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Twisted Isospectrality, Homological Wideness, and Isometry

A Sample of Algebraic Methods in Isospectrality

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A Sample of Algebraic Methods in Isospectrality

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Preface

The problems we deal with in this book date back at least to the Wolfkehl lectures *Alte und Neue Frage der Physik* delivered in Göttingen in 1910 by Hendrik A. Lorentz [63], who asked for the relation between spectra of operators (providing physically observable quantities, such as radiation) and geometric properties of the observed objects; more precisely, he asked—in modern terminology—whether for a geometric shape, the eigenvalues of the Laplace operator, with suitable boundary conditions, determine the volume of the shape, adding that "for several simple shapes where the computations can be done, this will be confirmed in a Leiden thesis".[1] That was the thesis of Johanna Reudler [82], published in 1912, done concurrently with Hermann Weyl's 1911 proof for the general case, using the theory of integral equations [100].

An even bolder but natural follow-up question is whether a geometric shape is completely determined by the eigenvalues.[2] Said more succinctly, "Does isospectrality imply isometry?", or—in the catchy formulation of Bers that was eternalised as title of the famous 1966 article by Mark Kac—"Can one hear the shape of a drum?". As the original question for planar domains with Dirichlet boundary conditions remained out of reach, the problem was considered for the spectrum of the Laplace-Beltrami operator on closed smooth oriented Riemannian manifolds. In that setting, the question was answered in the negative, first by John Milnor's construction of two isospectral, non-isometric tori of dimension 16 [71], then by Marie-France Vignéras' construction of such compact Riemann surfaces [97] and then by a general construction of Toshikazu Sunada [90]. The unifying principle behind these constructions is that they are *number-theoretical*, being based, respec-

[1] In [55], this is translated as "has been confirmed", complicating the search for the further unnamed thesis.

[2] Already in 1881, Arthur Schuster had pronounced that "It would baffle the most skilful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the desired direction" [85].

tively, on the arithmetic theory of quadratic forms, maximal orders in quaternion algebras and the theory of arithmetic equivalence. (As an aside, Sunada's result turned out to be a crucial ingredient in the final solution of the original question about planar domains in the negative by Gordon et al. [43]; see [26] for a popular account). This is the starting point for an approach to the problem of isospectrality that is rooted more in algebra (group theory, representation theory and number theory) than in analysis.

This book focusses on explaining and advancing that algebraic approach. The setup is the same as in Sunada's work: we are given a manifold (or, more generally, a developable orbifold) M_0 and two closed Riemannian manifolds M_1 and M_2 with a finite covering map to M_0. We find a G-Galois cover M of M_0 for some finite group G, such that M is a common finite cover of M_1 and M_2, corresponding to two subgroups H_1 and H_2 of G, respectively. Sunada shows that if H_1 and H_2 are weakly conjugate in G (meaning that the group representations of G corresponding to the permutation action of G on the cosets of H_1 and H_2, respectively, are isomorphic), then M_1 and M_2 are isospectral, but not necessarily isometric, or even covering equivalent (meaning that H_1 and H_2 are conjugate subgroups of G).

A new feature of this work is that we give a spectral characterisation of when M_1 and M_2 are equivalent Riemannian covers (in particular, isometric), assuming a representation-theoretic condition of "homological wideness", that involves the action of G on the first homology group of M. The condition holds, for example, when there exists a rational homology class on M whose orbit under G consists of $|G|$ linearly independent homology classes. We prove that, under this condition, Riemannian covering equivalence is the same as isospectrality of finitely many twisted Laplacians on the manifolds, acting on sections of flat bundles corresponding to specific representations of the fundamental groups of the manifolds involved. Using the same methods, we provide spectral criteria for weak conjugacy and strong isospectrality. In the negative curvature case, we formulate an analogue of our result for the length spectrum. We study examples where the representation theoretic condition does and does not hold. For example, when M_1 and M_2 are commensurable non-arithmetic closed Riemann surfaces of negative Euler characteristic, there is always such an M_0, and the condition of homological wideness always holds.

As indicated above, the proofs are inspired by number-theoretical analogues[3] and use representation theory, and are thus firmly rooted in the tradition of the work of Toshikazu Sunada as well as Hubert Pesce. The notion of homological wideness might be of independent interest to topologists.

The methods combine concepts and tools from different fields, and we have been rather discursive on details, background and recaps. We hope the book also

[3] Ironically, the Wolfkehl lectures were funded by an endowment that was available as long as Fermat's Last Theorem had *not* been solved, so in a sense, we owe both the origin as well as the solution of the isospectrality problem to number theory, albeit in one case due to our prolonged inability to solve a number theoretical problem.

serves as an introduction to basic concepts and results in the algebraic approach to isospectrality, namely: around fiber products of orbifolds, twisted Laplacians, zeta functions and spectra of operators, strong isospectrality, G-sets, weak and strong conjugacy and isospectrality, monomial representations, wreath product realisations, homology groups in Galois covers and homology representations. We have also made sure the text is full of examples, including an entire chapter devoted to examples and counterexamples. At the end of most chapters is a short list of open problems or possible future projects.

The introduction gives a technical description of the new results. After the introduction follows a *Leitfaden*, where we list some of the more folklore (but difficult to find) results to be found in the current text.

It is our pleasure to thank Durham University and Utrecht University for their hospitality. We thank Jens Funke for sparking this project by bringing us together at the 2017 BMC. At various points, we received help, triggers and feedback from Alex Bartel, Stefan Bechtluft-Sachs, Mikolaj Frączyk, Nigel Higson, Gerhard Knieper, Matthias Lesch, Juan Pablo Rossetti, Jeroen Sijsling and Harry Smit and we thank all of them wholeheartedly. We also thank the reviewers for their feedback on an earlier version.

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List of Symbols

General

$\mathrm{mult}_a(A)$	Multiplicity of an element a in a multiset A
$\|A\|$	Cardinality of a finite (multi-)set A; sum of the multiplicities of all elements of A
$A \cup B$	Disjoint union of multisets; i.e., if an element has multiplicity a in A and b in B, then it has multiplicity $a + b$ in $A \cup B$
nA	Disjoint union of n copies of a multiset A; i.e., the multiset in which the multiplicity of every element of A is multiplied with n
$\mathrm{Aut}(A)$	Automorphisms of A
$\mathrm{Hom}(A, B)$	Homomorphisms from A to B
$\mathbf{Q}, \mathbf{R}, \mathbf{C}$	Field of rational, real and complex numbers, respectively
$\overline{z}$	Complex conjugation of $z \in \mathbf{C}$; e.g., applied to representations and characters of groups, so if χ is a character on a group G, then $\overline{\chi}$ is the character defined by $\overline{\chi}(g) := \overline{\chi(g)}$ for all $g \in G$
e_i	Standard basis vector in a vector space $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a "1" in place i, and zeros otherwise
$\mathbf{Z}$	Ring of integers, where a decorated version $\mathbf{Z}_D$ for a boolean valued expression D on $\mathbf{Z}$ means the elements of $\mathbf{Z}$ that satisfy condition D; e.g., $\mathbf{Z}_{\geq 0}$ are non-negative integers
$\mathbf{Z}/n\mathbf{Z}$	Group or ring of integer residue classes modulo n
$\mathbf{F}_\ell$	Finite field with ℓ elements; if ℓ is prime, isomorphic to $\mathbf{Z}/\ell\mathbf{Z}$
gcd	Greatest common divisor
$R - \mathrm{mod}$	Category of finitely generated modules over a ring R
dim	Dimension
$\mathrm{ord}_{s=s_0}$	Order of a meromorphic function in s at a given point s_0

Manifolds

deg	Degree (of a map of manifolds)
∂M	Boundary of a manifold M
$M_1 \bullet_{M_0} M_2$	Compositum of two manifolds M_1 and M_2 that are both finite covers of a developable Riemannian orbifold M_0
$M_1 \times_{M_0} M_2$	Fiber product of two manifolds M_1 and M_2 that are both finite covers of a developable Riemannian orbifold M_0 (possibly disconnected, unequal to the compositum, and—in the orbifold setting—possible distinct from the set-theoretic fiber product)
$\widetilde{M}_0$	Universal covering manifold of a manifold or developable orbifold M_0
$\pi_1(M, x)$	Fundamental group of a connected manifold M with base point x; also denoted $\pi_1(M)$ if the base point is irrelevant
$*$	Concatenation of paths
$\mathrm{H}_1(M)$	Homology group of a manifold M, with integral coefficients, so $\mathrm{H}_1(M) = \pi_1(M)^{\mathrm{ab}}$
$\mathrm{H}_1(M, K)$	Homology group of a manifold M, with coefficients in a field K; by the universal coefficient theorem, $\mathrm{H}_1(M, K) = \mathrm{H}_1(M) \otimes K$
χ_M	Euler characteristic of a manifold M
$\chi_{M_0}^{\mathrm{orb}}$	Orbifold Euler characteristic of an orbifold M_0
h	(Rational) homology representation of a group G acting on a manifold M, i.e., $h\colon G \to \mathrm{H}_1(M, \mathbf{Q})$
Λ	Virtual Lefschetz character
$\mathbb{H}^n$	Hyperbolic n-space
$L(q; s_1, s_2, s_3)$	Five-dimensional lens space corresponding to the angles $2\pi s_i/q$
$\ell(\gamma)$	Length of a closed geodesic γ
h_M	Volume entropy of a negatively curved closed Riemmanian manifold M; equal to topological entropy

Groups

$\rtimes$	Semidirect product of groups (normal subgroup on the left)
$\mathrm{conj}_{\gamma_0}(\gamma)$	Conjugate element $\gamma_0\gamma\gamma_0^{-1}$ in a group containing γ and γ_0
$\mathrm{Out}(G)$	Outer automorphisms of a group G, so $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ where $\mathrm{Inn}(G)$ is the group of *inner* automorphisms, given by conjugation with a fixed element of G
$C_G(g)$	Centraliser of an element g of a group G
$[G : H]$	Index of a subgroup H in a group G
$[a, b]$	Commutator of two elements a, b of a group, i.e., $[a, b] = aba^{-1}b^{-1}$

G^{ab}	Abelianisation of a group G, i.e., $G/[G, G]$, where $[G, G]$ is the normal subgroup of G generated by all commutators; isomorphic to $\mathrm{Hom}(G, \mathbf{C}^*)$, the group of linear characters of G
$[g]$	Conjugacy class of an element g of a group; or free homotopy class of a loop
$\langle \Gamma_1, \Gamma_2 \rangle$	The subgroup generated by two subgroups Γ_1 and Γ_2 of a given group
$\Gamma_1 \Gamma_2$	The set of products $\gamma_1 \gamma_2$, where $\gamma_i \in \Gamma_i$, subgroups of a given group
$\mathrm{Comm}_G(\Gamma)$	Commensurator of the subgroup Γ of the group G
$\mathrm{GL}(N, K)$	Group of invertible $N \times N$ matrices over a field K
$\mathrm{U}(N, \mathbf{C})$	Group of complex unitary $N \times N$ matrices
tr	Trace

Representations

$K[G]$	Group ring of a group G over a field K
$\mathrm{Irr}(G)$	Set of irreducible representations of a group G
$\mathbf{1}_G$	Trivial group representation; also denoted $\mathbf{1}$ if the group G is fixed
$\rho_{G,\mathrm{reg}}$	Regular representation of a group G
$\chi_{G,\mathrm{reg}}$	Character of the regular representation of a group G
$\mathrm{Ind}_H^G \rho$	Representation of a group G induced from the representation ρ for the subgroup H of G
$\mathrm{Res}_H^G \rho$	Representation of a subgroup H of a group G induced from the representation ρ of G
$\langle \rho_1, \rho_2 \rangle$	Inner product of the characters corresponding to two representations ρ_1, ρ_2 of a finite group, in the space of class functions
$\mathscr{M}_G$	Coinvariants of a group G acting on a G-module $\mathscr{M}$
I	Kernel of augmentation map $R[H] \to R$ on a group ring $R[H]$
$\mathrm{H}^1(G, \mathscr{M})$	Group cohomology of a group G with values in a G-module $\mathscr{M}$

Operators

$C^\infty(M, E)$	Smooth sections of a bundle E on a compact differentiable manifold M
$L^2(M, E)$	Completion of $C^\infty(M, E)$ on a closed Riemannian manifold M w.r.t. the inner product corresponding to the volume form
$\sigma_M(A)$	Spectrum of A, i.e., the multiset of eigenvalues in $L^2(M, E)$ of a symmetric second order elliptic differential operator A acting on $C^\infty(M, E)$ for some Hermitian bundle E with non-negative eigenvalues; also denoted $\sigma(A)$ if the manifold is fixed

Δ_M Laplace(–Beltrami) operator on a Riemannian manifold M; also denoted Δ is the manifold is fixed

Δ_M^k Laplace operator on k forms a Riemannian manifold M; also denoted Δ^k is the manifold is fixed; e.g., $\Delta = \Delta^0$

σ_M Laplace spectrum of a Riemannian manifold M, i.e., $\sigma_M(\Delta_M)$

$\Delta_{M,\rho}$ Twisted Laplace(–Beltrami) operator on a Riemannian manifold corresponding to a unitary representation of the fundamental group $\pi_1(M)$; also denoted Δ_ρ when the manifold is fixed

$\sigma_M(\rho)$ Spectrum of the twisted Laplace(–Beltrami) operator Δ_ρ on a Riemannian manifold M, i.e., $\sigma_M(\Delta_{M,\rho})$, so $\sigma_M = \sigma_M(\mathbf{1})$; also denoted $\sigma(\rho)$ is the manifold is fixed

$\zeta_{M,A}$ Spectral zeta function of an operator A on a Riemannian manifold M; i.e., $\zeta_{M,A}(s) = \sum \lambda^{-s}$, where $\lambda \in \sigma_M(A)$ with $\lambda \neq 0$; also denoted ζ_A is the manifold is fixed

ζ_M Spectral zeta function of a Riemannian manifold M; so $\zeta_M = \zeta_{M,\Delta_M}$

Z_M Selberg zeta function of a negatively curved closed Riemannian manifold M

$L_M(\rho)$ L-series of a unitary representation ρ of the fundamental group of a negatively curved closed Riemannian manifold M

Chapter 1
Introduction

The chapter contains an overview of the main new results in the book: a spectral characterisation of covering equivalence of Riemannian manifolds using spectra of finitely many twisted Laplacians, a more detailed version for non-arithmetic Riemann surfaces, a version using the length spectrum. It also contains a spectral characterisation of weak conjugacy. The chapter contains pointers forward to other chapters that contain an exposition of the basic tools and methods, as well as detailed constructions, proofs and examples.

1.1 Setup and Conditions

Let M_1 and M_2 be a pair of connected closed oriented smooth Riemannian manifolds. There exist such M_1 and M_2 that are not isometric, but *isospectral*, i.e., they have the same Laplace spectrum with multiplicities. This leaves open the question whether equality of spectra of other geometrically defined operators on M_1 and M_2 is equivalent to M_1 and M_2 being isometric. In this text we investigate the use of twisted Laplacians (acting on sections of flat bundles constructed from representations of fundamental groups) in answering this question. We will see that under two conditions on M_1 and M_2, equality of finitely many suitably twisted Laplace spectra implies isometry of the manifolds. That at least some condition is necessary for such a result to hold is illustrated by the existence of simply connected isospectral non-isometric manifolds [87].

The **first condition** is the following: we suppose that the manifolds M_1 and M_2 are finite Riemannian coverings of a developable Riemannian orbifold M_0 (meaning that the universal covering of M_0 is a manifold), expressed through a diagram

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$$\begin{tikzcd} M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd} \tag{1.1}$$

Such a diagram may be extended to a diagram of finite coverings

$$\begin{tikzcd} & M \arrow[dl, "q_1"', "H_1"] \arrow[dr, "q_2", "H_2"'] \arrow[dd, "q"', "G"] & \\ M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd} \tag{1.2}$$

where M is a connected closed smooth Riemannian manifold M with $q_1 \colon M \twoheadrightarrow M_1 := H_1 \backslash M$, $q_2 \colon M \twoheadrightarrow M_2 := H_2 \backslash M$ and $q \colon M \twoheadrightarrow M_0 := G \backslash M$ Galois covers (Proposition 2.4.1); in particular, M_1 and M_2 are commensurable. Commensurability is in general weaker than the existence of a diagram (1.1) (see Proposition 2.5.4). However, if M_1 and M_2 are commensurable hyperbolic manifolds, then a diagram (1.1) (with an orbifold M_0) exists if the corresponding lattices are not arithmetic (Proposition 2.5.2).

The **second condition** is related to the action of G on the first homology group of M. Let $\mathbf{F}_\ell$ denote the field with ℓ elements. In terms of the data in diagram (1.2), we require that for some prime number ℓ not dividing $|G|$, the $\mathbf{F}_\ell[G]$-module $\mathrm{H}_1(M, \mathbf{F}_\ell) = \mathrm{H}_1(M) \otimes_{\mathbf{Z}} \mathbf{F}_\ell$ contains the permutation representation of G acting by left multiplication on the cosets G/H_i for either $i = 1$ or $i = 2$ (or both), i.e., $(\mathrm{Ind}_{H_i}^{G} \mathbf{1}) \otimes_{\mathbf{Z}} \mathbf{F}_\ell$ is an $\mathbf{F}_\ell[G]$-submodule of $\mathrm{H}_1(M, \mathbf{F}_\ell)$. Concretely, this means that if there are n cosets, then the $\mathbf{F}_\ell$-vector space $\mathrm{H}_1(M, \mathbf{F}_\ell)$ contains n linear independent vectors that are permuted in the same way as those n cosets under the action of any $g \in G$.

This second condition is implied by the stronger requirement that the G-action is $\mathbf{F}_\ell$*-homologically wide*, where, for a general field K, we say that the G-action is K*-homologically wide* if the regular representation $K[G]$ of G occurs in the homology representation $G \to \mathrm{Aut}(\mathrm{H}_1(M, K))$ (Lemma 8.1.2). This condition has the advantage of no longer referring to M_1 and M_2 (or H_1 and H_2).

An even stronger, geometrically tangible, condition is that the action of G on M is $\mathbf{Q}$-homologically wide; this means that there is a free homology class ω on M such that the elements $\{g\omega \colon g \in G\}$ are linearly independent in $\mathrm{H}_1(M, \mathbf{Q})$. This is diametric to the condition of homologically trivial group actions found more frequently in the literature (see, e.g., [99]). A $\mathbf{Q}$-homologically wide action is $\mathbf{F}_\ell$-homologically wide for any ℓ coprime to $|G|$ (Lemma 8.2.1).

In Chap. 9, we discuss $\mathbf{Q}$-homological wideness for certain low dimensional manifolds and locally symmetric spaces. For non-trivial fixed-point free group actions on orientable surfaces, $\mathbf{Q}$-homologically wideness is equivalent to any of the spaces M, M_0, M_1 or M_2 having negative Euler characteristic (Proposition 9.1.1).

A (not necessarily fixed point free) holomorphic group action on a Riemann surface M is $\mathbf{Q}$-homologically wide if the Euler characteristic of the quotient surface M_0 satisfies $\chi_{M_0} < 0$ (Proposition 9.1.2). In higher dimension, the picture can vary widely: for any $n \geq 3$, we use standard surgery methods to construct an n-manifold with a free action by any given non-trivial group that is or is not $\mathbf{Q}$-homologically wide (Proposition 9.3.1 and Corollary 9.3.3). Since locally symmetric spaces of rank ≥ 2 have trivial rational homology, no non-trivial group action on them can be $\mathbf{Q}$-homologically wide (see Sect. 9.4). On the other hand, in rank one, the condition relates to decomposition results for automorphic representations (see Sect. 9.5). A disadvantage of $\mathbf{Q}$-homological wideness is that it discards torsion homology; in Sect. 9.7 we give an example of some non-trivial $\mathbf{F}_5$-homologically wide group actions on the Seifert–Weber dodecahedral space (that has homology $\mathrm{H}_1(M, \mathbf{Z}) \cong (\mathbf{Z}/5\mathbf{Z})^3$).

1.2 Overview of Main Results

Our results involve spectra of *twisted Laplace operators* Δ_ρ corresponding to unitary representations $\rho\colon \pi_1(M) \to \mathrm{U}(N, \mathbf{C})$; these are symmetric second order elliptic differential operators acting on sections of flat bundles E_ρ over M; the sections are conveniently described as smooth vector valued functions on the universal cover that are equivariant with respect to the representation, and on these, Δ_ρ acts (componentwise) like the usual Laplacian of the universal cover (cf. Chap. 3). In fact, our representations will factor through a specific finite group, allowing for a very concrete description of the operators (cf. Sects. 3.6 and 3.7). Denote the spectrum of such an operator by $\sigma_M(\rho)$, where the index M indicates that the Laplacian is defined on sections over the space M.

Returning to the setup in diagram (1.1), we call M_1 and M_2 *equivalent Riemannian covers of* M_0 if there is an isometry between them induced by a conjugacy of the fundamental groups of M_1 and M_2 inside that of M_0 (since M_0 is developable, its universal covering is a manifold and we mean the subgroup of its isometry group that fixes M_0 pointwise, see Lemma 2.1.2). Our main result states that in this situation isometry can be detected via the spectra of finitely many specific twisted Laplacians.

Theorem 1.2.1 *Suppose we have a diagram* (1.1) *and suppose that the action of* G *on* M *in the extended diagram* (1.2) *is* $\mathbf{F}_\ell$*-homologically wide for some prime* ℓ *coprime to* $|G|$. *Then* M_1 *and* M_2 *are equivalent Riemannian covers of* M_0 (*in particular, isometric*) *if and only if the multiplicity of zero in the spectra of a finite number of specific twisted Laplacians on* M_1 *and* M_2 *is equal. In fact, at most* $2\ell|\mathrm{Hom}(H_2, \mathbf{C}^*)|$ *equalities suffice.*

For a precise formulation of the required twists and a more technical condition (denoted $(*)$) that is weaker than homological wideness, the reader is encouraged to glance at Theorem 6.4.1, which is a more detailed formulation of Theorem 1.2.1, our main result. The detailed formulation reveals that the representations occurring in the theorem are constructed explicitly using induction and restriction of special characters (termed "solitary" below) on groups corresponding to a specific finite Riemannian covering of M, whose existence is guaranteed by the assumption of homological wideness (or the weaker requirement $(*)$). Riemannian equivalence over M_0 is the same as conjugacy of H_1 and H_2 in G; a merit of the theorem is to show that this can be verified via a spectral geometric criterion using twists, where the twists on M_1 are constructed using information from M_2 and vice versa.

Sunada [90] showed that if we are given a diagram of the form (1.2), and H_1 and H_2 are *weakly conjugate* (meaning that the permutation representations given by the action of G by left multiplication on the cosets G/H_1 and G/H_2 are isomorphic), M_1 and M_2 are isospectral for the Laplace operator. The following example illustrates our theorem in such a situation.

