only possible using specific ways of reinterpretation of the measurement unit in situations, just like the one discussed above (see Chap. 5).

[15] This sentence itself is inappropriate because the ratio of different units cannot be added.

3.1.2.5 Using the Situation of Multiplication Only for the Attribute of the Object

In the 1950s, the Association of Mathematics Instruction (AMI), Japan, proposed to introduce the meaning of multiplication using the attribute of the object in relation to the binary operation with their theory of quantity (Kobayashi, 1986) and asserted that multiplication is not repeated addition (see Chap. 1). For example, two wheels are an attribute of a bicycle. In this situation, the row of 2 in the multiplication table is represented by the total number of wheels when the number of bicycles is given. The row of 3 is represented by the attribute of a tricycle. In this manner, AMI proposed to choose the specific situation in relation to the attribute of a specific object which cannot be divided for each row by the attribute of the specific object for the introduction of the multiplication table. Even the row of 0, which is normally not in the multiplication table, is explained with the belly button of a frog because the frog does not have it.

In Chap. 5, we will revisit the treatment of the attribute of an object for multiplication in the case of the Chilean approach with a discussion of making sense (or sense making) (McCallum, 2018).

3.1.2.6 In the Situation of Area, As for Extension to Decimals and Fractions

As it will be discussed in Chap. 4, for the extension of multiplication to decimals, conversion between measurement units such as 1.5 L and 15 dL is useful because it changes decimals into whole numbers, which can be seen as repeated addition, and the multiplication table can be applied. Area (diagram) is also used for the extension to decimals and fractions.

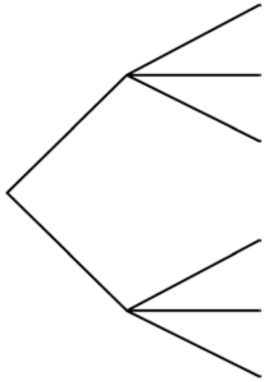
The area of a rectangle is defined by two perpendicular segments: $a \times b$, "length (longer side) × width," or "width × length." Before defining the area by multiplication, school textbooks usually introduce the dot array or block array diagrams[16] to explain multiplication (see Chap. 5). These array diagrams can be seen as a preparation to introduce the area (Mathematically, these can be seen as the idea for Cartesian Product: see 3.1.2.9). Conservation of the area of a rectangle in the dot array diagrams supports the commutative and distributive laws.

From the perspective of denominate numbers, the unit "1 cm^2" means the same area of the square as "1 (cm) × 1 (cm)." The number of unit squares in a rectangle with length 3 (cm) and width 2 (cm) is $3 \times 2 = 6$. Then, the area formula of a rectangle is "length × width." In the case of 2.5 (cm) × 1.2 (cm), it cannot be well represented by using the unit square "1 cm^2"; however, if we change the unit square to 1 mm^2 it means 25 (mm) × 12 (mm).[17] The area formula for a rectangle "length × width" supports the extension of multiplication from whole numbers to decimals

[16] The dot array diagram is also represented by parallel crosses.

[17] Japanese usually uses "dL" and "L" for the model diagram of decimals to show concepts such as $\frac{1}{10}$ L because 1 mm is too small for the model.

Fig. 3.2 Tree Diagram



and fractions through the permanence of form. In the case of 1.5 (cm) × 3 (mm) or 3 (mm) × 1.5 (cm), it has no meaning if they are expressed with different measurement units. Thus, we have to change them into 15 (mm) × 3 (mm) or 3 (mm) × 15 (mm). Changing the measurement units into the same quantity is a strategy for the extension of multiplication to decimals.

3.1.2.7 In the Situation of Tree Diagrams

In probability, multiplication can be applied in situations that can be explained by the tree diagram. In the tree diagram in Fig. 3.2, first there are two cases, then three cases that develop into six branches. If we use the term "splitting" for tree diagrams, one splits into two and then splits into three. This is written as 2 × 3. Based on the multiplication theorem of the probability for equally likely cases, it is written as $\frac{1}{2} \times \frac{1}{3}$. In tree diagrams, the operations 2 × 3 and 3 × 2 correspond to different diagrams and thus, area diagram is more preferable diagram explaining the commutativity of multiplication,

3.1.2.8 Seeing the Tree Diagram as an Operator

A multiplication on probability tree looks like an operator. In some situations, the symbol "×" shows processes such as "1 → (× 2) → 2" and then "2 → (× 3) → 6" in tree diagrams, and in situations of probability as "$1 \to (\div 2) \to \frac{1}{2}$" and then "$\frac{1}{2} \to (\div 3) \to \frac{1}{6}$" according to the multiplication theorem of probability in equally likely cases. Here, the process "→ (× 2) →" and "$1 \to (\div 2) \to \frac{1}{2}$" for indicating functions can be seen as operators. It implies that the "× 3" part of "2 × 3" or "1 × 2 × 3" and the "× $\frac{1}{3}$" part of "$\frac{1}{2} \times \frac{1}{3}$" or "$1 \times \frac{1}{2} \times \frac{1}{3}$" for showing the situations can be seen as operators.

In Indo-European languages, "× 2" should be written as "2×" and some prefer "2 ×" for showing the operator. Indeed, in $f(g(x))$, $f(x)$ is the operator for $g(x)$. However, "$1 \to (\div 2) \to \frac{1}{2}$" cannot be written as "$1 \to (2 \div) \to \frac{1}{2}$." In arithmetic, operators in "÷ 5" or "− 5" do not mean "5 ÷" or "5 −" because commutativity does not work for division and subtraction (except if it is an identity). For explaining the four arithmetic operations as operators, "× 2" is consistent usage with "÷ 2" in usage. Under the compartmentalization of knowledge, many Indo-European language users feel comfortable in using "2 ×" and "÷ 2" at the same time. However, preferring "× 2" is reasonable as long as it enhances the consistency of representations in the four arithmetic operations as operators. It is a kind of unary operator in mathematics. Indeed, even though European Language, the unary operator "^" is written in the right hand such as 3^2 (3 square). The matter of language will be discussed in the next section.

3.1.2.9 Activity of Elementary School and Cartesian Product

In Portugal's curriculum (Ministério da Educação e Ciencia, Portugal, 2013, p. 9) as mentioned in Chap. 2, the situations of multiplication for repeated addition are distinguished from those for combinatorics: "Solve one-step or two-step problems involving additive, multiplicative situations and combinatorial." The product of multiplication is also given by the counting activity in combinatorics: "Perform a given multiplication by fixing two disjoint sets and counting the number of pairs that can be formed with one element each by manipulating objects and by drawing." If we draw a diagram under this instruction, it should be a counting activity as shown in Fig. 3.3.

The combinatorial counting diagram in Fig. 3.3 can be seen as a part of tree diagram (Fig. 3.2). In the case of Portugal, it is introduced as another definition. In many countries, multiplication using a tree diagram is discussed after elementary school as combinatorics.

Watanabe (2003) explained Cartesian Product, A X B={(a,b): a∈A, b∈B}, as a meaning of multiplication. It can be seen from the perspective of probability tree because Fig. 3.3 can be seen from the perspective of ordered pares. On Cartesian Products, products by numbers of elements for A and B is a number of elements A X B. On set theory for Cartesian Products, commutativity and associativity do not work.

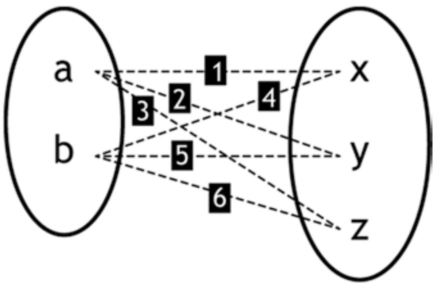


Fig. 3.3 Combinatorial counting

3.1.2.10 In Situations of Splitting as for Partitive Division

Confrey (1988) first presented splitting as a “multiplicative interpretation of partitive division” (p. 255) although repeated addition looks like a multiplicative interpretation of quotative division. Then, Confrey (1994, p. 292) defined splitting as “an action of creating simultaneously multiple versions of the original, which is often represented by a tree diagram.” Confrey focused on the development of ratios and proportional reasoning, including scaling, similarity, and exponentiation. All of these involve the coordination of two or more quantities or dimensions, which may or may not consist of levels of units that are commensurable.

Harel and Confrey (1994) point out that the idea of disaggregating or splitting is a powerful tool for teaching multiplication, which favors the extension of multiplication to decimals and fractions, providing a geometric, and not only an arithmetical, view of multiplication.

According to Steffe (2003, p. 240), the splitting operation is the simultaneous composition of partitioning and iterating, where partitioning and iterating are understood as inverse operations. Steffe (2003) and Hackenberg (2007) provide definitions focused on the unit (and coordination of a unit of units). Steffe’s splitting builds multiplication as repeated addition, based on counting, addition, and subtraction. The focus has been on the coordination of levels of units in students’ development of fractions, assuming equal-sized groups.

According to Harel and Confrey (1994), the operation that determines the total number of elements arranged in groups of equal quantity is of multiplicative character.

Following Confrey, in Fig. 3.4, equipartitioning/splitting indicates cognitive behaviors that have the goal of producing equal-sized groups (from collections) or pieces (from continuous wholes) as “fair shares” for each of a set of individuals. Equipartitioning/splitting is not breaking, fracturing, fragmenting, or segmenting in which there is a creation of unequal parts. Equipartitioning/splitting is the foundation of division and multiplication, as well as ratios, rates, and fractions (see Chap. 4).

Confrey maintains that the technique of splitting promotes early work with units that are not a singleton, diminishing the difficulty that children have in conceptualizing ratios and proportions and other areas of multiplicative structures. For Confrey, the appropriate conceptions regarding ratios and proportions are built not on the basis of multiplication as repeated addition but, rather, as a parallel numbering system that can be developed on the basis of a splitting operation. Confrey postulates that the foundation of the parallel system is developed naturally by children, and that the nature of such a system could have a powerful effect on the comprehension

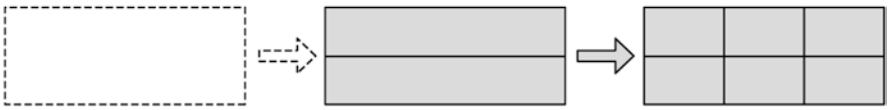


Fig. 3.4 Splitting equally: representation of 2×3 using splitting from the second rectangle to the third one. As well as the probability tree in Figs. 3.2 and 3.3, the splitting is consistent with multiplication as the operator: $1 \to (\times 2) \to 2, 2 \to (\times 3) \to 6$. Here the unit for counting number 6 is a smallest part of the rectangle in the right

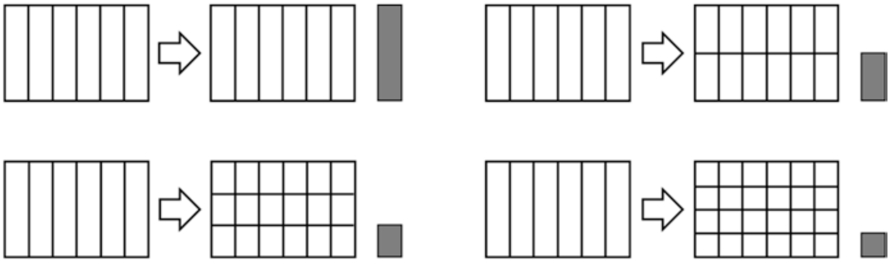


Fig. 3.5 Splitting changes the units' figures for products in the diagrams

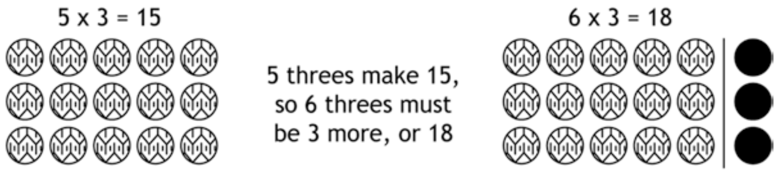


Fig. 3.6 Extending multiplication

of multiplicative concepts. Children could build a multiplicative world parallel to, complementary to, and interdependent on the additive world.

Splitting links multiplication and division because it includes the meaning of equal distribution (partitive division); however, it is inconsistent with repeated addition. Fig. 3.5 can be read as $6 \times 1 = 1 + 1 + 1 + 1 + 1 + 1$ and $6 \times 2 = 2 + 2 + 2 + 2 + 2 + 2$ and so on, but the basic units for counting the answers "6" and "12" are different. Splitting changes the unit of measurement before and after. In this context, the multiplicative world under the idea of splitting is consistent with equal division, partitive division, but independent of the additive world, as has been discussed regarding the rate of different units.Given this inconsistency with repeated addition, splitting in multiplication is inconsistent with definition by measurement according to the Japanese. Because splitting changes the units before and after multiplication, in Fig. 3.4, the whole rectangle on the left is 1 before the multiplication but is divided into 6 equal pieces after the second.

Considering this consequence, Portugal can be seen as a unique country as it introduces both meanings of multiplication (group of groups and combinatorics), as mentioned in the introduction to the discussion in Chap. 2.

3.1.2.11 Another Usage: Splitting in Relation to the Distributive Law

The terminology of "splitting" is also used in relation to the distributive law (van den Van den Heuvel-Panhuizen, 2001) but it is outside Confrey's claim in relation to partitive division. It is used in splitting, as in Fig. 3.6. Here, the knowledge of 5×3 (5 threes) helps to give meaning to 6×3: "If 5 threes make 15, how many are 6 threes?" For this expression, it is $5 \times 3 + 3$ and also can be seen as $(5 + 1) \times 3$.

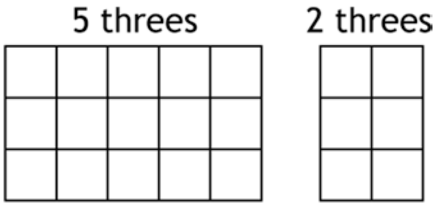


Fig. 3.7 Splitting and distribution

Table 3.1 Row of 2 and Row of 3 produce Row of 5

	1	2	3	4	5	6	7	8	9
Row of 2	2	4	6	8	10	12	14	16	18
Row of 3	3	6	9	12	15	18	21	24	27
Row of 5	5	10	6 + 9	8 + 12					

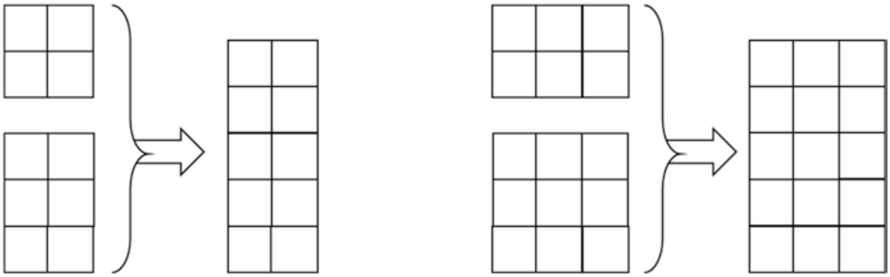


Fig. 3.8 Row of 5 from rows of 2 and 3 using the distributive law

On another usage of the word ‘Splitting’, it is used to explain the distributive law such as “If 5 threes are 15 and 2 threes are 6, then 7 threes must be 15 + 6, which is 21.” $5 \times 3 + 2 \times 3 = (5 + 2) \times 3$. The splitting on meaning of distribution is a key idea to extend the multiplication table and multiplication for multidigit numbers (Fig. 3.7).

In the multiplication table, (row of 3) + (row of 2) = (row of 5) if we adapt the distributive law (Table 3.1).

Here, splitting is used for inverse operation of distribution but not for equal division. It keeps the unit for counting. It is consistent with the array diagram and area. Japanese textbooks such as those from Gakko Tosho (Hitotsumatsu et al., 2005; Isoda, Murata & Yap, 2015; Isoda & Murata, 2011) use this idea to enable students to extend the multiplication table and adopt it by and for themselves (see Chaps. 6 and 7). The activity for this meaning of splitting can be explained by the theorem in action for the distributive law (Vergnaud, 1990; see also Tall, 2013, pp. 183–188) (Fig. 3.8).

3.1.2.12 Limitations of Every Model for Multiplication

According to Freudenthal (1983), multiplication is used to find a number, called the product, that is to the multiplier what the multiplicand is to the unit, such as 6:3 = 2:1 (6 is to 3 as 2 is to 1). It is related to proportionality and is consistent with definition

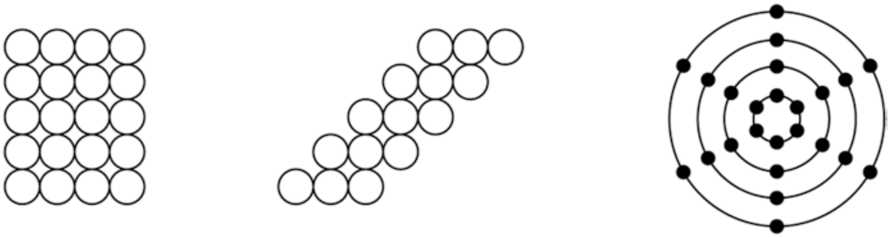


Fig. 3.9 Multiplication task variation (see Gakko Tosho textbooks—for example, Hitotsumatsu et al. (2005), Grade 2, Vol. 2, p. 12)

by measurement as used by the Japanese since the 1960s and Descartes's diagram (Fig. 3.1). In natural numbers such as 3×2, the multiplier 3 shows the number of repetitions of 2 for preferring multiplication in additive situations.

Even though the dot models in Fig. 3.9 can be used, there are various ways to find the units. In the models for repeated addition, all the units for counting should be seen as the same. Seeing the models from this perspective is possible when we have the idea of multiplication. At the same time, every model has its own nature as a mediational means. For example, the area diagram (model) for multiplication can be used for the extension of multiplication to decimals and fractions and positive (and 0) real quantities, and is appropriate to explain commutativity. However, it cannot be a model for multiplication of negative quantities. Descartes's constructions and proportional number lines, which are consistent with definition by measurement, can be applied for negative numbers, but commutativity cannot be seen instantly. Confrey links multiplication and partitive division; however, this is inconsistent with repeated addition because the unit for measurement changes.

From the viewpoint of magnitude, magnitude relationships (equivalent relations and order relations) can be illustrated by using models such as Descartes's construction, area diagrams, and dot diagrams[18] when their units of measurement are clearly embedded in the models. However, splitting and tree diagrams change their units. As Miwa (1983) mentioned, models function as a joint between mathematics and the real world. Gravemeijer (2008) discussed the model of a situation and the model for a form. Tall (2013) explained the conceptual difference and the development of the three worlds of mathematics by the terminologies "embodiment," "symbolism," and "formalism" and also the cognitive obstacles in one's development, which he termed "met-before." Freudenthal (1973) also explained the process of reorganization by mathematization. The Japanese use these inconsistencies as part of their curriculum content by explaining it as extension and integration for the opportunity to develop mathematical thinking (Chap. 1).

Depending on the context, the roles of the models are different. The number lines are bases for Cartesian coordinates to represent the changes in the graph of function and the figure defined by an equation. Descartes's construction of multiplication in his geometry is the origin of the Cartesian coordinates. Depending on the context, the roles of models change in the world of mathematics.

[18] Historically, Pythagorean schools used a dot diagram to represent properties of numbers.

Every model for multiplication has limitations in its nature. Every model for a specific situation is usually used for scaffolding. However, the reasoning when using the models is not the same as the formal reasoning even though they support mathematical–conceptual reasoning itself. In the case of Japan, the terminologies "concrete objects," "semi-concrete objects," and "abstract objects" have been used to discuss the different functions of the models and situations, and extension and integration have been the principles of the teaching sequence, corresponding to reorganization for mathematization.

3.1.2.13 Conceptual Fields for Multiplication

Vergnaud (1990) studied the conceptual field for multiplicative structures and distinguished three types of problems: isomorphism of measures, product of measures, and single measure space. This categorization provides a framework to distinguish conceptual difference in relation to multiplicative situations in teaching.

A problem of the first type, isomorphism of measure, is "A bag has 7 sweets. How many sweets are there in 6 bags?" A scalar resolution to the problem is "If there are 7 sweets per bag, in 6 bags there will be 42 sweets (7 sweets/bag × 6 bags)." A functional resolution is "If there are 6 bags, and in each bag, there are 7 sweets, then there will be 42 sweets (6 bags × 7 sweets/bag)." In the functional resolution, there is a movement from one measure (bags) to another (units of sweets). It is consistent with definition by measurement.

A problem of the second type, product of measures, is "We have 3 different shirts and 4 different skirts. How many combinations of shirts and skirts are possible?" This situation includes two fields of measurements that are composed without constituting a proportional function that associates the two fields. It is consistent with combinatorics and the probability tree.

A problem of the third type, unique measure space, is "Andres has thrice (3 times) the number of pencils that Jose has. How many pencils does Andres have if Jose has 4?" It is consistent with definition by measurement.

Vergnaud's categorization for multiplicative situations can be also seen in our terminologies for meanings of multiplication in situations (see Figs. 4.20 and 4.21 in Chap. 4).

This section has illustrated various meanings of multiplication; however, it has not discussed the curriculum design itself. As explained in Chap. 1, these terminologies distinguish the difference of content necessary for considering the curriculum and the task sequence. For example, the framework of the multiplicative structure must distinguish combinatorics and others, and combinatorics is consistent with splitting. Such discussions are bases to establish the sequence, but it does not explain well why only Portugal's curriculum introduces combinatorics from the beginning. The terminologies promote to distinguish conjectural difference but do not explain the curriculum sequence itself. The principle of extension and integration, or reorganization for mathematization, to develop mathematical thinking provide the sequence under the distinguished concepts (Chap. 1). The sequence will be discussed in Chap. 4 with further terminology.

3.2 Problems with Multiplication that Originate from Languages

Vygotsky (1934/1986, 1934/1987) and Wertsch (1991) enhanced the roles of mediational means to develop thinking. Under their perspective, children develop their mathematical thinking through mediational means used for communicating their language, such as speaking and writing, for making sense of what they learn.

Language enables us to verbalize numbers, such as "eleven, twelve, thirteen." However, writing does not always correspond with our way of speaking. For example, "twenty-five" is written as "25" under the base ten system and not as "205" (the way it is said). In English, as well as in other Indo-European languages, the way the four operations are spoken does not usually correspond to their algebraic expressions: "Add B to A" is A (augend) + B (addend), read as "A plus B." "Subtract B from A" is A (minuend) − B (subrahend), read as "A minus B." "Divide 12 by 4" is "12 (dividend) ÷ 4 (divisor)" but "multiplied 3 by 4" is "4 (multiplier) × 3 (multiplicand)": "–r" or "–d," which one is operato "–r"? Depending on Vygotskian claim, those inversions between grammatical structure and mathematical notation may set some limitations for learning mathematics in English, even though adults' users of English do not perceive any difficulties and inconsistencies in their usage. In Japanese, the grammatical expressions and algebraic expressions correspond well and there is no such inverted correspondence between their daily expression and mathematical expressions. In multilingual countries, the differences are more complicated. For example, the official language of Indonesia is inverted like English. However, like Japanese, the Javanese language of the central island of Java in Indonesia has no such inversion. In Javanese, 2×3 means "2, 3 times" as well as Japanese.

In the case of English and Spanish, the "×" symbol in the multiplication expression, which is read as "by" and *por* ("by"), respectively, does not necessary refer to the order of numbers. In English, there is no order if we say "multiply A and B."

However, if the expression is associated with the word "times" in English (or *veces* in Spanish) in real life—and, as such, the multiplier—the number of groups is placed to the left, as in Fig. 3.10. As for the language, there is a good ordinal correspondence.[19]

In Indo-European languages, when they introduce the multiplication table to be consistent with their languages, there is a syntactic contradiction between models A and B in Fig. 3.11 using "times."

The row of 2 in the multiplication table is usually shown below:

$2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 3 = 6$, $2 \times 4 = 8$, $2 \times 5 = 10$, $2 \times 6 = 12$, $2 \times 7 = 14$, $2 \times 8 = 16$, $2 \times 9 = 18$.

[19] As we discuss later, the daily usage of language and algebraic expression do not always correspond. For example, in English (Latin), the limited words for multipliers (such as "single," "double," "triple," and "quadruple") already include the meaning of "times" but are not applicable to the multiplication of any natural numbers. In real life, "double" in tea implies 2 cups of tea, with 1 cup as the unit. As in "half of something," the "of" implies the multiplication symbol "×". "Multiply 3 by 2 to get 6" in daily usage is "3 multiplied by 2 equals 6" in an algebraic sentence. However, "multiply 5 and 2" enhances commutativity and does not consider the order of the multiplier and the multiplicand.

Fig. 3.10 "×" as "times"

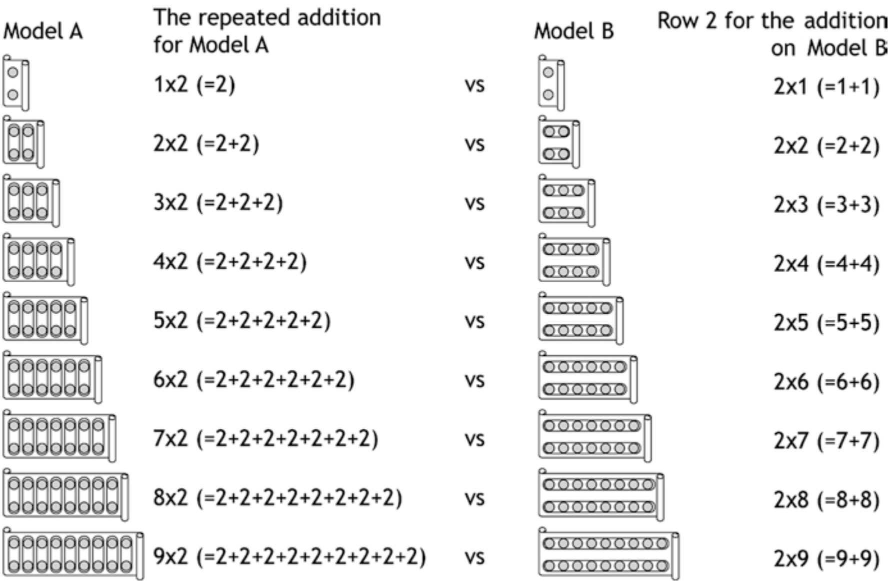


Fig. 3.11 What is repeated addition in a European language?

The difference between the consecutive products is + 2, the same as the multiplier. This property is used to proceduralize the sequence of the row of 2.[20] When we try to explain this constant difference with repeated addition, it will be understandable and reasonable for children to explain that "2×3 is 2 added 3 times, and 2×4 is 2 added 4 times, thus the difference corresponds to 2 added once." In model A, it should be written as 3×2 and 4×2. As long as we read the multiplication symbol "×" as "times," the appropriate interpretation based on the repeated addition is "2×3 [two times three] is $3 + 3$, and 2×4 is $4 + 4$, like model B. Every term increases by 1—that is, both 3s become 4s, and since there are two terms, the increase is 2." Adding 2 in repeated addition and the interpretation of the two terms will be contradictory for children as long as the definition of multiplication is repeated addition.

The reason for keeping the multiplier for the row number in the multiplication table—in this case, the multiplier for the row of 2—is based on multiplication in vertical form, which is called a multiplication algorithm in US English and the column method in UK English. For multiplying 43×2 in vertical form, the row of 2 is used for calculation from the lower line number 2 to the upper line number 43 (see Chap. 7). In multiplication in vertical form, multiplying from the lower number to the upper number is usually used not only in countries that speak Indo-European languages but also in countries that speak non-Indo-European languages, such as Japan.

In Fig. 3.11, the image of "increase by two in the row of 2" looks like model A. However, the row of 2 should be explained by model B. But model B cannot clearly explain the constant difference in the consecutive products. To avoid this contradiction that students may meet, there are two well-known traditional approaches:[21]

- The first approach enhances commutativity for applying repeated addition in model B: $2 \times 1 = 1 \times 2 = 2$, $2 \times 2 = 2 \times 2 = 2 + 2$, and $2 \times 3 = 3 \times 2 = 2 + 2 + 2$.

[20] As explained briefly in Fig. 1.1 of Chap. 1, this procedure is known as an automatized algorithm. Proceduralization means to produce an algorithm with meanings. In the Japanese approach, "thinking about how to calculate" is an objective, as well as understanding and achieving proficiency. Thus, it is recommended that the procedure is produced by students on the basis of the meaning they already know (see Isoda & Olfos, 2009, pp. 127–144). The Japanese use the meaningful pattern increase by the unit for memorizing the multiplication table. In Eastern culture, historically, the table should be memorized using the Chinese–Japanese abacus. In Western culture, memorization in mathematics education is usually discouraged because the word "memorize" often implies "without understanding" and the table is used for reference. From the Eastern cultural perspective, Western images of memorization look like a stereotype discussion. In East Asia, historically, people only used Chinese characters for academic subjects. Even if the word pronunciations were the same, they could reason by applying different characters to represent appropriate meaning. People were able to distinguish the meaning from the visible characters. The current simplified Chinese (pinyin) changed the tradition. Hangeul, and French-based Vietnamese alphabets become phonograms that have no intrinsic meaning for characters. However they still keep the tradition of meaningful memorization.

[21] Several approaches to vertical form will be discussed in Chap. 7.

- The second approach prefers that in the row of 2 defined by model A, the multiplicand is the constant, here 2, to be consistent with repeated addition, such as the following: $1 \times 2 = 2$, $2 \times 2 = 2 + 2 = 4$, $3 \times 2 = 2 + 2 + 2 = 6$, $4 \times 2 = 2 + 2 + 2 + 2 = 8$, ..., $9 \times 2 = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 18$.

Both approaches can be seen in countries that influenced from Indo-European languages and are considered to introduce the multiplication table reasonably. In the case of a number table without models and situations, the multiplication table is used to show the products of multiplication. In the multiplication table, if the multiplier is 2 and the multiplicand is 3, then the intersection, 6, is the product. In the multiplication algorithm in vertical form, mental calculation and mechanical writing of the products is necessary (see Chap. 7) and the first approach is preferred by many countries because it is well connected with the multiplication algorithm. Indeed, the order in the multiplication table, such as the multiplier and multiplicand, and the row and column, are related to multiplication and division in vertical form (the column method, algorithm, and long division) and the representation of proportion, $y = ax$ (see Chap. 4).

The reason why the second approach is not easily chosen is because it is inconsistent with the vertical form (column method), where multiplication is from the lower line to the upper line: If the row of 2 is "$1 \times 2 = 2$, $2 \times 2 = 4$, $3 \times 2 = 6$, $4 \times 2 = 8$, $5 \times 2 = 10$, $6 \times 2 = 12$, $7 \times 2 = 14$, $8 \times 2 = 16$, $9 \times 2 = 18$," 43×2 in vertical form becomes upper line to lower line and making decision of applying row of 2 from $2 \times [\ \ ]$ to $[\ \ ] \times 2$. And if so, the proportion changes to $y = xa$.

Against these two approaches, splitting has been proposed as an alternative approach in place of the traditional approaches. Indeed, Portugal considers both repeated addition and combinatorics (similar to the tree diagram) in introducing multiplication in the second grade. Due to the inconsistency between models A and B, it may be reasonable that Portugal introduces a number of cases from the second grade. It may be complicated for some of students if different situations cannot be seen as one operation for them. It will be supportive if students can use the idea of splitting to find the product, such as to split a rectangle horizontally into 2 and vertically into 3 (see Fig. 3.4). In English and other European languages, only splitting and the tree diagram are not complete approaches, unlike the others, because they change the meaning of the unit and thus are not consistent with repeated addition. On the other hand, in the Japanese syntax, the notation under the Japanese grammar does not produce such inconsistences (see Fig. 3.12) (Isoda, Arcavi, & Mena, 2007, p. 281). If the Japanese notation 3×2, which is written 3 [×] 2 here, is translated into English, it means "3, two times." In Fig. 3.12, "the difference in the row of 3 is the constant 3 (the constant property of the difference)" is explained consistently with repeated addition as follows: 3 [×] 1 = 3, 3 [×] 1 + 3 = 3 + 3 = 3 [×] 2, 3 [×] 2 + 3 = (3 + 3) + 3 = 3 [×] 3, 3 [×] 3 + 3 = (3 + 3 + 3) + 3 = 3 [×] 4, and so on.

Thus, in Japanese notation, there is no contradiction between repeated addition and the property of constant difference between consecutive products.

In some Indo-European-language-speaking countries that are supported by the Japan International Cooperation Agency (JICA), the Japanese notation is preferred for overcoming contradictions. Because as the discussion on Section 3.1.2.8, and Fig. 3.10, from the perspective of division operator, multiplier will be seen as the

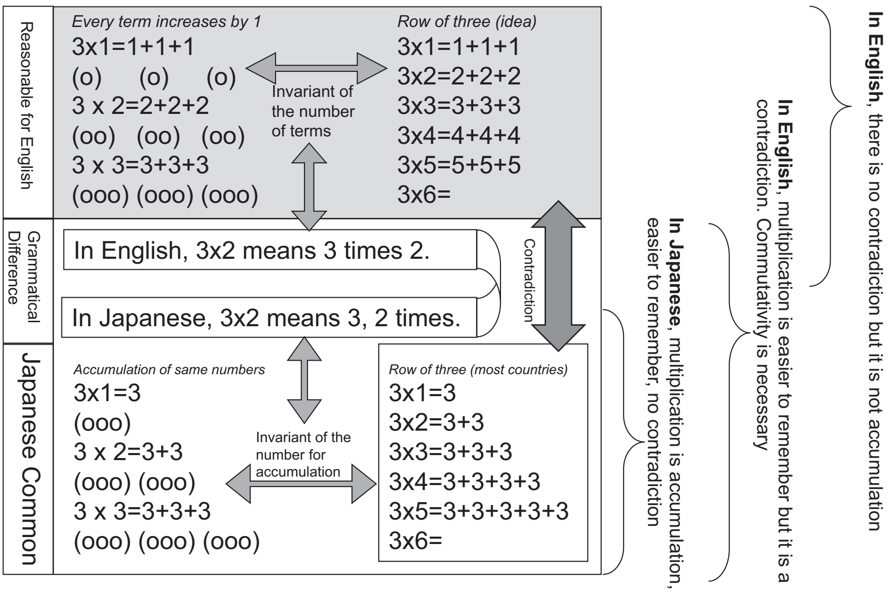


Fig. 3.12 Meanings and approaches of 3 × 2 in English and Japanese for applicable to traditional multiplication table and vertical form (column method) which multiply from the each digit in the bottom to the each digit in the top (see Chap. 7)

second number. For example, "'divide a by b' and then 'multiplied by c'" might be fine to be written as a÷b×c. It will be strange if we have to write is as c×a÷b in any time because it is read as 'c multiplied by a' and then 'divided by b'. In this approach, the terms "times" and *veces* create confusion in explaining and reading the multiplication symbol "×"; it should be read as "multiplied by," "by (por)," "of," or "and" instead of "times (*veces*)") because originally 3 [×] 2 meant "3, two times." Here, we cannot read the symbol "×" as "times." These syntactical changes are preferred by the curriculum departments in governments that have had deep discussions on historical tradition and current convenience. These countries use multi-languages on their histories and enhance the commutativity of multiplication.

The problem of inconsistency in English and Spanish originated from the difference between natural languages and mathematical notation.[22] Several difficulties might appear because the natural language should be preferred in school mathematics at the begging for referring to situations with quantities in real life. In the world of mathematics without situations, such confusion never appears.[23] Problematic appears in Indo-European languages but not in Japanese.

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[22] Fischbein, Deri, Sainati, and Sciolis (1985, p. 5), and Vergnaud (1990) also discussed the problematics of English but did not mention other languages. In this book, the roots of these contradictions are discussed in Chaps. 6 and 7.

[23] In informatics as a scientific language, mathematical notation itself can be changed. In programming language, "=" usually means substitution. In metanotation in informatics, there are Polish notations, reverse Polish notations, and others such as normal mathematical notations.

In theoretical arithmetic under normal mathematical notation, natural numbers are defined by the inductive definitions of Dedekind and Peano, and, in theory, the product is deduced inductively "$M \times (N + 1) = M \times N + M$" (Olfos, 2002). In this compete-inductive definition of multiplication, which is the same as the constant property in the table, the Japanese multiplication notation in Fig. 3.12 is consistent with the mathematical notation. The expression 3×4 refers to a group of three as the unit and 4 as the number of groups/units. Consequently, $3 \times 5 = 3 \times (4 + 1) = 3 \times 4 + 3$, as the unit is added to the initial groups. On the other hand, as previously mentioned, the English usage of "times" corresponds to $3 \times 5 = 5 + 5 + 5$. To see the sequence increase by three in English notation as for the repeated addition of 3, it must be changed, like $3 \times 5 = 5 + 5 + 5 = (4 + 1) + (4 + 1) + (4 + 1) = 3 \times 4 + 3$. It is an interpretation far from the inductive definition of multiplication.

The inconsistencies of expressions between natural language and mathematical notation in the Indo-European languages are problems not only for multiplication but also for the other three operations, as already mentioned. These inconsistencies produce difficulty for explanation of arithmetic in the said languages. As a consequence, there are projects that prefer the Japanese notation system in Central America, Thailand, and other places. To maintain consistency between language and mathematics, Japanese textbooks have established a sequence for extension that can be seen as attractive in being understandable (see Chap. 4).

If you feel uncomfortable about discussion of the Japanese notation of multiplication and not your notation, this is because of your familiarity with your mother tongue. However, we should note that our acquired usage itself can be seen as the result of our achieved curricula. There are various approaches for solving the matters in Fig. 3.12. There are further reasons why the Japanese approach is selected by some countries[24] as an alternative approach, like the idea of splitting in the US approach and combinatorics in Portugal. One reason is consistency of definitions with the extension of numbers and operations, and another reason is consistency with the multiplication table. Other reasons such as consistency of multiplication in vertical form and division and so on will be clearly illustrated in Chaps. 4, 5, 6, and 7 with explanations of the Japanese approach. The Japanese approach has rationality but it is one of the various existed approaches. The National Curriculum on Colombia introduce multiplication as 'multiplier x (multiplied by) multiplicand' at the lower grade and then,upper grades, treat 'multiplier' like an operator in relation to 'divisor' (see Section 3.1.2.8). Such an approach is normal for Latin America. On the next section, we would like to discuss the historical usages and influence to Chile.

[24] In Latin America, the countries of Honduras, Guatemala, El Salvador, Nicaragua, the Dominican Republic, and Mexico prefer Japanese notation based on Japanese textbooks.

3.3 European Languages and Their Historical Usages

Depending on historical origins such as languages and developments,[25] there have been various textbooks in different periods and regions that placed the multiplicand on the right, while others placed it on the left. In the thirteenth century, Ibn al-Bannā (from Almohades (Morocco), which included a part of Spain) explained a procedure for multiplying in columns (grids) by placing the multiplicand at the top of the column and the multiplier to the left or to the right of the column; this was later known as "Napier's bones" or "rods of Napier (1617). Ulloa (1706) indicated that the multiplier was placed below in the column algorithm. Before the predominance of modern mathematics, there were texts in Spain that presented the multiplier after the multiplicand, as the second number (Rey Pastor & Puig Adam, 1935). With the arrival of set theory, the language changed, and inconsistencies appeared in mixing arithmetic language with algebraic language. Prima-Luce (1976) stated, "We call a 'product' the cardinal of the Cartesian product. The second factor is called the multiplier. The first factor is called the multiplicand. $2 \times 3 = 3 + 3 = 6$." (Prima-Luce, 1976). There are two inconsistencies in the above description: maintaining names connected to the contexts together with formal language and exemplification with an inappropriate numerical representation.

The representation of "two hundred" can be seen as "2 times 100." Spanish grammar accepts this, saying *dos manzanas* for "two apples" although nouns usually come before adjectives in Spanish, as in *manzana roja* ("apple red" rather than "red apple"), which involves a kind of rupture. $2A$ is $A + A$ in algebraic notation. However, in the first grade, students learn arithmetic operations starting with situations like "Add something to A" or "Take something away from A." So, $A + B$ and $A - B$ initially are represented by situations that add B to A or take B away from A. A is the noun or the subject to be transformed, so A comes before B. If we adopt this approach to $A \times B$, it is possible for Spanish (Roman) to see "A" as the multiplicand, as in Japanese, because the action is done by B, the multiplier, as in the previous discussion of the operator. In reciting "2 times 3, 6," "2 times 4, 8," "2 times 5, 10" the number 2 can be seen as being multiplied by several numbers as the action. The sequence of results is 2, $2 + 2$, $2 + 2 + 2$, and so on. In this instance, it is like the probability tree that was discussed earlier in this chapter. In this manner, "2 times 3" implies "2, three times"; "three times" looks like part of the operator, and the first number 2 looks like the multiplicand in Japanese. In Spanish, $A \times B$ as "A times B" and "A multiplied by B" provide a polysemy, which affects the meaning of the

[25] European languages can be divided into Latin–Roman, Indian Europe (for example, German, English, and Nordic languages), and Slavic. Some languages such as Finnish and Hungarian are independent of these categories. Here, we are referring to Latin–Roman and Indian Europe, especially Spanish. Cajori (1928) explained that multiplication symbol "×" was introduced by Oughtred (1631, used Latin Edition, 1667). Oughtred used column multiplication for number and introduced "×" for his algebraic notation. He mentioned factor at introduction and discussed his column multiplication. He did not used symbol "×" for column multiplication. He discussed significance of multiplication for logistics and estimation of multiplicand and calculation of multiple on the column (p.8). It implicates that multiplicand comes upper and multiplier comes lower on column. See (Chap. 7).

expression for the multiplication table. Consequently, even students misinterpret the pattern $2 \times 6 = 2 \times 5 + 2$ as $(2 + 2 + 2 + 2 + 2) + 2$. It is not necessary to say "it should be 6 + 6" if they can see it like this in this situation. This is a possible reason why Central American countries such as Honduras, Guatemala, Nicaragua, and El Salvador prefer the Japanese notation of multiplication in JICA projects.

Freudenthal (1983) highlights the fact that the language of mathematics differs greatly from everyday language used in different countries where it has developed, and adds that the divergence between a natural language and mathematical language can in fact create learning difficulties. He also points out that "4 + 3" is a strange way to write the task "add 3 to 4," which mathematically indicates the sum of 4 and 3, and that everybody reads "four plus three" even though it does not agree with their language (English or German, and also Spanish or French). At the beginning of the twentieth century, "7 − 4" was read in German as *vier von sieben* ("4 from 7"). These antecedents are indications that in German and English, it would be natural to write the subtrahend and then the minuend, and by analogy the multiplier would precede the multiplicand.

With regard to Spanish, which originated in Castile, Vallejo (1841, p. 26) wrote, "The expression '5 − 3 = 2' means that after removing 3 units from 5, 2 are left, and is read 'five minus three equals (or is equal to) two.'"

Anglo-Saxon languages differ from Latin languages. Base twelve English measurement systems and base eight Spanish playing cards are remnants that predate the Indo-Arabic decimal system, which penetrated Europe through southern Spain. The Arab invasion of Spain during the eighth century brought the decimal system with its operative algorithms and modalities of oral expression, which surely conflicted with the existing European languages.

Research around 30 years ago by Fischbein, Deri, Sainati, and Sciolis (1985, p. 5) and Vergnaud (1990) revealed that differences between the multiplier and the multiplicand are at the root of different complexities presented by multiplication problems (which we mention in Figs. 3.11 and 3.12) and influence the decision of anticipating the operation that needs to be made.

3.3.1 The Transition in Chile

Chile inherited the Spanish language in the nineteenth century, along with textbooks that place the multiplicand first, on the left. Later, with North American influence and the universality of the International Commission on Mathematical Instruction (ICMI), Chilean mathematics programs in 1968 (MINEDUC, 1968) introduced multidigit multiplication in the fifth grade and used the term "factor" together with algebraic terminology with the idea of the multiplier on the left.

The current Chilean mathematics programs (MINEDUC, 2013b, 2013c) maintain the introduction of multiplication with the term "factor" and do not use the terms "multiplicand" and "multiplier." The current programs for the third and fourth grades identify the word "factor" as a key term, which is cited more than a dozen times in each program.

The mathematics program in the national curriculum for the second grade in Chile (MINEDUC, 2013a) presents multiplication as repeated addition, without

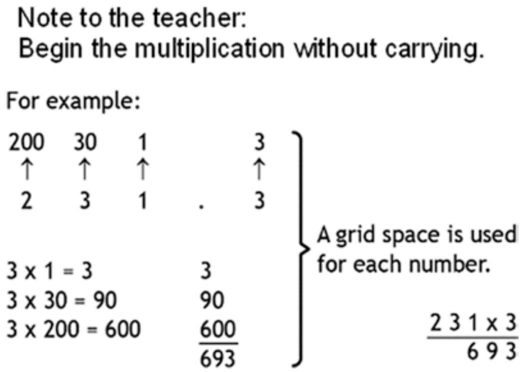

Fig. 3.13 Change the direction of multiplication from "left to right" to "right to left" for using the multiplication table

establishing associations between the factors and the meaning that each of them can take on. It does not establish connections between the number of groups and the term "times." It does not identify "times" as a pattern associated with a multiplier and a multiplicand.

The idea that the multiplier goes on the left is observed in the examples, without the term being mentioned, coinciding with the approach of textbooks from Singapore. In the second grade, the program says, "Demonstrate understanding of multiplication. using concrete and pictorial representations; expressing multiplication as the addition of equal addends. to construct the multiplication tables for 2, 5, and 10." The program for the third grade adds, "to construct the multiplication tables up to 10." The program for the fourth grade says, "Demonstrate understanding of multiplication of 3-digit numbers multiplied by 1-digit numbers" and the program for the fifth grade adds, "of 2-digit numbers multiplied by 2-digit numbers."

The same Chilean mathematics program for the fourth grade (MINEDUC, 2013c, p. 66) currently presents as an example the calculation "231 × 3," beginning the calculation on the right, although the multiplication table is introduced with the multiplier on the left, as can be observed in the bottom part of Fig. 3.13.

In Chile, textbooks and even curriculum standards have adapted influences from other countries and the tendencies in mathematical education of each period. Simultaneously, old textbooks still circulate in the country. In some textbooks and in the language of some parents and tutors, the teaching of the multiplication table—and, to a greater degree, the use of the procedure for multiplying from right to left—persist with the multiplier on the right. Despite all of these, for adults (and even for primary mathematics teachers), "3 × 2" means "3 times 2" and "3 multiplied by 2" without distinction, as the order of the factors does not change the product.

3.4 Final Remarks

This chapter has addressed the problematics in the conceptualization of multiplication in school mathematics—including definition of multiplication by measurement, various meanings of multiplication, and the problem of syntax in relation to languages and grammar—and has discussed historical transitions and adaptations to a country such as Chile.

The discussions of those problematics provide some answers to the related questions posed in Chap. 2; however, this chapter has not mentioned the curriculum and the task sequence themselves, which are necessary to consider for designing lessons. The mathematical terminology in this chapter provides a basis for the necessity to consider the Japanese approach in Chaps. 4, 5, 6, and 7. The terminology for the curriculum and the task sequence will be discussed in Chap. 4.

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Chapter 4
Introduction of Multiplication and Its Extension: How Does Japanese Introduce and Extend?

Masami Isoda and Raimundo Olfos

In Chap. 1, the Japanese theories related to lesson study which oriented the development of students who learn mathematics by themselves through the development of mathematical thinking were summarized by the aims and objectives under the national curriculum standards, the terminology to distinguish content, the task sequence to develop students, and the teaching approach. In Chap. 2, the questions to make clear the Japanese Approaches were posed through the comparison to other countries. In Chap. 3, the difficulties to learn multiplication from using national languages towards mathematical form was described. In this chapter, Japanese curriculum sequence will be over-viewed from the perspective to make clear the extension and integration process shown on Fig. 1.1 of Chap. 1 by using the terminology and task sequence related to multiplication. It also describes related content such as the unit, division, decimals, fractions, and proportionality, and how each content is embedded for the preparation of future learning for sense making. The necessity to distinguish multiplier and multiplicand will be explained to sequence of these contents. The significance for the definition of multiplication by measurement in Chap. 3 will be also confirmed in relation to proportional number line.

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4.1 The Introduction of Multiplication Using the Japanese Approach

As discussed at Chap. 2, the process of teaching multiplications are usually fixed depending on the national curriculum standards in every county. For example, Hulbert, E. T. et al (2017) brought their research for teaching multiplication and division to the classrooms in USA under the Common Core State Standards Mathematics (2010) and show their teaching process clearly. Here we would like to illustrate how Japanese sequence of teaching multiplication are consistent with the related content under the principle of Extention and Integration (see Chap. 1).

Under Japanese grammar, the definition of multiplication in the second grade consists of associating the mathematical sentence A [×] B,[1] "A, B times" with the answer C (A [×] $B = C$), which corresponds to the total number of elements, with A, the number of elements in each group, and B, the number of the same groups under the definition of multiplication by measurement in Chap. 3. The mathematical expression "3 [×] 2" codifies the operative procedure "three, two times", in English or Spanish. The Japanese syntax of multiplication gives independent meanings for A (*kakerareru-su*; "multiplicand" in Japanese) and B (*kakeru-su*; "multiplier" in Japanese), which, as they are associated with situations in context, make it necessary to consider that A [×] B and B [×] A refer to different situations, even though they give the same numerical result, C.[2]

On the other hand, in the case of English and Spanish notations, to avoid inconsistencies (which were discussed in Chap. 3), commutativity is enhanced from the beginning. Then, students do not care about the independent meanings of multiplier and multiplicand, thus $A \times B$ and $B \times A$ will be seen as the same from the introduction of multiplication. Some countries such as Brazil just call them factors (see Chap. 2). However, if we do or do not distinguish A (multiplier) and B (multiplicand) in multiplication, how will this influence other teaching content?

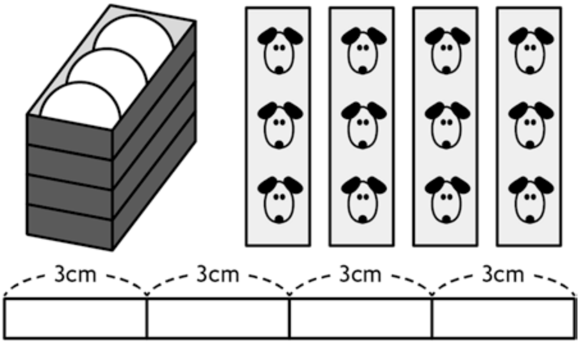
Here, we explain why the Japanese curriculum has a consistent teaching sequence, and then we explain the hidden inconsistencies seen in other countries. For explanation, we go back to the definition mentioned in the Japanese mathematics curriculum guide (Isoda, 2005, 2010; Isoda and Chino, 2006), which points out that multiplication is used to find the total based on "how many units there are when a unit is given." This was explained as definition by measurement in Chap. 3. For the second grade, the guide proposes the use of groups as a unit. Here a unit means an arbitrary measure wherein any number can be a unit in Descartes's definition (see Chap. 3). In the 1989 teaching guide, translated into English (Isoda, 2005), it is interpreted that:

- The study of multiplication begins as an efficient means to express a unit repeated several times. The unit can be the cardinality of a set or a group. So, if a group of

[1] In Japanese grammar, in the official placement of multiplication, the unit is on the left. In this chapter, we write Japanese multiplication using "[×]" instead of "×" to highlight this. In A [×] B, A is the multiplicand and B is the multiplier.

[2] In the ancient Mesopotamian language, Sumerian, the order of words is the same as that in Japanese (Muroi, 2017); it is represented as *A a-rá B túm*. Here, *túm* means "carry" (see Chap. 3).

Fig. 4.1 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, p. 18



3 elements is repeated 4 times, there are 3 + 3 + 3 + 3 elements, which is abbreviated as 3 [×] 4.
- The definition of multiplication arises from the assignation of the name of every quantity as an object of measurement; that is, the definition is set as the way of measurement in tape diagrams, which can be adopted into proportionality in the tape diagrams later.
- The meaning of multiplication is addressed gradually in the extension from restricted situations for repeated addition and times[3] up to decimals and fractions. The introduction of every row of the multiplication table begins with situations for quantities and extends multiplication up to 10 times. Units that are larger than 10 are discussed in the next grade. Continuous units are discussed, particularly with the centimeter as a unit of measurement[4] (they already know measurement by 1 cm), for extending it to decimals and fractions (quantities of length produce continuous numbers) in later grades (Fig. 4.1).

The introduction of multiplication in the second grade in Japan (See Situation A and Meaning A in Fig. 1.1 of Chap. 1) is based on the operation to get the total quantity when the unit quantity and the number of units are known. It means, for natural numbers, a number in every group (unit) and counting the number of the same groups (units). It is the definition of multiplication by measurement and the whole number at this stage—that is, a set of groups or a group of groups (see Chaps. 2 and 3). Two different quantities using denominate numbers are necessary; for example, each plate has 3 apples and there are 4 plates (see Chap. 3). This is significant in two ways. One is to explain the situation briefly, which enables students to distinguish ordinary addition. Another is repeated addition in situations which is the starting point in the proceduralization from repeated addition to using the multipli-

[3] The Japanese usage of "times" (*bai*) is not only limited to the number of repetitions. The number of repetitions is usually represented by *kai* instead of *bai*; *bai* in Japanese is used up to multiplication of decimals and fractions, and for proportionality in the context of enlargement and reduction of the given number. The idea of *bai* is the key idea in development of proportionality. Its usage is rather close to "of."

[4] In Japanese textbook, the symbol "×" is read as *kakeru* or *Kake*. It is close to *por* in Spanish and "by" in English. The tape diagram is introduced later after the redefinitions of "×" as *bai* (times). Bai is defined using the tape diagram. It is used for extension of numbers to decimals and fractions.

cation table. As discussed in Chap. 3, repeated addition is not multiplication as a binary operation, even though it is the only way to find the answer at the start. Multiplication as a binary operation begins with the multiplication table.

At the introduction of the multiplication table on Procedure A of Fig. 1.1, repeated addition is necessary. On the extension of the multiplication table in every row, the pattern of products increases by the unit of the row. This becomes the principle to construct the multiplication table (see the discussion on permanence of form in Chap. 3 and Table 1.1). On the teaching of the multiplication table, the pattern of products is introduced and then all rows are combined to form the multiplication table. Knowing the properties of the multiplication table promotes the change in the meaning of multiplication from conceptual to procedural without concrete situations (see Chap. 1 and 3, especially Fig. 1.1). The properties of the table itself provide the procedural meaning of multiplication in the world of mathematics, which exists as patterns for sets of products without quantities. Chapter 7 of this book shows teaching of multidigit multiplication in the third grade in Japan. The proceduralized table with patterns also results in the conceptual meaning of multiplication (this emerges as a procedure in the second grade; see Chap. 1, Fig. 1.1), which is used for developing the procedure in vertical form (the column method). The multiplicative procedure of multidigit numbers can be created based on the conceptual and procedural knowledge of the multiplication table and the base ten system up to the second grade.

The dualities through conceptualization and proceduralization in the Japanese teaching sequence for the conceptual development of multiplication are not limited to multiplication but also apply to all teaching sequences for mathematics in Japan (Isoda, 1996; Isoda and Olfos, 2009, pp. 127–144). Those gradual conceptual development processes are well illustrated in the Gakkotosyo textbooks (Hitotsumatsu, 2005; Isoda and Murata, 2011; Isoda, Murata, and Yap, 2015).

4.1.1 The Way to Initiate the Situation for Multiplication Before Repeated Addition in the Japanese Approach[5]

The first task consists of challenging the students with multiplicative situations so they are able to distinguish them from additive situations (see Figs. 4.2 and 4.3). Figure 4.3 is discussed at the introduction of the symbol “×” (multiplied by) with the expression “multiplied by” (*kakeru* in Japanese) or just “by” (*kake* in Japanese). For future extensions to fractions and decimals, the Japanese textbooks prefer the multiplication symbol to be read as “multiplied by” or just “by” instead of times.

The introduction of multiplication is enhanced to form groups (sets) of an equal quantity (set) and to determine the total number based on the number of groups. It is a simple activity for an adult. However, to distinguish it as a binary operation from ordinal additive situations, these tasks are necessary for second-grade students. Thus, the second-grade textbooks include various situations for multiplica-

[5] This section explains the outline. In Chaps. 5 and 6, it will be explained more concretely.

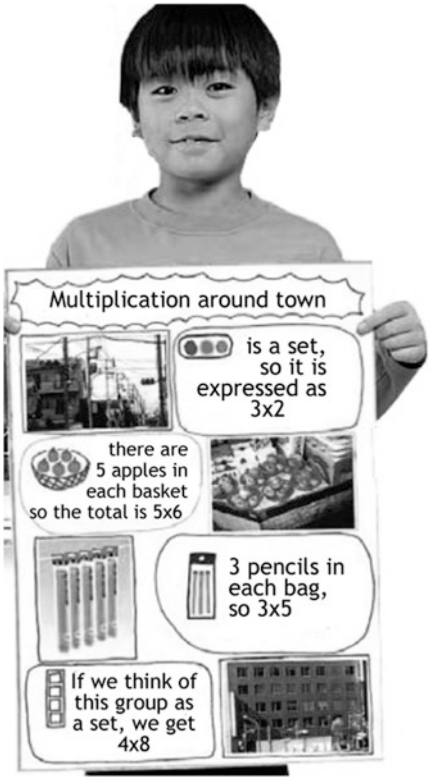


Fig. 4.2 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, p. 47

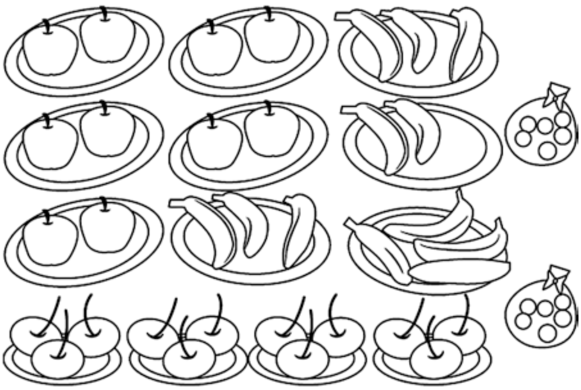


Fig. 4.3 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, pp. 2–3, "Look at the banana plates. Is it a multiplicative situation?"

tion based on the number of groups, where the group represents the unit. For example, in Fig. 4.2, if each stoplight has 3 lights, 2 stoplights have 6 lights. In Fig. 4.3, if we move one banana to another plate, we produce a situation where there are 3 bananas for every plate and 4 plates. Based on those tasks, teachers enable students to see the situation as a multiplicative situation by seeing the repeated

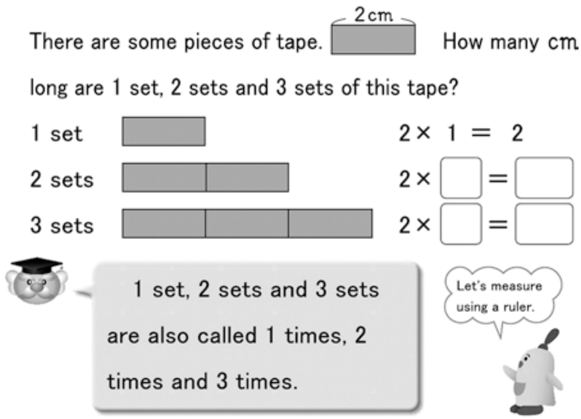


Fig. 4.4 Gakkotosyo (Hitotsumatsu, 2005), Grade 2, Vol. 2, p. 10., pp. 2–3

quantity as the unit, and the expression of multiplication is introduced. When the students are able to distinguish the repetition of these situations with other additive situations, we can say that they are able to see the world by multiplication, as in Fig. 4.2.

In the beginning, the students will find the product by repeated addition or counting; however, in the addition form, it is necessary to make clear the number of repetitions. For this necessity, a multiplication expression is introduced. For knowing this reasonable and effective way of representation, the Japanese textbooks show various situations to count the number of times (*kai* in Japanese), for the students to be aware of its reasonableness and the simplicity of its form.

As shown in Fig. 4.4, a unit length of tape such as 3 cm is introduced. The tape model (diagram) is necessary for later extension to continuous numbers in relation to proportionality. After the definition of multiplication with a situation as a binary operation based on definition by measurement (see Chap. 3), the tape diagram in Fig. 4.4 is introduced and the term *bai* looks the same as “times” (in English) and *veces* (in Spanish) at this stage. The Japanese usage of *bai* (“times”) is not the same as the English and Spanish usage of the symbol “×”; in Japanese, *bai* is the terminology used to explain proportionality. Later, the term *bai* can be used for extension from whole numbers to decimals and fractions by using proportional number lines (see Sect. 4.3), and then it becomes the base to define proportions in relation to ratio.

For finding the answer or product for a binary operation, the term *bai* is also connected to repeated addition (*kai* in Japanese). As mentioned in Chap. 3, the repeated addition meaning of “times” in English and *veces* in Spanish is inconsistent with the multiplication table in relation to the order of expression.

After this, the multiplication table is introduced with the rows of 2 and 5, which have already been learned as ways of counting. Those rows are convenient for students because they already knew the answers from their experience of counting by 2s and 5s. Recognizing that the products can be increased by the units becomes the basis to extend multiplication in each row up to 9. After the exploration of the properties of the multiplication table, a project to find multiplicative situations (as in Fig. 4.2) is done for students to explain the significance of multiplication.

4.1.1.1 Repeated Addition and Challenges to Difficulty

When the students begin the study of multiplicative situation, it is normal for them to see the situations as a kind of addition because they have learned only addition and subtraction, and they are recommended to use what they have already learned in the Japanese approach. To distinguish multiplication from ordinary addition, they need to reorganize their knowledge from counting by one to counting by the unit number (such as 2, 3, . . . , 9) as sets in the manner shown in Fig. 4.3 (see also Chap. 5, Sect. 5.2). Up to the introduction of multiplication in the Japanese approach, the Japanese textbooks provide opportunities for students to learn that any magnitude can be a unit for counting. If the teachers have only discussed counting by one in the first grade, this becomes an obstacle for learning multiplication in the second grade. Thus, the first-grade Japanese textbooks enhance the activity of measurements to set the tentative unit for measuring as well as counting by 2s and 5s. The introduction of multiplication from verbalizing of the grouping such as in Fig. 4.3 looks like repeated addition to adults. However, this verbalizing activity enhances the ability of the students to explain the situation by a number in every group (unit) and to count the number of the same groups (units), by using different denominate numbers such as “3 apples for each plate and 4 plates.”

During the introduction of multiplication, if the teachers do not include the denominations of quantity (see Chap. 3), such as apple and plate, and just say “3 for each and 4,” the students lose the point of learning at the beginning even though it is routine for those students who have learned it well. If we compare it with “3 apples for each dish and 4 dishes,” they may understand what the object of counting is. The number of dishes should be clearly mentioned for showing the unit of counting. First, the Japanese textbooks ask the students to explain the situation of grouping to develop the notion of the unit (Fig. 4.3) and, later, shift to repeated addition. In this context, “How many apples are there? And how many dishes are there?” and “Which have the same number of fruits in the dishes?” are not the same because the first questions can be a question of counting and the second one is a question for explaining a multiplicative situation. To clarify such differences, the Japanese textbooks consider tasks that distinguish the situations by sets of groups.

Repeated addition in situations is necessary for developing the multiplication table. However, as was mentioned in Chap. 3, it is not so much reasonable to represent repetition of 1 and 0 in situations such as $1 + 1 + 1$ and $0 + 0 + 0 + 0$ by multiplication. If the counting unit is 1, it is not addition but just looks like counting. Japanese textbooks introduce the row of 1 by the permanence of form: the situation “2 apples for every dish and 3 dishes” is 2 [×] 3; if it is “1 apple for every dish and 3 dishes” how shall we express it? This is the question for the permanence of form. In the case of 0, the Japanese textbooks in the second grade discuss 10 times (*bai*) instead of multiplying 0.

4.1.1.2 Use of the Multiplicand and Multiplier for Students to Think of Division Situations by and for Themselves

When Japanese students begin multiplication with a situation of the type "2 multiplied by 3" in English, they learn that this is *3 ga 2 ko* ("3, 2 times")—writing it as "3 [×] 2"—and they learn to read it as *3 kakeru 2 ha 6*. This Japanese notation may be misread by English readers, however; as mentioned in Chap. 3, the notations of multiplication in English and Spanish include contradictions.[6]

The multiplicand as the first element in multiplication and the multiplier as the second element in Japanese are introduced in the multiplication table and used in the extension of multiplication to fractions, decimals, and negative numbers (see the discussion of Descartes in Chap. 3).

In the Japanese curriculum sequence, the "multiplicand [×] multiplier" in multiplication is directly connected to division. Students who are able to define division by themselves are expected to use the idea of multiplication in situations involving division. In this context, teachers have to enable students to distinguish the first and second numbers by identifying the multiplicand and multiplier. The two different situations in division, which are called partitive and quotative division, can be distinguished by this identification.