Example 1.2.2 We provide the explicit data for what is maybe the oldest example of weakly conjugate subgroups, due to Gaßmann [38]:

$$G = S_6,\ H_1 = \langle (12)(34), (13)(24) \rangle,\ H_2 = \langle (12)(34), (12)(56) \rangle,$$

with both H_1 and H_2 isomorphic to the Klein four-group, but not conjugate inside S_6 [38, pp. 674–675]. As in [90, p. 174], choose a compact Riemann surface M_0 of genus 2 and a surjective group homomorphism $\pi_1(M_0) \twoheadrightarrow G$. This leads to a diagram of the form (1.2) [90, §2], and homological wideness is immediate from Proposition 9.1.1. In this case, M_1 and M_2 are inequivalent covers of M_0, but they have the same Laplace spectrum [90, §2]. We can set $\ell = 7$ in the main theorem, and, with the group G having order 720, inequivalence of M_1 and M_2 may be verified purely spectrally by checking 56 equalities of multiplicities of zero in the spectrum of twisted Laplacians corresponding to representations of dimension 180. $\diamondsuit$

To show the scope of our result, we discuss several more examples in Chap. 11. Our result should be contrasted with [4, Thm. 1.1], where it is shown that large non-arithmetic hyperbolic manifolds M admit arbitrary large sets of (strongly) isospectral but pairwise non-isometric finite Riemann coverings.

Our method of proof for Theorem 1.2.1, presented in Chaps. 2–6, is based on a similar construction of Solomatin [89] for algebraic function fields, which in turn is based on number theoretical work of Bart de Smit in [28]. The analogy between number theory and spectral differential geometry was pioneered by Sunada [90] (see also the survey [93]), and the importance of representation theoretical techniques was pointed out early on by Sunada [91] and Pesce [79]. We have given a self-contained presentation, with references to the number theory literature when appropriate. Our construction uses a certain wreath product of G with a cyclic group; in Sect. 7.4, we describe a universality property of such wreath products that should make their appearance look less surprising.

We have formulated our results using spectra, but the analogy to number theory becomes most apparent by instead using spectral zeta functions of (twisted)

Laplacians; that this is equivalent is explained in Sects. 3.2 and 3.10, pointing to some subtleties concerning the multiplicity of zero in the spectrum of more general operators.

From our earlier brief discussion of the two conditions in the theorem, we find the following specific result in dimension two.

Corollary 1.2.3 *Let M_1, M_2 be two commensurable non-arithmetic closed Riemann surfaces. Then they admit a diagram* (1.1) *and, assuming the corresponding orbifold M_0 satisfies $\chi_{M_0} < 0$, isometry of M_1 and M_2 can be checked by computing the multiplicity of zero in at most $4((\chi_{M_1}\chi_{M_2}/(\chi_{M_0}^{\mathrm{orb}})^2)!)^2$ twisted Laplace spectra, where $\chi_{M_0}^{\mathrm{orb}}$ is the orbifold Euler characteristic, defined in* (9.1).

The detailed formulation and proof can be found in Corollary 9.1.4. If one believes in a positive answer to the (open since several decades) question whether commensurability of such Riemann surfaces is implied by their isospectrality, then a purely spectral formulation of the corollary is possible, replacing "commensurable" by "isospectral". Intriguingly, the only cases where a positive answer to the open question is known are arithmetic [81], precisely the ones excluded in the corollary.

In Chap. 12, we study the analogue of Theorem 1.2.1 for the *length spectrum* on negatively curved manifolds. The statement about agreement of multiplicities of zero in certain spectra is changed into equality of the pole order of certain L-series (details are found in Theorem 12.2.4).

Theorem 1.2.4 *Suppose we have a diagram* (1.1) *of negatively curved Riemannian manifolds with (common) volume entropy h, and suppose that the action of G on M in the extended diagram* (1.2) *is $\mathbf{F}_\ell$-homologically wide for some ℓ coprime to $|G|$. Then M_1 and M_2 are equivalent Riemannian covers of M_0 if and only if the pole order at $s = h$ of a finite number of specific L-series of representations on M_1 and M_2 is equal.*

Sunada's result [90] quoted above says that weak conjugacy implies isospectrality, but the converse is not necessarily true; this leaves open the question to characterise weak conjugacy of H_1 and H_2 in G in a spectral way using the associated manifolds M_1 and M_2. One of our intermediate results answers this question using induction and restriction from the trivial representation $\mathbf{1}$.

Proposition 1.2.5 *If we have a diagram* (1.2)*, then H_1 and H_2 are weakly conjugate if and only if the multiplicity of the zero eigenvalue in $\sigma_{M_i}(\mathrm{Res}^G_{H_i}\,\mathrm{Ind}^G_{H_j}\,\mathbf{1})$ is independent of $i, j = 1, 2$.*

This result, reformulated in Proposition 4.2.4, is proven by an adaptation of a number theoretical argument of Nagata [73]. The crucial differential geometric ingredient is the spectral characterisation of the multiplicity of the trivial representation in any given representation (Lemma 3.9.1). As a corollary, we get a spectral characterisation of *strong isospectrality* (cf. Definition 4.2.2 and Corollary 4.2.5).

Open Problem

The most pressing question that remains open concerns the case where there is not necessarily a common finite orbifold quotient: are *twisted isospectral* manifolds M_1 and M_2 (meaning that there is a bijection of all unitary representations of their fundamental groups such that the spectra of the corresponding twisted Laplacians are equal) with *large* fundamental groups (i.e., containing a finite index subgroup with a non-abelian free group as quotient) isometric?

Project

Develop the theory when M_1 and M_2 are also orbifolds.

Project

Develop an analogous theory for graphs instead of manifolds.

Leitfaden

The *Leitfaden* contains a graph of dependencies between chapters, as well as a list of folklore results that can be found in the main text, of possible independent interest.

Interdependency Graph

The dependencies between chapters is indicated in the graph below. In boldface is the minimal path to reach the main results (Theorems 1.2.1 = 6.4.1 and Theorem 1.2.4 = 12.2.4), in regular font a geometric instead of purely group-theoretical construction, and in boxed font material on homological wideness and examples. A dashed arrow represents a weak dependence.

$$
\begin{tikzcd}
2 \arrow[dd] \arrow[drr, dashed, bend right] & \mathbf{3} \arrow[d, Rightarrow] & \mathbf{5} \arrow[d, Rightarrow] & & \\
& \mathbf{4} \arrow[r, Rightarrow] & \mathbf{6} \arrow[r, Rightarrow] \arrow[d] \arrow[dll, bend left] \arrow[drr, dashed] & \mathbf{12} & \\
7 \arrow[r] & 10 & \boxed{8} \arrow[l] \arrow[r] & \boxed{9} \arrow[r] & \boxed{11}
\end{tikzcd}
$$

"Folklore" Results with Forward Pointers

Apart from the main results, we have included several "folklore" results that are hard to find in the literature. We list some of these.

- The relation between (connected components of the) fiber product, compositum and commensurability (Chap. 2).
- The relation between the spectrum and the spectral zeta function for general symmetric second order linear differential operators acting on sections of Hermitian line bundles (i.e., the question whether the non-zero spectrum determines the multiplicity of zero in the spectrum); we show that they mutually determine each other in the odd dimensional case (Proposition 3.2.1), that there are obstructions to this result in the even dimensional case (in Proposition 3.2.2, we give an example where this is due to a non-vanishing $\widehat{A}$-genus, and in Remark 3.2.3 we exhibit an example of such two-dimensional orbifolds due to Gordon and Rossetti), as well as a proof that for twisted Laplacians on isospectral manifolds, they do determine each other also in even dimension (Proposition 3.10.1).
- The alternative proof (by Sunada in [91]) of his main theorem (that manifolds corresponding to a pair of weakly conjugate subgroups in a given group are isospectral) depending on a relation between spectra of twisted Laplacians (proof of Theorem 3.8.2).
- The theory of monomial structures for group representations, including Bart de Smit's result on the existence of characters whose induced representation has a unique monomial structure up to isomorphism (Chap. 5).
- A representation-theoretic characterisation of conjugacy of two subgroups of a group (Remark 5.3.4).
- The relation between the action of an isometry on the first homology group and the action of the corresponding outer automorphism on the fundamental group (Lemma 6.2.3).
- The universal property of the wreath product for manifold covers (Sect. 7.4).
- How to use the Lefschetz fixed point theory to determine the representation of a group of isometries acting on the first homology group (Sects. 9.1 and 9.2).
- A method of Cooper and Long to construct certain homology representations in dimension 3 (Proposition 9.3.1).
- Determination of the modular (Brauer-)character for Mednykh's computation of the homology representation of the isometry group of the Seifert-Weber dodecahedral space (Proposition 9.7.2).
- An analytic proof of the existence of split geodesics in a covering of negatively curved manifolds using the Ruelle zeta function, modelled on the corresponding proof of existence of split primes using the Dedekind zeta function in number theory (Proposition 10.2.1), without using Parry and Pollicott's much more general Chebotarev-style theorem.

Chapter 2
Manifold and Orbifold Constructions

In this chapter, we collect some information on various constructions of manifolds, orbifolds, and their covers. Notably, we discuss the notions of fiber product (in the sense of Thurston) and compositum of manifolds over a common developable orbifold, the difference with the set-theoretic fiber product, the connected components of the fiber product, compositum of Galois covers, normal closure of a cover, and the relation between commensurability, arithmeticity, and the existence of certain covers in relation to Mostow rigidity.

2.1 Riemannian Coverings and Their Equivalence

We fix the following notation for the remaining of the text: we assume that M_1 and M_2 are connected smooth oriented closed (i.e., compact with empty boundary) Riemannian manifolds that are *finite* Riemannian coverings of a developable Riemannian orbifold M_0, i.e., there is a diagram

$$\begin{tikzcd} M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd}$$

of the form (1.1). Recall that the connected space M_0 being a *developable orbifold* means that it is the quotient of a connected and simply connected Riemannian manifold $\widetilde{M}_0$ by a cocompact discrete group of isometries Γ_0, that is the (orbifold) fundamental group of M_0. For the theory of (good/developable) orbifolds, see, e.g., [95, Ch. 13] [25] [33, §1]. Given this setup, we can write $M_i = \Gamma_i \backslash \widetilde{M}_0$ for finite index subgroups Γ_i of Γ_0 acting without fixed points on $\widetilde{M}_0$.

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Remark 2.1.1 In dimension 2, there is a relation with the theory of branched coverings of surfaces [78, §3] (compare [95, Thm. 13.3.6]). Consider an orbifold cover $M_1 \to M_0$ of degree n, where M_0 is a closed developable 2-orbifold with r elliptic (cone) points $\{x_i\}$: the local orbifold structure is that of rotation over an angle $2\pi/e_i$. Assume that M_1 is a closed 2-manifold. To such an orbifold cover corresponds a branched degree n cover of the corresponding underlying closed surfaces $\Sigma_1 \to \Sigma_0$ branched over the r given points such that x_i has exactly n/e_i points above it with ramification index e_i (in particular, e_i divides n). A diagram (1.1) corresponds to a diagram of branched coverings of the corresponding surfaces

$$\begin{tikzcd} \Sigma_1 \arrow[dr, "p_1"'] & & \Sigma_2 \arrow[dl, "p_2"] \\ & \Sigma_0 & \end{tikzcd}$$

where, if p_j has degree d_j ($j = 1, 2$), we have the added data of r branch points $\{x_i\}$ on Σ_0 and r integers e_i dividing $\gcd(d_1, d_2)$ such that there are exactly d_j/e_i points above x_i in Σ_j for $j = 1, 2$ and $i = 1, \dots, r$.

Lemma 2.1.2 *Suppose we have a diagram of manifolds/orbifolds*

$$\begin{tikzcd} & M \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, "q"', "G"] \arrow[dl, phantom, "H_1", near start] \arrow[dr, phantom, "H_2"', near start] & \\ M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd}$$

of the form (1.2) *and* $\Gamma_0, \Gamma_1, \Gamma_2$ *are as above. Suppose* Γ *is such that* $M = \Gamma\backslash\widetilde{M}_0$. *Then the following are equivalent:*

(i) H_1 *and* H_2 *are conjugate in* G*;*
(ii) Γ_1 *and* Γ_2 *are conjugate in* Γ_0*;*
(iii) M_1 *and* M_2 *are equivalent Riemannian covers of* M_0*, i.e., there is an isometry* φ *such that following diagram commutes*

$$\begin{tikzcd} M_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd} \tag{2.1}$$

Proof With $H_i = \Gamma_i/\Gamma$ and Γ normal in Γ_0, we have that there exists a $g \in G$ satisfying $gH_1g^{-1} = H_2$ if and only if for any lift $\gamma_0 \in \Gamma_0$ of g, $\gamma_0\Gamma_1\gamma_0^{-1} = \Gamma_2$. This shows the equivalence of (i) and (ii). To show the equivalence of (ii) and (iii), we argue as follows. If $gH_1g^{-1} = H_2$ for $g \in G$, a (finite) group of isometries of M, then $\varphi\colon M_1 \to M_2$ given by $\varphi(H_1x) = H_2gx$ (for $x \in \widetilde{M}_0$) satisfies the requirements of (iii). Conversely, if M_1 and M_2 are equivalent Riemannian covers of M_0, then the corresponding subgroups Γ_1 and Γ_2 of the orbifold fundamental group Γ_0 are conjugate. □

2.2 Compositum

We set $M_{00} := \Gamma_{00}\backslash\widetilde{M}_0$ for $\Gamma_{00} := \Gamma_1 \cap \Gamma_2$. This is a finite connected common Riemannian cover of M_1 and M_2 (finiteness follows since $[\Gamma_0 : \Gamma_1 \cap \Gamma_2] \leq [\Gamma_0 : \Gamma_1] \cdot [\Gamma_0 : \Gamma_2]$), which we call the *compositum* $M_1 \bullet_{M_0} M_2$ of $p_1\colon M_1 \to M_0$ and $p_2\colon M_2 \to M_0$. Note that M_{00} is a manifold since Γ_i acts properly discontinuously without fixed points on $\widetilde{M}_0$. It is important to notice that the construction of M_{00} and the covering maps $M_{00} \to M_i$, $i = 1, 2$, depend on the actual maps p_1, p_2, not just the spaces M_0, M_1, M_2, but it is customary to leave out the maps from the notation if they are clear. If necessary, we will write $M_1 \; {}_{p_1}\!\bullet_{M_0,p_2} M_2$ (or $M_1 \bullet_{M_0,p_2} M_2$ if p_1 is clear).

2.3 Fiber Product

Given a diagram (1.1), we can also form the (orbifold) *fiber product* $M_1 \times_{M_0} M_2$ (or, more precisely, $M_1 \; {}_{p_1}\!\times_{M_0,p_2} M_2$, where we use the same convention as before concerning including or leaving out the maps from the notation). We follow the construction of Thurston [95, 13.2.4] as explained in [25, 4.6.1] (but where we are using left actions instead of right actions):

$$M_1 \times_{M_0} M_2 = \Gamma_0\backslash \left(\widetilde{M}_0 \times \Gamma_1\backslash\Gamma_0 \times \Gamma_2\backslash\Gamma_0\right),$$

where the left action of $\gamma_0 \in \Gamma_0$ is given by

$$\gamma_0(x, \Gamma_1\gamma_0^{(1)}, \Gamma_2\gamma_0^{(2)}) = (\gamma_0(x), \Gamma_1\gamma_0^{(1)}\gamma_0^{-1}, \Gamma_2\gamma_0^{(2)}\gamma_0^{-1})$$

for $x \in \widetilde{M}_0, \gamma_0^{(i)} \in \Gamma_0$ ($i = 1, 2$). This has the universality property required for fiber products. Using the bijection

$$(\Gamma_1\backslash\Gamma_0 \times \Gamma_2\backslash\Gamma_0)\,/\,\Gamma_0 \to \Gamma_1\backslash\Gamma_0/\,\Gamma_2\colon [(\alpha, \beta)] \to [\alpha\beta^{-1}],$$

we find an isometry

$$M_1 \times_{M_0} M_2 = \bigsqcup_{\gamma\in\Gamma_1\backslash\Gamma_0/\,\Gamma_2} (\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1})\backslash\widetilde{M}_0.$$

In our situation, where M_1 and M_2 are manifolds, it follows that $M_1 \times_{M_0} M_2$ is also a (possibly disconnected) manifold, since elements of $\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1}$ act without fixed points on $\widetilde{M}_0$. The fiber product is a Riemannian manifold for the metric inherited from the universal covering (manifold) $\widetilde{M}_0$.

The fiber product contains the compositum as a connected component. However, it is not necessarily connected (similar to the tensor product of two number fields not always being isomorphic to their compositum); in fact, we see from the above that $M_1 \times_{M_0} M_2$ has $|\Gamma_1\backslash\Gamma_0/\Gamma_2|$ connected components. The components need not all be isometric to the compositum, but if $M_i \to M_0$ are Galois covers, then they are (since then, $\gamma\Gamma_i\gamma^{-1} = \Gamma_i$ for all $\gamma \in \Gamma_0$, $i = 1, 2$).

There are two projections

$$M_1 \times_{M_0} M_2 \twoheadrightarrow \Gamma_0\backslash \left(\widetilde{M}_0 \times \Gamma_i\backslash\Gamma_0\right) \cong M_i,$$

where the latter isometry is given by

$$(\{\gamma_0(x), \Gamma_i\gamma_0^{(i)}\gamma_0^{-1}\}_{\gamma_0\in\Gamma_0}) \mapsto \Gamma_i\gamma_0^{(i)}x,$$

which is bijective, since Γ_i has no fixed points on $\widetilde{M}_0$.

Remark 2.3.1 There is a surjective map from the orbifold fiber product to the *set-theoretic fiber product*

$$\begin{aligned} M_1 \times_{M_0} M_2 &\to \{(x_1, x_2) \in M_1 \times M_2 \colon p_1(x_1) = p_2(x_2)\}, \\ (\{\gamma_0(x), \Gamma_1\gamma_0^{(1)}\gamma_0^{-1}, \Gamma_2\gamma_0^{(2)}\gamma_0^{-1}\}_{\gamma_0\in\Gamma_0}) &\mapsto (\Gamma_1\gamma_0^{(1)}x, \Gamma_2\gamma_0^{(2)}x). \end{aligned} \tag{2.2}$$

If the action of Γ_0 on $\widetilde{M}_0$ has fixed points, then this map is not necessarily injective, so the set-theoretic description of fiber product cannot be used in the orbifold setting, even if M_1, M_2 are manifolds. However, if M_0 is itself a manifold (so Γ_0 acts without fixed points), then the map in (2.2) is an isometry of Riemannian manifolds (in particular, bijective), where the right hand side is a manifold since the projection maps are Riemannian submersions.

We can complete the diagram (1.1) into diagrams of Riemannian coverings

$$\begin{tikzcd} & M_1 \times_{M_0} M_2 \arrow[dl] \arrow[dd] \arrow[dr] & \\ M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd} \qquad \begin{tikzcd} & M_1 \bullet_{M_0} M_2 \arrow[dl] \arrow[dd] \arrow[dr] & \\ M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd}$$

Lemma 2.3.2 *If $M_i \to M_0$ are Galois covers, then $M_1 \times_{M_0} M_2$ is connected and hence isometric to the compositum $M_1 \bullet_{M_0} M_2$ if and only if $\Gamma_0 = \langle \Gamma_1, \Gamma_2 \rangle$, the subgroup of Γ_0 generated by Γ_1 and Γ_2. This holds if the degrees of $M_i \to M_0$ are coprime.*

Proof The number of components of the compositum is the cardinality of the double coset space $\Gamma_1 \backslash \Gamma_0 / \Gamma_2$, and this is 1 if and only if $\Gamma_0 = \Gamma_1 \Gamma_2$. Since Γ_i are normal subgroups of Γ_0, the product $\Gamma_1 \Gamma_2$ is a subgroup; in fact, $\Gamma_1 \Gamma_2 = \langle \Gamma_1, \Gamma_2 \rangle$. Hence the first statement holds. To see the second, note that we have a sequence of finite index group inclusions

$$\Gamma_i \hookrightarrow \langle \Gamma_1, \Gamma_2 \rangle \hookrightarrow \Gamma_0,$$

so we find that $[\Gamma_0 : \langle \Gamma_1, \Gamma_2 \rangle]$ divides $\gcd([\Gamma_0, \Gamma_1], [\Gamma_0, \Gamma_2])$. □

Lemma 2.3.3 *If $M_i \to M_0$ are G_i-Galois of coprime degree, then*

(i) *$M_1 \bullet_{M_0} M_2 \to M_0$ is $(G_1 \times G_2)$-Galois;*
(ii) *$M_1 \bullet_{M_0} M_2 \to M_1$ is G_2-Galois and $M_1 \bullet_{M_0} M_2 \to M_2$ is G_1-Galois.*

Proof We always have an exact sequence

$$1 \to \Gamma_1 \cap \Gamma_2 \to \Gamma_0 \xrightarrow{\varphi} \Gamma_0/\Gamma_1 \times \Gamma_0/\Gamma_2.$$

Now since $[\Gamma_0 : \Gamma_1 \cap \Gamma_2]$ is divisible by the coprime integers $[\Gamma_0 : \Gamma_1]$ and $[\Gamma_0 : \Gamma_2]$, the map φ is surjective. This proves the first statement. The second statement follows from

$$\Gamma_1/(\Gamma_1 \cap \Gamma_2) \cong (\Gamma_0/(\Gamma_1 \cap \Gamma_2))/(\Gamma_0/\Gamma_1) \cong \Gamma_0/\Gamma_2,$$

and similarly with the indices 1 and 2 interchanged. □

2.4 Normal Closure

If $M_{00} \twoheadrightarrow M_0$ is a general finite covering of connected spaces, then there exists a connected space M and a sequence of coverings $M \twoheadrightarrow M_{00} \twoheadrightarrow M_0$ such that $M \twoheadrightarrow M_0$ and $M \twoheadrightarrow M_{00}$ are finite and Galois (i.e., the corresponding subgroup of the fundamental group is normal), cf. [101, Thm. 1]. We call the minimal such M the *normal closure* of $M_{00} \twoheadrightarrow M_0$. In terms of fundamental groups, if M_{00} corresponds to the subgroup Γ_{00} of Γ_0, then M corresponds to the subgroup Γ of Γ_0 given as the intersection of all Γ_0-conjugates of Γ_{00}, the so-called *(normal) core* of Γ_{00} in Γ_0. Alternatively, the normal core Γ is the kernel of the action of Γ_0 by permutation on the cosets of Γ_{00} in Γ_0. In particular, since the index of Γ_{00} in Γ_0 is finite, so is the index of the normal core, and $M \to M_0$ is indeed finite. In fact, by the above alternative description, the degree of $M \to M_0$, the index of the normal core in Γ_0, is bounded by the order of the group of permutations of the set $\Gamma_{00}\backslash\Gamma_0$. The number of elements of this set is the degree of the map $M_{00} \twoheadrightarrow M_0$, and hence we find a bound

$$[\Gamma_0 : \Gamma] \leq \deg(M_{00} \twoheadrightarrow M_0)! = [\Gamma_0 : \Gamma_{00}]!. \tag{2.3}$$

Proposition 2.4.1 *Given manifolds M_1 and M_2 fitting into a diagram* (1.1)*, there exists a diagram of the form* (1.2)*.*

Proof We start with a diagram (1.1) and add the normal closure of the compositum, to find a diagram of the form (1.2), where M corresponds to the normal core of $\Gamma_1 \cap \Gamma_2$:

$$M := \Gamma\backslash\widetilde{M}_0 \text{ for } \Gamma := \bigcap_{\gamma\in\Gamma_0} \gamma(\Gamma_1 \cap \Gamma_2)\gamma^{-1}.$$

2.5 Commensurability

Two manifolds M_1 and M_2 are called *commensurable* if they admit a common finite covering. Proposition 2.4.1 implies that if M_1 and M_2 have a common finite (developable orbifold) quotient as in diagram (1.1), then they are commensurable. We briefly look at the converse statement.