Students are able to think by and for themselves, and are able to reorganize what they already know by using their daily language as well as their learned mathematical language. In this curriculum sequence, Japanese primary school teachers in the lower grades ask students to codify the situation for the expression "2 [×] 4" and distinguish to represent it as "4 [×] 2." Japanese teachers try to develop students to develop mathematical sense to make sense by and for themselves based on what they have learned and to elaborate the definitions in their classes (see Chap. 5). They are asked to analyze the situations and formulate them by using their everyday language (see Fig. 4.5).

4.1.1.3 Commutativity and Order in Expression

In Japan, it is expected that all students will memorize the multiplication table in the second grade. For developing the table, the property "increase by 3 if the row is 3" is used.[7] For memorizing the multiplication table, the teachers shorten an expression such as *3 kakeru 2 ha 6* to *3 kake 2 ha 6* and to *3 2 ga 6*[8] by characteristic abbrevia-

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[6] In Japan, only primary school teachers recognize the difference between 3 × 2 and 2 × 3, explaining the meaning of multiplication in each situation. In secondary school, teachers never distinguish these two because they do not feel any necessity to do so in their teaching. Primary teachers have to consider it on their curriculum sequence.

[7] Here, we call this a "procedure with meaning" (Isoda and Olfos, 2009). Students memorize the table using properties (patterns), meaningfully.

[8] The term *ga* is only used in the event that the product is less than 10. If it is more than 10, even *ga* is omitted, such as 3 [×] 4 = 12 ("3 4, 12").

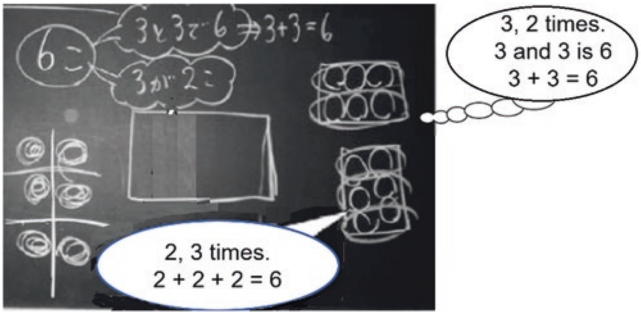


Fig. 4.5 How we read and express from Mr. Satoshi Natsusaka's lesson

tions in the Japanese language which in English mean "three one, three" (3 [×] 1 = 3) and "three two, six" (3 [×] 2 = 6)."

Once the students have completed and memorized the multiplication table up to 9 × 9,[9] they lose the need to use repeated addition to get the product within the range of the multiplication table. They get the product of the binary operation automatically, without referring to situations and repeated addition. In the task to find the properties of the multiplication table, without considering situations in multiplication, the students can find many of them. The numbers in the table have a symmetrical property on the diagonal. For example, 2 × 3 and 3 × 2 are the same value since the answer does not change even though the order of the multiplicand and multiplier is changed. This discussion is on multiplication expression. On the other hand, in a concrete problem-solving situation, teachers and students continue to distinguish which one is the multiplicand and which is the multiplier in situations such as partitive division and quotative division in the third grade (Isoda, 2010). It is useful up to ratios and rates for considering which one is the base unit quantity in the situation.

4.1.1.4 Differences in the Multiplier and Multiplicand in an Array and a Block Diagram

In Japanese classes, the teachers usually ask how to read the array or block diagram like those in Figs. 4.5 and 4.6 (see also Chap. 3). This is an opportunity to identify and distinguish the multiplicand and multiplier.

Figure 4.7 is a representation of 4 plates (the unit or group). Each plate (unit or group) has 2 sweets—that is, "four times two" in English—and this is codified using the mathematical expression "4 × 2." In Japanese, it is represented as "2 [×] 4" ("2, 4 times"). In Chap. 3, the Japanese notation is consistent with the property

[9] The Japanese numeral system follows the base ten numeral system. The base ten numeral system can be well recognized from twenty in the case of English and from hundred in the case of Spanish.

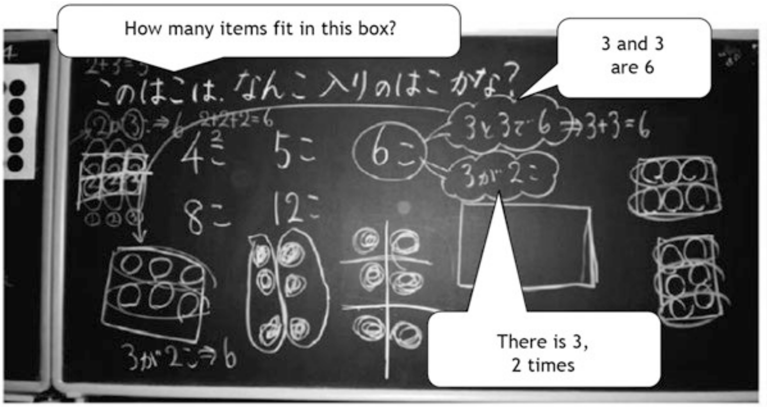


Fig. 4.6 How we read and express from Mr. Satoshi Natsusaka's lesson (see Chap. 5)

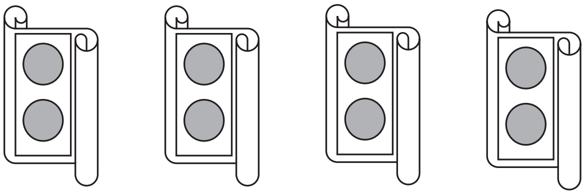


Fig. 4.7 4 × 2 in English and 2 × 4 in Japanese

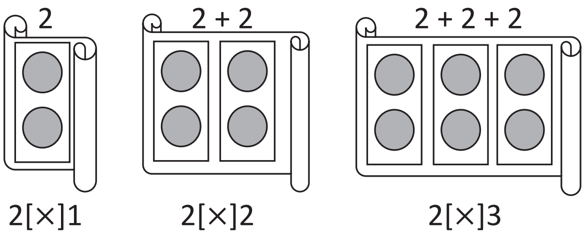


Fig. 4.8 Increase by 2 on row 2

of row 2 (Fig. 4.8). In this comparison, how to see the array is the point of the discussion, such as vertically and horizontally in Fig. 4.9.

Figure 4.9 is a model to illustrate commutativity—why it produces the same products and added the information for the order of multiplier and multiplicand in relation to Fig. 3.12.

4.1.1.5 Revisiting Which Notation Is Better and Why

"3 apples on each plate, and 2 plates" is 3 [×] 2 (3 apples, 2 times) in Japanese; the multiplier is on the right. In English, it is 2 × 3 (2 times 3 apples); the multiplier is on the left.

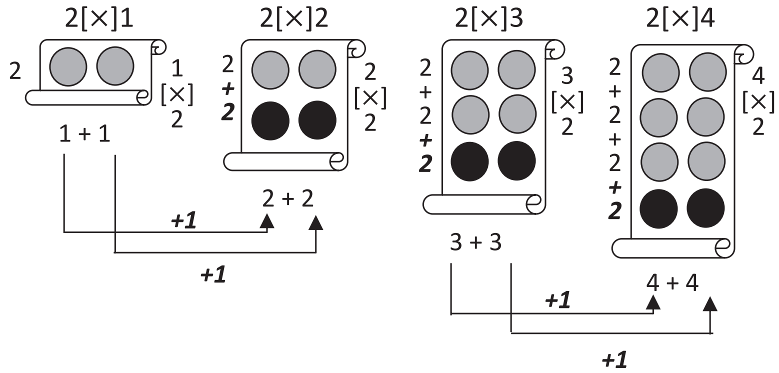


Fig. 4.9 How to read the array diagram horizontally and vertically

There is a grammatical–syntactical difference. As long as preferring the teaching language is fixed first, there is no choice. Thus, the discussion on which is better is unsolvable because it is a cultural matter; however, the question is necessary to design the curriculum sequence. Some usages of daily language such as "times," "dividing into equally," and "equally likely" are also learned in mathematics class at first. In this context, some Latin American countries already prefer the Japanese notation of multiplication for themselves.[10]

The Japanese form has the following significance (see Chap. 3):

- It facilitates the construction of the multiplication table: if the multiplier increases by 1, the product increases by the quantity of the multiplicand.
- The multiplication table is consistent with the multiplication algorithm.
- The meaning of multiplication consistently applies to the two meanings of division in situations.
- The meaning of multiplication and the multiplication table is consistent with the algebraic expression (constant) × (variable).

Additionally:

- It agrees with the traditional Spanish arithmetic book by Rey Pastor and Puig Adam (1935) on the use of the terms "multiplicand" as the first factor and "multiplier" as the second factor.
- It first presents the multiplicand, the unit, the size, or the quantity of the elements in each group, which the students consider in order to be able to decide if multiplication is appropriate in this situation.

The English form has the following significance:

- It agrees grammatically with the use of the term "times" in English and the terms used in most of European languages, such as *veces* in Spanish.

[10] Many Central and South American countries use Spanish as their national language; however, they use multilanguage in relation to their mother tongues.

As will be explained in the next section, the Japanese notation produces consistency in the curriculum sequence. In terms of this consistency, the Japanese teaching sequence in textbooks is considered more deeply as compared with that used in Chile.

4.2 Preparation for Multiplication in the Japanese Curriculum and Textbooks

One of the features of the Japanese course of study (the national curriculum standards) and authorized textbooks is that the sequence is well prepared for future learning for sense making (see also Chap. 5), which is explained by the extension and integration principle.[11] In terms of this principle, the following sections describe the teaching sequence beginning in the first grade, the extension of multiplication to new numerical domains through proportionality, and the extension to other content such as division, rates, and fractions, up to proportions, in the Japanese context.

4.2.1 Preparation for Introduction of Multiplication in the First Grade

The teaching sequence of Japanese textbooks is well prepared for future learning, which means that each part of the teaching content includes preparation of the necessary underlying ideas for use in the future, such as the idea of the "number of units," according to the principle of learning based on what the students have already learned. In the following sections, the four preparations for introducing multiplication in the first grade are explained.

4.2.1.1 Composition and Decomposition of Cardinal Numbers for Binary Operations

First, the textbooks from the publisher Composition and decomposition of numbers in Gakkotosyo (Isoda and Murata, 2011) intensify the development of the idea of a number as the cardinal of a set up to 10. They teach composition and decomposition

[11] As explained in Chap. 1, the extension and integration principle was used in the course of study in 1968 (Ministry of Education, 1968). The meaning is almost the same as the reorganization of experience which was defined by Freudenthal (1973) with his terminology of "mathematization" under his reinvention principle, although the term "mathematization" has been used officially in Japan since 1943 (Sugimura, Simada, Tanaka, and Wada, 1943). Preparation for future learning, conversely, is done using learned knowledge and skills from the perspective of students. However, the students do not know which of them should be used. Students have to know how to extend or to use the known. It is a source of problematics which should be solved in the lesson (See Fig. 1.1 in Chap. 1). It is also a source from which the students produce misconceptions by their own overgeneralization of their learned knowledge and skills. It is the task for a dialectical style of communication between appropriate and inappropriate use of what they have learned in the classroom (Isoda, 1996).

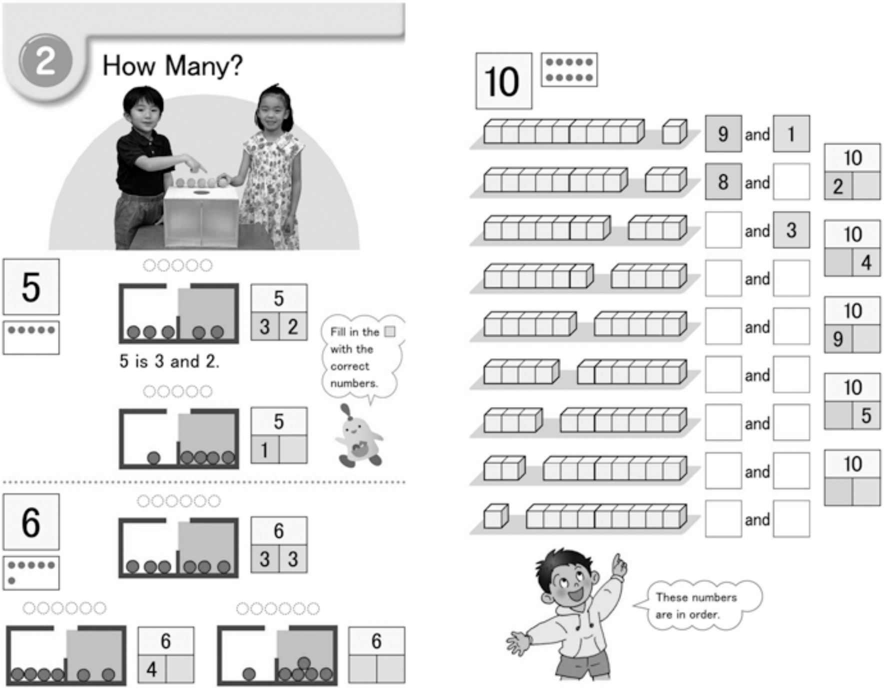


Fig. 4.10 Gakkotosyo (Hitotsumatsu, 2005), Grade 1, p. 26 and p. 29

of numbers before dealing with addition and subtraction (p. 26 in the first-grade textbook; see Fig. 4.10). This is done with the purpose of teaching the association of a number (cardinal) with a set and preparing for mental calculation for addition and multiplication.

If composition and decomposition of numbers are not taught before addition and subtraction, the students can only obtain the result of addition and multiplication by counting. If these are taught before addition and subtraction up to 10, the students can also obtain the answers as sets of objects and not necessarily by counting. Addition up to 10 is based on the composition of numbers, while subtraction up to 10 is based on the decomposition of numbers.

When the students encounter addition of more than 10 up to 20, they will be able to use manipulatives, using the idea of making 10, such as 8 + 3 = 8 + (2 + 1) = (8 + 2) + 1 = 10 + 1 = 11. Here, 3 = 2 + 1 is a decomposition of the number 3, and 8 + 2 = 10 is a composition of the number 10. On this learning trajectory, addition is extended/reorganized from the composition of numbers to the combination of decomposition and composition of numbers for making 10 in relation to carrying.

Mathematically, addition and subtraction are both binary operations. When students study addition on this trajectory, they can see addition as a binary operation and the sum as the value obtained. Otherwise, they can only use counting in instances such as "3 + 2 is three, four, five." It becomes 3 + 1 + 1 when we represent the process by using the plus sign. Getting the sum by counting is not a binary operation even though it is a strategy to get the sum. Here, counting is used as a method to justify the answer.

To learn addition as a binary operation, it is necessary that the two numbers refer to two sets. If we do not prefer this trajectory, counting will still remain as the method used to find the answer. The students may keep on counting as long as they can count. If students learn composite and de-composite of numbers.

4.2.1.2 Counting by Twos or by Fives as the Base for the New Unit to Count

Second, in the extension of numbers beyond 10, students are taught to count by 2s or by 5s as "ways of counting". Here, the students become proficient in the number sequence for counting by 2s or by 5s. It becomes the basis for learning the multiplication table and, for this reason, Japanese textbooks address the multiplication table starting with the rows of 2 and 5 in the second grade. Base 10 system itself is the base for column multiplication by using distribution in the later grade (See Chap. 7 and Meaning of B in Fig. 1.1, Chap. 1).

4.2.1.3 Polynomial Notation

Third, multiplication is a binary operation and the answer is given by repeated addition at the introduction of multiplication (Fig. 4.11). To get the answer in multiplication, the students have to know polynomial notation first before they can interpret the meaning of polynomial notation.

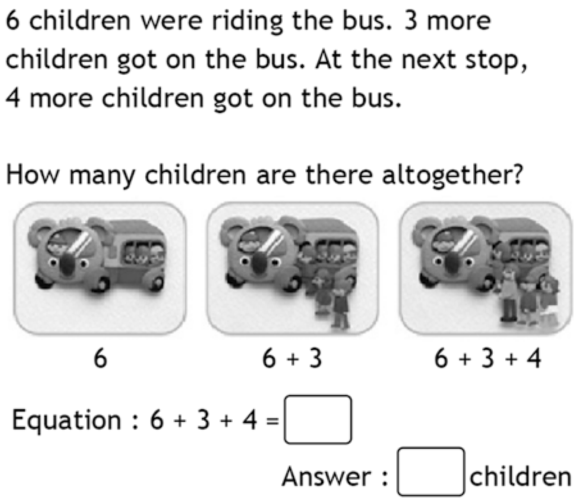


Fig. 4.11 Preparation of repeated addition, Direct comparison, In direct comparison, and Arbitrary Unit, Gakkotosyo (Hitotsumatsu, 2005), Grade 1, p. 97

4.2.1.4 Production of Tentative/Arbitrary Units

Fourth, in the introduction of "measurement" in the first grade (Fig. 4.12), the students learn "how to compare" and not the measurement quantity itself such as "cm." For "how to compare," students study direct comparison, indirect comparison, and arbitrary units. For indirect comparison, through the comparison of *A*, *B* and *C*, students make an order and visualize transitivity, clearly: if *B* is smaller than *A* and *B* is smaller than *C*, then *C* is smaller than *A*. For direct and indirect comparisons, the differences are usually discussed. These are necessary to produce arbitrary units. The differences can produce a unit for measuring (a Euclidean algorithm). In Japan, students learn how to produce arbitrary units in this way. The standard units for measurement quantities such as "cm" are introduced in later grades. Those activities are the bases to understand that any object can be seen as a unit (See Table 1.1 in Chap. 1: the idea of unit). And this processes are prepared for students who are able to learn how to produce the necessary unit. It can be seen as learning trajectory by Szilagyi, Clements, & Sarama (2013).

Those preceding four preparations are the bases for the introduction of multiplication in the first grade. In addition, there are other preparations. For example, multiplication is the base for proportionality. The number line is a key preparation for representing times and extending it to proportionality. In the first grade, it is implicitly introduced as a line of numbers by using repetitions of the unit tape (Fig. 4.13).

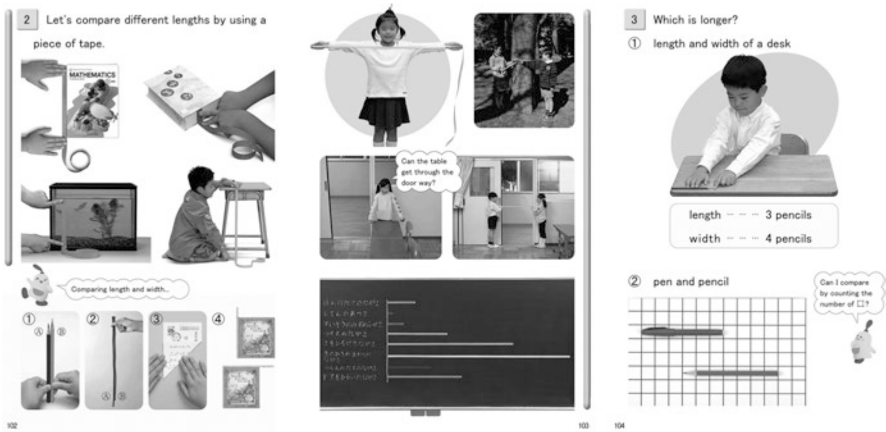


Fig. 4.12 Gakkotosyo (Hitotsumatsu, 2005), Grade 1, pp. 103–104

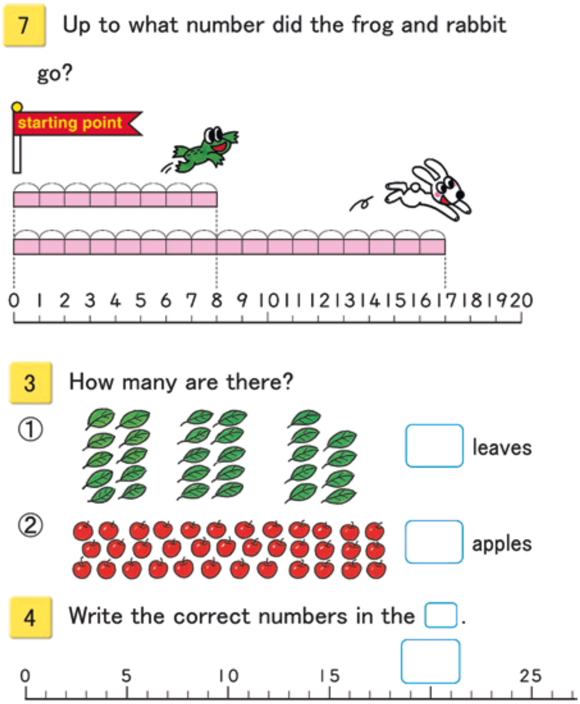


Fig. 4.13 Preparation of Number line, Gakkotosyo (Hitotsumatsu, 2005), Grade 1, p. 67, p. 70

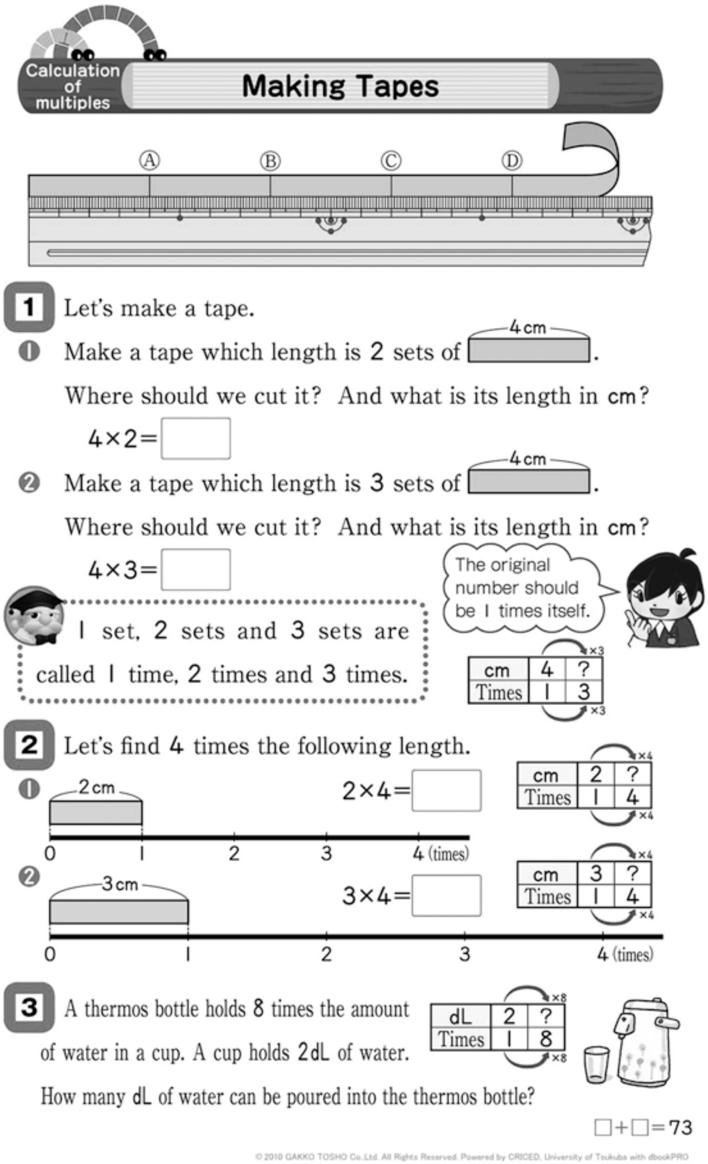
4.3 Proportionality for Extension of Multiplication

Extension of numbers is part of the curriculum sequence in any country. In Japan, the key idea for multiplication in extension of numbers is proportionality (properties of proportion) using a tape diagram and a table with the rule of three before teaching the formal definition of proportions. Since the 1960s, proportionality has been embedded in the Japanese textbooks by the following sequence for extension of numbers.

In the introduction of multiplication in the second grade, times (*bai*) is introduced with a tape diagram (see Fig. 4.4) and repeated addition of the unit length tape, which corresponds to the constant difference in the multiplication table. The tape diagram is used for the extension of numbers to decimals and fractions as proportional number lines (see Meaning C of Fig. 1.1 in Chap. 1).

4.3.1 Introduction of Proportional Number Lines and Their Adaptation for Extension

In the third grade, the Gakkotosyo textbooks (Isoda and Murata, 2011; Isoda, Murata, and Yap, 2015) extend this tape diagram to two-dimensional lines, which the Japanese call "proportional number lines" (see Figs. 4.14 and 4.15). The proportional number line is a model that represent the definition of multiplication by measure-



Calculation of multiples

Making Tapes

Ⓐ Ⓑ Ⓒ Ⓓ

1 Let's make a tape.

❶ Make a tape which length is 2 sets of [4 cm].

Where should we cut it? And what is its length in cm?

4×2=☐

❷ Make a tape which length is 3 sets of [4 cm].

Where should we cut it? And what is its length in cm?

4×3=☐

1 set, 2 sets and 3 sets are called 1 time, 2 times and 3 times.

The original number should be 1 times itself.

cm	4	?
Times	1	3

×3

2 Let's find 4 times the following length.

❶ 2 cm   2×4=☐

0 1 2 3 4 (times)

cm	2	?
Times	1	4

×4

❷ 3 cm   3×4=☐

0 1 2 3 4 (times)

cm	3	?
Times	1	4

×4

3 A thermos bottle holds 8 times the amount of water in a cup. A cup holds 2dL of water. How many dL of water can be poured into the thermos bottle?

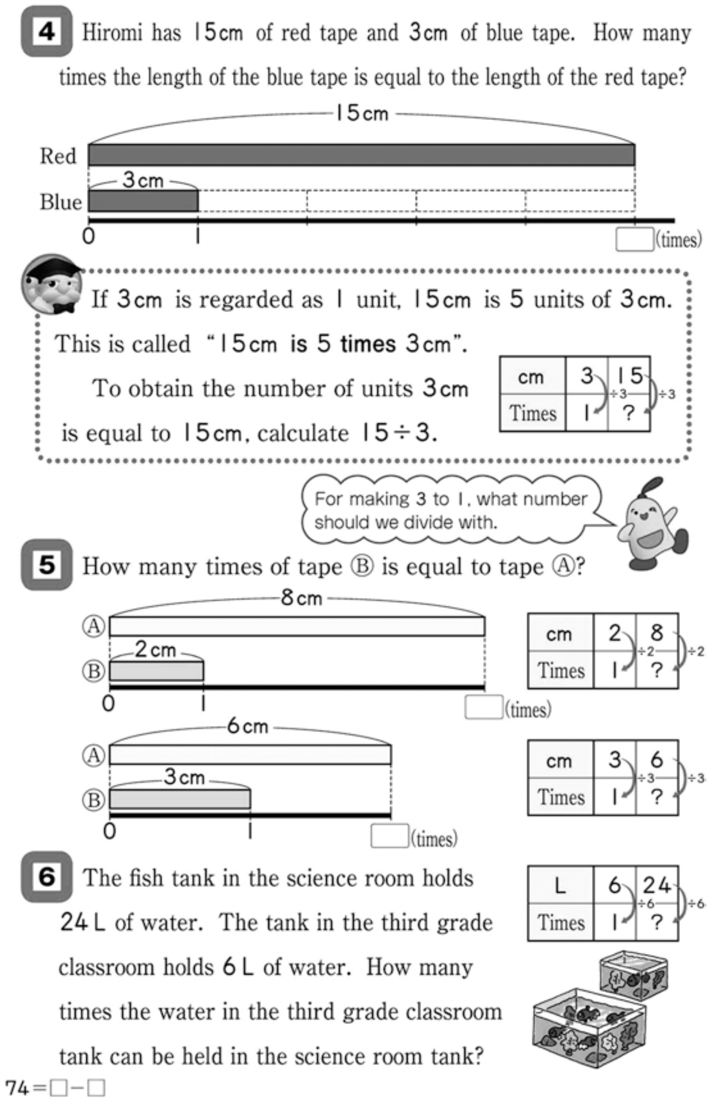
dL	2	?
Times	1	8

×8

☐+☐=73

Fig. 4.14 Proportional number line is introduced by the number line with tape diagram in the case of Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 73

ment in all Japanese primary schools' mathematics textbooks (see Chap. 3). In particular, the Gakkotosyo textbooks enhance the rule of three by using arrows to show the pattern on the table (see the four-column tables in Figs. 4.14 and 4.15). Proportionality is also embedded in these tables.



4 Hiromi has 15cm of red tape and 3cm of blue tape. How many times the length of the blue tape is equal to the length of the red tape?

If 3cm is regarded as 1 unit, 15cm is 5 units of 3cm. This is called "15cm is 5 times 3cm".

To obtain the number of units 3cm is equal to 15cm, calculate 15÷3.

cm	3	15
Times	1	?

5 How many times of tape Ⓑ is equal to tape Ⓐ?

cm	2	8
Times	1	?

cm	3	6
Times	1	?

6 The fish tank in the science room holds 24 L of water. The tank in the third grade classroom holds 6 L of water. How many times the water in the third grade classroom tank can be held in the science room tank?

L	6	24
Times	1	?

Fig. 4.15 Proportional number line is introduced by the number line with tape diagram in the case of Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 74

Those two types of representations—proportional number lines and the table for the rule of three—are not necessary to find the answer in situations involving multiplication at Grade 3. However, they are necessary to prepare for the extension of multiplication from whole numbers to decimals and fractions in upper grades (See Extension of B to C, Fig. 1.1, Chap. 1). Thus, the model representations in Figs. 4.14 and 4.15 are the preparations made in the third grade for future learning in later grades as for sense making.

4.3.2 Extension of Multiplication by Using Proportional Number Lines

The following are the first four pages of the fifth-grade textbook on the extension to decimals (Isoda and Murata, 2011):

In Figs. 4.16, 4.17, and 4.18, there are tape diagrams to show proportionality and, at the same time, the rule of three in the table in Figs. 4.16 and 4.18. There are also further strategies that can be seen for extensions using the properties of multiplication sentences at 10 times and 1/10 in Fig. 4.18 and area diagrams in Fig. 4.19. When students discuss the mutual relationship of their ideas in Fig. 4.18, it is an opportunity for them to develop the idea of proportionality.[12]

In this manner, Japanese textbooks prepare for future learning by consistently developing and using the same representations. In these preparations, students are able to challenge further learning such as proportion (see Sects. 4.3.4 and 4.4) by and for themselves.[13]

4.3.3 Partitive and Quotative Divisions Using Multiplication

In case of divisibility (with no remainder), division is represented by (dividend) ÷ (divisor) = (quotient). Division is the inverse operation of multiplication, which is (dividend) = (divisor) × (quotient) or (quotient) × (divisor). If multiplication is repeated addition of the same number, then division can be seen as repeated subtraction of the same number. However, we should note that commutativity does not hold in division.

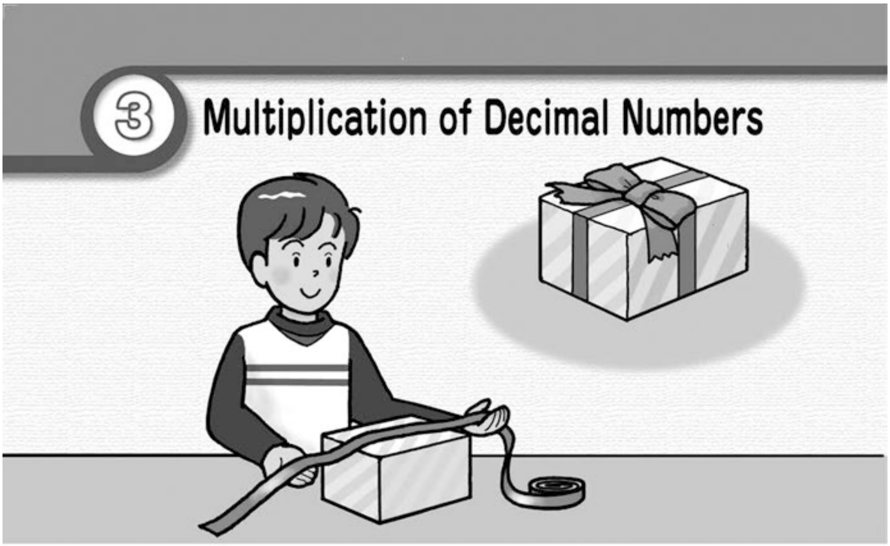
Two meaning of division are shown in Fig. 4.20.

The situations of the two activities on division in Fig. 4.20 are different. The situation of the partitive division activity establishes the number of equal partitions. The situation of the quotative division activity distributes the same amount recursively until there is no more left to distribute. However, if we compare only the left part of the diagram showing partitive division with that showing quotative division in Fig. 4.20, it looks the same as repeated subtraction. This correspondence provides a reason for students to explain these different situations to be integrated as one operation.

---

[12] The term "proportion" is learned in the fifth grade in the 2011 edition and in the sixth grade in the 2005 edition.

[13] As explained in Chap. 1, in Japan, the problem-solving approach is enhanced based on sequential preparations of applying already-learned knowledge to unknown tasks for extension and integration. Learned knowledge is not limited to the procedure but also includes ways of meaningful representations. Such preparations are beyond the strategy of teaching in Pólya's articles. On this basis, the Japanese problem-solving approaches are very far from just the solving of nonroutine problems under the Pólya framework. In this context, Japanese teachers try to develop students' mathematical thinking every day.



1 Calculating (Whole Numbers) × (Decimal Numbers)

▶▶ Keita is thinking about wrapping the box with a ribbon around it. He needs 2.4m of ribbon.

1 The price of the ribbon is 80 yen per meter. Let's find out how much it would cost for 2.4m.

❶ Draw a number line with a taped diagram.

❷ Write an expression.

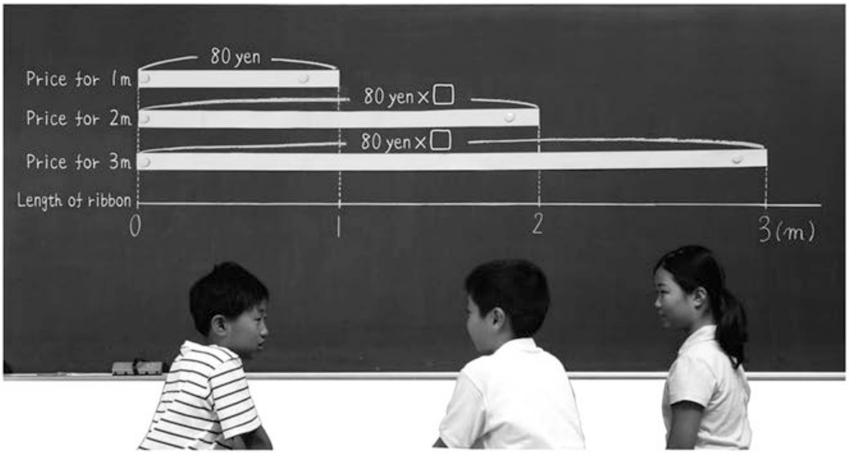
Price (Cost)	80	?
Length of ribbon (m)	1	2.4

Expression : $\square$

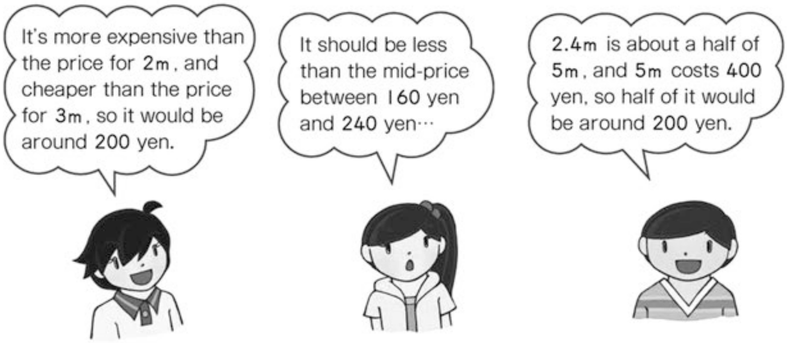
$30 = \square \times \square$

Fig. 4.16 Extension to decimals, Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 30

Multiplication in those situations is (number of each unit) [×] (amount of unit) = (product; total number) in Japanese notation, and (amount of unit) × (number of each unit) = (product; total number) in English notation. The partitive division situation corresponds to finding the amount of each unit (per dish or child), where the quotient means the amount of each unit. On the other hand, the quotative division situ-



❸ Approximately, how much would the cost be?

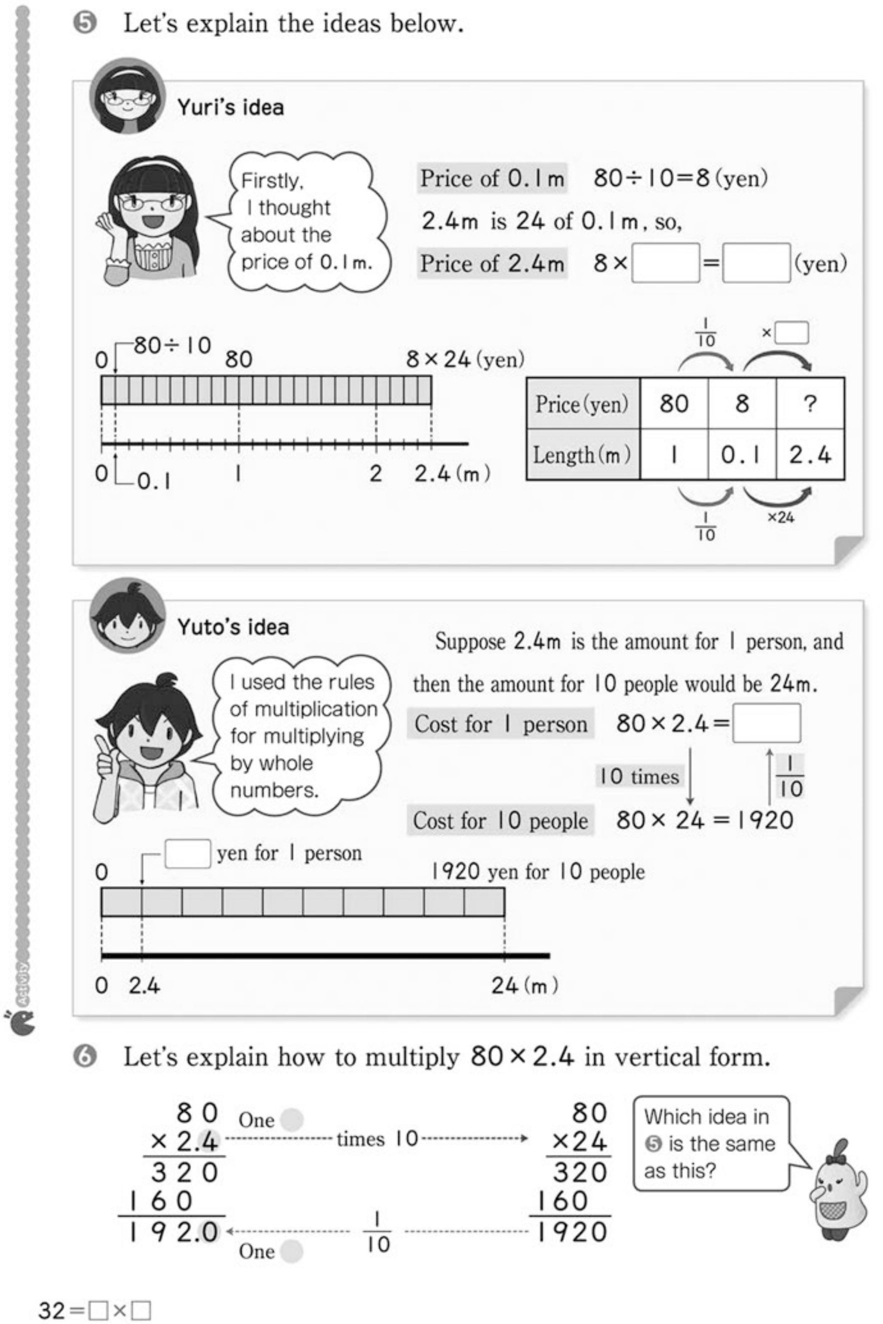


As shown with the length of ribbon, when the multiplier is a decimal numbers instead of a whole number, the expression is the same as for multiplication of whole numbers.

❹ Let's think about how to calculate.

□÷□=31

Fig. 4.17 Right page of Fig. 4.16 (continuous), Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 31



❺ Let's explain the ideas below.

Yuri's idea

Firstly, I thought about the price of 0.1 m.

Price of 0.1 m 80÷10=8 (yen)

2.4m is 24 of 0.1 m, so,

Price of 2.4m 8× ☐ = ☐ (yen)

Price (yen)	80	8	?
Length (m)	1	0.1	2.4

Yuto's idea

I used the rules of multiplication for multiplying by whole numbers.

Suppose 2.4m is the amount for 1 person, and then the amount for 10 people would be 24m.

Cost for 1 person 80×2.4= ☐

10 times 1/10

Cost for 10 people 80×24=1920

❻ Let's explain how to multiply 80×2.4 in vertical form.

```
   8 0   One ○                       80
 × 2.4 ·········· times 10 ········› ×24
   3 2 0                            320
 1 6 0                             160
 1 9 2.0 ‹········ 1/10 ··········  1920
         One ○
```

Which idea in ❺ is the same as this?

32=☐×☐

Fig. 4.18 Continuous from Figs. 4.16 and 4.17, Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 32

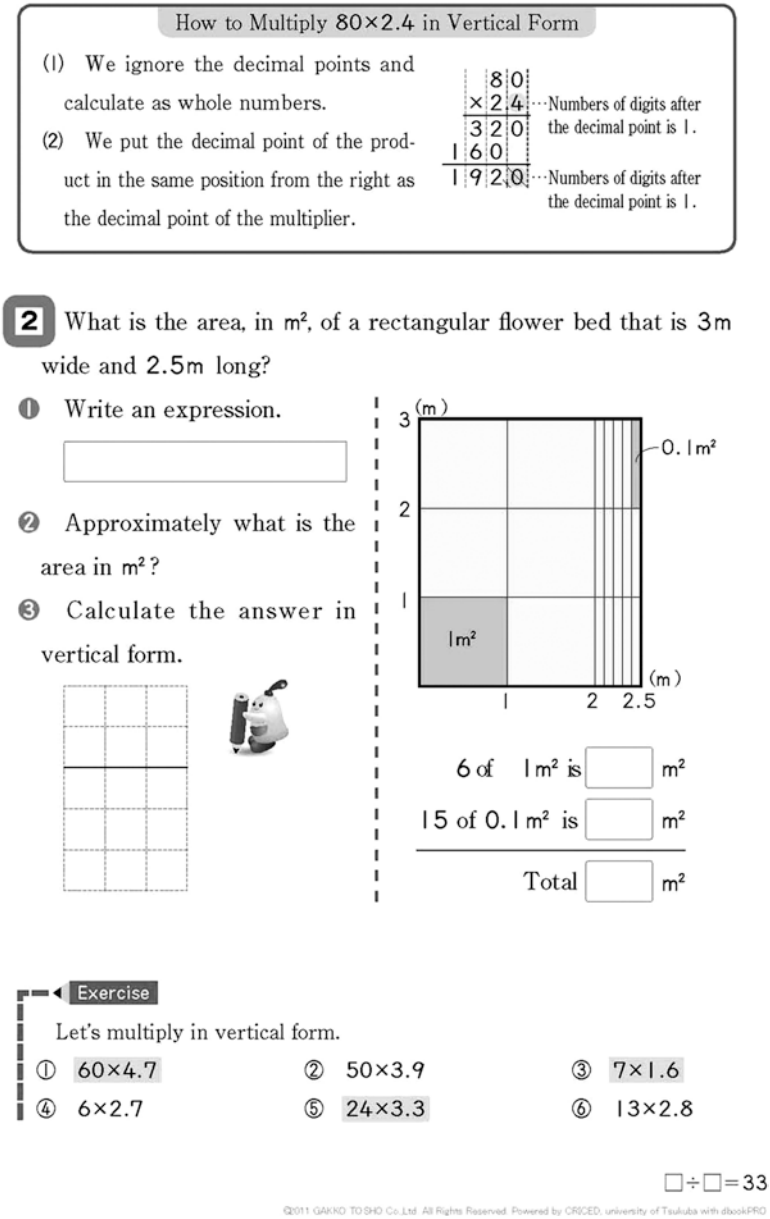


Fig. 4.19 Right page of Fig. 4.18, Gakkotosyo (Isoda and Murata, 2011), Grade 5, Vol. 1, p. 33

ation corresponds to finding the number of units, where the quotient means the number of units. Thus, the partitive division situation is represented by (total number) ÷ (amount of unit) = (number of each unit), which is the representation of the inverse operation (total number) = (*number of each unit*) [×] (amount of unit). The quotative division situation is represented by (total number) ÷ (number of each unit) = (amount

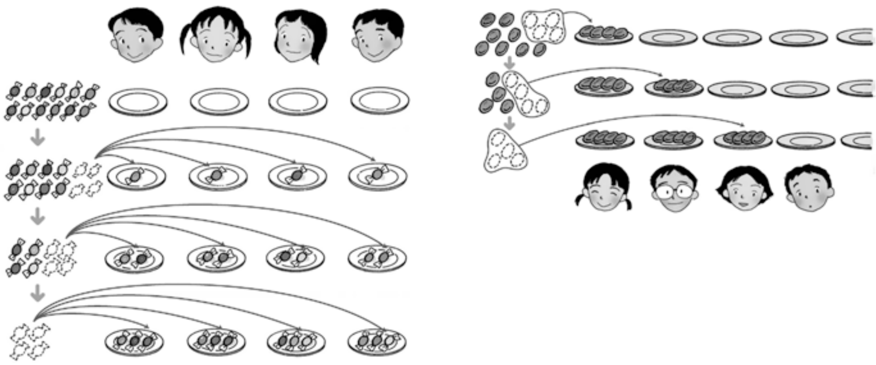


Fig. 4.20 Partitive division (*left*) and Quotative division (*right*), Gakkotosyo (Hitotsumatsu, 2005), Grade 3, Vol. 2, p. 4 and p. 8

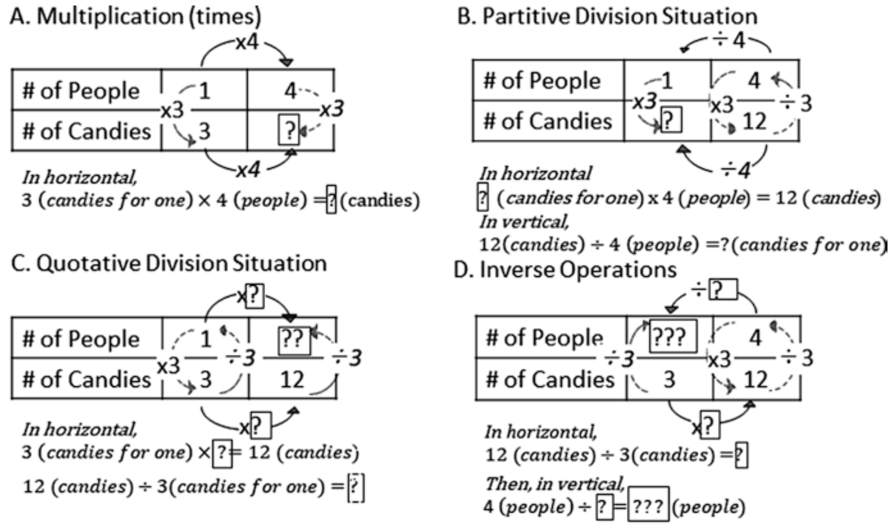
of unit), which is the representation of the inverse operation (total number) = (number of each unit) [×] (*amount of unit*). Those are two different meanings of quotient, depending on the situation. Left vertical arrows in partitive division on the left of Fig. 4.20 and left vertical arrows in quotative division on the right of Fig. 4.20 both, can be seen as repeated subtraction of the same numbers. Repeated subtraction is a key to seeing both situations as the division operation for integration.

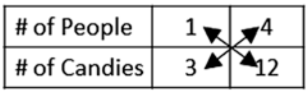
For teachers teaching mathematics in Indo-European languages, this discussion is not so clear because they do not use the terms "multiplier" and "multiplicand" (see Chaps. 2 and 3) and may not feel the necessity to do so because it is customary for them to use commutativity in their minds and expressions for finding the answer in multiplication. From the viewpoint of the Japanese approach, teachers who do not feel any necessity to do so can be seen as teachers who are less likely to teach mathematics by using what their students have already learned. If teachers are able to see that the two meanings are not exactly the same, they may understand the difficulty that students have in seeing the different situations as one operation. If they can make the distinction, they can really understand what content they should teach. The Japanese distinguish it clearly based on consistency of multiplication.[14]

4.3.4 Relationships Among the Rule of Three, Multiplication, and Division

The well-memorized and proceduralized multiplication-table is adapted to the tables that embed proportionality by multiplications and division rules. For filling in the blanks in the table, division is treated as the inverse operation of multiplication, as shown in different contexts in Fig. 4.21.

[14] The Japanese use two different characters for division. Partitive division is *waru* ("splitting") which implies dividing equally. Quotative division is *jyo* ("subtraction") which implies repeated subtraction. Due to this difference, both partitive division and quotative division are necessary terminologies to specify what they teach even though they do not teach these words to students. In Japan, division using the abacus was introduced in the sixteenth century.





The product of the inner terms is equal to the product of the outer terms on ratio. The cross shows the line for multiplication. It is not the origin of multiplication symbol (Cajori, 1928).

Adaptation of proportionality to different situations for the ratio of the same quantity

First Usage of Ratio (find ratio (times); table C treatment):

How many times (M) equals (N) when base amount (M) and comparing amount (N) are known; "Taro has 30 balls and Hiro has 45 balls. How many times of number of balls does Hiro have when we compared his number of balls with Taro's?"

ratio	1	?
balls	30	45

Second Usage of Ratio (find the comparing amount; table A treatment):

When the based amount (M) and ratio (p) are known, to find the amount (N) to be compared with (M); "Mie has 20 dolls and the number of Chie's dalls is 1.5 times of the number of Mie's dolls. How many dolls does Chie have?"

ratio	1	1.5
dolls	20	?

Third Usage of Ratio (find the base amount: table B treatment):

When the compered amount (N) and ratio (p) are known, to find the base amount (M); "Taro's height is 156cm which is 1.2 times of the height of Mie. How tall is Mie?

ratio	1	1.2
height	?	156

Table arrangement changes depending on how we draw a table and ways to read. In Japanese textbooks, normally, ratio (times) comes the bottom row such as Figures 4.14 and 4.15.

Fig. 4.21 Various preparation for ratio, rate and proportion by using the table in relation to rule of three, multiplication and division

Historically, these calculations using proportionality are done under the name of the “rule of three.” The “?” blanks in tables A, and “??” in B and C in Fig. 4.21 can be represented by multiplication as follows: (A) $3 \times 4 = 12$, (B) $(?)\times 4 = 12$, (C) $3 \times (?) = 12$.[15] If students learn division as inverse operation of multiplication, tables can be seen in various ways by using the idea of proportionality like Fig. 4.21. For finding these answers using multiplication, it is not necessary to refer to the original situations as long as the numbers in every table are placed in the appropriate columns under the proportionality. In Japan, the three usages of the ratio on the situations likely Fig. 4.21 are summarized as kinds of formulas and such discussions were existed before World War II. If teacher teach those different usages just different formulas, it produce difficulty for students. To recognize the proportionality, multiplication and division in the table treatments like Fig. 4.12 make it meaningful (see Fig. 1.1). Thus, Gakkotosyo textbook introduce it from Grade 3 in relation to multiplication and division and prepare the extension of multiplication and division into decimals and fractions, and ratio and proportion in Grades 5 and 6.

4.3.5 From Division to Ratios and Rates Using the Multiplicative Format

Division by a different quantity (partitive division) results in the rate, which is the unknown quantity and is represented by a quantity per another quantity, such as speed (km/h = distance (km)/duration (hour)) or population density (population/km^2).

Division by the same quantity, which corresponds to quotative division, results in the ratio of the same quantity which produces the ratio value to show the coefficient.[16] However, if we carefully read the task for quotative division, we may recognize that it is not exactly division of the same quantity (denomination). It can be represented by (number of students) = (candies) ÷ (candies per student). Students usually see both the numbers 12 and 4 as the same candies and thus do not easily recognize them as different quantities in the sentence. Knowing the unit as a quantity per another quantity in those situations is a key to deriving expressions. In Japanese, terminology “per” is used for ratio on Grade 5, thus at Grade 2, Japanese use terminology “for each amount” instead of “per” to be expendable to ratio.

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[15] In Chap. 3, we referred Vergnaud (1983) to explain the situations for multiplication. Currently, the rule of three is explained by algebraic expressions. However, if we ask students to distinguish the expressions as different formula, it produce difficulties. Historically, the rule of three existed as methods to find the answers with the three numbers alignment before the emergence of algebraic expressions. With regard to ratios, the “× 4” arrows in the tables in Fig. 4.21 are explained as *bai* (“times”). In Japanese terminology, *bai* is a key word to explain proportionality. Division is alternated reciprocal number of multiplication. The division treatments on tables in Fig. 4.21 are alternated to reciprocal-number times at Grade 6.

[16] This is the Japanese usage of “times” (*bai*) like multiple.

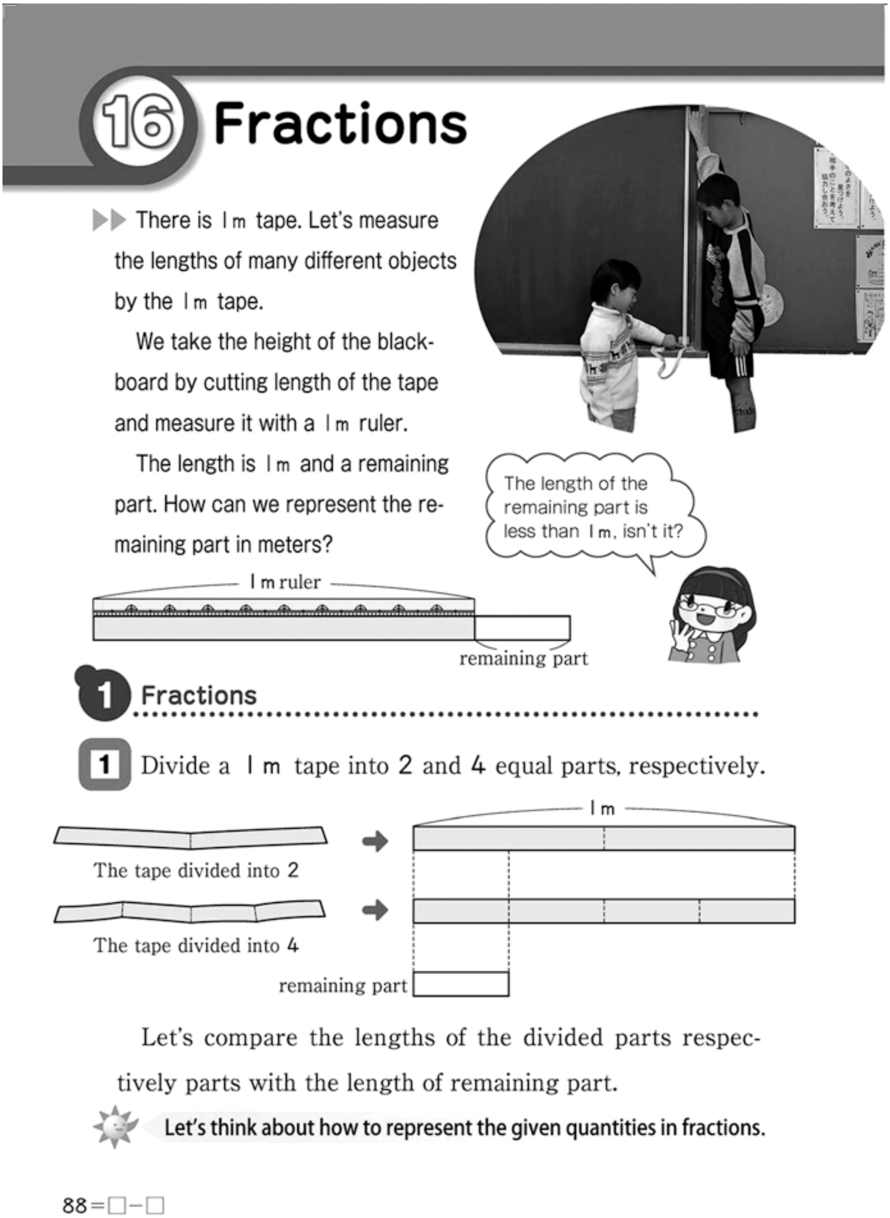
4.4 Various Meanings of Fractions Embedding the Meanings of Division Situations

Fraction in situations can be distinguished in various perspectives and contexts (see Isoda, 2013).

A fraction as a part–whole relationship usually corresponds to the partitive division situation; the Japanese call this a "dividing fraction." A fraction can also be seen as quotative division. The situation of a fraction in quotative division is called an "operational (taking away, measuring) fraction." A fraction with a denomination to indicate the unit quantity by the denomination is called a "fraction with a quantity." A fraction used for showing the number and used for the value of division as a quotient is called a "quotient fraction." A fraction as a ratio is called a "fraction as the value of a ratio". Japanese distinguish these five meanings of fractions.

Before providing explanations, first we describe the existing English-language terminologies used to distinguish the meanings of fractions. In English, the following meanings of fractions are distinguished: according to Reys, Lindquist, Lambdin, and Smith (2012), from the USA, the meanings of fractions in situations are distinguished as part–whole, the quotient, and the ratio. Part–whole corresponds to the Japanese "dividing fraction" and the "operational (measuring) fraction" means the measuring by unit such as the measuring by using reminder like 4.22. However, it is not so much clear to distinguish these two ideas. According to the USA Common Core State Standards for Mathematics (CCSSM) (2010) "Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$. (CCSS.MATH.CONTENT.3.NF.A.2.A)." It is a dividing fraction. And CCSSM continues "Represent a fraction a/b on a number line diagram by marking off a length $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line. (CCSS.MATH.CONTENT.3.NF.A.2.B)." This can be seen as an "operational fraction" if the scaling is done by $1/b$; however, it is unclear if it is operational fraction. It implicates that CCSSM does not use terms like Japanese to establish conceptual consistency between the two division meanings and meanings of fractions even though it embedded the ideas. Indeed, Watanabe (2006) in USA makes clear the activity of the operational fraction and explains the uniqueness of the Japanese way to introduce fractions. According to Haylock (2010), also from the USA, a fraction is (a) a part of a whole or unit, (b) a part of a set, (c) a modeling division problem, (d) a ratio and it is unclear if this is an operational fraction. According to Van de Walle, Karp, Bay-Williams, and Wray (2015), from the UK, there is no such manner to distinguish different types of fractions in different situations. According to Kupferman (2017), from Israel, there are various analyses. Petit, M. et al (2016) also explained various models of fraction under CCSSM well and we can read the ideas in it with Japanese terminology of fraction but their wording is following the CCSSM. These articles implicate that the Japanese approach clearly adopts two meanings of divisional situations into meanings of fraction in situations as "dividing fractions" and "operational fractions."

The following is the introduction to the fraction in Gakkotosyo textbooks (Isoda and Murata, 2011) using a situation for an operational fraction (Figs. 4.22 and 4.23).



16 Fractions

There is 1 m tape. Let's measure the lengths of many different objects by the 1 m tape.

We take the height of the black-board by cutting length of the tape and measure it with a 1 m ruler.

The length is 1 m and a remaining part. How can we represent the remaining part in meters?

1 Fractions

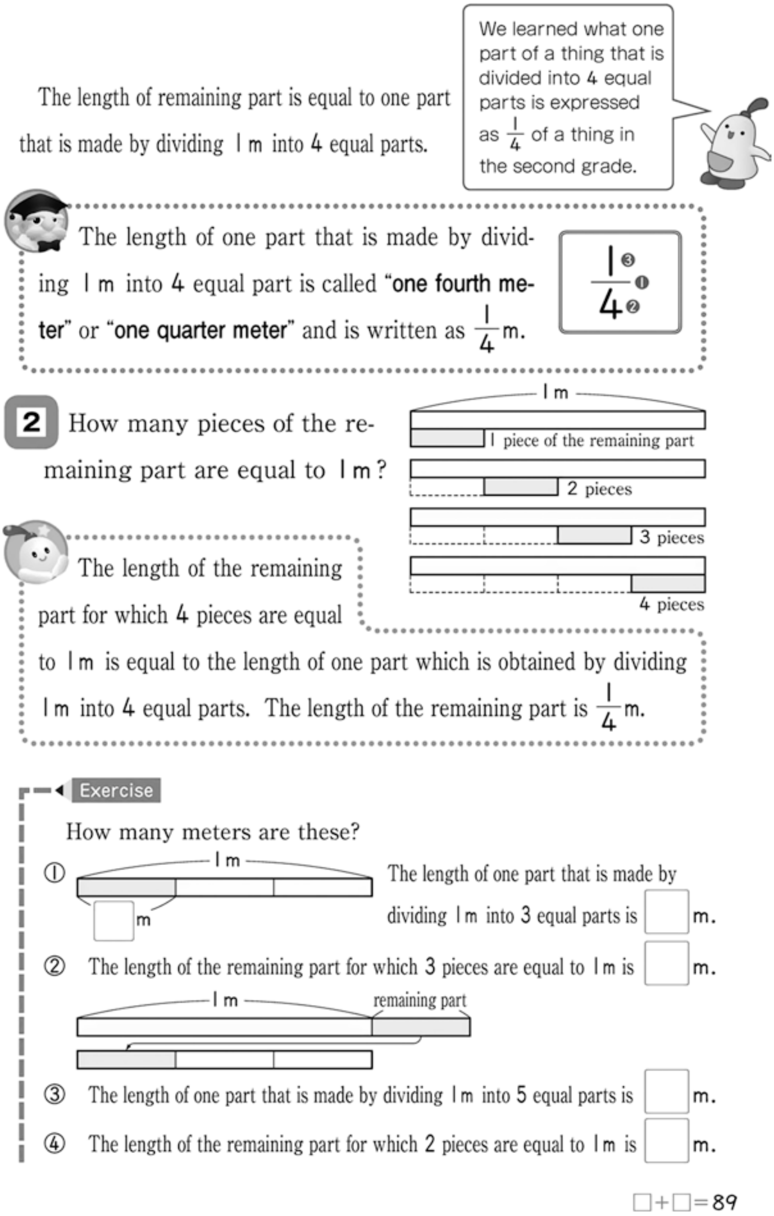
1 Divide a 1 m tape into 2 and 4 equal parts, respectively.

Let's compare the lengths of the divided parts respectively parts with the length of remaining part.

Let's think about how to represent the given quantities in fractions.

88=□−□

Fig. 4.22 Measuring by the remaining part. Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 88



The length of remaining part is equal to one part that is made by dividing 1 m into 4 equal parts.

We learned what one part of a thing that is divided into 4 equal parts is expressed as $\frac{1}{4}$ of a thing in the second grade.

The length of one part that is made by dividing 1 m into 4 equal part is called "**one fourth meter**" or "**one quarter meter**" and is written as $\frac{1}{4}$ m.

$\frac{1}{4}$ ❸ ❶ ❷

2 How many pieces of the remaining part are equal to 1 m?

1 m

1 piece of the remaining part

2 pieces

3 pieces

4 pieces

The length of the remaining part for which 4 pieces are equal to 1 m is equal to the length of one part which is obtained by dividing 1 m into 4 equal parts. The length of the remaining part is $\frac{1}{4}$ m.

Exercise

How many meters are these?

① The length of one part that is made by dividing 1 m into 3 equal parts is ☐ m.

1 m ☐ m

② The length of the remaining part for which 3 pieces are equal to 1 m is ☐ m.

1 m remaining part

③ The length of one part that is made by dividing 1 m into 5 equal parts is ☐ m.

④ The length of the remaining part for which 2 pieces are equal to 1 m is ☐ m.

☐+☐=89

Fig. 4.23 Continued from Fig. 4.22, Gakkotosyo (Isoda and Murata, 2011), Grade 3, Vol. 2, p. 89

In Figs. 4.22 and 4.23, we can see three different meanings of a fraction. One is an operational fraction, which is measured by the remaining part. This remaining part as the unit for measurement is called a "unit fraction."[17] The other one is fraction with quantity, here 1m tape.[18]

In this context, the size of the fraction cannot be compared without fixing the unit such as "m." The size of a half a pizza in a dividing fraction cannot be compared without fixing the size of the original (whole) pizza before dividing it. According to this meaning, dividing and operational fractions cannot be explained well as comparative sizes of numbers without fixing what the whole is. Especially, in dividing fractions, fixing the whole is easily forgotten than operational fraction. If forgot, the sizes of the fractions cannot be compared. To see a fraction as a number to compare sizes, Japanese textbooks show that meaning of the fraction in a context with a quantity such as 3/4 L and 1/2 L. It is a "fraction with quantity." The fraction with quantity enables it to be put on the number line, clearly, because it fixes the unit for the magnitude.

The answer to division is called a quotient, which is a number without a denomination (quantity). The equivalence of fractions can be seen on the number line. The fraction which is the answer to division is called a "quotient fraction" in the Japanese approach. By introducing the quotient fraction and equivalence of fractions, a fraction becomes a number because it begins to function as part of the operation of other numbers—that the division of numbers should have their answers on the number line.[19]

A fraction as the "value of a ratio" or the rate is not always a part–whole relationship. Even the ratio of the width to the length of a rectangle with the same quantity is not a part–whole relationship physically because the width never belongs to the length of the rectangle directly.[20] Contextually, the value of the ratio can be seen as a quotient fraction if it is a fraction. It is usually used for (a/b) times (quantity). The Japanese consider this fraction as a ratio such as "half of a bottle." Here, "half" is the value of the ratio and "of" implies multiplication. In Japanese, *bai* ("times") is usually used in this context ("of"). The rate is represented by the division of different quantities and results in a new quantity such as "km/h." It is related to partitive division but not to part–whole relationships because of the differences in quantities. In the Japanese teaching sequence, ratios and rates can be seen as the extended adaptation of multiplication and partitive and quotative divisions in relation to proportion-

[17] The unit is the unit for measurement which is based on the definition of multiplication by measurement. The unit can be a decimal or a fraction.

[18] In the case of Gakkotosyo (2005), dividing fraction appeared the same pages. In the case of 2005 edition, fraction is introduced from this page at the same time because students are not yet learned how to fold the 1m tape into 4. In the case of 2011 edition, paper folding for fraction is learned at 2nd grade. Those differences originated from the difference of national curriculum standards.

[19] Numbers can be seen as numbers when this becomes a number system which discusses existence, magnitude (greater or less, equality, comparison), and the four operations. The quotient fraction is the answer of division. Division is defined by the inverse operation of multiplication. Before the quotient fraction, Japanese textbooks already addressed addition and subtraction of fractions.

[20] In the Euclidean algorithm for finding the greatest common divisor of two segments of a rectangle, we have to move the width to length by using a compass. As mentioned in the introduction of measurement, it functions to find the common unit for the measurement and also addition with different denominators.

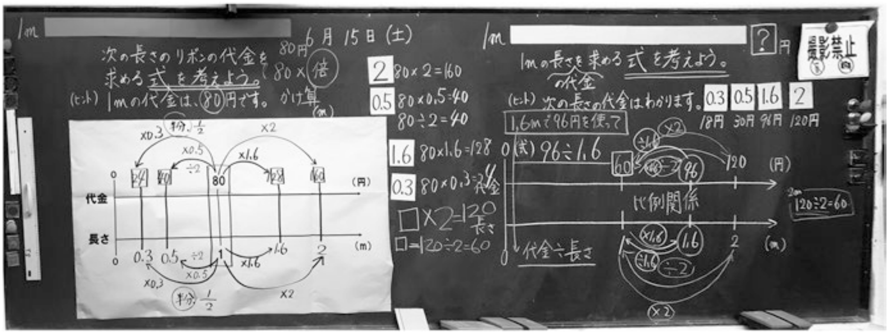


Fig. 4.24 Open class by Takao Seiyama (June 15, 2019 at the Elementary School at the University of Tsukuba). Learned task is multiple (see Fig. 4.21, A) (*left*): "Let's find the price of ribbon when the price of 1 m is given." Unknown task for today (see Fig. 4.21, B; *right*): "Let's find the price of 1 m of ribbon when the price for 2 m is given." Before this class, students have already learned about proportions using proportional number lines. The major objective of today's class is as follows: Using the proportional number line, recognize that division is the inverse operation of multiplication, such as 0.5 times is one half times or division by 2 (as review; *left*), and adopting learned to the new division task which can be seen as multiplication through finding the unit price at first and then finding the value by multiplication (*right*). By using the proportional number line, multiplication and division are integrated on the tasks for ratio like tables in Fig. 4.21

ality (See Fig. 4.21). In the Japanese curriculum, the different meanings of fractions are well distinguished and sequenced in the curriculum up to ratios and proportions, and up to Integration of multiplication and division to multiplication by representing division as multiplication of reciprocal number.[21]

In the Japanese curriculum sequence, the definition of multiplication by measurement is consistent with the multiplication table, division, fractions, and ratio and proportion. This consistency is supported by the model representations, proportional number lines (see Fig. 4.24) and the table in relation to the rule of three (see Fig. 4.21). This is the reason why Japanese distinguishes the multiplier and multiplicand at the introduction of multiplication. Actually, if we do not distinguish both, we can not distinguish partitive and quotative division, then, can not distinguish dividing and operational (measurement) fraction. After this consistency and extended adaptation of multiplication, the formal definition of proportions is introduced. To introduce Proportion formally, Gakko Tosyo Textbooks evolve the proportional number line from tape diagrams on Fig. 4.4 to a tape and a number line on Fig. 4.14, and apply it for the extension of numbers on Fig. 4.16, and replace it to the parallel number lines on Fig. 4.24 as for the preparatory representations of the proportion. It illustrates the sequence to develop sense making for using multiplicative reasoning and proportionality on the principle of extension and integration. This sequence and representations make possible to apply the definition of multiplication by measurement to different teaching content (see Chap. 3). With this

[21] On this integration, "÷3" becomes "× (1/3)" which changes the view of multiplier from the first number to likely second number as an operator in the case of India-European language (see Chap. 3).

meaning, Japanese can introduce multiplication as preparation for division, fractions, and proportions. In the Japanese approach, the teaching content and sequence are usually preparation for future learning. It is not only just making sense for reasonable explaining at every moment but also to develop sense making for extending and integrating by and for themselves.

4.5 Further Challenges to Distinguish Additive and Multiplicative Structures

There are a number of misconceptions between additive and multiplicative structures in relation to ratios, rates, and proportions, such as misusing addition in multiplicative or divisional situations. An origin of this type of misconception is originated from the properties of the multiplication table (see Fig. 4.25).

When students learn multiplication in the second grade, they find and use this additive property. On the other hand, the multiplicative property is not easy for students because they have just learned the table and still use the additive property for explaining the table, Explaining such as two times of multiplier produces two times (double) products, and three times of multiplier produces three times (triple) product, and so on are not easy because the symbol "×" itself can be read "times." On the multiplicative property there are "×" meaning of times and "left arrows and right arrows between expressions" meaning of times appeared at once on the multipliers in the table and the number of times (see Fig. 4.25; Isoda, 2015). If students extend multiplications to fractions in the upper grades, they can realize the difference of these two properties, such as half of two in Fig. 4.26. Thus, in the second grade, they

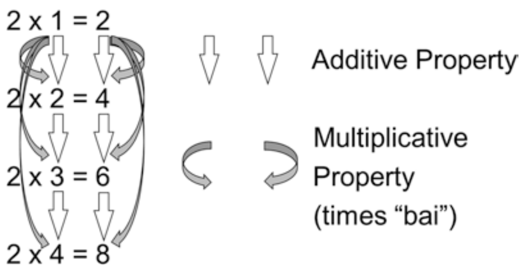


Fig. 4.25 Additive property and multiplicative property of the multiplication table

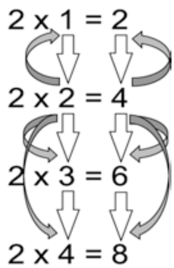


Fig. 4.26 After the extension of multiplication to proportionality of numbers such as decimals and fractions, students are able to discuss ×0.5 and ×(1/2)

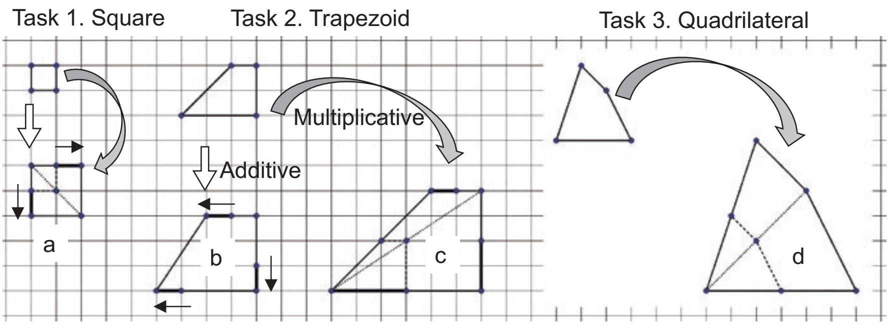


Fig. 4.27 The sides of the square, trapezoid, and quadrilateral are enlarged two times on 1-cm-squared paper (by Suzuki in Isoda (1996), revised by Isoda)

cannot easily distinguish additive and multiplicative structures in the table like the one shown in Fig. 4.25, but not like the one shown in Fig. 4.26.

In Japan, a proportion is defined in the fifth grade by using "2 times (*bai*), 3 times (*bai*), . . . , of *x* corresponds to 2 times (*bai*), 3 times (*bai*), . . . , of *y*, respectively" and the constant of the quotient is the property of the proportion. Even though they have learned the use of "times" (*bai*) in proportions in the fifth grade, students are still confused, depending on the daily context; for example, the usage of *bai-bai* means "three times" or "quadruple."[22] Even though proportionality is learned in the fifth grade, students have to extend its usage and meet the problematic (see Chap. 1 and Tall, 2013) to distinguish additive and multiplicative properties on every occasion. In the sixth grade, students extend it to the enlargement of figures. In the task sequence shown in Fig. 4.27, both additive and multiplicative strategies appear.

Figure 4.27 shows a task sequence to develop students who learn mathematics by and for themselves by using what they have learned through extended tasks.

In task 1, all students easily draw "a." The students use three drawing strategies: just adding 1 cm (↓), doubling of sides, and using the diagonal. They cannot distinguish these three on the drawing in task 1. In task 2, there are two drawings ("b" and "c") and they meet the problematic. The students who draw "b" use the strategy of "adding 1 cm." The students who draw "c" use the strategy of "doubling of sides" or "using the diagonal." In task 3, which is posed on nonsquared paper, students draw "d" by "using the diagonal" and compare it with the other two strategies ("b" and "c"). Then, the teacher asks the students to summarize what they have learned through this task sequence and continues the lesson on how to draw by using diagonals with the idea of proportions.

[22] In Japanese, just *bai* without a number implies "double." Thus, *bai-bai* means "quadruple." In English grammar, there are several types of numerals: cardinal or set numbers such as one, two, and three; ordinal numbers such as first, second, and third; multiplicative numbers such as once, twice, and thrice; multipliers such as single, double, and triple; and fractional numbers such as half and quarter. In Japanese, ordinal numbers, multipliers, and multiplicative numbers are expressed with denominations to the number such as first (1 *banme*), second (2 *banme*), and third (3 *banme*); single (1 *bai*), double (2 *bai*), and triple (3 *bai*); and once (1 *kai*), twice (2 *kai*), and thrice (3 *kai*).

This task sequence for the problem-solving approach (See Chap. 1) was produced under the curriculum and task sequence theory of Isoda (1992, 1996) based on the theory of conceptual and procedural knowledge by Hiebert (1986) (see Chap. 1, Fig. 1.1). In Task 1, the students already know the word "enlargement" in the daily context with images. At the beginning, it functions as the meaning. Thus, "a" is appropriate based on this meaning and it recognizes additive and multiplicative procedures. In Task 2, based on this meaning, "b" is inappropriate because it is an overgeneralization of additive procedure, and "c" (and "d") is appropriate because it is an adaptation of a multiplicative (proportional) procedure. While comparing the procedures, students are able to recognize the difference in additive and multiplicative procedures. The task sequence functions to reconceptualize it with an appropriate procedure. From Tasks 2 to 3, the procedure "using diagonals" is the only one that works. It is used to reconceptualize the meaning of enlargement without an additive strategy, and students are able to define enlargement based on proportion with the point (homothetic center) of enlargement by using diagonal. After this, the students continue to learn the case of other figures such as enlargement of polygons. Consequently, using diagonals becomes the procedure for enlargement of figures.

4.5.1 Redefinition of Proportionality at Junior High School

In Japan, proportionality is extended to negative numbers and redefined as a function at the junior high school level. The difference in additive and multiplicative structures is discussed using positive and negative numbers from four arithmetic operations to two operations by using the reciprocal and inverse (see Chap. 3). The equation of the function $y = ax$ can be seen algebraically as a generalized equation of the multiplication table such as in Figs. 4.25 and 4.26. The variable x is consistent with the multiplication table (see the discussion of Figs. 3.11 and 3.12 in Chap. 3). For $y = ax + b$ and $y = ax$, only $y = ax$ keeps the multiplicative property, and both keep the additive property.

The enlargement of figures in the sixth grade is the base for the definition of similar figures in relation to the center of similarity at the junior high school level. The line and point similarity of figures also becomes the source of problematics on this redefinition.

4.6 Final Remarks

The Japanese approach to multiplication at the elementary levels provides a consistent sequence for preparing future learning in the curriculum in the context of extension and integration, up to proportions. The introduction of multiplication in the second grade is a preparation for division in the third grade and a preparation for proportions in the fifth and sixth grades. The reason why the Japanese try to maintain a consistent sequence to develop and extend ideas is based on the aims of education, which tries to develop students who think and learn by and for themselves.

It is the process to develop the sense for making sense. The curriculum sequence enables students to think about what they already know and how they can challenge themselves to extend ideas by and for themselves. In the Japanese curriculum and textbooks, the problem-solving approach is enhanced based on a well-designed task sequence for applying already-learned knowledge to unknown tasks as well as preparation for future learning. Learned knowledge is not limited to the procedure but includes ways of thinking, the methods of representations, and values and attitudes regarding mathematics.

Solving non-routine problems under the Pólya framework is not the same in usage as the Japanese problem-solving approach in Chap. 1 because the task sequence, which is designed by the teacher, is basically defined under the curriculum sequence for enabling students to think by and for themselves in future learning. The second-grade tasks are unknown problems for the students; however, they become a routine problem after they have learned them or in later grades. The objectives of the tasks which produce unknown and problematic situations can be explained by using these terminology in the curriculum sequence. Under the shared curriculum, Japanese teachers has been engaging in lesson study by using these terminology to explain and distinguish the objective of every class and produceing the textbooks for enabling them to practice. For developing students' mathematical thinking and ideas for future learning through problem solving, the terminology is necessary to explain task sequence beyond just using a method of teaching simply changing every task to be open ended for just solve an independent problem. Similar terminology is existed in teacher education in the world but it is not always functioning among classroom teachers. Beside, Japanese teachers have to use it on their lesson study activities systematically.[23]

The Japanese approach, which is based on the consistency of the curriculum sequence under explained terminology described in this chapter, is preferred for projects in Central America, Southeast Asia, Central Asia, Africa, and so on. In relation to this, the Japanese definition and notations of multiplication are also preferred because of the following consistencies:

- Consistency among situations, repeated addition, and the multiplication table
- Consistency with other content such as measurement, division, fractions, ratios, and proportions in relation to distinguishing the multiplier and multiplicand
- Expandability to decimals and fractions by using consistent representations such as dot–area diagrams, proportional number lines, and tables for the rule of three
- Consistency in proportionality

The specified consistencies support the process of extension and integration at future learning. Multiplication provides strong bases up to proportions in this process. This specified feature is one of the reasons why the Japanese approach has been preferred by other countries. Such an approach, including the terminology,

[23] For example, Izak & Beckmann (2019) systematically explained the role of proportional number line however, in Japan, it was introduced for lesson study, already systematically embed the into textbooks 50 years ago (see Chap. 1) and now, it progressively changes its representations beyond the grades.

which is explained in this chapter, existed in the textbooks until the curriculum of 1968 and has been used in several Japanese official development assistance (ODA) programs around the world, from Singapore in the late 1970s to Africa and Central and South America from the 1990s onward.

There is an additional discussion on consistency in relation to the vertical form of multiplication in Chap. 7.

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Chapter 5
Japanese Lesson Study for Introduction of Multiplication

Raimundo Olfos and Masami Isoda

In Chap. 2, we posed questions about the differences in several national curricula, and some of them were related to the definition of multiplication. In Chap. 3, several problematics for defining multiplication were discussed, particularly the unique Japanese definition of multiplication, which is called definition of multiplication by measurement. It can be seen as a kind of definition by a group of groups, if we limit it to whole numbers. In Chap. 4, introduction of multiplication and its extensions in the Japanese curriculum terminology were illustrated to explain how this unique definition is related to further learning. Multiplicand and multiplier are necessary not only for understanding the meaning of multiplication but also for developing the sense to make sense the future learning. The curriculum sequence is established through the extension and integration process in relation to multiplication. In this chapter, two examples of lesson study illustrate how to introduce the definition of multiplication by measurement in a Japanese class. Additionally, how students develop and change their idea of units—that any number can be a unit in multiplication beyond just counting by one—is illustrated by a survey before and after the introduction of multiplication. After the illustration of the Japanese approach, its significance is discussed in comparison with the Chilean curriculum guidebook. Then, the conclusion illustrates the feature of the Japanese approach as being relatively sense making for students who learn mathematics by and for themselves by setting the unit for measurement (McCallum, W. (2018). Making sense of mathematics and making mathematics make sense. *Proceedings of ICMI Study 24 School Mathematics Curriculum Reforms: challenges, changes and Opportunities* (pp. 1–8). Tsukuba, Japan: University of Tsukuba.). A comparison with Chile is

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given in order to demonstrate the sense of it from the teacher's side. In relation to lesson study, this is a good exemplar of how Japanese teachers develop mathematical thinking. It also illustrates the case for being able to see the situation based on the idea of multiplication (Isoda, M. and Katagiri, S. (2012). *Mathematical thinking: How to develop it in the classroom*. Singapore: World Scientific; Rasmussen and Isoda Research in Mathematics Education 21:43–59, 2019), as seen in Figs. 4.2 and 4.3 in Chap. 4 of this book.

5.1 Lesson Study for the Introduction of Multiplication

The introduction of multiplication to students does not demand much time. Teaching the meaning of multiplication demands 3 or 4 hours of lessons or sessions[1] of 45 minutes each in the three Japanese textbook series we analyzed (Gakko Tosyo,[2] Tokyo Shoseki,[3] Keirinkan, and PROMETAM[4]) for multiplicative situations.[5] The terms "multiplicand" and "multiplier" are introduced to create the mathematical sentence appropriate for a given situation. Enabling students to see multiplicative

[1] Japanese usually teach mathematics with the whole class and use terminologies in Chap. 4 on the unit plan. A mathematics lesson in Japan corresponds to a session in a subunit of the unit plan. The subunit is usually called a "phase." Another usage of the term "session" refers to one class hour. The term "lesson" refers to the topic addressed by the lesson plan and is sometimes not limited to one class hour. The lesson plan usually refers to a part of the phase in the unit plan, which means a section in the textbook. On the other hand, based on research in mathematics education, sessions usually use the context of the topic sequence. Here we have used the term "session" for one class hour. The lesson plan for lesson study by the group usually has a study theme and the objective of the class with the content topic as the teaching material (see Chap. 1; Isoda, 2015).

[2] The English-translated edition of *Study with Your Friends: Mathematics* (Hitotsumatsu, 2005; Isoda and Murata, 2011; Isoda, Murata, and Yap, 2015; Isoda and Murata, 2020). Thai translated edition (Inprasitha and Isoda, 2010) is from the 2005 edition. Spanish-translated edition (Isoda and Cedillo, 2012) is from the 2005 edition. Indonesian adapted edition (Isoda et al) is from the 2011 edition. Chilean adapted edition (Isoda et al., 2020; Isoda and Estrella, 2020) are the 2005 and 2011 editions.

[3] The English-language edition of *New Mathematics* is used (Hironaka and Sugiyama, 2006).

[4] PROMETAM is the Project for Improving Technical Education in the Area of Mathematics in Honduras, with technical assistance from the Japanese International Cooperation Agency (JICA). The JICA-supported projects PROMESAM in the Dominican Republic, PROMECEM in Nicaragua, GUATEMATICA in Guatemala, and COMPRENDO in El Salvador were also implemented in the period in which the framework for the development of the texts was elaborated.

[5] For analyzing those textbooks, we also referred to the framework of Vergnaud (1990) to describe the concept of multiplication in a situation, the invariant, and the representation. However, we did not explain the lesson using his terminologies because we would not be able to clearly explain the significance of the teaching sequence based on his framework. Actually, the teaching sequence was never discussed on his framework. Instead of using his analytical terminology, we illustrated the real lesson study classroom in Japan and compared the Japanese approach with the Chilean approach to show its significance.

situations with the idea of multiplication is a particular feature of Japanese education.[6] In a later section, it will be compared with the Chilean approach using the terminology "sense making" or "making sense."

5.1.1 Lesson Study on the Meaning of Multiplication, by Mr. Natsusaka

In relation to the subtheme of this book, this section presents an exemplar of lesson study with the lesson plan, implementation of the lesson (an open class), and discussion of the implementation, carried out in June 2008. The open class for the lesson study was implemented by Mr. Satoshi Natsusaka from the Elementary School at the University of Tsukuba. The implementation corresponds to the first of the three lessons that introduce the meaning of multiplication to second-grade students.

5.1.1.1 Description and Plan of the Lesson Being Investigated

The topic to be studied in this lesson was "the meaning of multiplication," developed by Mr. Natsusaka. The goal of the study was to consider lessons that would allow for developing students' competency to use multiplication by linking the situation with multiplication expressions, taking advantage of how students would understand the situation.

[Lesson plan by Mr. Satoshi Natsusaka]

1. *Unit name*: Multiplication (1).[7]
2. *Research theme of lesson study*: To develop the eyes to see the situation mathematically.

(a) From "counting" and "discovering" activities to "expressing" activities: When there are a number of groups with the same quantity of elements (a unit of measurement)—say, balls—it is expressed as the "number of balls in a group times

[6] In Japan (as explained in Chap. 1), development of mathematical values, attitudes, ways of thinking, and ideas have been the objectives of mathematics teaching since 1968. Mathematical ideas usually change the way to see the situation. In research on mathematics education, this is sometimes referred to using terms such as "intuition" and "insight" (see van Hiele, 1986).

[7] He is an author of Gakko Tosho textbooks (2005). It has four chapters on multiplication for grade 2. Multiplication (1) provides an introduction and definitions with the meanings of situations. Multiplication (2) covers development of the row of 2, the row of 5, the row of 3, and the row of 4, and learning how to develop the multiplication table. The discussion between rows is used to produce the idea of distribution for extension of the table. Then, Multiplication (3) discusses extension of the multiplication table to include the rows of 6 to 9 and the row of 1. It is expected that students are able to extend it. Multiplication (4) explores the properties of the multiplication table. Finally, the book discusses the making of a project by students (see Fig. 4.2 in Chap. 4, and see Chap. 6). The lesson being analyzed here is his original work. Textbook authors usually try to offer new challenges in their classes to produce innovative ideas for teaching and further revision of textbooks.

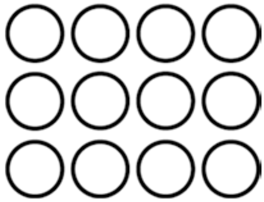


Fig. 5.1 The way to explain the array diagram such as 4 columns of 3 balls and so on

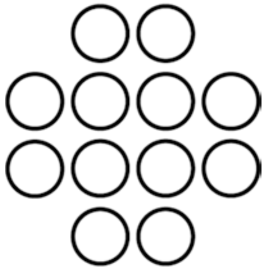


Fig. 5.2 Grouping the dots for multiplication

the number of the same groups" (in Japanese) which corresponds to the mathematical expression of multiplication. The students express such a situation using phrases like "There are *n* balls in each group (set) and there are *m* groups (set)" even though they do not know the mathematical expression for multiplication (see Fig. 4.3, Fig. 4.8 in Chap. 4). For example, when balls are placed in a box as shown in Fig. 5.1,[8] some students may express this situation by saying, "There are 4 columns of 3 balls." This expression can be considered to identify groups of 3 balls aligned vertically and to show that there are 4 columns with this quantity of balls. There are no lines that separate or encircle groups of 3 balls, but students who use this expression are imagining these lines.

Similarly, some students may observe the same situation from other points of view, such as "3 rows of 4 balls" or "2 groups of 6 balls." In any case, they will try to calculate the total number of balls by identifying groups with the same quantity of elements. If it is understood that there are "4 columns of 3 balls," the total number of balls can be found by making the calculation "3 + 3 + 3 + 3 = 12." It is appropriate to lead the students to the multiplication expression, obtaining the expression from them and confirming what the expression "3 [×] 4" represents.[9]

(b) The competency to see the situation as multiplication:[10] As shown in Fig. 5.1, the quantity of balls in Fig. 5.2 is 12. The students who realized that in Fig. 5.1

[8] This was discussed as the array diagram in Chaps. 2 and 3.

[9] In this chapter we use the Japanese notation "3 × 4" instead of the English notation "4 × 3" because we quote Japanese textbooks and photos in the classrooms, and we could not change original photos, and so on. Thus, "[×]" is written as "×" from here onward.

[10] In Japan (as discussed in Chap. 1), seeing the situation through mathematical ideas has been emphasized as subject matter of teaching to develop mathematical thinking since 1958 (see Isoda and Katagiri, 2012; Rasmussen and Isoda, 2019).

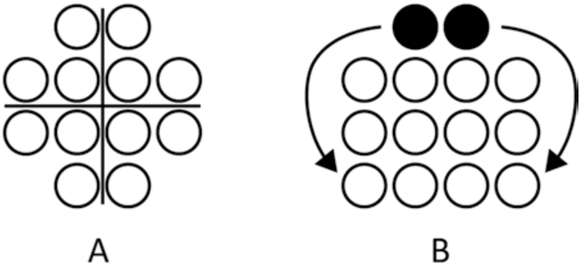


Fig. 5.3 To see the shape for multiplication: (**a**) 4 corners of 3 balls. (**b**) If we move two balls to appreciate places, it changes to 4 columns of 3 balls

there were 4 groups of 3 balls are asked if they can also see that there are 4 groups of 3 balls. Then, some students may think of separating the balls as shown in Fig. 5.3a or moving the 2 balls placed in the upper part to the corners of the lower part as shown in Fig. 5.3b. As such, the way of placing the balls is changed so that it is the same as in Fig. 5.1.[11]

The custom of observing the figure and determining the quantity of balls per unit or group will increase students' competency to see the situation as a multiplication expression or a model of multiplication. Also, listening to how other students interpret the figure and recognizing the model will allow them to enrich their points of view.

3. *Unit goals*:

(a) To understand the meaning of multiplication through concrete situations.
(b) To be able to formulate the multiplication expression for situations that can be expressed as such.

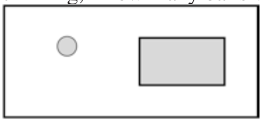
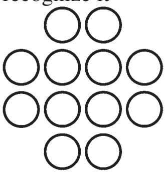
4. *Unit plan* (4 hours):

(a) First phase: The meaning of multiplication (2 hours); this is the first of the 2-hour lesson.
(b) Second phase: Applying multiplication (2 hours).

5. *Lesson outline:*

(a) Goal (objective of the class): Learn to express that "there are *m* groups of *n* quantities" considering groups of the same quantity when the number of elements is counted.
(b) Development of the lesson.

[11] This is the activity that enhances seeing the situation as a multiplicative situation (see Figs. 4.2 and 4.3 in Chap. 4).

Main learning activity	Considerations
Situation 1: observing the box and thinking, "How many balls will fit in it?"  2 + 2 + 2 = 6 (balls) 6 balls because 2 balls are placed in 3 columns 6 balls because 3 balls are placed in 2 rows *Situation 2*: observing the shape of a box where 12 balls can fit, and thinking, "How many balls will fit in it?" 6 sets of 2 balls (2 + 2 + 2 + 2 + 2 + 2) 4 sets of 3 balls (3 + 3 + 3 + 3) 2 sets of 6 balls (6 + 6) Observing the balls placed as (Fig. 5.4) from the same point of view as in situation 2 if they are moved 2 + 4 + 4 + 2 = 12 4 sets of 3 balls are seen 6 sets of 2 balls are seen	Thinking by considering the role of the rectangular drawing of the box It is desirable for the students to realize that if the number of rows and columns is known, the total number can be determined without putting all the balls in the box Try to get verbal expressions like "there are so many groups of so many balls" from the students or expressions through the additive model It is desirable to take advantage of the point of view of situation 1 Make the expression correspond to the words If the numbers are added from the first row downward, it can be expressed using the equation 2 + 4 + 4 + 2 = 12 If there are students who try to change the way the balls are placed by moving some, they could also recognize it 

[End of lesson plan]

5.1.1.2 A Public Lesson (Open Class) by Mr. Natsusaka

The following is a translation of a transcript of the notes taken during the implementation of the lesson by Mr. Natsusaka with a class of 39 second-grade students from the Tsukuba School in Tokyo on June 19, 2008. These notes were taken in Spanish based on the simultaneous translations from Japanese that were offered to Central American teachers observing the lesson.

The lesson took place in the Elementary School Theater at the University of Tsukuba in Tokyo. Fig. 5.4 shows the arrangement of the desks between the stage and the first row of seats in the theater.

At 9:18 a.m. the students went up to the stage in two lines and received a round of applause from the audience. There were more than 300 people present, the majority of whom were teachers from different parts of Japan. Some of the guardians (parents) participated in recording the class to support Mr. Natsusaka. Without a doubt, the lesson being observed was an important occasion not only for the teachers watching but for the students as well.

After Mr. Natsusaka guided the students in greeting the audience and ceremonially opening the lesson, he flashed on the interactive screen questions about types of triangles and polygons. On various occasions, the students went to the screen and touched a part of it as a way to answer the question asked, such as "Which of these

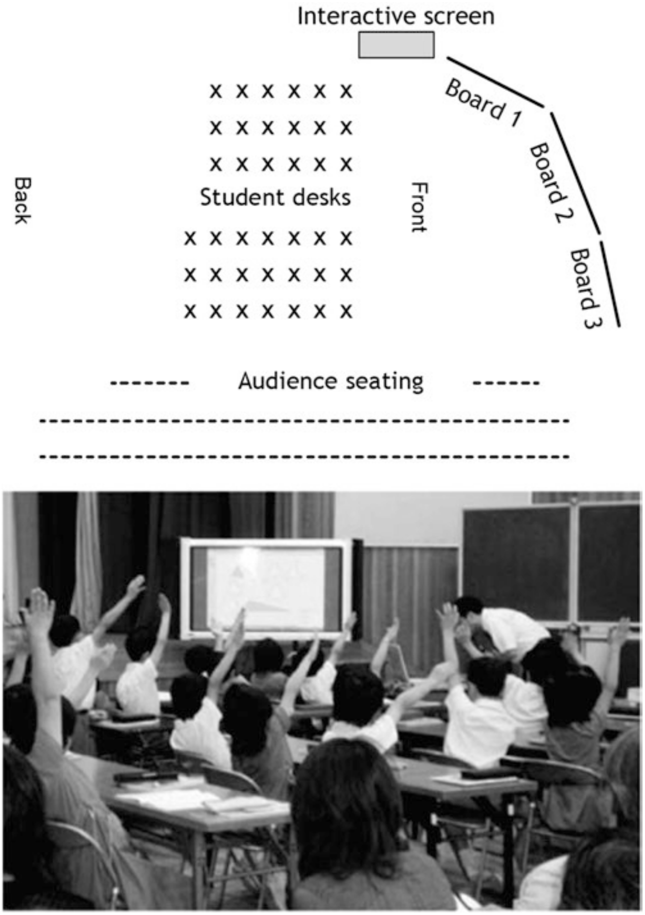


Fig. 5.4 Open class given by Mr. Natsusaka

figures is a triangle?" and "Which of the triangles is equilateral?" When a student touched the correct answer, the figure was filled in with the color green. Otherwise it was filled in with the color red. The activity let the students recognize 3 triangles, 2 rectangles, 1 pentagon, and 1 hexagon among the figures. Next, Mr. Natsusaka proceeded on to new questions using the interactive software, changing the content flashed on the screen, such that 4 triangles and 6 rectangles appeared. At this point, all the students raised their hands, and it could be seen that they had become comfortable and were involved in the dynamics with the teacher.

9:36 a.m. (Teacher, Mr. Natsusaka, presented on the interactive screen a rectangle with circular pastries in it. He never mentioned that this was an introduction to multiplication. It should be noted that he used the interactive screen (see Fig. 5.5) and not paper or a chalkboard to present the problem situation, as was indicated in the lesson plan.)

Fig. 5.5 Mr. Natsusaka uses the interactive screen to present the problem situation

Teacher: "How many sweets will fit in the box?" (A rectangle shape and one circular[12] sweet were shown on the screen.) "Make guesses about how many pink sweets will fit in the blue box."

(A student came up to the electronic screen and demonstrated on the screen what he understood. Teacher did not say whether this was good or not; he delayed reacting on purpose to give time for the students to think by themselves.)

9:37 a.m. Teacher: "How many pink sweets will fit?"

9:38 a.m. Teacher: "Open your notebooks and write. How many sweets will fit?" (Teacher observed that some students were not working, then he added the following for those who hadn't thought of it.) "Look at the screen. I would like to know your predictions." (Teacher walked around the classroom from desk to desk and quickly looked at the students' notebooks.) "Now, I'm going to write here," (using the left part of the second board) "some of your answers in your notebooks: 4 . . . , 5 . . . , 6 . . . ; from what I can see, some of you have written '4,' others '5,' and others '6.' One student wrote '8' and another '12.' Which of these answers seem possible to you? Which seem impossible?" (Teacher provided ambiguous situations and let the students fix the necessary conditions by asking questions that made them think. Indeed, the students began to critique.)

Student 1: "It can't be 4; 2 more would fit."

Student 2: "If we look at it, 6 would fit."

9:40 a.m. Student 2: "Can the sweets be placed on top of each other?"

Student 3: "One layer of 6 and another of 6. I don't think that only 6 will fit. If you want it to be a box with 6 on the bottom and 6 on top, it has to be a taller box."

Teacher: "What are the bases for your conjectures?"

Student 4: "I think that 12 will fit: 6 on top and 6 on the bottom." (looking at the box from the top view)

Student 5: "Observing it from above, then it would be 12: 3 layers of 4."

9:45 a.m. Mr. Natsusaka: "We will exclude that case. The box has to have all the pastries visible." (The boxes with layers viewed from the top were excluded.)

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[12] They know the circle as a shape; however, they do not know the property of a circle. Thus, it is an ambiguous figure.

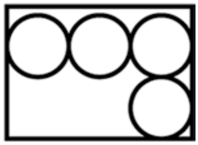


Fig. 5.6 On the Monitor Screen: Mr. Natsusaka confirmed 3 or 4

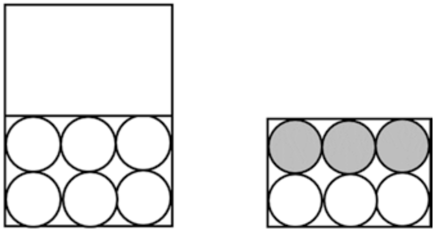


Fig. 5.7 Show the students ideas on the screen

Student (going to the board and showing with his fingers how the length of the diameter of a circle was contained three times in the length of the rectangle): "3 on the top row and 3 on the bottom row, so 6 fit."

(Mr. Natsusaka put another box on the screen under the first. With the mouse and software tools, he placed 4 pastries in the box (see Fig. 5.6).)

Teacher: "It's the same as what you did with cardboard. We have to prove . . . ," (after drawing) "so 6 fit. There is enough space. If there are 3 in the first row, then . . . 3 fit on top."

Student 1 (using the software's copy option to draw another rectangle on the screen and commenting as follows): "Since 6 fit the first time, if the box is tall enough, 6 more will fit." (In the left part of Fig. 5.7, the 6 balls became a unit, which was the side view of the layer.)

9:50 a.m. Student 2 (speaking from his desk and pointing at the 3 balls in the right part of Fig. 5.7): "The balls are superimposed."

Teacher (trying to lead them to see it as 6): "Do 6 fit? Raise your hand if you think that 6 fit." (Several students raised their hands.)

Student 4: "3 fit in 1 row. 3 + 3 is 6."

Teacher (writing the expression "3 + 3 = 6" on the board): "3 + 3 = 6."

Student (pressing the software's buttons, visible on the interactive screen, and drawing as he spoke): "Then there are 6. There is a group of 3 and there is another group of 3."

Teacher: "How did you divide it?" (on the screen)

Student: "Days ago," (Mr. Natsusaka did not conduct the class for multiplication, yet) "we made a drawing like this," (pointing to the ovals drawn on the board (see Fig. 5.8)) "we changed the shape, but is it the same?"

Student: "If we think of 2 groups of 3, there are 6."

9:59 a.m. Teacher: "First, listen to what your classmate said," (repeating the student's idea) "yesterday, someone separated it like that and said there were 2 groups of 3. Could I also say that there are 3 groups of 2?" (See Fig. 5.9.)

Fig. 5.8 Board Writhing (Bansho), Japanese teacher listen students' idea through questioning and note on the board. See Fig. 1.2, Chap. 1

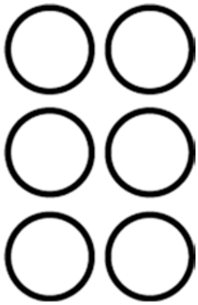


Fig. 5.9 Read the diagram and explain vertically and horizontal

Fig. 5.10 Mr. Natsusaka, Teacher, and a student interacting on the board

(Teacher no longer drew with the software on the electronic board; he drew on board 2 with colored chalk, in the upper left-hand portion.)

Teacher: "So . . . there are 3 groups of 2. So, in a row of 2, there are 3 groups. So, 3 groups of 2, there are 3 groups of 2. Do you all see it like that?"

Student: "If we take 3, twice, then it will be 6."

Teacher: "3 + 3 = 6." (See Fig. 5.10.) "So, in this case, how can you express 2 + 2 + 2? If you express it using addition, how can we express it?" (Only half the class raised their hands, and a student asked Mr. Natsusaka a question).

Student: "It's 2, like it's grouped that way. So, is it about groups?"

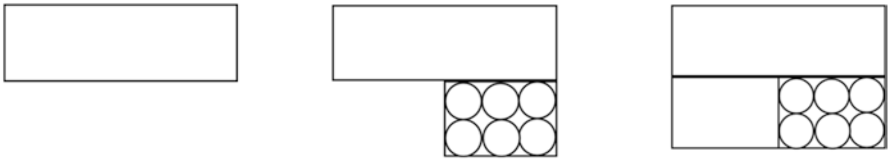


Fig. 5.11 How many boxes? Here the box is a unit for counting: The inverse idea of splitting (see Chap. 3).

Teacher: "Your classmate asked what this number 2 represents." (Teacher paused and waited for the students to raise their hands, then he spoke to the student who had just spoken): "Could you repeat what you said before?"

Student (returning to the board and explaining it as follows): "A group of 2 repeated 3 times."

Teacher (speaking to another student to evaluate his understanding and focus the discussion): "Can you repeat what your classmate said?"

(The student did not answer, so Teacher did.)

Teacher: "So, 2 represents the number that divides. There are 3 groups of 2. The number that divides $2 \times 3 = 6$ is 2. 3 indicates how many times."

Student: "I can say it more simply: 2 is the number that is going to be multiplied."

Teacher: "He said it in a way that is easier to understand."

Student: "This number, 2, of 2 times 3, leads to 6. 3 shows how many there are."

10:03 a.m. Teacher (drawing 2 pastries inside a circle on the board in blue): "So, 3 indicates 'how many circles.' One way is 2 groups of 3, and another is 3 groups of 2; that is, it can be said in different ways." (He then returned to the interactive screen.) "Now, in this box," (see the left part of Fig. 5.11) "how many will fit?"

Student: "Can you show the previous box again?" (See Fig. 5.6.)

Student: "Can you show both boxes?"

Teacher used the mouse to copy the box, showing both.)

Student: "Can you move one box under the other?"

10:06 a.m. Teacher: "Yes, I can move it." (Since the student came up with arguments, Teacher asked him to go over to the interactive board.) "Come here."

Student: "In this [the box underneath], 6 fit." (He used the software to move the boxes and line them up (see the middle part of Fig. 5.11).)

Teacher: "It looks like it marked it there. Do you know what it's doing?"

Student: "It's covering it up."

Teacher: "He adjusted, marked, and moved. Think, what is Lu's intention?"

(Teacher gave the students 30 seconds to talk in groups of three.)

Teacher: "Are the rectangles below of the same width?"

Teacher (showing that the rectangles had the same width by placing one rectangle beside the other): "So, how many sweets fit in this big box?"

Students (all responding together): "12."

Teacher: "Who thinks that it's not 12?"

Student: "I'm not sure, but it has to be even."

Fig. 5.12 6 + 6 = 12 on the third board; is there another way to express the total?

10:12 a.m. Student (in front of the interactive screen, and moving the lower rectangle): “It fits twice, so 12 fit.” (See the right part of Fig. 5.11.)
Student 2: “6 fit in the small box. I marked it there, and the space is equivalent to the box. So, the big box is equivalent to two small boxes. So, the total is obtained by adding two sixes.”
10:15 a.m. Teacher (writing on the board): “You all say that twice 6 is 12.” (See Fig. 5.12.)

$$6+6 \rightarrow 12$$

Teacher: “So, 2 groups of 6, 4 groups of 3, 2 groups of 6. Is there another way to express the total?”
Student: “4 + 4 + 4.”
Teacher: “4 and 4 is 8, and 8 and 4 is 12.”
Student: “We can divide 3 times 4 in another way. 4 + 4 + 4 is a new way.”
Teacher: “Are there other ways?” (Teacher then decided to end the lesson.) “I had planned to have you try with stickers, but it’s time to end, so we will have to leave that for the next lesson on Monday. Now we’re going to say goodbye to the teachers visiting us.” (They looked at the visiting teachers.) “I’ll go with you all in a little bit. I’ll catch up.”
(A student asked if they would have another special activity the next day.)
Teacher: “No, you’ll have your normal classes. Tomorrow, there is music class. Don’t forget to bring your pencils, textbooks, and PE uniforms.”
10:19 a.m. (The students left.)

5.1.1.3 Post–Open Class Discussion

Once the students had left the theater in a line, the teacher spoke to the audience to give justifications for his actions according to the goal proposed for the lesson.

Mr. Natsusaka (Teacher): “Thank you; I would like to receive your comments. We just witnessed a second-grade lesson of introducing the meaning of multiplication, in which the students made conjectures about how many pastries fit in a box.

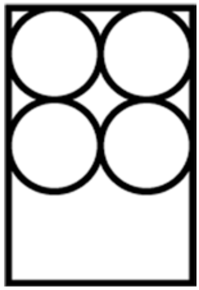


Fig. 5.13 Estimation of how many balls in the box

My intention was that the students would present at least the number per column. More than half already knew the word ‘multiplication’ although I don’t know if they understand it. But my intention was that the students would learn the meaning of multiplication, so although I heard the word ‘multiplication’ many times or expressions like ‘2 times 3’ I didn’t repeat them because I wanted them to understand. So, I avoided introducing the expression ‘multiplication’ on purpose. I tried to use terms they all know.”

10:31 a.m. Mr. Natsusaka: “In the lesson, the first student said ‘4 + 2 is 6,’ expressing the total as a sum (See Fig. 5.13.). The second student said, writing vertically, that a group of 3 and another group of 3 is 6.” “This introductory part lasted for 10 minutes. There were two expressions that came out of this: ‘2 groups of 3’ and ‘2, 3 times.’ Maybe the students didn’t realize this. But I wanted them to understand. A student said ‘2 + 3’ but this sum cannot be used, so multiplication appeared as something important and necessary. Expressing verbally ‘in 2 groups there are 3’ indicates that there is another way to see it. So, I changed the color from blue to red because it represented something different. I wanted them to learn a new arithmetic operation. So, here,” (pointing to a diagram made during the lesson) “there are 12 units. Then a child explained the situation thinking of figures of 4 objects, 3 groups of 4, separating it in different ways. My intention was for the students to discover different groupings. How many groups could there be? I wanted them to group the objects in different ways before using the term multiplication.”

Visiting teacher from Central America: “Why didn’t you use concrete materials?”

Mr. Natsusaka: “I decided not to use tokens or concrete materials as I had already shown this to the students. Also, we already did that in first grade. That is used in Japan, but this time they didn’t use tokens. Honestly, I was thinking of using a blank piece of paper and different colored stickers to stick on the boxes. In the fourth grade, we study area and dimensions.”

Teacher in the audience: “I am using this program and I see the usefulness of the program. But why didn’t they use the real conjectures in three dimensions and see the height, as maybe it could be shown in different ways? Comparing would be easier for the students with something more real.”

Mr. Natsusaka: “This time I showed 2 vertically and 3 horizontally. What do you suggest?”

Teacher in the audience: “A student showed 2 times 3 vertically. But using a real box would be more efficient. It would be possible to have various boxes and adapt to the students’ answers.”

Mr. Natsusaka: “Before class, I practiced with the software. Maybe more drawings could be included. I didn’t think of using a real box because it could be 6 or 12 that can fit. I didn’t think they would reach that point [three dimensions].”

Teacher in the audience: “I come from an island. You achieved the goal, but I’m lost. You always ask, ‘Why do you think so? Write the reason.’ This kind of behavior was visible and developed reasoning and imagination. But why did the students know how to answer?”

Mr. Natsusaka: “I follow the guidelines of the new program [the national curriculum standards] which places greater emphasis on verbal expression: expressive competency, comprehensive competency, and textual formulations. It is important to ask for the reason or ‘Why can it be written like that?’ Teachers tend to assume what the student expresses. It is important to let the student say why so that the teachers will know how much they have understood and where their limit of understanding is. It is important to know how far they have really understood. The students fail in verbal expression, so they do it with diagrams. The students want to communicate their ideas. There are cases when students have difficulty expressing their ideas verbally, so they just draw diagrams. I understood that one student could not say what he understood, so I asked him to express it in another way or with more words—that is, to paraphrase. It is also important to promote the competency for listening—that they know how to interpret what others are trying to tell them. I intend to listen well to be able to communicate. If I express the ideas ambiguously or unclearly, then more time and situations are needed to communicate. If the students understand, then they ask questions. I asked them to express their ideas in another way by drawing. When the students try it, you have to evaluate what they understand. In the beginning, we played with these shapes.” (He indicated the triangles and rectangles on the interactive screen.) “There is an open polygonal chain. I asked, ‘Why isn’t it a shape, or can it be considered a shape?’ I get the children to think, ‘Why?’ Back to the topic, it is interesting to develop the competency for interpretation. I observe the students’ faces to see if they are listening to me. Looking at their faces, I can see if they don’t understand. As my colleague Mr. Tanaka says, ‘I look at the back of the classroom, and I go to the middle of the room to see if they are listening and understanding.’ It is important to see the students’ faces.”

Teacher in the audience: “As you speak of the meaning of multiplication, you mentioned that in the fourth grade they will study area (dimensions). Regarding area, how would you introduce it? Because now you are drawing pastries (circles) . . .”

Mr. Natsusaka: “It’s a difficult question. Today, for example, I used pastries of the same color; maybe one row could be one color and the row underneath another color. But my intention, which I wanted to develop among the students, was that by looking at the same drawing, they could see various forms. Colors are useful for area; 1 rabbit, 2 ears; 3 rabbits, 6 ears. Such attribute models are in another discussion because it fixes the view to every rabbit. But my intention was that in this lesson, the students would learn to group in different ways by themselves. In the case of rabbits, it’s obvious that they have 2 ears; it cannot be changed.”

Fig. 5.14 Establish entangler shape to recognize the situation for multiplication

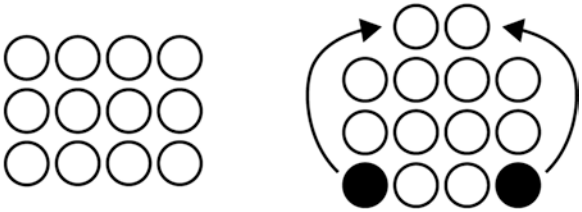


Fig. 5.15 To find the various unit for multiplication

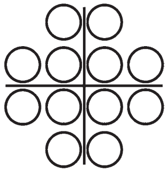
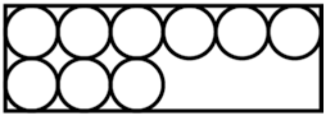


Fig. 5.16 Change the box for changing the number of balls in it



11:05 a.m. Mr. Natsusaka: "How many circles are there? I thought of expressions like '4 + 4 + 4,' but there are students who thought, '3 in the corners, 4 times.'" (See Figs. 5.14 and 5.15.) "It is important to develop this competency for discovering different groupings. After learning the meaning of multiplication, it can also be applied to this drawing. I moved two circles to give shape to the image." (See Fig. 5.14.) "It's not that they already know, but, rather, that before starting with the multiplication table, they have already learned to group in different ways."

Visiting teacher: "I come from a distant province. You insisted on the competency to group in different ways. But when you said to a student that the big box (pointing to the rectangle drawn on the interactive screen) was the same as the 2 small boxes, he said that it wasn't the same. Maybe he said that 2 boxes of 6 isn't the same as 1 box of 12."

Mr. Natsusaka: "Maybe a student said 3 rectangular boxes in the big box." (See the right part of Fig. 5.16.) "There were students who saw the big box as 2 small boxes. But there were students who saw it as 3 boxes of 3 circles each. Later, in the next lesson, the students can continue with representation. There were a few who thought that there were 3 boxes."

Visiting teacher: "Your lesson gave me ideas for my lesson. Sometimes, comparing with my class, I intervene too much. But what were you trying to do? Also, the board was not used very well."

Mr. Natsusaka: "I asked the students to express their ideas verbally. They didn't do it. I tried to get them to formulate something before coming to the board, as some of them forget when they try to express it verbally. I wrote slowly so that the students could keep up. I also intervened when it was something important. When I posted the four problems, there wasn't enough space on the board for the fourth problem. It depends on the students. Some of them try to economize in their notebooks. The use of the board and the notebooks have a lot to do with each other. There are teachers who insist on writing down the class goals. I don't agree, because the goals are not static. The students' goals, hidden goals, or maybe apparent goals can appear. I think this is wrong. On the other hand, writing down the goals leads the weaker students to understand better. Depending on the nature of the goals, it may be best to write them down or not, as some will be explored and discovered. If I write down the goals and I want them to discover regularities, then the lesson is already over, because if I write it, then they already know that there is regularity; thus, there is no more exploration. There are topics for which the lesson goals cannot be written. They are understood during the development of the lesson. Maybe halfway through the lesson, the intention can be written from the students' perspective."

11:20 a.m. Teacher in the audience: "We were observing the first lesson for understanding multiplication. What will the next lesson be like?"

Mr. Natsusaka: "It continues with the topic on expressing multiplication. For example, '2 × 3 is 6,' and it will show multiplication directly, no longer using 3 + 3 + 3 but, rather, the multiplication expression."

Teacher in the audience: "One student said '1 unit 12 times.' How do you deal with this student?"

Mr. Natsusaka: "I would use the idea of 1 × . . . ; the multiplication table only goes up to 9, but maybe 12 groups of 1 can be expressed, or 1 unit 12 times, even though for now we only express up to 1 × 9. But it can be done."

Teacher in the audience: "For the students to use multiplication, do you think it is important that they see different shapes or groupings?"

Mr. Natsusaka: "The students grouped in different ways, as I have said. In the following lessons, we will use multiplication and the students will learn the multiplication table. To familiarize them with the multiplication table, I use the method of practicing with a written record of their progress in the multiplication table." (He shows notebooks made by the students that they use as a support for memorizing the table (see Fig. 5.17).) "The student learns the multiplication table for each

Fig. 5.17 Students' homework notebook (journal) for multiplication to show the group as unit and a number of groups

number and is asked to check his progress. Then, the teacher or a family member (parent) signs after checking the memorization of the table. Then the student advances with the multiplication table of 2, of 3, etc. In the following lessons, I have the students write their ideas, then I have them do exercises. Following that, we look at some of the properties of multiplication. I ask them, ‘If I add these two rows,’ (referring to 2 and 3 in 3 × 2 and 3 × 3) ‘is it the same as 3 × 5? How much is it? If there are 6 and 9, then there is 15.’ That way, the students in the second grade discover that the results for the row of 5 are the sums of the results of the rows of 2 and 3. Now I cover the part of the multiplication table, and I ask them to say the sum. What I am using is the distributive law. That way, they think of, look for, and discover patterns in the multiplication table. I can cover four numbers at a time. Many things can be learned from the multiplication table, which is why it is good for them to know how to use it well.”

11:27 a.m. (Dr. Isoda introduces himself as a professor at the University of Tsukuba.)

Dr. Isoda: “The students learn ‘How many more?’ but ‘How many times?’ is something different. Now, the students do not know how to multiply, but through grouping, it is possible that multiplication expressions present themselves. Multiplication, as an arithmetic operation, is important for students to learn how to express relations with a meaning different from that of addition. In multiplication, the first number represents something totally different from the second number. This was not mentioned in the lesson.”

Mr. Natsusaka (thanking Dr. Isoda for his contribution and closing the comments and question time): “Thank you for your attention.”

(While the audience is leaving the theater, a group of Central American teachers stays in the hall and asks Dr. Isoda some questions.)

Observing teacher from Central America: “How did the teacher carry out the evaluation of the lesson? The students have a tendency to count, and the teacher’s intention is that they group.”

Dr. Isoda: “Assessment for teaching and rating of students should be distinguished. As a confirmation, the teacher usually assesses students’ learning within the lesson, such as observing whether or not the students raise their hands and if they understand. Based on such assessment, teachers make decisions on what is necessary activity and needs to share the ideas or ask students to imagine other’s ideas, and so on.”

Observing teacher: “Do they all have computers? What was the importance of the use of the interactive screen?”

Dr. Isoda: “In this lesson, only the program with interactive software was used. There is a tendency to use it. Today the software’s advantages were not seen well. We can do the similar activity by using cards and so on. It is being experimented with now. It is a good interactive tool as well as other manipulative. There is a tendency to use the interactive screen, to learn Information, Communication and Technological (ICT) tool, not to do something new but, today, it was used rather, as a teaching tool.” (See Fig. 5.18.)

Fig. 5.18 Dr. Raimundo Olfos (*left*), Dr. Masami Isoda (*middle*), and a translator (*right*)

5.1.2 Lesson Plan on Applying the Meaning of Multiplication After Learning the Multiplication Table, by Mr. Tsubota

In the second grade in Japan, seeing situations in various ways with multiplication are usually learned both in the introduction to the meaning of multiplication and in the application after students have learned the multiplication table. The following lesson plan uses the meaning of multiplication and was developed by Mr. Kozo Tsubota (2007), Vice Principal of the Elementary School at the University of Tsukuba. Please note that the lesson study usually has a research theme. The proposed research theme in this case is “Representing Ideas Using Expressions and Interpreting Expressions” for solving problems using multiplication, which is related to finding the unit for multiplication. However, in this exemplar, students have already learned the multiplication table. This is a good task for students in the next grade. Thus, interpretation between an expression and a situation is the main study theme. It should be noted that in lesson study in Japan, the lesson study theme and the goal/objective of the class should be distinguished (see Chap. 1 and Isoda, 2015a).[13] The study theme is the

[13] Around the world, there have been a number of research on lesson study and some misconceptions about Japanese lesson study. They are related to the research of M. Yoshida (see Fernandez and Yoshida, 2004), who focused on school-based lesson study, which was a very unique activity in the world for professional development 20 years ago. In English, international researchers did not have the opportunity to understand the various meanings of Japanese lesson study (see Chap. 1). School-based lesson study at the elementary school level usually enhances a limited lesson study group as a learning community in the school; this is true. On the other hand, there are several types of lesson study communities in Japan as we mentioned Fig. 1.4 in Chap. 1. A good example is the subject-based lesson study that originated from the Elementary School at the University of Tsukuba in 1873 (see Isoda, Stephens, Ohara and Miyakawa, 2007). It is used for curriculum development too. In subject-based lesson study, the teacher usually focuses a lot on both personal research activity in the research society beyond his or her school and demonstration activity to show his or her practice at several schools as an invited consultant. Indeed, every teacher at the Elementary School at the University of Tsukuba has his or her community of lesson study in his or her subject.

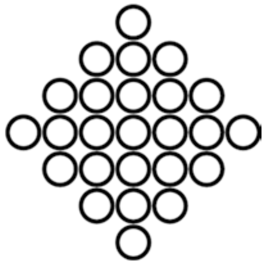


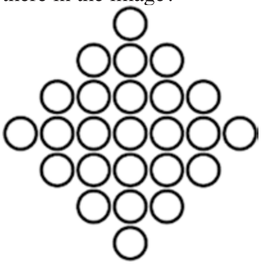
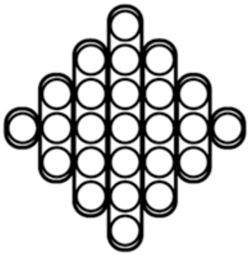
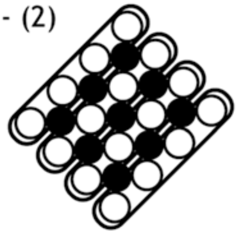
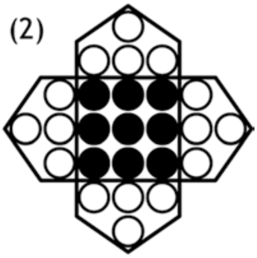
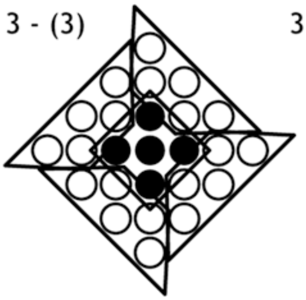
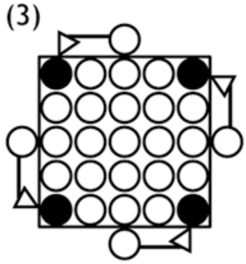
Fig. 5.19 Various unit for grouping and which is easier to get the number

theme proposed by the teacher who teaches the class and is written as a general issue. The objective of the class is defined for the specific content in the curriculum sequence.

1. *Study theme of lesson study*: Representing ideas using expressions and interpreting expressions.
2. *About the theme*: In this lesson, the students find the number of dots in a collection (arrangement) of dots (see Fig. 5.20), and find ways of counting the number of dots in the arrangement. Some students represent their ways of counting using expressions, and others interpret the meaning of each expression. Through these activities, the students can find unexpected interpretations for their own expressions, and other ways of counting can emerge. We want to use these experiences to encourage students to value learning from each other in studying mathematics. In particular, for each expression presented by a student regarding Fig. 5.19, another student interprets the meaning of the expression. This activity provides an extension of the students' ways of thinking about the expressions.
3. *Goal (objective of the class)*: To understand how to solve problems using multiplication.
4. *Duration of the lesson*: Special 1-hour lesson.
5. *Development of the lesson*:

   (a) Lesson goal: To find ways of counting the total number of dots in a square with 4 dots on each side, represent each way of counting as an expression, and to interpret the meaning of the expressions.
   (b) Development:

The mathematics study group at the school has its own *Journal of Elementary School Mathematics Education* in Japan. Every mathematics teacher at the school edits at least one issue of the journal in a year in collaboration with his/her study group. At its annual meeting, more than 1000 teachers participate in studying new research issues for lesson study. All of them are professional leading teachers in Japan. The quality of school-based lesson study is maintained by such subject-based lesson study. In this context, if a lesson plan does not have a study theme, it is just preparation for a class and is not for lesson study to show others in Japan. Having only an objective without a study theme, the lesson plan looks like just a preparation of teaching. However, Japanese lesson study theme usually related to develop students who learn mathematics by and for themselves (Isoda and Nakamura, 2010, Isoda 2015a, Isoda 2015b). Thus, the lesson study theme usually focuses on teaching mathematical values, attitudes, ideas, and ways of thinking. Under the same theme, every teacher can develop different exemplars to share what to teach (see Chap. 1).

Content	Considerations
Look at the image below and think, "How many dots are there in the image?"  Confirm that there are 25 dots and think about how to represent the way of counting them, using an expression The students represent their ways of counting using expressions, and other students interpret the meaning of each expression: $1 + 3 + 5 + 7 + 5 + 3 + 1 = 25$ $(3 \times 3) + (4 \times 4) = 25$ $5 \times 5 = 25$ $6 \times 4 + 1 = 25$ $3 \times 8 + 1 = 25$ etc. Confirm that there are various ways of counting	Show the image to the students briefly for them to construct a mental image of the arrangement of dots in the image Each student should try to represent his or her own way of counting, using an expression The students look at the expressions made by other students and think about the interpretation of these expressions 3 - (1)  3 - (2)  3 - (2)  3 - (3)  3 - (3)  Confirm that there are various ways of counting by grouping and that for each expression there are various possible interpretations, as shown in the images

This activity is developed in the textbook "Item 2" from Shogaku Publishers (2008), including possible student responses, as proposed by Mr. Tsubota during lesson study (see Fig. 5.20)

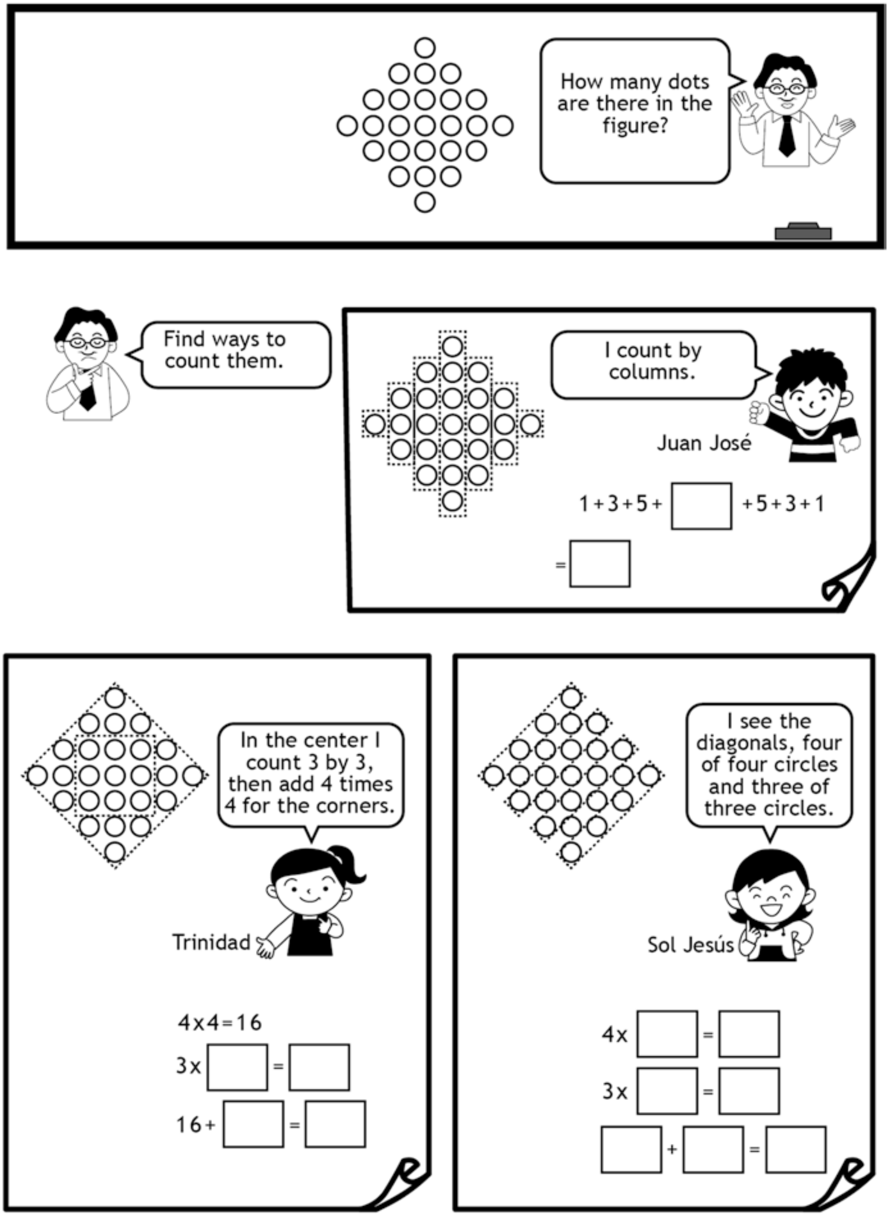


Fig. 5.20 Possible student responses, as proposed by Mr. Tsubota during lesson study

5.2 Evidence to See Any Number as a Counting Unit

The 1989 curriculum standards (Isoda, 2005) reinforced the variety of the types of grouping and that any number can be seen as a unit and every unit is not limited to the base ten (decimal) place value system. The standards are reflected in the second grade at the beginning of the study of multiplication. In Japan, teachers use textbooks, approved by the Ministry of Education, that follow the standards. For example, the number of unit squares in Fig. 5.21 is 27 by counting, by adding 10 + 9 + 8, and by multiplying 9 × 3.[14] The Japanese standards ask teachers to develop students to choose the appropriate unit for counting, depending on what they have learned. This approach has been implemented since 1992 (based on the 1989 curriculum standards).

Isoda and Odajima (1992) researched the development of the cardinal number among students from the viewpoint of grouping strategies. They studied how students' competency for grouping is reorganized, depending on the content of their learning, by comparing the grouping strategies offered by first-, second-, and third-grade students in a survey (see Fig. 5.22).