Lemma 2.5.1 *Assume that M_1 and M_2 are commensurable with common finite covering M, let $\widetilde{M}$ denote the universal cover of M with isometry group $\mathscr{I}$, and let $\Gamma, \Gamma_1, \Gamma_2$ denote the subgroups of $\mathscr{I}$ corresponding to M, M_1, M_2 respectively. Then a diagram of the form* (1.1) *exists if and only if Γ_1 and Γ_2 are of finite index in the subgroup of Γ generated by Γ_1 and Γ_2.*

Proof If M_0 exists and corresponds to a subgroup Γ_0 of $\mathscr{I}$, then $\widetilde{M}_0 = \widetilde{M}$ and $\langle\Gamma_1, \Gamma_2\rangle$ is a finite index subgroup of Γ_0, and by assumption Γ_1 and Γ_2 have finite index in Γ_0. Conversely, if $\langle\Gamma_1, \Gamma_2\rangle$ is of finite index in Γ, we can set M_0 to be the corresponding orbifold $\langle\Gamma_1, \Gamma_2\rangle\backslash\widetilde{M}$. □

Proposition 2.5.2 *If $\widetilde{M}$ is a homogeneous space for a connected semisimple non-compact real Lie group with trivial center and no compact factors and M_1 and M_2 are quotients of $\widetilde{M}$ corresponding to commensurable irreducible uniform lattices Γ_1 and Γ_2, then there exists a diagram of the form* (1.1) *if at least one of Γ_1 and Γ_2 is non-arithmetic. In this case both Γ_1 and Γ_2 are non-arithmetic.*

Proof Let $\mathscr{I}$ denote the isometry group of $\widetilde{M}$. Since Γ_1 and Γ_2 are commensurable, their commensurator in $\mathscr{I}$,

$$\mathscr{C} := \mathrm{Comm}_{\mathscr{I}}(\Gamma_i) = \{g \in \mathscr{I} : [g\Gamma_i g^{-1} : \Gamma_i \cap g\Gamma_i g^{-1}]\cdot[\Gamma_i : \Gamma_i \cap g\Gamma_i g^{-1}] < \infty\}$$

is the same (indeed, if Γ_1 and Γ_2 are commensurable and Γ_1 is commensurable to $g\Gamma_1 g^{-1}$, then also $g\Gamma_1 g^{-1}$ and $g\Gamma_2 g^{-1}$ are commensurable and, therefore Γ_2 and $g\Gamma_2 g^{-1}$ are commensurable since commensurability is an equivalence relation). Margulis' theorem [66, Thm. (1), p. 2] (see also [102, Props. 6.2.4, 6.2.5 and Thm. 6.2.6]) states that either Γ_i is not arithmetic and of finite index in $\mathscr{C}$ or Γ_i is arithmetic and $\mathscr{C}$ is dense in $\mathscr{I}$ (to connect to the more general formulation in [66]: we consider a single semisimple group over the reals, and the "anisotropy condition" in loc. cit. is satisfied since we assume the group is not compact). This directly implies that either both lattices Γ_1 and Γ_2 are arithmetic or they are both not arithmetic.

Consider the sequence of inclusions

$$\Gamma_i \hookrightarrow \langle \Gamma_1, \Gamma_2 \rangle \hookrightarrow \mathscr{C} \tag{2.4}$$

for $i = 1, 2$.

If neither of Γ_i is arithmetic, the composed inclusion is of finite index. Hence the same holds for the first inclusion, and we can set $M_0 := \langle \Gamma_1, \Gamma_2 \rangle \backslash \widetilde{M}$. □

Example 2.5.3 The proposition applies in particular to compact hyperbolic manifolds $M_i = \Gamma_i \backslash \mathbb{H}^n$, where $\mathbb{H}^n$ is hyperbolic n-space. ♢

Proposition 2.5.4 *There exist commensurable compact hyperbolic Riemann manifolds M_1 and M_2 of dimensions* 2 *and* 3, *for which a diagram of the form* (1.1) *does not exist.*

Proof The uniform arithmetic isospectral, non-isometric lattices constructed by Vignéras satisfy this property [97]; for commensurability, see [96, Ch. IV]. Additional information can be found in [24, Prop. 3]. □

Remark 2.5.5 More generally, Alan Reid has shown that isospectral manifolds corresponding to arithmetic lattices are commensurable ([81], compare [61] for a quantitative statement). It is an open problem whether isospectral Riemann surfaces are always commensurable. Through work of Lubotzky, Samuels and Vishne, it is known that isospectrality does not imply commensurability [64] in general.

Open Problem

Find further relations (other than Proposition 2.5.2) between commensurability and existence of a common finite quotient for special types of spaces, e.g., spaces whose universal covering is fixed and has infinite fundamental group. Example 2.5.3 concerns the case where the common universal covering is hyperbolic n-space.

Open Problem

Find a criterion to decide precisely which pairs of arithmetic manifolds are common finite cover of some developable orbifold. See Remark 2.1.1 for a relation between this problem for surfaces and branching data.

Chapter 3
Spectra, Group Representations and Twisted Laplacians

We're gonna do the twist and it goes like this
— "Let's twist again", written by Kal Mann and Dave Appell

In this chapter, we review basic notions about spectra, group representations, and twisted Laplace operators. We first recall how to define the spectrum and the spectral zeta function for a general symmetric second order elliptic differential operator acting on smooth sections of a Hermitian line bundle. We prove that the non-zero spectrum (i.e., the spectral zeta function) determines the entire spectrum on an odd-dimensional manifold, but also give an example showing that this is not always true for even-dimensional manifolds; the example is obstructed by the non-vanishing of some topological genus. After setting up some notation from representation theory, we discuss G-sets and weak conjugacy ("Gaßmann equivalence") of subgroups of a group, explaining the interrelations. In the final sections, we introduce twisted Laplacians, corresponding to unitary representations of the fundamental group. After this, we focus on the case of a twisted Laplacian arising from a finite Galois cover of manifolds and we relate the spectrum on the top manifold to that of the induced representation on the bottom manifold. We relate the multiplicity of zero in the spectrum to the multiplicity of the trivial representation in the given representation, and finally we show that, contrary to the general case, the multiplicity of zero in the spectrum of a twisted Laplacian is determined from the non-zero spectrum, provided one also knows the usual Laplace spectrum of the manifold.

As basic background references for this chapter, we use [54, 83, 88].

3.1 Spectrum and Spectral Zeta Function

Let $M = (M, g)$ denote a connected closed oriented smooth Riemannian manifold with Riemannian metric g. Let E denote a Hermitian bundle on M and A a symmetric second order elliptic differential operator acting on smooth sections

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$C^\infty(M, E)$ of E with non-negative eigenvalues. The operator extends to the corresponding space $L^2(M, E)$ of L^2-sections where it has a dense domain. The *spectrum* $\sigma_M(A)$ *of* A (or $\sigma(A)$ is M is fixed) is the multiset of eigenvalues of A, where the multiplicities of the elements in the set are given by the multiplicities of the eigenvalues.

We make the following convenient notational conventions: if S_1 and S_2 are multisets, we let $S_1 \cup S_2$ denote the multiset consisting of elements of S_1 or S_2, where the multiplicity of an element is the sum of the multiplicities of that element in S_1 and S_2, and for a multiset S and an integer n, we mean by nS the multiset of elements of S where all multiplicities are multiplied by n.

Example 3.1.1 To (M, g) is associated a *Laplace(–Beltrami) operator* Δ_M, acting on the space $C^\infty(M)$ of smooth functions on M, given in local coordinates (x^i) as

$$\Delta_M(f) := -\det(g)^{-1/2} \sum_{i,j} \frac{d}{dx^i}\left(\det(g)^{1/2} g^{ij} \frac{d}{dx^j} f\right).$$

An intrinsic definition follows from the more general next example. $\diamondsuit$

Example 3.1.2 More generally, for every $k \geq 0$, there is such a *Laplace operator* Δ_M^k *acting on (the space of) k-forms* $C^\infty(M, \bigwedge^k T^*M)$, defined as follows. If $d^k \colon C^\infty(M, \bigwedge^k T^*M) \to C^\infty(M, \bigwedge^{k+1} T^*M)$ denotes the exterior derivative on k-forms, and $\delta^{k+1} \colon C^\infty(M, \bigwedge^{k+1} T^*M) \to C^\infty(M, \bigwedge^k T^*M)$ the adjoint of d^k for the inner product induced by the metric g, then $\Delta_M^k = \delta^{k+1} d^k + d^{k-1}\delta^k$ acting on $C^\infty(M, \bigwedge^k T^*M)$ with $d^{-1} = 0$ by convention.

For $k = 0$, Δ_M^k equals $\Delta_M^0 = \delta^1 d = \Delta_M$ as defined in Example 3.1.1. The kernel of Δ_M^k consists of so-called *harmonic k-forms*, which, by a theorem of Hodge, is identified with $\mathrm{H}^k_{\mathrm{dR}}(M)$, the k-th de Rham cohomology group of M. De Rham's theorem identifies $\mathrm{H}^k_{\mathrm{dR}}(M) \cong \mathrm{H}^k(M, \mathbf{R})$ with the usual real-valued (singular) cohomology of M. Thus, the multiplicity of 0 in the spectrum of Δ^k equals the k-th Betti number of M:

$$\mathrm{mult}_0(\sigma_M(\Delta_M^k)) = \dim_{\mathbf{R}} \mathrm{H}^k(M, \mathbf{R}). \tag{3.1}$$

We refer to [83, Chapter 1] for details. $\diamondsuit$

The *spectral zeta function* of A as above is defined as

$$\zeta_{M,A}(s) = \zeta_A(s) := \sum_{0 \neq \lambda \in \sigma(A)} \lambda^{-s}$$

with the sum not involving the zero eigenvalues. The function can be meromorphically continued to the entire complex plane [83, Thm. 5.2]. Since $\zeta_A(s)$ is a (generalised) Dirichlet series, the identity theorem for such Dirichlet series [46, Thm. 6] implies that it is determined by its values on a countable set with an accumulation point, e.g., by its values at all sufficiently large integers.

3.2 Spectrum Versus Spectral Zeta Function

We will formulate all results using the spectrum, rather than the spectral zeta function. For odd-dimensional manifolds, these give exactly the same information, as the following proposition shows.

Proposition 3.2.1 *If M is an odd-dimensional manifold, the multiset $\sigma_M(A)$ and the function $\zeta_{M,A}(s)$ mutually determine each other.*

Proof It is clear that the function $\zeta_A(s)$ determines $\sigma(A) - \{0\}$, so we only need to show that if M is of odd dimension, the multiplicity of zero in the spectrum is also determined by ζ_A; this multiplicity is $\dim \ker A$, which equals $-\zeta_A(0)$ if M has odd dimension (see [83, Thm. 5.2]). □

The result in Proposition 3.2.1 does not hold in general if M is of even dimension n, as the example in the following proposition shows.

Proposition 3.2.2 *There exists a 4-dimensional manifold M (in fact, M may be chosen as a complex quartic surface) and two second order bundle operators $\Delta^\pm$ on M such that $\zeta_{\Delta^+} = \zeta_{\Delta^-}$ but $\sigma(\Delta^+) \neq \sigma(\Delta^-)$.*

Proof The basic idea is that two commuting operators have the same non-zero spectrum, but that the difference of the dimensions of their kernels can have a topological interpretation as an index, that might be non-vanishing. The proof will use the theory of spin manifolds and the index theorem for the Dirac operator, for which we refer to [11] or [39].

Suppose M is an even dimensional spin manifold with Dirac operator

$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix},$$

where the spinor bundle is decomposed into eigenspaces for the chirality operator as $S^+ \oplus S^-$, with $D^+ \colon S^+ \to S^-$ and $D^- \colon S^- \to S^+$, $D^- = (D^+)^*$ adjoint to D^+. Then the second order operators

$$\Delta^\pm := D^\mp D^\pm$$

have the same non-zero spectrum (this is true in general for the non-zero spectrum of the products AB and BA of two operators A and B); hence $\zeta_{\Delta^+} = \zeta_{\Delta^-}$.

On the other hand,

$$\ker \Delta^+ = \ker (D^+)^* D^+ = \ker D^+$$

since if $\Delta^+\varphi = (D^+)^* D^+\varphi = 0$, then

$$||D^+\varphi||^2 = \langle D^+\varphi, D^+\varphi\rangle = \langle \varphi, (D^+)^* D^+\varphi\rangle = 0.$$

Hence with $m^\pm := \mathrm{mult}_0(\sigma(\Delta^\pm))$ the multiplicity of 0 in the spectrum of $\Delta^\pm$, we find that

$$\begin{aligned} m^+ - m^- &= \dim\ker \Delta^+ - \dim\ker \Delta^- = \dim\ker D^+ - \dim\ker D^- \\ &= \text{index } D = \int_M \widehat{A}(M) \end{aligned}$$

is, by the Atiyah–Singer index theorem, the $\widehat{A}$-genus of M, which may be non-zero (compare [11, 3.4, 4.1]). For example, if M is a complex quartic surface (of real dimension 4) in $\mathbf{CP}^3$, then index $D = \int \widehat{A}(M) = 2$ (see, e.g., [39, p. 727]). This shows that $\sigma(\Delta^+) \neq \sigma(\Delta^-)$. □

Remark 3.2.3 If one is willing to consider orbifolds instead of manifolds, there exist two-dimensional orbifolds that are isospectral for the Laplace operator acting on 1-forms (cf. Example 3.1.2), as shown by the following example of Gordon and Rossetti. Consider the quotient of the standard flat torus $\mathbf{Z}^2\backslash\mathbf{R}^2$ by the involutions induced by the following maps on $\mathbf{R}^2$ (with coordinates (x, y)):

(i) $(x, y) \mapsto (x, -y)$, leading to the cylinder C;
(ii) $(x, y) \mapsto (y, x)$, producing the Möbius strip M;
(iii) $(x, y) \mapsto (-x, -y)$, leading to the pillow orbifold $\mathscr{O}$.

Then the non-zero spectra of the Laplace operators acting on the space of 1-forms on C, M and $\mathscr{O}$ agree. However, 0 is not an eigenvalue on 1-forms for C and M, whereas it is for $\mathscr{O}$ [42, Example 2.5 and Theorem 3.1].

Interestingly, our main results, such as Theorem 1.2.1 and Proposition 1.2.5, are formulated in their strongest possible form using *precisely* the multiplicity of zero in the spectrum of certain twisted Laplacians. For these operators, it turns out that also in even dimension this multiplicity (and hence the zeta function) is fixed by the non-zero spectrum of the usual and the twisted Laplacian, cf. Proposition 3.10.1 below.

3.3 Group Representations

If G is a finite group, let $\check{G} = \mathrm{Hom}(G, \mathbf{C}^*)$ denote the group of linear characters of G, and let $\mathrm{Irr}(G)$ denote the set of inequivalent irreducible unitary representations of G. We consider complex representations as $\mathbf{C}[G]$-modules $\mathscr{M}$ or group homomorphisms $\rho\colon G \to \mathrm{Aut}(V) \cong \mathrm{GL}(N, \mathbf{C})$ with $V = \mathbf{C}^N$ and freely mix these concepts, writing expressions such as "$\mathscr{M} \cong \rho$". By further slight abuse of notation, if ρ_1 and

ρ_2 are representations of G, we write

$$\langle \rho_1, \rho_2 \rangle = \langle \mathrm{tr}(\rho_1(-)), \mathrm{tr}(\rho_2(-)) \rangle = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(\rho_1(g)) \overline{\mathrm{tr}(\rho_2(g))}$$

for the inner product of the corresponding characters in the space of class functions. The multiplicity of an irreducible representation $\rho' \in \mathrm{Irr}(G)$ in the decomposition into irreducibles of a general representation ρ of G is then $\langle \rho, \rho' \rangle$.

The *regular representation* $\rho_{G,\mathrm{reg}}$ corresponds to the $\mathbf{C}[G]$-module $\mathbf{C}[G]$. It decomposes as

$$\rho_{G,\mathrm{reg}} = \bigoplus_{\rho_i \in \mathrm{Irr}(G)} \dim(\rho_i) \rho_i.$$

If H is a subgroup of G and ρ a representation of H, then $\mathrm{Ind}_H^G \rho$ denotes the representation induced by ρ from H to G: if ρ corresponds to the $\mathbf{C}[H]$-module V, then $\mathrm{Ind}_H^G \rho$ corresponds to the $\mathbf{C}[G]$ module $W := \mathbf{C}[G] \otimes_{\mathbf{C}[H]} V$. In coordinates, this means the following: since G permutes the cosets of H in G, if we choose coset representatives

$$G/H = \{g_1 H = H, \ldots g_n H\},$$

then for any $g \in G$ we have

$$g g_i = g_{g(i)} h_{g,i}$$

for some $h_{g,i} \in H$ and some permutation $i \mapsto g(i)$ of $\{1, \ldots, n\}$. If we write

$$W = V^{G/H} = \bigoplus_{i=1}^{n} g_i V$$

using g_i as placeholder, then with $v_i \in V$, we have

$$\mathrm{Ind}_H^G \rho(g) \left(\sum_i g_i v_i \right) = \sum_i g_{g(i)} \rho(h_{g,i})(v_i). \tag{3.2}$$

Let $\mathrm{Res}_H^G \rho$ denotes the restriction of ρ, a representation of G, from G to H. If ρ is a representation, $\overline{\rho}$ denotes the complex conjugate representation. Recall the standard calculation rules

$$\langle \rho_1 \otimes \rho_2, \rho_3 \rangle = \langle \rho_1, \rho_3 \otimes \overline{\rho_2} \rangle \text{ and } \langle \mathrm{Ind}_H^G \rho_1, \rho_2 \rangle = \langle \rho_1, \mathrm{Res}_H^G \rho_2 \rangle,$$

(the latter is known as "Frobenius reciprocity").

More generally, the above theory applies *mutatis mutandis*, replacing $\mathbf{C}$ by an algebraically closed field of characteristic coprime to the order $|G|$ of the group G. For a non-algebraically closed field K of characteristic coprime to $|G|$, an irreducible representation might decompose over the algebraic closure into a sum of irreducible (Galois-conjugate) representations, but the above theory remains valid, with the caveat that a rational character is not always the character of a rational representation, but a multiple is. If the characteristic of K divides $|G|$, the category of $K[G]$-modules is not semisimple, so complementary modules for submodules do not always exist. For us, it will be important that, in general, the regular representation is defined over $\mathbf{Q}$, and induction and restriction turn K-representations into K-representations.

3.4 G-Sets

If G is a group, a *G-set* is a set that admits a left G-action. An example is the left cosets G/H of a subgroup H with the action of left multiplication by G. A *morphism* of G-sets is a G-equivariant map of the sets. We say a G-set is *transitive* if G acts transitively on it. If X is a transitive G-set and H the stabiliser of any point in X, then X is isomorphism to G/H as G-set.

3.5 (Weak) Conjugacy

We let $\mathbf{1} = \mathbf{1}_G$ denote the trivial representation of a group G. If, as before, $\{g_1, \dots, g_n\}$ is a set of representatives for the (left) H-cosets in G, then $\mathrm{Ind}_H^G \mathbf{1}$ is the permutation representation (i.e., the action of each $g \in G$ is given by a permutation matrix, a matrix having exactly one non-zero entry 1 in each row and column) given by the action of G on the vector space

$$\mathbf{C}[G/H] := \bigoplus_{i=1}^{n} \mathbf{C} g_i H$$

spanned by the cosets of H in G. We have the following.

Proposition 3.5.1 *Let H_1 and H_2 denote two subgroups of a finite group G.*

(i) *The following properties are equivalent:*

 (a) *The representations $\mathrm{Ind}_{H_1}^G \mathbf{1} \cong \mathrm{Ind}_{H_2}^G \mathbf{1}$ are isomorphic.*
 (b) *Each conjugacy class c of G intersects H_1 and H_2 in the same number of elements.*
 (c) *There exists a set-theoretic bijection $\psi : H_1 \to H_2$ such that h_1 and $\psi(h_1)$ are conjugate in G for any $h_1 \in H_1$.*

If any of these holds, we say H_1 and H_2 are weakly conjugate *in G.*

(ii) *The stronger property that the groups H_1 and H_2 are* conjugate *in G is equivalent to the cosets G/H_1 and G/H_2 being isomorphic as G-sets.*

Weak conjugacy is sometimes called "almost conjugacy", and also known as "Gaßmann equivalence" in number theory, cf. [77].

Proof

(i) Representation isomorphism is the same as isomorphism of characters, and the character of the representation $\mathrm{Ind}_H^G \mathbf{1}$ is

$$\psi(g) = |[g] \cap H| \cdot |C_G(g)|/|H|, \tag{3.3}$$

where $[g]$ is the conjugacy class of g and $C_G(g)$ is the centraliser of g in G (compare, e.g., [18, §1]). The final equivalent statement is proven in [24, Lemma 2].

(ii) The existence of a G-isomorphism $\phi : G/H_1 \to G/H_2$ implies that the G-stabiliser H_1 of the coset eH_1 equals the G-stabiliser gH_2g^{-1} of some coset $gH_2 = \phi(eH_1)$, and hence $H_1 = gH_2g^{-1}$. Conversely, if $H_2 = g_0^{-1}H_1g_0$, the map $\phi : G/H_1 \to G/H_2, \phi(gH_1) = gg_0H_2$ is a well-defined isomorphism of G-sets. □

Remark 3.5.2 If we have a diagram (1.1) and M_1 and M_2 have the same Laplace spectrum (viz., the same spectral zeta function) and the same dimension n, then they have the same volume (from Weyl's law, or, equivalently, from the value of the residue of their zeta functions at $s = n/2$). Since the covering degrees $\deg(p_i)$ of p_i are $\mathrm{vol}(M_i)/\mathrm{vol}(M_0)$, these are also equal, and hence in diagram (1.2), we find from $|H_i| = |G|/\deg(p_i)$ that $|H_1| = |H_2|$. So in this case, there is *always* a set-theoretic bijection $\psi : H_1 \to H_2$.

3.6 Twisted Laplacian

Suppose $\rho : \pi_1(M) \to \mathrm{U}(N, \mathbf{C})$ is a unitary representation of the fundamental group $\pi_1(M)$ of M. Let $\Pi : \widetilde{M} \to M$ denote the universal covering of M, and set $E_\rho := \widetilde{M} \times_\rho \mathbf{C}^N$, where the subscript ρ indicates equivalence classes for the relation

$$(z, v) \sim (\gamma z, \rho(\gamma)^{-1} v)$$

for any $z \in \widetilde{M}$, $v \in \mathbf{C}^N$ and $\gamma \in \pi_1(M)$. Now E_ρ is a flat vector bundle of rank N over M, whose global sections $f \in C^\infty(M, E_\rho)$ correspond bijectively to smooth ρ-equivariant vector-valued functions $\vec{f} : \widetilde{M} \to \mathbf{C}^N$, i.e., functions with $\vec{f}(\gamma z) = \rho(\gamma)\vec{f}(z)$. The *twisted Laplacian*

$$\Delta_{M,\rho} = \Delta_\rho : C^\infty(M, E_\rho) \to C^\infty(M, E_\rho)$$

is defined as

$$\overrightarrow{\Delta_\rho(f)} := \Delta_{\widetilde{M}} \vec{f}(z).$$

Henceforth, we will also denote the spectrum $\sigma_M(\Delta_\rho)$ simply by $\sigma_M(\rho)$ or by $\sigma(\rho)$ if the underlying manifold M is fixed.

Notice that when $\rho = \rho_1 \oplus \rho_2$, then Δ_ρ admits a block decomposition $\Delta_{\rho_1} \oplus \Delta_{\rho_2}$ on $E_\rho \cong E_{\rho_1} \oplus E_{\rho_2}$, and thus, the spectrum satisfies (as multisets)

$$\sigma_M(\rho_1 \oplus \rho_2) = \sigma_M(\rho_1) \cup \sigma_M(\rho_2).$$

3.7 Twisted Laplacians on Finite Covers

In case ρ factors through a finite group G, there is no need to use the universal covering. Let $M' \to M$ denote a (fixed-point free) G-cover and $\rho : G \to \mathrm{U}(N, \mathbf{C})$ a unitary representation. The vector space $C^\infty(M, E_\rho)$ is canonically isomorphic to the vector space of smooth ρ-equivariant vector-valued functions on M' given by

$$C^\infty_\rho(M', \mathbf{C}^N) := \{\vec{f} \in C^\infty(M', \mathbf{C}^N) \mid \vec{f}(\gamma x) = \rho(\gamma)\vec{f}(x), \forall x \in M', \gamma \in G\}.$$

In this case,

$$\overrightarrow{\Delta_\rho f} = \Delta_{M'} \vec{f}, \tag{3.4}$$

where $\vec{f}$ is the ρ-equivariant function in $C^\infty_\rho(M', \mathbf{C}^N)$ corresponding to f. Note that $\Delta_{M'} \vec{f}$ is again a ρ-equivariant function in $C^\infty_\rho(M', \mathbf{C}^N)$ and therefore represents an element in $C^\infty(M, E_\rho)$.

3.8 Twisted Laplacians for Induced Representations

The following lemma is stated in [91, Lemma 1]; we write a proof using our notation.

Lemma 3.8.1 *If $M \to M_1 \to M_0$ is a tower of finite Riemannian coverings and $M \to M_0$ is Galois with group G, $M \to M_1$ with group H, and $\rho\colon H \to \mathrm{U}(N, \mathbf{C})$ a representation, then*

$$\sigma_{M_0}(\mathrm{Ind}_H^G \rho) = \sigma_{M_1}(\rho).$$

Proof Write $\rho^* := \mathrm{Ind}_H^G \rho \colon G \to \mathrm{U}(Nn, \mathbf{C})$ and let

$$G/H = \{g_1 H = H, \dots, g_n H\}$$

denote representatives for the distinct cosets of H in G. Define two maps

$$\begin{tikzcd} C^\infty_\rho(M, \mathbf{C}^N) \arrow[rr, bend left, "\Phi"] & & C^\infty_{\rho^*}(M, \mathbf{C}^{Nn}) \arrow[ll, bend left, "\Psi"] \end{tikzcd}$$

by

$$\Phi(\vec{f})(x) = (\vec{f}(g_1^{-1}(x)), \dots, \vec{f}(g_n^{-1}(x)))$$

and

$$\Psi(\vec{F}) = \Psi((\vec{f_1}, \dots, \vec{f_n})) := \vec{f_1}.$$

Recall that by definition

$$\rho^*(g)((\vec{f_i}(x))_{i=1}^n) = (\rho(g_{g(i)}^{-1} g g_i)\vec{f}_{g(i)}(x))_{i=1}^n$$

where $g(i)$ is given by

$$g g_i H = g_{g(i)} H.$$

This allows one to check that Φ and Ψ are well defined and mutually inverse bijections. Recall that $\overrightarrow{\Delta_\rho(f)} = \Delta_M(\vec{f})$ and $\overrightarrow{\Delta_{\rho^*}(F)} = \Delta_M(\vec{F})$ with Δ_M applied componentwise. Since the g_i are isometries, Φ is a unitary operator in L^2 and $\Delta_M \Phi = \Phi \Delta_M$, so that we have the intertwining

$$\Phi \circ \Delta_\rho = \Delta_{\rho^*} \circ \Phi,$$

and the equality of spectra follows. □

As is shown in [91], the main theorem of Sunada [90] follows easily from this.

Theorem 3.8.2 (Sunada's Theorem [90, Theorem 1]) *If we have a diagram of the form* (1.2) *and H_1 and H_2 are weakly conjugate in G, then M_1 and M_2 are isospectral.*

Proof We apply Lemma 3.8.1 to the trivial representation $\rho = \mathbf{1}$ for $M \to M_i \to M_0$ with $i = 1$ and $i = 2$. Since H_1 and H_2 are weakly conjugate, $\mathrm{Ind}_{H_1}^G \mathbf{1} \cong \mathrm{Ind}_{H_2}^G \mathbf{1}$ (cf. Proposition 3.5.1), and so

$$\sigma_{M_1} = \sigma_{M_1}(\mathbf{1}) = \sigma_{M_0}(\mathrm{Ind}_{H_1}^G \mathbf{1}) = \sigma_{M_0}(\mathrm{Ind}_{H_2}^G \mathbf{1}) = \sigma_{M_2}(\mathbf{1}) = \sigma_{M_2},$$

finishing the proof. □

Remark 3.8.3 The original proof used a trace formula [90, Lemma 1], that is now hidden in the computations with induced representations and their characters. The trace formula proof has the advantage to apply to all "natural" operators alike, such as the Laplace operators on k-forms (cf. Example 3.1.2); see Sect. 4.2 for more on natural operators and strong isospectrality. For another exposition in the style of the original argument and background information, see [23, Chapter 11].

Remark 3.8.4 The converse of the theorem is not true: isospectral manifolds fitting into a diagram of the form (1.2) do not necessarily have H_1 and H_2 weakly conjugate in G. See Corollary 4.2.5 for a spectral characterisation of weak conjugacy using twisted Laplacians.

Remark 3.8.5 The main application of Theorem 3.8.2 is to the construction of isospectral, non-isometric manifolds. For this, one needs to realise a diagram of manifolds as in (1.2) *and* guarantee that M_1 and M_2 are not isometric, for example, by making sure that $\mathrm{H}_1(M_1)$ and $\mathrm{H}_1(M_2)$ are distinct.

3.9 Multiplicity of Zero in Twisted Laplace Spectra

Decomposing a general representation $\rho : G \to \mathrm{U}(N, \mathbf{C})$ into irreducibles as $\rho = \bigoplus \langle \rho_i, \rho \rangle \rho_i$, we have

$$\sigma_M(\Delta_\rho) = \bigcup \langle \rho_i, \rho \rangle \sigma_M(\Delta_{\rho_i}). \tag{3.5}$$

Applied to the regular representation, we find a relation between the spectra of the usual Laplacian on the cover M' and of the twisted Laplacians on the original manifold M, as follows:

$$\sigma_{M'}(\Delta_{M'}) = \sigma_M(\Delta_{\rho_{G,\mathrm{reg}}}) = \bigcup \dim(\rho_i) \sigma_M(\Delta_{\rho_i}). \tag{3.6}$$

(The first equality follows from Lemma 3.8.1 since $\mathrm{Ind}_{\{1\}}^G \mathbf{1} = \mathbf{C}[G] = \rho_{G,\mathrm{reg}}$). We see in particular that the eigenvalues of any twisted Laplacian are also eigenvalues of the usual Laplace operator of the corresponding cover.

Lemma 3.9.1 *Let G be a finite group acting by fixed-point free isometries on a closed connected Riemannian manifold M' with quotient $M = G \backslash M'$. If ρ is any unitary representation of G, then the multiplicity $\langle \rho, \mathbf{1} \rangle$ of the trivial representation* *in the decomposition of ρ into irreducibles equals* $\dim\ker\Delta_\rho$, *the multiplicity of the zero eigenvalue in $\sigma_M(\rho)$.*

***Proof** (First Proof of Lemma 3.9.1)* Since M and M' are connected, the multiplicity of zero in $\sigma_{M'}(\Delta_{M'})$ and $\sigma_M(\Delta_M)$ is one. It follows from the decomposition of multisets (3.6) that for any irreducible representation $\rho_i \neq \mathbf{1}$ of G, $\sigma_M(\Delta_{\rho_i})$ does not contain zero. If we now decompose ρ as a sum of irreducibles, the decomposition of multisets (3.5) implies that the multiplicity of zero in $\sigma_M(\Delta_\rho)$ is indeed the multiplicity with which $\mathbf{1}$ occurs in ρ. □

We can also give a "direct" proof, as follows.

***Proof** (Second Proof of Lemma 3.9.1)* Let $\rho : G \to \mathrm{U}(N, \mathbf{C})$. A function

$$f \in \ker\Delta_\rho \subseteq C^\infty(M, E_\rho)$$

corresponds to a function $\vec{f}$ on M' with $\Delta_{M'}\vec{f} = 0$ and $\vec{f}(\gamma z) = \rho(\gamma)\vec{f}(z)$ for all $\gamma \in \Gamma$. Since M' is closed and connected, this implies that $\vec{f} = \vec{f}_0$ is a constant vector in $\mathbf{C}^N$, and the equivariance condition translates into

$$(\rho(\gamma) - 1)\vec{f}_0 = 0 \quad \text{for all } \gamma \in G.$$

Hence each such linearly independent vector $\vec{f_0} \in \mathbf{C}^N$ can be used to split off a one-dimensional invariant subspace in ρ, and we find the result. □

3.10 Spectrum Versus Spectral Zeta Function for Twisted Laplacians

In Proposition 3.2.2 we showed that, in general, on an even-dimensional manifold, knowledge of the spectrum is stronger than that of the spectral zeta function, i.e., of the non-zero spectrum. The operators in Proposition 3.2.2 were not twisted Laplacians. For twisted Laplace operators, the situation is better, as the following proposition shows.

Proposition 3.10.1 *Let G be a finite group acting by fixed-point free isometries on a closed connected n-dimensional Riemannian manifold M' with quotient $M = G\backslash M'$. If ρ is any unitary representation of G, then on the one hand the pair of multisets $\sigma_M(\Delta_\rho)$ and $\sigma_M(\Delta)$, and on the other hand the pair of zeta functions $\zeta_{M,\Delta_\rho}(s)$ and $\zeta_{M,\Delta}(s)$ mutually determine each other. In fact, the multiplicity of zero in $\sigma(\Delta_\rho)$ is given by*

$$\dim\ker\Delta_\rho = (\zeta_\Delta(0) + 1)\left.\frac{\zeta_{\Delta_\rho}(s)}{\zeta_\Delta(s)}\right|_{s=\frac{n}{2}} - \zeta_{\Delta_\rho}(0). \tag{3.7}$$

Remark 3.10.2 As an illustration consider the easy situation when $|G| = 1$, so $M' = M$ and $\rho \sim N\mathbf{1}$ for some N. Then $\sigma_M(\Delta_\rho) = N\sigma_M(\Delta^N) = N\sigma_M$, $\zeta_{\Delta_\rho} = N\zeta_\Delta$ and $\mathrm{mult}_0(\sigma(\Delta^N)) = N$ is equal to $(\zeta_\Delta(0)+1)\left.\frac{\zeta_{\Delta_\rho}(s)}{\zeta_\Delta(s)}\right|_{s=\frac{n}{2}} - \zeta_{\Delta_\rho}(0) = 2N - N$.

Proof It suffices to prove that ζ_Δ and ζ_{Δ_ρ} determine the multiplicity of zero in the spectrum (i.e, the dimension of the kernel) of Δ_ρ. If the dimension n of M is odd, this follows from the stronger Proposition 3.2.1. For even n and a general operator A as in Sect. 3.1,

$$\zeta_A(0) = -\dim\ker A + (4\pi)^{-n/2}\int_M u_{n/2}(A), \tag{3.8}$$

where $\mathrm{tr}(e^{-tA}) \sim (4\pi t)^{-n/2}\sum_{k=0}^{\infty}\int_M u_k(A)t^k$ is the asymptotic expansion of the heat kernel of A as $t\downarrow 0$ [83, Thm. 5.2].

We apply this in our situation, with

$$A = \Delta_\rho = \bigoplus_{i=1}^{N}\Delta_{M'}$$

acting on $C^\infty(M, E_\rho) = C^\infty_\rho(M', \mathbf{C}^N)$. Denote by 1_N the identity matrix of size $N \times N$. Recall that the principal symbol $p(\Delta_M)$ of a Laplace operator Δ_M on a Riemannian manifold (M, g) is determined by the metric tensor g (more accurately, it is the quadratic form on the cotangent bundle dual to g). Since $M' \to M$ is a Riemannian covering, the metric tensor of M pulls back to that of M', and hence the principal symbol of $\Delta_{M'}$ is the same as that of Δ_M. Therefore, Δ_ρ is a "Laplace-style" operator in the sense of [40, 1.2]: it has (matrix) principal symbol the diagonal matrix $p(\Delta_\rho) = p(\Delta_M)\cdot 1_N$. Such operators have a description depending on a connection on the bundle and an endomorphism of the bundle as in [39, Lemma 1.2.1], and in our situation, for E_ρ, the bundle connection is flat (curvature $\Omega \equiv 0$) and the endomorphism e is trivial.

The coefficients $u_k(\Delta_\rho)(x)$ (as a function of $x \in M$) are of the form

$$u_k(\Delta_\rho)(x) = \mathrm{tr}_{E_{\rho,x}}(e_k(\Delta_\rho)(x)),$$

where $\mathrm{tr}_{E_{\rho,x}}$ denotes the fiberwise trace in the fibers $E_{\rho,x} \cong \mathbf{C}^N$, and where $e_k(\Delta_\rho)$ is a linear combination with universal coefficients (independent of the dimension n of the manifold and the rank N of the bundle) of covariant derivatives of $\underline{R}\cdot 1_N$ (where $\underline{R}$ is the covariant Riemann curvature tensor of M), the bundle curvature Ω and the endomorphism e [39, §3.1.8–3.1.9]. Since the latter two are identically zero in our situation, we can write $e_k(\Delta_\rho) = P_k \cdot 1_N$ with P_k only depending on the covariant derivatives of $\underline{R}$, in particular, not depending on ρ. We conclude that

$$(4\pi)^{-n/2} \int_M u_{n/2}(\Delta_\rho)(x) = (4\pi)^{-n/2} \int_M \mathrm{tr}_{E_{\rho,x}}(P_{n/2}(x) \cdot 1_N) = NU,$$

where $U = (4\pi)^{-n/2} \int_M P_{n/2}$ is independent of ρ.

Therefore, applying (3.8) to Δ and Δ_ρ, we find

$$\dim \ker \Delta_\rho = NU - \zeta_{\Delta_\rho}(0) = N(\zeta_\Delta(0) + 1) - \zeta_{\Delta_\rho}(0). \tag{3.9}$$

We can compute the rank N in terms of the first coefficients in the asymptotic expansions: using $e_0(\Delta_\rho) = 1_N$, we find $N = \int_M u_0(\Delta_\rho) / \int_M u_0(\Delta)$. On the other hand, $\zeta_{\Delta_\rho}(s)$ has a simple pole at $s = n/2$ with residue $\Gamma(n/2)^{-1} \int_M u_0(\Delta_\rho)$ (see, e.g., the proof of [83, Thm. 5.2]), so that the function $\zeta_{\Delta_\rho}(s)/\zeta_\Delta(s)$ is holomorphic at $s = n/2$ and takes value N there:

$$N = \left. \frac{\zeta_{\Delta_\rho}(s)}{\zeta_\Delta(s)} \right|_{s=\frac{n}{2}}. \tag{3.10}$$

Combining Eqs. (3.9) and (3.10) gives the desired expression (3.7) for the multiplicity of zero in the spectrum in terms of spectral zeta functions only. □

Remark 3.10.3

(i) The above argument also shows that for a twisted Laplacian Δ_ρ corresponding to a unitary representation on a fixed manifold M, the value

$$\dim \ker \Delta_\rho + \zeta_{\Delta_\rho}(0)$$

only depends on the dimension of the representation ρ.

(ii) Weyl's law for Δ_ρ says that if $\mathrm{N}(\Delta_\rho, X)$ denotes its number of eigenvalues $\leq X$, then

$$\lim_{X \to +\infty} \frac{\mathrm{N}(\Delta_\rho, X)}{X^{n/2}} = N \cdot \frac{\mathrm{vol}(M)}{(4\pi)^{n/2} \Gamma\left(\frac{n}{2} + 1\right)},$$

so that on a fixed manifold, the dimension N of the representation ρ can be read off from the asymptotics of the spectra of Δ_ρ and Δ_M:

$$N = \lim_{X \to +\infty} \frac{\mathrm{N}(\Delta_\rho, X)}{\mathrm{N}(\Delta_M, X)}; \tag{3.11}$$

formulas (3.10) and (3.11) are equivalent through Karamata's version of the Tauberian theorem (compare [11, pp. 91–92]).

Open Problem

Find a "geometric" formula for the difference in multiplicities of zero for two operators A and B on a manifold of the general type considered here that have identical non-zero spectrum.

Open Problem

Is the multiplicity of zero in the spectrum of a twisted Laplacian determined by the non-zero spectrum of the twisted Laplacian alone, without assuming knowledge of the spectrum of the usual Laplacian?

Open Problem

Study how *disjoint* the spectra of the different Δ_{ρ_i} in (3.6) are, similar to the question how disjoint zeros of number theoretic L-series are (cf. [84]): the so-called *grand simplicity hypothesis* says that the imaginary parts of the zeros of all Dirichlet L-series for primitive characters are linearly independent over $\mathbf{Q}$. From the above decomposition results, it is clear that if ρ' is a subrepresentation of ρ, then $\sigma(\rho') \subseteq \sigma(\rho)$ (this even holds for infinite amenable groups if ρ' is weakly contained in ρ, cf. [92]); here, we are asking for a kind of converse result.

Chapter 4
Detecting Representation Isomorphism Through Twisted Spectra

In this chapter, we give a spectral characterisation of isomorphism of induced representations. We also discuss strong isospectrality in the sense of Pesce (which, by a result of Sunada, is implied by weak conjugacy of subgroups), discuss an illustrative example of lens spaces due to Ikeda, and use the first result to give a spectral characterisation of weak conjugacy.

4.1 Spectral Detection of Isomorphism of Induced Representations

In this section, we assume again that we have a diagram (1.2)

$$\begin{tikzcd}
& M \arrow[dl, "q_1"', "H_1"] \arrow[dd, "q"', "G"] \arrow[dr, "q_2", "H_2"'] & \\
M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\
& M_0 &
\end{tikzcd}$$

of finite coverings. We start with a proposition that allows us to detect isomorphism of representations induced from linear characters purely from spectral data.

Proposition 4.1.1 *For two linear characters $\chi_1 \in \check{H}_1$ and $\chi_2 \in \check{H}_2$, the following are equivalent:*

(i) $\operatorname{Ind}_{H_1}^G \chi_1 \cong \operatorname{Ind}_{H_2}^G \chi_2$.
(ii) *The spectrum $\sigma_{M_i}(\overline{\chi}_i \otimes \operatorname{Res}_{H_i}^G \operatorname{Ind}_{H_j}^G \chi_j)$ is independent of $i, j = 1, 2$.*
(ii') *Condition* (ii) *holds for the pairs (i, j) given by* $(1, 1), (2, 1)$ *and* $(1, 2), (2, 2)$.

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(iii) *The multiplicity of the zero eigenvalue in* $\sigma_{M_i}(\overline{\chi}_i \otimes \operatorname{Res}^G_{H_i} \operatorname{Ind}^G_{H_j} \chi_j)$ *is independent of* $i, j = 1, 2$.

(iii') *Condition* (iii) *holds for the pairs* (i, j) *given by* $(1, 1), (2, 1)$ *and* $(1, 2), (2, 2)$.

Remark 4.1.2 Condition (ii') is

$$\begin{cases} \sigma_{M_1}(\overline{\chi}_1 \otimes \operatorname{Res}^G_{H_1} \operatorname{Ind}^G_{H_1} \chi_1) = \sigma_{M_2}(\overline{\chi}_2 \otimes \operatorname{Res}^G_{H_2} \operatorname{Ind}^G_{H_1} \chi_1); \\ \sigma_{M_1}(\overline{\chi}_1 \otimes \operatorname{Res}^G_{H_1} \operatorname{Ind}^G_{H_2} \chi_2) = \sigma_{M_2}(\overline{\chi}_2 \otimes \operatorname{Res}^G_{H_2} \operatorname{Ind}^G_{H_2} \chi_2). \end{cases}$$

In this form, the statement of the proposition is similar to a number-theoretical result of Solomatin [89] that inspired our proof below.