The results, expressed in percentages, are shown in Table 5.1.

As shown in Fig. 5.23 and Table 5.1, in the first grade, some students use counting or grouping to add. In the second grade, before studying the multiplication table, coins are used in forming groups to add. Some students can use grouping to multiply after their introduction to the meaning of multiplication in the classroom. In the third grade, after all the students have studied the multiplication table and the algorithm with the column method in vertical form, more than half of them use the idea of grouping to add or multiply. This task is not like the one shown in Fig. 5.22. At the time of this survey of student development, the teachers were not yet implementing the new curriculum. Even though it is not easy to find the unit to

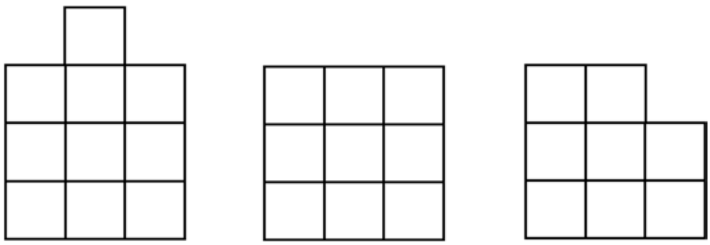


Fig. 5.21 How many unit squares are there?

[14] Figure 5.21 is used by Prof/Dr. Satoshi Kodo to explain the significance of mathematical ideas in how students change their view to see objects through the learning of mathematics (Personal communication in 1984, see Chap. 1, Mathematical Thinking).

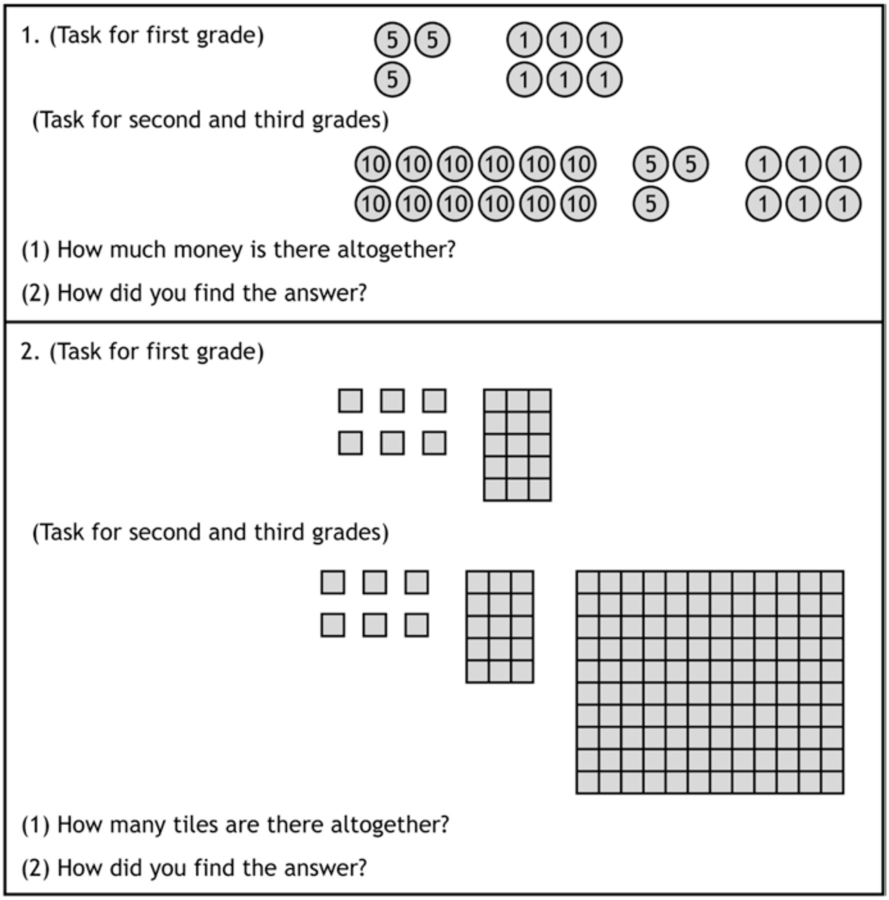


Fig. 5.22 Grouping strategies offered by first-, second-, and third-grade students in a survey

Table 5.1 Difference of the ways of counting by setting the various units for counting

	197 first-grade students (%)		214 second-grade students (%)		167 third-grade students (%)	
Method	Coins	Tiles	Coins	Tiles	Coins	Tiles
Count one by one	36	47	36	23	10	12
Count by 2s or 5s	3	8.6	4	4	1	1
Count by 10s	3	5.4	10	22	8	8
Simple addition	10	2	3	4	1	0
Group to add	48	37	38	31	48	43
Group to multiply	0	0	9	16	32	36

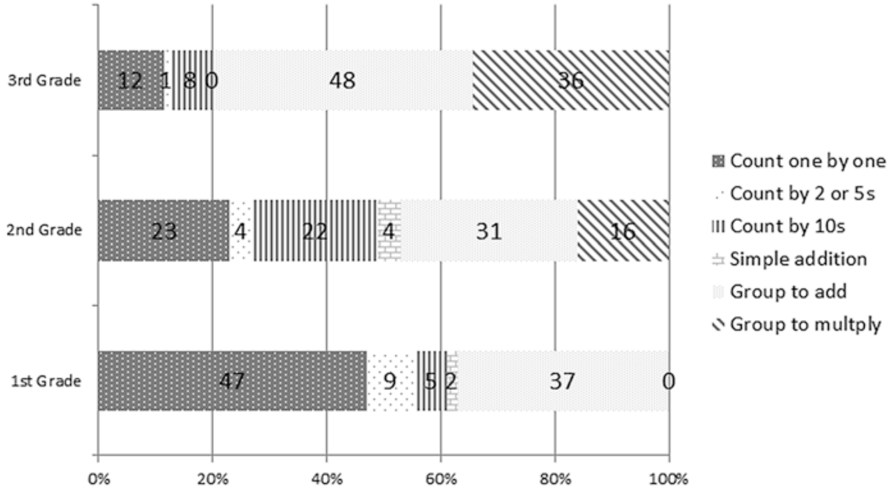


Fig. 5.23 Students' competency for grouping tiles. (Note: The first data row in Table 5.1 is shown at the bottom row of the graph)

multiply in the tasks and the students are not asked to think about it, they applied multiplication.[15]

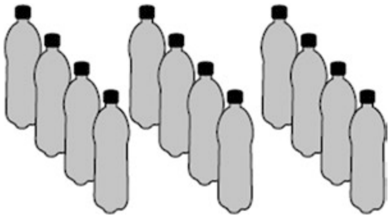
This result shows that learning multiplication develops the idea of grouping. Base ten units such as ones, tens, and hundreds are not the only units used for counting. In the Japanese approach, the students should learn how to choose the appropriate unit, set, or group for counting by using multiplication. In relation to vertical form, students should also fulfill the necessity to reorganize multiplication under the base ten place value system, which will be discussed in Chap. 7. The process of extension and integration is explained in Chap. 1.

[15] This survey included data collected at another school. That school did not follow the national curriculum but used the methods and textbooks proposed by the Toyama group (see Kobayashi, 1986), as explained at Chap. 1. The data showed that the students taught under the Toyama group (AMI) approach did not change their view as in Table 5.1. A limited number of schools preferred the AMI approach at that time. The data (which were taken from two classes) were insufficient to compare the difference with other schools that did follow the national curriculum. However, the data that showed no change represent the critical point for the discussion in the discussion of attribute on the next Sect. 5.3. In that section, the Japanese and Chilean approaches are compared. The Toyama approach is similar to the Chilean approach.

5.3 Comparison of the Japanese and Chilean Approaches

This section illustrates the feature of the Japanese introduction of meaning, which was explained in Sect. 5.1, in comparison with Chile. In Chile, multiplication is illustrated by repeated addition with seven sample activities (MINEDUC, 2017, pp. 151–154). If we prefer the activities closest to the Japanese approach, the following sample activities can be quoted:

In activity 1, the students are asked to transform sums in expressions with the word "times" (*veces* in Spanish), asking the following questions in these situations:

	(i) How many times do you repeat the 2 in the case of the number of ears of the 5 children in the image shown on the left?
	(ii) Answer the following questions that relate to the groups of 4 bottles in the image shown on the left: How many times is the row of 4 bottles repeated? How many times is 4 bottles repeated?

In these examples, it is clear that there are no discussions to set the unit of measurement by students because every pair of ears is fixed to the faces of the students, and counting the number of students corresponds to counting by 2s. Having two ears is an attribute of humans.[16] Thus, instead of counting each ear, we count the number of students. In (ii), the number of 4-bottle sets is asked. Then, the children have to see the set likely to be an attribute of humans. Here, (i) will be a metaphor for (ii) to see the set as an attribute. Thus, the metaphor of the attribute can be seen as a model for the binary operation to introduce multiplication, which is discussed in Chap. 3.

In activity 2, the students are asked:

(i) To draw a situation explaining what they understand about it and answering the question: "I have 5 cats and each one has 4 legs. How many legs are there in total?"

(ii) To complete the following story, drawing what they are told: "5 friends go to a store and each one buys 2 figurines . . ."

[16]This Chilean approach using attributes is the same idea as Toyama's approach (see Kobayashi, 1986, Sect. 5.2, Chaps. 1 and 3). It tries to express the meaning of multiplication by using a specific model for every row.

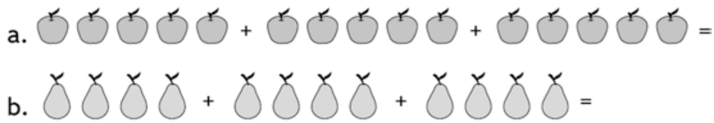


Fig. 5.24 Repeated addition or counting by each: MINEDUC, 2017, pp. 151–154

Four legs are an attribute of a cat. This can be generalized to a situation like one person for two figurines, using the example of the cat as the metaphor for the attribute. It is also enhanced to see the situation as the base for the binary operation.

The third example follows the reverse scheme postulated by the definition.

In activity 3, the students are asked to express the quantities in Fig. 5.24 as a repeated sum and then as multiplication in the form of “times” and then give the answer.

In a of Fig. 2.54, for example, the students are expected to calculate the quantity by using the expression “3 times 5 [for ‘3 · 5’ because a dot (‘·’) is used for ‘×’ in Chile] is 5 + 5 + 5 = 15.”

Here, multiplication can be seen as repeated addition. However, the answer can also be obtained by counting by each in the diagram and not necessarily by adding. In relation to activities 1 and 2, the task sequence implies that multiplication is introduced by the metaphor of the attribute of the object and reorganized as repeated addition.

From the discussion in Chap. 3, we can explain the reason why the Chilean program enhances the sequence from activity 1 to activity 3. If we represent the denomination of quantity clearly, “3 × 5 = 15” means “3 (dishes) × 5 (apples/dish) = 15 (apples).” However, students cannot directly understand the meaning of “apples/dish” as ratio. Thus, the Chilean program introduces the part of “apples/dish” using the metaphor of the attribute such as 5 apples for each dish. The attribute of the dish is 5 apples. For using the attribute as a metaphor for the binary operation, multiplication has to be introduced, such as the ears of humans and the legs of cats. It can be seen as an effort to make sense of multiplication as a binary operation and as repeated addition.

However, as long as we use the attribute of animals, we encounter the difficulty of asking students to overgeneralize the attribute of the original model because we do not discuss a person with six fingers or with two heads. If it is an attribute, students cannot change the unit of measurement. In addition, even though we introduce the unit of measurement by attributes, it cannot be connected well to repeated addition because it should be understandable if we write it as follows:

$$\underbrace{5\text{ (apples)} + 5\text{ (apples)} + 5\text{ (apples)}}_{3\text{ (dishes)}} = 15\text{ (apples)}$$

As explained in Chap. 3, it is not the same as “3 (dishes) × 5 (apples/dish) = 15 (apples).”

Through the comparison of the Chilean and Japanese approaches, we can recognize well why the Japanese approach enhances the setting of various units of measurement by students and asks them to count the number of units for setting the multiplication expression under the definition of multiplication by measurement (see Chap. 3). The most necessary activity for the introduction of multiplication is to see the situation by various units of measurement and find or set the number of units. As well as the usage of times (*bai* in Japanese) directed for proportionality represented by the proportional number lines (see Chap. 4), is a key idea of mathematical thinking to see the situation with mathematical ideas—in this case, ideas of a set (a group) and multiplication (see the idea of set and unit in Chap. 1, Table 1.1). For setting the unit of measurement, we can move the object (as in Figs. 5.3, 5.14, and 5.16, and in Fig. 4.3 in Chap. 4). In the case of Chile, the attribute of a given object is used to let students see the number as a unit.

5.4 Final Remarks

On the comperison of Chilean and Japanese Approaches, the Chilean approach analyzed to make sense of multiplication in the situation from the teachers’ side, and the Japanese approach analyzed to develop the sense-making activity of students who are able to set the measurement unit and to try to make clear the number of units by and for themselves, as well as making sense.[17] This chapter has illustrated this feature with two lesson study exemplars, a survey of student development from the first grade to the third grade before and after introduction of multiplication in Japan, and a comparison with Chile. Even we conclude Chilean Approach using attribute is an approach for making sense rather than sense making like Japanese approach, we should note the differences were originated from the behind school system and teaching culture. For example, Ministry of Education Chile distribute the different companies’ textbooks to the different grades as for the national textbooks. For example, first grade textbooks are published from the company A and second grades’ textbooks are published from company B. On this setting of Chile, it is difficult to teach based on what students already learned and preparing future learning. Indeed, if the textbooks are different depending on the grades, students’ sense making beyond grades is difficult because the ways for make sense are not the

[17] McCallum (2018) explained the sense-making stance as “the process perspective: mathematics as pattern seeking, mathematics as problem solving, big ideas have in common what I call the sense-making stance” (p. 2). He also mentioned, “Where the sense-making stance sees a process of people making sense of mathematics (or not), the making-sense stance sees mathematics making sense to people (or not). These are not mutually exclusive stances; rather they are dual stances jointly observing the same thing. The making-sense stance is related to the content perspective described by Schoenfeld, without the unappetizing ‘carving content into bite-sized pieces.’ It views content as something to be actively structured in such a way that it makes sense” (pp. 2–3). Both perspectives are necessary for curriculum development.

same amongst several textbook companies. If Chile try to shift to the sense making stance from making sense stance, it have to change the textbook free distribution system itself.[18] On this setting, Chilean make sense approach for multiplication can be seen as a best consideration on the current Chilean setting. In the countries such as England and USA, textbooks are not referenced as the minimum essentials but functioning as the one of the sources for the worksheets which teachers prefer every day. Such countries might be much more difficult to establish consistent sense making teaching sequence beyond the grades like Japan as we discussed at Chap. 4.

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[18] On 2019, the ministry of education chile decided to adapt Gakkotosho textbooks (Isoda and Murata, 2011) for them and Chilean using the Chilean adapted edition (Isoda et al., 2020; Isoda and Estrella, 2020).

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Chapter 6
Teaching the Multiplication Table and Its Properties for Learning How to Learn

Raimundo Olfos and Masami Isoda

Why do the Japanese traditionally introduce multiplication up to the multiplication table in the second grade? There are four possible reasons. The first reason is that it is possible to teach. The second reason is that Japanese teachers plan the teaching sequence to teach the multiplication table as an opportunity to teach learning how to learn. The third reason is that memorizing the table itself has been recognized as a cultural practice. The fourth reason is to develop the sense of wonder with appreciation of its reasonableness. The second and the fourth reasons are discussed in Chap. 1 of this book as "learning how to learn" and "developing students who learn mathematics by and for themselves in relation to mathematical values, attitudes, ways of thinking, and ideas." This chapter describes these four reasons in this order to illustrate the Japanese meaning of teaching content by explaining how the multiplication table and its properties are taught under the aims of mathematics education. In Chap. 1, the aims are described by the three pillars: human character formation for mathematical values and attitudes, mathematical thinking and ideas, and mathematical knowledge and skills.

6.1 Revisiting the Japanese Educational Principle

For explaining the Japanese content of teaching, we have to revisit Chap. 1 of this book first and provide some necessary information on the manner of teaching. The Japanese educational principle in mathematics (MEXT, 2008; Shimizu, 1984) is to develop students who learn mathematics by and for themselves based on what they

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have already learned. In accordance with this principle, learning how to learn in itself becomes the content of teaching. Indeed, Japanese students learn how to extend the multiplication table after they have been introduced to the meaning of multiplication in the same grade. The extension of the multiplication table is one of the best opportunities to develop students in accordance with this principle.

Learning the multiplication table is a facilitated activity which includes extension of the table and coordination of the processes of memorization and application. For students to be able to learn mathematics by and for themselves, Japanese teachers plan well-sequenced activities and think of several strategies for teaching. In the given teaching sequence (task sequence), the students are able to engage in activities in which they need to remember what they have learned and appreciate the advantage of those methods for development in the lesson. Major activities in class usually include solving a given unknown task, with a discussion of the unknown as a problematic, and communication of ideas to solve the problematic by challenging the unknown to be known.[1] At every necessary moment throughout the class, the teachers provide opportunities for students to compare what is learned and what is unknown, and to reflect on what they have learned before and during the class. In the classroom, the teachers hang posters or printouts on the walls in an organized way showing content related to what has already been learned as hints so that the weaker students can use them as needed. This way, the students not only learn knowledge and skills but also learn how to learn, including values, attitudes, ideas, and ways of thinking in mathematics. From this process, the students gain a rich opportunity for understanding and connecting various ideas.

6.2 A Survey of Appropriate Grades to Introduce the Multiplication Table

In Japan, after World War II, under the USA occupation through the General Headquarters (GHQ) of the Allied Powers, there was a discussion on whether to introduce multiplication and the multiplication table in the second or third grade. Traditionally, the Japanese used to introduce it in the second grade; however, the GHQ recommended the third grade or upper grades in relation to the experience in the USA, known as progressivism. In 1957, Tatsuya Matsubara surveyed the appropriate grade for memorizing the multiplication table in relation to mental age with the support of Yoshinobu Wada.[2]

[1] Japanese teachers call problem solving for a problematic originating from a given unknown task a "problem-solving approach" (Isoda, 2015; Isoda and Katagiri, 2012).

[2] Yoshinobu Wada, a professor at the Tokyo University of Education, was known as a curriculum specialist in the Ministry of Education, who tried to defend the order from the General Headquarters of the Allied Powers. He introduced mathematical activity as the reorganization of the organism by J. Dewey (1916) which is currently known as the radical constructivism by Glasersfeld (1995) in relation to Piaget (1970). The mathematical activity at that time was the new view of the mathematization principle described in the 1943 textbooks during World War II. It was revised as the mathematical thinking and attitude principle, and the extension and integration principle in Japanese national curriculum reform (Isoda, 2018, 2019).

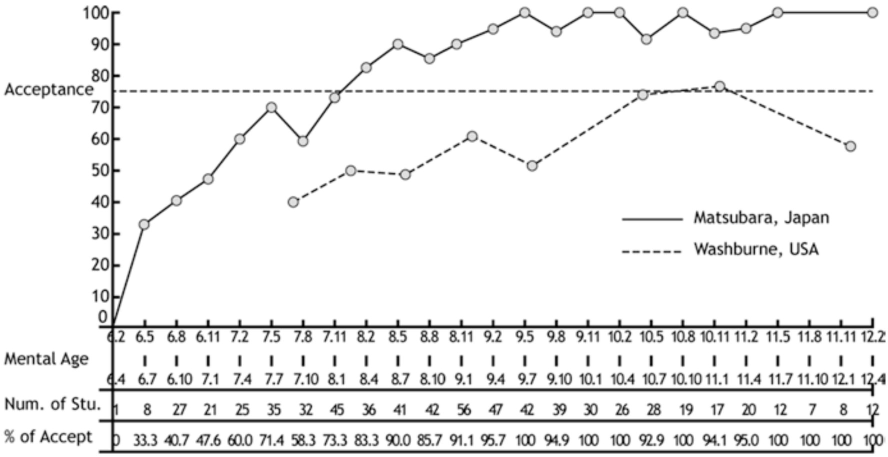

Fig. 6.1 The mental age for successful learning of the multiplication table according to the 75% acceptance (*Accept*) line among Japanese students and USA students (*Stu.*). *Num.* number

In his survey, he adopted the research by Carleton Washburne[3] (1931) in Japanese settings, such as the ways of teaching, and he compared the difference in students' success between Japan and the USA, as shown in Fig. 6.1. The Japanese setting meant the Japanese method of teaching under the cultural tradition of memorizing the multiplication table.[4] The teaching content and methods involved 36 hours of lessons which were developed under the supervision of Wada and the teachers from the Elementary School at the Tokyo University of Education.[5] The US setting studied by Washburne was the Winnetka schools in the USA which were influenced by progressivist education.

From the obtained results, shown in Fig. 6.1, Matsubara (1969) concluded that a mental age of 8.1 years is a possible age to learn multiplication, which implies that it might be suitable to teach the multiplication table from the later semester in the second grade. From the viewpoint of curriculum reform, the USA setting was influenced by progressivism. The results were related to differences in the curriculum and teaching culture. This implies that the lower achievements in the USA at an older age may have been relevant to the curriculum and education in that setting in that era.

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[3] Washburne was known for the Winnetka Plan for progressive education and was the president of the Progressive Education Association.

[4] Since the sixteenth century, the Japanese "3Rs" (reading, writing, and arithmetic) have included memorization of the multiplication table. In the East, it was normal to memorize the division table in the past. See also Chap. 7 in this book.

[5] It is a kind of lesson study under the collaboration of Wada Group and the Elementary School Mathematics Group of Tokyo university of Education. Tokyo University of Education was the predecessor of the University of Tsukuba, which originated lesson study in 1873. The elementary schools established a national-level lesson study group as a society in 1904. Wada also established his own lesson study group as a society and its still exist after Wada passed away.

6.3 The Multiplication Table in Japanese Textbooks for Learning How to Learn

This section illustrates how Japanese teachers teach the multiplication table and learning how to learn in order to develop students who learn mathematics by and for themselves. In the case of Japan, elementary school mathematics textbooks are part of the results of lesson study as well as a major reference for lesson study.[6] Here, these textbooks are preferred for illustration of the teaching.[7]

The four sets of textbooks analyzed were Gakko Tosho (Hitotsumatsu, 2005; Isoda and Murata, 2011),[8] Tokyo Shoseki (Hironaka and Sugiyama, 2006), and PROMETAM (2005)[9] (Secretaría de Educación, 2007).[10] The objective of the analysis was to know the aims of constructing, extending, memorizing, and applying the multiplication table of the numbers from 1 to 9.

For teaching the meaning of multiplication and the multiplication table, around 33–35 hours of lessons with exercise and tests are allotted, which is distributed as described in the sample shown in Table 6.1.

The activities employed in the various books for teaching the multiplication table are similar. For example, Gakko Tosho textbooks present seven activities for introducing the multiplication table of 2, and these same activities are used with minimal variation in addressing the tables of 5, 3, and 4. The activities proposed in the Gakko Tosho books for presenting the multiplication table of 2 are shown in Fig. 6.2.

Table 6.1 Sample for teaching multiplication in the second grade

Content of subunits	Number of hours of lessons
1. Meaning of multiplication	4 hours of lessons
2.1 Multiplication tables of 2, 5, 3, and 4	9 hours (including time for memorizing) + 3 hours of exercise, application
2.2 Multiplication tables of 6 to 9	9 hours + 1 hour of practice
2.3. Multiplication by 1	1 hour
3. Properties of the multiplication table	3 hours + 2 hours of practice and challenges

[6] In Japan, teachers must preferentially use textbooks authorized by the national government.

[7] This alternative might be not understandable for countries that usually use worksheets for teaching, such as the USA and the UK. The roles of textbooks differ depending on the country. Worksheet culture has originated from textbooks that are applicable for different curriculum and do not only follow the official curriculum. In the East, textbooks traditionally represent the official curriculum well.

[8] The chapters on multiplication for the second grade in the 2005 and 2011 editions are similar.

[9] Here, we chose various textbooks to shows the similarity and consistency on the different curriculum standards in Japan and the country in Japan Overseers Cooperation. See the English translation of three generations of Japanese Curriculum Standards: Isoda, 2005, Isoda and Chino, 2006 and Isoda, 2010.

[10] PROMETAM [Proyecto Mejoramiento en la Enseñanza Técnica en el Área de Matemática] was a textbook developmental project conducted in Honduras by JICA [the Japan International Cooperation Agency].

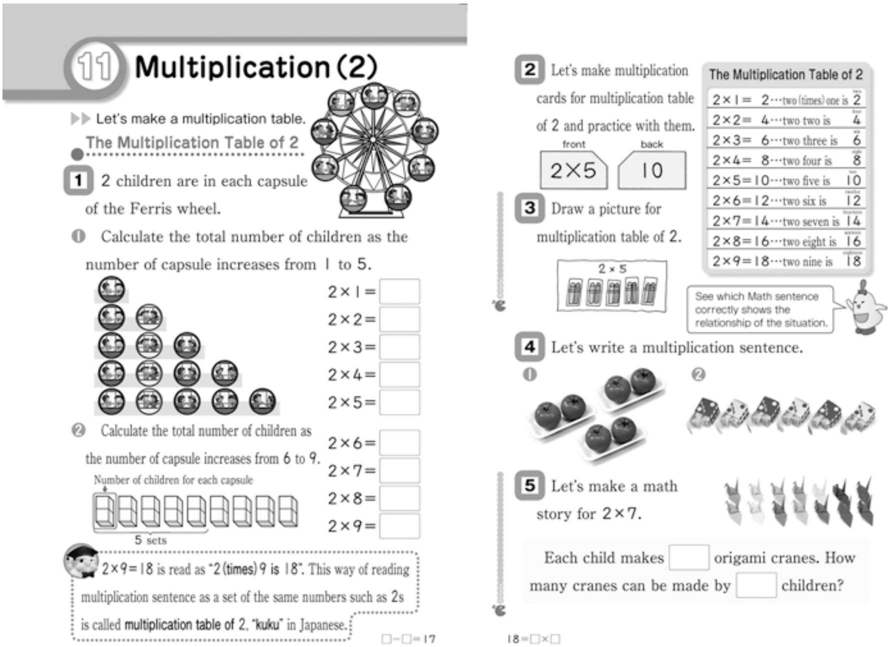


Fig. 6.2 Isoda and Murata (2011), Grade 2, Vol. 2, pp. 17–18, Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 13–14

As exemplified in Fig. 6.2, both editions are almost the same and include the following activities:

1. A situation with discrete quantities which can be extended
2. An activity for extension with a diagram and tape (consecutive antiquity) at the back, so that students can extend it with a block model and see the pattern, and can continue by reading the expression and its interpretation (the expression of multiplication and the multiplication table)
3. The manner of reading the row of 2 for comparing expressions and memorizing
4. Using cards with the product written on the back for memorizing
5. Representing situations as multiplication
6. Determining multiplication from the picture
7. Developing a situational problem from an expression like 2 × 7

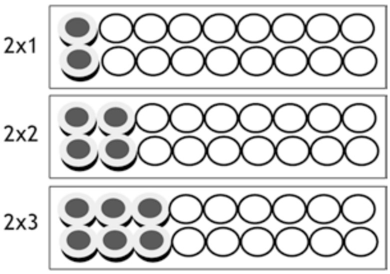
The activities proposed in the Gakko Tosho textbook for presenting the multiplication tables of 2, 5, 3, and 4 are shown in Table 6.2. Similar teaching of content and sequence are repeated in every extension of each row for enabling students to learn how to extend the multiplication table.

Table 6.3 shows that Gakko Tosho, Tokyo Shoseki, and PROMETAM have chosen the same manner of presenting the multiplication table. The similarity between the learning activities and problem situations in the books from the different publishers implies consistency of the Japanese approach. The reason is explained in the next section.

Table 6.2 Gakko Tosyo teaching sequence

Activity	Multiplication table of 2	Multiplication table of 5	Multiplication table of 3	Multiplication table of 4
1. A situation	Yes	Yes	Yes	Yes
2. Finding products and extending	Yes	Yes, variation	Yes, variation	Yes, variation
3. Continuing the row for memorizing	Yes	Yes	No	No
4. Using cards	Yes	Yes	Yes, variation	Yes, variation
5. Representing with drawings	Yes	No	Yes, variation	Yes, variation
6. Determining the expression	Yes	Yes	Yes, variation	Yes, variation
7. Constructing problems	Yes	Yes	Yes	Yes
8. Others	No	Yes	Yes	Yes, variation

Table 6.3 Comparison of Gakko Tosyo, Tokyo Syoseki, PROMETAM

Activities for learning the multiplication table of 2	Publisher		
	Gakko Tosho	Tokyo Shoseki	PROMETAM
1. A situation	Yes	Yes	Yes
2. Finding products and extending	Yes	Yes	Yes
3. Continuing the row for memorizing	Yes	Yes	Yes
4. Using cards	Yes	Yes	Yes
5. Representing with drawings	Yes	No	No
6. Determining the expression	Yes	Yes	Yes
7. Constructing problems	Yes	Yes	No
Practicing with rows of an array 	No (instead of the array, it uses blocks with covering and uncovering sheet to show it as variable like Fig. 6.9)	Yes	Yes

6.3.1 Developing Multiplication Tables for the Rows of 2, 5, 3, and 4

There is consistency in developing the rows of the multiplication table in Japanese textbooks, which is the repetition of the format shown in Fig. 6.2 from the row of 2 to the other rows. The repetition provides the students with the opportunity for learning how to construct and extend the rows: Students are able to imagine the ways of learning at the next rows.

The teaching sequence for the rows of 2 to 5 is 2, 5, 3, and 4, instead of 2, 3, 4 and 5 because the products in the rows of 2 and 5 are known through counting by twos and fives. Students feel the necessity for memorization of the products in the rows of 3 and 4, likely through counting by 2s and 5s.

Uniquely, the Gakko Tosho (Hitotsumatsu, 2005) textbook for the second grade has the following activity between the rows of 2–5 and the rows of 6–9 (see Table 3.1 in Chap. 3). The idea embedded in Fig. 6.3 is the distribution which makes it possible for the students to extend the rows of 2–5 to the rows of 6–9. For example, addition of the row of 2 and the row of 4 produces the row of 6. Students can predict further rows for extension of the table by themselves. The way of extending multiplication based on their prediction encourages them to develop further rows by and for themselves.

In the case of the PROMETAM project for the Central American country of Honduras, the teachers' guide recommends that students need to practice for about 5 minutes each day without fail. For example, they can recite the table being studied when they arrive at school, before starting class, before leaving for recess, before leaving school, etc. The students should memorize the tables appropriately to solidify the base for understanding multidigit multiplication, which will be discussed in the next grade.

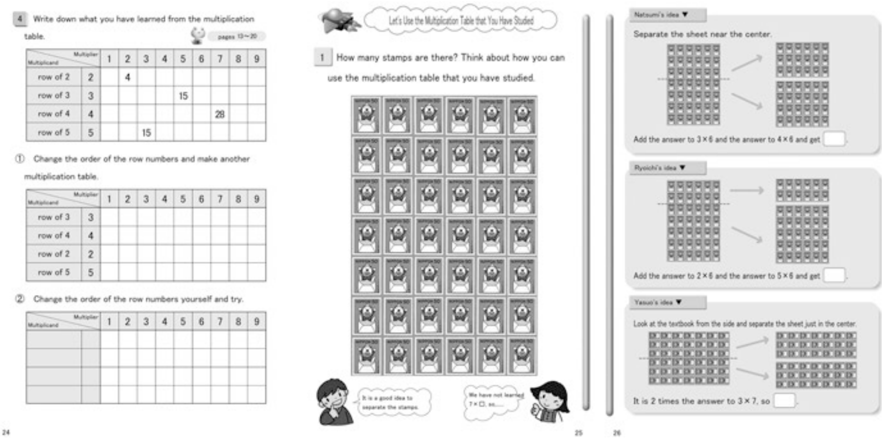


Fig. 6.3 Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 22–24

6.3.2 Transferring the Responsibility for Construction and Memorization of the Multiplication Table

The responsibility for the construction and memorization of the table is transferred from teachers to students in the following teaching sequence and materials (see Brousseau, 1997).

The study of the multiplication tables of 2 and 5 guided by the teacher includes the way to learn. Based on counting by 2s and 5s, the students can easily know the product of the rows of 2 and 5. Then, the study process for the tables of 3 and 4 should be planned so that the students will manage concrete situations and build these tables by applying what they have learned. The students can find each product by adding the multiplicand to the previous product in the table, so they do not need to add from the beginning to find the next product in the table. By repetition of the same ways of learning (Fig. 6.2), the students are able to imagine what they need to do next. As shown in Fig. 6.3, the students have a hypothesis for the extension of the table, which they want to check by themselves. By repetition in Table 6.2 and use of the hypothesis, they are able to generate and confirm new rows in a learned manner.

As shown in Fig. 6.2 and in Tables 6.1 and 6.2, the teacher and the students can use arrays or blocks, multiplication cards, and manner of reading pattern for every row as a means for constructing, extending, practicing, and memorizing the multiplication table. To make the students responsible for constructing, extending, and memorizing, the teaching sequence and materials are prepared in the textbooks and by the teachers.

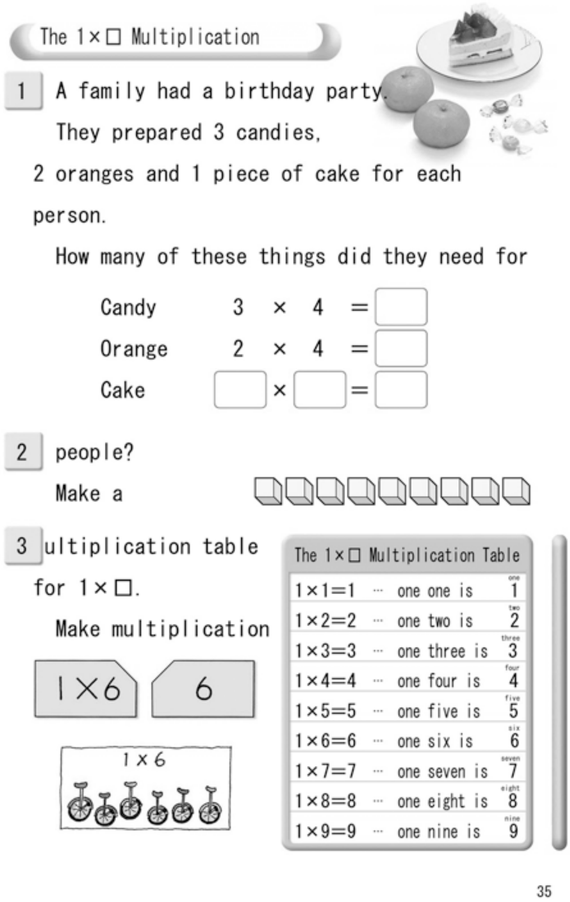
6.3.3 Extension of the Multiplication Tables of 6–9 and 1

Based on learning how to learn by repetition of the same learning sequence for the multiplication tables of 2 to 5 and the expectation of extension, students can extend the multiplication tables of 6 to 9 in every two class hours by themselves. In every class, the teachers ask the students to develop every row in the same manner.

The row of 1 is not easy to learn in the same manner because students do not feel any necessity for learning it. In the Gakko Tosho textbook, it is introduced as shown in Fig. 6.4. The necessity of the row of 1 exists for permanence of form (see Chaps. 3 and 4, Peacock (1880)). As long as the students use their previously learned knowledge, the numbers of candies and oranges should be expressed by multiplication. In this context, the piece of cake on the dish is expressed as 1×1. Realizing its necessity, the students can develop the row of 1 in the same manner for permanence of form.

Japanese teachers usually allot about two class hours for every row because it takes time for memorization as well as construction of the multiplication table by the students.

Fig. 6.4 Hitotsumatsu (2005), Grade 2, Vol. 2, p. 35



6.3.4 *Properties of the Multiplication Table for Discovering the World of Multiplication with a Sense of Wonder*

After the construction of every row and memorization, the Japanese textbook treats the multiplication table as a world of multiplication and as an operation without situations (Figs. 6.5 and 6.6). It is remarkable difference when we compared it with other countries such as Chile, Mexico and Singapore which use several grades to extend multiplication table up to row of 9 (see Table 2.4 in Chap. 2). Even if the students have not memorized the multiplication table well, they can fill in the products using the property of every row by adding the same number to the next column. After completing the table, the students can find several patterns hidden in the multiplication table. Commutativity of multiplication is discovered at this moment. As discussed in Chap. 3, there is no contradiction in the Japanese definition and the

Fig. 6.5 Patterns in the multiplication table of 3, as demonstrated by Y. Yamamoto (Rasmussen and Isoda, 2019)

What can you find?

$3 \times 1 = 3$	$3 \times 11 = 33$
$3 \times 2 = 6$	$3 \times 12 = 36$
$3 \times 3 = 9$	$3 \times 13 = 39$
$3 \times 4 = 12$	$3 \times 14 = 42$
$3 \times 5 = 15$	$3 \times 15 = 45$
$3 \times 6 = 18$	$3 \times 16 = 48$
$3 \times 7 = 21$	$3 \times 17 = 51$
$3 \times 8 = 24$	$3 \times 18 = 54$
$3 \times 9 = 27$	$3 \times 19 = 57$

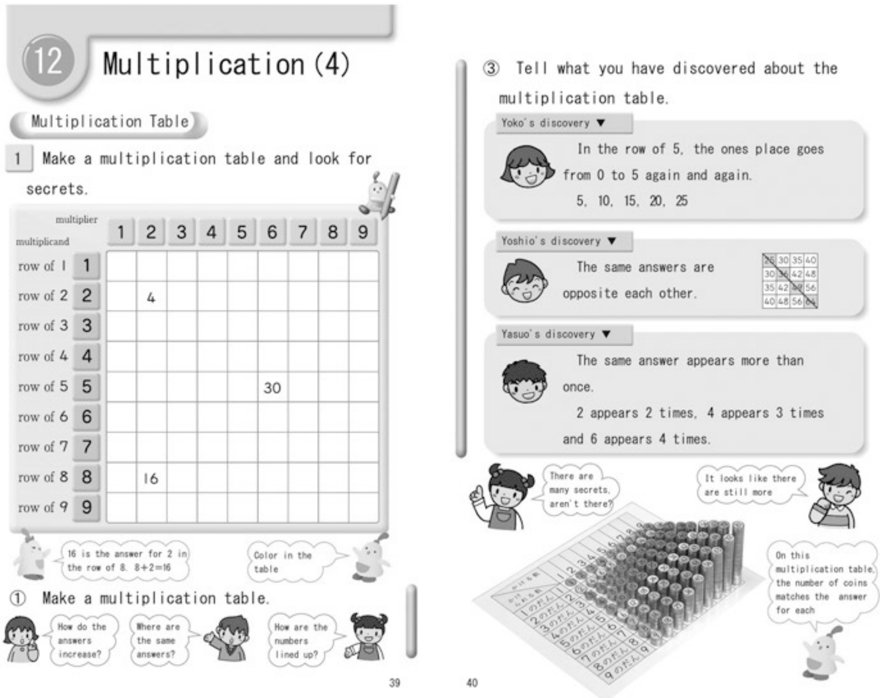
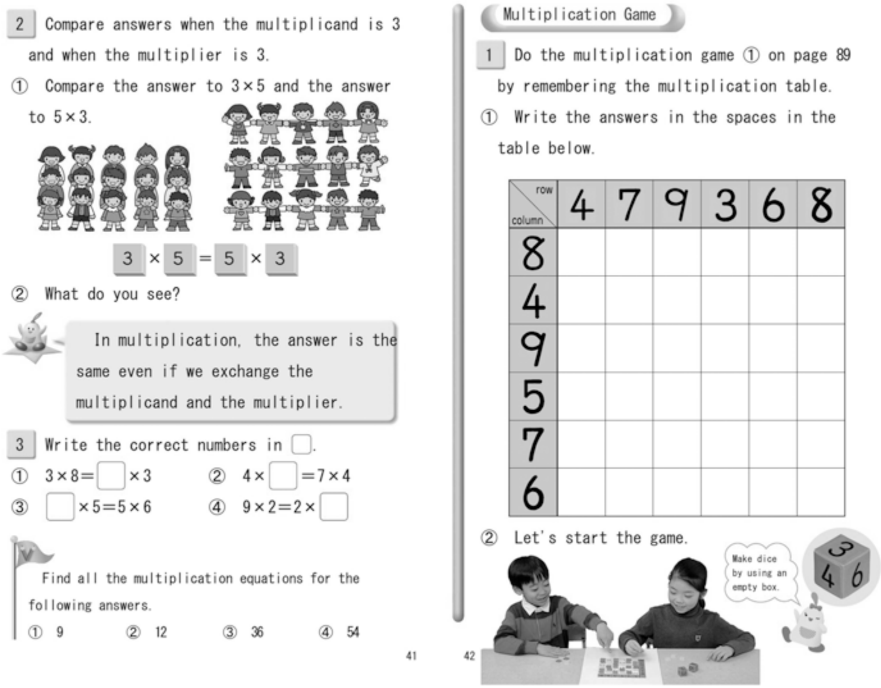


Fig. 6.6 Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 39–40

multiplication table; thus, it is not necessary to discuss commutativity from the introduction of multiplication (see Chap. 5).

Students find a number of different properties in the multiplication table and feel a sense of wonder.[11] Such mathematical structures of multiplication table enable

[11] It is related with mathematical value in relation to mathematical thinking (see Chap. 1, Table 1.1 and Mangao, Ahmad, and Isoda (2017).


2 Compare answers when the multiplicand is 3 and when the multiplier is 3.

① Compare the answer to 3×5 and the answer to 5×3.

3 × 5 = 5 × 3

② What do you see?

In multiplication, the answer is the same even if we exchange the multiplicand and the multiplier.

3 Write the correct numbers in □.

① 3×8=□×3 ② 4×□=7×4

③ □×5=5×6 ④ 9×2=2×□

Find all the multiplication equations for the following answers.

① 9 ② 12 ③ 36 ④ 54

41

Multiplication Game

1 Do the multiplication game ① on page 89 by remembering the multiplication table.

① Write the answers in the spaces in the table below.

column \ row	4	7	9	3	6	8
8						
4						
9						
5						
7						
6						

② Let's start the game.

Make dice by using an empty box.

42

Fig. 6.7 Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 41–42 (Row and column should be alternate)

students to recognize the existence of the world of multiplication with the harmonious feeling of beautifulness.[12] Some of them are revealed in later grades, as discussed below (Fig. 6.7).

Example 1 In the third grade (grade 2 in the 2017 curriculum), multiplication will be extended beyond 9×9. In Fig. 6.5, various patterns in the numbers can be found: products in the tens place: 0 0 0 (blanks of tens), 1 1 1, 2 2 2, 3 3 3, 4 4 4, and 5 5 5; $3 \times 18 = 3 \times (10 + 8) = 3 \times 10 + 3 \times 8 = 30 + 3 \times 8$); products in the units place (3, 6, 9, 2, 5, 8, 1, 4, 7 (if we change the order, we see 1 in the 7th line, 2 in the 4th line, 3 in the 1st line, 4 in the 8th line, 5 in the 5th line, 6 in the 2nd line, 7 in the 9th line, 8 in the 6th line, and 9 in the 3rd line). How do we explain these patterns? Can we find similar properties in other rows? (This example was provided by Yoshikazu Yamamoto from the Elementary School at the University of Tsukuba (Rasmussen and Isoda, 2019).)

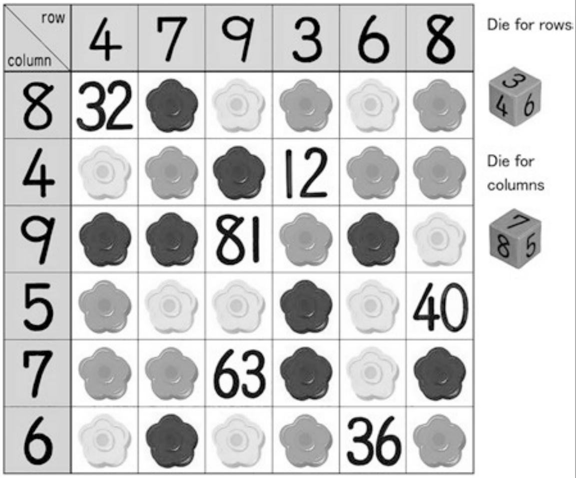
[12] In Japan, students have the opportunity to learn the world of addition using an addition table and the world of subtraction using a subtraction table by finding their properties. See Isoda and Katagiri (2012), Dizon M., D., Ahumad, N., J., Isoda, M. (2017), and the lesson study videos by Takao Seiyama at https://www.youtube.com/watch?v=7TY_SHTgmFQ, https://www.youtube.com/watch?v=TR34ZdBXmz8, https://www.youtube.com/watch?v=NNWtmIQ7YNs&t=426s, and https://www.youtube.com/watch?v=njNK6xoAkwQ.

Example 2 In the upper grades, after students have learned the concept of averages, some teachers ask the students to find the total products in the multiplication table up to 9 × 9. There are various ways to find the total value of the multiplication table. Two beautiful and wonderful ways are 45 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) and 5 × 5 × 81.[13] The explanation of this property requires the ability to see the decomposition of a number with factors (multiplication) and addition.

Rasmussen and Isoda (2019) have analyzed example 1 using anthropological theory and noted that the Japanese extension of the multiplication table is fruitful teaching material to develop mathematical thinking.

After they have studied the multiplication table, the students are engaged in a game to know the significance of memorizing the table. Table 6.4 is a sample lesson plan given in the Annex of the Gakko Tosho textbook. And then, on the 2nd grade

Table 6.4 A Lesson for enjoying to use memorized table: Row and column should be alternate

<table>
<tr><td colspan="2">Objective: That students have fun using the multiplication tables of 6 to 9 and learn
Point of assessments: Do the students enjoy playing by using the multiplication tables from memory? In the game, can students predict the values of the dice from the remaining numbers on the game board?</td></tr>
<tr><td>Teacher: We are going to play with a game board and two dice. We will use 30 chips to cover some of the spaces on the board
Today we are going to play first with dice and then with cards


Teacher: To play, roll the dice (or use a substitute). Multiply the two numbers and say the answer. If the answer is correct, you win the chip from that space. If there is no chip in that space, you have to put one of yours in that space. Decide how many times you will roll the dice. Play by taking turns. The student who gets the most chips wins</td><td>Specific materials are needed. The game board can be made by the students in their notebooks, or a photocopy can be used. The dice can be made from pieces of wood. A spinner or a deck of multiplication cards made by the students can be used as substitutes.
To make the game board, write the numbers in the first row and the first column. In the inner spaces of the game board, write the products of the numbers in the first row and the first column.
Then, place 30 chips on the game board, leaving some spaces uncovered. Students realize that the patterns of multiplication table do not appear if we change the order of numbers on the multiplication. They cannot fill in without memorizing the multiplication table. To answer this task they recognize the significance of memorizing</td></tr>
</table>

[13] 5 × 5 is at the center of the multiplication table.

Gakko Tosyo textbook, the 2005 edition extend the multiplication to the case of ten times and the 2011 edition additionally extend the multiple beyond the multiplication table 9 × 9 to multiplied simple two digit numbers by ones.

6.4 Memorizing the Multiplication Table as a Cultural Practice

Memorization of the multiplication table is a cultural practice that favors learning the multiplication table. In some countries, memorizing has a negative meaning because it seems to be forced by teachers without appropriate understanding and express it as a part of number sense instead of explaining it as memorizing. However, it does not have such a negative meaning in the East. The Japanese have been engaging in this cultural practice since the sixteenth century for using the abacus. In the sixteenth century, even though the knowledge of the division table for the abacus was necessary to be an accountant. *Jinkoki*, by Yosoda (1627), as shown in Fig. 6.8, was the most popular and standard textbook until the middle of the nineteenth century which mentioned up to extraction of the square root and Pythagoras theorem. It became popular for everyone to memorize the multiplication table like songs. In this book, the multiplication table was read as *ni ni no shi* (“2 2, 4”), *ni san no roku* (“2 3, 6”), etc. In English, this means “two multiplied by two equals four” (in short, “two two is four”) and “two multiplied by three equals six” (in short, “two three is six”). There were no algebraic expressions yet in that era.

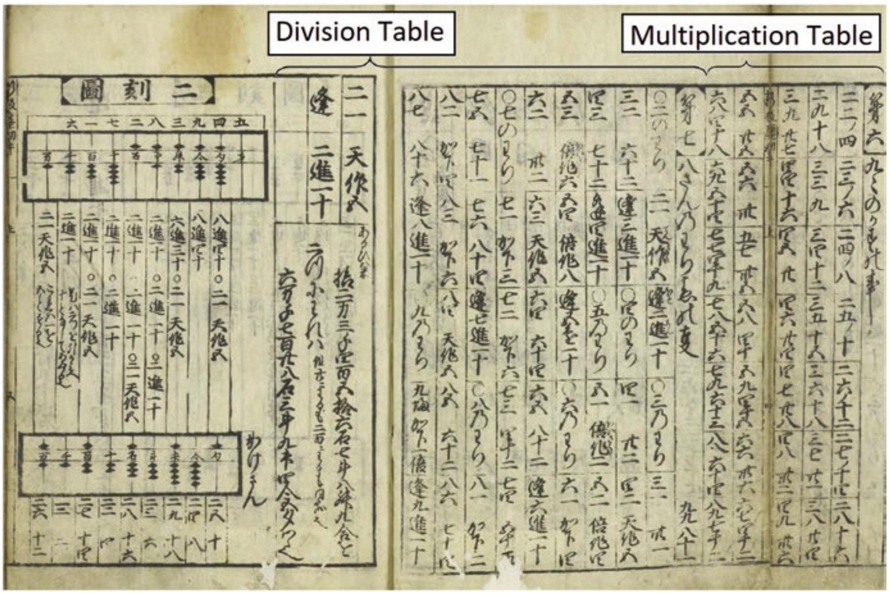


Fig. 6.8 Yoshida, M. (1627) *Jinkoki*, pp. 3–4

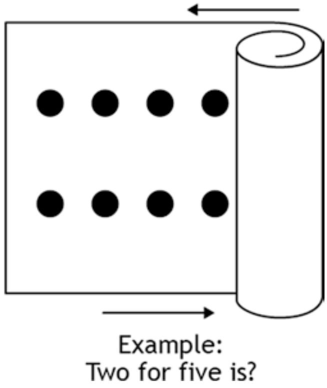


Fig. 6.9 A paper role model to extend the row of 2 to see the multiple as variable

At present, the recitation begins *ni ichi ga ni* ("2 1, 2"), *ni nin ga shi* ("2 2, 4"), *ni san ga roku* ("2 3, 6"), etc. For the row of 2, the students can recite it like a song in 10 seconds. The majority of 8- or 7-year-old students can memorize it, as already mentioned. As for second-grade students, it is a milestone for their learning in their culture. Historically, there was a tradition to memorize not only the multiplication table but also the division table, memorizing multiplication table was basics and the people who mercerized the division table recognized experts for using the abacus.[14] In the case of the division table, the practice of memorization was lost because we do not need it if we know multiplication and we do not use the abacus anymore for calculation.

In Chaps. 4 and 5 of this book, we mentioned that the multiplication table is introduced with the rows of 2 and 5 because the products of both can be found through counting by 2s and 5s. Additionally, teachers use some sequences for memorization practice. The following is an example from a Japanese class:

1. After constructing the row of 2 with meaning, ask the students to say and repeat it from "2 × 1" to "2 × 9" on the board.
2. Cover the product of "2 × 1" with a piece of paper and ask them to say what it is (then lift the piece of paper to verify the answer).
3. Ask the students: Two multiplied by one? Two one is two. Ask them to visualize and repeat the sequence, counting by 2s up to 10 to promote memorization.
4. As in Fig. 6.9, covering the products of "2 × 1" and "2 × 2", get the students to recite the multiplication table from "2 × 1" to "2 × 9" with counting by 2s and adding 2 every time. Repeat the activity, covering up more products, until they are all covered.
5. Ask the students to stand up and recite the multiplication table quietly and to sit down once they are finished. (The teacher observes who among the students takes a longer time, who is faster, and who needs additional practice.)

When the lesson ends, sometimes the printout of the multiplication table with the products covered can be left on the classroom wall. The students can practice freely

[14] *Jinkoki* did not address addition and subtraction because it is a kind of visualized counting if we use an abacus.

and with satisfaction at confirming their answers by uncovering the products. This practice is competitive but enjoyable for second-grade students.

In Eastern culture, teachers have the responsibility to make students memorize the multiplication table. Thus, teachers place a lot of opportunity for providing activities to support the students. An array sheet like that shown in Fig. 6.9 is used in building the multiplication table and also in practicing it. The amount of the vertical array diagram represents the multiplicand or the quantity in each group. The situation and the product can be presented by moving the paper that covers the groups horizontally.

Practicing the multiplication table includes four activities: (1) correctly recite the table observing the expression or the collection of arrays; (2) reciting from 2×1 to 2×9; (3) reciting the table from the bottom up and from the top down; and (4) reciting the table in random order.

Teachers assess students' degrees of understanding by observing whether they can relate the mathematical expression to the meaning of other expressions. The group of groups represented by collections of balls also suggests plates with fruit, columns with cubes, etc. The student gains understanding by relating each expression to the expression in the table; for example, $2 \times 4 + 2$ is 2×5. (Mr. Tsubota's class in Chap. 5 of this book is also an exemplar.)

6.4.1 Using the Cards

As shown in Figs. 6.2 and 6.4, each card has on its front the expression (binary operation) of multiplication and on its back the product. The Gakko Tosho textbooks include them in the Annex. Otherwise, the teachers or students prepare them in an appropriate size. They are used not only to practice memorization but also to find patterns in the multiplication table. The fundamental ways of memorizing the multiplication table using multiplication cards are as follows:

Individual use: (A) Place the cards in random on the table. Say the product while looking at the expression on the front of the card. (The students can place a mark on the cards they have incorrectly answered and practice more with them.) (B) Place the cards in random. Say the expression while looking at the product on the back of the card. (C) Carry out the practice of (A) or (B) with various multiplication tables.

Use in pairs: (A) One student shows the front of a card to another student, who gives the product while looking at the expression on the front of the card. Repeat this activity, taking turns. (The roles can also be changed when one student answers incorrectly, or each student can continue until he or she has correctly answered five times). (B) Each student prepares cards for one of the multiplication tables in random order. (It is best that they use cards for only one or two rows.) Each student places a card face up on the table at the same time, reads the expression, and gives the product while looking at the card. The student with a greater product wins. (C) Place the cards on the table, face up. A student chooses one, reads the expression, and gives the product. To check the answer given, look at the product on the back of the card. If the student has answered correctly, he or she can keep this card and continue with another card. If he or she has answered incorrectly, he or she lose his turn and does not keep the card (they can also take turns). The student who collects the greatest number of cards wins.

Use in pairs or in a group: (A) Place the cards face down on the table. A student quizzes his or her classmates by saying an expression from the multiplication tables in use. The others look for the product of this expression and pick up the cards that have this product. The student who gets the greatest number of cards wins. (B) Place the cards face up on the table. A student quizzes his or her classmates by saying a product from the multiplication tables in use. The others look for the expression of this product and pick up the cards that have this expression. The student who gets the greatest number of cards wins.

The teachers should help the students to invent other ways and to use the cards considering the students' real situation (see Fig. 4.2, Chap. 2). For this kind of activity, Japanese teachers usually use the first 3–5 minutes of each class to practice all together. Enjoyable daily cultural practice is the key to memorization.

When we say real situation, some of teachers and math-educators usually imagine the dichotomy to distinguish mathematics and real world. However, as explained Chap. 1, Japanese Approach usually consider on the curriculum sequence under the extension and integration principle (see Fig. 1.1). It is the reorganization process of mathematization. On this context, Japanese Approach enhance sense making (see Chap. 5), and it means change the intuition (see Fig. 5.22) and reality itself. What is the reality for students in these activities on memorizing and using multiplication table? To the terminology of horizontal and vertical mathematization by Treffers, A. (1987), Freduental, H. (1994) expressed uncomfortableness from his perspective of mathematization (1973) and redefined mathematization with levels by the terminology of living and life. He also mentioned mathematical object as entity (1983) to explain existence. On these context, reality, here, means the reality for second grade students on their life. For second grade students, reality is also existed on their enjoying games to think about and explore the rule and the behind structure for winning the game, as well as their narrow economical experience. With comparison of second grade students' economical-arithmetical life, these kinds of games provide the real situation for their world of multiplication within classroom. On this reality, these activities to memorize multiplication table is a kind of cultural practice with enthusiasm in Japanese classroom. The tools for these cultural practices has been developed by teachers. Followings are further examples.

6.4.2 Using Area-Array Cards

Mr. Hiroshi Tanaka (2007) designed new illustrated multiplication cards which include area-array images (see Fig. 6.10).

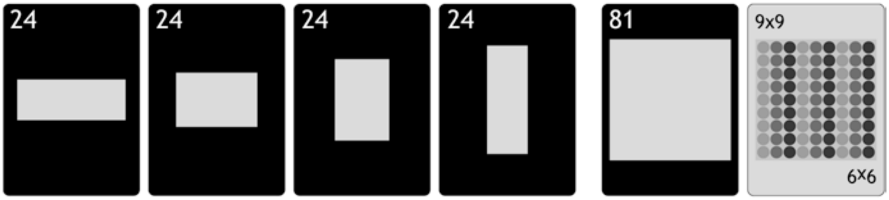


Fig. 6.10 Multiplication cards by Hiroshi Tanaka

Fig. 6.11 A notebook: The left is student activity and the right is progress of every row and three step assessment with stamps and signs

6.4.3 Using a Notebook and Journal Writing at Home

In Japan, to develop children's custom of self-learning at home, teachers usually use a notebook for homework and have them exchange journals/diaries.

These activities are not only for memorization but also for making it enjoyable for students, as shown in Fig. 6.11.

6.5 The Sense of Wonder in the Multiplication Table

During the middle of the second semester in three semesters per year, the second-grade students in all schools in Japan can be seen reciting the multiplication table in front of their teachers. What kind of actual practice does the teacher provide when the students are learning the multiplication table?

The following lesson plan was developed by Mr. Kozo Tsubota (2007), a teacher at the Elementary School at the University of Tsukuba. It uses the voice and ideas of a real teacher—the one who is designing and leading the lesson study community. The theme of his lesson study on the multiplication table is "Teaching the properties of the multiplication table to encourage students to discover patterns in the multiplication table with a sense of wonder and to appreciate the patterns in the table." The task is related with judicious using of calculator if we ask it at second grade students, and if not it become upper grade task.

6.5.1 Focusing on Beautiful Patterns with a Sense of Wonder and Appreciation

When the multiplication table is being taught, it is usually with the following sequence of steps:

1. The meaning of multiplication is built through known situations: ways of counting and iterated sums.
2. The multiplication table is developed up to 9. It is extended up to 9 × 9, through explorations.
3. Students are asked to recite the multiplication table and apply it.
4. The multiplication table as a whole is used with the goal of identifying patterns of addition, subtraction, and multiplication.

In these activities, many teachers usually focus on step 3. However, students should not simply memorize the multiplication table as if it were a song. In step 4, students should be given activities so they can discover the beautiful patterns in the numbers—in several rows of results—that make up the multiplication table. For example, the sum of the digits in the units place and in the tens place for any product of 9 is equal to 9; thus, $9 \times 7 = 63$ and $6 + 3 = 9$. Moreover, if we take any product from the first half of the row of 9 and add it to the corresponding product from the opposite side of the second half of the row, the result will be 90; for example, $9 \times 1 = 9$ and $9 \times 9 = 81$, and $9 + 81 = 90$. Similarly, $9 \times 2 = 18$ and $9 \times 8 = 72$, and $18 + 72 = 90$.

6.5.2 Preparing a Problematic: "Why"

Students develop a sense of wonder based on the awareness of problematics in relation to given tasks (see Chap. 1). The lessons should be designed to allow the students to follow up on these kinds of questions and investigate the "why."

Materials have been developed so that students can see two multiplicative expressions in class and be amazed by the fact that the results are the same. They ask why, carefully observing the expressions, transforming them and hypothesizing a response, and find relationships among the numbers. The expressions given to the students are:

$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$
$8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$

The students are asked which expression will give a larger result.

The answer is not easy for them to find, even when they make the calculations on paper. The students are allowed to use a calculator to find the answer. At this point, they can use the repetition function for arithmetic operations. The function consists of pressing 4 × = = . . . and 8 × = = ... When the calculator displays the results, it is confirmed that they are exactly the same. The result of both expressions is 16,777,216.

At this moment, the question "Why are the results the same?" appears in the students' minds. The students spend the rest of the lesson trying to answer the question and discussing the problems among themselves.

The teacher should allow interaction among the students and guide the discussion toward mathematical thinking. For example, the teacher should try to get the students to reach an understanding of the numbers 4 and 8. The students should realize that $4 \times 4 \times 4 = 64$ and $8 \times 8 = 64$ are equal, or that the numbers can be decomposed into $4 = 2 \times 2$ and $8 = 2 \times 2 \times 2$. The structure of this problem uses the power that $4^{12} = 8^8$; in other words, $4^{12} = (2^2)^{12}$, and $8^8 = (2^3)^8$.

6.5.3 How to Begin the Class?

"Now I will write two mathematical expressions on the board. As soon as I finish, I will ask you which of the two gives a larger result. I want you to give an intuitive prediction, so raise your hand for the expression you think is greater."

The teacher then writes the two following expressions silently on the board. The students look attentively at the board while the teacher writes the expressions. They are thinking about the results of the two addition problems:

(A) $4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4$
(B) $8 + 8 + 8 + 8 + 8 + 8 + 8 + 8$

After writing on the board, the teacher says, "OK, now I will ask. First, who thinks the result of A is greater?"

A few students raise their hands. The teacher continues with "Who thinks that B has a greater result?" Now, many students raise their hands. The majority of the students think that 8 is greater. The teacher then asks, "Why do you think so?" The students will probably give many different answers. The teacher asks one of the students who raised his hand.

Student: "I calculated the answer. I thought of a simple addition."

The teacher asks: “Good, so, how did you calculate the answer?” The student replies that he used multiplication. When the teacher asks them to write the expression, the students write:

(A) $4 \times 12 = 48$
(B) $8 \times 8 = 64$

The majority of the students agree that this is correct. The teacher then asks, “Any other reason?” Another student gives another reason. He goes to the board and tries to explain it by drawing line segments between the two expressions.

(4 + 4)	+	(4 + 4)	+	(4 + 4)	+	(4 + 4)	+	(4 + 4)
|		|		|		|		|
8	+	8	+	8	+	8	+	8

This is fast. After grouping and connecting the numbers, he asks the other students to explain it. Can anyone explain the meaning of the groupings?

After some interactions, the teacher says: “Now comes the principal question of the lesson. I will change the plus signs to multiplication signs, and you must respond quickly to the initial question: Which of the expressions do you think is greater?”

6.6 Final Remarks

In Chap. 2 of this book, we confirmed that the multiplication table is taught in different grades around the world and posed the question as to the choice of grade for introducing it. In Japan, it is taught in the second grade, and this chapter has explained four reasons for this. The first reason is that it is possible. The second reason is that students are able to extend the multiplication table by themselves in an appropriate teaching sequence. To do so, they study ways to produce the table for the rows of 2 to 5 at first, and then they adapt ways of extension to other rows. They learn the meaning of a situation, producing the row with models and patterns, and creating situations for multiplication expressions. At the last stage, the structure of the multiplication table is analyzed and the properties of the table are established. The third reason is that memorizing the table is an enjoyable activity for students. The fourth reason is to develop a sense of wonder by exploring the patterns in the table and appreciate the reasonableness of the world of multiplication.

In the Japanese approach, students are able to learn the skill to extend what they have learned and the significance of their learning. Japanese teachers try to set the tasks and activities for memorizing and using the table through the various activities for sense making on the world of multiplication as a part of enjoyable cultural practice. The consequence of further Japanese students’ achievements in relation to the number sense, but not only limited multiplication, are known by surveys such as Reys, Reys, Nohda, Ishida, Yoshikawa, & Shimizu (1991) and Reys, Reys, Nohda and Emori (1995).

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Chapter 7
The Teaching of Multidigit Multiplication in the Japanese Approach

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This chapter illustrates the process of the teaching multi-digit multiplication in relation to Chap. 1, Fig. 1.1 as follows. Firstly, the diversity of multiplication in vertical form is explained in relation to the multiplier and multiplicand, and the Japanese approach in comparison with other countries such as Chile and the Netherlands is clearly illustrated. Secondly, how a Japanese teacher enables students to develop multiplication in vertical form beyond repeated addition is explained with an exemplar of lesson study. Thirdly, the exemplar illustrates a full-speck lesson plan under school-based lesson study which demonstrates how Japanese teachers try to develop students who learn mathematics by and for themselves including learning how to learn (see Chap. 1). Fourthly, it explains the process to extend multiplication in vertical form to multidigit numbers by referring to Gakko Tosho textbooks.

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7.1 Diversity of Column, Algorithm, and Vertical Form Methods for Multiplication

There is a diversity of column multiplication in vertical form around the world; the terminology itself differs, such as "column methods" in UK English and "algorithm" or "long multiplication" in US English. As part of algebra, the expression a × b is standardized around the world even though some countries, such as Chile, prefer to write "3 • 4" for 3 × 4. On the other hand, there is no universal standardized form for multiplication in vertical form, as well as other operations in vertical form. For example, in Chile, Japan, and the Netherlands, 23 × 7 is written as shown in Fig. 7.1.

In Fig. 7.1, all approaches use row 7 of the multiplication table. Japan and the Netherlands do multiplication from the lower to the upper columns. The Chilean method is consistent with algebraic expressions. It is not exactly vertical, and it looks like a kind of memo if we compare it with others. The Chilean method calculates the ones first. The Japanese method asks students to devise various methods by themselves at the beginning and then later reduces the adding (intermediate) part in the process of extension to 2-digit multiplication. In the Japanese curriculum standards, thinking about how to calculate the operation is one of objective as well as understanding the meanings and getting proficiency. At the last moment, they compare and discuss about easiness or fastness. Students communicate and explain that, 14 means 140 because of place value; It is not read as "one hundred forty" but as "fourteen" as an adaptation of the multiplication table. If it just means 7 × 2 = 14 instead of 70 × 2 = 140, the way of calculation can be seen as an algorithm using the multiplication table on the place value. This is the reason why the column method is called as an algorithm. To get the answer, it is necessary to use the multiplication table but not repeated addition (see Meaning of B, Fig. 1.1 in Chap. 1). Both the Japanese and the Netherlands forms calculate from the lower digit to the upper digit. However, in the Netherlands, 7 × 23 means to apply the multiplication table and calculate from tens, which is also the way to avoid a contradiction in Indo-European languages (See Chap. 3). In the case of Japan, there are some students who calculate from the largest place value in vertical form even though it is a way for mental estimation which follows the east culture cultivated by their abacus. Students prefer to calculate from the ones as well as the case of addition and subtraction in vertical form[1].

Here, these methods of multiplication are called column multiplication, an algorithm, or vertical form. For understanding of all kinds of column multiplication with such huge diversity, we provide a historical perspective and set conditions for what vertical form in multiplication is.

Chile:	2	Japan:	23	23	23		23	Netherlands	23
	23x7		x 7	x 7	x 7	⇒	x 7	(Freudenhtal	7x
	161		140	21	21		161	Institute):	140
			21	140	14				21
			161	161	161				161

Fig. 7.1 Various vertical forms for multiplication

[1] Hulbert, E. T. et al (2017) also illustrated how progress students' mathematical writing of multiplication algorism under CCSSM in USA.

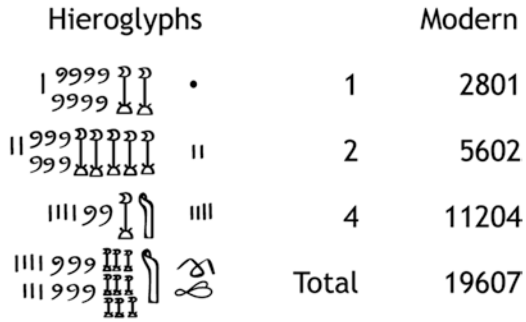


Fig. 7.2 Arcavi and Isoda (2007) from Eric Peet (1923)

7.1.1 Historical Illustration of Diversity

The roots of vertical forms in calculation can be found in ancient civilizations. Historically, various vertical forms appeared before the unified algebraic notation for arithmetic operations.

For example, ancient Egyptians wrote numbers in vertical form with the idea of doubling (2 times). However, it is not our meaning of multiplication because it was not necessary for them to memorize the whole multiplication table for doubles (see Fig. 7.2 for revision of Problem 79 from the Rhind Papyrus, 1650 BCE).

From the modern perspective, the idea of proportional reasoning can be found between the lines of this Egyptian writing. However, the Egyptians used doubles.

In Euclid's Elements, there was a theory of proportion with measurement and multiples for proportional reasoning, in general. However, there was no current meaning of multiplication even though some English translations of Euclid's Elements have used that term. Current historians explain it by the term "multiple/multiplicity" (see Chap. 3). For example, we can find the same figure as Descartes's definition of multiplication (see Chap. 3 and Elements Chap. 6, Proposition 11). It was *not* the same as the current meaning of multiplication, which allows multiplication of different quantities, but a way of measurement such as to find a segment of a geometric mean.

Fibonacci's *Liber Abaci* (1202) in English edition (Sigler, 2002) is known as a book that influenced calculations in vertical form from East Asia and India through Arabia with Arabic numerals during that era. It is done by the base ten place value notation system using Arabic numerals. We should note that most people used counting boards before *Liber Abaci* because they provide the answer by manipulative counting. Arabic numerals were introduced in that era and the book of Fibonacci is known as a book that influenced innovative movements on arithmetic in Europe with the base ten place value notation system and column methods. Algebraic expression and the multiplication symbol "×" were invented after the Renaissance, especially the symbol "×" was introduced by William Oughtred (1631; see Cajori, 1928). From that era, column calculation and the multiplication table gradually spread in Europe (Fig. 7.3).

Fig. 7.3 Gregorio Reisch (1504), *Margarita Philosophica*. Argentineñ: Opera Joannis Schotti. Arabic column methods versus a counting board. (Chapter title page for arithmetic; no page numbers in this book)

First 4 12 12		Second 44 12 12		Third 144 12 12
(Explanation: 2 × 2 on ones, which corresponds to the product of ones)	→	(Explanation: 1 × 2 + 2 × 1 between ones and tens, which corresponds to the product of tens)	→	(Explanation: 1 × 1 on tens, which corresponds to the product of hundreds)

Fig. 7.4 Fibonacci's *Liber Abaci* (Sigler, 2002, p. 24)

The first chapter of *Liber Abaci* explained addition and multiplication tables as well as the base ten place value notation system with Arabic numerals in comparison with Roman numerals. The multiplication table begins from row 2: 2 times 2 make 4, 2 3 6 (as 2 times 3 make 6), 2 4 8, up to 10 times 10 make 100 (no symbol between numbers). The second chapter is about multidigit multiplication in vertical form. The first example used to introduce multidigit multiplication was 12 × 12, the same 2-digit multiplication, which was explained by the process shown in Fig. 7.4.

Fig. 7.5 Fibonacci's manner of one digit number multiplied by two digits number

392	(Explanation:
8	$8 \times 9 = 72$, 7 in mind.
49	$8 \times 4 = 32$, add 7 in mind)

Fig. 7.6 Gregorio Reisch (1504), *Margarita Philosophica*. Argentineñ: Opera Joannis Schotti. (No page numbers in this book), The multiplication table, left of the figure, is not the whole table but a half and no row of 1

These steps show why it begins with multiplication of the same 2-digit numbers. It is for explaining how to set the place value for the product with the algorithm using the multiplication table. Thus, base ten place value system is the bases for vertical form. After such an example of the same 2-digit numbers, in the next section, multiplication of a 1-digit number by a 2-digit number is explained with 8×49 as an example (Fig. 7.5). In vertical form, 8 was written at first, then 49 was written below 8, under the row (line) of 49, and the answer (product) was written at the top. If we write the product in the bottom row (line) instead of in the top row (line), the format becomes the same as that of the Japanese (Fig. 7.1). On the other hand, if we read it from the bottom row to the top row, it looks like the reverse of the Netherlands method.

In *Margarita Philosophica* by Gregorio Reisch (1504), which was known as an essential textbook for liberal arts in the sixteenth century, the explanation of vertical form and the multiplication table shown in Fig. 7.6 can be seen. Before multiplication, it explains addition and subtraction of column methods. On addition in vertical form, it states, "augend upper line plus addend lower line." In the same manner, in Fig. 7.6, the multiplicand is in the upper line and the multiple is in the lower line.

Before the Fig. 7.9, the multiplication section in *Margarita Philosophica* began as follows:

> What is multiplication? Magnificent! It is to produce the proportional number correspond to multiplicand. It is multiple of unit. For example, a 3 by (per) 4, multiplier make number 12. It is the same proportion (ratio) 12 to 4 as 3 to unit. Because the ratio (proportion) of both, triple (thrice).[2]

[2] *Verbi gratia 3 per 4 multiplicare est numerum 12 pro creare. Qui se in eadem proportione ad 4 habet sicut 3 ad unitatem. Quia utro bigs est proportio tripla.* (No page numbers in this book).

This definition of multiplication based on proportionality, as is Descartes's, which is the definition of multiplication by measurement (see the discussion in Chap. 3). In this book, multiplication was explained for people who already knew about ratios and proportions because there is a chapter of Geometry before this chapter for Arithmetic. Thus, their usage of terminology is not the same as today's. There was no algebraic expression but only vertical form with the base ten place value system and a multiplication table without algebraic symbols. The vertical form and table were the form for expressions. In this book, multiples[3] and (set/cardinal) numbers are distinguished in the explanation. A number is represented by Arabic numerals and a multiple is represented by spelling out, such as "twice" (double), not represented as "2 times" by using Arabic numerals. The text sentences use a multiple such as "triple" which means 3 times. In a multiplication table such as "2 4 8" (see Fig. 7.6), it is read *bis 4 sunt 8* ("twice 4 is 8") which means the multiplier functions as "number of times." At the rows on Fig. 7.6, right, the first number was used to be read as multiplicative numeral such as bis (twice). If multiplication is to produce the proportional number corresponding to the multiplicand, a further interpretation of the multiplication table on the right side of Fig. 7.6 could be to understand it as "3 4 12" corresponding to "1 to 3 is 4 to 12"; for example, in the table, "2 2 4" implies "1:2 = 2:4" and "2 3 6" implies "1:2 = 3:6" (the algebraic expression did not exist in the text).[4] On this notation, if first numbers on the table were read as multiplicative numeral, they may not feel necessary to use the symbol "×" because the number of times such as "two times" is represented by multiplicative numeral "twice".

On the basis of this understanding, we would like to return to the problem of the multiplier and multiplicand which has been discussed since Chap. 2. If we compare the left and right sides of Fig. 7.6 and consider the correspondence, we find that the multiplication on the left corresponds to (multiplier) [space (×)] (multiplicand) [space] (product) in the horizontal table on the right. In the table, the multiplier (numeral) is read as a multiple and the multiplicand is read as a number. It is a calculation in vertical form as (lower line: multiplier) × (upper line: multiplicand), which means it calculates from the lower row to the upper row in vertical form and the product is written under the lower row. The vertical form as a column method and the horizontal multiplication table function as mathematical forms instead of an algebraic expression at this era.

[3] In English, the difference between multiplier (single, double, and triple: adjectives) and multiplicative numbers (adverbs: once, twice, and thrice) are existed. Japanese does not these numerals (Ramsey, 1892, p339). In English, the multiple (number of times) is based on the natural number. In Japanese, multiplicative numeral is represented by times (bai) and bai is not limited to use natural number but decimal and fraction (see Chap. 4). Proportionality is not limited to discrete numbers but is extended to real numbers and extend. The definition of multiplication by measurement (Chap. 3) is based on proportionality.

[4] This interpretation of multiplication under proportionality is the only possible in Western culture. Under the influence of Euclid, Western Arithmetic is known as being ratio–proportion oriented. Eastern Arithmetic is known as being digit–calculation oriented, under the influence of the calculation matrix (table) and abacus.

The forms in *Margarita Philosophica* did not contain contradictions. However, in this format, we can find the origin of the contradictions and confusion about multiplication in Europe, which are discussed in Chaps. 3, 4, and 6. The contradiction will appear if we add the multiplication symbol "×" into the vertical form as well as the algebraic expression. If we rewrite the vertical form shown on the left in Fig. 7.6 as an expression from the top row to the bottom row, it is 7954 × 642 which means 7954 (as the multiplier) × 642 (as the multiplicand) in the manner of the table. This contradicts the explanation given by *Margarita Philosophica* from the lower row to the upper row. If we rewrite the vertical form as 7954 × 642 and read the original method in Fig. 7.6, it is 7954 (multiplicand) [×] 642 (multiplier) which looks the same as the Japanese notation. In *Margarita Philosophica*, it is recommended that a large number is written in the top line and a small number is written in the lower line. Instead of using the multiplication symbol "×" and reading it as "multiplied by", it uses "per (by)" or "multiple (numeral)." At that time, there was no contradiction. However, the current difficulty may have appeared in the process of reorganization with algebraic notation.

Under the Universal Mathematics by Deacartes which integrate various mathematical subjects under the algebra, algebraic notation had spread in Europe (see Fig. 3.1 of Chap. 3). Oughtred introduced the symbol "×" as or algebraic notation and he never used it to represent the column method. He explained the necessity and usefulness of multiplication for logistics. In Oughtred on later 1694 Edmond Halley edition, he called numbers in multiplications by factores, products, rectangle, and plane and not mentioned multiplier or multiplicand.[5] On Gilberto Clark commentary for Oughtred's Clavem mathematicam (Key of the Mathematics) in 1682, the rectangle area diagram is added and both numbers of multiplier and multiplicand in column multiplication were called by factors, It implies that to avoid the confusion between multiplier and multiplicand in vertical form and expression they might preferred their rectangle and factors. Indeed, today, the area formula is length (longer side) × width (shorter side) as well as column multiplication which locate larger number top line. Rectangle is the model to explain commutativity from the era.

In that era of Margarita Philosophica in Europe, to define multiplication, they needed proportions. On the other hand, in China, arithmetic meant various methods of the numerical calculation on situations which had more than four operations from an early stage. In ancient China, arithmetic operations were written in the *Suàn Shù Shū* [*A Book on Numbers and Computations*] (186 BCE; English translation by Cullen, 2004), a bamboo book (Dauben, 2008). The multiplication table was necessary to memorize for using rods[6] on a calculation matrix which represented the base ten place value system, like the column methods. In *Jiǔzhāng Suànshù* [*The Nine Chapters on the Mathematical Art*], anonymous authors in the tenth to second

[5] In 1667 edition of Oughtred, there was no symbol for operation on vertical form to calculate numbers. The symbol for operations were appeared to explains the algebraic operation for the operation of letters. Thus, originally there were no operation symbols on vertical form.

[6] Red rods represent positive numbers and black rods represent negative numbers.

Fig. 7.7 Zhū Shìjié (1299), *Suànxué Qǐméng* (used (元)朱世傑「新編筭學啓蒙 3巻坿總括1巻」李朝初期) multiplication table (*left*) and division table (*middle*); and Yáng Huī (1274, 1275) *Yanghuī Suan Fǎ* (*right*) (used (宋)揚輝編「宋揚輝筭法 7巻」慶州府, 宣德8 [1433]). In these books, the tables are to be memorized for calculations. In the case of *Suanxue Qǐmeng (left)*, the multiplication table is half and second number is constant like $1 \times 1 = 1$, $1 \times 2 = 2$, $2 \times 2 = 4$, $1 \times 3 = 3$, $2 \times 3 = 6$, $3 \times 3 = 9$, $1 \times 4 = 4$, ..., $4 \times 4 = 16$, $1 \times 5 = 5$, ..., $5 \times 5 = 25$ and so on. See Jinkoki on Fig. 6.8 in Chap. 6, Chap. 6 for comparison: $2 \times 2 = 4$, $2 \times 3 = 6$, $2 \times 4 = 8$, …, $3 \times 3 = 9$, $3 \times 4 = 12$ and so on. Jinkoki's table, Fig. 6.8, is similar as Fig. 7.6 *right, Margarita Philosophica* but different with *Suànxué Qǐméng*

centuries BC (10th–2nd centuries BC) had already discussed equations in a matrix.[7] Later, in the Yen Dynasty, Suànxué Qǐméng (1299) began a book with a multiplication table (Fig. 7.7). It also included a division table[8] which may imply that they used an abacus for calculations.

At the end of the Sòng Dynasty, Yáng Huī asked learners to memorize a multiplication table before studying his book *Yánghuī Suàn Fǎ* [*Yáng Huī Algorithms*] (1274, 1275), which is known as an introductory book (Jochi, 2003)[9]. This Chinese tradition was thought to have influenced the Middle East and reached Europe through Fibonacci.[10]

The Chinese did not necessarily invent algebraic expression itself because their calculations were well done on a matrix sheet[11] up to positive and negative numbers and algebra. The Japanese extended it to solve equations using the abacus (Seki, 1674). Even during the era of Descartes in the early seventeenth century, the vertical form, not the expression, was still the major form used to represent arithmetic operations in Europe. Today, European algebraic representations have became a

[7] It can be seen as a kind of vertical form of the sweep-out method (thirteenth century), because it is based on the base ten system for the rod arrangement.

[8] A division table cannot be understood without using an abacus.

[9] http://www.osaka-kyoiku.ac.jp/~jochi/jochi2003b.pdf.

[10] 0 was established in India; however, the vacant place in the calculation matrix meant 0.

[11] The Chinese matrix sheet, horizontally, represents the base ten place value system, the coefficients of a polynomial, and so on; vertically it represents the process of operations.

universal language for mathematics around the world. However, various vertical forms have been used in arithmetic.

Most of these forms, except those in ancient Egypt, were written vertically, using both the idea of base ten place value in columns and the multiplication table. Ancient Egypt did not use place value numerals but doubling—row 2 in the multiplication table. Here, we would like to focus on multiplication in vertical form by using the base ten place value system and the multiplication table. Under these conditions, the Egyptian method is not multiplication in vertical form. The Chinese–Japanese abacus[12] has place value but the numbers are represented by beads. The abacus is a manipulative, thus the given numbers are lost in the process of manipulation and only the product remains. On the other hand, multiplication in vertical form retains the multiplier, multiplicand, and product. Vertical form is a kind of expression that preserves the relationships among the multiplier, multiplicand, and product.

Calculation on the abacus is usually done from the largest place value. In the case of multiplication of 35×24, a way of manipulation is done by the following sequence: 3×2, 3×4, 5×2, 5×4. If we do a calculation in this manner with an abacus, there is no contradiction between the multiplier and the multiplicand (see Chap. 3) because the order of (multiplier) × (multiplicand) never changes and their multiplication table was half which means that their table itself existed under the commutativity. Thus, the Chinese who invented the abacus did not encounter a contradiction like the European people who imported multiplication in vertical form with tables from the East, invented algebraic expression, and later, re-embedded the expression symbol "×" into the column methods.

Quoted historical books usually begin with or referred the multiplication table. The multiplication table can be seen as a historical root of expression of multiplication as a binary operation. Before algebraic expression, multiplication used the table and the column. There was no necessity to explain multiplication as repeated addition because algebraic expression did not exist at that time.

7.1.2 Revisiting the Confusion Between the Multiplier and Multiplicand, and the Need to Differentiate Them

As explained in the historical roots, the confusion as to which one is the multiplier and which is the multiplicand in $a \times b$ was appeared in relation to algebraic expression. In English, "a" is the multiplier and "b" is the multiplicand. We should note

[12] Originally, the Chinese abacus, which used to have two five-beads on top and five one-beads for every place value in bottom, could be used for both base ten and base sixteen systems under their measurement system. The Japanese revised it into one five-bead and four one-beads as an adaptation of the base ten system for educational and industrial objective to adapt the base ten French-European unit-quantities system (see Fig. 6.8). Ministry of Education fixed Japanese-style abacus in 1935, officially. The Japanese-style abacus influenced all East Asia before World War II. Currently, it is not easy to find the original Chinese traditional style abacus in East Asia. The Chinese–Japanese abacus is a tool to support mental calculation; it is not just for counting tools like other abacuses in the world.

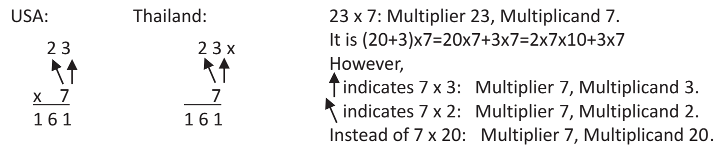


Fig. 7.8 Confusion of multiplier and multiplicand

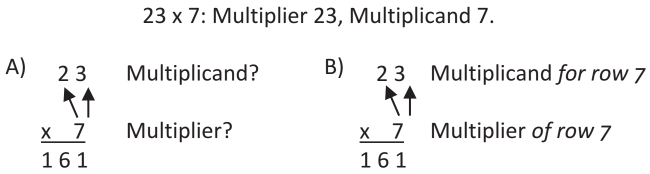


Fig. 7.9 To distinguish multiplier and multiplicand

that the algorithm in the vertical form of multiplication proceeds from the lower digits to the upper digits using a multiplication table such as in *Margarita Philosophica* (Fig. 7.6). In the expression "a × b" the first number "a" is the multiplier but "b" is usually explained as the multiplier of the row of 7. The problem might have originated from seeing the vertical form as as for the presentation of algebraic notation expression because the historical representation does not have algebraic symbols such as "×" and "="; indeed, if we put the symbol "×" into the vertical form, the following contradiction will happen.

If we do not have the multiplication symbol in Fig. 7.8, it is just to support mental arithmetic. The source of confusion originated from seeing the vertical form by algebraic expression. It was identified as an overgeneralization of algebraic expression in the historical manner of arithmetic. Actually, it produces confusion even for teachers because they are likely to explain the vertical form from *Margarita Philosophica* as A, instead of B (Fig. 7.9).

Writing "A" is the source of confusion because 23 is the multiplier in 23 × 7. There are five ways to avoid this confusion: the first is to be careful of expressions like "B"; the second is to change the format of the vertical form, as in the Netherlands (Freudenthal Institute)z; the third is to change the format of multiplication, which was mentioned in Fig. 3.11 (Model A) in Chap. 3; the fourth is to change the names such as the naming of the first number (factor) and the second number (factor) instead of "multiplier" and "multiplicand," and the fifth way is to enhance commutativity. In Table 2.3 in Chap. 2, Chile, Mexico, Portugal, Singapore, and the USA (but not Brazil and Japan) do not use the terms "multiplier" and "multiplicand" (to avoid confusion) and just call them factors which do not imply the order of the two numbers.

In the case of factors with enhancing commutativity, there is no order in the expression. If students do not pay attention to the difference between the multiplier and multiplicand in situations, the students may lose the meaning of multiplication, as to which number is the unit (later it become the base for rate) and which number is the number of units. Students do not pay attention regarding the difference between 5 candies for each dish and 3 dishes, or 3 candies for each dish and 5 dishes. They also cannot distinguish situations of division as partitive division or quotative division (see Chap. 4). As we discussed in Fig. 4.20, they cannot produce the correspondence of meanings in both divisional situations as different interpretations of multiplication, multipliers, and multiplicands in situations. And at the later grade, "for each dish" becomes "per dish" which is a necessary terminology as for the bases of ratio and rate.

7.1.3 Terminology for Teaching Column Multiplication

Multiplication in vertical form is not repeated addition. For clear understanding, here we would like to confirm some basic technical terms for multiplication in vertical form, considering various approaches depending on the country.[13]

Mental Arithmetic Mental arithmetic is done by calculating mentally using memorized arithmetic. For vertical forms of addition and subtraction, it is necessary to memorize composition and decomposition of numbers for making 10 which is necessary for carrying and borrowing by place values. For multiplication in vertical form it is also necessary to memorize the multiplication table. In the diversity of vertical forms (Fig. 7.1) the Chilean method needs more mental arithmetic than those of Japan and the Netherlands.

Mental arithmetic is a necessary part of number sense to devise numbers and operations judiciously. For example, if students recognize 4 times in comparing 25 and 100, they have a sense of the quadruple. In the Japanese approach, the relationship between two expressions such as 80×2.4 and 80×24 (see Fig. 4.18 in Chap. 4) are formally learned as a part of number sense.[14]

Multiplication Table In relation to a numeral system such as in English, a multiplication table sometimes includes numerals up to 12 or more, depending on the country and culture. In the case of Spanish, the numerals up to 15 have specific names, then from 16 onward they are written as *dieciséis* ("ten and six"), etc., but after 100, the numbering in Spanish is well configured as the base ten system. On

[13] The terminologies in English for teaching elementary school mathematics were locally systematized by various scholars such as Treffers, Nooteboom, and de Goeij (2001) and Reys, Lindquist, Lambdin, and Smith (2012). The Freudenthal Institute provides the necessary ideas to describe the learning trajectory (see van den Heuvel-Panhuizen, 2001). Clements & Sarama (2004) defined learning trajectory by three aspects: a learning goal, developmental progressions of thinking and learning, and a sequence of instructional tasks. However, these terminologies are not unified around the world under the teaching culture which teachers prefer their own work sheets for teaching (see Sect. 4.4, Chap. 4). Terminology for elementary school mathematics teaching also exists in Japan (Isoda and Nakamura, 2010) and it is more precise in the shared curriculum sequence as explained in Chap. 4.

[14] In Japan, this has been formally enhanced, as an objective from the 1998 curriculum, as one of the ways to think about how to calculate, and the think about how to calculate based on the number sense are necessary for the bases of symbol sense in junior high school in the 2009 curriculum. Symbol sense was discussed by Abraham Arcavi (1994).

the other hand, many Spanish-speaking countries use multiplication tables up to 9. This implies that those countries may have more difficulty engaging in multiplication as mental arithmetic. The French numeral system is also complicated.

In some countries such as Singapore, memorization is explained as development of number sense or proficiency in operations. In some countries such as Mexico and Chile, advanced students are able to use their partially memorized table with possible strategies to find the answer in multiplication.

Standard or Formal Algorithm An algorithm[15] is a fixed sequential step-by-step calculation or procedure which usually includes recursive process. The terms "standardized algorithm" or "formal algorithm" in vertical form can be fixed in every country but are not necessarily the same as those in other countries because there is no universal format likely algebraic expression (see Fig. 7.1). The Japanese curriculum asks students to think the ways of calculation.[16] In the case of vertical form, it means selecting the standard algorithm in comparison with other possible approaches and appreciating every idea, especially the reasonableness of the standard algorithm. In Japanese textbooks, an algorithm similar to the Netherlands one (Freudenthal Institute) also appeared as a student's idea before the Japanese standard algorithm was set. Here, "formal" and "informal" are relative because the likely Netherlands algorithm also appeared in Japanese textbooks as a student's idea. In classroom, students ideas can be seen as informal ideas however on the Japanese textbooks such possible ideas are formally treated. Japanese teachers are expected to treat them as ways of meaningful calculation in the process to select simpler, faster and easier one (see Chap. 1 Mindset, Table 1.1 in Chap. 1).

Decomposition Decomposing a number with base ten by using the distributive law enables students to consider the way of multiplication beyond the multiplication table. Before the introduction of column multiplication, the known product of multiplication was within the table. If we multiply by 10 times (*bai* in Japanese), it is easier to find the product of multiplication by 20 times, 30 times, and so on. In 20 [×] 3 (20, 3 times), if we decompose the multiplication, it is $2 \times 10 \times 3 = 2 \times 3 \times 10$. Decomposing numbers with base ten by using the distributive law such as $23 \times 7 = 20 \times 7 + 3 \times 7$ is a key idea to produce column multiplication to distinguish

[15] In relation to developing the competency for coding and computational thinking (National Research Council, 2011; Araya, Isoda, Rafael, Inprasitha, To appear), finding and creating the algorithm itself enhances the objective of multiplication in vertical form.

[16] In Japan, as well as understanding of the meaning and acquisition of the skill, thinking about ways of calculation, or thinking about how to calculate, which asks students to consider various ways of calculation, is a key objective. It includes a variety of vertical forms. It was introduced in the 1998 reforms. In the newest Japanese curriculum (MEXT, 2017a, 2017b), it is explained as follows. (1) Teachers should help students: (a) to understand that multiplication of 2- and 3-digit numbers by 1- and 2-digit numbers is based on basic multiplication of 1-digit numbers, and to understand how to calculate, using algorithms in a column form; (b) to multiply accurately and to use multiplication appropriately; and (c) to understand simple properties that hold for multiplication. (2) Teachers should help students to acquire the following abilities of thinking, making decisions, and expressing: (a) focusing on mathematical relations, thinking about ways of calculation; (b) exploring properties that hold for calculations; and (c) calculating simply and checking the result of a calculation by making use of the properties.

tens and ones using the row of 7 in the multiplication table. In Chap. 3, splitting as another usage (Figs. 3.6 and 3.7) is a representation of the distributive law and originally meant dividing equally (Fig. 3.4). In Fig. 7.1, the Japanese and the Netherlands vertical forms for multiplication clearly use decomposition which requires addition of an intermediate process for multiplication in vertical form. However, the Chilean vertical form requires mental arithmetic for the intermediate addition part and is not clear on how students do the intermediate part. Teachers may have to teach it through giving exercises. The Japanese approach enables students to think about how to calculate the intermediate addition part at first, and later this part will be reduced in relation to the progress in mental arithmetic.

7.2 Lesson Study for Introducing Multiplication in Vertical Form

As discussed at Fig. 1.1 in Chap. 1, for the extension of multiplication, students have to reintegrate multiplication table with base ten place value system by using decomposition of numbers, opposite direction of distribution, instead of repeated addition. Even though the distributive law itself will be learned later, Gakko Tosho textbooks already introduced the idea at the second grade as for the extension of multiplication table (see Fig. 6.3, Chap. 6). Here, the way a Japanese teacher introduces multiplication in vertical form for Grade 3 students, especially how to introduce the idea of decomposing with the distributive law, is illustrated with the full format of the lesson plan, as follows. The first steps are to watch the video for understanding of the lesson and then to show the whole lesson plan to share how it was carefully prepared in the case of school-based lesson study for developing students in Japan. The first part is intended to illustrate decomposition of numbers to prepare for multiplication in vertical form. It is an exemplar showing how Japanese students produce their ideas, some of which are necessary for further learning based on what they have already learned. The second part is detailed in the next section as an Annex for explaining school-based lesson study with the full format of the lesson plan which includes a unit plan for introducing multiplication in vertical form beyond repeated addition.

7.2.1 Lesson Study Video Introducing Vertical Form

This lesson was taught based on the 1998 curriculum by Mr. Hideyuki Muramoto, with the assistance of Prof. Kazuyoshi Okubo (Muramoto and Okubo, 2007), in the third grade, on the topic of multiplication algorithms. It was video recorded for the Asia–Pacific Economic Cooperation (APEC) project "Innovations in the Classroom Through Lesson Study" (Isoda, Shimizu, Loipha, and Inprasitha, 2007).[17] The list of

[17] Report retrieved on June 29, 2019, from http://www.criced.tsukuba.ac.jp/math/apec/apec2007/progress_report/; video of Mr. Muramoto's class and lesson plan retrieved on June 29, 2019, from http://www.criced.tsukuba.ac.jp/math/apec/apec2007/index.html#video.

Fig. 7.10 Participants in the Asia–Pacific Economic Cooperation (*APEC*) Lesson Study Project for 2007 who observed Mr. Muramoto class

episodes and clips was developed by David Tall (Tall, 2013), and video can be seen too at the following URL: https://youtu.be/7tG_UDbQnmo.

The lesson is an example of the lesson study process for teaching mathematics. This 50-minute research lesson was planned and taught at Maruyama Primary School in Sapporo, Japan, to a grade 3 class of 40 students. It is the fourth class in a sequence of 13 sessions (see the last part of the next section). The task sequence in the 13 sessions begins from 20×3 which can be solved by repeated addition, and is then extended to 23×3 which is not easy to solve by repeated addition but is easy to solve by decomposition under the base ten place value system.[18] Finally, decomposition is used in multiplication in vertical form, followed by exercises. The fourth class discussed 23×3, which participants observed (Figs. 7.10 and 7.11).

The previous lesson considered the product of 20×3 and encouraged students to calculate the number of black circles (marbles) in the arrangement shown in Fig. 7.12,[19] where the total of (10, 3 times) plus (10, 3 times) is $30 + 30$, which is 60.

The detailed lesson plan can be found in the next section. Please note that the array diagrams used here can be read in two directions. As discussed in Fig. 4.9 in Chap. 4, the diagram does not consider the order of operation.

In this lesson, the students are encouraged to use their learned knowledge to solve the problem of calculating how many circles there are in a new arrangement (in which they will find 23, 3 times). The plan is to find various ways of doing it and consider which ones are more complicated and which ones are easier. The long-term goal is to make the students aware of the advantages of constructing column

[18] About Japanese task sequence for extension and integration principle, please see Chap. 1 and Chap. 4 such as Fig. 4.10 for the introduction of decomposition and composition of numbers and making 10, and Fig. 4.27 for task sequence for viable arguments by extension. Extension and Integration principle is a key principle for "Construct viable arguments and critique the reasoning of others." (CCSS.MATH, 2010).

[19] This is the diagram for decomposing multiplication. For learning vertical form, Japanese teachers enable students to use the diagram and never use concrete objects for this task because concrete objects merely enhance counting which is not multiplication.

Fig. 7.11 The summary part of the lesson study by Mr. Muramoto

Fig. 7.12 A diagram for 20 × 3 (20, 3 times)

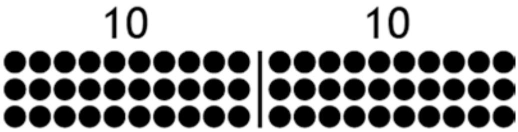
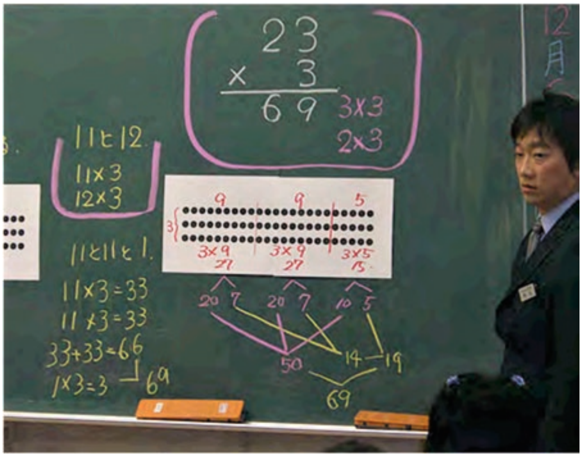


Fig. 7.13 Which one among the decompositions is better?



multiplication through a meaningful experience related to practical examples (Fig. 7.13).

When watching the video, take note of how the teacher begins at the left side of the chalkboard with the problem, prepares the development of the lesson, and indicates important points in yellow chalk so the structure of the entire lesson is visible on the chalkboard.

The objective of this lesson is to help the students think about how to multiply 2-digit numbers by 1-digit numbers.[20] As soon as they see the mathematical expression (that is, 23×3), many of them feel that the problem cannot be solved directly using the multiplication table. If the students can see the structure of the problem with an arrangement (split) diagram, they will realize they can calculate this problem using the results of the multiplication they have already learned. "I want to make sure the students can see that they can use the idea of how many times a quantity contains the unit quantity,"[21] Mr. Muramoto indicates.

In this lesson, the students will decompose the 2-digit numbers that are easy to use with the multiplication table. Through this investigation, the students will carry out the decomposition of a 2-digit number into various ways to make the calculation possible. Finally, based on simplicity, decomposition by tens and units (that is, 23 into 20 and 3) is preferred to use for the vertical form. Additionally, they will learn that this idea is the foundation of the multiplication algorithm (the method for calculating with pencil and paper).

The crucial point of this lesson is that the students consider the way of calculation by themselves. They investigate the ways to decompose the number 23 so they can use what ever they have already learned. For example, students learned to set various groups as for the unit to study the every row of multiplication. To understand the algorithm, it is necessary that the students recognize the significance of decomposing 23 into 20 and 3 such as simplicity. In this lesson, the teacher wants the students to observe a diagram in order to decompose the 2-digit number for use of the multiplication table.

In the following description, the teacher's intention for this class, the actual teaching phases for the class, and the teacher assessment views are illustrated to provide the minimum knowledge needed to follow the video. The precise information for understanding the theme of the lesson study is provided in the next section as an Annex based on Mr. Muramoto's lesson plan.

7.2.2 Mr. Muramoto's Objectives for This Class

At the start of the postclass discussion, after the class observation, Mr. Muramoto restates his purpose as follows:

> Since the beginning of the school year (April), I have taught the students to draw a diagram of the problem situation in order to think about how to deduce expression and calculate. Also, I have emphasized the importance of mathematical learning in class, so the students can use the diagram to explain their logical thinking processes.

---

[20] "Think about how to calculate" is a key objective of teaching operations in the Japanese national curriculum standards, as well as understanding of the meaning and acquisition of skill (see footnote 1). With this objective, Japanese teachers do not just try to make sense by putting the meaning into students but provide preparations for sense making that students may make sense of by and for themselves, by using their learned knowledge (see Chap. 5).

[21] This is the definition of multiplication by measurement (see Chap. 3).

> There are some students in the class who already know how to multiply using the algorithm. Even though they already know the algorithm, it is not clear if they really understand its meaning. The students can understand it by looking at the diagram. They recognize the meaning and the value of decomposing the 2-digit number to calculate and generalizing the idea of "how many times a certain quantity contains the unit quantity."
>
> The solution to the problem 23 × 3 is always 69, independently of how the number 23 is decomposed to make the calculation. The students will realize how diverse ideas for making the calculation can be used, learning from each other in the classroom.
>
> Doubtful students or those who have difficulty with 2-digit multiplication may not be able to grasp the idea of decomposing the 2-digit number, and instead they might use addition (23 + 23 + 23 = 69). By learning from each other in the classroom and presenting various ideas, they can begin to think, "If I decompose 23, I wonder if the calculation would be easier."
>
> A diagram that shows how the number 23 is decomposed in various ways and the mathematical expressions that go along with each different method will help these students to compare ideas and think of a better method.

This is his commentary after the class. The observers observe the class with a lesson plan. The lesson plan will be explained later. The illustration of the real class activity shows how the students are able to think of decomposing the number for multiplication in vertical form instead of repeated addition. The original lesson plan for school-based lesson study is too long and is shown in the next section.

7.2.3 Description of Actual Lesson Episodes

The lesson plan by the teacher, Mr. Muramoto, can be found in the Annex. The following table describes the seven principal episodes of the lesson, which were produced by David Tall. The total lesson video was retrieved on June 30, 2019, from https://youtu.be/7tG_UDbQnmo.

Description of the content of each of the principal episodes of the class (available in the videos)	Identification of the episodes in a video clip
In this class, Mr. Muramoto introduces a new problem, and the students try to guess what it is, based on their prior experience. The problem is presented in the video clip, and at the end, the students wait for a copy of the problem to calculate Video at https://youtu.be/tolkfvBbDRM	The problem (Video 1 of 7) Begins at 01:58, duration 1:20
After establishing the problem of calculating 23 × 3, Mr. Muramoto encourages the students to work on their own, then he walks around the classroom while they work for about 5 minutes. He takes note of who has finished and who has not, then invites the students to explain their ideas. Initially, all the ideas are related to decomposing 23 into 20 plus 3, or into 10 plus 10 plus 3. The video clip shows the first answer Video at https://youtu.be/Qk6gJRIw9rY	The student Amon sees 23 as 20 + 3 (Video 2 of 7) Begins at 16:45, duration 2:18

(continued)

Description of the content of each of the principal episodes of the class (available in the videos)	Identification of the episodes in a video clip
Each answer is received with approval, except possibly that of one student, who sees the entire arrangement as 30 + 30 + 9; he has seen the whole problem as two subarrangements of 3 rows of 10, which is 30, and a subarrangement of 3 rows of 3, which is 9. The teacher explains to him calmly that he has not yet finished and must write it down in his notebook Video at https://youtu.be/Di2xz4hoJgk	Amano has not finished (Video 3 of 7) Begins at 21:56, duration 1:01
One answer suggests that the 2 in 23 can be considered as two 10-yen coins Video at https://youtu.be/ef_5eHYv4nI	Using 10-yen coins (Video 4 of 7) Begins at 25:18, duration 2:00
After about 17 minutes dedicated to the examples of decomposing 23 into 20 and 3—or into 10, 10, and 3—a student suggests that no one has proposed anything different Video at https://youtu.be/a6IUrFlynL8	"I realized something" (Video 5 of 7) Begins at 33:53, duration 1:02
After this intervention, various possibilities come up, including 11 + 12, 9 + 9 + 5, and 11 + 11 + 1. The teacher encourages the students to talk among themselves. The video shows the difficulty of decomposing it as 9 + 9 + 5 Video at https://youtu.be/Qk6gJRIw9rY	3 × 9, 3 × 9, 3 × 5 (Video 6 of 7) Begins at 38:13, duration 2:15
The teacher finds that some students have used the standard vertical form for presenting the problem. In the video, he encourages one of them to explain his idea. Then, he connects the vertical form to the other methods using posters and puts up a poster of the vertical sum for direct comparison. After the episode is shown, there is a detailed 5-minute session that summarizes the class, in which Mr. Muramoto gets the students to read the purpose of the class from the chalkboard and suggest phrases to describe the class. The entire chalkboard presents the lesson's principal ideas from left to right, allowing the students to consider the whole discussion and make their own notes Video at https://youtu.be/i0m_K9wqJxE	Vertical form (Video 7 of 7) Begins at 42:08, duration 4:02

The previous table shows seven episodes. The following table refers to the identification of a 40-episode sequence. Each episode is associated with a position in the sequence, a duration, a name that identifies it (as well as an introduction to the problem, class activity, discussion, and summary), and a brief reference to the content of the episode.

Multiplication algorithm for the third grade. Teacher: Mr. Hideyuki Muramoto December 6, 2006; 1:35–2:20 p.m.; Maruyama Elementary School, Sapporo		
Time	Length	Episode
00:15.8	01:35.6	*Introduction* [8 min, 36 s] The teacher begins by writing down the problem for the students to copy in their notebooks. Note that the students know the format: a problem is presented on the chalkboard and they wait for a photocopy to do the calculations themselves

(continued)

02:21.6	01:28.1	*The problem* How many circles are there? Showing the circles row by row, the students guess how many there are (often based on the previous class with 20 circles in each row, before realizing that now there are 23)
03:19.5	01:56.9	Handing out the photocopies
05:16.4	00:06.6	Finding the answer by calculating (adding or counting)
05:23.0	00:27.3	23 circles
05:50.3	00:22.4	How many are there in the top row?
06:12.7	02:40.6	And in the next row?
08:53.3	00:38.6	*Lesson activity* [5 min, 13 s] Think about how to calculate 23 × 3
09:31.9	04:34.6	[Lesson activity]
14:06.5	01:48.3	*Discussion* [first half: 17 min, 39 s] The teacher begins the discussion by suggesting that they cannot calculate 23 × 3 using the ideas they have already learned
16:44.9	00:50.3	Who has written it down? The teacher first checks who among the students have written an answer, who knows how to do it, and who still doesn't have any idea
19:04.2	02:19.2	The student Amon decomposes 23 as 20 + 3. The teacher writes 3 × 3 = 9, 20 × 3 = 60, 9 + 6 = 69, corrected (as observed below) to 9 + 60 = 69
19:51.5	00:47.3	23 × 3 is complicated. Shinjo presents the same idea
20:41.0	00:49.5	My idea might be the same. Another girl does the same thing; 3 × 3 is 9 and 20 × 3 is 6 (corrected to 60)
21:24.7	00:43.7	I use what we have learned. Another girl explains how she has used what she learned from the previous lesson
21:40.7	00:31.2	Tens and units. The girl explains in terms of units places and tens places
21:56.9	01:10.5	Amano has not finished. (Amano sees the arrangement as 9, 30, 30. The teacher speaks softly to him and does not make it public)
23:06.4	02:12.1	Chiba is similar to Amano but makes an error. He decomposes 23 into 20 and 3, multiplies 3 × 3, and adds 20, obtaining 29. Other students discuss
25:18.5	25:18.5	Using 10-yen coins, they once again decompose 23 into 20 and 3, but now they explain 20 as two 10-yen coins, so 2 × 3 is 6
27:19.2	01:34.2	2 groups of 10, one of one (the teacher divides the poster into rows of 10, 10, and 3, and writes what the students have explained): 10 × 3 = 30, 10 × 3 = 30, 30 × 2 = 60, 3 × 3 = 9, 60 + 9 = 69
28:53.4	00:34.1	Tell me why. (Another boy explains how easy it is to make groups of 10. The book says that)
29:27.5	01:21.6	Who has the same idea? (Another student gives a similar explanation)
30:49.1	00:56.4	20 × 3 is easier. I know something. Multiplication by 10 is easier
31:45.5	01:42.4	*Another method* [14 min, 27 s] A boy suggests a new distribution, decomposing 23 into 10 + 3 + 10 with 3 × 3 in the middle. There are surprised whispers in the classroom
33:27.9	00:25.3	Now use 10 and 5. A girl returns to yesterday's calculations of 20 × 3 which are still displayed at the side
33:53.2	01:02.6	I noticed something. A student suggests that all the methods end with 60 and 9; none calculated 39 and 30
34:55.8	00:54.6	23 × 3 is complicated. As we said before, 23 × 3 is very complicated and confusing. We have learned to calculate 20 × 3, numbers that end in zero

(continued)

35:50.4	01:01.1	23 × 3 does not end in zero. A boy explains that this is why we decomposed 23 into 20 and 3 or 10, 10, and 3
36:51.5	00:07.9	A different way? The teacher asks if anyone has decomposed it in a different way
36:59.4	01:13.6	11 + 12. A boy says he decomposed 23 into 11 and 12 to calculate 11 × 3 and 12 × 3. The teacher says, "We haven't studied that yet." The students talk about the difficulty of that
38:13.0	02:18.1	3 × 9, 3 × 9, 3 × 5. The teacher writes the students' calculations on the chalkboard in a complex manner. The teacher approves and asks if they are similar to the other calculations.
40:31.1	01:37.3	23 is 11, 11, and 1. A boy makes a calculation with a small error, which is corrected
42:08.4	01:37.4	Vertical form. The teacher notices that Mai writes the problem in vertical form using the standard algorithm. He asks her to share her idea. There is discussion about tens and units, with some use of the idea of 10-yen coins
43:45.8	01:43.6	Is it totally different? Yamada talks about the relationship between the poster and the calculation that 3 × 3 is 9 and 3 × 20 is 60. In particular, he focuses on 3 × 2, which is 3 × 20 with the answer in the tens place. The teacher explains it in terms of 10-yen coins
45:29.4	00:43.5	Watching carefully. The teacher takes the paper with calculations using vertical rows of circles
46:12.9	00:07.2	*Summary* [5 min] The time runs out. A boy says, "I want to do more!"
46:20.1	02:08.3	Any good ideas? Takashi thinks it is good to think of two 10-yen coins. He explains that some people use numbers like 60 and add numbers that are not round numbers (in the units place)—round numbers that end in zero. Tsubota expands the idea
48:28.4	00:23.8	Let's read. The teacher asks the students to read what they have written on the chalkboard. "We thought about how to calculate 23 × 3"
48:52.2	02:20.4	What should we write? The teacher encourages the students to say what to write. He takes the phrase "the vertical calculation form" and writes a phrase selected from the students' suggestions to end the class
50:36.5	01:35.6	*End and credits*

For watching the video, please note the questions for formative assessment, written in the lesson plan (and listed in the next section), which will provide focal points.

7.2.4 Criteria for Formative Assessment in the Lesson Plan

The lesson plan, which is explained in the next section, plans to promote the students' capacity for logical explanation. The teacher plans to pay attention to the following points and help the students to recognize them individually and as a class.

- Do the students use diagrams to understand the problem situation?
- Can students show their own thinking using diagrams?

- Can they reflect on, justify, and analyze their thinking using diagrams?
- Can they express their thinking or thought process using words like "because," "as such," "for example," "if . . . , then . . . ," and "while . . . , then . . ."?
- What point of view do the students have for comparing various ideas?
- How different are their answers?
- How different are their expressions?
- What are the reasons behind their thinking?
- How much do they use prior knowledge?
- Can they recognize the value of comparing different ideas and appreciate the new questions that result from this comparison?
- Can they relate their knowledge to the problem being discussed?

These explanations support the content in the video for establishment of decomposition of numbers to prepare for multiplication in vertical form beyond repeated addition.

The video illustrates well how Mr. Muramoto's students actively participate in and contribute to the lesson by and for themselves. His deep consideration to develop students is explained in his original lesson plan in the next section.

7.3 Annex for Sect. 7.2: Excerpts of the Lesson Plan by Mr. Muramoto, Illustrating Why and How a Japanese Teacher Prepares School-Based Lesson Study

The previous exemplar with the video is an ordinary Japanese method to initiate multiplication in vertical form. It is the subtheme of this book. The subtheme explains the Japanese approach with the various theories behind lesson study which is mentioned Chap. 1. For lesson study, Japanese teachers usually have a research (study) theme and an objective for the lesson (Isoda, 2015a, 2015b), as discussed in Chaps. 1 and 5. The objective of the lesson is written for the specified teaching content in the curriculum sequence. The research theme is usually related to higher-order thinking skills such as mathematical thinking, values, and attitudes. In Japan, these are written as the general aim in the mathematics curriculum such as development of mathematical thinking and appreciation of simplicity. There are various Japanese theories[22] behind this, such as mathematical thinking for making clear the objectives of the teaching materials such as value, attitude, mathematical ideas and

[22] The Japanese theory for mathematics education has been oriented toward designing mathematics class for developing children who learn mathematics by and for themselves, and trying to explain, specify, and share the objectives and aims of every class; they also function as the assessment standards for teaching. The theory is used for designing mathematics classes to carefully recognize the aims of mathematics education in every lesson and its task, and how well embedded the aims are into every lesson and task sequence over several lessons in every teacher's planned curriculum. When compared with other countries, major differences can be seen in the curriculum and task sequences which have been prepared for enabling students to learn value and ways of thinking, and so on. The sequence is prepared to support extension and integration, which means reorganization of learned knowledge for extended situations (Chap. 1).

ways of thinking (Isoda, 2012, 2016; Managao, Ahmad, & Isoda 2017), and theories to establish the task sequence to set the opportunity for students to think by and for themselves.

In Japan, school-based lesson study (see Chap. 1, Fig. 1.5; and Chap. 5, footnote 13) is usually done for research and development in the school on the setting and targets of the school and under the subject groups under theoretical discussion. It clarify the comprehensive objectives of their mathematics teaching in the school. If non-Japanese teachers just observe the video, they may recognize some differences in the teaching methods from the activities of teachers and students. If they try to copy the activities as a method of teaching, they may experience difficulty and attribute this to cultural differences and so on. Such impressions may come from overlooking and missing perspectives such as the teaching materials with clear objectives, the established task sequence for the unit level, and the long-term sequence for human character formation. The Japanese approach is a cultural practice based on the theories behind these perspectives (Chap. 1).[23] Here, to illustrate how lesson study is carefully planned, excerpts from Mr. Muramoto's lesson plan as a part of school-based lesson study are presented.[24] The research theme of the school, the lesson study group and the teacher, and part of the lesson plans will be presented in the following order: the school and lesson study group vision in the setting of the school in relation to the research (study) theme, the unit plan with its objective, and the lesson plan[25] with its objective and assessments. The followings sited in small fonts are half of the excerpts from the original documents provided by Mr. Muramoto as for school-based lesson study. Here, the term "we" means his lesson study group at Maruyama Elementary School. In the followings, small font sentences are quotations or resume from his complete-specification lesson plan and normal fonts are commentaries.

7.3.1 Maruyama Elementary School Mathematics Group Vision and Mathematics Lesson Study Group's Goals

Japanese lesson study is oriented toward the aims and objectives of education in the curriculum. Mr. Muramoto explains his school's lesson study vision as follows:

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[23] It is not just a method of teaching that can be alternate other methods because it is proffered to realize the specified objective.

[24] As explained at Fig. 1.4 in Chap. 1, school-based lesson study is managed by the research department at every school. Subject-based lesson study is usually managed by teachers' societies for specified subjects/disciplines. National- or regional-level lesson study is usually supported by laboratory schools affiliated with universities. The subject-based and national levels lead the national curriculum reform and the establishment of theories for designing the school curriculum with known theories. Here, this is school-based lesson study and discusses a school mathematics curriculum. The Japanese aims of education are discussed in Chap. 1.

[25] In the school-based lesson study approach (*konaikenkyu* in Japanese), which produces a learning community of teachers under the leadership of the principal, the description given in this section is necessary as part of the lesson plan for the school-based lesson study.

The mathematics group's goals are those of elementary mathematics from the first grade through the sixth grade; that is:

- To establish learning with clear and systematic connections throughout the learning content
- To help children to acquire basic knowledge and technical skills regarding numbers, quantities, and geometric figures through mathematical activities; to promote the capacity for creative and logical thinking; and to promote the attitude of enjoying the activity and appreciating the value of mathematical manipulation, and its use in daily life[26]

7.3.1.1 Actual Setting of the Students in Maruyama

When we, the mathematics lesson study group, analyzed the students' scores on the achievement test in our school, we found that our students were above the national average in every domain in elementary school mathematics, although the drop in student achievement in the international context has become a topic of discussion in Japan.

7.3.1.2 Research Theme for Lesson Study

What kind of lessons develop students who can use what they have learned before to solve problems in new learning situations by making connections? For this question, preparation of teaching materials is the key.

7.3.1.3 Focal Points for *Kyozaikenkyu* (Preparation of Teaching Materials According to the Objective/Research on the Subject Matter) for Implementation of the Research Theme

We think that encouraging problem solving through mathematical activities will help us to reach this goal.

We think that teachers need greater clarity about how the topics of study are connected to one another. We need to think about how students can use previously learned content to solve problems in new situations and how different problem-solving situations require various forms of prior learning, and we need to use these ideas in the development of units and lessons.

To help the students to be responsible for their own problem-solving process, we think that students should be more aware of their own problem-solving processes and be able to articulate how they have made connections to prior learning and how they have used the ideas to solve problems in new situations.

Students acquire the capacity to think about their own diagrams and the number line, reflecting on their own problem-solving processes, determining what they understand and what they do not, and comparing their solutions with those of their classmates.

We think that students should not only focus on the accuracy or inaccuracy of their answers but also reflect on their own problem-solving processes. They have to understand that it is important to feel the genuine enjoyment of learning mathematics as well as getting correct or incorrect answers.

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[26] See Chap. 1.

7.3.1.4 Thinking About Assessments That Help Students to Be More Precise in Their Problem-Solving Processes

> We need to think about what points to pay attention to in assessing students' learning in the teaching process in order to help them develop the mathematical thinking that is necessary to carry out meaningful and effective problem-solving activities (see Sect. 7.2.4).

7.3.2 Support for Other Teachers in School to Improve Students' Learning

In school-based lesson study, teachers work as a team. The mathematics lesson study group in school also supports other teachers. Mr. Muramoto describes this as follows:

> We will administer tests to understand the current state of student learning. Giving tests not only is a way to understand the current state of learning but also can be useful if teachers use them to reflect on and improve their own teaching.

7.3.2.1 Necessary Communication with Other Teachers

> We share our essential approach with other teachers by demonstrating it through an open class. For example, it is important to encourage students to express themselves mathematically on what they have learned from each other in the classroom. Some examples of the capacities we want to develop are:
>
> - To be able to describe ideas using number lines and diagrams
> - To be able to manipulate concrete materials and explain their ideas to others
> - To be able to think about and understand the meanings of numbers and operations, expressing them in mathematical expressions
> - To be able to take notes that reflect students' thinking and points of view
>
> It is necessary for the mathematics group to engage in good communication with other groups in the school. Our assessment and vision of teaching and learning in the classroom is discussed because all staff members in the school can provide a consistent and systematic approach in educating our students as a whole.

7.3.3 To Promote Human Character Formation with Strong Hearts and Minds, Students Who Acquire This Kind of Competency Can Participate in the Classroom in the Following Ways

In the Japanese national curriculum standards, mathematics is a subject for human character formation, as well as other subjects. In relation to the theme of the lesson study, Mr. Muramoto and his study group teachers describe the subject as the progressive development of logical thinking, as discussed in the following sections.

7.3.3.1 Planning Consistent Development of Proficiency in Logical Thinking

At the end of the second grade, students begin to use expressions like "because . . ." to describe their reasons and support their ideas.

In the third grade, they begin to compare their own ideas with those of others, and the expressions they use are "My idea is similar to that idea, so . . ."

In the fourth grade, students use expressions like "for example . . ." and "because . . ." more frequently. Also, they begin to use hypothetical declarations like "If this is so, then . . ."

In the fifth grade, they can be more sophisticated in their statements—for example, "If this is . . . , then it will be . . . , but if it is . . . , then I think we can say . . ." under certain conditions.

Finally, in the sixth grade, students can begin to describe things like "It can be said that this is so, but in the situation . . . , . . . is much better" and begin to make decisions about how to choose a better idea.

We hope to see this capacity of expressing oneself mathematically more often in the classroom, and, as such, we would like to examine the current state of student learning more carefully.

We believe that feelings and emotions need to be incorporated into students' learning. The feelings and emotions we refer to here are the students' hopes and desires, as well as their feelings and emotions that are derived from their particularities, all of which are necessary for students to autonomously and actively involve themselves in their own learning. This includes feelings and emotions expressed through phrases like "I wonder why . . . ," "If that's so, then . . . ," "Is this always true?" and "There, I found it!"

These are some of the things we hope for and are trying to achieve. We believe that knowledge is gained through feelings and emotions, and that these will really help students to acquire solid capabilities and strong hearts and minds.

7.3.4 Survey of Students for Preparation and Challenges

In the School Based Lesson Study, teachers usually survey current status of their students for knowing reform direction, improvements and progress:

We carried out a survey about mathematics learning among third-grade students at Mayurama Elementary School for preparing lessons, and the responses were as follows:

Do you like mathematics?

- 50%: yes
- 44%: sometimes yes
- 5%: sometimes no
- 1%: no

The students who answered "Yes":

- I like calculations and enjoy them.
- Yes, I understand, it's entertaining.
- Because the answers are clear.
- Because I can listen to various ideas.

The students who answered "Sometimes yes":

- I like calculations but not problems.
- It is very difficult to construct mathematical expressions for the problems.

The students who answered "Sometimes no" or "No":

- I don't like problems.
- The tests are difficult.
- It is very difficult to construct mathematical expressions.

Maruyama's Elementary School third-grade students like calculations, but many of them feel they are not good at constructing mathematical expressions for the problems. Thus, the following ideas are used to develop units and lessons:

We would like to increase the number of students who think logically and provide them with the capacities they need to understand the structures of the problems using diagrams and the number line.

We would like to increase the number of students who are interested in listening to other students' problem-solving processes, thinking about whether the problem-solving process is similar or different, and being able to communicate it.

7.3.5 Exploring Topics That Students Learn in the Third Grade

The topics that students learn are the following:

- Addition and subtraction (3-digit numbers in vertical form)
- Multiplication (2- and 3-digit numbers multiplied by a 1-digit number using the algorithm)
- Division (its meanings and remainders)
- Large numbers (up to 10 million)
- Time and duration (meaning)
- Volume, length, and weight
- Characteristics of rectangles and squares
- Box forms (characteristics and nets)
- Tables and bar graphs (categorized data and construction of tables and bar graphs)

The key mathematical ideas and thinking that students learn in almost all domains of third-grade mathematics are to think about quantities in terms of how many times the unit of measurement is contained in the quantity.[27]

- In addition, subtraction, and large numbers, we take 1, 10, 100, etc., as the unit.
- In multiplication and division, we look at how many times a quantity contains the unit of measurement, and we look at dividing something by a number of units.
- In time and duration, volume, length, and weight, we see how many times something contains the unit of measurement.

  Using the big mathematical idea of how many times a quantity contains the unit quantity as the governing principle, we develop lessons that help to emphasize this idea, as well as thinking of daily lessons that will help to nourish this idea. For example:

- We develop lessons that help students to be aware of the connection between what they have learned before and what they are learning now, and to use previously learned knowledge to overcome obstacles in a new situation.

[27] This is the definition of multiplication by measurement (Chap. 3).

- We representing a problem situation with diagrams based on the idea of how many times a quantity contains the unit quantity consistently, helping students to understand the situation and the solution with greater clarity, and developing lessons that incorporate this idea to help them use the diagram to think logically about the solution to the problem.
- We develop lessons that help students to understand what they need to compare in various ideas, previous ideas, and representations such as diagrams.

  These students' understanding will be enriched through lessons that pay attention to the problem-solving process in which prior knowledge is used.

7.3.6 Challenging Issues for the Lesson Study Group with Viewpoints

Although the achievement of students at Maruyama Elementary School appears to be good, we recognize that there are many students who wait to receive instructions from teachers about how to solve the problems instead of doing that by themselves.

We do not think there are many students who indicate a strong desire to address challenging problems, saying, "I want to solve this problem on my own, even if it takes me a long time." Also, there are not many students who enjoy solving problems by trial and error.

We think this is the result of lessons that have not provided pleasant experiences in which the students reach solutions on their own, see interesting regularities or patterns in their investigations, think about this, and share questions that come up during learning with their classmates.

To develop students who can enjoy learning mathematics and acquire capacities for logical reasoning, which are the aims of the national curriculum standards, we decided to develop lessons with three viewpoints.

These viewpoints are discussed in the following.

7.3.6.1 Viewpoint 1: Teaching Material to Connect Unknown Content with Learned content

To develop teaching materials that pay attention to the connections between previously learned content and new content

It is necessary to clarify the mathematical thinking that the students have learned in the 6 years of primary school, by researching teaching materials and the students' processes of development. To do so, one must understand how previously learned content is necessary for learning new content, and how useful it is.

What students learned about multiplication in the second grade is precisely useful for calculation. The idea they learned regarding "how many times the unit of measurement a quantity is" is a fundamental idea of mathematics.

Also, in the second grade the students learn "length" by direct comparison, indirect comparison, and measurement with arbitrary units. So, the students who recognize the necessity of measuring with a universal unit can learn "weight" in the third grade using similar thinking.

The students who think about the "why" of the problem-solving process can begin to make connections between the problem and what they need to think about it, as well as what they need to think.

7.3.6.2 Viewpoint 2: Knowing the Significance of Own Ideas Through Comparison with Others' Understanding

Students can learn from each other and this helps them to think conscientiously about their own problem-solving processes.

There are many new things students can learn from each other in the classroom when they experience the real value of mathematics, its beauty, and its importance.

- Students can clarify their own problem-solving processes and participate in discussions to learn from each other.
- Students can learn through discovery by comparing their own thinking with that of others.
- Students can reflect and evaluate what they understand and what they do not.
- Students can clarify how they solve problems.

Learning experiences in the classroom that promote learning from each other not only improve student learning but also develop strong bonds among students.

7.3.6.3 Viewpoint 3: Prepare the Task Sequence with Formative Assessments

Assessment that promotes students' capacity for logical thinking

For the students to be capable of thinking logically, we think they need to clarify their own problem-solving processes when they are doing problem-solving activities.

First, so that students enrich their learning, we think it is very important that the teacher provides help in organizing the chalkboard and highlighting the lesson's important points.

Second, we want to plan appropriate help so that students feel the need to think about what prior knowledge they need to remember and can make connections to the new problem situation. Also, we want to include support questions to encourage students to think deeply about their problem-solving processes, understand each idea they produce (including the similarities and differences of these ideas), and expand the knowledge they can gain through working together.

Finally, we want to prepare a second problem that helps us to understand student learning during the lesson to support understanding of the effect of what students learn from each other in the lessons.

Considering the current state of learning of Maruyama's students and the content of the topic, we think it is important to develop units and lessons with these viewpoints in order to achieve the overall goal of developing students who can use what they have learned previously to solve problems in new learning situations by making connections.