Proof of Proposition 4.1.1 We start by proving that (i) implies (ii). Let ρ denote any irreducible representation of G; then for any $i = 1, 2$, we have

$$\begin{aligned} \langle \operatorname{Ind}^G_{H_i} \left(\overline{\chi}_i \otimes \operatorname{Res}^G_{H_i} \operatorname{Ind}^G_{H_i} \chi_i\right), \rho\rangle &= \langle \overline{\chi}_i \otimes \operatorname{Res}^G_{H_i} \operatorname{Ind}^G_{H_i} \chi_i, \operatorname{Res}^G_{H_i} \rho\rangle \\ &= \langle \overline{\chi}_i, \operatorname{Res}^G_{H_i} \rho \otimes \overline{\operatorname{Res}^G_{H_i} \operatorname{Ind}^G_{H_i} \chi_i}\rangle \\ &= \langle \overline{\chi}_i, \operatorname{Res}^G_{H_i} \left(\rho \otimes \overline{\operatorname{Ind}^G_{H_i} \chi_i}\right)\rangle \\ &= \langle \overline{\operatorname{Ind}^G_{H_i} \chi_i}, \rho \otimes \overline{\operatorname{Ind}^G_{H_i} \chi_i}\rangle. \end{aligned}$$

By assumption (i), this final expression is independent of $i = 1, 2$, and hence the same holds for the initial expression. Since this holds for any ρ, we find that

$$\operatorname{Ind}^G_{H_1} \left(\overline{\chi}_1 \otimes \operatorname{Res}^G_{H_1} \operatorname{Ind}^G_{H_1} \chi_1\right) = \operatorname{Ind}^G_{H_2} \left(\overline{\chi}_2 \otimes \operatorname{Res}^G_{H_2} \operatorname{Ind}^G_{H_2} \chi_2\right).$$

By Lemma 3.8.1, we find

$$\begin{aligned} \sigma_{M_1}(\overline{\chi}_1 \otimes \operatorname{Res}^G_{H_1} \operatorname{Ind}^G_{H_1} \chi_1) &= \sigma_{M_2}(\overline{\chi}_2 \otimes \operatorname{Res}^G_{H_2} \operatorname{Ind}^G_{H_2} \chi_2) \\ &= \sigma_{M_2}(\overline{\chi}_2 \otimes \operatorname{Res}^G_{H_2} \operatorname{Ind}^G_{H_1} \chi_1), \end{aligned} \tag{4.1}$$

the last line again by assumption (i). This is condition (ii) for $(i, j) = (1, 1)$ and $(i, j) = (2, 1)$. Using assumption (i), one may replace $\operatorname{Ind}^G_{H_1} \chi_1$ by $\operatorname{Ind}^G_{H_2} \chi_2$ in formula (4.1) on one or both sides, and this shows condition (ii) for all other choices of i, j.

Passing from stronger to weaker statements, (ii) implies (ii') and (iii), and (iii), as well as (ii'), imply (iii'). Hence we only need to prove that (iii') implies (i). Consider, for different i, j,

$$a_{i,j} := \langle \overline{\chi}_i \otimes \operatorname{Res}^G_{H_i} \operatorname{Ind}^G_{H_j} \chi_j, \mathbf{1}\rangle = \langle \operatorname{Ind}^G_{H_j} \chi_j, \operatorname{Ind}^G_{H_i} \chi_i\rangle, \tag{4.2}$$

where the last equality follows by Frobenius reciprocity. Setting ψ to be the class function $\psi := \mathrm{Ind}_{H_1}^G \chi_1 - \mathrm{Ind}_{H_2}^G \chi_2$, this allows us to compute that

$$\langle \psi, \psi \rangle = a_{1,1} + a_{2,2} - a_{1,2} - a_{2,1},$$

and since in (iii') we are assuming $a_{1,1} = a_{2,1}$ and $a_{1,2} = a_{2,2}$, it follows that $\langle \psi, \psi \rangle = 0$, so $\psi = 0$, which is condition (i). □

4.2 Strong Isospectrality and Spectral Detection of Weak Conjugacy

Isospectrality of manifolds M_1 and M_2 in a diagram of the form (1.2) does not in general imply that H_1 and H_2 are weakly conjugate.

Example 4.2.1 Consider the situation where $M = S^5$, and $M_1 = L(11; 1, 2, 3)$ and $M_2 = L(11; 1, 2, 4)$ are lens spaces $L(q; s_1, s_2, s_3)$ defined as the quotient of S^5 by the block diagonal 6×6 matrix given by three 2×2 blocks representing planar rotations over respective angles $2\pi s_i/q$. In this case, the two groups $H_i \cong \mathbf{Z}/11\mathbf{Z}$ are not equal as subgroups of the isometry group of S^5; they commute, and we can set

$$M_0 = (H_1 \times H_2)\backslash M.$$

Ikeda has shown that M_1 and M_2 are isospectral for the Laplace operator on functions (see [52, p. 313], observing that since $8 = -3 \bmod 11$, by Ikeda [52, Thm. 2.1] M_1 is isometric to $L(11; 1, 2, 8)$, where the latter parameters are the ones used by Ikeda). However, H_1 and H_2 are not weakly conjugate: since G is abelian, conjugacy classes are singletons and $|\{g\} \cap H_i|$ is 0 or 1 depending on whether g is in H_i or not. ♢

Sunada [90, Lemma 1] proved that weak conjugacy of H_1 and H_2 implies *strong isospectrality*, defined as follows in the sense of Pesce [79, §II].

Definition 4.2.2 Two Riemannian manifolds M_1 and M_2 admitting a common cover M are called *strongly isospectral* if the spectra of $q_{i*}A$ acting on $L^2(M_i, q_{i*}E) \cong L^2(M, E)^{H_i}$ are equal for any natural operator A on M. Here, $q_i : M \twoheadrightarrow M_i$ are the corresponding covering maps, and an operator A as above on M is *natural* if G acts isometrically on the fibers of the bundle E and A commutes with the action of G.

The Laplace operators acting on k-forms (see Example 3.1.2) are natural for all k, so if H_1 and H_2 are weakly conjugate, then the spectra of all of these are equal (sometimes, "strong isospectrality" is used to mean that precisely these operators are isospectral, but we will follow Pesce's definition as above).

Example 4.2.3 The lens spaces in Example 4.2.1 are isospectral for the Laplacian on functions, but not on all k-forms, cf. [60, Remark 3.8]. In fact, M_1 and M_2 from Example 4.2.1 are isospectral on functions, but not on 1-forms, as is explained in [53, Example 1, p. 416] (in the notation of loc. cit., $M_1 = \overline{L}_2$ and M_2 is isometric to $\overline{L}_1 = L(11; 1, 2, 5)$).

Strongly isospectral lens spaces are isometric [60, Proposition 7.2]. There exist lens spaces isospectral on k-forms for all k, while at the same time not strongly isospectral (i.e., isometric) in the sense of Definition 4.2.2; for example, $M_1 = L(49; 1, 6, 15)$ and $M_2 = L(49; 1, 6, 20)$. This follows from the characterisation of k-isospectrality for all k in terms of lattice norms with an extra geometric condition in [60], which permits the construction of many examples, as well as infinite families of pairs of lens spaces with this property, e.g., [60, Table 1 and Theorem 7.1]. $\Diamond$

The next Proposition 4.2.4 provides a spectral criterion that is equivalent to weak conjugacy, and is an immediate corollary of Proposition 4.1.1. It is analogous to a number theoretical result of Nagata [73].

Proposition 4.2.4 *Suppose M is a connected smooth closed Riemannian manifold, G a finite group of isometries of M and H_1 and H_2 are two subgroups of fixed-point free isometries in G with associated quotient manifolds $M_1 := H_1\backslash M$ and $M_2 := H_2\backslash M$. Then H_1 and H_2 are weakly conjugate if and only if the multiplicity of the zero eigenvalue in $\sigma_{M_i}(\mathrm{Res}^G_{H_i} \mathrm{Ind}^G_{H_j} \mathbf{1})$ is independent of $i, j = 1, 2$.*

Proof The groups H_1 and H_2 are weakly conjugate precisely when there is an isomorphism of permutation representations $\mathrm{Ind}^G_{H_1} \mathbf{1} \cong \mathrm{Ind}^G_{H_2} \mathbf{1}$. By Proposition 4.1.1, this is equivalent to the claim. $\square$

One may vary the condition of twisted isospectrality using the equivalent conditions in Proposition 4.1.1. For example, using Remark 4.1.2, we deduce the following (adding some redundant information).

Corollary 4.2.5 *If a diagram as in (1.2) is given, and the following twisted spectra agree:*

$$\begin{cases} \sigma_{M_1}(\mathrm{Res}^G_{H_1} \mathrm{Ind}^G_{H_1} \mathbf{1}) = \sigma_{M_2}(\mathrm{Res}^G_{H_2} \mathrm{Ind}^G_{H_1} \mathbf{1}); \\ \sigma_{M_1}(\mathrm{Res}^G_{H_1} \mathrm{Ind}^G_{H_2} \mathbf{1}) = \sigma_{M_2}(\mathrm{Res}^G_{H_2} \mathrm{Ind}^G_{H_2} \mathbf{1}), \end{cases}$$

then the manifolds M_1 and M_2 are strongly isospectral.

Remark 4.2.6 Mackey's theorem describes how, for two subgroups K_1 and K_2 of a group G, a representation of the form $\mathrm{Res}^G_{K_2} \mathrm{Ind}^G_{K_1} \rho$ splits into irreducibles (see, e.g., [88, Prop. 22]). In the situation of Proposition 4.2.4, with $K_i \in \{H_1, H_2\}$, we find that $\mathrm{Res}^G_{K_2} \mathrm{Ind}^G_{K_1} \mathbf{1}$ splits as the direct sum of the permutation representations corresponding to the action of K_2 on the cosets of $sK_1s^{-1} \cap K_2$ for $s \in K_2\backslash G/K_1$, and the occurring spectra are the (multiset-)union of the spectra corresponding to these representations; for example,

$$\sigma_{M_i}(\mathrm{Res}^G_{H_i}\,\mathrm{Ind}^G_{H_i}\,\mathbf{1}) = \bigcup_{s\in H_i\backslash G/H_i} \sigma_{M_i}(\mathrm{Ind}^{H_i}_{sH_is^{-1}\cap H_i}\,\mathbf{1})$$

contains the usual Laplace spectra $\sigma_{M_i}(\Delta_{M_i})$ (setting s to be the trivial double coset).

Project

It is possible that, given an integer p_0 satisfying $0 < p_0 \leq n$, n-dimensional manifolds are isospectral for the Laplacian on p-forms for all $p < p_0$, but not for $p = p_0$, and one could say that the larger p_0/n, the "more strongly isospectral" the manifolds are. Can one give a geometric meaning to p_0/n, given two manifolds (possibly of special type)? For example, this problem is solved by Lauret [59] for lens spaces by encoding some geometric properties in a generating series.

Chapter 5
Representations with a Unique Monomial Structure

In this chapter, we recall the notion of monomial structures (and their isomorphism) on a representation, show a natural monomial structure on induced representations, and introduce solitary characters (characters whose induced representation has a unique monomial structure up to isomorphism); these characters may be used to detect conjugacy of subgroups. We also recall a specific type of wreath product construction and state and prove Bart de Smit's theorem on the existence of solitary characters for these (and a follow-up result of Pintonello for characters of degree two)—these were previously formulated and used in the context of number theory, but we present them abstractly. We give an application to covering equivalence in a very specific setup of manifolds, and also count the number of required characters, based on a formula for the commutator of a wreath product.

5.1 Monomial Structures

Definition 5.1.1 Suppose $\rho\colon G \to \mathrm{Aut}(V)$ is a representation, and

$$V = \bigoplus_{x \in \Omega} \mathscr{L}_x$$

is a decomposition of V into one-dimensional spaces ("lines") $\mathscr{L}_x$ for $x \in \Omega$, with Ω some index set. If the action of G on V permutes the lines $\mathscr{L}_x$, we say that the G-set

$$L = \{\mathscr{L}_x : x \in \Omega\}$$

is a *monomial structure* on ρ.

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Equivalently, in a basis having precisely one element from each line $\mathscr{L}_x$, the action of any $g \in G$ is given by a matrix having exactly one non-zero entry in each row and column. Note that, contrary to the case of permutation matrices, the non-zero entry in the matrix need not be 1.

An *isomorphism of monomial structures* L and L' on two representation of the same group G is an isomorphism of L and L' as G-sets.

Example 5.1.2 An induced representation $\mathrm{Ind}_H^G \chi$ of a linear character $\chi \in \check{H}$ admits (by definition) a monomial structure where $\Omega = \{g_1, \ldots, g_n\}$ is such that $g_i H$ are the different cosets of H in G, and $\mathscr{L}_x = \mathbf{C} \cdot xH$. The corresponding matrices have as non-zero entries n-th roots of unity if χ is a character of order n. We call this monomial structure the *standard monomial structure* on $\mathrm{Ind}_H^G \chi$. This standard monomial structure is isomorphic to G/H as G-set. ♢

Definition 5.1.3 A linear character Ξ on a subgroup H of a group G is called G-*solitary* if $\mathrm{Ind}_H^G \Xi$ has a unique monomial structure up to isomorphism.

Lemma 5.1.4 *Let G denote a group with two subgroups H_1 and H_2, and suppose $\Xi \in \check{H}_1$ is a G-solitary linear character. There exists a linear character $\chi \in \check{H}_2$ for which there is an isomorphism of representations* $\mathrm{Ind}_{H_1}^G \Xi \cong \mathrm{Ind}_{H_2}^G \chi$ *if and only if H_1 and H_2 are conjugate subgroups of G.*

Proof In this situation, $\mathrm{Ind}_{H_2}^G \chi$ carries two monomial structures: the standard one and the one induced from the standard one on $\mathrm{Ind}_{H_1}^G \Xi$ through the isomorphism of representations. Hence these monomial structures have to be isomorphic. But as G-sets, they are G/H_1 and G/H_2, respectively (see Example 5.1.2). By Proposition 3.5.1(ii), this means precisely that H_1 and H_2 are conjugate in G. □

5.2 Wreath Product Construction

Definition 5.2.1 Let G denote a finite group and H a subgroup of index $n := [G : H]$ with cosets

$$\{g_1 H = H, g_2 H, \ldots, g_n H\}$$

of cardinality n. For a prime number ℓ, let $C = \mathbf{Z}/\ell\mathbf{Z}$ denote the cyclic group with ℓ elements, and let

$$\widetilde{G} := C^n \rtimes G$$

denote the *wreath product*; this is by definition the semidirect product where G acts on the n copies of C by permuting the coordinates in the same way as G permutes the cosets $g_i H$. In coordinates, this means that if we let $e_1, \ldots, e_n$ denote the standard basis vectors of C^n, and, as before, define the permutation $i \mapsto g(i)$ of $\{1, \dots, n\}$ by $gg_iH = g_{g(i)}H$, then the semidirect product is defined by the action

$$G \xrightarrow{\Phi} \operatorname{Aut}(C^n)\colon g \mapsto \Phi(g) = \left[\sum_{j=1}^{n} k_j e_j \mapsto \sum_{j=1}^{n} k_j e_{g(j)}\right] \tag{5.1}$$

where $k_j \in \mathbf{Z}/\ell\mathbf{Z}$. This is the (left) action of $g \in G$ on C^n given by

$$C^n \ni (k_1, \dots, k_n) \mapsto (k_{g^{-1}(1)}, \dots, k_{g^{-1}(n)}) \in C^n.$$

Define

$$\widetilde{H} := C^n \rtimes H$$

to be the subgroup of $\widetilde{G}$ corresponding to H. The cosets of $\widetilde{H}$ in $\widetilde{G}$ are of the form

$$\{\widetilde{g}_1\widetilde{H} = \widetilde{H}, \widetilde{g}_2\widetilde{H}, \dots, \widetilde{g}_n\widetilde{H}\},$$

where for $g_i \in G$, we have a corresponding element $\widetilde{g}_i := (0, g_i) \in \widetilde{G}$.

Remark 5.2.2 Recall that $\operatorname{Ind}_H^G \mathbf{1}$ is the $\mathbf{Z}[G]$-module corresponding to the permutation representation of G acting on the G-cosets of H. Thus, if we identify C with the additive group of the finite field $\mathbf{F}_\ell$, the action of G on $C^n \cong \mathbf{F}_\ell^n$ corresponds to the $\mathbf{F}_\ell[G]$-module $(\operatorname{Ind}_G^H \mathbf{1}) \otimes_{\mathbf{Z}} \mathbf{F}_\ell$.

Proposition 5.2.3 (Bart de Smit [28, §10]) *For all $\ell \geq 3$, there exists a $\widetilde{G}$-solitary character of order ℓ on $\widetilde{H}$.*

Proof Define Ξ by

$$\Xi\colon \widetilde{H} \to \mathbf{C}^*\colon (k_1, \dots, k_n, g) \mapsto e^{2\pi i k_1/\ell}. \tag{5.2}$$

Let $L = \{\mathscr{L}_x\}$ and $L' = \{\mathscr{L}'_x\}$ denote two monomial structures on $\rho := \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \Xi$, where L is the standard one (see Example 5.1.2). The action of $G \leq \widetilde{G}$ on L is that of G on G/H and (after rearranging) the action of $C^n \leq \widetilde{G}$ is given by

$$(k_1, \dots, k_n) \cdot \mathscr{L}_j = e^{2\pi i k_j/\ell} \cdot \mathscr{L}_j, \tag{5.3}$$

where we used the simplified notation $\mathscr{L}_j := \mathscr{L}_{g_j\widetilde{H}}$. The character ψ of ρ can be computed using as basis any set of vectors from the lines in L or L'. From the above,

$$|\psi((1, 0, \dots, 0))| = |e^{2\pi i/\ell} + \underbrace{1 + \cdots + 1}_{n-1}| > n - 2,$$

where the last inequality is strict since $\ell \geq 3$. On the other hand, computing the same trace using a basis from L', we get a sum of some number, say, m, of ℓ-th roots of unity, where m is the number of lines in L' that are mapped to itself by $(1, 0, \dots, 0)$. If there is a line not mapped to itself (a zero diagonal entry in the corresponding matrix), then there are at least two (since every row/column has precisely two non-zero entries), so $m = n$ or $m \leq n - 2$. In the latter case, $|\psi((1, 0, \dots, 0))| \leq n - 2$, which is impossible. Since C^n is generated by G-conjugates of $(1, 0, \dots, 0)$, we find that C^n fixes all lines in L'. Hence $L' \subseteq L$, but since $|L| = |L'| = [\widetilde{G} : \widetilde{H}]$, we have $L = L'$. □

Pintonello [80, Theorem 3.2.2] has shown that for $\ell = 2$, there does not always exist a solitary character as in Proposition 5.2.3. However, he also proved the following result, of which we give a self-contained proof.

Proposition 5.2.4 (Pintonello [80, Theorem 2.3.1]) *Given a group G with two subgroups H_1 and H_2, consider the corresponding wreath products $\widetilde{G}$, $\widetilde{H}_1$ and $\widetilde{H}_2$ with $C = \mathbf{Z}/2\mathbf{Z}$. Set $\Xi \colon \widetilde{H}_1 \to \mathbf{C}^* \colon (k_1, \dots, k_n, g) \mapsto (-1)^{k_1}$, and assume that both*

$$\operatorname{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \mathbf{1} \cong \operatorname{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \mathbf{1} \text{ and} \tag{5.4}$$

$$\operatorname{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi \cong \operatorname{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \chi, \tag{5.5}$$

for some linear character χ on $\widetilde{H}_2$. Then $\widetilde{H}_1$ and $\widetilde{H}_2$ are conjugate in G.

Proof Equality (5.5) induces two monomial structures L_1 and L_2 on $\rho := \operatorname{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi$, where L_i is isomorphic to $\widetilde{G}/\widetilde{H}_i$. As in (5.3), $\varepsilon := (1, 0, \dots, 0) \in C^n \leq \widetilde{G}$ fixes all lines in L_1. Note that the number of lines in L_i fixed by ε is the value of the character of $\operatorname{Ind}_{\widetilde{H}_i}^{\widetilde{G}} \mathbf{1}$ at ε, given in (3.3), and by (5.4), these are equal for $i = 1$ and $i = 2$. Hence all lines in L_2 are fixed by ε, and as in the previous proof, we conclude that C^n fixes all lines in L_2. Hence $L_2 \subseteq L_1$, but since $|L_1| = |L_2| = [\widetilde{G} : \widetilde{H}_i]$, we have $L_1 = L_2$. □

5.3 Application to Manifolds

We deduce the following intermediate result.

Corollary 5.3.1 *Suppose we have a diagram (1.2). Let $C := \mathbf{Z}/\ell\mathbf{Z}$ denote a cyclic group of prime order $\ell \geq 3$. Let $\widetilde{G}$ and $\widetilde{H}_1$ denote the wreath products as in Definition 5.2.1 (with $H = H_1$) and $\widetilde{H}_2 := C^n \rtimes H_2$ (with the same action defined via the H_1-cosets), and assume that there exists a diagram of Riemannian coverings*

$$\begin{tikzcd}
& M' \arrow[d, "C^n"] \arrow[ddl, bend right, "\widetilde{H}_1"'] \arrow[ddd, bend left=60, "\widetilde{G}"] \\
& M \arrow[dl, "H_1"'] \arrow[dd, "G"] \\
M_1 \arrow[dr, "p_1"'] & \\
& M_0
\end{tikzcd} \tag{5.6}$$

Then M_1 and M_2 are equivalent Riemannian covers of M_0 if and only if for a $\widetilde{G}$-solitary character Ξ on $\widetilde{H}_1$ and for some linear character χ on $\widetilde{H}_2$, the multiplicity of zero is equal in the two spectra

$$\sigma_{M_1}(\overline{\Xi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_1} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi) \text{ and } \sigma_{M_2}(\overline{\chi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_2} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi)$$

and in the two spectra

$$\sigma_{M_1}(\overline{\Xi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_1} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi) \text{ and } \sigma_{M_2}(\overline{\chi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_2} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi).$$

Proof First of all, since $\ell \geq 3$, a $\widetilde{G}$-solitary character Ξ on H_1 exists, by Proposition 5.2.3. By Proposition 4.1.1, the equalities of multiplicities of zero is equivalent to $\mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi \cong \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi$. Since Ξ is $\widetilde{G}$-solitary, we conclude by Lemma 5.1.4 that $\widetilde{H}_1$ and $\widetilde{H}_2$ are conjugate in $\widetilde{G}$. As C^n is normal in $\widetilde{H}_2$ with quotient H_2, we find that $\widetilde{H}_2 \backslash M' = H_2 \backslash M = M_2$ and hence this conjugacy defines an isometry from M_1 to M_2 that is the identity on M_0. □

Since χ runs over linear characters of $\widetilde{H}_2$, the "less abelian" the extension is, the less spectra need to be compared. A more precise statement is the following, where we use the abelianisation H_2^{ab} of H_2, defined as the quotient of H_2 by the subgroup generated by commutators (equivalently, the largest abelian quotient of H_2; equivalently, $H_2^{\mathrm{ab}} \cong \mathrm{Hom}(H_2, \mathbf{C}^*)$). The two extremes are then: if H_2 is abelian, H_2^{ab} is as large as H_2; but if H_2 is non-abelian simple, then $|H_2^{\mathrm{ab}}| = 1$.

Proposition 5.3.2 *In the setup of Corollary 5.3.1, the dimension of the representations of which the spectra are being compared is the index $[G : H_2]$. Furthermore, the number of spectral equalities to be checked in Corollary 5.3.1 by using all possible linear characters on $\widetilde{H}_2$ is bounded above by $2\ell \cdot |H_2^{\mathrm{ab}}|$.*

In Corollary 5.3.1 and Proposition 5.3.2, one may interchange the roles of H_1 and H_2, which could lead to tighter results.

Proof The dimension of the representations we are considering, as induced representations, is the index $[\widetilde{G} : \widetilde{H}_2] = [G : H_2]$.