7.3.7 Unit and Lesson Plans

The teachers intend to carry out:

- Lessons that prepare students to think conscientiously about the connection between what they have learned before and what they are leaning now
- Lessons in which students learn from each other and that help them to think conscientiously about their own problem-solving processes
- An assessment that helps to strengthen students' capacities for logical thinking

The specific unit goals are:

- To think about how to calculate the multiplication of 2- and 3-digit numbers by 1-digit numbers using the ideas about multiplication that have been learned previously (calculations with 2- and 3-digit numbers multiplied by 1-digit numbers using the idea of decomposing numbers in the base ten system)
- To be able to carry out the calculation of 2- and 3-digit numbers multiplied by 1-digit numbers using the algorithm

The content that the students learned before this unit includes:

- Multiplication of 1-digit by 1-digit numbers (second grade)
- Multiplication that involves zero, multiplying by tens (third grade)
- Using the idea of the distributive law of multiplication to create the multiplication table (for example, the multiplication table of 7 can be developed using the tables of 5 and 2)

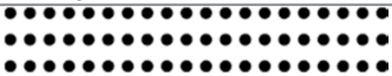
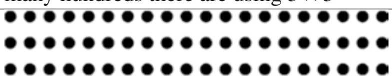
The lesson topic is:

- Third-grade mathematics lessons that promote students' capacity to use what they have previously learned and make connections for solving problems in new learning situations

The lesson learning goal is:

- To be able to think about how to carry out the calculation of a 2-digit number multiplied by a 1-digit number using what was previously learned about multiplication (mathematical thinking)

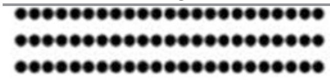
Unit Plan for 13 Sessions

	Learning activities
1	 How many • are there? Let's find out by calculating! Because we have 3 groups of 20 circles, I wonder if we can use multiplication. To calculate 20×3 or $20 + 20 + 20$, 20 is two tens. We can discover how many tens there are using 2×3
2	Let's think about the statement of the problem that shows the mathematical expression 20×3 "Each chocolate costs 20 yen. We buy three. What is the total price?"
3	If the price of an item is 300 yen, what is the mathematical expression? 300×3 This time we can think about how many groups of 100 there are. We can discover how many hundreds there are using 3×5
4	 How many • are there? Let's find out by calculating! This time a group has 23 circles. There are approximately 60 circles The mathematical expression should be 23×3. We cannot calculate it easily using the multiplication table. If we decompose 23 into smaller parts, then we could use the multiplication table. We can use an algorithm (a method of calculating with paper and pencil) to calculate. 9×3, 9×3, 5×3, together is 69. 10×3, 10×3, 3×3, together is 69. 20×3, 3×3, together is 69. Which of these ideas is easiest to calculate? They all decompose 23 into smaller parts

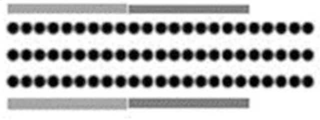
(continued)

	Learning activities	
5	Let's find out how to calculate using the algorithm (a method of calculating with paper and pencil). Think of 23 as 20 and 3. Put 3 × 3 and 20 × 3 together. Calculate using the multiplication table	23 × 3 69
6	How many • are there? Let's find out by calculating! The mathematical expression is 16 × 4. It should be greater than 40. It looks like it is greater than 40. We can make this calculation by decomposing 16 into 10 and 6 like we did before. Let's make this calculation using the algorithm. 6 × 4 = 24. We cannot write 24 in the units place. I wonder how I should write the number . . . We can write the 2 in 24 in the tens place	
7	Let's do a bunch of problems like ••• × •! Let's think about all the problems using the algorithm. Some of the answers give 3-digit numbers. There are answers where 0 appears in the tens place. There are problems that imply regrouping twice	
8	The price of a meter of ribbon is 312 yen. We buy 3 meters of ribbon. How much does the ribbon cost? What would an estimate for the answer be? It should be more than 900 yen. The mathematical expression is 312 × 3. I wonder if I can use the algorithm again for this. . . . If we decompose 312 into smaller numbers, we can calculate . . . 300 × 3, 10 × 3, 2 × 3, together is 936	
9	Let's do some problems like ••• × •! I do a problem in which the answer is a 4-digit number. I do a problem that implies regrouping	
10	Let's practice calculating with the algorithm!	
11	We can begin to calculate however we want. The price of a cake is 60 yen. There are four cakes in each box. If we buy two boxes, what will the total price be? I think we will need two mathematical expressions to solve this problem. First, we find the price of a box. 60 × 4 = 240. We have two 240-yen boxes; 240 × 2 = 480. First, we find the total number of cakes; 4 × 2 = 8. A cake costs 60 yen, so 60 × 8 = 480	
12	Let's practice!	
13	Let's review what we have learned in this unit	

Lesson Plan

Learning activities and anticipated student reactions and thoughts	Points to consider
 How many • are there? Let's find out by calculating!	To understand the task, help students to see the circles as "how many in a group" and "how many groups"
There are 23 circles in each row There are 3 groups of 23 circles There are more than 60 circles. We can discover the number of circles by counting or adding I wonder if we could use what we have already learned about multiplication. The mathematical expression should be 23 × 3	Before calculating, encourage students to estimate the answer
We cannot simply use the multiplication table to make the calculation. What should we do?	Praise them when they remember what they have already learned. Try to understand the students' various ideas by walking around the classroom

(continued)

Learning activities and anticipated student reactions and thoughts	Points to consider
	When you encounter students solving the problem using addition, ask them "Can you use multiplication to make this calculation?" Make sure you use the diagrams to represent how the calculations were carried out
S1: Decompose 23 into 9, 9, and 5 $9 \times 3 = 27$, $9 \times 3 = 27$, $5 \times 3 = 15$ $27 + 27 + 15 = 69$; 69 circles *S2*: Decompose 23 into 10, 10, and 3 $10 \times 3 = 30$, $10 \times 3 = 30$, $3 \times 3 = 9$ $30 + 30 + 9 = 69$; 69 circles *S3*: Decompose 23 into 20 and 3 $20 \times 3 = 60$, $3 \times 3 = 9$, $60 + 9 = 69$; 69 circles	Make sure you encourage students to share their various ideas and help them to make a conscious effort to make their own value judgment regarding the various ideas. If a student uses an algorithm to calculate, ask him or her to think about how this calculation method is related to the diagram
If we decompose 23 into smaller parts, we can use different multiplications from the multiplication table to make the calculations. Which of these do you think is a good idea? What similarities are there among the different solutions? All the methods decide to decompose 23 into smaller parts. There are methods that imply decomposing 23 into 3 parts and into 2 parts. The numbers used in the mathematical expressions are different. If we use the multiplication 20×3 that we learned before, we have two mathematical phrases. I use an algorithm (calculating with paper and pencil) to make the calculation 23×3. If we compare this method and the diagram, this method also decomposes 23 into 20 and 3. If we decompose a number into smaller parts, then we can use the multiplication table, making the calculation in today's problem. The idea we use in the algorithm (calculating with paper and pencil) is similar to the idea of decomposing 23 into 20 (2 in the tens place) and 3 (3 in the units place)	Make sure you highlight the idea of "making the calculation easier using the multiplication table and other ways of multiplying that we have already learned." if a student uses the algorithm, help him or her to consciously connect the idea of the algorithm to this idea

From the excepts of Mr. Muramoto's full-speck lesson plan, it is clear that the lesson plan is not written for illustrating the methods of teaching for copying; instead, it is written for answering why and what questions, such as why we need teaching materials and what teaching materials are needed for the specified students. If we share why, we can develop teaching materials with appropriate task sequences and clear objectives. These are the theories behind the explanation of the teaching activities in the Japanese problem-solving approach. It is not a method of teaching but a method to achieve the objectives with well-configured and sequenced teaching materials (see Chap. 1). What is necessary for the approach is a set of objectives and teaching materials that can be defined by the content and the task sequence with the aims and the objectives.

7.4 Multidigit Multiplication in Vertical Form: Task Sequence for Extension and Integration in the Case of Gakko Tosho

The previous section illustrates how a Japanese teacher introduces multiplication in vertical form with the example of 23 × 3. In this section, the task sequence (see Chap. 4) of multidigit multiplication after learning multiplication of a 2-digit number by a 1-digit number is illustrated to explain how Japanese teachers develop students who are able to extend their ideas by and for themselves by using what they have already learned.

In this section, the Gakko Tosho textbooks *Study with Your Friends: Mathematics* are referred to because these have been preferred and used in Thailand,[28] Mexico,[29] Indonesia,[30] and Papua New Guinea[31] on well-configured task sequences for extension and integration. This is the outstanding feature of the Gakko Tosho edition. The following sections include excerpts from a Gakko Tosho textbook for illustration of task sequences to explain the manner of extension and integration by students. Every task has an exercise for proficiency, but that is not described here.

7.4.1 Task Sequence for Extension

In the Gakko Tosho textbooks (Isoda and Murata, 2011; Hitotsumatsu, 2005), multidigit multiplication introduced Grade 3 in the following.

[28] This is the Open Approach Project by Maitree Inprasitha. By using the Thai edition of the Gakko Tosho textbook (Inprasitha and Isoda, 2010), he and his colleagues in Thailand produced a number of research articles under the name Open Approach such as in the Psychology of Mathematics Education (PME) and others. Their reteaches follows the Gakko Tosho textbook sequence under the Japanese national curriculum; it is called the problem-solving approach in Japan. When the Japanese say "open approach" this implies that the class is working with open-ended tasks (see Nohoda's open approach in Chapter 1). With regard to the task for the problem-solving approach defined by the task sequence in the textbooks in relation to the objectives under the unit plan in the curriculum, it is not necessarily the task should be an open-ended task; however, it produces various solutions like an open-ended task because it is posed as an unknown task for students in the task sequence. within students' reach (zone of proximal development (ZPD), Vygotsky, 1962). Students can challenge as long as they well learned the previous tasks under the curriculum and textbooks.

[29] The Pre-service Teacher Education Project for all teacher education colleges under the Ministry of Education, Mexico, by Marcela Santillan Nieto and Tenoch Cedillo Ávalos (Isoda and Cedillo, 2012).

[30] The Curriculum Center Project by the Ministry of Education, Indonesia (ongoing).

[31] The Japan International Cooperation Agency (JICA) Improving the Quality of Mathematics and Science Education (QUIS-ME) project by the Department of Education, Papua New Guinea (ongoing).

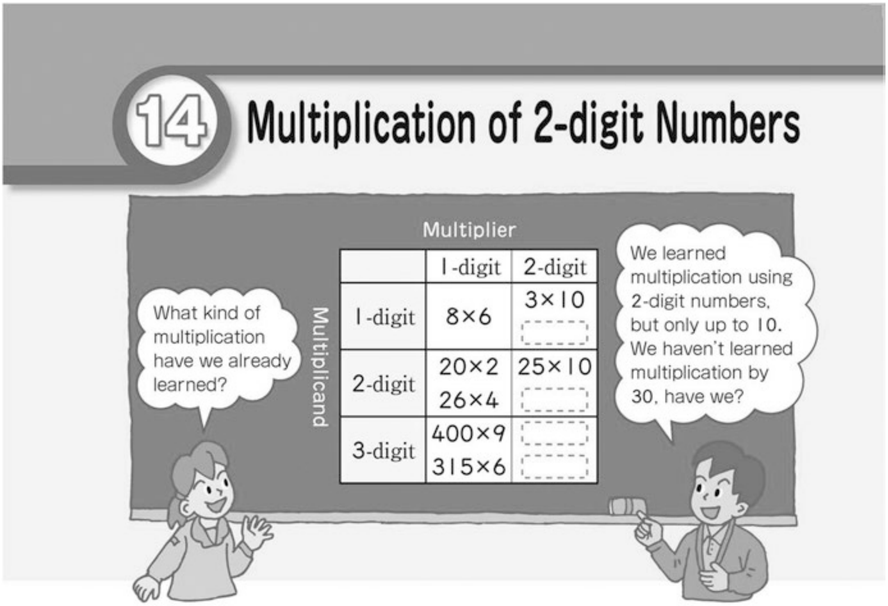


Fig. 7.14 Gakko Tosho (Hitotsumatsu, 2005), Grade 3, Vol. 2, p. 59; and Gakko Tosho (Isoda, Murata, 2011), Grade 3, Vol. 2, p. 63

7.4.1.1 Task 1: Extension by Students

In this task (Fig. 7.14), a teacher provides a two-dimensional table with empty boxies, and asks students to fill in examples and discuss what they have learned (such as when and how they learned it, and what they have not yet learned. In Fig. 7.14, there are the empty boxes indicate things they have not yet learned. The other filled expressions indicate things the students have already learned when they studied multiplication in vertical form at Grades 2&3. From this contrast, students recognize the necessity to extend the numbers for multiplication in vertical form to multidigit numbers. Some of the leading teachers ask students to plan and discuss their learning sequence too.[32]

7.4.1.2 Task 2: 4 × 30

The textbook provides an opportunity for thinking about how to calculate, which has been an aim in the national curriculum standards since 1999, as well as comprehensive understanding and fluency of operation. To meet this objective, the task sequence is established by the extension and integration principle (see Chap. 1).

[32] The Japanese approach oriented to develop students by and for themselves. Thus, leading teachers of Lesson Study usually demonstrate their ways to develop students to learn mathematics by and for themselves.

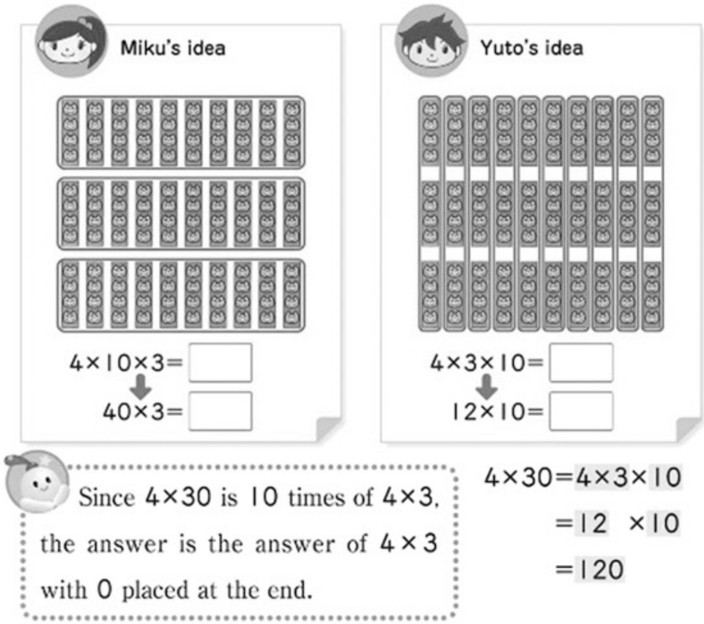


Fig. 7.15 Gakko Tosho (Hitotsumatsu, 2005), Grade 3, Vol. 2, p. 60; and Gakko Tosho (Isoda, Murata, 2011), Grade 3, Vol. 2, p. 64

At Grade 2, in Fig. 7.15, the task 2 for Grade 3 can be solved using groups of groups, 10 times. At Grade 2, students already learned T0 × U. The associativity of multiplication has already been learned (see Chap. 6).

7.4.1.3 Task 3: 21 × 13

Task 3 (Fig. 7.16) is a case without carrying. It is extended with carrying as shown in Fig. 7.17.

7.4.1.4 Tasks 4 and 5: With Carrying and with 0

In the case of these tasks (Fig. 7.17), students also use block diagrams to be able to explain to others; however, they are never expected to use manipulatives because manipulatives usually provide the opportunity for counting. From the early stages such as grades 1 and 2, students should be developed to be able to draw the diagrams. The task sequence continues on to multiplication of 3-digit numbers such as 123 × 32 and 385 × 35. The last task is 508 × 40 which needs to consider the treatment of 0.

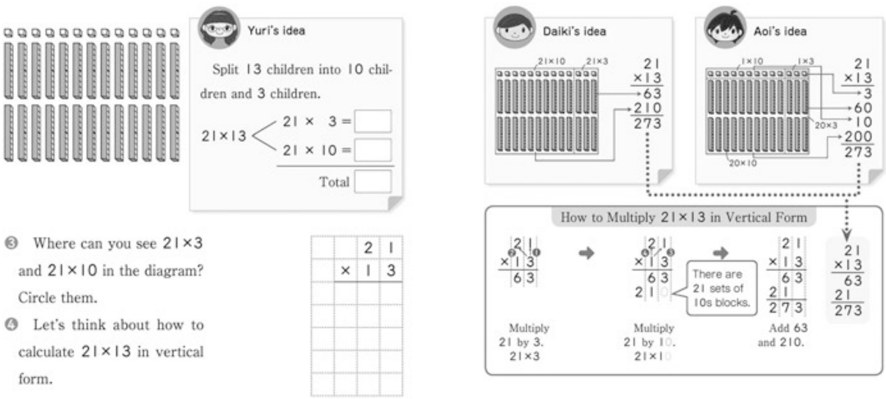


Fig. 7.16 Gakko Tosho (Hitotsumatsu, 2005), Grade 3, Vol. 2, pp. 61–62; and Gakko Tosho (Isoda, Murata, 2011), Grade 3, Vol. 2, pp. 65–66

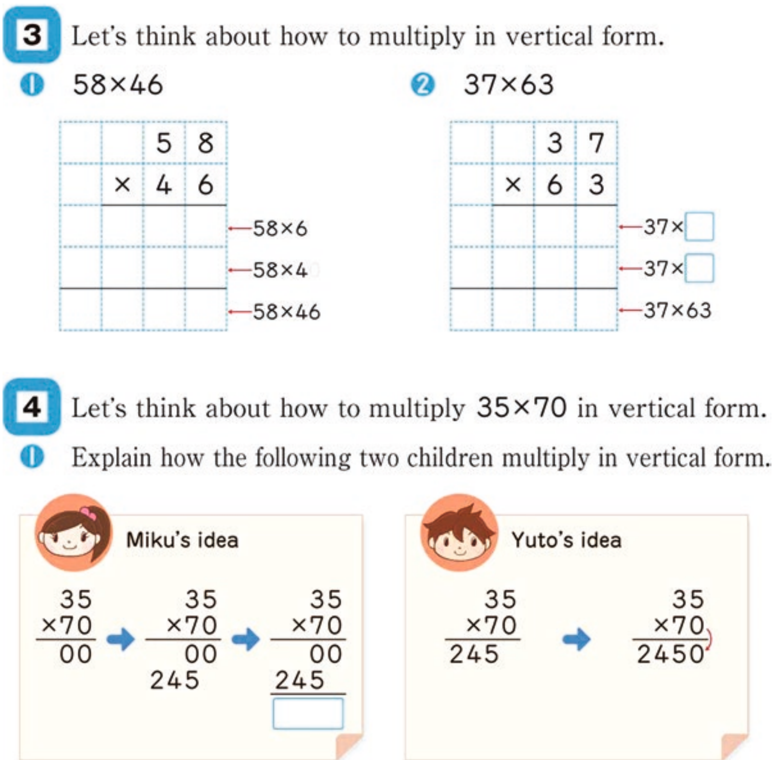


Fig. 7.17 Gakko Tosho (Hitotsumatsu, 2005), Grade 3, Vol. 2, p. 63; and Gakko Tosho (Isoda, Murata, 2011), Grade 3, Vol. 2, p. 67

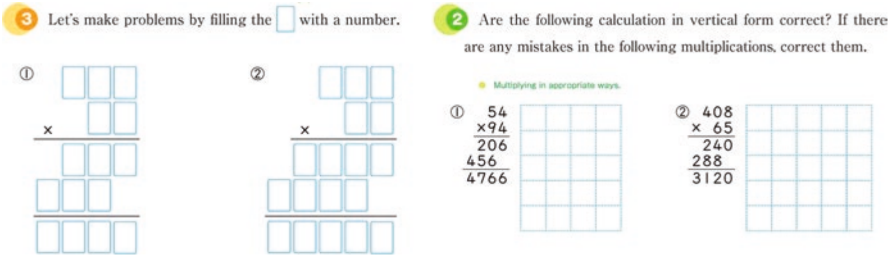


Fig. 7.18 The task on the *left side* asks students to pose the question to others, and the task on the *right side* asks students to find mistakes in others' answers. The tasks itself include the objective: Construct viable argument and critique reasoning of others on CCSS.MATH. (2010), USA

These tasks can be solved by using what the students have already learned from the previous tasks. The following Fig. 7.18 are parts of the last exercise (Fig. 7.18).[33]

In this task sequence, teachers ask students to use what they have already learned to justify their ways of calculation. For developing students who learn mathematics by and for themselves, a Task 1–style task is well known to focus their mind-set on the inquiry of extension. Tasks are sequenced for enabling students to extend their ideas by using what they have already learned. Posing question to others and critiquing other students' ideas are also enhanced in the textbooks.

7.5 Final Remarks

This chapter has illustrated the extension and integration of multiplication from single digit to multidigit by using vertical form with the base ten system and a multiplication table (see Fig. 1.1 in Chap. 1 from Meaning of B to Procedure B). It has also discussed how students are able to integrate the definition of multiplication by measurement (a groups of groups; see Chap. 3), which supports repeated addition, with the base ten place value system (see the allow '↑' on Meaning B n Fig. 1.1

[33] As mentioned in Chap. 1, the task sequences in Japanese textbooks are written under the extension and integration principle. Ordinal task sequence is from specific to general like Fig. 4.27 in Chap. 4. This process is also explained as the processes of both conceptualization of procedures and procedurization of concepts (see Chap. 1, Fig. 1.1). Gaining proficiency in the procedure is necessary for further conceptualization. Thus, there is a rich set of exercises at the end of every chapter in the textbook which maximize the proficiency for operations. The task sequence of the exercise for proficiency can be written from general to specific instead of from specific to general in cases; for example, after learned long division, the task sequence 85÷7, 68÷3, 54÷5 in exercise, is written in a general-to-specific form as for adaptation of an algorithm (Fig. 7.18, Gakko Tosho, 2011, Grade 4, Vol. 1, p. 46): On 85÷7, 8–7×1=1, 15–7×2=1; On 68÷3, 6–3×2=0, 8–3×2=2. On 54÷5, 5–5×1=0, 4. Isoda learned this from Prof. Tadao Kaneko at the textbook editorial meeting.

coming from the outside of figure). The students are confident of what they have learned about addition and subtraction in vertical form (column methods) under the base ten system (see the allow '↑' on Procedure B in Fig. 1.1).

Section 7.1 discussed historical possibilities explaining why multiplication in English-speaking and Spanish-speaking countries involves a contradiction, which has been recursively discussed since posed questions in Chap. 2 and it is the answer from the origin of contradiction for the questions.

In relation to the Japanese approach and lesson study discussed since Chap. 1, the explained task sequences in Sects. 7.3 and 7.4 were developed through experiences of lesson study by a number of teachers, and some of them were embedded into textbooks with the following hidden principles.[34] The first principle is the sequence of extension from the special/simple case to the general/complex case which enables students to use learned knowledge and develop their learning of mathematics by and for themselves (See Chap. 1, Fig. 1.1). The second principle is the task sequence of mathematical necessity[35] for enabling students to solve mathematical tasks by themselves, having further expectations of mathematical development and its integration. The third principle is the task sequence that enables students to appreciate their progress through collaborative problem solving with others.[36] Last principle is related with the objective: Construct viable argument and critique reasoning of others on CCSS.MATH. (2010), USA. These principles were explained by the general principle "Extension and Integration" on Japanese curriculum. On the context of re-invention principle by Freudenthal (1973), it can be said as mathematization because it asking students to reorganize mathematics by and for themselves. These are also seen in Sect. 7.3 which was written by Mr. Muramoto's lesson study group, prepared for his school-based lesson study.

Sections 7.2, 7.3, and 7.4 are an illustration of the Japanese approach to designing lesson study which is mentioned in Chap. 1 with its theoretical background. The example provided by Mr. Muramoto illustrates that Japanese lesson study is a reproducible

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[34] These principle is subsequents of general principle: extension and integration (see Chap. 1). The first principle can be recognized if readers read students' textbooks on the principle. The second and third principles are usually explained in teachers' guidebooks. The following textbooks are clearly written from the students' perspective: Isoda and Tall (2019), *Junior High School Mathematics*, Vols. 1–3, Tokyo, Japan: Gakko Tosho.

[35] In French didactics (Artigue, 2014), a priori analysis is also discussed to make clear the significance: in so many cases, it is based on pure mathematics. On the other hand, the Japanese terminology orients teachers to be able to distinguish conceptual differences in teaching content in the curriculum, such as different meanings of fractions. The mathematical necessity of introducing fractions is explained by using the terminology "dividing fraction, operational (measurable) fraction, and fraction with quantity" (See Chap. 4, Figs 4.22 and 4.23).

[36] In Mr. Muramoto's lesson, the students discuss different ideas for multiplying 23×3, such as $9 \times 3 + 9 \times 3 + 5 \times 3$, $10 \times 3 + 10 \times 3 + 3 \times 3$, and $20 \times 3 + 3 \times 3$. They appreciate every idea and the last one is more economical, being related to the base ten system and the memorized multiplication table. These insights are possible under the task sequence in his unit plan and the curriculum with acquisition as learned knowledge and skills.

science[37] (Isoda, 2015a, 2015b) that produces better practice through exploring "why" for objective and "what" for teaching materials with a sequence under the shared curriculum. In Chap. 1, the Japanese theories used to design lessons are categorized as follows: the theory to clarify the aims and objectives in every class such as the national curriculum standards and mathematical thinking, the terminologies to distinguish conceptual differences in teaching content, the theory to establish the curriculum sequence and task sequence, and the theory to manage the lesson such as Problem Solving Approach. Mr. Muramoto's lesson plan (described in Sects. 7.2.2, 7.2.4, and 7.3) was written based on these theories as background knowledge and shows how Japanese teachers deeply plan their lessons through the year.[38]

Sect. 7.5 illustrated the trajectory for enabling students to develop multiplication in vertical form beyond repeated addition using what they have learned such as the definition of multiplication by measurement, addition and subtraction in vertical form with the base ten place value system, and block diagrams with splitting for decomposing numbers. It is the exemplar for how Japanese teachers plan and teach learning how to learn as a part of human character formation by using what students have learned with considering how students extend their ideas for performing extended tasks. Japanese teachers who are engaging in subject based lesson study usually try to develop their lessons to develop the students who construct viable argument and critique of others. In Chap. 1, it is explained as Dialectic Approach on Fig. 1.4. Stigler and Hiebert (1999) en-lighted Japanese Problem Solving Approach through the comparison of classroom videos among USA, Germany and Japan. On the context of learners perspective study by Clarke, Keitel, and Shimizu, Sekiguchi, (2006) illustrated clearly from the perspective of Japanese classroom culture by using his analytical framework for classroom norms. Through the part I from Chap. 1 to Chap. 7, this book informed the unknown Japanese theories which teachers use for designing their practice to realize their objectives for developing students. It is the originality of this book as well as the meanings and roles of multiplication in elementary mathematics curriculum and its historical origin.

[37] Even though in this case, teachers use the same textbook and task sequence to minimize their preparations, the teachers have to reinvent the objective for the teaching content. By sharing Japanese theories through lesson study, they usually find ideas for teaching such as the meaning of a task that is really problematic for children and is the subject of discussion (See Chap. 1, Fig 1.3).

[38] This is the case in subject-based lesson study. Mr. Muramoto is a member of several subject-based lesson study groups, such as Isoda's lesson study group, based on the meaning (concept) and procedure mentioned in Chap. 1, Fig 1.1. In Japan, in the context of lesson study, teachers do not use the custom of writing references on their lesson plans like academic research papers even though they have studied various theories. In the teacher training program provided by the teacher training center, universities, and so on, they have a lot of opportunities to study design theories, which are explained in Chap. 1.

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Part II
Ibero and Ibero-American Contributions for the Teaching of Multiplication

Chapter 8
An Ethnomathematical Perspective on the Question of the Idea of Multiplication and Learning to Multiply: The Languages and Looks Involved

Claudia Georgia Sabba and Ubiratan D'Ambrosio

8.1 Introduction

One of the most delicate tasks performed by students and teachers continues to be learning about the objects and relationships of mathematics in school. Apart from this discipline having been referred to as difficult to understand for centuries, students are divided between those who fear it and those who adore it.

This passion for numbers is nurtured early on, not only in the school environment, but it must also be encouraged and developed through the guidance of elders who already perform more complex operations. The elders demonstrate to the young people the need to elaborate mathematical reasoning for resolution of daily problems, to use it as an object of leisure through hobbies, and to interconnect it with different areas of knowledge such as art, the exact sciences, the humanities, and the social sciences.

Mathematics is present in our daily routines. This causes all of us to use it consciously or unconsciously for the accomplishment of our tasks.

In this sense, one of the mathematical concepts used by all of us involves multiplying quantities. It is present in almost all of our tasks and in the classroom full of students trying to learn and their worried teacher. However, most children and adults perform this activity easily out of the school context, although some of them have not attended school or even have not learned to develop some mental calculation techniques; nor do they comprehend the use of algorithms for multiplication.

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It is also worth emphasizing that thinking about the learning of mathematical concepts in the scope of educating the human being (a noun) to be (a verb) human requires a process in which mathematics is worked on as a whole—that is, by including and relating to it during the learning of content by the students. From the teacher, this demands an attitude of an educator involving himself or herself more and more in the ethnomathematics program, since, in education today, we believe that the process goes beyond teacher–student dialogue. It involves knowledge of mathematics, social relations, and the use of technology, which are present in all our actions, even though they are not available to all people of the world. This is one of the great challenges of the present century.

The computerization of societies, contact with technology through social networks, and the use of technology as a form of registration through the image, and also as a way of expressing what is thought, are parts of the universe of the young people of today. The act of teaching using new technologies or virtual environments is just another way of bringing the mathematics that exists in the world into their objects and relationships, packaged in a language that is easily accessible for the young.

The choice of appropriate language to extend the understanding of mathematical content makes the students more interested in learning something that matches the realities in which they exist. The use of art through symbols, icons, and photographs is getting closer and closer to young people.

In this context, we want to report here some mathematical ways and thoughts about this task, which involves the most basic as well as advanced calculations, but which are part of the daily life of some professionals and students.

The elaboration of the concept of multiplication happens naturally in the history of man. The elaboration of an algorithm to facilitate the solution of a problem was a human creation. However, these practical devices are not yet in the domain of the entire world population, although, in practice, everyone has a way to perform a necessary or requested calculation.

Perhaps this difference in looking at it causes a gap between practical and theoretical knowledge. Since the practice of some teachers is increasingly related to their classroom, this may make education decontextualized.

In this way, some groups of educators are increasingly placed in a macroperception between scientific knowledge and the cultural knowledge of the social groups in which teachers and students exist. According to gestalt theory, the perception of the whole context in which the individual exists gives him or her elements to better understand mathematical knowledge.

Another way of thinking about this involves the concept map of a website, where anyone can have access to all of the concepts involved and only perceive it in a linear way in which they relate to it. Weaving through the web in all dimensions will require the interaction of the individual in his or her learning.

8.2 Alternative Modes

8.2.1 Project Learning

Escola Municipal de Ensino Fundamental (EMEF) Desembargador Amorim Lima was initially inspired by the Escola da Ponte to promote a major change in its political pedagogical project. Today, with its own identity, this Brazilian school has a slightly more flexible political calendar, thus allowing newly proposed workshop activities to be added to the basic curriculum, which is composed of research and execution of scripts elaborated by the school.

These completed scripts work as miniprojects. They are integrated into the different areas of knowledge about the researched subject in the sense of giving autonomy to different ways of learning and discovering the world at the rate at which students develop their activities.

The school's physical plan has been remodeled to carry out all the activities provided for in the political pedagogical project. The main building consists of three floors. On the ground floor, there is the computer room (which is well equipped with microcomputers and the internet), the capoeira room, the literacy room for first-grade children, and a repository of the teaching materials available to teachers. In the courtyard, there is a stage. On its wall are the principles of the school, written by teachers and students together. At the end of the inner courtyard, there is the canteen and, at its side, the stairs that lead to the classrooms. Through a corridor next to the canteen are the library and the art room. Accessing the school by the service entrance, the first room is for the secretaries of the school, the rooms of the teachers, the classroom of the first grade, the room of the pedagogical coordination, the boardroom (where meetings with teachers are held), and finally, the bathrooms (Fig. 8.1).

Going up the stairs from the inner courtyard on the first floor, there is a room for English workshops, two rooms for project realization, and the hall—inspired by the Escola da Ponte, formed by three linked rooms—where there are students from second to fourth grades. This hall has three living spaces and six computers, which are used by students doing research. The second floor is laid out similarly to the first floor; the only difference is the occupants. The activities here are for students in grades 5–8. Throughout the school building, there are works by the students. The walls are well painted. The rooms and halls are clean and organized between one activity and another. Green curtains complement the spaces, provide protection from the sun, and brighten the environment. Around the building, there is an outdoor area with courts, a skating rink, a vegetable garden (planted together by teachers and students; the food produced is used to prepare lunch in the canteen), a white event tent (which was donated to the school), and a wooden tipi (created by Guarani natives from the village Morro da Saudade), which was part of an exchange with the school.

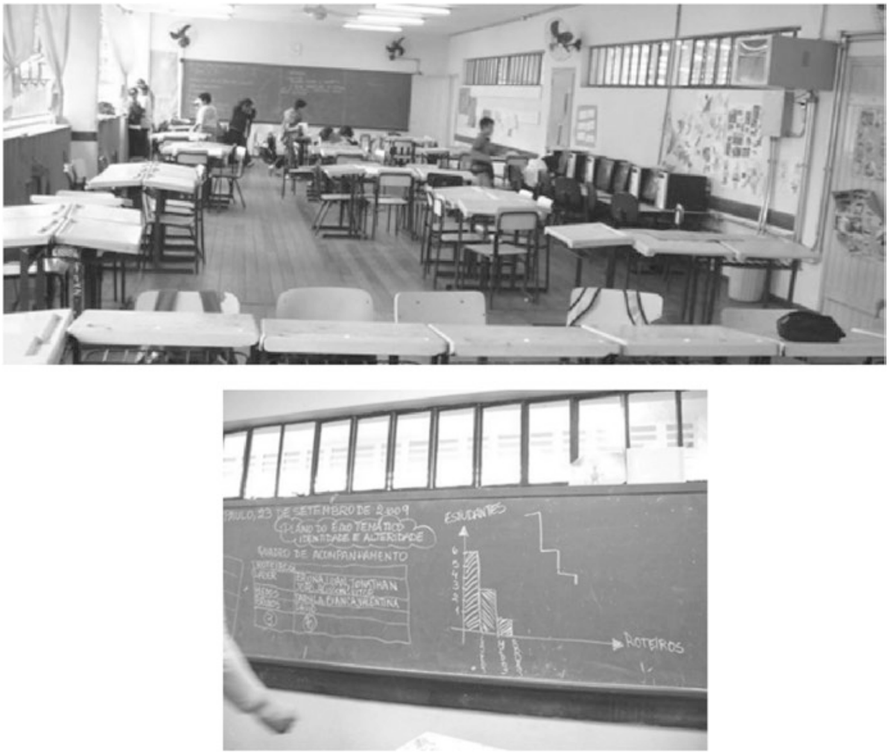


Fig. 8.1 The hall (*top*) and a blackboard (*bottom*) at Escola Municipal de Ensino Fundamental (EMEF) Desembargador Amorim Lima (2004)

As of February 2009, Prof. Dr. Geraldo Tadeu Souza introduced the fourth thematic axis: our world. In 2010, a fifth axis was added, so the work is organized as follows by the coordinating teachers of these axes:

1. *Alterity and identity*: This uses theatrical language, a resource that has been elaborated since 2007 with the help of a professor from the Faculty of Education (FE) at the University of São Paulo (USP), Brazil.
2. *Life*: This axis has been built through experiments and their documentation, under the scientific eye; for this, the school has spaces that allow this type of activity with groups.
3. *Our planet*: In this axis, the focus is on chemical experiments that are performed and recorded.
4. *Our world*: In this axis, art is predominant. Through activities that involve the construction of models and going to the cinema, the students produce reports in elaborate scripts.
5. *Literacy*: This axis was added to the project in 2010. It arose from the need for experience by practice and to give more support to the activities of documentation and communication through literacy. As described by D'Ambrosio (1999):

- LITERACY: It is the ability to process written and spoken information, which includes reading, writing, calculation, dialogue, eclogue, the internet in daily life [COMMUNICATION INSTRUMENTS].
- MATHERACY: It is the capacity to interpret and analyze signs and codes, to propose and use models in daily life, to elaborate abstractions about representations of the real [INTELLECTUAL INSTRUMENTS].
- TECHNOCRACY: It is the ability to use and combine simple or complex instruments, including the body itself, evaluating its possibilities and limitations and its suitability to diverse needs and situations [MATERIAL INSTRUMENTS].

It is important to emphasize that this proposal, as a whole, seeks to build knowledge based on the freedom to comprehend the world and to educate for responsibility and freedom.

At this pedagogical moment, the traditional teacher is replaced by a tutor teacher responsible for 20 students, arranged into groups of five. The purpose is to organize the research work to be done in the resolution of the scripts aside from assisting classmates to overcome some doubts.

8.2.2 Thinking of Multiplication Through Research Scripts

Although the mathematical content is not perfectly integrated into the projects in this school, in general, the questions cover all areas of knowledge. It is possible to think of the following topics for work:

Theme: Bees (second and third grades).
Questions
1. How are the bees arranged in the hive?
2. How are the combs arranged in the hive?
3. When observing the work of bees, can we say that there is some kind of organization and hierarchy in this environment?
4. Draw a piece of honeycomb on a 10 cm × 15 cm wooden frame. Describe some ways to organize mathematical thinking to count all the honeycomb cells.
5. How can we count the bees that are in the combs? Exemplify ways of counting by drawing it. Does knowing the number of cells help calculate the number of bees?
6. What are the most common types of bees? Where do they come from?
7. Would there be another geometric figure that could be used by the bees to build a honeycomb? Which one do you consider the best? Does the hexagon suit the honey bees?

This example makes it possible to explore with groups not only the ways of counting the number of cells but also the beginning of the organization of mathematical thinking by introducing some notions to begin the discussion of the concept of multiplication. So, if we are able to think of groups of 2 cells, we can count how many there are and carry out multiplication. Likewise, as we increase the group size to 3, 4, and so on, we will reach a maximum group where the children by themselves will understand that it leads to the need to create an algorithm.

As we can see in the schematic drawings, when we count by 2s, there is a large group to be marked.

When you choose groups of 3, the number starts to decrease, and so on. Sometimes the group of students themselves suggests choosing groups of 10 so that the counting becomes much easier.

However, when we choose groups larger than 10, it is easier to visualize the groups, but the counting begins to be difficult mentally. At this point, one may suggest or even begin to question the need to use a process or a way of performing calculations for numbers in this situation, so next we show how the multiplication algorithm works in order to facilitate the counting to be performed (Fig. 8.2).

Evaluation and self-evaluation: Elaboration of the activities and later complementation of the activities start in the classroom. At the end of each activity, it is recommended that students have a prearranged space to write their self-assessment of the knowledge presented. In this way, the student will say in his or her own words what he or she has learned that day, what still needs to be learned, and the curiosities or doubts that the lesson has generated, in order to research them later with the help of an older person, the internet, or the teacher.

It is important to emphasize that it is not only the work of observation, analysis, control, and evaluation that the tutor performs. At the same time, he or she also initiates the construction of deep links bound both to the learning of scripts and to the personal and collective responsibilities of the group.

At this school, it is possible to perceive the affection and the concern of the teachers for the students. Besides recognizing each one by name and always looking directly in their eyes when talking and explaining the activities and content to them, the teacher guides the groups during analysis and evaluation of the results. Perhaps this is one of the most humanitarian facets of education, which puts an end to indiscipline and allows students to respect themselves as well as their teachers and all who contribute to the administration of the school.

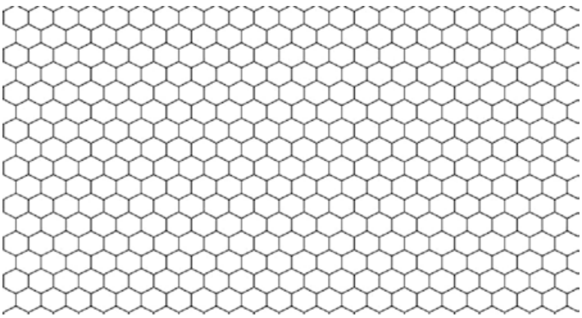


Fig. 8.2 Honeycomb

8.3 Multiplication: Tables with Polygons

In general, mathematical educators are very concerned with learning and teaching operations, especially with the concept of multiplication. According to Isoda and Olfos (2009a, b, pp. 46–55), in *Guías para la Enseñanza de la Matemática*, the concept of multiplication is approached by showing variations of how to represent a quantity that is repeated several times, going through concrete examples until it is abstracted for the calculation that one wishes to make.

In the Waldorf Schools of São Paulo, the concept of multiplication is built together with the geometry of flat figures through the elaboration of mathematical thought in conjunction with string figures created on a wooden table with nails (geoplan) and strings. These are designed especially for each child to be able to manipulate and understand the concept (Fig. 8.3).

Tag: Mathematical knowledge of the product of two natural numbers.
Topic: Learn and teach the product of two natural numbers.
Duration: For each multiplication table, it is necessary for the teacher to work through the mathematical content with the students. It is hoped that five lessons will be enough for each of them; however, this will depend on the knowledge of the group.
Content to be developed: Multiplication tables for numbers 1 to 12, polygons, relation of similarity, and direction.
Objectives: To give support for learning of multiplication tables, polygons, and similarity.
Methodology adopted: This class should be used after the teacher has already talked with the students and has exemplified and worked on individual activities on the concept of multiplication with an experimental class. This is followed by discussion in groups, formed by two or three students, for the elaboration of a geoplan, where the figures that geometrically represent the multiplication tables will be constructed. It is expected that from such discussion, geometric forms will emerge that will help the student to remember the results that are found.
Teaching resources: White board, pencil suitable for use on the board, kraft paper, piece of wood 25 cm × 25 cm, nails, string, hammer, thick wool, notebook, pencil, crayons.

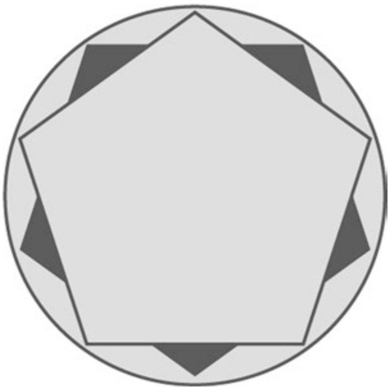


Fig. 8.3 Image of geoplan (Table 2)

Description of activities: 10 nails are hammered with equal spaces between them on the circle. With a string or woolen ball, we begin the multiplication table and with each number found, we take a turn on the nail. For the table of 2, $2 \times 1 = 2$, so we mark position 2; $2 \times 2 = 4$, so we mark position 4. This goes up to $2 \times 6 = 12$, so we take one turn and two more. In the same way, $2 \times 7 = 14$, so we reach position 4. We continue until $2 \times 10 = 20$, which falls in the tenth position. In this way, the figure formed is a pentagon, which is associated with the results of this table.

We consider this idea relevant because it works with the results of the multiplication tables associated with the number they represent and gives an opportunity to work with the geometric figures.

8.4 Multiplication using Art and Technology

Leonardo da Vinci, a Renaissance artist and scientist, developed interfaces between art and science, not only to transform painting into a faithful representation of the world around us, through the use of perspective and nature as templates, but also to invent machines to change some things in nature. In this way, he conceptualized many machines, such as flying machines and machines to move large volumes of land or to defend the nobles for whom he worked at that time. He explored, studied, designed, and reproduced, on canvas, the geology of soils and the structure of various plants and flowers. In his records and drawings of the human body, we can analyze his explanations and clear observations of how the human being is constituted, from the development of the fetus until birth.

In this context, bringing the figure of Leonardo into the classroom attracts the attention of young and old alike, both for the artistic value of his works of art and for the concepts explored in his inventions that inspire the desire to understand everything around us.

Sabba (2004) studied the works and the life of Leonardo, which show the integration between scientific knowledge and art, besides listing categories that could help the development of the human being as a whole by valuing emotions and sensations through categories such as corporality (*corporalità*) and sensations (*sensazione*) using art and drawings. These categories are already developed in schools such as the Waldorf Schools and the Sakura School in Tsukuba municipality, according to Isoda and Olfos (2009a, b, p. 55).

A good example of working in the classroom in this sense is the area of mathematics represented by the features that make up Leonardo's work *Vitruvian Man*. This presents a man inside a square and a circle, such that it is inscribed in the figures, besides having the body marked by proportions that make the human being so harmonious in our eyes.

Figure 8.4 shows the relationships that exist in our body—how it is possible to construct the height of the individual by knowing the size of his arm span and how there are symmetries between the parts in a certain direction.

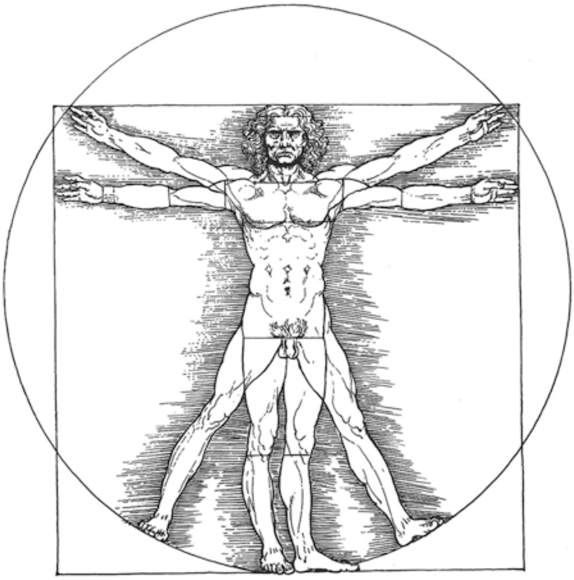


Fig. 8.4 Leonardo da Vinci's *Vitruvian Man*

An activity to be developed would involve pieces of kraft paper, the size of which would be slightly larger than the height of the student. In this way, a young man would lie on top of the sheet while another one would draw his outline with a marker pen. In this drawing, we would mark the student with his feet together and with his arms stretched out, without moving the upper body. Then, his outline is drawn in another position with his feet and arms spread half open, just like the position shown in Leonardo's *Vitruvian Man*.

With the drawings at hand, we would ask them to draw the proportions that exist in the body and, after that, a square and a circle around the body. First, we would observe the ideas of students on how they would perform the activity, and then explore the geometric relationships that would involve the square, the circle, and also where the square is inscribed or circumscribed.

When the teacher decides to use this drawing, some students may feel uncomfortable; thus, the activity may not achieve its purpose. In order to avoid this, we can substitute a different approach that children always use that would also produce the same effect. For example, using a cell phone for taking pictures, instead of drawings, does not pose any problems for young students; on the contrary, this procedure shows them how mathematics is part of our universe and how it is possible to work with mathematical content within the world of information technology in which the students exist.

Perhaps a teacher who was born before the 1990s may feel uncomfortable using this approach. However, the students will feel confident in photographing their friends in the requested poses and can still work with a ruler and compass to perform the calculations. It is possible to work out the proportions that exist on the face, as well as the relations between the body parts. For example, the foot is the size of the forearm, and the perimeter of a closed hand also corresponds to this measure; these are among many such relationships that exist in the human body.

Tag: Mathematical knowledge of division and products, construction of a square and circumference.
Theme: Learning and teaching of division and multiplication of integers.
Duration: It is expected that two lessons will be enough for the student to work through the mathematical content, with the help of the teacher in the second class; however, the teacher will decide, depending on the knowledge of the group.
Content to be developed: Relation of similarity, construction of the square and the circumference.
Objectives: To give support for learning of multiplication tables, polygons, and similarity.
Methodology adopted: This lesson should be used after the teacher has already talked with the students about Leonardo da Vinci and decided with them what methodology to use. Such activity is expected to show how the proportions construct the whole and make the symmetries of the figure more beautiful.
Teaching resources: Kraft paper roll, marker pen, whiteboard, pen suitable for use on the picture, notebook, pencil, colored pencil, mobile phone, computer, printer.

8.4.1 Multiplication Using the Calculator

Students are still prohibited by some teachers from using a calculator in tests and school activities, which at the end of their training entails a lack of knowledge of how to use and exploit such a tool that is necessary to facilitate the daily routines of many engineers, administrators, vendors, and others.

We believe that many activities can be done in order to facilitate conscious learning of the algorithms that we use to perform operations and also knowledge of the resources available to facilitate day-to-day calculations.

For example, the teacher usually asks students to solve problems that offer two numbers to multiply and find the result. Actually, in that case, the calculator solves everything and there is no room for many interactions.

However, if the educator offers the result of the product and one of the numbers involved in multiplication, the student will have to find the other number, which will lead to an inverted table exercise. For example, to solve the problem proposed in Fig. 8.5, without using the division algorithm, we must investigate the results of the three-dimensional table to deduce that it is a 3-digit number.

In this way, the student should think about which number, when multiplied by 3, will have a 2 in the units place. They will remember that if the number is 4, the result is 12. One group of 10 is carried over to the tens place, and 2 is in the units place. Again, remembering the results of the table of number 3, they will look for a number in the tens place such that, when multiplied by 3 and with the addition of 1 to the product, will give 2. This number is 7. Using the same analogy, the student would find the number 6. Therefore, the number sought is 674 (Fig. 8.6).

Still on this subject, after teaching the algorithm on multiplying 2-digit numbers or 3-digit numbers, the teacher could propose exercises, as shown in the Fig. 8.6, suggesting some of the numbers to help the others search Fig. 8.6.

Fig. 8.5 Multiplication algorithm

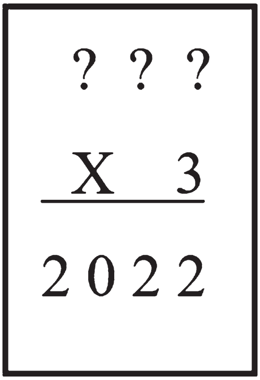
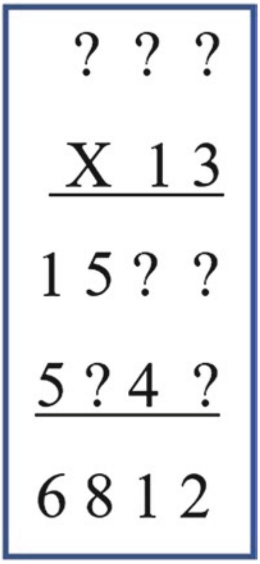


Fig. 8.6 ??? × 13 = 6812, finish the count



In the same way, the activities would allow an investigation on the part of the student, allowing him or her to think of several methods to solve a single problem as well as reminding him or her of the importance of the positional numeration that the numerical system possesses.

Tag: Mathematical knowledge of the product of natural numbers by units and by numbers greater than 10.
Theme: Learn and teach the product of natural numbers.
Duration: It is expected that each algorithm model will be worked on in three classes after the students have been familiarized with it; however, the teacher will decide, depending on the knowledge of the group.
Content to be developed: Multiplication of natural numbers, multiplication tables.

Objectives: To give support for learning of multiplication tables and algorithms for multiplication, developing logical reasoning.
Methodology adopted: This class should be taught after the teacher has explained each of the algorithms.
Teaching resources: Whiteboard, pen suitable for use on the picture, notebook, pencil, colored pencils.

In all activities—both those described here and those that the teacher has already incorporated into his or her classroom practice—it is possible to ask the student to make a self-assessment by writing a few lines about their improvement, presenting their ideas in front of the class, and requesting them to write down the questions or the important points learned that day. We believe that honest self-assessment on the part of the student—and, likewise, reading and adaptation of the teacher's practices—will aid in the success of teaching and learning outcomes.

8.5 Some More Ideas About Learning and Teaching of Mathematical Knowledge

Although the learning and teaching of mathematics are generally decontextualized from the reality of the students and teachers for different reasons, it is important to observe that there is a worldwide increase in the number of both teachers and students seeking ethnomathematics as a way to improve mathematical education and also to increase their self-confidence in the learning of educational content.

The ethnomathematics program allows integration of everyday knowledge to the scientific knowledge presented in schools and in unification the knowledge, as the human being is directly responsible for comprehension of their knowledge as well as for interfering in the environment around them and in reality, through the use of this knowledge. It is important for the teacher to contextualize each new topic in order to articulate this new knowledge in the context of the knowledge has already been acquired.

In the sense of articulating mathematical knowledge as a whole, the gestalt theory opens up a new vision. Thus, this theory applied in teaching shows how important macrovisualization of the object under study is, as well as visualization of its parts. It raises an important question in showing that the sum of the parts is different from the interaction of the parts. The whole—that is, the totality that gestalt refers to—can be understood as the articulation of various mathematical theories or examples which are sometimes presented without apparent connections but have the same theoretical basis. We call attention here mainly to the use of gestalt in mathematics.

For example, a student should imagine the object or the problem in question as a photograph. It is noted that the focus—the attention—of seeing the picture is not in the details but in the overall aspect that it conveys. The photograph of the whole is important, as is the visualization of the parts and their interaction.

The presentation of the "photograph" gives the students a sense of location—where I am and where I should go—in order to acquire/construct knowledge.

This indication is sometimes neglected mainly in mathematics and contributes to its decontextualization, since the students end up learning mathematics (the whole) as a list of rules and topics (the parts) to be assimilated and reproduced in a test without understanding that it is part of a greater whole. The same thing happens in the area of Portuguese language when a student writes an essay in which their use of periods is grammatically correct although there is no coherence between them; the writing will not make much sense even if its parts are perfectly correct.

For better understanding of the interaction of the parts, it is worth remembering the idea of a movie. Analyzing a movie frame by frame does not show the movement and dynamism that the movie provides unless the frames are shown at a certain speed. In the same way, the articulation of mathematical content provides the idea of other mathematics.

An example of development in the practice of this vision would be the study of relations involving the volume of a cylinder, a sphere, and two cones, all with the same height, as shown in Fig. 8.7.

Figure 8.8 shows the articulation between the volumes (V) contained in half of a sphere, a cone, and half of a cylinder, all with height R.

$V_{\text{sphere}} = (2/3)\ V_{\text{cylinder}}$

$V_{\text{cylinder}} = 3V_{\text{cone}}$

$V_{\text{sphere}} = 2V_{\text{cone}}$

This shows the proportion between the volumes.

In general, the teaching of geometry in elementary school is based on a narrative that starts from simple elements—points, lines, and planes—in search of constitution of an image as the whole photograph.

The fundamental message that gestalt theory suggests is that the perception of the photograph gives rise to an interest in the points that constitute it. Beyond all that, the interaction between the points and the photograph can be only minimally understood if we limit it to a one-way direction that leads from the points to the photograph.

From what has been considered up to now, in all activities—both those described here and those that the teacher incorporates into his or her class practice—it is possible to ask students to make a self-evaluation by writing a few lines about their improvement, presenting their ideas in front of the class, and requesting them to write down the questions or the important points learned that day. It is important for students to learn not only to multiply and understand mathematics as techniques that relate numbers, points, and lines, but also to use this knowledge to solve practical problems in life.

Fig. 8.7 Geometry

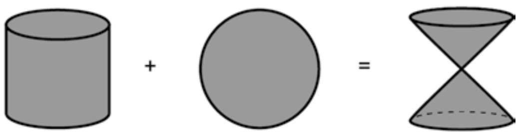
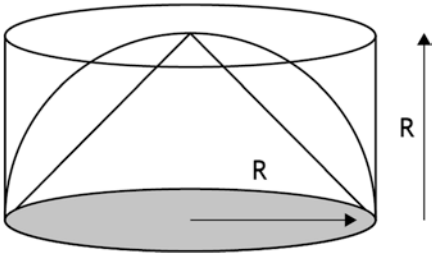


Fig. 8.8 Articulation between volumes



Foot notes

1. Geraldo Tadeu Souza is a professor at the Federal University of São Carlos (UFSCAR), Sorocaba Campus, and a doctor in linguistics from the Faculdade de Filosofia, Letras e Ciências Humanas (FFLCH) at USP.
2. Hernández (1998) explains that in a work project, the integrative aspect of knowledge construction violates the model of traditional education, so the transmission of knowledge compartmentalized and chosen by the teacher takes a much broader and dynamic context due to the interaction of meanings, resulting in active production of meanings and knowledge—that is, "a variety of actions of understanding that show an interpretation of the theme, and, at the same time, an advance on it." (Hernández, 2000, p. 184). The author further emphasizes that the project is not a methodology but a way of reflecting on the school and its function, which will present differences in each context.
3. EMEF Desembargador Amorim Lima is a public school in the city of São Paulo, SP, Brazil. Escola da Ponte is a Portuguese public school in Vila das Aves in the District of Porto, Portugal.

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Chapter 9
"Necklaces": A Didactic Sequence for Missing-Value Proportionality Problems

Laura Reséndiz and David Block Sevilla

9.1 Introduction: The Little Math Problem Factory

Multiplication or division problems that seemingly establish a relationship among three values are, in fact, "missing-value"[1] proportionality problems where a fourth value becomes involved (Vergnaud, 1988, 1990). For example, "If a pencil costs 3 pesos, how much would 5 pencils cost?"

Pencils	Pesos
1	3
5	x

Three problems (one multiplication and two divisions) can be obtained, based on these four values, by shifting the position of the unknown value.

Multiplication	
Pencils	Pesos
1	3
5	x

Partitive Division	
Pencils	Pesos
1	x
5	15

Quotative Division	
Pencils	Pesos
1	3
x	15

Problems solved through division present different relationships between numbers: in partitive division, 15 pesos are distributed equally among 5 pencils; quotative division consists of finding out how many 3 pesos groups can be made

[1] Also known in the old ratio and proportion theory as "fourth proportional" problems.

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from 15 pesos. Various studies have shown that young children see considerable differences between each type of division. When unaware of which problems require division, children approach these problems in very different ways (Nesher, 1988; Martínez & Moreno, 1996, and others).

The three problems above show the unit value (the price of one pencil), as a given value (for multiplication or partitive division) or as the object of a question (quotative division). There is a fourth type of problem where the unit value is neither requested nor provided, as shown in the table below.

Division/multiplication	
Pencils	Pesos
3	15
7	x

This is a typical missing-value proportionality problem. There are several ways to solve it, as will be shown later on, but all methods imply one division and one multiplication. In general, these problems are more complex than the previous ones.

Primary school students are expected to learn to solve all four types of problems, which, along with other types of problems, belong to the conceptual field of multiplicative structures.[2] While some circumstances may require students to approach these problems separately, simultaneous learning is convenient in other circumstances. The "necklaces" sequence presented in this article explores this possibility. Next is a description of the sequence followed by the results of its application with a 4th grade primary school group (9 and 10 years old).

9.2 "Necklaces": a Didactic Sequence

The following sequence is an adaptation of Guy Brousseau's original idea as developed in B. Mopondi's doctoral thesis[3] (1986).

(a) The setting

The setting is a factory that produces necklaces based on an initial "sample necklace." Each sample necklace has a certain number of different-colored beads. The samples vary depending on the number of beads of each color. For example (see Fig. 9.1), 1 necklace has 2 blue beads, 1 red bead, 4 green beads, and 3 yellow beads.

Before making n necklaces from a given sample, the factory requires a purchase order listing the exact number of each type of bead. Both the number of beads used in the sample necklace and the ones in the order can be organized into tables such as the following one.

[2] The conceptual field of multiplicative structures is made of "situations that can be analyzed as simple or multiple proportion problems and that usually require multiplication or division" (Vergnaud, 1988).

[3] In Mexico, an adaptation of this situation has been published in Block, Martínez & Moreno (2013).

Fig. 9.1 Sample necklace

1 necklace	15 necklaces
4 blue	60 blue
9 red	135 red
7 green	105 green

(b) The type of problem

In missing-value problems, every element in what we will now call the "initial" set is matched to an element in the "final" set. For example, a given number of pencils corresponds to a certain amount of money—to a cost. In the necklace problem, each element or each number of necklaces in the set is matched with the numbers in the final set that represent the number of each color of bead required for that number of necklaces. For example, 15 necklaces require 60 blue beads, 135 red beads, and 105 green beads. Meanwhile, the unit value for this necklace is composed of several values: 1 necklace → (4 blue, 9 red, 7 green). We call this a "one-to-many" relationship.

One-to-one relationship	
Initial set pencils	Final set pesos
3	15
7	x

One-to-many relationship	
Initial set necklaces	Final set beads
1	(*4 blue, 9 red, 7 green*)
15	(x, y, z)

A known example of this kind of relationship is the typical school problem where a certain number of students are matched to certain amounts of ingredients in a cooking recipe.[4] However, necklaces are more tangible and familiar to students than recipes and provide empirical ways of verifying results.

As we discuss later in this chapter, one-to-many relationships lead to a greater wealth of relationships among elements than problems with "one-to-one" relationships.

(c) Didactic variables and situation sequences

When modified, didactic variables may increase the difficulty or trigger changes in the strategy or procedure used to solve a situation (Chevallard, Bosch, & Gascón, 1997, p. 216). In the "necklaces" situation sequence, the main variable is the presence or absence of unit values (beads in a necklace). During the first stage, situations provide the unit value, while situations in the second stage do not. Other variables are as follows: (1) if unit values are absent, the relationship between two numbers

[4] A proportional relationship between a number of boxes and a number of objects, when all boxes have the same quantities, is probably the first relationship that students recognize and use in primary school. For more on the role of contexts or settings in identifying proportionality, see Burgermeister and Coray (2008).

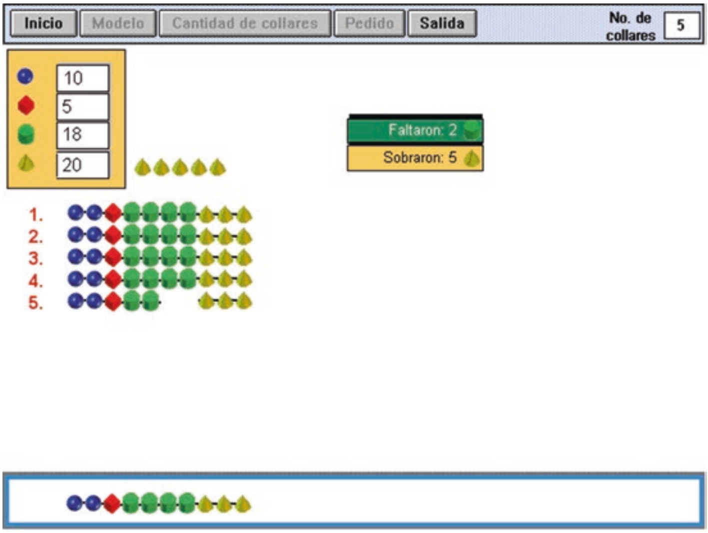

Fig. 9.2 An example of a mistaken resolution displayed in the computer software

of necklaces could be whole or not; and (2) the number of beads could be small or relatively large. The table provided in the Annex offers examples of both types of problems and their characteristics.

(d) Verifying results using computer software

These didactic situations are designed to give students means of empirically verifying their results while assisting them in finding errors and ways to correct them.[5] At the end of each exercise, students enter the following information into a computer program[6] specifically designed for this purpose: the number of each color of bead in the sample necklace, the number of necklaces they are making, and, finally, the number of each color of bead in the order. The computer then displays the sample necklace, all completed necklaces, and any leftover or missing beads (see Fig. 9.2).

(e) The purposes of the sequence

The sequence has two intended objectives: (1) to improve knowledge of division and multiplication with natural numbers by placing them in problems implying distribution (division), multiplicative comparison between quantities (division), and addition of given quantities several times (multiplication); and (2) to enable students to develop different procedures for calculating missing values in relationships between proportional quantities, specifically the unit value procedure.[7]

[5] For more on validation procedures in didactic situations theory, see Brousseau (1998).

[6] The software was designed by a team from General Academic Computing Services (Dirección General de Servicios y Cómputo Académico (DGSCA)) at the National Autonomous University of Mexico (Universidad Nacional Autónoma de México (UNAM)).

[7] Unit value procedures are those where a value corresponding to a single unit is calculated in order to get the value that corresponds to any number of units.

9.3 Applying the Sequence

9.3.1 Methodology

(a) Theoretical framework

This study is based on a methodology known as "didactic engineering" (Artigue, 1995; Chevallard, et al. 1997, p. 213), which belongs to didactic situations theory (DST). Didactic engineering is known for its experiment structure "based on conceiving, creating, observing, and analyzing sequences in teaching" and also for its way of achieving results by confronting data obtained through experimentation with previously formulated hypotheses. Artigue (1995, p. 36–37) calls it "internal validation."

(b) Conditions

School and Students We worked with a 35 students group (aged 9 and 10) of 4th grade primary school, in Mexico City. According to official Mexican curricula, year 3 and year 4 mathematics is focused on learning multiplication and division. An expected outcome throughout the sequence was for students to strengthen their grasp on multiplication and division. On the other hand, during the second stage of this research, students were asked to calculate intermediate values, which can be significantly more complex than simple multiplication and division. According to official school programs (Secretaría de Educación Pública, 1993), these problems are meant for Fifth grade curriculum. However, given the outcomes of the first stage of this research and given the context (based on a factory where necklaces are made with beads), it was decided that Fourth grade students would be able to approach these problems. As we will learn in the following pages, this was not the case for all students.

Duration The experiment took place during nine 60- to 90-minute sessions over a period of 2 months. The project leader guided the activities in the classroom. Five observers were assigned to log each session and to follow more or less seven participating pairs of students as well as group activities.

Development Some aspects were consistent in every situation: the teacher read the instructions, students worked in pairs for between 15 and 30 minutes, then results were verified followed by a group discussion. Result verification took place in two stages: (1) after completing their tables, students would review their partners' results; and (2) any remaining doubts could be verified by entering the quantities into the computer.

The Available Technology Two portable computers and one projector for all sessions.[8]

[8] Two computers were insufficient for the number of students we worked with. Despite implementing measures to help the process, time was lost while students waited for the two computers in the class to turn on.

9.3.2 Results

These results have been reported in Block (2001) and in Reséndiz (2005).

This section is divided into three parts: first, the students' answers to problems where the relationship was 1 necklace to *n* necklaces; second, solutions to problems where the relationship was *n* to *m* necklaces (where *n* and *m* were greater than 1); and, finally, a brief discussion of the feedback from each situation, which turned out to be both problematic and interesting at the same time.

9.3.2.1 Problems Where the Relationship Is Between the Number of Beads in One Necklace and the Number of Beads in *n* Necklaces (the Unit Value Is Given or Present in a Question)

During sessions 1 and 2, students were asked to solve basic tables, which allowed most of them to understand the problem and become familiar with table formats and computer software.

Table 1-B	
1 necklace	Order for ___ necklaces
4 blue	16 blue
6 red	24 red
5 green	20 green

Table 2-A	
1 necklace	Order for ___ necklaces
3 blue	___ blue
___red	48 red
___green	56 green

We will focus on situations solved during sessions 3–5, where six tables were presented and each table led to three problems: one partitive division, one quotative division, and one multiplication.

Table 3-A	
1 necklace	__ necklaces
3 blue	15 blue
6 red	__ red
__ green	30 green

What to Calculate First?

When solving this variant, the first challenge is knowing where to start calculating the number of necklaces in the order. While most students figured this out on their own, some took longer than others, as seen in the following example.

Table 3-B	
1 necklace	___ necklaces
3 blue	24 blue
9 red	___ red
___green	8 green

Silvia: “Do we have to put 3 beads in each necklace? I don’t understand how the sample is made.”
(The observer drew the sample with 3 blue beads and 9 red beads, and told Silvia that they still didn’t know the number of green beads needed.)
Silvia: “So each necklace has 3 [blue beads]?” (They drew necklaces with 3 beads until they reached 24 beads. They answered that there were 8 necklaces in the order.)

Finding the Number of Necklaces: Different Approaches to Quotative Division

Students attempted a range of procedures for finding the number of necklaces, from iterated necklace drawings to using the conventional division algorithm.

Iterating the Blue Beads from the Sample Necklace with Drawings Claudia and Yadira iterated a sample necklace that had 3 blue beads. They did it over and over while counting in their minds until all 15 blue beads stated in the order had been used. They reached the result by counting the number of times they iterated 3 blue beads. This procedure was common during the first tables in the exercise but later disappeared.

Repetitively Adding the Number of Blue Beads in the Sample This procedure took a step further than the previous one because it occurred on a numerical level: knowing that the sample had 4 blue beads and the order had 36, students added the number 4 several times until they reached 36 (see Fig. 9.3), then they obtained the number of necklaces by counting the number of addends.

Juan Carlos or Alejandro (one of them): “If we multiply 4 times 10, we get 36.”
(The observer asked them to try again and pointed to each 4 as they added them up. She stopped when they reached 36, and they counted the 4s they had used.)
Students: “9.”

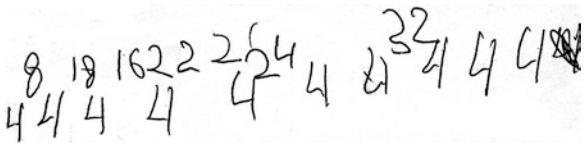

Fig. 9.3 Repetitively adding 4 beads to reach 32

Finding the Unknown Factor in Multiplication This procedure became more and more frequent. Some students recognized that the problem required division. Students used different approaches to find the unknown factor, depending on the size of the quantity in question: successive approaches, multiplication tables, decomposition of known factors, and others. As we will see in the examples, these forms of division were also an opportunity to practice multiplication.

- Successive approaches

Table 3-B	
1 necklace	__ necklaces
3 blue	24 blue
9 red	__ red
__ green	8 green

Francisco (doing "times 3" multiplications out loud): "3 times 5 is 15; 3 times 6, 18; 3 times 7, 21; you can make 7 necklaces and have beads left over."
[. . .]
Observer: "Remember that you should have the least number of beads left over or preferably none at all."
Francisco: "Let's see, 3 on each necklace? We can make 7 and have 3 leftover beads. Oh, no, we can make 1 more, so it's 8 necklaces." (He wrote down "8 necklaces.")

This procedure was frequently used in written form by students completing the last two tables, which was likely due to an increase in the number of beads used in these tables. For example:

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	__ red
__ green	126 green

(Fig. 9.4 shows what the students wrote on the back of their Table 3-E and 3-F worksheets.):

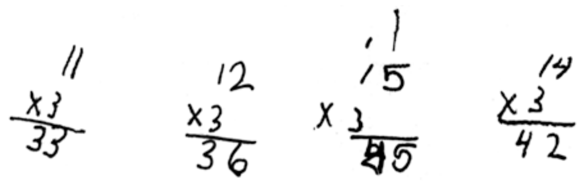

Fig. 9.4 Successive multiplications to find the unknown factor

Some students started by multiplying the number of blue beads in the necklace 10 times. Below is another example.

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	__ red
__ green	126 green

[. . .]

Roberto: "First I did 3 times 10 and got 30, then 3 times 11, and then 3 times 12 until I got to 14; 14 times 3 is 42."

- Using multiplication charts

Looking up quotients on multiplication charts is not a simple task. It requires an understanding of connections between quantities. The following example shows how Francisco and Adriana found the number of necklaces in Table 3-C by searching for a number that when multiplied by 5 would equal 30.

Table 3-C	
Sample	__ necklaces
5 blue	30 blue
3 red	__ red
__ green	24 green

Francisco and Adriana: "It's 6 necklaces because 5 times 6 equals 30." (They showed the observer their multiplication chart and said they used it to find the result.)

Observer: "How did you look for the results on the chart?"

Adriana: "On the '5' chart," (she ran her finger along the 5 row until she reached 30 then moved up along the corresponding column and showed the observer the 6) "so it's 5 times 6 equals 30 . . . then 3 times 6 is 18. We can see it on the '3' chart," (she ran her finger along the 3 row until she reached 18 then moved up along the corresponding column and showed the observer how it intersected with the number 6) "and for the other one, 4 times 6 is 24."

- Using the conventional division algorithm

Some students recognized that the problem was asking them to divide. As the quotient became larger, some of these students used the conventional division algorithm, though not always successfully (see Fig. 9.5).

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	__ red
__ green	126 green

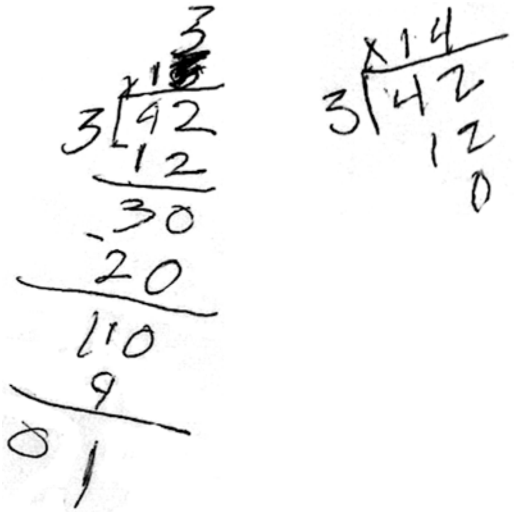

Fig. 9.5 Conventional division algorithm

Finding the Number of Beads in a Necklace from the Numbers in ***n*** Necklaces: Partitive Division

Students' informal procedures reflected subtle differences in comparison to quotative division procedures employed during the previous stage, which confirmed their awareness of differences in relationships between quantities (Martínez & Moreno, 1996). Let's look at an example of cyclical distribution.

Claudia and Yadira's worksheet showed the following procedure (see Fig. 9.6).

Table 3-A	
1 necklace	__ necklaces
3 blue	15 blue
6 red	__ red
__ green	30 green

The pair started by drawing 5 necklace "strings" (horizontal lines) and distributing the green beads one at a time. When they finished, they counted the green beads on each string. There was a significant difference between this procedure and their procedure for the previous problem where they drew strings with 3 beads each until they used up all 15 beads and then counted the number of strings.

Another observation was that more students attempted to solve these problems (partitive division) using the division algorithm than in previous problems: at least five pairs used it while only two had used it to solve quotative division problems.

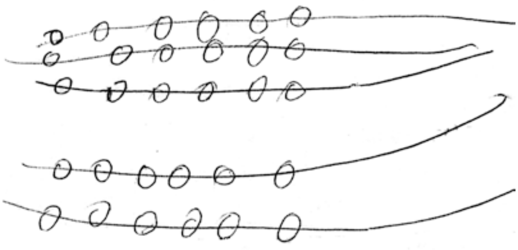

Fig. 9.6 Distributing the green beads one by one

This increase may have been a consequence of elementary schools using more partitive division problems to teach division.

Finding the Number of Beads in *n* Necklaces from the Beads in a Single Necklace: Different Ways of Solving Problems That Require Multiplication

The following problem required simple multiplication. Again, students solved these problems through a variety of methods, from iteration and counting drawings to repeated addition and multiplication. Even without having previously used the algorithm, students could distinguish multiplication problems more often than division problems. These students used the algorithm most, especially when trying to find larger quantities. In this section, we will look specifically at the explicit use of multiplication.

Multiplying the Number of Red Beads in the Sample Necklace by the Number of Necklaces Students used this procedure more than any other. As they progressed toward the final tables, they found themselves needing to write their calculations on paper. Some students used multiplication charts.[9] In the following example, students went as far as decomposing a multiplication factor.

When multiplying 6 by 14, Corelma and Luis implicitly decomposed a multiplication factor as follows: $14 \times 6 = (10 + 4) \times 6 = (10 \times 6) + (4 \times 6) = 60 + 24$.

Table 3-E	
Sample	__ necklaces
3 blue	42 blue
6 red	___ red
___green	126 green

$$\begin{array}{r} 60 \\ \underline{+24} \\ 84 \end{array}$$

[9] Also known as a "Table of Pythagoras."

Incorrect Procedures Only a few of the students who correctly calculated the number of necklaces displayed procedural errors when attempting to solve the second problem. Errors were apparently the result of not being able to discern what quantities were represented in the table. The following example shows what happened when David and Karina attempted to solve Table 3-B.

Table 3-B	
Sample	__ necklaces
3 blue	24 blue
9 red	__ red
__ green	8 green

David (looking at the worksheet): "It's 8 necklaces because 3 times 8 is 24." (They wrote "8 necklaces" in the table.)
Karina (looking at the worksheet): "8 necklaces? 9 times . . . 9 times 3 is 27, then it's 27, and we need 9 green beads."
Observer: "Are you sure that 8 necklaces require 27 red beads in total?"
David and Karina: "Yes . . . can we check it on the computer?" (They verified their results and saw that their necklaces were incorrect.)

Karina misread the number of necklaces (she said "3" instead of "8") despite David writing down the number of necklaces and despite both of them repeating the number out loud. What happened? She seemed unsure of what each quantity represented. In general, students expressed difficulty adapting to changes in table structure and therefore in the position of the missing value.

However, the results obtained from this set showed that these problems were suitable for 4th grade, as the students were able to approach these problems, develop procedures, and improve them in the process within a context rich in multiplicative relationships.

9.3.2.2 Problems Where the Relationship Is Between the Number of Beads in *m* Necklaces and the Number in *n* Necklaces (the Unit Value Is Neither Presented as a Known Quantity nor Requested)

In the final two sessions, the students solved tables like the ones below:

4 necklaces	7 necklaces
8 blue	
12 red	
20 green	
4 yellow	

4 necklaces	7 necklaces
12 blue	
8 red	
4 green	
20 yellow	

The First Challenge: Understanding that the Number of Beads in the Sample (the Unit Value) Is Not Provided and Is Not Being Requested

During the first of the final two sessions, 11 of 17 pairs—70% of the group—solved the problem as if the four-necklace order (first column) was the sample. They repeated the procedure that had led them to solve previous tables (where the unit value was explicit), which were presented in a very similar format (two columns with missing data).

> "We did the same as usual and multiplied these numbers," [the numbers in the column under the order for 4 necklaces] "times this number" [7—the number of necklaces in the order on the right side of the table]. "I don't get it . . . we have to find this order," [on the right side of the table, 7 necklaces] "with this one?" [on the left side of the table, 4 necklaces].

They were unable to distinguish between the sample and the order. The instructions were repeated using drawings as visual aids. However, some students repeated the same incorrect interpretation the second time. What caused these difficulties? It appears that the problem was not in the instructions but in the notion that both orders came from the same sample, which was unknown and needed to be figured out—or, as was expressed by a student when he finally understood the situation:

> "Oh, I get it! This," [the 4-necklace order] "is not the sample, and we need the sample. How do we find it if it's not there anymore? Do we need to find other numbers?"

As a didactic variable, withholding the problem's unit value considerably increased the degree of difficulty.

Procedures

Once they understood the problem, the students were able to calculate the sample and the order without expressing further difficulties. Informal procedures were common throughout the two sessions: some students used graphic representation for assistance with divisions and, on a lesser scale, for support with multiplication. Most students estimated using multiplication to solve divisions, while very few used the conventional algorithm.

Before analyzing correct procedures, we will look at two incorrect procedures identified during the session.

Incorrectly Reinterpreting the Problem The following procedure was most common during the first attempts.

4 necklaces	7 necklaces
12 blue	84 blue
8 red	56 red
4 green	28 green
20 yellow	140 yellow

Observer (explaining the task): "What do you need to know before making an order for 7 necklaces?"
Pablo: "We must multiply 8 seven times to find the order."
Observer (speaking to Jorge): "And what are you going to do?"
Jorge: "The same." (He multiplied mentally and wrote the result in his table.)

Using Additive Constants At least one pair of students used this procedure. At the start of the activity, Jair and Ixami simplified the problem (this error was explained earlier), then looked for an additive constant, and, finally, understood the need to find unit values during the group discussion. Their worksheet was quite smudged (see Fig. 9.7). Below the Fig. 9.7 is the observer's register.

4 necklaces	7 necklaces		
12 blue	20	24	21
8 red	16	20	14
4 green	12	16	7
20 yellow	28	35	35

Observer's version
(The numbers in cursive were erased.
The numbers on the right remained
and wereconsidered correct.)

Apparently, to find the number of beads for 7 necklaces, they started by adding 8 beads to each number of beads in the 4-necklace order. Then they added 12 beads to each quantity in the order for 4 necklaces. They became aware of their mistakes when they verified their results on the computer. In the end, they noted the correct results, after the group discussion.

Next, we will review examples of the many correct procedures that appeared in the session.

Using Internal Relationships In Table 4-F, the relationship between the number of necklaces is one to two, or double. At least one student, David, based his procedure on this relationship. With this procedure, he implied that the relationship between the two numbers of necklaces was the same relationship for all pairs of

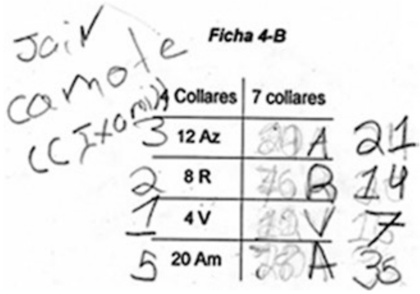

Fig. 9.7 Use of an additive constant

beads of each color in this table. In fact, this relationship presented a proportionality constant. It should also be said that, as expected, this procedure was rare and appeared only in cases where relationships were whole (double).

double

6 necklaces	12 necklaces
18 blue	
60 red	
42 green	
6 yellow	

Obs2: "How did you find it?"
David: "I multiplied. I doubled that (points to the chart where it says '6 necklaces')"

Using the Relationship Between the Number of Necklaces and the Number of Beads of a Certain Color (or the Relationship Between Quantities of Beads) At least two pairs attempted this procedure for Table 4-B by finding the relationship between one quantity of necklaces and one quantity of beads.

Enrique: "Look, it's just that, I think here," [in the order for 7 necklaces] "it's 21, because 4 times 3 is 12 and 7 times 3 is 21."
Adriana: "I think we need to do 12 times 7."

Table 4-B

4×3=12 7×3=21

4 necklaces	7 necklaces
12 blue	
8 red	
4 green	
20 yellow	

The use of these factors appeared to be intuitive and probably did not originate from an understanding of the problem. For example, they did not make any connection with the unit value. It also became evident when students who attempted this procedure were asked, “What are the values for the sample necklace?” while verifying results on the computer.[10]

Observer: “What is the sample [in this problem]?”
Adriana: “Mmm, there isn’t one.”
Enrique: “7.”
Observer: “No, you are going to make 7 necklaces. And the sample?” (pointing to the sample on the computer).
Student: “21 blue.”
Adriana: “This is the model” (pointing to the column with 4 necklaces).
Observer: “No, that’s the order for 4 necklaces.”
Enrique: “No, this one,” (pointing to the order for 7 necklaces) “you write in the order.”
Observer: “No, we’re still looking at the sample—look, in the sample.”
Enrique: “Ah!”
Observer: “What should we write here?” (pointing to the space for the sample in the computer). “How did you find these numbers [the order for 7 necklaces]?”
Enrique: “It’s 3 here [blue], 2 here [red], 1 here [green]. Adriana, how many did we say here?” (Both students stopped to think about the number of yellow beads in the order required.) “It’s 5 here.”

At first, Adriana and Enrique tried to answer the sample necklace question by using all the numbers they encountered. It may be worthwhile asking: Did these students realize that the factors they identified (times 3, times 2, etc.) corresponded to the number of beads of each color per necklace? Did the students find the term “sample” (*modelo* in Spanish) confusing (as opposed to using “number of beads per necklace”)? In the end, Enrique seemed to grasp the meaning of the sample and provided the answers that had been requested.

By Recognizing the Need to Know the Sample (the Unit Value) before Obtaining the Quantities in the Order Once the students realized they had to figure out the sample necklace, they completed the implied calculations, dividing then multiplying, using procedures like those mentioned before, both canonical and noncanonical. See the examples below.

- By finding the model through cyclical distribution, with the help of iconic representations

Two teams followed this procedure. Below is an example where the observer helped students by suggesting the pertinence of finding the sample.

Table 4-B

4 necklaces	7 necklaces
12 blue	
8 red	
4 green	
20 yellow	

[10] When entering quantities, students needed to know the values for the sample necklace—that is, the number of beads of each color in one necklace.

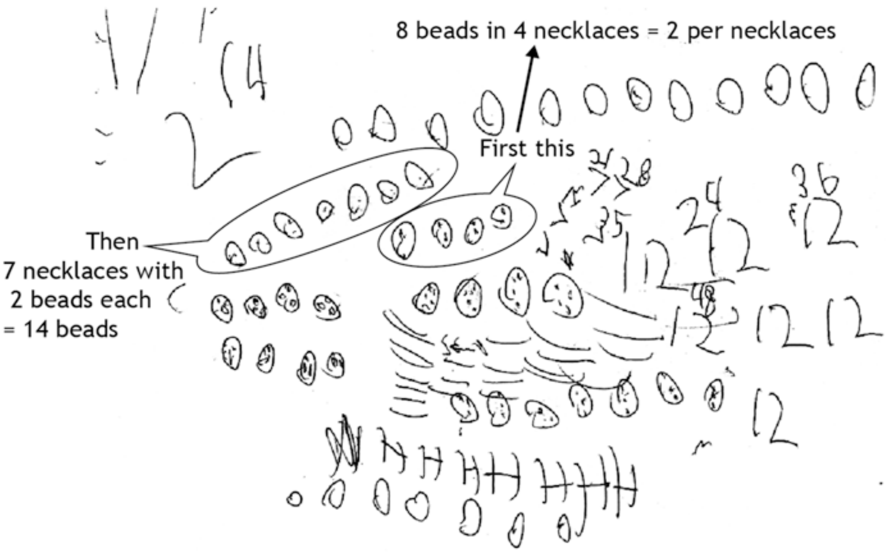

Fig. 9.8 Finding the model through cyclical distribution, with the help of iconic representations

Before this procedure, the pair had simplified the problem.

Observer: "It says here that 12 blue beads were needed to make 4 necklaces. Can you use that information to figure out the number of blue beads in the sample?"
Juan Carlos (drawing 4 small circles then distributing 12 little dots one by one and counting the ones that remained inside one of the circles): "There are 3."
Observer: "So, then, to make 7 necklaces, how many blue beads do you need?"
Juan Carlos (drawing 7 circles with 3 little dots in each circle; there were 21): "OK, I get it."

This procedure was apparently used to find several results. The Fig. 9.8 shows an excerpt from the student's worksheet, demonstrating how he found the answer for the red beads (8 red beads in a 4-neklace order).

- By searching for the unknown factor in a multiplication

Several students decided to calculate the number of beads corresponding to 1 necklace (the sample) by looking for the unknown factors in multiplication where "number of necklaces times '×' beads per necklace = number of beads." Below is an example that demonstrates the difficulties that even the most skilled students encountered with this task. Also, for Table 4-A, Karina knew she needed to find a sample or, as she called it, "what goes on each necklace." This is how Karina and Cecilia solved Table 4-A.

Table 4-A	
4 necklaces	7 necklaces
8 blue	
12 red	
20 green	
4 yellow	

Karina: “OK, let’s see what goes on each necklace. Since there are 8 [blue beads in 4 necklaces], each necklace needs 2 [blue beads].”

On the multiplication chart (see Fig. 9.9), Karina searched down the 4 column for the number 8 where it intersected with the 2 row; 2 was the quotient of 8 divided by 4 and the number of blue beads per necklace. Then, to find the number of blue beads in the order, she multiplied 2 by 7: she looked for the intersection of the 2 row and the 7 column, which was 14.

Karina started by figuring out the sample, then the order (See Fig. 9.9).

Karina demonstrated her skills using the chart for both dividing and multiplying. She quickly completed the two calculations needed to go from 4 to 7 necklaces by figuring out the quantities required in a single necklace.

Observer: “Karina, what’s in the sample? What do we place in the sample?”
Karina: “What do you mean by ‘place’?”
Observer: “Remember how the software asks us to create a sample? You wrote 14 [blue], 21 [red], 28 [green], and 28 [yellow], but that is the order for 7 necklaces. If we want to verify this on the computer, we need a sample, right?”
Karina: “Which one is the sample?”
Observer: “How did you find these quantities [for the 7-necklace order]?”
Karina: “Oh! . . . Um, um, that’s right. Here, on the blue sample, we need 2 blue beads for each necklace; here [on the red beads] we need 3 per necklace, I think.” (They used the multiplication chart to finish the sample.)

Karina’s original way of finding the numbers of blue and red beads led, in fact, to finding each unit value. However, she seemed to have forgotten that those values corresponded to the “sample.”

Fig. 9.9 Use of the chart for searching the unknown factor in a multiplication

0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27

9.3.2.3 Feedback

During the first stage, where the problems included the unit value (as a given value or as a question), the software fulfilled its purpose of enabling students to understand the problem and empirically verify their results, which saved time and resources. When the problem was solved correctly, the software became a functioning validation tool. When typing errors had been made (miscalculations, for example), the software detected where the error had taken place. Below is a scenario where Irving and Carlos suggested that the order was asking for 3 necklaces (in Table 3) during the group discussion.

The teacher requested the results for Table 3 during the group discussion.

Table 3	
1 necklace	__ necklaces
4 blue	16 blue
6 red	24 red
5 green	20 green

Teacher: “How many necklaces in this order?”
Carlos: “4.”
Irving: “3.”

The teacher asked each student to explain how they reached their results, but the rest of the class disagreed with their explanations. Then, the teacher suggested verifying their results on the computer.

Teacher: “Irving, maybe we can try something; we can tell the computer that the answer is 3.”
Francisco: “There will be leftover beads.”

The computer made 3 necklaces, leaving 4 blue beads, 6 red, and 5 green unused. There were enough beads to make another necklace.

Teacher: “See? There were leftover beads.”
Francisco: “Enough for another one.”
Teacher: “Enough for another necklace, to make the fourth necklace.”

During the second stage, where the unit value was not included, verifying results on the computer became considerably more difficult. The most widespread difficulty occurred when students attempted to verify their results on the computer without knowing the sample. They had approached the problem through addition, solved the problem without stopping to find the sample necklace (by identifying regularities), or solved it by other means. When the computer asked students to enter the numbers for the sample, they did not understand what they were meant to do. As a result, some felt bewildered and returned to their seats; others attempted to find a sample for each order, which was a step away from having one sample for both orders. Some students did not realize the pertinence of the sample until the computer requested it.

The program sometimes validated incorrect answers. It happened mainly when students, knowing the numbers for one sample, verified only the values for one of the orders and omitted the second one. Let's analyze an example. Yadira, Claudia, and Alejandro solved Table 4-C using drawings. However, they entered their data incorrectly.

Table 4-C

Sample	5 necklaces	11 necklaces
2	10 blue	22
4	20 red	44
4	15 green	44
1	5 yellow	11

The students drew their sample instead of writing it.

Observer: "What is the sample?"
Yadira: "2 blue, 4 red, 4 green, and 1 yellow."
Observer: "What's in the order?"
Yadira: "22 blue, 44 red, 44 green, and 11 yellow."
Student: "I think the numbers are wrong."
Observer: "Does the sample have the same number of red beads and green beads?"
Students: "Yes."
Observer: "How did you find them, Yadira?"
Yadira: "Where it says 5 necklaces, 1 has 2 blue, so then, where it says 11 necklaces, I placed 2, 2, 2, and got 22," (showing me the drawing on her chart; she had drawn 12 necklace strings and added 2 beads on each of the first strings) "and these [the other numbers in the order]; they found them."
Observer: "OK, let's see if it works, OK?" The necklaces were correct.

The computer accepted the quantities for the sample and the order, and it validated the result without detecting any errors. However, the observer aptly reminded the students that the sample should be the same for both orders.

Observer: "It turned out all right, didn't it? Just remember that the necklaces in this order are identical to the ones in the other [five-necklace] order. . . . Should we test it? . . . Let's use the same sample for the other number of necklaces, and they should be identical. This will tell us if we're right. . . . The sample is the same: 2, 4, 4, and 1, because the necklaces are identical. Can someone read me the numbers in the 5-necklace order?"
Yadira: "10 blue, 20 red, 15 green, and 5 yellow."
Observer: "Let's see if it's true. If they come out the same as these [11 necklaces], then we're fine, but if they don't, there must be some mistake, right?" Three necklaces were made correctly, but the fourth and fifth were incomplete; 5 green beads were missing.

After noticing the missing beads, the students realized they had made a mistake. In this case, however, given the number of relationships at stake, they could not find the error in their procedure.

Other Validation Resources Aside from using the software to validate results empirically, some students were seen verifying results through a range of inherent properties embedded in the problem's relationships. This might have been the case for the student who, seeing the quantities in the following table, stated, "I think the numbers are wrong."

5 necklaces	11 necklaces
10 blue	22 blue
20 red	44 red
15 green	44 green
5 yellow	11 yellow

The student probably thought that if an order presented two equal quantities of different-colored beads, then the number of beads in those two colors should also be the same in other purchase orders.

Others also tried keeping the order: if the number of beads of a certain color was greater than the number of another color in the set (for example, in the sample), that same relationship must carry over to the other set (for example, in the order). See the examples below.

Example 9.1

Table 2	
1 necklace	8-necklace order
3 blue	__ blue
__ red	48 red
__ green	56 green

For the red beads:

Yadira: "6 red beads." (The observer thought she found the answer by testing numbers that when multiplied by 8 would equal 48. Then, she drew 8 lines and added 8 beads only to the first 2 lines.)

For the green beads:

Yadira: "It can't be less than 6." (She realized that now she needed more beads than before and that the number of necklaces was the same. She wrote "7 green" on her worksheet.)

Example 9.2
(That example was worked during a group discussion)

Table 3-A	
Sample	5 necklaces
3 blue	15 blue
6 red	30 red
__ green	30 green

Tea1cher: "Finally, how did you know you needed 6 green beads?"
Student: "Because of 6 times 5."
Luis: "Because if the line above said 6 red becomes 30, the line below is the same because the result is there [in the row for red beads]."

Students favored this type of semantic validation (Brousseau, 1998) in problems where plural relationships were in play. These resources could have been shared more during group sessions in a way that would encourage the students to find and reflect on relationships. For example, if one quantity in the sample was larger than another, the quantities in the order would maintain the same relationship; if the relationship between beads in the sample was "double," the same applied to the order. Meanwhile, additive relationships, where one quantity was "larger than another by 3 beads" would not be reflected as such in the order.

9.4 Final Remarks

Most fourth-grade students were comfortable approaching relationships among three or four quantities of beads on 1 necklace and the corresponding quantities on n necklaces, and they could solve all three implied problems (two divisions, one multiplication), usually through noncanonical procedures. It can be said that one-to-many relationships provided richer contexts than those presented in problems with four values.

The relationships between the number of beads in n necklaces and m necklaces (where n and m were greater than 1) were significantly more difficult. However, at least 50% of the pairs in this 4th grade group could understand one or more of the problems presented during the last two sessions. It seemed that the greatest challenge was understanding that (1) each set of beads in the table matched a different "order" rather than a single necklace and a single order, and (2) both orders came from the same sample. Once students understood this, they encountered far fewer difficulties calculating the number of beads in the sample based on one of the orders, to then generate the second order. From the perspective of understanding multiplicative relationships, this was a remarkable achievement.

On the other hand, considering their school level, this was probably the first time that most of the students had needed to calculate information that was not directly requested in the problem (the unit value) to solve it. Potentially, more students would be able to approach this variable when they reached 5th grade, which might be a better time to introduce it.

Finally, the computer software, as a “virtual means of interaction” (Mariotti, 2002), supported the students’ understanding of problems and, more specifically, of relationships between the broad range of quantities used in the “necklace factory.” The computer software also provided a way of verifying the results. However, the program showed some limitations. When verifying the relationship between *n* and *m* necklaces, it asked for the sample necklace even when students had not figured out its pertinence for finding the solution. The program did not always enable students to identify the source of their errors and did not invalidate certain types of erroneous procedures. It would be worthwhile to explore other forms of empirical verification for more difficult cases where the relationship in question was between *n* and *m* necklaces. For example, the program could display a graphic representation of the two orders that came from a certain sample. On the other hand, the wealth of relationships in play enabled a variety of semantic forms of verification, which were valuable from a didactic point of view.

Annex

Problem Chart

	Stage 1: unit value provided or requested						Stage 2: unit value not provided			
	Small numbers				Larger numbers		Nonwhole relationships between numbers		Whole relationships between numbers	
Variables	First type of situation		Second type of situation		Third type of situation		Fourth type of situation		Fifth type of situation	
Table example	Table 1		Table 2-A		Table 3-F		Table 4-A		Table 4-F	
	1 necklace	12 necklaces	1 necklace	__ necklaces	1 necklace	__ necklaces	4 necklaces	7 necklaces	6 necklaces	12 necklaces
	__ blue beads	__ blue beads	4 blue beads	12 blue beads	4 blue beads	60 blue beads	8 blue beads	__ blue beads	18 blue beads	__ blue beads
	__ red beads	__ red beads	6 red beads	18 red beads	9 red beads	__ red beads	12 red beads	__ red beads	60 red beads	__ red beads
	__ green beads	__ green beads	5 green beads	15 green beads	__ green beads	105 green beads	20 green beads	__ green beads	42 green beads	__ green beads
	__ yellow beads	__ yellow beads					4 yellow beads	__ yellow beads	6 yellow beads	__ yellow beads
Implied knowledge	Multiplication		Division		Division or multiplication		Division and multiplication Unit value		Division and multiplication Unit value 12C/6C reason	
Number of similar tables applied	One table		Two tables		Six tables		Six tables			
Number of sessions dedicated	One session		One session		Three sessions		Two sessions			

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Chapter 10
Building Opportunities for Learning Multiplication

Fátima Mendes, Joana Brocardo, and Hélia Oliveira

10.1 Introduction

There have been fewer research studies on multiplication than studies on addition and subtraction (Fuson, 2003; Verschaffel, Greer, and De Corte, 2007). Verschaffel et al. (2007) state that there is really a scarcity of research regarding strategies used by students to solve multiplication and division problems.

Brocardo and Serrazina (2008) point out a curricular "big idea" to approach numbers and operations with the perspective of number sense development, articulating numbers, operations, and applications. For example, the authors note that decomposition of numbers can be learned through number expertise articulating addition and multiplication: 80 is 20 + 20 + 20 + 20, four times 20, or 4×20. They also refer to the importance of articulating the meanings and the structures of the operations.

Focusing on number sense and the role of mental calculation, Brocardo (2011) stresses the importance of being able to look at numbers as the center of a web of relationships. For example, the number 48 may be represented as $2 \times 2 \times 12$, 2×24, 4×12, $50 - 2$, $100/2 - 2$, or 6×8. When solving numerical problems, students with number sense use the representation of 48 that is more suitable to mental manipulation of the numbers or that best suits the context of the problem.

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The comprehension-based approach for learning multiplication introduced in this chapter includes these ideas of curricular integration and working with numbers and operations in the perspective of number sense development, which are briefly referred to here and will be detailed in the following sections.

10.2 Learning Trajectories

Planning multiplication teaching implies more than structuring the mathematical ideas involved in this operation. It is equally important to think how students can learn and progress in their learning, and to bear in mind that not all of them learn at the same pace and in the same way.

Simon (1995) uses the sailor metaphor to explain the concept of learning trajectory, which we consider paramount for thinking about teaching multiplication. The sailor has a global plan that includes precise milestones and a clear definition of the place of arrival at the end of the trip. However, this plan has to be successively adjusted according to different events—weather conditions, boat performance, or unforeseen situations that may arise. Such adjustments may also include unanticipated stages. Like the sailor, teachers also need a global plan to guide the proposals they prepare for students. They have to change their global plan to take into account the capacity of each student to learn, the ideas or doubts that arise, and the unforeseen situations they encounter. Like the sailor, teachers plan each stage of their journey, bearing in mind the hypothetical trajectory and the conditions resulting from the implementation of the previous stages.

Setting the global plan of the "journey" (learning multiplication) involves starting by clarifying the key milestones that determine the stages of a nonlinear path. At a macro level—the global plan of the journey—the hypothetical learning trajectory includes setting the progression of mathematical ideas and of strategies and models associated with multiplication. It also includes a flexible sequential vision, since the trajectory that is actually undertaken determines the adjustments and the paths to follow at the next stage. Lastly, it includes progression and interconnection as aspects that always underlie the design/selection of tasks for students.

10.3 A Hypothetical Trajectory in the Third Grade

The Portuguese educational curriculum states that in the third grade, (8- to 9-year-old) students should complete their studies on multiplication tables, develop their knowledge of whole and decimal numbers, and learn to build multiplication algorithms. During the previous grades, they have started the transition from repeated addition to multiplication, the exploration of multiplication meanings, and comprehension and memorization of facts—namely, the ones arising from study of the multiplication tables of 2, 5, and 10.

In the third grade, the key milestones for multiplication learning are the following (Fosnot, 2007; Fosnot and Dolk, 2001):

- Consolidation of understanding of a group as a unit
- Distributive property of multiplication in relation to addition and subtraction
- Commutative property of multiplication
- Position values pattern associated with multiplication by 10
- Associative property of multiplication
- Understanding of the inverse relationship between multiplication and division
- Understanding of the proportional reasoning of multiplication

This last one—although emerging in the third grade—is further developed in later grades.

The models associated with multiplication that students can build by exploring each task are also important milestones for setting the hypothetical learning trajectory. They are closely related to the models and procedures used in addition (see Figs. 10.1 and 10.2):

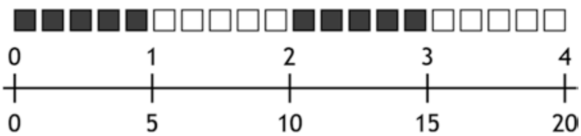
- Decomposition of terms used in repeated addition allows moving from a linear model to a two-dimensional model- the array model.
- The linear model, which supports repeated addition by "jumps," becomes a proportional model, such as the double line or proportion tables.

Considering that our focus is the third grade and that the work with proportions is developed in later grades, the trajectory we introduce here favors the use of the array model. The choice of the array model is supported by authors such as Barmby, Harries, Higgins, and Suggate (2009), who consider it an important support in the evolution of multiplicative reasoning. It should be noted that this is the model that helps to build and consolidate the use of distributive and associative properties, as Figs. 10.3 and 10.4 illustrate.

The rectangular model also allows understanding of the commutative property (Fig. 10.5), which cannot be understood from the linear model of successive addition: 4 rows of 5 elements have the same number of elements as 5 rows of 4.

Fig. 10.1 From a linear model to an array model

Fig. 10.2 From a linear model to a double line model



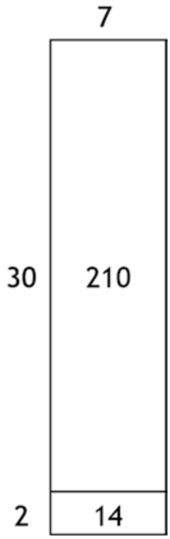


Fig. 10.3 Array model for supporting the use of the distributive property of multiplication in relation to addition in calculation

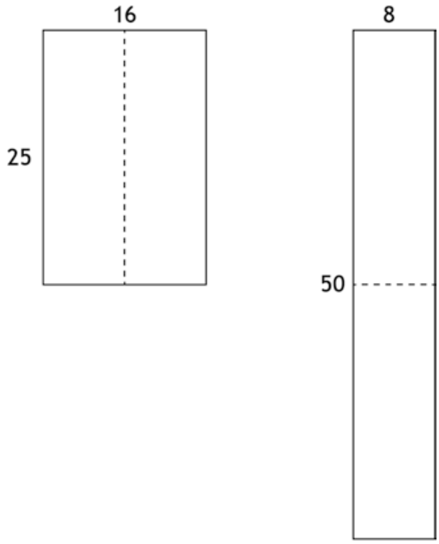


Fig. 10.4 Array model for supporting the use of the associative property of multiplication

Combining the learning milestones with the array model, the numerical universe that is used, and the order of magnitude of the numerical values, several learning trajectories can be built according to choices depending on the specific curricular nature, the characteristics of the students, and also the specific context of each school.

The hypothetical trajectory introduced below (see Table 10.1) is thus one among many possible trajectories. This is an example of an actual trajectory implemented in a third-grade class (Mendes, Brocardo, and Oliveira, 2013), which includes adjustments resulting from an experiment in a classroom of ten sequences of tasks and some particular conditions of the class and the school. We also point out the contexts used in the tasks, which take into account aspects related to multiplication learning.

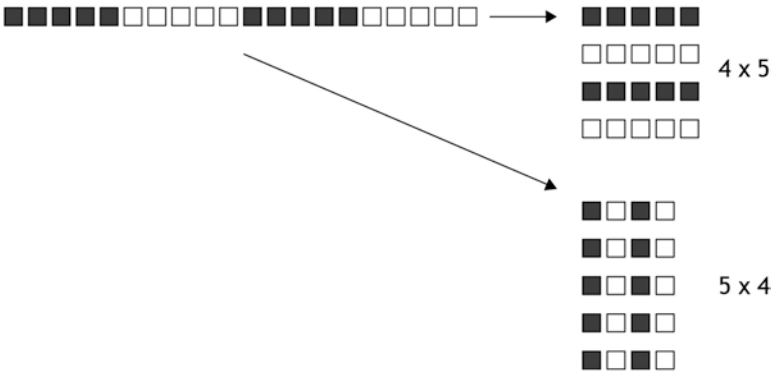


Fig. 10.5 Array model for supporting the commutative property of multiplication

Table 10.1 Multiplication learning trajectory in the third grade

Sequences of tasks	Learning milestones	Contexts and numbers
Sequences 1 and 2: multiplication tasks where the calculation by groups is made evident (6 tasks)	Consolidation of understanding a group as one unit Distributive property of multiplication in relation to addition and subtraction Commutative property of multiplication	Items displayed in a grocery store interconnected with use of multiples of 5, 3, and 6 Packs with 4, 6, and 12 stickers
Sequences 3 and 4: tasks whose context is related to the rectangular array (6 tasks)	Consolidation of understanding a group as one unit Distributive property of multiplication in relation to addition and subtraction Commutative and associative properties of multiplication	Patterns in curtains and yard pavements interconnected with use of multiples of 5 and 10 Stacks of boxes interconnected with use of multiples of 5 and 10
Sequences 5 and 6: tasks with numbers in decimal representation (6 tasks)	Distributive property of multiplication in relation to addition and subtraction Commutative property of multiplication	Filling and emptying of bottles, relating their capacities to use of reference decimal numbers and relating them to each other (0.5, 1.5, 2.5, and 0.25) Using and relating reference decimal numbers associated with values of different coins (0.1, 0.2, 0.5, and 0.99)
Sequences 7 and 9: division tasks where multiplication is favored, revealing the relation between two operations (8 tasks)	Understanding of the inverse relationship between multiplication and division	Collecting cards and using beverage vending machines to divide using multiplication, using multiples of 6 and 8
Sequence 10: multiplication tasks where multiplication is favored, revealing the relation between two operations (3 tasks)	Understanding of proportional reasoning	Filling in and using prices from tables for grocery item costs and a trip to the theater, using multiples of 1.25, 1.10, 1.60, and 0.99

Looking at Table 10.1 we can see that the learning milestones are the support for the trajectory and they emerge from the contexts of the tasks. When a new numerical set is introduced, the learning milestones are "revisited": by starting the study of multiplication with decimal numbers, the learning milestones previously considered when studying the natural numbers are reworked.

This "revisiting" process is also present in the numbers included in each task. It starts by using situations involving multiples of 2, 3, 5, and 6. Then, it "revisits" the use of those multiples in order to work with multiples of 4, 10, and 12. This numerical "revisiting" is a sequential chain that repeats itself when introducing new learning milestones: when developing the idea of the inverse relationship between multiplication and division, the numerical set is restricted to the natural numbers and the groups of 6, 8, and 10 are used again (sequences 7 and 9). When introducing the proportional sense of multiplication, 1.25 and multiples of 10 are used, which are numbers that were previously considered as a reference (sequence 10). This is the starting point to build relationships with new numerical values.

10.4 Specifying the Hypothetical Trajectory: A Sequence of Tasks

Setting a learning trajectory like the one we showed in the previous section implies paying great attention to the specific characteristics of each of its sequence of tasks. We will now analyze sequence 4, composed of four tasks, as shown in Figs. 10.6, 10.7, 10.8, and 10.9.

Task 1: Stacks of Boxes

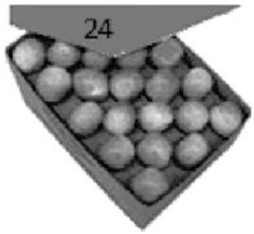
The Piedade Grocery Store received boxes, each containing 24 apples as shown in Fig. 10.8. The 25 boxes were stacked as shown in the Fig. 10.6.

In total, how many apples are there?

Fig. 10.6 25 boxes with 24 apples each

Fig. 10.7 25 boxes with 48 apples each

Fig. 10.8 Box with 24 apples



Task 2: Stacks of Boxes
In the Bairro Supermarket, there is also a stack of 25 apple boxes. These boxes are bigger, and each contains 48 apples as shown in the Fig. 10.7.

In this supermarket, how many apples are stored in the boxes?

Task 3: Stacks of Boxes
In the Girassol Supermarket, the total number of apples is the same as that in the Bairro Supermarket, but each box contains only 24 apples as shown in Figure.

In total, how many boxes of 24 apples are there in the Girassol supermarket?

10.4.1 Connected Calculations

Regarding mathematical ideas about multiplication, with this sequence it is intended that students progressively drop the idea of repeated addition and evolve toward multiplicative reasoning. It is also intended that they use the properties of multiplication to calculate products. Therefore, the context of tasks 1 and 2 facilitates a progression towards the array use, in combination with use of the properties of multiplication. The stacks of boxes can be seen in different ways, as different groups of rows or columns. For example, the stack with 25 boxes in task 1 may be seen as having 5 columns, each one with 5 boxes. It may also be seen as having 2 rows of 5 boxes, plus another 2 rows of 5 boxes, plus 1 row of 5 boxes. In the first case, we "see" the stack of boxes organized into 5 columns, and it is the calculation of the number of apples in each column that sets the total number of apples. In the second case, we observe that in 2 columns there are 10 boxes, and we look at the stack with the biggest possible number of groups of 2 columns. The total number of apples is obtained from the number of apples in each of the groups (of 2 columns and of 1 column) that are considered.

When students use these two strategies and make groups to calculate the requested value, they do not think about the properties of multiplication, nor the array model. However, the analysis of these strategies and the solution for other situations based on the same type of context can lead to an understanding of the properties, drawing the conclusion that $5 \times 24 + 5 \times 24 = 10 \times 24$ and that $25 \times 24 = 10 \times 24 + 10 \times 24 + 5 \times 24$. Besides exploring the different groups of boxes and the corresponding use of the distributive property of multiplication, students can also associate the total number of rows and columns with the total number of boxes. They start using the array model in similar situations, where each "cell" of the rectangle corresponds to a set of objects—in this case, a set of apples—and not just to an object, as happened at an earlier stage of learning multiplication.

Two important ideas regarding the learning trajectory, progression, and interconnection are achieved either in the numerical values involved or in the possibility of using results and relationships from previous tasks:

- In task 2, each box has twice the number of apples as each box in task 1.
- In tasks 2 and 3, the total number of apples is equal and the quantity that fits in the boxes in task 2 is twice that in task 3.
- In task 2, by moving 2 boxes (the ones in the last column), we get an arrangement similar to the one in task 1.
- Task 4 helps to consolidate relations and properties used in the previous tasks—double, distributive, commutative, and associative properties—and the use of numerical values present along the chain, such as groups of 24 and 48, and products of factors that result in 600 and 1200.

In parallel with matters of progression and interconnection between the tasks in each sequence, which are essentially oriented by the fundamental ideas linked to multiplication learning and to the overall design of the global hypothetical trajectory, it is also important to bear in mind other aspects like diversity and the characteristics of each task. We will analyze those aspects in the next section.

10.5 The Tasks

In this hypothetical learning trajectory, we have included tasks of different natures: not just problems and investigations, nor just exercises. Each type of task has its own potential. It is fundamental to select the more appropriate ones according to the teaching objectives.

The sequence shown in Figs. 10.6, 10.7, and 10.8 includes problems (tasks 1, 2, and 3) and exercises (task 4). The stickers packs task (Fig. 10.10) is an example of another type of task (investigation) which can also be included when building a learning trajectory.

10.5.1 Task 4: Stickers Packs

Eva, Luís, and Leandra collect stickers. The stickers are sold in packs of 4, 6, and 12 stickers. The 12-stickers packs are sold out. Raquel bought stickers, and she got 48.

Which stickers pack might she have bought? Explain your thoughts.

The selection of problem and investigation contexts—i.e., the characteristics of the situations that may be mathematized by the students (Fosnot and Dolk, 2001)—should be oriented so that the contexts (i) allow construction of models, (ii) make it possible for students to really understand and act upon them, and (iii) inspire students to ask questions and find solutions.

Tasks 1, 2, and 3 (Figs. 10.7 and 10.8) explore a context of fruit boxes and the different ways they can be stacked. There are other contexts that also allow students to progressively build and refine the models underlying the multiplication [characteristic (i)]. They are based on organizing groups of objects in packages (eggs, balls, drink cans), rectangular patterns in curtains, or collecting and organizing the necessary data to inventory the objects stored in a certain place. The boxes and the way they are stacked (Fig. 10.6) can help certain ways of thinking associated with the rectangular arrangement; i.e., they allow students to model situations using such an arrangement. Earlier, students should have had the opportunity to explore contexts that allowed them to model a situation such as repeated addition on a numerical line. At a later stage of multiplication learning, they should, for example, explore situations whose context allows them to model multiplication as an area or a proportion, using a corresponding double numerical line.

By analyzing the tasks included in Figs. 10.6, 10.7, 10.8, 10.9 and 10.10, we can easily see that they suggest that students seek ways to find solutions using different knowledge and relations [characteristic (ii)] in accordance with their level of mathematical development. For example, in task 1, to calculate 25×24, they can observe the image and start to calculate the total number of apples in each column, determining 5×24. Others can make groups of 10 boxes, calculating 10×24. Yet others, less familiar with multiplication procedures, can repeatedly add 24.

Fig. 10.9 Task 4: Three numerical chains

50 × 10 =	10 × 60 =	12 × 50 =
25 × 20 =	20 × 30 =	24 × 50 =
25 × 4 =	40 × 15 =	50 × 24 =
25 × 24 =	40 × 30 =	25 × 48 =
50 × 12 =	20 × 60 =	50 × 48 =

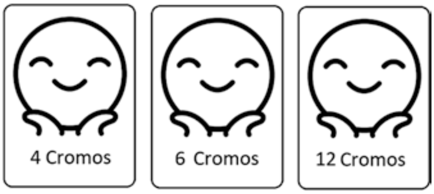


Fig. 10.10 The stickers packs task

The context of the tasks should also challenge students to analyze possibilities, find patterns, ask questions, or compare different forms of reasoning [characteristic (iii)]. Therefore, the tasks should "refer to" situations that students know or can imagine. They should also allow analysis from different points of view. For example, in the stickers packs task (Fig. 10.10), students could discuss several purchase possibilities for Raquel and decide which one would be the most inexpensive.

When designing and implementing a learning trajectory, tasks that focus on appropriation of certain facts and numerical relations should be included—usually called practice exercises. We highlight the ones we call numerical chains, as described by Fosnot and Dolk (2001). They aim to develop students' mental calculation using important properties and relations of multiplication. Considering the specific characteristics of numerical chains (and according to the authors above), when exploring them, teachers should maintain a lively pace (not spending more than 15 minutes on them), and should favor oral skills (instead of written records).

The sequence shown in Fig. 10.9 includes three chains aiming to highlight powerful strategies of mental calculation based on application of the associative property of multiplication in the particular case of relations with doubles and halves, and the distributive property of multiplication in relation to addition, using reference numbers. Each chain should be explored on different days, since the aim is to focus on relations one at a time.

The exploration of a numerical chain has particular characteristics that we illustrate with the case of a teacher, Isabel, when she was working on the second chain in task 4 (Fig. 10.9). Isabel wrote an expression on the board and gave some time for students to think about it. She first wrote "10 × 60" on the board. Only after several students had raised their hand, stating that they already knew what the result of 10 × 60 was, did she ask one of them to give his or her answer. After analyzing it, Isabel moved on to the next numerical chain, writing "20 × 30" on the board. The following dialogue shows how Isabel explored the various students' answers for the 10 × 60 calculation.

Leandra: "It is 10 × 60 or 60 × 10; it is 600."
Isabel (writing "20 × 30" on the board): "And now?"
Duarte: "20 × 30 is 600 because it is 20 × 10 × 3. And 20 × 10 is 200, and × 3 is 600."
Bernardo: "And it can also be 10 × 30 times 10 × 30, which is 300 plus 300."
Raquel: "It is 600 because it is equal to the last one! 40 is the double of 20 and 15 is half of 30."
Gustavo: "We can also do 40 × 10 plus 40 × 5. It is 400 + 200, which is 600."
(Isabel wrote "20 × 60" and a lot of arms are raised). Isabel - "And now?"
Guilherme: "It is 1200 because 60 × 10 is 600 and plus 60 × 10 is 600, so it is 1200."
David: "I thought of 20 × 30 two times."
José: "It is 1200 because it is the same as 40 × 30."
Duarte: "We also can do 60 × 2 and then × 10."

According to the objectives of the chain and the way the students reacted, the teacher would decide the level of freedom for justifications for different procedures to be analyzed and which processes to highlight. In the previous episode, Isabel chose not to ask for a justification for the 10 × 60 result since it was an answer most students already knew by heart. Regarding other calculations, she gave them opportunities to explain their ways of thinking that revealed application of different properties of multiplication.

10.6 Enacting the Tasks: Planning and Exploring

After selecting each task, teachers still must consider two very important moments: planning how to organize the class and putting such planning into action by exploring the task in the classroom (Stein, Engle, Smith, and Hughes, 2008). These two moments should be oriented by the learning trajectory, taking a global and flexible route to be followed according to the learning purposes and the students' reactions.

At these two moments, teachers' attention should be focused on the students. Keeping the learning trajectory always in mind, teachers should be able to plan and explore from what students can understand, do, and ask.

10.6.1 Planning the Tasks' Enactment

This is a stage in teachers' work that can involve different aspects. We consider that the aspects involving the preparation of the tasks' enactment in the classroom—bearing in mind what students will be able to do and the doubts they might have—are particularly relevant. Therefore, we give great importance to anticipation of their strategies and difficulties.

Anticipating the strategies associated with a task involves in-depth knowledge of their potential and mainly thorough knowledge of the way students think. Teachers have to put themselves in the place of their students and foresee the ways they find solutions at different levels of sophistication, according to different levels of learning and distinct ways of reasoning. This anticipation will help teachers recognize and understand the strategies used in the classroom and understand which ones are related to their teaching objectives, i.e., the mathematical ideas they intend students to learn (Stein et al., 2008).

By anticipating students' strategies, teachers will be able to identify their difficulties in the classroom according to the solutions that are found and understand why these exist. Thus, it will be easier to find a way to help students to overcome such difficulties. At the same time as they foresee students' strategies, teachers should also anticipate possible difficulties linked to the interpretation of the task itself.

We will now show the strategies students could use in task 2 of the sequence illustrated in Fig. 10.7. In parallel with this anticipation, we will also identify some difficulties that students may have in each strategy.

We will organize the possible ways to find solutions into three categories: (i) ones based on additive reasoning, (ii) ones that use multiplicative reasoning and that take into account the context of the task, and (iii) ones that use multiplicative reasoning but do not take into account the context of the task. For each of these categories, we sequentially list the strategies from the least to the most sophisticated.

(i) *Strategies based on additive reasoning*: Task 2 is included in sequence 4, so it is expected that students will use strategies underlying the properties of multiplication. However, some students can still use additive strategies like the ones listed in Table 10.2.

(ii) *Strategies using multiplicative reasoning and taking into account the context of the task*: The context "stacks of boxes" favors use of strategies that take advantage of the properties and multiplicative relations, like the ones listed in Table 10.3.

(iii) *Strategies using multiplicative reasoning but not taking into account the context of the task*: Students may use multiplicative strategies and not be influenced by the way the boxes are organized. Still, some expected strategies take advantage of the previous task (see task 1 in Fig. 10.6) by establishing numerical relations between them (Table 10.4).

Table 10.2 Additive strategies and expected difficulties

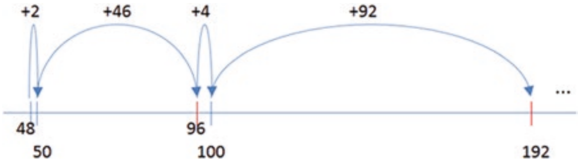
Expected strategies	Possible difficulties
Thinking of 25 boxes, each one with 48 apples, and using additive procedures: Horizontally representing addition, by doing 48 + 48 + . . . + 48 (25 addends) Repeatedly adding 48, by doing 48 + 48 = 96, 96 + 48 = 144, . . . Adding addends 2 by 2, by doing 48 + 48 = 96, 96 + 96 = 192, 192 + 192 = 384, . . .	In doing calculations with large numbers correctly, taking into account the number of addends (25) and the number we need to add (48)
Thinking of 25 boxes, each with 48 apples, and using the numerical line to sum them 	In doing calculations with large numbers correctly, it is easy to lose count of the number of times 48 is added
Adding the number of boxes to the number of apples in each box, by doing 25 + 48 (an incorrect strategy)	Interpreting and understanding the task

Table 10.3 Multiplication strategies taking into account the context and expected difficulties

Expected strategies	Possible difficulties
Observing how the boxes are stacked and from there calculating by rows using multiplication, thinking: 2 rows, each with 6 boxes, is 12 boxes 2 rows, each with 5 boxes, is 10 boxes 1 row with 3 boxes $12 \times 48 + 10 \times 48 + 3 \times 48$ Or thinking row by row: $6 \times 48 + 6 \times 48 + 5 \times 48 + 5 \times 48 + 3 \times 48$	Calculating the product 12×48 Forgetting to add some partial products
Observing how the boxes are stacked and from there calculating by columns using multiplication, thinking: 2 columns, each with 4 boxes, is 8 boxes 3 columns, each with 5 boxes, is 15 boxes 1 column with 2 boxes $8 \times 48 + 15 \times 48 + 2 \times 48$ Or thinking column by column: $4 \times 48 + 4 \times 48 + 5 \times 48 + 5 \times 48 + 5 \times 48 + 2 \times 48$	Calculating the product 15×48 Forgetting to add some partial products
Observing how the boxes are stacked, mentally understanding that such an arrangement is the same as having a rectangular layout with 5 columns and 5 rows of boxes, then calculating by columns or by rows using multiplication, thinking: $5 \times 5 \times 48$ or 25×48 First calculating 5 x 48 and then multiplying by 5, which is the same as $5 \times (5 \times 48)$ or $(5 \times 5) \times 48$ Calculating 25×48, by doing 20×48 plus 5×48	Doing the calculations linking factors to multiples of 10

Table 10.4 Multiplication strategies not taking into account the context and the expected difficulties

Expected solutions found by students	Possible difficulties
Identifying the situation as being multiplicative and the numerical values to use, then calculating using the decimal decomposition of 48: $25 \times 48 = 25 \times 40 + 25 \times 8$	Doing the calculations linking factors to multiples of 10
Identifying the situation as being multiplicative and the numerical values to use, then calculating using the decimal decomposition of 25: $25 \times 48 = 20 \times 48 + 5 \times 48$	Doing the calculations linking factors to multiples of 10
Relating this task to the previous one and thinking that the number of boxes is the same, but now each box has 48 apples; i.e., it has double the number of apples that were in the boxes in the previous task. If the total of apples was 600 before, now it is doubled: $2 \times 600 = 1200$	
Relating this task to the previous one and thinking that the number of boxes is the same, but now each box has 48 apples; instead of immediately thinking of doubling it, duplicating the number of apples in each box, by doing: $25 \times 48 = 25 \times (2 \times 24)$ $25 \times (2 \times 24) = 2 \times (25 \times 24)$—i.e., $2 \times 600 = 1200$	
Identifying the situation as being multiplicative and the numerical values to use, then using doubles and halves relations: $25 \times 48 = 50 \times 24$ because 50 is the double of 25 and 24 is half of 48 $50 \times 24 = 100 \times 12$ because 100 is the double of 50 and 12 is half of 24 $100 \times 12 = 1200$ because I know how to multiply by 100	Doing the calculations linked to doubles and halves
Identifying the situation as being multiplicative and the numerical values to use, then using reference numbers (in this case, the number 50, close to 48) and compensating: $25 \times 48 = 25 \times 50 - 25 \times 2$	Forgetting that the compensation implies subtracting 25×2 and not just 2

Foreseeing the strategies students will use to find solutions for a certain task is very demanding and difficult for teachers. However, as this practice progresses, anticipation of different solutions becomes easier since the level of knowledge of the way students think about multiplicative reasoning is increasingly deeper.

Besides improving the knowledge of expected solutions when using this practice, it is also fundamental that teachers be able to list and discuss with other teachers the possible solutions to a given task. Possibly the task has already been explored in previous years, so it would be interesting to see the solutions found by those students, and to interpret and understand them, thus increasing the knowledge of the way students reason and what different representations they use to explain it.

Although teachers try to list, as thoroughly as possible, the expected solutions students may use, it is possible that unexpected strategies emerge in the classroom. Still, the fact that teachers have thought about different task solutions in advance may later prove useful for recognizing and understanding the ones that have not been thought about before in the classroom.

10.7 Exploring and Discussing Tasks

All work carried out in class has to consider two fundamental aspects. The first one is related to the teacher's purpose for exploring a given task, considering the mathematical ideas they expect students to develop and without losing sight of the task's objectives and the learning trajectory set. The second aspect, directly related to predicting the strategies to be used by students, is how teachers manage the interactions between them, ensuring that "bridges" are built between strategies with different levels of sophistication. In this way, it is possible for students who use less powerful strategies to be able to understand the more efficient strategies of their colleagues, which will allow them to progress in their learning.

The two aspects identified above prove that the teacher's action in the classroom is strongly supported by the preparation that has been done beforehand regarding selection of tasks and prediction of the strategies that students may use.

In the classroom, after a brief presentation of the selected task, students start to solve it individually or in pairs. At that moment, the teacher's role is to monitor the students' work, which is facilitated by the preparation made in anticipating the students' strategies. So, teachers initially have to have an idea if students understand the task and interpret it correctly. From there, each one works at his or her own level of knowledge.

As students develop their work, and facing the different strategies that come out, teachers should be able to relate such strategies to the ones they have anticipated. The way students represent and explain their reasoning is not always clearly noticeable, even when teachers have foreseen a strategy based on similar reasoning. To facilitate their action at this exploration stage, it is important to ask themselves questions such as:

- "Do most students understand the problem? Are there any difficulties?"
- "Are the strategies used in line with the ones I anticipated?"
- "Are there any strategies I did not foresee?"

In this particular case, when monitoring the students' work, the teacher Isabel realized that they were not using additive strategies. In fact, although these strategies were expected due to the previous experience of the students in other tasks covered by the multiplication trajectory, they only used procedures whose underlying reasoning was a multiplicative one. Based on the foreseen strategies, Isabel identified that a pair of students had suggested a way of solving the task that she had not thought of beforehand.

While monitoring the students' work, teachers begin to prepare the collective discussion. They ask themselves questions about the objectives that have been set, and they identify the potential of the strategies that are used in order to select those that should be presented and discussed with the whole class. They ask themselves questions such as:

- "Considering the purpose of the task I have chosen and the strategies I have anticipated, which solutions will be presented and discussed with all students?"
- "In what order will they be presented and discussed?"

Isabel's aim was that students use the array model associated with the context and relate it to the properties of multiplication. So, looking at the students' strategies, she chose two of them in line with the ideas she intended to highlight. The choice she made was facilitated by the work she had done in advance regarding the strategies, since this allowed her to make a quick decision in class and in accordance with her intentions. It is interesting to note that one of the chosen strategies had not been initially anticipated by Isabel, to her surprise. However, when questioning the students about the way they were thinking, she decided this solution was worth sharing with all class.

The two solutions chosen by Isabel were from the pair Eva and Guilherme (Fig. 10.11) and from the pair Duarte and Tiago (Fig. 10.12).

While Eva and Guilherme showed a sketch to support their reasoning, Duarte and Tiago did not explicitly show something that could ground the reasoning they made.

In order to decide the order of the presentations and their discussions, Isabel used the criterion of progressive presentation of the strategies from less to more abstract. Therefore, Eva and Guilherme were the first pair to do their presentation, followed by Duarte and Tiago. The pairs were supported by the A3 sheet of paper on which they had solved the task, which was put up on the blackboard.

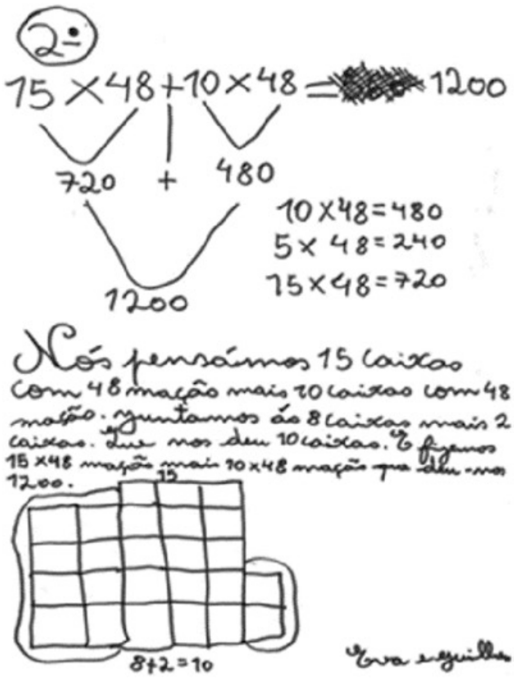


Fig. 10.11 The solution found by Eva and Guilherme

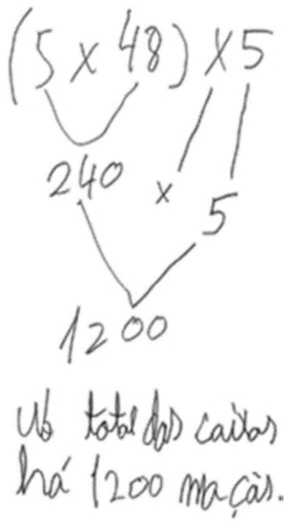


Fig. 10.12 The solution found by Duarte and Tiago

After choosing which students' solutions will support the discussions with the whole class, teachers have a decisive role in the key moments that follow. Indeed, the moment when the teacher guides the discussion with the whole class—facilitating the interactions between the students—is fundamental in the whole process. This is when the ideas associated with the learning trajectory set are pointed out, and "bridges" should be built at several levels: between different solutions with more or less sophisticated levels of reasoning; between solutions, ideas, and mathematical concepts; and also between the solutions that have been found and the purposes of the class. Teachers can guide their actions by asking themselves questions such as:

- "How should I guide the presentations and the sharing of the solutions that have been found, so as to facilitate the interactions between students?"
- "How should I manage the collective discussions in order to build 'bridges' between different solutions—some more informal and others more powerful?"
- "How should I guide the discussion so that students at lower levels of learning may evolve?"
- "How should I manage the collective discussion so that all students may learn in light of the class objectives?"