The spectral criterion in the proposition requires testing of 2 equalities of spectra for each linear character on $\widetilde{H}_2$, so there are at most $2|\widetilde{H}_2^{\mathrm{ab}}|$ equalities to be checked.

The commutator subgroup of a wreath product $\widetilde{H}_2 = C^n \rtimes H_2$ is computed in [69, Cor. 4.9], and we find that in our case, with $\Omega = \{g_1, \ldots, g_n\}$ a set of representatives for the cosets,

$$|[\widetilde{H}_2, \widetilde{H}_2]| = |[H_2, H_2]| \cdot |\{f : \Omega \to C \colon \sum_{y \in \Omega} f(y) = 0\}|;$$

where, with $|C| = \ell$, the second factor is $\ell^{|\Omega|-1}$. Hence we find $|\widetilde{H}_2^{\mathrm{ab}}| = |H_2^{\mathrm{ab}}| \cdot \ell$, and the result follows. □

Remark 5.3.3 Using Proposition 4.1.1 to reformulate spectrally the extra assumption in Proposition 5.2.4 (where $\ell = 2$), we find that in this case, the number of equalities to check is at most $2 + 4|H_2^{\mathrm{ab}}|$.

Remark 5.3.4 By Lemma 3.9.1, the multiplicity of zero in the spectrum $\sigma_M(\rho)$ can be computed purely representation theoretically as the multiplicity of the trivial representation in ρ, which is in principle possible by Mackey theory (cf. Remark 4.2.6), but this would be going in reverse (from spectra to group theory instead of the other way around). Knowing the group G and its subgroups H_1 and H_2, Riemannian equivalence of M_1 and M_2 over M_0 can be checked by a finite computation, verifying that H_1 and H_2 are conjugate in G. Corollary 5.3.1 translates this into a spectral statement (in the special setup where the group $\widetilde{G}$ is realised as indicated there).

Remark 5.3.5 One may strip all geometric analysis from the results so far, and formulate the following purely group theoretical result. *Given a finite group G and two subgroups H_1 and H_2, then*

$$H_1 \text{ and } H_2 \text{ are conjugate in } G \text{ if and only if } \mathrm{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi = \mathrm{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \chi$$

for some linear character χ on $\widetilde{H}_2$. Here, $\widetilde{G}$ denotes the wreath product corresponding to the action of G on the G-cosets of H_1, and Ξ denotes a solitary character of order 3 on $\widetilde{H}_1$ (which exists by Proposition 5.2.3). The proof is immediate from Lemma 5.1.4 and the final sentence in the proof of Proposition 5.3.1. Observe that the construction of the wreath products and of Ξ is completely explicit, and the linear characters on $\widetilde{H}_2$ can be described in terms of those on H_2 via the results used in the proof of Proposition 5.3.2.

In the next chapters, we study under which circumstances we have a cover as in Corollary 5.3.1, i.e., we deal with the realisation problem for the wreath product as isometry group of a cover, given an isometric free action of G on a closed manifold

M. This is analogous to the inverse problem of Galois theory, realising the wreath product as Galois group of a number field. In manifolds, some condition is necessary on M for such an extension to be possible at all.

Chapter 6
Construction of Suitable Covers and Proof of the Main Theorem

In this chapter, we set up some theory concerning the action of isometries on fundamental groups and first homology groups. More specifically, we show that the action of a finite group of isometries on the first homology group of a manifold corresponds to conjugation by a lift to the orbifold fundamental group of the quotient. This result is then used to give a necessary and sufficient condition for realising a certain wreath product by a Galois cover (by an explicit construction of the corresponding subgroup of the fundamental group)—a problem similar to the algebraic problem of inverse Galois theory. The required condition, called $(*)$, is a representation-theoretic property of the action on homology. Using the representation theoretical results from the previous chapter, this allows us to finish the proof of the main theorem of the text, describing covering equivalence of manifolds through spectra of twisted Laplacians.

6.1 Fundamental Group and First Homology

We first fix some notations and constructions. Let M denote a connected closed oriented smooth Riemannian manifold. Fixing a point $x \in M$, the universal covering $\widetilde{M}$ is described as the set of homotopy classes $[w]$ of paths $w : [0, 1] \to M$ with $w(0) = x$. This provides a projection map

$$\Pi : (\widetilde{M}, \widetilde{x}) \to (M, x),\ \ \Pi([w]) = w(1),$$

where $\widetilde{x} \in \widetilde{M}$ represents the homotopy class of the constant path at x. If we equip $\widetilde{M}$ with the pull-back of the Riemannian metric on M then the group Γ acts isometrically by deck transformations on $\widetilde{M}$, and M is identified with the quotient $\Gamma\backslash\widetilde{M}$.

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Let $*$ denotes path-concatenation read from left to right, that is, $[a] * [b]$ is the homotopy class of the path obtained by first traversing a and then b. Letting $\widetilde{x}$ denote the constant path at x, we have an identification

$$\Phi_{\widetilde{x}} : \Gamma \to \pi_1(M, x)$$

via the map $\Phi_{\widetilde{x}}(\gamma) = \gamma(\widetilde{x})$, with $\gamma(\widetilde{x})$ representing a homotopy class of a closed loop starting and ending at x. More generally, any homotopy class of a path $[w]$ in $\widetilde{M}$ induces a map $\Phi_{[w]} : \Gamma \to \pi_1(M, w(1))$ via

$$\Phi_{[w]}(\gamma) = [w^{-1}] * \Phi_{\widetilde{x}}(\gamma) * [w]. \tag{6.1}$$

We denote the first homology group of M (with integer coefficients) by $\mathrm{H}_1(M) = \mathrm{H}_1(M, \mathbf{Z})$. The universal coefficient theorem for homology [48, §3.A] implies that for any field K, we have an isomorphism

$$\mathrm{H}_1(M, K) = \mathrm{H}_1(M) \otimes_{\mathbf{Z}} K.$$

Also, since M is a connected manifold, it is path-connected, so that we have a Hurewicz homomorphism inducing an identification

$$\mathrm{H}_1(M) = \pi_1(M, x)^{\mathrm{ab}} \cong \Gamma^{\mathrm{ab}}.$$

The map is given by considering a (homotopy class of a) loop as a concatenation of oriented 1-cells and mapping it to the (homology class of the) signed sum of those cells.

Composing the maps, we have a homomorphism $\Psi_0 : \Gamma \to \mathrm{H}_1(M, \mathbf{F}_\ell)$ given as the composition of $\Phi_{[w]}$ with the abelianisation map and the Hurewicz isomorphism, followed by reduction modulo ℓ:

$$\begin{tikzcd} \Gamma \arrow[r, "\Phi_{[w]}"] \arrow[rrr, bend right, "\Psi_0"'] & \pi_1(M, w(1)) \arrow[r, ".^{\mathrm{ab}}"] & \mathrm{H}_1(M, \mathbf{Z}) \arrow[r, "\otimes \mathbf{F}_\ell"] & \mathrm{H}_1(M, \mathbf{F}_\ell). \end{tikzcd} \tag{6.2}$$

Since by (6.1) the homotopy classes of loops $\Phi_{[w]}$ for different w are freely homotopic, the composed map Ψ_0 is independent of the choice of w, as notation indicates. The standard choice is $w = \widetilde{x}$, but we will naturally encounter others.

6.2 First Homology and Galois Covers

Suppose now that G is a finite group of isometries acting on M, and let $q : M \to M_0$ denote the orbifold quotient, with Γ_0 the covering group of $\Pi_0 : \widetilde{M} \to M_0$. Let Γ

be the normal subgroup of Γ_0 corresponding to $\Pi\colon \widetilde{M} \to M$, so that there is a short exact sequence of groups

$$1 \to \Gamma \to \Gamma_0 \xrightarrow{F} G \to 1 \tag{6.3}$$

giving an isomorphism $G \cong \Gamma_0/\Gamma$. The setup is summarised in diagram (6.4).

$$\begin{tikzcd} \widetilde{M} \arrow[d, "\Pi"', "\Gamma"] \arrow[dd, bend right=60, "\Gamma_0"', "\Pi_0"] \\ M \arrow[d, "q"', "G"] \\ M_0 \end{tikzcd} \tag{6.4}$$

Definition 6.2.1 For $\gamma_0 \in \Gamma_0$, let $\mathrm{conj}_{\gamma_0} : \Gamma \to \Gamma$ be conjugation by γ_0, that is

$$\mathrm{conj}_{\gamma_0}(\gamma) = \gamma_0 \gamma \gamma_0^{-1}.$$

Remark 6.2.2 If $g \in G \cong \Gamma_0/\Gamma$ satisfies $g = \gamma_0\Gamma$, this represents the usual map

$$G \to \mathrm{Out}(\Gamma)\colon g \mapsto \mathrm{conj}_{\gamma_0}$$

induced by the exact sequence (6.3), where $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ is the quotient of the group of automorphisms of Γ by the group $\mathrm{Inn}(\Gamma) = \{\mathrm{conj}_\gamma : \gamma \in \Gamma\}$ of inner automorphisms.

In this setup, the action of the isometry g on M corresponds to the action of $\mathrm{conj}_{\gamma_0^{-1}}$ on Γ, so that there is a commuting diagram

$$\begin{tikzcd} & \widetilde{M} \arrow[dl, "\Gamma"'] \arrow[dr, "\gamma_0^{-1}\Gamma\gamma_0"] & \\ M \arrow[rr, "g"] & & M \end{tikzcd} \tag{6.5}$$

The action of G on M by isometries induces a linear action of G on $\mathrm{H}_1(M, \mathbf{F}_\ell)$, providing an $\mathbf{F}_\ell[G]$-module structure on this homology group. The following lemma describes the relation between this action and the above outer conjugation on Γ: they commute by the Hurewicz map. The argument is similar to the one used in the proof of Hopf's formula [21, (5.3)].

Lemma 6.2.3 *Let $\gamma_0 \in \Gamma_0$ and $g \in G \cong \Gamma_0/\Gamma$ such that $g = \gamma_0\Gamma$. Then the following diagram commutes:*

$$\begin{tikzcd} \Gamma \arrow[r, "\mathrm{conj}_{\gamma_0}"] \arrow[d, "\Psi_0"'] & \Gamma \arrow[d, "\Psi_0"] \\ \mathrm{H}_1(M, \mathbf{F}_\ell) \arrow[r, "g\cdot"] & \mathrm{H}_1(M, \mathbf{F}_\ell) \end{tikzcd} \tag{6.6}$$

where Ψ_0 *is as in* (6.2) *and the bottom line indicates the action of* $g \in G$ *on the first homology group.*

Proof The vertical maps are given by picking a base point $x \in M$, considering $\gamma \in \Gamma$ as a homotopy class of a closed loop in M based at x via $\Phi_{\tilde{x}}$, rewriting γ as a concatenation of oriented 1-cells, and mapping these to the corresponding sum of 1-cells in homology. The crucial observation that makes the proof work is that we can decompose into 1-cells, not just 1-cycles, and the image is independent of the choice of base point. In this way, the closed loop corresponding to $\mathrm{conj}_{\gamma_0}(\gamma)$ decomposes as a (left-to-right) concatenation of the 1-cells

$$(\gamma_0 : x \to gx) * (g \cdot \gamma : gx \to gx) * (\gamma_0^{-1} : gx \to x),$$

that is mapped by Ψ_0 to the sum of homology classes $[\gamma_0] + g \cdot [\gamma] - [\gamma_0] = g \cdot [\gamma]$, proving the commutativity of the diagram. □

Remark 6.2.4 In fact, there is a larger commutative diagram:

$$\begin{tikzcd} \Gamma \arrow[r, "\mathrm{conj}_{\gamma_0}"] \arrow[d, "\Phi_{\tilde{x}}"'] \arrow[dd, bend right=60, "\Psi_0"'] & \Gamma \arrow[d, "\Phi_{\gamma_0\tilde{x}}"] \arrow[dd, bend left=60, "\Psi_0"] \\ \pi_1(M, x) \arrow[r, "g\cdot"] \arrow[d] & \pi_1(M, gx) \arrow[d] \\ \mathrm{H}_1(M, \mathbf{F}_\ell) \arrow[r, "g\cdot"] & \mathrm{H}_1(M, \mathbf{F}_\ell) \end{tikzcd} \tag{6.7}$$

The detailed proof goes as follows: we have argued before that both vertical maps (from top to bottom) on the left and right hand side of (6.7) agree and are equal to Ψ_0. Naturality of the Hurewicz isomorphism guarantees commutativity of the lower square in (6.7) and it therefore suffices to prove commutativity of the upper square in (6.7).

Elements $g \in G$ are isometries $g : M \to M$ and the groups Γ_0 and Γ acting by isometries on $\widetilde{M}$ can be described via deck transformations as follows

$$\begin{aligned} \Gamma &:= \{I : \widetilde{M} \to \widetilde{M} \text{ isometry} \mid F(I([w])) = F([w]) \,\forall\, [w] \in \widetilde{M}\}, \\ \Gamma_0 &:= \{I : \widetilde{M} \to \widetilde{M} \text{ isometry} \mid \exists\, g \in G : F(I([w]) = gF([w]) \,\forall\, [w] \in \widetilde{M}\}. \end{aligned} \tag{6.8}$$

In this description, the map $F : \Gamma \to G$ from (6.3) is given by mapping $I \in \Gamma_0$ to the (uniquely determined) corresponding element $g \in G$ in (6.8).

Let $\pi_1(M, x, y)$ denote the set of homotopy classes of paths in M from x to y. We identify the elements in Γ_0 with homotopy classes of paths starting at x via the following bijective map:

$$\begin{aligned} \pi_1(M, x, gx) &\to F^{-1}(g) \subset \Gamma_0, \\ [a] &\mapsto I_a \colon [w] \mapsto [a] * [gw]. \end{aligned} \tag{6.9}$$

It can be easily checked that $I_a^{-1}([w]) = [g^{-1}a^{-1}] * [g^{-1}w]$. To prove commutativity of the upper square of diagram (6.7), we go around the square both ways.

- *Computing* $\Phi_{\gamma_0 \widetilde{x}}(\mathrm{conj}_{\gamma_0}(\gamma))$. We first describe the conjugation action in terms of concatenation. Using (6.9), we identify γ_0 with a map $I_a : \widetilde{M} \to \widetilde{M}$ for a path a satisfying $a(0) = x$ and $a(1) = gx$. Similarly, we identify γ with a map $I_c : \widetilde{M} \to \widetilde{M}$ for a path c satisfying $c(0) = c(1) = x$. For any $\widetilde{y} = [w] \in \widetilde{M}$ with a path w in M starting at $w(0) = x$, we have

$$\mathrm{conj}_{\gamma_0}(\gamma)(\widetilde{y}) = I_a(I_c(I_a^{-1}([w]))) = [a] * [gc] * [a^{-1}] * [w]$$

and, in particular,

$$\mathrm{conj}_{\gamma_0}(\gamma)(\widetilde{x}) = [a] * [gc] * [a^{-1}].$$

Now we evaluate $\Phi_{\gamma_0 \widetilde{x}}(\mathrm{conj}_{\gamma_0}(\gamma))$. We first note that

$$\gamma_0 \widetilde{x} = I_a(\widetilde{x}) = [a] * (g\widetilde{x}) = [a],$$

since $g\widetilde{x}$ is the homotopy class of the constant path at gx in M. This implies

$$\begin{aligned} \Phi_{\gamma_0 \widetilde{x}}(\mathrm{conj}_{\gamma_0}(\gamma)) &= [a^{-1}] * \Phi_{\widetilde{x}}(\mathrm{conj}_{\gamma_0}(\gamma)) * [a] \\ &= [a^{-1}] * \mathrm{conj}_{\gamma_0}(\gamma)(\widetilde{x}) * [a] \\ &= [gc]. \end{aligned}$$

- *Computing* $g \cdot \Phi_{\widetilde{x}}(\gamma)$. We have $g \cdot \Phi_{\widetilde{x}}(\gamma) = g \cdot \gamma(\widetilde{x}) = g \cdot I_c(\widetilde{x}) = g \cdot ([c] * \widetilde{x}) = [gc]$.

Since the results of the two computations agree, the proof is finished.

6.3 Realisability of the Wreath Product

The following result gives an exact topological criterion for realisation of the wreath product, in terms of the $\mathbf{F}_\ell[G]$-module structure of $\mathrm{H}_1(M, \mathbf{F}_\ell)$.

Proposition 6.3.1 *Suppose that we have a diagram of Riemannian coverings*

$$\begin{tikzcd} & M \arrow[dl, "H_1"', "q_1"] \arrow[dd, "G"] \\ M_1 \arrow[dr, "p_1"'] & \\ & M_0 \end{tikzcd} \tag{6.10}$$

with M_0 a developable orbifold and M, M_1 manifolds. Fix a prime ℓ, set $C = \mathbf{Z}/\ell\mathbf{Z}$ and consider the wreath products $\widetilde{G}$ and $\widetilde{H}_1$ as in Definition 5.2.1 (with $H = H_1$). Let $\{g_1H_1 = H_1, g_2H_1, \ldots, g_nH_1\}$ denote the cosets in G/H_1. For any $g \in G$, define the permutation $i \mapsto g(i)$ on $\{1, \ldots, n\}$ and the element $h_{g,i} \in H_1$ via $gg_i = g_{g(i)}h_{g,i}$.

Define the $\mathbf{F}_\ell[G]$-module $\mathscr{N}$ as $\mathscr{N} := (\mathrm{Ind}_{H_1}^G \mathbf{1}) \otimes_{\mathbf{Z}} \mathbf{F}_\ell$ (cf. Remark 5.2.2). Then the diagram (6.10) can be extended to a diagram of the form (5.6) if and only if $\mathscr{N}$ is a $\mathbf{F}_\ell[G]$-quotient module of $\mathrm{H}_1(M, \mathbf{F}_\ell)$.

Proof We fix the following for the duration of the proof.

- Set $\mathscr{M} := \mathrm{H}_1(M, \mathbf{F}_\ell)$.
- Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in C^n \cong \mathbf{F}_\ell^n$ denote the standard "basis vectors".
- As in Remark 5.2.2, we identify $\mathscr{N} \cong C^n \cong \mathbf{F}_\ell^n$ as $\mathbf{F}_\ell[G]$-modules, where the action of G on C^n given by the permutation representation of the cosets implemented by the map (5.1) used to define the wreath product, i.e., $\Phi\colon G \to \mathrm{Aut}(C^n)\colon g \mapsto (e_i \mapsto e_{g(i)})$.
- $\widetilde{M}$ is the universal covering of M, $M_0 = \Gamma_0\backslash\widetilde{M}$, $M_1 = \Gamma_1\backslash\widetilde{M}$ and $M = \Gamma\backslash\widetilde{M}$.
- F is the map from (6.3).

In one direction, assume the extended cover exists, corresponding to a quotient of Γ. Now $M' \to M$ is a Galois cover with group the elementary abelian ℓ-group C^n. Using the universal property of abelianisation, the corresponding map $\Gamma \to C^n$, whose kernel we denote by Γ', factors through to a map of $\mathbf{F}_\ell$-vector spaces

$$\varphi\colon \mathscr{M} \twoheadrightarrow C^n \cong \mathscr{N}.$$

We claim that this is a map of $\mathbf{F}_\ell[G]$-modules. For each $g \in G$ consider the corresponding element $\widetilde{g} = (0, g) \in \widetilde{G}$. Let $\gamma_0 \in \Gamma_0$ denote any lift of $\widetilde{g}$ (i.e., $\widetilde{g} = \gamma_0\Gamma' \in \widetilde{G} = \Gamma_0/\Gamma'$). By Lemma 6.2.3 applied to both M' and M, $\widetilde{g}$ acts on Γ' and g acts on Γ as conj_{γ_0}. Hence $\widetilde{g}$ acts on $\Gamma/\Gamma' = C^n$ as conjugation by $\gamma_0\Gamma' = \widetilde{g}$. Now in the semidirect product $\widetilde{G}$, the action of $\widetilde{g}$ on $c \in C^n$ is given by conjugation $c \mapsto \widetilde{g}c\widetilde{g}^{-1}$, which corresponds to the action of G on C^n via Φ, i.e., gives the $\mathbf{F}_\ell[G]$-module structure $\mathscr{N}$ to C^n. Therefore, $\varphi(gm) = g\varphi(m)$, as desired.

Conversely, suppose there exists a map $\varphi\colon \mathscr{M} \twoheadrightarrow \mathscr{N}$ of $\mathbf{F}_\ell[G]$-modules. Precomposing with Ψ_0 as in (6.2), we get a map $\Psi\colon \Gamma \to C^n$, and we set

$$\Gamma' := \ker \Psi \lhd \Gamma \text{ and } M' := \Gamma' \backslash M. \tag{6.11}$$

With this map we can also extend the commutative diagram (6.6) as follows:

$$\begin{tikzcd}
\Gamma \arrow[r, "\mathrm{conj}_{\gamma_0}"] \arrow[d, "\Psi_0"'] \arrow[dd, bend right=60, "\Psi"'] & \Gamma \arrow[d, "\Psi_0"] \arrow[dd, bend left=60, "\Psi"] \\
\mathrm{H}_1(M, \mathbf{F}_\ell) \arrow[r, "g\cdot"] \arrow[d, "\varphi"'] & \mathrm{H}_1(M, \mathbf{F}_\ell) \arrow[d, "\varphi"] \\
C^n \arrow[r, "\Phi(g)"] & C^n
\end{tikzcd} \tag{6.12}$$

where the commutativity of the bottom square is guaranteed precisely by our assumption that φ is a map of $\mathbf{F}_\ell[G]$-modules.

It remains to verify that the manifold M' fits into a diagram (5.6). For this, we need to prove that Γ' is normal in Γ_0 (hence also normal in Γ and Γ_1) such that there are induced group isomorphisms $\Gamma_0/\Gamma' \cong \widetilde{G}$, $\Gamma_1/\Gamma' \cong \widetilde{H}_1$ and $\Gamma/\Gamma' \cong C^n$, where $\widetilde{G} = C^n \rtimes G$ is the wreath product introduced in Definition 5.2.1 (with $H = H_1$). For this, we show that the group Γ' has the following properties (i)–(v).

(i) Γ' *is normal in* Γ_0; indeed, commutativity of diagram (6.12) implies that if $\gamma_0 \in \Gamma_0$ with $F(\gamma_0) = g$ and $\Psi(\gamma') = 0$ (i.e., $\gamma' \in \Gamma'$), then $\Psi(\mathrm{conj}_{\gamma_0}(\gamma')) = \Phi(g)\Psi(\gamma') = 0$ (i.e., $\gamma_0\gamma'\gamma_0^{-1} \in \Gamma'$).

Fix a set-theoretic section $G \to \Gamma_0\colon g \mapsto \overline{g}$ of the map F, i.e., for any $g \in G$ fix any $\overline{g} \in \Gamma_0$ such that $F(\overline{g}) = g$. Fix an element $c \in \Gamma$ with $\Psi(c) = e_1$ (which exists since Ψ is surjective), and for $i = 1, \dots, n$, let $c_i := \overline{g}_i c \overline{g}_i^{-1} \in \Gamma_0$. Notice that

$$\Psi(c_i) = \Phi(g_i)e_1 = e_i. \tag{6.13}$$

Then we have the following properties:

(ii) $c_i c_j \Gamma' = c_j c_i \Gamma'$ *for all* $1 \le i, j \le n$. This follows since $\Gamma/\Gamma' \cong C^n$ is commutative.

(iii) *The cosets of* Γ' *in* Γ_0 *are represented by* $\overline{g} c_1^{k_1} \cdots c_n^{k_n} \Gamma'$ *with* $g \in G$ *and* $0 \le k_1, \dots, k_n < \ell$. From (6.13) we see that $c_1^{k_1} \cdots c_n^{k_n} \Gamma'$ are the cosets of Γ/Γ', and this combines with $G \cong \Gamma_0/\Gamma$.

(iv) $\overline{g} c_i \Gamma' = c_{g(i)} \overline{g} \Gamma'$ *for all* $g \in G$. Indeed,

$$\Psi(\overline{g} c_i \overline{g}^{-1}) \overset{(6.12)}{=} \Phi(g)\Psi(c_i) \overset{(6.13)}{=} \Phi(g)(e_i) \overset{(5.1)}{=} e_{g(i)} \overset{(6.13)}{=} \Psi(c_{g(i)}).$$

It follows that $\overline{g}c_i\overline{g}^{-1}\Gamma' = c_{g(i)}\Gamma'$ and, since Γ' is normal in Γ_0, we can interchange left and right cosets and multiply on the right with $\overline{g}$ to find the result.

By (iii) and (iv), the cosets of Γ_0/Γ' are given by $\overline{g}c_1^{k_1}\cdots c_n^{k_n}\Gamma' = c_{g(1)}^{k_1}\cdots c_{g(n)}^{k_n}\overline{g}\Gamma'$ with $g \in G$ and $0 \le k_1, \ldots, k_n < \ell$. Define $\widehat{\Psi} : \Gamma_0 \to \widetilde{G} = C^n \rtimes G$ by

$$\widehat{\Psi}(c_1^{k_1}\cdots c_n^{k_n}\overline{g}\Gamma') = (k_1, \ldots, k_n, g).$$

(v) Using the commutativity property in (ii), we have for $k_i, k'_j \in \{0, 1, \ldots, \ell-1\}$, and $g, g' \in G$,

$$\left(c_1^{k_1}\cdots c_n^{k_n}\overline{g}\Gamma'\right)\left(c_1^{k'_1}\cdots c_n^{k'_n}\overline{g}'\Gamma'\right) = \left(c_1^{k_1}\cdots c_n^{k_n}c_{g(1)}^{k'_1}\cdots c_{g(n)}^{k'_n}\right)(\overline{g}\cdot\overline{g}')\Gamma',$$

which follows immediately from (i) and (iv).

(vi) *Via a modification of the section $G \to \Gamma_0$, $g \mapsto \overline{g}$, the map $\widehat{\Psi}$ becomes a surjective group homomorphism with kernel* Γ'. This follows immediately from (v) if we can choose the section in such a way that $(\overline{g}\cdot\overline{g}')\Gamma' = \overline{g\cdot g'}\Gamma'$. Consider the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma/\Gamma' & \longrightarrow & \Gamma_0/\Gamma' & \xrightarrow{\alpha} & \Gamma_0/\Gamma & \longrightarrow & 1, \\ & & \wr\| & & & & \wr\| & & \\ & & C^n & & & & G & & \end{array} \tag{6.14}$$

with the canonical maps. Note that $\Gamma/\Gamma' \cong C^n$ is a G-module via the action

$$g\cdot\left(c_1^{k_1}\cdot c_n^{k_n}\Gamma'\right) = c_{g(1)}^{k_1}\cdot c_{g(n)}^{k_n}\Gamma'.$$

Since the order of G is coprime to that of C^n, we have the vanishing of group cohomology:

$$\mathrm{H}^1(G, \Gamma/\Gamma') = \mathrm{H}^2(G, \Gamma/\Gamma') = 0$$

[21, IV 2.3, 3.12, 3.13] and hence (6.14) splits; let $\jmath$ denote a group homomorphism $\jmath : G \to \Gamma_0/\Gamma'$ such that $\alpha\circ\jmath = \mathrm{id}_G$. Redefine the section $G \to \Gamma_0$ in such a way that $\overline{g}\Gamma' = \jmath(g)$. Then we have

$$(\overline{g}\cdot\overline{g}')\Gamma' = \overline{g}\Gamma'\cdot\overline{g}'\Gamma' = \jmath(g)\cdot\jmath(g') = \jmath(gg') = \overline{g\cdot g'}\Gamma'$$

and $\widehat{\Psi} : \Gamma_0 \to \widetilde{G}$ is a surjective group homomorphism with kernel Γ'.

It follows that $\widehat{\Psi}$ induces an isomorphism $\Gamma_0/\Gamma' \cong \widetilde{G}$. We have already seen the isomorphism $\Gamma/\Gamma' \cong C^n$ in the proof of (ii). To show that $\Gamma_1/\Gamma' \cong \widetilde{H}_1$ and finish the proof, note that $c_1^{k_1} \cdots c_n^{k_n}\overline{h}\Gamma'$ with $k_i \in \{0, \ldots, \ell-1\}$ and $h \in H_1$ are the cosets of Γ_1/Γ' and that the quotient Γ_1/Γ' is isomorphic to the subgroup $\widetilde{H}_1$ of $\widetilde{G}$ under the restriction of the isomorphism induced by $\widehat{\Psi}$. □

6.4 Main Result

We can now prove the main result.

Theorem 6.4.1 *Suppose M_1 and M_2 are two connected closed oriented smooth Riemannian manifolds such that there is a diagram*

$$\begin{tikzcd} M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd}$$

of finite covers of a developable Riemannian orbifold M_0, as in (1.1) *. Then*

(i) *The diagram* (1.1) *may be extended to a diagram of finite coverings*

$$\begin{tikzcd} & M \arrow[dl, "q_1"', "H_1"] \arrow[dr, "q_2", "H_2"'] \arrow[dd, "q"', "G"] & \\ M_1 \arrow[dr, "p_1"'] & & M_2 \arrow[dl, "p_2"] \\ & M_0 & \end{tikzcd}$$

as in (1.2)*, where M is a connected closed smooth Riemannian manifold M with three Galois covers*

$$q_1 \colon M \twoheadrightarrow M_1 := H_1\backslash M,$$
$$q_2 \colon M \twoheadrightarrow M_2 := H_2\backslash M$$
$$q \colon M \twoheadrightarrow M_0 := G\backslash M.$$

Suppose furthermore that there exists a prime number $\ell \geq 3$ such that

$(*)$ $(\mathrm{Ind}_{H_1}^{G} \mathbf{1}) \otimes_{\mathbf{Z}} \mathbf{F}_\ell$ *is an* $\mathbf{F}_\ell[G]$*-quotient module of* $\mathrm{H}_1(M, \mathbf{F}_\ell)$.

Then

(ii) *There exists a diagram of Riemannian coverings*

$$
\begin{tikzcd}
 & M' \arrow[d, "C^n"] \arrow[ddl, bend right, "\widetilde{H}_1"'] \arrow[ddd, bend left=60, "\widetilde{G}"] \\
 & M \arrow[dl, "H_1"'] \arrow[dd, "G"] \\
M_1 \arrow[dr, "p_1"'] & \\
 & M_0
\end{tikzcd}
$$

as in (5.6)*, where* $C = \mathbf{Z}/\ell\mathbf{Z}$*,* $\widetilde{G} = C^n \rtimes G$ *is the wreath product where* G *permutes the copies of* C *in the same way as it permutes the cosets of* H_1 *in* G*, and* $\widetilde{H}_i = C^n \rtimes H_i$ *are subgroups of* $\widetilde{G}$ *corresponding to the groups* H_i*,* $i = 1, 2$*.*

(iii) *Consider the linear character*

$$\Xi\colon \widetilde{H}_1 \to \mathbf{C}^*\colon (k_1, \dots, k_n, h_1) \to e^{2\pi i k_1}/\ell$$

with $(k_1, \dots, k_n) \in C^n$ *and* $h_1 \in H_1$*. Then the manifolds* M_1 *and* M_2 *are equivalent Riemannian covers of* M_0 *if and only if there exists a linear character* $\chi\colon \widetilde{H}_2 \to \mathrm{U}(1, \mathbf{C})$ *such that the multiplicity of zero in the following two pairs of spectra of twisted Laplacians on* M_1 *and* M_2 *coincide:*

$$\sigma_{M_1}(\overline{\Xi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_1} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi) \textit{ and } \sigma_{M_2}(\overline{\chi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_2} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_1} \Xi)$$

and

$$\sigma_{M_1}(\overline{\Xi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_1} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi) \textit{ and } \sigma_{M_2}(\overline{\chi} \otimes \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}_2} \mathrm{Ind}^{\widetilde{G}}_{\widetilde{H}_2} \chi).$$

There are $\ell|H_2^{\mathrm{ab}}|$ *linear characters* χ *on* $\widetilde{H}_2$*, and the dimension of the representations involved is the index* $[G : H_2]$*.*

Proof Part (i) holds by Proposition 2.4.1. Part (ii) is shown in Proposition 6.3.1. Then (iii) holds by Corollary 5.3.1 and Proposition 5.3.2, using that the character Ξ given in (iii), is the one constructed in the proof of Proposition 5.2.3 (cf. formula (5.2)), and is $\widetilde{G}$-solitary on $\widetilde{H}_1$. □

Remark 6.4.2 Using Proposition 3.10.1, the condition on the multiplicity of zero in the spectrum of the indicated twisted Laplacians in Theorem 6.4.1 may be replaced by an equality of their spectral zeta functions, if we assume in addition that M_1 and M_2 are isospectral, i.e., $\zeta_{M_1,\Delta_{M_1}} = \zeta_{M_2,\Delta_{M_2}}$.

Condition ($*$) in the main theorem can be varied, as we will see in the next two chapters. This will also produce a geometric realisation of M' and a set of examples where the condition holds.

Project

Describe a theory of (homological) conditions for the realisability of general extensions of finite groups $1 \to A \to G \to H \to 1$ with abelian kernel A, extending a given H-covering; note that a theory of abelian coverings of a given manifold—the case where H is trivial—is encoded almost tautologically in the first homology group, since it is the abelinization of the fundamental group.

Chapter 7
Geometric Construction of the Covering Manifold

In this chapter, we provide a geometric construction of a manifold extending a given Galois cover to a wreath product, using composita and fiber products. For this to be possible, a certain assumption on the homology, previously called $(*)$, needs to be strengthened to a new condition $(**)$ (equivalent in most cases). To motivate and use this new condition, we first recall the connection between homology of a quotient and coinvariants. Apart from geometric tools, the construction is also based on the vanishing of certain group cohomology, which is used to prove the existence of certain isometries of manifolds. In the final section, we give a universal property of the wreath product in relation to coverings of manifolds, just like there is such a universal property in the theory of Galois extensions of fields.

7.1 From Quotient to Submodule

If ℓ is a prime number coprime to $|G|$, by Maschke's theorem, any short exact sequence of $\mathbf{F}_\ell[G]$-modules splits, and condition $(*)$ from Theorem 6.4.1 is equivalent to

$$(**)\ (\mathrm{Ind}_{H_1}^G \mathbf{1}) \otimes_{\mathbf{Z}} \mathbf{F}_\ell \textit{ is an } \mathbf{F}_\ell[G]\textit{-submodule of } \mathrm{H}_1(M, \mathbf{F}_\ell).$$

7.2 Homology of a Quotient as Coinvariants

We recall the following tool from invariant theory, see, e.g. [21, II §2]. If R is a commutative ring (for us, R is $\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{F}_\ell$), H a finite group, and $\mathscr{M}$ is a (left) $R[H]$-module, its *coinvariants* are defined as the R-module $\mathscr{M}_H := \mathscr{M}/I\mathscr{M}$ where

G. Cornelissen, N. Peyerimhoff, *Twisted Isospectrality, Homological Wideness, and Isometry*, SpringerBriefs in Mathematics,
https://doi.org/10.1007/978-3-031-27704-7_7

I is the kernel of the augmentation map $R[H] \to R\colon \sum k_h h \mapsto \sum k_h$. An explicit description is given by

$$I\mathscr{M} = \langle h(x) - x\colon h \in H, x \in \mathscr{M}\rangle$$

(by linearly, it suffices to let x run over a set of generators of $\mathscr{M}$). Denote the projection map by

$$\underline{\mathrm{t}}_R\colon \mathscr{M} \to \mathscr{M}_H = \mathscr{M}/I\mathscr{M}. \tag{7.1}$$

When R is clear from the context, we will leave it out of the notation and simply write $\underline{\mathrm{t}}$ for this map.

This map is particularly easy if $\mathscr{M} = \bigoplus R[H]x_i$ is *free* as an $R[H]$-module with generators x_i; then $\mathscr{M}_H = \bigoplus Rx_i$ with the obvious map, i.e.,

$$\underline{\mathrm{t}}_R\colon \bigoplus R[H]x_i \to \bigoplus Rx_i\colon \sum_i \sum_h k_h h x_i \mapsto \sum_i \left(\sum_h k_h\right) x_i, \tag{7.2}$$

cf. [21, (2.3)].

One may use "transfer" to prove the following (the case of a free action is also in [21, II.(2.4)]).

Lemma 7.2.1 ([16, III.2.4]) *If H is a finite group of isometries of a closed smooth manifold M with quotient map*

$$q\colon M \to H\backslash M,$$

and the order of H is coprime to the characteristic of the field K, then the first K-homology of the quotient, $\mathrm{H}_1(H\backslash M, K)$, is isomorphic to the coinvariants $\mathrm{H}_1(M, K)_H$ of the first K-homology of M, and under this identification, the map q_* that q induces on the first homology groups is the map $\underline{\mathrm{t}}_K$ from (7.1), i.e., we have a diagram

$$\begin{tikzcd}
 & \mathrm{H}_1(H\backslash M, K) \arrow[d, "\cong"] \\
\mathrm{H}_1(M, K) \arrow[ur, "q_*"] \arrow[dr, "\underline{\mathrm{t}}_K"'] & \\
 & \mathrm{H}_1(M, K)_H
\end{tikzcd}$$

7.3 Geometric Construction

We refer back to the situation of diagram (6.10), and keep our assumption that $\mathbf{F}_\ell$ is a field of order coprime to $|G|$. By condition $(**)$, we have a decomposition of

$\mathbf{F}_\ell[G]$-modules

$$\mathrm{H}_1(M, \mathbf{F}_\ell) = \mathscr{N} \oplus V \cong \bigoplus \mathbf{F}_\ell \omega_i \oplus V$$

(for some $\mathbf{F}_\ell[G]$-submodule V), where the G-action on $\mathscr{N}$ is given in terms of the permutation of cosets as $g\omega_i = \omega_{g(i)}$, with the convention that $i = 1$ corresponds to the trivial H_1-coset in G. We also let

$$V' := \bigoplus_{i \geq 2} \mathbf{F}_\ell \omega_i$$

denote the vector space complement of $\mathbf{F}_\ell \omega_1$ in $\mathscr{N}$.

The quotient map $q_1 \colon M \to M_1 = H_1 \backslash M$ induces a surjective map

$$q_{1*} \colon \mathrm{H}_1(M, \mathbf{F}_\ell) \to \mathrm{H}_1(M_1, \mathbf{F}_\ell),$$

and we define $\omega_1' := q_{1*}(\omega_1)$.

Let Γ_1 denote the subgroup $\Gamma_1 \leq \Gamma_0$ for which $M_1 = \Gamma_1 \backslash \widetilde{M}$.

Lemma 7.3.1 *Suppose ℓ is coprime to $|G|$ and condition* $(*)$ (equivalently, $(**)$) *holds. Then we have a well-defined and commutative diagram:*

$$\begin{tikzcd}
\Gamma \arrow[r, two heads] \arrow[d, "\iota"] \arrow[rr, bend left, "\Psi"] & \mathrm{H}_1(M, \mathbf{F}_\ell) \arrow[r, two heads, "\varphi"'] \arrow[d, "q_{1*}"] & C^n \arrow[d, "r_1"] \\
\Gamma_1 \arrow[r, two heads] \arrow[rr, bend right, "\chi_0"'] & \mathrm{H}_1(M_1, \mathbf{F}_\ell) \arrow[r, two heads, "\varphi_0"] & C
\end{tikzcd} \tag{7.3}$$

where

- *ι is the embedding of* Γ *in* Γ_1*;*
- *$r_1 \colon C^n \to C$, $(k_1, \ldots, k_n) \mapsto k_1$ is projection onto the first coordinate;*
- *φ_0 is defined by*

$$\varphi_0 \colon \mathrm{H}_1(M_1, \mathbf{F}_\ell) \xrightarrow{\cong} \mathbf{F}_\ell \omega_1' \oplus W \to \mathbf{F}_\ell \cong C$$
$$k_1 \omega_1' + w \mapsto k_1 \qquad (k_1 \in \mathbf{F}_\ell, w \in W).$$

with $W := q_{1*}(V \oplus V')$ a complementary vector space to $\mathbf{F}_\ell \omega_1'$ *in* $\mathrm{H}_1(M_1, \mathbf{F}_\ell)$.

Proof To see that this is well defined and the right square commutes, we need that ω_1' is linear independent of $W = q_{1*}(V \oplus V')$; so suppose that there are $a_1, a_2 \in \mathbf{F}_\ell$ such that $a_1 \omega_1' + a_2 q_{1*}(v) = 0$ for some $v \in V \oplus V'$. This means that

$$a_1\omega_1 + a_2 v \in \ker(q_{1*}). \tag{7.4}$$

By Lemma 7.2.1, the kernel of q_{1*} is equal to the kernel of $\underline{t}_{\mathbf{F}_\ell}$, and by definition this kernel is spanned by elements $h_1(\omega_i) - \omega_i$ $(i = 1, \ldots, n)$ and $h_1(v) - v$ for $v \in V$ and $h_1 \in H_1$. Now

- for any $h_1 \in H_1 \le G$, $h_1(\omega_i) - \omega_i = \omega_{h_1(i)} - \omega_i$; if $i = 1$, this element is zero, since that index corresponds to the trivial conjugacy class of H_1 in G, whereas if $i \neq 1$, this element belongs to V', since then also $h_1(i) \neq 1$;
- since $\mathscr{N} \oplus V$ is a decomposition as $\mathbf{F}_\ell[G]$-modules, $h_1(v) - v \in V$ for all $v \in V$ and all $h_1 \in H_1$. □

It follows that $\ker(q_{1*}) \subseteq V \oplus V'$, and by (7.4), $a_1\omega_1 \in V \oplus V'$. Since ω_1 is linearly independent from $V \oplus V'$, we conclude that $a_1 = 0$, as desired. This guarantees that if $\omega = \sum k_i\omega_i + v \in \mathrm{H}_1(M, \mathbf{F}_\ell)$ with $v \in V$ (so $\varphi(\omega) = (k_1, \ldots, k_n)$), then $q_{1*}(\omega) = k_1\omega_1' + w \in \mathrm{H}_1(M_1, \mathbf{F}_\ell)$ with $w \in W$, so

$$\varphi_0(q_{1*}(\omega)) = k_1 = r_1(\varphi(\omega)).$$

Just like we defined $\Gamma = \ker \Psi$ in (6.11), we now set

$$\Gamma_1' := \ker \chi_0 \lhd \Gamma_1 \;\text{ and }\; M_1' := \Gamma_1' \backslash \widetilde{M} \text{ with covering map } q_1' \colon M_1' \to M_1. \tag{7.5}$$

The following lemma describes the relationship between the group $\Gamma = \ker \Psi$ used in Chap. 6, and $\Gamma_1' := \ker \chi_0$, the group used in this chapter.

Lemma 7.3.2 *Suppose ℓ is coprime to $|G|$ and condition* $(*)$ (equivalently, $(**)$) *holds. The group $\Gamma' = \ker \Psi$ can be expressed in terms of the group $\Gamma_1' = \ker \chi_0$ and a set $\{\overline{g}_1, \ldots, \overline{g}_n\}$ of lifts of $\{g_1, \ldots, g_n\}$ to Γ_0, as $\Gamma' = \Gamma'_{\text{new}}$, where*

$$\Gamma'_{\text{new}} := \bigcap_{i=1}^{n} \overline{g}_i \Gamma_1' \overline{g}_i^{-1} \cap \Gamma = \bigcap_{i=1}^{n} (\Gamma \cap \overline{g}_i \Gamma_1' \overline{g}_i^{-1}) = \bigcap_{i=1}^{n} \overline{g}_i (\Gamma \cap \Gamma_1') \overline{g}_i^{-1}. \tag{7.6}$$

Proof The equalities in (7.6) follow since Γ is normal in Γ_0. It remains to prove $\Gamma'_{\text{new}} = \ker \Psi$. Notice that it follows from diagram (7.3) that

$$\Gamma_1' \cap \Gamma = \ker \chi_0 \cap \Gamma = \{\gamma \in \Gamma \mid r_1 \circ \Psi(\gamma) = 0\} = \Psi^{-1}(\{0\} \times C^{n-1}). \tag{7.7}$$

Since Ψ is surjective, this implies $\Psi(\Gamma_1' \cap \Gamma) = \{0\} \times C^{n-1}$. Since by definition

$$\Phi(g_i)(\{0\} \times C^{n-1}) = C^{i-1} \times \{0\} \times C^{n-i},$$

from diagram (6.12), we conclude that

$$\Psi(\overline{g}_i(\Gamma_1' \cap \Gamma)\overline{g}_i^{-1}) = \Phi(g_i)\Psi(\Gamma_1' \cap \Gamma) = C^{i-1} \times \{0\} \times C^{n-i}, \tag{7.8}$$

and therefore

$$\Psi(\Gamma'_{\text{new}}) \subseteq \bigcap_i C^{i-1} \times \{0\} \times C^{n-i} = \{0\},$$

so $\Gamma'_{\text{new}} \subseteq \ker \Psi$.