Isabel chose to alternate the presentations by the selected pairs with discussions with the whole class. She started by asking Eva and Guilherme to explain their solution. Eva's oral presentation was very close to the written records made by the pair.

Eva: "We thought of 15 boxes with 48 apples plus 10 boxes with 48 apples. We added to the 8 boxes 2 more boxes," (here she pointed to the sketch they made) "which gave us 10 boxes. And we did $15 \times 48 + 10 \times 48$."

Fig. 10.13 Representation of a stack of 10 boxes plus 15 boxes (*caixas* in Portuguese)

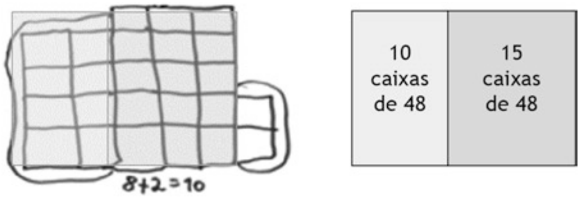


Fig. 10.14 Representation of 10 × 48 + 15 × 48 = 25 × 48

These students took advantage of the rectangular arrangement, transforming the "stack of boxes" into two "rectangles" with 10 and 15 boxes (see Fig. 10.13), and then calculated the corresponding partial products—10 × 48 and 15 × 48—considering that each box had 48 apples.

Isabel stressed that the way these students had used the rectangular layout to calculate the total number of apples by adding the two products 15 × 48 and 10 × 48 was the same as calculating 25 × 48.

This relation allowed comparison between the strategy used by Eva and Guilherme and those used by other students who determined the total number of apples by calculating the product of 25 × 48 (see Fig. 10.14). Supported by a sketch, students could understand that calculating the number of apples in 10 boxes plus the number of apples in 15 boxes is the same as calculating, all at once, the number of apples in 25 boxes.

Confronting strategies enables the teacher to point out mathematical ideas that are key to multiplication learning. The fact that 10 × 48 + 15 × 48 and 25 × 48 are equal illustrates the distributive property of multiplication in relation to addition.

Encouraging students to orally explain their way of thinking, along with their written records, may help other colleagues with different levels of understanding about multiplication to progress in terms of ideas and relations that can be established.

For example, the part of the solution from Eva and Guilherme that is shown in Fig. 10.11 could also help support a collective discussion, where it is pointed out that the calculations made were based on numerical relations.

Using the knowledge of multiples of 10, 10 × 48 is mentally calculated first. Considering that 5 is half of 10, the corresponding product is also half of the prior product. Lastly, underlying the distributive property of multiplication in relation to addition, 15 × 48 is calculated by adding the previous partial products (see Fig. 10.15).

Fig. 10.15 Part of the solution found by Eva and Guilherme

$$10 \times 48 = 480$$
$$5 \times 48 = 240$$
$$15 \times 48 = 720$$

We will now see how Isabel guided the collective discussion about the presentation by the other selected pair, Duarte and Tiago. The level of abstraction of their strategy, which was noticeable in their written records and in the way they explained it, initially led their classmates to ask for clarifications. The episode transcribed below shows how difficult it was for their classmates to understand this solution and how Isabel guided the pair in order for them to explain it in other words, considering that the first attempt had not been successful.

Gustavo: "I don't understand! Can you explain it better?"

Isabel: "Can one of you two try to explain it in another way so your classmates can understand it?"

Duarte: "We took these two boxes," (he pointed to the two last boxes on the right of the figure) "and we put them over here," (he pointed to the upper layer on the left) "and they disappeared from here," (he pointed to the two last boxes on the right) "then we did 5 × 48 because they were the boxes in one column. Since there were 5 columns, we then did times 5." (He wrote on the blackboard "(5 × 48) × 5)".)

Unlike the previous pair, these students did not draw a sketch to support the visualization of the transformation of the box stack into a rectangle; they only did it mentally. Then they calculated the number of apples by column, by doing 5 × 48. As they identified 5 columns, they then calculated 5 times the number of apples in each column. However, because they wrote their calculations as they were reasoning, they put factor 5 corresponding to 5 columns on the right since they wrote sequentially from left to right.

In case there were still students who did not understand this way of representing and thinking, it was important to clarify the expression "(5 × 48) × 5." The intermediate calculation of 5 × 48 allowed its translation according to the context of the task. Considering a rectangle with 5 columns (and 5 rows), the teacher might ask students for the meaning of 240, i.e., the number of apples in each column of boxes. From there, the meaning of 5 × 240 might be quickly associated with the total number of apples since there were 5 columns, each one with 240 apples. The relation between 5 × 240 and 240 × 5 (the expression used by Duarte and Tiago), which was not supported by the context itself, can be understood if we take into account these students' previous experiences. In mathematical terms, the equality between 5 × 240 and 240 × 5 is justified by the commutative property of multiplication, which the students already knew about—namely, when they did calculations associated with multiplication tables.

Still focusing on this solution, Isabel encouraged the class to ask for clarifications from Duarte and Tiago:

Enzo: "I would like to know how Duarte and Tiago did 240 × 5 so quickly."
Duarte (answering, thinking of 5 × 240): "We know that 5 × 4 is 20, so 5 × 40 is 200. And as we know that 5 × 2 is 10, we know that 5 × 200 is 1000. That's why we wrote 1200."

Duarte's explanation, besides underlying the distributive property of multiplication in relation to addition, is related to another fundamental idea of multiplication learning: the use of multiples of 10. By stimulating questions about powerful strategies of calculation and its corresponding explanation, Isabel promoted the students' development in terms of their level of learning multiplication.

Isabel's action led to possible answers to the questions that could guide the collective discussions mentioned above. Regarding the presentation and sharing of the selected solutions, the teacher organized two moments associated with each presentation. After the presentation by the first pair, she generalized the discussion with the whole class, giving an opportunity for students to contribute to it. The teacher highlighted aspects she considered relevant to the targeted solution. After the presentation by the second pair, Isabel organized a second collective discussion, which was another important moment of interaction and in which she had also a key role.

Regarding "bridge" building (Stein et al., 2008), Isabel picked up Eva and Guilherme's presentation and related it to other students' solutions, highlighting the equality between the two expressions that were the basis for each group to initiate the calculation. In the case of Duarte and Tiago, she encouraged them to clarify their solution and explain the way they thought it was associated with the rectangular arrangement. She was trying to understand if the other students understood it and, when doubts remained, she guided the discussion so the context could be used to facilitate the explanation. She also used the students' previous experience with the commutative property.

To allow students at lower learning levels to evolve, Isabel requested Eva and Guilherme to explain orally how they did some of their calculations, where powerful numerical relations associated with properties of multiplication were evident. She also encouraged Duarte and Tiago to clarify, when asked by a classmate, how they "quickly" did a certain calculation, highlighting important relations associated with multiples of 10.

The objectives of the task—to use the array model associated with the context and relate it to properties of multiplication—were highlighted throughout the discussion. Therefore, the teacher selected some solutions and, using the students' voices, related the strategies to the array model. From there, important multiplication ideas related to its properties emerged. These were explained by the students or highlighted by the teacher.

10.8 Implementing Learning Trajectories and Lesson Study: Perspectives on the Teacher's Role

By building a multiplication learning trajectory, we have sought to exemplify central elements of the teachers' action. Next, we will show some convergent aspects between the approach introduced here and the lesson study approach (Isoda and Olfos, 2009) to the teacher's role, and we will discuss the possible contributions of such contexts to teachers' professional development.

10.8.1 The Teacher's Role

One central aspect in both approaches is the importance given to careful lesson planning. Within the scope of lesson study, Isoda and Olfos (2009) set six challenges associated with lesson planning, where we can see some similarities with the learning multiplication trajectory developed by us: (i) description of the mathematical situations in context to be addressed in the lesson, (ii) characterization of the different tasks assigned to students and to teachers at different moments of the lesson, (iii) time limits and organization of the different moments of the lesson, (iv) anticipation of the students' behaviors and products, (v) preparation of possible interventions by the teacher to guide the class toward the proposed goal, and (vi) selection and preparation of the materials and means for the lesson. Next, we will explain how each of these challenges set by Isoda and Olfos (2009) are similar to the perspectives guiding our work.

In connection with the first challenge, and in the case of the set learning trajectory, both the sequences of the mathematical tasks and each task itself are carefully thought through. For each task, there is a clear description of its objectives, with identification of the learning milestones and models involved in the solution of a problem in context. Their preparation follows the criterion of the interconnection between tasks, considering, for example, the numbers involved from task to task and the contexts promoting the use of certain models or strategies. The contexts of the tasks are also important, since they should be a challenge for the students and lead them to want to explore them.

Teachers bring well-planned and previously explored tasks to the classroom. We also see these aspects reflected in a similar way in lesson study—namely, in challenges (i) and (vi). As shown earlier in this chapter, planning the learning trajectory involves, among other aspects, anticipation of the strategies used by students to find solutions. This aspect is also included in the characteristics referred to in challenge (iv), as well as the possible difficulties the students may face, according to their different levels of mathematical development during the trajectory in question.

Another similarity between the two approaches is related to the nature of the mathematical tasks proposed. In both cases, the selection of suitable problems and how they are explored in the classroom demands a very particular focus from the teacher. In fact, according to Isoda and Olfos (2009), by solving good problems, the students may gain new knowledge by applying previously learned knowledge. This characteristic of progressing in knowledge by solving problems is also included in the set trajectory, since the aim is to integrate, in each new task, knowledge and strategies developed in previous tasks. Therefore, in both approaches, problem solving is seen not simply as application of knowledge but as an opportunity to generate new knowledge.

However, for this to happen, we have to consider the way the problem is explored in class and how teachers and students' activities are organized (challenges (iii) and (v)). Both approaches have a social dimension, since, for each task, time is reserved for presenting and discussing the students' solutions. These moments are seen as opportunities to deepen the students' learning, favoring the understanding of concepts. Thus, in both approaches there is clearly a time limitation and organization of the different moments of the lesson (challenge (iii)) and characterization of the different tasks assigned to students and to teachers at different moments of the lesson (challenge (ii)), particularly at the moment of the discussion of students' exploration of the task.

This crucial moment in the lesson has to be prepared beforehand. Considering the task's objectives and the mathematical ideas that students develop in finding a solution, the teacher will select and order the solutions that will be presented in front of the class. So, as Isoda and Olfos (2009) state, the teacher has to "study the students' possible answers beforehand in order to ensure a flow and a progression pace and avoiding inactivity" (p. 166). As shown in the previous section, these options also aim to promote communication between students presenting strategies with different levels of mathematical sophistication, favoring not only progression of those still at less developed stages but also learning improvement of the other students. In fact, by becoming aware of the various possible solution processes and by reflecting on them under the teacher's guidance, the students can develop a deeper understanding of the mathematical knowledge involved.

In this respect, we stress again the importance of the teacher's role, since teachers have to lead students to make connections among the various solutions found in class and to highlight the most powerful representations and the mathematical ideas underlying the strategies presented. In one of the examples shown above, we saw clearly how the teacher could relate a less sophisticated solution to one of the most powerful ideas associated with the topic: the distributive property of multiplication in relation to addition. This is an aspect also pointed out by Isoda and Olfos (2009) regarding the teacher's role at this stage of the lesson: "The main task of the teacher is to listen to students, understand their point of view, connect it to the class objective, and guide the next moments" (p. 165). Thus, when assuming this role, teachers focus on listening and understanding students' reasoning, building "bridges" with the learning milestones of the ongoing trajectory.

10.8.2 Opportunities for Professional Development

The implementation of the learning trajectory presented here occurred in a collaborative context between one primary classroom teacher and one researcher. We could imagine a similar scenario in a curricular development project, a research project, or a teacher training program where, despite the different roles, collaborative work could be developed (Goodchild, 2014). We also find here some overlapping points with lesson study, which (according to Isoda, Arcavi and Mena (2007)) usually includes a cycle with the following steps: planning, the research lesson, and the reviewing lessons. These can then be repeated in two or more implementation cycles with other teachers. As pointed out by these authors, all these processes occur in collaboration with other teachers, higher education teacher educators, and, possibly, supervisors from local educational authorities.

In the case presented here, the teacher and the researcher undertook a very meaningful and extended process to adapt and constantly improve the learning trajectory. They met every week to reflect on each class and to plan the next ones. The researcher attended the classes and followed up on the students' work, also contributing to the teacher's decision making in the lessons—namely, the decisions related to the collective discussion moments. The teacher is a professional who was always willing to learn and reflect on topics she considered could improve her performance and her students' learning quality, seeing this experience as an important opportunity for professional development.

Still, we have to consider that the workload involved in the preparation and implementation of a learning trajectory like the one shown here is huge. It is an ambitious project that needs to be accomplished with the support of one or more experts. While recognizing that this work cannot be developed by the teacher alone, teachers may adapt and put this idea into practice under certain conditions; namely, they can collaboratively develop learning trajectories that are more limited in time, involving the preparation of fewer tasks or tasks already tested by themselves or others, which will allow acquisition of knowledge of students' strategies and difficulties.

The set of ideas for teaching multiplication developed in this chapter can be adapted to particular contexts and to the specific curricular guidelines of each country and each grade. Furthermore, we consider that the materials presented here could be used in initial and in-service teacher education. These can allow future teachers to get to know several students' strategies and reasoning, helping them to understand all of their potential. Such materials can also lead teachers to question and discuss the learning of multiplication, thus becoming a starting point for reflection about their practice and for motivating themselves to discuss and improve it.

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Chapter 11
Can We Explain Students' Failure in Learning Multiplication?

Maria del Carmen Chamorro

11.1 Problem Presentation

Volumes have been written about teaching multiplication, and no didactic manual is without at least one chapter dedicated to this issue. Precisely for this reason, it is paradoxical that the teaching of multiplication, to which much time is dedicated, continues to be so deficient, and the results of students' learning of it is so mediocre. This issue is not trivial if one considers that it is knowledge that should be acquired in compulsory elementary education and that is aimed at giving future citizens the necessary general education to deal with common problems in everyday life.

The problems students encounter, at least in Spain, are of four kinds.

First, to give up on memorizing results, long considered an outdated and aberrant pedagogical method that has resulted in poor mastery of the multiplication table, which makes students take a long time to carry out multiplication of, for example, three digits by two digits, making the activity tedious, as well as leading to many errors in the results. This circumstance seems to exceed the limits of a given country. As such, at a conference held in Santiago, Chile, in February 2003, Guy Brousseau stated that:

> In recent times, French teachers made students (and thus their parents) responsible for learning the multiplication table, given that they considered learning it to be too repetitive and non-technical. When teachers today assume this responsibility again, they do so, using the same methods as parents (simple repetition). Emptied of content and of mathematical supports, this learning loses part of its interest and efficacy.[1]

[1] Conference presentation by G. Brousseau in Santiago, Chile, in February 2003.

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Research on brain function has shown that memorization through oral recitation has a high cost, as well as not being mathematically pertinent:

> Starting school supposes a radical change in mental arithmetic. One moves from an intuitive knowledge of numerical quantities, in which counting dominates, to arithmetic learned by memory. This great change, not coincidentally, is concurrent with the first difficulties in mathematics. Often, progressing in mathematics implies storing in memory great quantities of numerical information, a task that our brains are not prepared for. Children adapt to this as well as they can, but, as we will see, they often lose all intuitive understanding of arithmetic operations (Dehaene, 2003).

Dehaene (1997) postulates that to master elemental arithmetic, our brains use at least two formats to represent numbers: a symbolic format, based on our language faculties, which is used for manipulating symbols and numerical algorithms; and a kind of language-independent representation that is located in brain circuits associated with visual and spatial processing, which is used for approximate calculation of numerical quantities. Elemental arithmetic capacities are obtained as the result of the dynamic integration of these two kinds of representations.

Second, the multiplication algorithm universally taught and used socially—the Fibonacci algorithm—is not precisely the most adequate and it presents innumerable inconveniences: the necessity of retaining in memory the amount carried while a result from the multiplication table is being found; placement of the partial results obtained by multiplying the multiplicand by each digit of the multiplier, in a way that is difficult for students to understand and is often unjustified; errors in placement when there are intermediate zeros in the multiplicand or multiplier; lack of control, when an error is produced, in finding its origin; etc.

Third, the understanding of the meaning of multiplication is not worked on enough, which leads to not identifying situations that can be solved with a multiplicative calculation. So, we find ourselves with schoolchildren who can apply the multiplication algorithm but are unable to resolve a simple multiplication problem, and ask their teachers the classic questions “Is it with addition?” “Is it with multiplication?” etc.

Finally, it must be said that we have practically never seen schoolchildren taught, simultaneously with the operative techniques, control mechanisms that allow them to evaluate, with the teachers’ sanction, if the result obtained when carrying out multiplication has an aspect of verisimilitude or, on the contrary, is clearly incorrect or even ludicrous. The reigning didactic contract indicates that the responsibility of the student ends when he or she provides a number as a result of the multiplication, without ever including, as part of the student’s work, deciding whether or not it is correct, which is a competency only of the teacher.

Classical learning of multiplication is based on mechanization; this mechanization reaches both the multiplicative repertoire and the learning of the algorithm—an algorithm given to the student ready-made, without an express concern for the student discovering the usefulness and pertinence of the intervening mechanisms, which necessarily leads to lack of motivation and interest. The wide array of alternative algorithms (lattice multiplication or gelosia multiplication, Egyptian multiplication, Russian multiplication, etc.) are not contemplated to give the student the choice of the algorithm that is most understandable or best suited to the numbers in question.[2]

It is evident that these four problems, the causes of which we analyze below, are interrelated and reinforce each other, and that one cannot be competent in calculation when conceptual understanding is not guaranteed and the calculation methods utilized are not understood.

11.2 Multiplication of Natural Numbers in the Curriculum

Operations with natural numbers have made up part of the elementary education curriculum in all countries of the world since long ago, and, as such, the contents are fixed and not up for discussion, although the same does not occur with the issue of how they should be taught.

The National Council of Teachers of Mathematics (NCTM, 2003) indicates in its curricular standards—as goals from the third grade to the fifth grade, in the part regarding understanding of the meaning of operations—the following:

- Understand diverse meanings of multiplication and division.
- Understand the effects of multiplying and dividing natural numbers.
- Identify and utilize the relations among operations (division as the inverse operation of multiplication, for example) to solve problems.
- Understand and utilize properties of the operations, for example, the distributive property of multiplication with respect to addition.
- With regard to fluency and estimation of calculations, it indicates:
- Develop fluency in the basic combinations of multiplication and division and utilize them to mentally carry out calculations related to them, for example, multiplying 30 times 50.
- Develop fluency in the four basic operations with natural numbers.
- Develop and utilize strategies for estimating the results of calculations with natural numbers and judge the reasonableness of these results.
- Choose and use appropriate methods and tools (mental calculation, estimation, calculators, pencil and paper) to calculate with natural numbers, according to the context and nature of the calculation in question.

These indications would be accepted today in almost all countries, although it does not follow from this—and this is what is curious—that the methodology applied in classrooms leads in all cases to achieving these goals.

[2] This fact is so notorious that during the initial education of future elementary teachers, the teaching students are amazed when they are told that there are other ways to multiply and confess that they have always thought that there is only one way to do it: the traditional way that they learned in school. Also, they tend to be unable to justify the placement of the partial results and need to be convinced that the multiplication of the digits of the multiplier must always be done in the order units, tens, hundreds . . .

Entering a bit more into a vision of the future of what teaching calculation can lead to, the results of the Kahane Commission, created by the French Ministry of Education to reflect on mathematics teaching, have been published. One of the chapters in this nearly 300-page study by Kahane (2002) is dedicated to teaching calculation, and some of its recommendations that we consider most insightful are the following:

- Mental calculation can play an important role in linking calculation and reasoning, and exact calculation and approximate calculation in elementary school.[3] If we want to achieve this role, it should not be the result of routine and memorization but should be associated with diverse calculations strategies.
- For mental or written calculation to be effective, it must be supported by a minimum memorized repertoire.
- Working on thinking calculation[4] is essential for developing mathematical properties and concepts.
- The importance given to calculation algorithms is in decline, as exact numerical calculation done today with a pencil and paper is very limited, so it does not seem reasonable for the school to dedicate so much time to it, nor to demand a high level of competency from students in this area. Having available a reliable algorithm for simple cases seems sufficient.
- Greater interaction between calculation with a calculator and calculation with a pencil and paper, as a function of the goals of each situation, is desirable.

The reductionist image of calculation as a mechanical, automatable, and unintelligent activity must be fought against, as well as the idea that learning it is a purely repetitive process. Calculation should be thoughtful, beginning with initial education, and related to reasoning and proof.

The Spanish curriculum is regulated by Royal Decree 1513/2006, which establishes educational minimums in primary education[5] and defines mathematical competency regarding number algorithms as follows:

> Mathematical competency implies the ability to follow certain processes of thinking . . . and apply some calculation algorithms.

Later, in block 1, dedicated to numbers and operations, it gives the following methodological indications:

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[3] In the work previously cited, Dehaene explains how the human brain is gifted with continuous and approximate representation. When our brains are presented with a number in a symbolic form such as “8” they immediately make an effort to convert it into a continuous quantity, and do so automatically and unconsciously. In this way, our brains allow us to find meaning in the symbol “8” as a quantity contained between 7 and 9, closer to 10 than to 2.

[4] Thinking calculation is not the same as mental calculation; it is halfway between mental and written calculation. Intermediate steps can be written, but procedures more similar to mental calculation than written calculation tend to be used.

[5] It can be obtained online at http://www.educacion.es/educacion/que-estudiar/educacion-primaria/contenidos.html.

> Numbers should be used in different contexts, with the knowledge that understanding of the processes developed and the meaning of the results is a prior and priority context compared to skill in calculation. Of principal interest is the ability to calculate with different procedures and the decision in each case regarding which is the most adequate.

In each of the cycles, the corresponding contents are detailed:

First cycle (first and second grade):

- Utilization of multiplication in familiar situations to calculate the number of times
- Oral expression of the operations and the calculation
- Construction of the multiplication tables for 2, 5, and 10, based on the number of times, repeated sum, arrangement in grids . . .
- Development of personal strategies for mental calculation . . . for calculating doubles and halves of quantities
- Approximate calculation; estimation and rounding of the result of a calculation to the nearest ten, choosing among various solutions and evaluating reasonable answers

Second cycle (third and fourth grade):

- Utilization of multiplication as an abbreviated sum, in rectangular arrangements, and combinatorics problems in familiar situations
- Additive and multiplicative decomposition of numbers; construction and memorization of the multiplication tables
- Utilization of standard algorithms for adding, subtracting, multiplying, and dividing in problem-solving contexts
- Utilization of personal strategies for mental calculation
- Estimation of the result of an operation on two numbers, evaluating whether or not the answer is reasonable

We can conclude that the Spanish curriculum follows the fundamental recommendations of the NCTM, although we appreciate that certain issues that we consider vital to the understanding of the meaning of the operation are not given the weight they deserve (understanding diverse meanings of multiplication, understanding the effects of multiplying and dividing natural numbers, identifying and utilizing relationships among operations—division as the inverse operation of multiplication, for example—to solve problems). Also, few indications are given regarding how to construct multiplication tables or how to arrive at the calculation algorithm, nor are the advantages of teaching one algorithm or another analyzed.

As strengths of this curriculum, we recognize the references to the need for working on mental calculation and estimation, as well as the use of the calculator. While it is accepted that students create personal calculation procedures, these seem to be limited to the domain of mental calculation and not applicable to written calculation.

If we compare this to the Chilean curriculum, it can be appreciated principally that the latter is more detailed and explicit, providing more indications regarding what to do and how to do it. We consider the strong points of the Chilean curriculum to be the proportionality approach to multiplication and its simultaneous treatment with division. We also find the learning order of the multiplication tables reasonable (2, 5, and 10 first, as the first thing children learn is to count by twos, by fives, and by tens). We share practically all the indications in the teaching guide that we have been able to read,[6] which give very precise indications of how to proceed in the classroom to reach the definitive algorithm, and, as such, we believe that if teachers follow these indications rigorously, it will lead to the success of the students. We can summarize by saying that it is a good curriculum, and, as such, the causes of scholastic failure must be looked for in other areas—for example, in how teachers apply this curriculum or in the training they have for its concrete interpretation.

Our knowledge of the Japanese curriculum is limited to what is described by Isoda and Olfos (2009), and we have been amazed to see the degree of detail and meticulousness in the Japanese government's mathematics teachers' teaching guide in the development of content related to multiplication. We appreciate, as a distinctive feature of the Japanese curriculum, the importance granted to the manipulation of material, often considered "not very mathematical" in other cultures (e.g., in Spain), as well as to graphic representations (in particular, to numerical patterns) and how much time is dedicated to ensuring student comprehension of the meaning of an expression, without ignoring the acquisition of calculation procedures. We regard the disciplined participation of students in the development of the lesson as definitive for achieving the stated results, but we consider it difficult to extrapolate to Latin societies, where, unfortunately, the intrinsic motivation of mathematics itself is not usually enough to stimulate the desire to learn.

11.3 Contributions to Didactics

Recent research in the didactics of mathematics gives emphasis to considering multiplicative calculation, and arithmetic in general, as a means for comfortably and effectively resolving problems that present themselves in students' daily lives, giving more importance to the meaning of operations than to the speed reached in using calculation algorithms. Currently, a universally accepted methodological principle is that more time and attention should be dedicated to dealing with situations that give meaning to multiplication, with less time dedicated to memorization and repetition of the corresponding standard algorithm, as numerical competency cannot exist if it is not based on conceptual competency (Fig. 11.1).

In Gerard Vergnaud's words:

> Mathematical competency can be defined with relatively variable criteria:
>
> (a) Someone who knows how to deal with situations and solve problems is more competent than those who do not;
> (b) Someone who solves problems in the most efficient, most reliable, fastest, most general, or conceptually most elaborate way is more competent;
> (c) Someone who has a variety of alternative means for solving problems of a certain category and can choose the appropriate method as a function of the values of certain parameters of the situation is more competent (Vergnaud, 2001).

[6] We have been able to access only what is described in the book by Isoda and Olfos (2009), and not the original documents.

Fig. 11.1 Students discuss in pairs in the math lab

If we apply the previous case to multiplication, the result is that we should aspire for students to be able to distinguish in the case or when they encounter a situation that demands a multiplicative calculation, to know which is most appropriate (as a function of the numbers that appear), a) and to use a calculator, b) to use an algorithm that requires a pencil and paper, or c) to use thinking or mental calculation. From this, it is easily deduced that standard learning of multiplication—which dedicates many hours to learning the traditional algorithm and does not provide or teach alternative, personal calculation methods, ignores the existence of mental calculation, and dissociates problem solving and calculation—cannot educate schoolchildren with the necessary numerical competency.

11.3.1 What Does the Theory of Conceptual Fields Teach Us?

One of Vergnaud's most important contributions in his theory of conceptual fields (Vergnaud, 1990) has been to effectively show how some concepts relate to others and the necessity of considering the different contexts in which a concept appears. In the case at hand, it refers to not dissociating (as habitually happens) work with multiplication, division, and proportionality, as the situations that demand their use form part of the same conceptual field.

Gerard Vergnaud (1981), as early as his first texts, made manifest the necessity of carrying out an exhaustive study of the different types of situations in which multiplicative calculation participates, and which, as such, give meaning to the operation, leading to his well-known classification of multiplicative problems as isomorphism of measures, product of measures, and single measure space. This classification not only informs us about the level of difficulty of each of these types—which on its own helps us to explain many students' errors and difficulties and the different rates of success and failure in one type or another—but also shows us the different contexts in which the necessity of multiplying appears. It is necessary for the teacher to be familiar with these contexts in order to be able to provide students with all the variety of situations that give meaning to the concept of multiplication, as it must not be forgotten that recognition of situations that can be dealt with using multiplication is much more important than having an effective multiplication algorithm. Making students face this variety of situations will obligate them, in the best of cases, to adapt, modify, and generalize problem-solving procedures, and to abandon them and construct new ones in other cases. What we call learning is nothing other than an individual's capacity to decontextualize a concept or procedure and then recontextualize it again, and in doing so make necessary adaptations or changes.

Working and systematically observing the different problem-solving procedures for a multiplicative situation ("approximative" in Piaget's language, or "working on schema" in Vergnaud's[7]) helps students to discover operative invariants and is useful for the teacher not only to be able to determine with greater precision the levels of skill reached by the students, but also to follow a logical teaching progression adapted to the students' competencies, as:

> The cognitive function of a subject or of a group of subjects in situation is based on the repertoire of previously formed schemata available to each of the subjects considered individually.

As a consequence of this, there is unanimous agreement in didactics regarding the necessity of making students face, from the very beginning, situations of a multiplicative character, without needing to wait for students to have available algorithms or advanced procedures for numerical resolution. Thus, emerging techniques like drawing the situation and then counting will lead to the iterated sum of equal summands, which is useful for giving meaning to the calculations, so that students always know what they are calculating in order to respond to a concrete question. This is the technique that is developed in many situations observed and studied by Isoda and Olfos (pastries in a box with various layers, and knowing how many there are in each layer; pastries in various boxes, and knowing how many there are in each box; pastries that fit in a box, and knowing the size of the pastries; balls in various containers, and knowing how many fit in each container; pencils in stacked boxes of pencils; etc.).

[7] Vergnaud defines the concept of a schema as "the invariant organization of behavior for a given class of situations." Subjects' knowledge in action should be investigated in a schema—that is, the cognitive elements that allow for a subject's action to be operative. The expressions "knowledge in action" and "theorem in action" designate knowledge contained in a schema, which can also be designated by the more global expression "operative invariants."

11.3.2 Developing Didactic Progressions for Teaching Multiplicative Calculation

Students should experience in class something that is intrinsic to mathematics: the need to debate the truth or falsity of an affirmation, the search for more effective solutions for solving a problem, and practicing debate as a means to answering these questions. The first edition of the book by Isoda and Olfos (2009) for teaching multiplication expertly shows something that many do not consider, but that has enormous importance in learning mathematics: that mathematical knowledge is built collectively in the little society of the class, which is why student motivation is needed.

Guy Brousseau, considered the father of the modern didactics of mathematics, says this on the topic (Brousseau, 1995):

> As a social practice, proof is the legitimate method of convincing an interlocutor: the interlocutor should be respected, using nothing except his or her repertoire (logical, mathematical, scientific . . .) and the information he or she currently has available, and other means of pressure—rhetorical (formal ability), psychological (such as seduction, authority, or compassion) or material (threats, violence, etc.)—should be avoided.
>
> In mathematics, knowing how to prove an affirmation, justify a result, etc., is part of one's own learning of the material, but the practice of proof is constructed here very differently than how it tends to occur socially. There is a series of psychological barriers to overcome, as the person who is correct is not always the most powerful or the most socially valued, but rather the one who can prove their arguments to be valid, so our self-esteem is often compromised. The truth in mathematics is not associated with power, which conflicts with social habits. Even the teacher is obligated to demonstrate that what he or she says is true. Authority is not enough. Nor are things true based on voting, nor can we support our friends' answers based on loyalty if these answers are not correct. The instrument of this initiation is learning proofs, not only as official knowledge, but also as a way of practicing proofs (and of limiting them to their domain of relevance). It forms part of the individual and particularly of the rational individual, just as much as the most essential social relations do. Democracy cannot exist without a social organization that integrates the role of knowledge in decision making and without shared and correct management of knowledge, truth, and proof. In primary school, this fundamental civic formation is not formulated, but it first happens in mathematics (see Fig. 11.2).

Fig. 11.2 Researchers take notes from pairs' discussions

For debate to arise naturally in class, and not as an imposition by the teacher, a problem should be proposed that makes sense to the students and allows them to use personal or group strategies that can be compared and validated. The situations have to be designed so that the knowledge the students possess at that moment allows them, if not to solve the problem completely, at least to understand the solution and an outline of the solution (a base strategy). We should consider that whatever is being learned, students always possess prior knowledge, which is often partial or incorrect, and one of the teacher's tasks is precisely to begin with this prior knowledge and make compatible something that is very important in calculation: the use of these personal procedures and the acquisition of faster and more effective universal algorithms.

If the students' knowledge were sufficient to resolve the situation, we would be in a situation of application of prior knowledge, not a learning situation. As such, the students' base strategies must be shown to be insufficient or not very effective, and the students should progress to be able to successfully solve the problem proposed in the situation (modification of schema, generalization, or construction of new schema).

As we have seen in the text from Isoda and Olfos, numerical learning requires considerable periods of time, and, as such, a family of interconnected situations must be designed—that is, didactic engineering (see Chamorro, 1999, 2003, 2004).

One of the first examples of didactic engineering—developed at COREM (Centre d'Observation por la Recherche en Enseignement des Mathématiques de Bordeaux) in 1985 and, as such, under the supervision of Guy Brousseau himself—is about multiplication and is clearly based on his theory of situations. Despite the 25 years that have passed, and everything that has happened in didactics in that time, some of its guiding principles remain relevant today:

Part I

Introducing multiplication through the need for rapidly counting the number of elements in a collection structured, or susceptible to being structured, in equal parts. The multiplicative structure $a \times b$ appears as a comfortable and effective way of designating the total number of elements in this collection, a manipulable collection at first, and later a represented collection. The need for using writing is connected to a situation of communication between teams: sending a message with a written multiplicative expression allows the receiver to form the corresponding collection (3 sessions).

Designation as a product of a collection arranged in the form of a table, using the number of elements per row and per column (1 session) (see Fig. 11.3 and 11.4).

Designation of products in the form $a \times b$ (4 sessions).

Part II

Comparison of numbers (near 250) written in the form $a \times b$ (1 session).

First calculation methods for $a \times b$ based on a multiplicative repertoire (3 sessions).

For example (see Fig. 11.5), find the value of 7×15 using the following repertoire: $4 \times 6 = 24$, $3 \times 6 = 18$ . . . $4 \times 6 = 28$, $7 \times 7 = 49$ . . .

$2 \times 7 = 14$
15

Fig. 11.3 Collections arranged in the form of a table

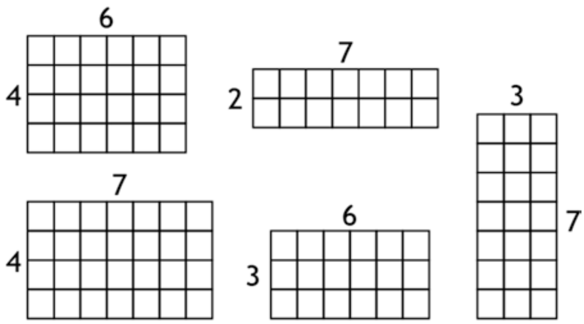


Fig. 11.4 Products as tables of rows and columns

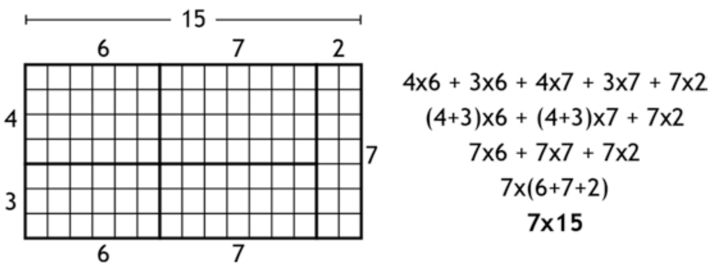


Fig. 11.5 Distributive property applied to arrays

Part III: Abandoning Graph Paper

First sessions (2 or 3). Find the total number of squares in a grid, for example 24 × 18 (see Fig. 11.6), using only blank paper and a multiplicative repertoire.

Second group of sessions (3 or 4). Progressive elaboration of a complete solution based on parts, using fundamentally the dimension 10 (see Fig. 11.7).

$$24 \times 18 = 11 \times 7 + 11 \times 8 + 11 \times 3 + 13 \times 7 + 13 \times 11$$

Last group of sessions (3 or 4). Search for the results of products provided by the teacher, without using graph paper, which can only be used for checking results.

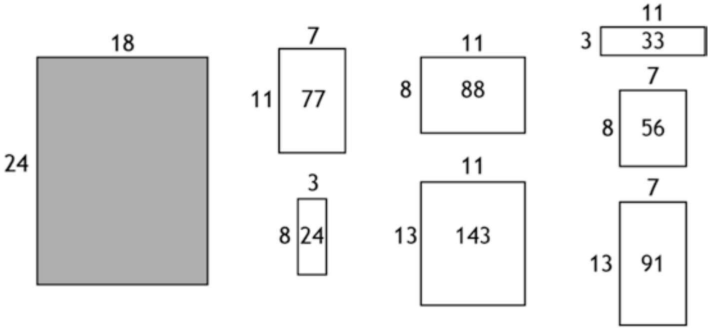
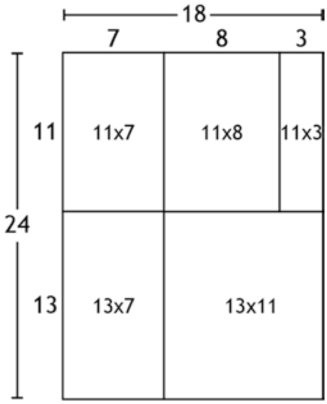


Fig. 11.6 Task visual information

Fig. 11.7 Representation of a solution



Part IV: Fine Tuning an Algorithm (Lattice Multiplication)

- Organization and observation of the product of one-digit numbers (tables) (1 session).
- Rule of zeros: calculate in 1-step products like 20 × 30, 7 × 80, general rule (4 to 7 sessions).
- Organization and arrangement of calculations, connected through additive decomposition of the factors (tens and units) and the distributive property.
- Reduction of the decomposition (3 to 6 sessions).
- Institutionalization of the algorithm, preferably lattice multiplication (1 session).

In parallel, mental calculation and solving multiplication problems are worked on.[8]

Part V

- Counting a collection (4 sessions): squares in a grid (43 × 32, 46 × 32, 56 × 37, 234 × 526 . . .) posted on the board, using blank paper or graph paper.

[8]The problem statements are of the following type: "A train has 9 cars, each with 18 seats and 4 wheels. How many children can sit in the train?" or "A construction worker wants to put tiles in a bathroom. The tiles come in boxes of 10. The worker places 5 rows of 6 tiles each. How many tiles has the worker placed?"

We could say that the ERMEL group (Equipe de Recherche des Mathématiques de l'Enseignement Elementaire)[9] continues the paths introduced by the Bordeaux IREM (Institut de Recherche pour l'Enseignement des Mathématiques), which, at the same time, was heavily influenced by research carried out by Guy Brousseau.

In the guidelines in ERMEL's latest edition, some principles to be followed in teaching multiplication can be observed:

- Reinforce what has been learned about decimal numeration.
- Introduce multiplication through iteration situations in which collections formed of subcollections of the same number of elements participate, or situations whose resolution requires a repetition of actions that imply adding or subtracting repeatedly the same quantity.
- Build the meaning of multiplication through the set of problems that belong to the multiplicative conceptual field.
- Give preference to multiplicative problems of the direct proportionality type in which the student can use known procedures that should evolve and adapt to new situations.
- Abandon graph paper, despite its advantages (easy geometric observation of the commutative property, easy management of the decomposition of products, multiplicative writing of $a \times b$ as a designation of a number and not as a calculation, etc.) due to the long and difficult process that must be followed to reach the Fibonacci algorithm if all the steps are followed.
- Do not separate multiplicative problems from the associated division problems.
- Construction of multiplication tables based on a series of multiples: discovery of the rule of zeros (using commutativity, iterated summation, or multiplicative decomposition of the numbers).
- Construction of the multiplication technique by the students.
- Insist on processes that allow for solving products through mental calculation (successive doubling, using multiples of 10, decomposing a number, etc.).
- Encourage the use of processes that can solve products through mental arithmetic (successive duplications, the use of multiples of 10, decomposition of numbers, etc.).
- Construction of the operatory multiplication technique through summing of multiples of the multiplicate of the type ×10, ×20, etc.

The above treatment is achieved over 2 years (in the third and fourth years) by presenting several different situations that must be resolved using multiplication procedures, as well as games aimed at the acquisition and memorization of sets or the use and discovery of mental arithmetic techniques (dominoes, battle games, bingo, etc.).

[9] Since 1977, this group has been publishing various manuals aimed at preschool teachers, elementary teachers, and teacher trainers—manuals that have collected practically all the research results in didactics of mathematics at the elementary level, and that, as such, constitute an obligatory reference (see https://forums-enseignants-du-primaire.com/topic/78945-ermel/). Through the various successive editions, one can appreciate the evolution that has occurred in the teaching of different mathematical concepts.

In our opinion, although the models and underlying multiplication structures are mathematically clear, there are unresolved questions in all known didactic approaches to multiplication, implying the need for in-depth study, analyzing the proposals given by the teacher in this regard. For example, if the teacher begins with situations that demand repeated addition, how can we justify that $a \times b$ is equal to $b \times a$ when one of the factors is measurement with dimensions? To find the process of calculating 4 bags of flour that cost €2 a bag, the correct answer is to do 2 + 2 + 2 + 2, since calculating 4 + 4 would be absurd and make no sense, even though 4 + 4 = 2 + 2 + 2+ 2. Nevertheless, something that can easily be seen, even without wanting to see it, is that the number of objects arranged in 2 rows of 4 is the same as when they are arranged in 4 rows of 2.

Despite this, this difficulty is mainly seen in solving problems in which it is necessary to maintain the meaning of the operations being carried out, keeping the connection with what is represented by the problem data. Thus, in the calculation of a multiplication ($2 \times 4 = 4 \times 2$),[10] the pupil must search for the best way to solve the problem, meaning that the commutative property is greatly helpful.

Perhaps the only possible solution is always to propose the answer to a problem using the form that makes the most sense, clearly separating it from the calculation stage of actual multiplication, though it is then necessary to recontextualize the result obtained in order to ensure it makes sense.

11.4 Informal Arithmetic Methods

For many years, several researchers have questioned the importance of the common practice of teaching arithmetic algorithms, relative to the lack of consideration of informal arithmetic procedures used by pupils in daily life, often in parallel with the usual algorithms from school. The result is that in the eyes of the pupils, the school has a different way of doing things from daily life, and they are unable to realize that they are dealing with procedures that aim to find solutions to the same problem.

Resnick and Ford (1990) use data obtained by Lankford to conclude the following:

1. The thought patterns/arithmetic strategies pupils develop when studying basic mathematics are highly individual, and they often do not follow orthodox models from textbooks or the classroom.
2. Differences can be seen in . . . the arithmetic strategies of pupils that are successful and those that are not.

[10] The operation written as "2 × 4" is not read the same in all cultures. In Japan it would mean "2 + 2 + 2 + 2" whereas in Spain it is read as "2 times 4" and would therefore be written as "4 + 4." We understand that although one of the two forms is more advantageous to the construction of tables, cultural tradition is far stronger, and it would be wrong for a school to go against social/mental norms.

Fig. 11.8 Grouping in five groups of five. (Reproduced from Tsubota, 2007)

3. Indications can be found for teaching that support arithmetic ability based on pattern observation . . . by pupils who do incorrect calculations.

It can also be said that the use of informal arithmetic procedures is mostly among the pupils, and not all those who use them make mistakes.

For example, for counting the quantity of spots (see Fig. 11.8) some children will see the five spots of a die in five locations on the tile, while others will move the spots from the four corners into a new location, turning the tile into a 5 × 5 square.

Many of the informal multiplication methods used by pupils are based on a common pattern: counting (2 by 2, 3 by 3, etc.). It is therefore important to include this type of exercise in mental arithmetic work. We tend to think that this method, which can appear simple and primitive, is only used by first-year pupils, but the reality is that it remains in use by pupils in later years. Lankford found that of 176 seventh-year pupils, 63 (36%) used counting when doing multiplication. It is precisely the use of the counting algorithm that causes pupils to have difficulty in dealing with and retaining multiplication results in memory work, while also making them take longer in finding the results.

Therefore, the idea of pupils memorizing multiplication tables lies in the aim of transitioning from the counting algorithm to recalling numerical facts from long-term memory; i.e., numerical facts can be recalled from long-term memory almost immediately, thus freeing up resources in the working memory for immediate results and consequently decreasing the number of errors. This does not mean that the pupil will no longer make mistakes, as it is known that remembering a numerical result is more complex than recalling it from long-term memory, since numerical facts are strongly connected, even when they apply to different operations, and it is easy to activate an incorrect result such as "2 + 7 = 14" or "2 × 7 = 9."

It is also known that the mistakes made by pupils are more systematic than random. They respond to a certain logic, and this often originates from a lack of understanding of the procedures implied in algorithms, which are therefore applied incorrectly (see Fig. 11.9). It is precisely this logic that makes many errors persistent, since the same incorrect procedure is repeatedly applied. It is therefore important for teachers to take time to observe these errors and identify which procedures they come from. If this is not done, they will be unable to help their pupils overcome the errors.

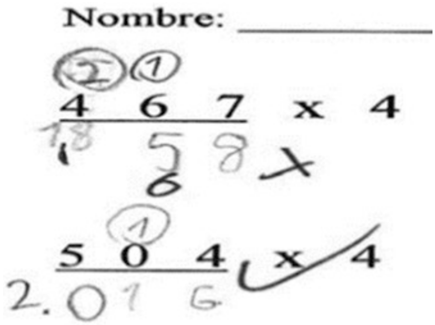


Fig. 11.9 Student' work and mistake

If the aim of teaching arithmetic is the development of understanding, research must be done into the informal procedures that pupils use in daily life to ensure that the teacher can help the child make a connection between their formal mathematics learning in school and their everyday practices. Baroody (1988) recommends that any formal expression of the type 3 × 4 = 4 + 4 + 4 + 4 is always linked to real experiences that have meaning for the pupil. The pupil can then establish connections with her/his own informal knowledge, and the formal symbolism of the mathematics avoids becoming something hollow.

This becomes all the more significant when considering some results of research into how the human brain functions. For example, when asking why the results of memorizing multiplication tables are so mediocre (a lot of time is spent on memorization and repetition of the tables, with very poor results, as pupils get confused and forget many of the multiplications despite the number of hours spent on it), Dehaene provides some very interesting clues, such as the very structure of the multiplication tables themselves. Furthermore, to make the difficulty experienced by children when learning this for the first time more understandable to adults, he replaces the list of numbers 0, 1, 2, 3, . . . with a list of names and replaces the multiplication with a workplace, giving a table such as the following:

- Carl David works in Richards-Brown Street (3 × 4 = 12)
- Carl William works in Brown-Richards Street (3 × 7 = 21)
- William Pierce works in Carl-Pierce Street (7 × 5 = 35)

It is obvious that memorizing the results above is a very difficult task. This is because our memory is not structured like that of a computer. It is associative[11] and it weaves several different connections between very different pieces of information; this is at the same its strength and weakness.

[11] Since human memory is associative, it weaves innumerable connections between very different pieces of information that in turn are activated regardless of whether they proceed or not, which happens from a very early age. When learning multiplication tables, it is vitally important that numerical facts are not mixed with facts relating to other operations, giving the result of a sum or a difference instead of a product. However, the human memory has difficulty saving the results of different operations separately. As a result, it is easier to notice that "2 × 4 = 9" is wrong than that "2 × 3 = 5" is wrong.

As Dehaene says, it is interesting to recall the behavior of a lion when we see a tiger, but it is disastrous to activate knowledge of $7 + 6$ or 7×5 when we want to know 7×6. Interference and inappropriate association are the basis of the failure to memorize the multiplication tables.

The errors are not random, and incorrect answers are always numbers that are on the multiplication tables somewhere, often in the same line or column as the result the pupil is looking for. Considering that our brains use continuous and approximative representation, it is reasonable that when searching our long-term memory for the answer to 7×8, the results of 7×9 and 6×8 are also activated. The brain also has difficulty saving additions and multiplications separately. This explains why we are quicker to see that "$2 + 4 = 8$" is wrong and slower to spot whether "$2 \times 4 = 6$" is wrong. Similarly, it is easier to see that "$2 \times 4 = 7$" is wrong than to see that "$2 \times 4 = 6$" is wrong. It is also known that the difficulty in recalling a numerical fact from long-term memory depends particularly on the number of associations that cause interference—so-called interfering associations (Bideau and Lehalle, 2002)—which vary with the development of the individual and are activated when looking for an answer to a problem.

About 80% of errors arising when learning the multiplication tables are of the type described above, and the so-called distance effect can be seen (van Hout, Meljac, and Fischer, 2005). For example, the error "$7 \times 8 = 42$" has a result from the adjoining table ($7 \times 6 = 42$). Only 13% of errors are not related to inverted numbers, such as "$8 \times 7 = 54$"; since 54 is 6×9, it is not in the 8 table or the 7 table. When multiplication and some other operation appear together in the same class or in the same problem, the errors that appear are consistent with swapping the operations (e.g., "$8 \times 7 = 15$" or "$4 + 2 = 8$"), and this type of error can account for up to 30% of errors. Only 7% of errors are those of the type where the answer is not related to the numbers or the operations, for example: "$5 \times 9 = 26$."

Since it is known that the brain is associative, if the table has been built and learned by the pupils by establishing connections between the results—as shown in the guidelines of PROMETAM [Proyecto Mejoramiento en la Enseñanza Técnica en el Área de Matemática] (Secretaría de Educación, Honduras, 2007) and described in classes by Professor Tsubota (2007) or in the texts of ERMEL (1993, 1995)—activating 7×5 can be helpful if we know that the next answer is found by adding another 7. This means that if we want to be more effective with less effort, we should adapt the way tables are taught to what is known about how the brain stores and recalls information in the long-term memory, favoring semantic learning of multiplication tables.[12] However, pupils in primary education do not in general spontaneously seek out this type of method, meaning that it is necessary to encourage discovery of the properties of multiplication.

[12] The term "semantic" assumes comprehension and serves to differentiate between rote learning (which is the most common type and is not based on comprehension or connections) and associations between products. Semantic learning of multiplication tables uses the associative characteristic of memory, which is useful to find, for example, 6×9, using procedures such as $(6 \times 10) - 6$, or $(6 \times 6) + (6 \times 3)$.

Of the multiplication tables, special mention must be made of the 1 and 0 tables, since they can be learned by the general rules: anything multiplied by 0 is 0, and anything multiplied by 1 is itself. It has been shown that access to numerical facts does not work in this case, such that results presented as $n \times 1 = n$ and $m \times 0 = 0$ are recalled from memory through selective rules that can be lost or confused, meaning that errors affect all the answers in the table and not just certain numerical facts, as is the case with the other numbers.

Some researchers, such as McCloskey and Macaruso (1994), posit that the cognitive system related to numerical treatment is structured into modules and comprises:

- A comprehension system
- A production system
- An arithmetic system

The first two can in turn be divided into two subsystems, one related to Arabic numbers and the other to verbal names. The third has three components: knowledge of the operation symbols, the arithmetic procedures, and the numerical facts saved in long-term memory. According to this model (see Fig. 11.10), each operation has a network of different representations, which can easily explain the disassociation between operations in the minds of many schoolchildren.

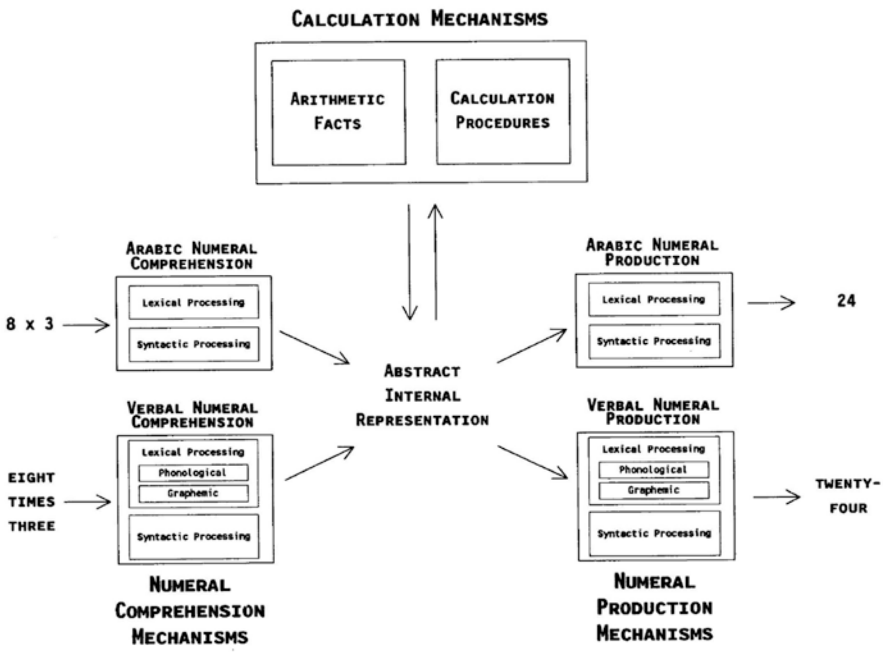


Fig. 11.10 The McClosky model. (From McCloskey, 1992, p. 113)

For Dehaene, each number is represented by an analogous code in the form of a number line, an audiovisual code, and a visual–Arabic code, and each of these codes is used for different tasks. Specifically, multiplications and some simple additions, learned routinely by memory by some pupils, are coded verbally, while the results of subtractions and divisions are learned and solved by the application of rules that involve semantic manipulation (e.g., $68 - 17 = 68 - 20 + 3 = 48 + 3 = 48 + 2 + 1 = 51$), and therefore the analogous representation of the quantities. This fact is neurologically linked with tasks carried out by each of the two hemispheres of the brain; thus, arithmetical operations are only possible for the left hemisphere, while both hemispheres can recognize whether two numbers are identical and perform counting, though the latter is done more easily by the left hemisphere.

For the treatment of calculation difficulties, educators/teachers should insist on the presentation of situations (to provide activities for students) in the varieties of codes, use verbal, written and Arabic numerals interchangeable.

The Japanese method of teaching the multiplication tables, as is done in schools in many countries, also involves memorization through repetition—i.e., using verbal memory to store phrases such as "three times four is twelve" easily in the memory. It should be noted that the verbal memory stores this phrase on the same level as the phrase "two thousand bees appeared on the honeycomb"—i.e., a sentence without any numerical meaning.

There are many studies, dating from 1967, that confirm Asian superiority in mathematics and, in particular, that of Japanese[13] students. Some factors that explain this superiority are the following:

- Schoolwork is of a large quantity and high quality, with pupils dedicating considerably more extracurricular time to schoolwork than South American students. In particular, as stated in the text cited above, they spend a lot of time not only on systematic work to learn arithmetic but also on solving situations that require the application of that arithmetic.
- The attitude of parents, being more demanding and ambitious with the progress of their children.
- The culture of competition within schools (the text by Isoda and Olfos (2009) describes this aspect very well), putting additional pressure on students to obtain good results in school.
- Motivation based on the idea that work and effort are important virtues that are absolutely necessary for success in later life. These aspects are clearly seen in the classroom studies described in the aforementioned text.

In addition to these factors, there is also the numerical language. The uniqueness of the Japanese[14] language allows for much shorter sentences than those possible in Spanish, since they omit the word "times," thus facilitating memorization.

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[13] In the text by Fischer (2002, pp. 215–237), several studies can be found on the implementation of these aspects, particularly in the USA.

[14] See Isoda and Olfos (2009), p. 50.

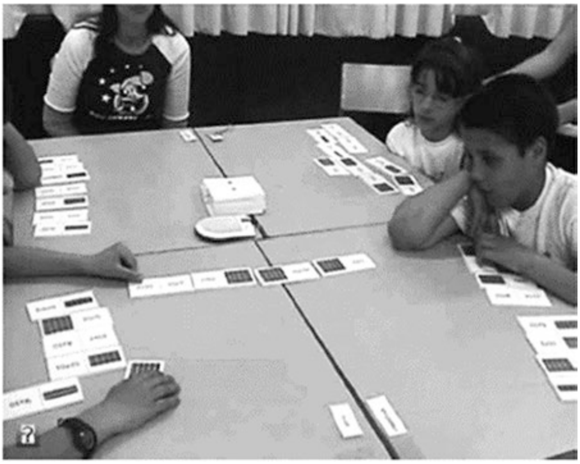

Fig. 11.11 Dominoes for connecting arrays with products

With regard to oral numeration, Asian systems of number words are fully regular, while oral numeration in Spanish is very irregular and uses the different powers of ten as its basis, with each one given a specific word: *diez* ("ten"), *cien* ("one hundred"), *mil* ("one thousand"), *diez mil* ("ten thousand"), *cien mil* ("one hundred thousand"), *un millón* ("one million"), etc. Furthermore, a different word is needed to designate each of the numbers from zero to fifteen. The words *once* ("eleven"), *doce* ("twelve"), *trece* ("thirteen"), *catorce* ("fourteen"), and *quince* ("fifteen") are plainly irregular, as are *veinte* ("twenty"), *treinta* ("thirty"), *cuarenta* ("forty"), *cincuenta* ("fifty"), *sesenta* ("sixty"), *setenta* ("seventy"), *ochenta* ("eighty"), and *noventa* ("ninety"). Asian systems, on the other hand, are fully regular and the composition of a number is clearly apparent in its name; for example, the word for "eleven" transliterates as "ten one," the word for "twenty-five" as "two tens and five," etc. All of these make the names of the numbers easier to learn for Asian pupils in general and for the Japanese in particular. This oral construction of numbers also makes it easier to avoid many errors that commonly arise in arithmetic, by combining the cardinal meaning with the name of the number.

However, doing arithmetic in Spanish requires a pre-established connection between the written number and the number words used in oral numeration (since the multiplication tables are learned orally)—i.e., understanding the quantitative meaning of the written form, which is evidently more complex (see Fig. 11.11).

11.5 Do We Have to Teach Algorithms?

It is evident that learning arithmetic algorithms is more costly in terms of classroom hours and the effort and failure of pupils, leading us to ask the question as to whether this effort is worthwhile in mathematics teaching.

Before answering this question, we would like to examine one of the most significant causes of pupils' failure in arithmetic, which goes unnoticed by many teachers: the lack of understanding of decimal numbers.

The positional principle that governs decimal numbers[15] is based on a considerable mathematical apparatus. It should not be forgotten that although all numbers involve an expression of a polynomial of 10 to different powers, their normal abbreviated written form, which removes the powers and leaves only the coefficients, works with a norm for reading and writing the numbers based on the value of the position—i.e., it is the position that allows us to interpret the value of the number.

Kamii conducted an experiment to determine children's level of comprehension of the place value (Kamii, 1985). Basically, the test consisted of asking children to associate the number 16 with a number of corresponding tokens and then indicate how many tokens each of the numerals 1 and 6 represented in a drawing. The results were surprising: only 51% of the fourth-year pupils, 60% of the sixth-year pupils, and 78% of the eighth-year pupils drew ten tokens to represent the 1 in 16.

It is clear that, as such, a large percentage of pupils have difficulties understanding place value, even in older age groups. The number of pupils who will fail in arithmetic, particularly in applying classical arithmetic algorithms, will also be very large, since they are almost all based on the properties of decimal numbers. The solution that many pupils find to this problem is rote learning without understanding the steps of the algorithm; thus, they lose control over what they are doing. For them, the path, the act of placing numbers in classical multiplication, going to one place if there is a zero in the multiplier, etc., are purely mechanical acts, lacking explanation; it is done like that merely because it is, and, as Baroody states:

> Although children recall basic information learnt by memory, this does not guarantee intelligent use of that information. Deep down, many of them learn arithmetic but do not learn mathematics. These problems are made worse when the exercises and repetitions lack any interest and meaning. All too often, mass teaching becomes an obstacle to meaningful learning, thought, and problem solving (Baroodi, 1988, p. 55).

We work to help students learn automated procedures mechanically as arithmetical algorithms (learning arithmetic), but they do not know how they are built and what they are for (learning mathematics). The pupils accumulate easily assessable knowledge, but they cannot use it in a meaningful way because it is not part of their interests or the solution to any problem. Adding to the difficulties that children have, for the reasons detailed above, when relating the name of a number to its cardinal meaning, the panorama facing teachers when teaching algorithms is not promising.

We should also ask ourselves about the usefulness of algorithms in daily life and their frequency of use. Many of us have never done a multiplication with a three-digit number after leaving school, and when it has been necessary, we have used a calculator or an estimate, depending on whether an exact answer was needed. This cannot be denied, but it is not sufficient to conclude that schools should adopt measures to promote other type of arithmetic, both mental and use of a calculator, instead of spending time on learning algorithms.

[15] See Isoda and Olfos (2009), p. 50.

In our opinion, using a written arithmetic algorithm to multiply numbers with three digits is a waste of time and takes a great deal of effort for most pupils. However, these same pupils can acquire knowledge of number theory solely through observation, which can be done simply by letting them use a calculator freely. This can be enhanced by discoveries guided by questions proposed by the teacher: magic squares, numbers whose squares are palindromes, numbers whose products do not change when the numbers are written backward (e.g., 36 × 84 = 63 × 48), random numbers, the pole of a number, etc.

For multiplication of two-digit numbers by two-digit numbers, we recall the use of the distributive property and the automation of simple results, mainly multiples of 10, later adding the results without the need for putting them in place, as is done with the Fibonacci method.

To calculate 36 × 28, we can do the following:

$$
\begin{aligned}
&36\times 28=(30+6)\times(20+8)=30\times 20+30\times 8+6\times 20+6\times 8\\
&30\times 20=3\times 2\times 10\times 10=600\\
&30\times 8=3\times 8\times 10=24\times 10=240\\
&6\times 20=6\times 2\times 10=12\times 10=120\\
&6\times 8=48\\
&600+240+120+48=960+48=1008
\end{aligned}
$$

or use mental arithmetic strategies, depending on the level achieved by the pupils—for example, using doubles which are often automated easily.

$$
\begin{aligned}
&36\times 28=(36\times 30)-(36\times 2)=(36\times 20)+(36\times 10)-(36\times 2)=\\
&720+360-72=1080-721008\\
&36\times 28=(40\times 28)-(4\times 28)=(28\times 2\times 2\times 10)-(28\times 2\times 2)=\\
&(56\times 2\times 10)-(56\times 2)=1120-112-1008
\end{aligned}
$$

Other nonconventional algorithms, such as Egyptian multiplication, are based on the process of doubling. In the case of the multiplication above, we have the following:

1	36	1	28
2	72	2	56
4	144	4	112
8	288	8	224
16	576	16	448
___	____	32	896
28	1008	____	_____
		36	1008

The Russian peasant algorithm.

* Write each number at the head of a column.
* Double the number in the first column, and halve the number in the second column.
* If the number in the second column is odd, divide it by two and drop the remainder.
* If the number in the second column is even, cross out that entire row.
* Keep doubling, halving, and crossing out until the number in the second column is 1.
* Add up the remaining numbers in the first column.
* The total is the product of your original numbers.

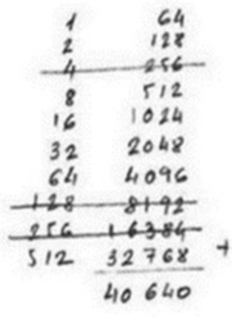

Fig. 11.12 Russian peasant multiplication algorithm

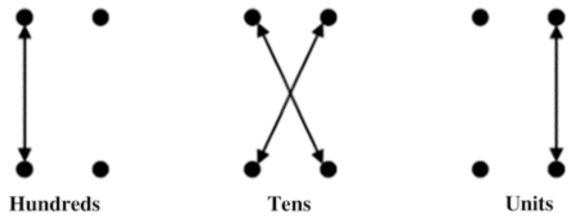

Fig. 11.13 Diagrammatic explanation of Russian multiplication

In the first case, we double 36, obtaining 4, 8, and 16 times 36 (in bold text), which can then be summed to find 28 times 36. On the right we can see that the result is the same if we double 28, obtaining 4 and 32 times 28, which are summed to give 36 times 28.

The diagram below is also useful, Fig. 11.13, as it can be followed mentally to find the product of two two-digit numbers (c.f. Fig. 11.12). For numbers with more than two digits, we believe that mental arithmetic is not appropriate; a calculator is.

If we apply the diagram above to 28 × 36, we have:

- Units: 8 × 6 = 48; we write the "8" and carry the 4 to be added to the tens figure.
- Tens: 2 × 6 = 12, 3 × 8 = 24, 12 + 24 = 36, 36 + 4 = 40; we write the "0" and carry the 4 to be added to the hundreds.
- Hundreds: 2 × 3 = 6, 6 + 4 = 10; we write the "10".

The result is 1008. The process can be done mentally, noting only the final result, but we can aid the process with a pencil and paper, writing down the intermediary steps, as described above.

In conclusion, we are left only to underline one of the ideas already described above: that arithmetic is not an end in itself but a means of solving problems quickly and effectively. Therefore, learning numerical facts or algorithms to the detriment of understanding and the meaning of the operation should be avoided at all costs. Attaining speed with arithmetic should not be an objective in school, and using fingers or other objects should not be seen as embarrassing or something to be discouraged in pupils. We can learn from the results of neuropsychology research, making us more understanding and tolerant of pupils' mistakes, allowing us to adapt our teaching methods to how the brain actually works, as this is the root of many failures in learning arithmetic.

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