To prove the reverse inclusion, assume that $\Psi(\gamma) = 0$ for some $\gamma \in \Gamma$. Then by diagram (6.12) we also have $\Psi(\gamma_0^{-1}\gamma\gamma_0) = 0$ for any $\gamma_0 \in \Gamma_0$, so

$$\gamma_0^{-1}\gamma\gamma_0 \in \Psi^{-1}(0) \subseteq \Psi^{-1}(\{0\} \times C^{n-1}) \overset{(7.7)}{=} \Gamma_1' \cap \Gamma.$$

Therefore $\gamma \in \gamma_0(\Gamma_1' \cap \Gamma)\gamma_0^{-1}$ for all γ_0, showing that $\gamma \in \Gamma'_{\text{new}}$, so $\ker \Psi \subseteq \Gamma'_{\text{new}}$. □

Remark 7.3.3 Standard expressions for the kernel of the restriction and induction of representations (see, e.g., [54, Lemma 5.11]) allow one to give a representation-theoretic description of Γ'_{new}. Namely, let $\widetilde{\chi}_0$ denote the linear character on Γ_1 given by $\widetilde{\chi}_0(\gamma) = e^{2\pi i \chi_0(\gamma)/\ell}$ where χ_0 is as in diagram (7.3). Then, with $\ker \widetilde{\chi}_0 = \ker \chi_0 = \Gamma_1'$, we have

$$\ker \operatorname{Res}_{\Gamma}^{\Gamma_0} \operatorname{Ind}_{\Gamma_1}^{\Gamma_0} \widetilde{\chi}_0 = \Gamma \cap \ker \operatorname{Ind}_{\Gamma_1}^{\Gamma_0} \widetilde{\chi}_0 = \Gamma \cap \bigcap_{\gamma_0 \in \Gamma_0} \gamma_0 \ker(\widetilde{\chi}_0)\gamma_0^{-1} = \Gamma'_{\text{new}}.$$

We now perform the following 2-step geometric construction:

(a) For $g \in G$, "twist" the cover $q_1 \colon M \to M_1$ by defining $q_1^g \colon M \to M_1$ by $x \mapsto q_1(g^{-1}x)$, and set

$$M_g'' := M_1' \times_{M_1, q_1^g} M;$$

corresponding to the following diagram:

$$\begin{tikzcd}
& M_g'' \arrow[dl] \arrow[dr] & & \\
M_1' \arrow[dr, "q_1'", swap, "C"'] & & M \arrow[dl, "q_1^g"'] \arrow[r, "g^{-1}\cdot"] & M \arrow[dll, "q_1"] \\
& M_1 & &
\end{tikzcd} \tag{7.9}$$

The two different M in the diagram are in fact identical, but the maps to M_1 are different.

(b) Iteratively construct the fiber product

$$M'_{\mathrm{new}} := M''_{g_1} \times_M M''_{g_2} \times_M \cdots \times_M M''_{g_n}, \tag{7.10}$$

where $\{g_1, \ldots, g_n\}$ is the chosen set of representatives for G/H_1; this is presented in the following diagram:

$$\begin{tikzcd}
 & & M'_{\mathrm{new}} \arrow[dll] \arrow[dl] \arrow[drr] & & \\
M''_{g_1} \arrow[drr, "C"'] & M''_{g_2} \arrow[dr, "C"] & & \cdots & M''_{g_n} \arrow[dll, "C"] \\
 & & M & &
\end{tikzcd} \tag{7.11}$$

We will prove that this manifold M'_{new} is the same as M', the one constructed in the previous chapter.

Proposition 7.3.4 *Suppose ℓ is coprime to $|G|$ and condition* $(*)$ (equivalently, $(**)$) *holds.*

(i) *The fiber product M'_{new} in* (7.10) *is represented as*

$$M'_{\mathrm{new}} = \Big\{(x_1, \ldots, x_n, x) \in M'_1 \times \cdots M'_1 \times M \mid \\ q'_1(x_i) = q_1(g_i^{-1}x),\ i = 1, \ldots, n\Big\}, \tag{7.12}$$

and in these coordinates, the projection $M'_{\mathrm{new}} \to M''_{g_i}$ is given by

$$M'_{\mathrm{new}} \ni (x_1, x_2, \ldots, x_n, x) \mapsto (x_i, x) \in M''_{g_i}.$$

M'_{new} is a connected manifold and corresponds to the subgroup Γ'_{new}, so that in fact $M'_{\mathrm{new}} = M'$.

(ii) *Geometrically, the action of $\widetilde{G}$ on M'_{new} is expressed as follows in the coordinates used in* (7.12)*: there exists an isometry $\iota\colon M'_{\mathrm{new}} \to M'_{\mathrm{new}}$ that conjugates the action of $\widetilde{G}$ into*

- *$\underline{c} = (c_i) \in C^n \le \widetilde{G}$ acts componentwise on each factor M''_{g_i}, i.e.,*

$$\iota^{-1}\underline{c}\iota \cdot (x_1, x_2, \ldots, x_n, x) = (c_1x_1, \ldots, c_nx_n, x); \tag{7.13}$$

- *$g \in G \le \widetilde{G}$ acts on M'_{new} by*

$$\iota^{-1}g\iota \cdot (x_1, x_2, \ldots, x_n, x) = (x_{g^{-1}(1)}, x_{g^{-1}(2)}, \ldots, x_{g^{-1}(n)}, gx), \tag{7.14}$$

where $g^{-1}(i)$ is defined, as before, via $g^{-1}g_i \in g_{g^{-1}(i)}H_1$. Colloquially, this means that, up to an isometry, in diagram (7.11), g act naturally on the "base" manifold M, while the points in the various M''_{g_j} above a given point in M are permuted across these different manifolds in the same way as g^{-1} permutes the cosets G/H_1.

Proof Since the group homomorphism $\chi_0 \colon \Gamma_1 \to C$ in diagram (7.3) is surjective, $\Gamma_1' := \ker \chi_0 \lhd \Gamma_1$ is of index ℓ in Γ_1, and $q_1' \colon M_1' \to M_1$ is a C-Galois cover.

(a) Since M_1 is a manifold, the compositum is described as

$$M''_g = \{(x_1, x) \in M_1' \times M \colon q_1'(x_1) = q_1(g^{-1}x)\}.$$

Since the degrees of the covers $q_1' \colon M_1' \to M_1$ and $q_1 \colon M \to M_1$ are coprime, Lemma 2.3.2 implies that M''_g is connected and equal to the compositum. As in (6.5), the action of g^{-1} on M_0 and M_1 corresponds to the action on Γ_0 and the subgroup Γ_1' by conjugation with $\overline{g}^{-1}$, where $\overline{g}$ is a lift of g to Γ_0. Hence the corresponding group is the intersection $\Gamma''_g := \Gamma \cap \overline{g}\Gamma_1'\overline{g}^{-1}$, i.e., $M''_g = \Gamma''_g \backslash \widetilde{M}$. By Lemma 2.3.3 and coprimality of the degree, the covering $M''_g \to M$ is C-Galois.

(b) Since M is a manifold, the underlying set of the fiber product is indeed the set theoretic fiber product in (7.12). We next argue that M'_{new} is connected, agrees with the compositum, and indeed corresponds to the group Γ'_{new} (and hence Γ') in (7.6), i.e., $M'_{\text{new}} = M'$. This will finish the proof of (i). To see the connectedness, we use induction with respect to the number of factors. So suppose we have already proven that $M''_{g_1} \times_M \dots M''_{g_{N-1}} \to M$ is a connected C^{N-1}-cover corresponding to the group $\bigcap_{i=1}^{N-1} \overline{g}_i(\Gamma \cap \Gamma_1')\overline{g}_i^{-1}$. By Lemma 2.3.2, the product with the next factor M''_{g_N} is connected if and only if

$$\Gamma = \langle \bigcap_{i=1}^{N-1} \overline{g}_i(\Gamma \cap \Gamma_1')\overline{g}_i^{-1}, \overline{g}_N(\Gamma \cap \Gamma_1')\overline{g}_N^{-1} \rangle. \tag{7.15}$$

To prove this, we notice that is true after applying Ψ, using (7.8): the image of left hand side is C^n, and the image of the right hand side is the subgroup of C^n spanned by $\{0\}^{N-1} \times C^{n-N}$ and $C^{N-1} \times \{0\} \times C^{n-N}$, which equals the whole of C^n. Hence Eq. (7.15) is true up to $\ker \Psi$, and from Lemma 7.3.2, it follows that $\ker \Psi$ is contained in both the left hand side and the right hand side of the equality, proving that (7.15) holds on the nose.

To prove (ii), note that $M'_{\text{new}} \to M$ is a C^n-Galois cover by Lemma 2.3.3, with one copy of C acting componentwise on each factor M''_{g_i}, and this is the same as the action of C^n on M'. The claim about the action of $g \in G \leq \widetilde{G}$ can be proven as follows: the action of G on M' is given by considering G as a subgroup of $\widetilde{G}$, and as such it acts by isometries on $M_{\text{new}} = M'$. We know that, in the geometric representation (7.12) for M'_{new},

$$g\underline{x} = \underline{y} \text{ with } \underline{x} = (x_1, \ldots, x_n, x), \underline{y} = (y_1, \ldots, y_n, y)$$

for some *unique* $y_i \in M'_1$ and $y \in M$ with $q'_1(y_i) \overset{\text{(I)}}{=} q_1(g_i^{-1}y)$. We only need to determine what y_i and y are. Since the action of G on M is as given, $y \overset{\text{(II)}}{=} gx$. Recall also that $g^{-1}g_i = g_{g^{-1}(i)}h_{g,i}$ for some $h_{g,i} \in H_1$. In particular, with $q_1 : M \to M_1$ the covering with group H_1, for $x \in M$ we have $q_1((g^{-1}g_i)^{-1}x) \overset{\text{(III)}}{=} q_1(g_{g^{-1}(i)}^{-1}x)$. We collect this information to compute

$$q'_1(y_i) \overset{\text{(I)}}{=} q_1(g_i^{-1}y) \overset{\text{(II)}}{=} q_1(g_i^{-1}gx) = q_1((g^{-1}g_i)^{-1}x) = q_1(g_{g^{-1}(i)}^{-1}x) \overset{\text{(III)}}{=} q'_1(x_{g^{-1}(i)}).$$

Since $q'_1 : M'_1 \to M_1$ is a C-cover, this shows that $y_i = c_i x_{g^{-1}(i)}$ for some $\underline{c} = (c_i) \in C^n$, that a priori depends on g and $\underline{x}$, i.e., it is a map

$$\underline{c} : G \times M' \to C^n.$$

Let us first prove that it does not depend $x \in M'$. Denote the dependency on $\underline{x}$ by $\underline{c}(\underline{x})$. Let $d(\cdot, \cdot)$ denote the distance on a manifold induced from the Riemannian metric. Since C acts properly discontinuously on M'_1 there is a $\delta > 0$ such that, for any two elements $c, c' \in C$ and $x \in M'_1$, if $d(cx, c'x) < \delta$, then $c' = c$. If $\underline{x}'$ is at distance ε from $\underline{x}$ in M', then so is $g\underline{x}$ from $g\underline{x}'$, and hence so is $c_i(\underline{x})x_{g^{-1}(i)}$ from $c_i(\underline{x}')x'_{g^{-1}(i)}$ for all i. Hence

$$\begin{aligned} d(c_i(\underline{x})x_{g^{-1}(i)}, c_i(\underline{x}')x_{g^{-1}(i)}) \\ &\le d(c_i(\underline{x})x_{g^{-1}(i)}, c_i(\underline{x}')x'_{g^{-1}(i)}) + d(c_i(\underline{x}')x'_{g^{-1}(i)}, c_i(\underline{x}')x_{g^{-1}(i)}) \\ &= d(c_i(\underline{x})x_{g^{-1}(i)}, c_i(\underline{x}')x'_{g^{-1}(i)}) + d(x'_{g^{-1}(i)}, x_{g^{-1}(i)}) \le 2\varepsilon, \end{aligned}$$

(the equality in the above formula holds since $c_i(\underline{x}')$ is an isometry) and thus $c_i(\underline{x}) = c_i(\underline{x}')$ as soon as $\underline{x}$ and $\underline{x}'$ are at distance $< \delta/2$. We conclude that $\underline{c}(\underline{x})$ is locally constant in $\underline{x}$, and since M' is connected, $\underline{c}$ is actually independent of $\underline{x}$. so that we have a map

$$\underline{c} : G \to C^n. \tag{7.16}$$

Now denote the dependence on g by $\underline{c}(g)$. We will prove that this is a cocycle; note that we write the group operation on C^n multiplicatively. We observe that for two elements $g, h \in G$,

$$
\begin{aligned}
(c_i(gh)x_{(gh)^{-1}(i)}, ghx) = gh\underline{x} &= g(c_i(h)x_{h^{-1}(i)}, hx) \\
&= (c_i(g)c_{g^{-1}(i)}(h)x_{h^{-1}g^{-1}(i)}, ghx),
\end{aligned}
$$

so $\underline{c}(gh) = \underline{c}(g)\underline{c}(h)^g$, where the action of g on $\underline{c} = (c_i)$ is given by $\underline{c}^g := (c_{g^{-1}(i)})$. This shows that the map $\underline{c}$ in Eq. (7.16) is a cocycle from G to C^n, and the corresponding first group cohomology class lies in $\mathrm{H}^1(G, C^n)$. Since $|G|$ and $|C^n| = \ell^n$ are coprime, the latter cohomology group is zero [21, III.(10.1)], proving that $\underline{c}$ is a coboundary, i.e., there exists $\underline{v} \in C^n$ (independent of g) such that $\underline{c}(g) = \underline{v}^{-1}\underline{v}^g = (v_i^{-1}v_{g^{-1}(i)})$. Consider the isometry

$$
\iota \colon M'_{\mathrm{new}} \to M'_{\mathrm{new}} \colon \underline{x} = (x_i, x) \mapsto (v_i^{-1}x_i, x).
$$

Now

$$
\iota^{-1}g\iota(\underline{x}) = \iota^{-1}(c_i(g)v_{g^{-1}(i)}^{-1}x_{g^{-1}(i)}, gx) = \iota^{-1}(v_i^{-1}x_{g^{-1}(i)}, gx) = (x_{g^{-1}(i)}, gx),
$$

as was claimed. Note also that conjugating by ι commutes with the action of C^n, so it does not change that action. □

Remark 7.3.5 The action of $\widetilde{G}$ on M'_{new} ties up with the group theoretical construction from the previous chapter, as follows. The group $\widetilde{G} \cong \Gamma_0/\Gamma'$ acts naturally on $M' = \Gamma'\backslash\widetilde{M}$ via

$$
(\gamma_0\Gamma') \cdot (\Gamma'\widetilde{x}) = \Gamma' \cdot (\gamma_0\widetilde{x}). \tag{7.17}
$$

The explicit identification between M' and M'_{new} is given by the map

$$
M' = \Gamma'\backslash\widetilde{M} \ni \Gamma'\widetilde{x} \mapsto (\Gamma'_1\overline{g}_1^{-1}\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}\widetilde{x}, \Gamma\widetilde{x}) =: (x_1, \dots, x_n, x) \in M'_{\mathrm{new}}, \tag{7.18}
$$

where $\{g_iH_1\}$ represent the cosets of H_1 in G and $G \to \Gamma_0 : g \mapsto \overline{g}$ is a section such that we have $\overline{e}_G = e_{\Gamma_0}$, $\overline{g^{-1}} = \overline{g}^{-1}$ and $\overline{g}\Gamma' = \jmath(g)$ with the homomorphism $\jmath : G \to \Gamma_0/\Gamma'$ representing the splitting of (6.14). The action of $\widetilde{G} \cong \Gamma_0/\Gamma'$, transferred from M' to M'_{new} is then

$$
(\gamma_0\Gamma') \cdot (\Gamma'_1\overline{g}_1^{-1}\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}\widetilde{x}, \Gamma\widetilde{x}) = (\Gamma'_1\overline{g}_1^{-1}\gamma_0\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}\gamma_0\widetilde{x}, \Gamma\gamma_0\widetilde{x}). \tag{7.19}
$$

Let $c \in \Gamma$ be an element satisfying $\Psi(c) = e_1$ and set $c_i := \overline{g}_i c \overline{g}_i^{-1} \in \Gamma$, as in Sect. 6.3. Utilising the diagrams (6.12) and (7.3), we see that $c_i \in \overline{g}_j\Gamma'_1\overline{g}_j^{-1}$ for all $j \neq i$. Thus, (7.19) implies

$$
\begin{aligned}
(c_i\Gamma') \cdot (x_1, \dots, x_n, x) &= (\Gamma'_1\overline{g}_1^{-1}c_i\widetilde{x}, \dots, \Gamma'_1\overline{g}_n^{-1}c_i\widetilde{x}, \Gamma c_i\widetilde{x}) \\
&= (\Gamma'_1\overline{g}_1^{-1}\widetilde{x}, \dots, \underbrace{\Gamma'_1 c\overline{g}_i^{-1}\widetilde{x}}_{i\text{-th entry}}, \Gamma'_1\overline{g}_n^{-1}\widetilde{x}, \Gamma\widetilde{x}). \qquad (7.20)
\end{aligned}
$$

Let $g \in G$; by definition, we have $\overline{g}^{-1}\overline{g}_i\Gamma_1' = \overline{g}_{g^{-1}(i)}c^{-k_i(g)}\Gamma_1'$ for some $k_i(g)$ modulo ℓ. This implies that

$$\begin{aligned}
(\overline{g}\Gamma')\cdot(x_1,\dots,x_n,x) &\overset{(7.19)}{=} (\Gamma_1'\overline{g}_1^{-1}\overline{g}\widetilde{x},\dots,\Gamma_1'\overline{g}_n^{-1}\overline{g}\widetilde{x},\Gamma\overline{g}\widetilde{x})\\
&= (\Gamma_1'c^{k_1(g)}\overline{g}_{g^{-1}(1)}\widetilde{x},\dots,\Gamma_1'c^{k_n(g)}\overline{g}_{g^{-1}(n)}\widetilde{x},\Gamma\overline{g}\widetilde{x})\\
&\overset{(7.20)}{=} (c_1^{k_1(g)}\cdots c_n^{k_n(g)}\Gamma')\cdot(x_{g^{-1}(1)},\dots,x_{g^{-1}(n)},g\cdot x).
\end{aligned}$$

Now $g \mapsto (c_i^{k_i(g)}\Gamma')_{i=1}^n$ is a cocycle from G to $C^n = \Gamma/\Gamma'$, and since $\mathrm{H}^1(G,C^n) = 0$, there exists $(m_1,\dots,m_n)$ with $k_i(g) = m_{g^{-1}(i)} - m_i$ (modulo ℓ). Using the commutativity of the elements $c_i\Gamma'$, this implies that if we set $c_0 := \prod_{i=1}^n c_i^{m_i}$, then

$$(c_0\jmath(g)c_0^{-1})\cdot(x_1,\dots,x_n,x) = (x_{g^{-1}(1)},\dots,x_{g^{-1}(n)},g\cdot x)$$

for all $g \in G$ and all $(x_1,\dots,x_n,x) \in M'_{\mathrm{new}}$. This shows that a copy of G in $\widetilde{G} \cong \Gamma_0/\Gamma'$, namely $c_0\jmath(G)c_0^{-1}$, acts on M'_{new} via permutation of the first n entries. In other words, it is possible to conjugate the subgroup G in $\widetilde{G}$ to realise the specific action (7.14) on M'_{new}.

7.4 Universal Property of the Wreath Product

The appearance of the wreath product in our constructions becomes less of a surprise given the following universal property, showing that the minimal Galois cover that "contains" a G-cover and a C-cover as in our situation arises from this wreath product (the analogous result in the theory of field extensions is well known, compare [37, 13.7]).

Proposition 7.4.1 *Let G and C denote finite groups with C cyclic of prime order ℓ not dividing the order of G. Suppose that we are given Riemannian manifolds M, M_1, M_1' and a developable Riemannian orbifold M_0 such that $M \to M_0$ is G-Galois with subcover $M_1 \to M_0$, and $M_1' \to M_1$ is C-Galois. If $N \to M_0$ is a Galois cover of minimal degree admitting Riemannian covers $N \to M$ and $N \to M_1'$, then the Galois group G' of N over M_0 is the wreath product $\widetilde{G} := C^n \rtimes G$, where n is the degree of the cover $M_1 \to M_0$ (see Figure (7.21).)*

$$\begin{tikzcd}
 & N \arrow[dl, dashed] \arrow[dr, dashed] \arrow[ddd, dashed, bend left=70, "G'"] & \\
M_1' \arrow[dr, "C"'] & & M \arrow[dl] \arrow[ddl, bend left=10, "G"] \\
 & M_1 \arrow[d, "n"'] & \\
 & M_0 &
\end{tikzcd} \tag{7.21}$$

Proof Writing the manifolds M_0, M, M_1, M_1', N as quotients of he universal cover $\widetilde{M}_0$ of M_0 by the respectively group Γ_0, Γ, Γ_1, Γ_1', Γ_N, the defining properties of N imply that it is the normal closure of the compositum of M_1' and M over M_0, and hence

$$\Gamma_N = \bigcap_{\gamma_0 \in \Gamma_0} \gamma_0(\Gamma \cap \Gamma_1')\gamma_0^{-1}.$$

First of all, for $g \in G$, choose one element $\overline{g} \in \Gamma_0$ that maps to $g \in \Gamma_0/\Gamma \cong G$. We claim that

$$\Gamma_N = \bigcap_{i=1}^{n} \Gamma_{g_i} \text{ where } \Gamma_{g_i} := \overline{g}_i(\Gamma \cap \Gamma_1')\overline{g}_i^{\,-1},$$

for $\{g_i\}$ a set of coset representatives for H_1 in G. Indeed, for any $\gamma_0 \in \Gamma_0$ we can write $\gamma_0 = \overline{g}_i\gamma_1$ for some $i \in \{1, \dots, n\}$ and some $\gamma_1 \in \Gamma_1$, since the cosets of $H_1 \cong \Gamma_1/\Gamma$ in $G \cong \Gamma_0/\Gamma$ are $g_1H_1, \dots, g_nH_1$ and the cosets of Γ_1 in Γ_0 are therefore $\overline{g}_1\Gamma_1, \dots, \overline{g}_n\Gamma_1$. The statement now follows from the fact that $\gamma_0 \in \Gamma_0$ must lie in one of these cosets $\overline{g}_i\Gamma_1$; since both Γ and Γ_1' are normal in Γ_1, we have

$$\begin{aligned}
\gamma_0(\Gamma \cap \Gamma_1')\gamma_0^{-1} &= (\overline{g}_i\gamma_1)(\Gamma \cap \Gamma_1')(\overline{g}_i\gamma_1)^{-1} = \overline{g}_i\gamma_1(\Gamma \cap \Gamma_1')\gamma_1^{-1}\overline{g}_i^{-1} \\
&= \overline{g}_i(\Gamma \cap \Gamma_1')\gamma_1\gamma_1^{-1}\overline{g}_i^{-1} = \overline{g}_i(\Gamma \cap \Gamma_1')\overline{g}_i^{-1}.
\end{aligned}$$

Now since Γ is normal in Γ_0, $\Gamma \geq \Gamma_N$, and we find an exact sequence

$$1 \to \Gamma/\Gamma_N \to \Gamma_0/\Gamma_N \to \Gamma_0/\Gamma \cong G \to 1.$$

The natural map $\varphi\colon \Gamma \to \prod_{i=1}^{n} \Gamma/\Gamma_{g_i}$ has kernel $\bigcap_{i=1}^{n} \Gamma_{g_i} = \Gamma_N$. Next, $\Gamma/\Gamma_{g_i} \cong C$ since the index is the prime number ℓ. Finally, we claim that φ is surjective. For this, it suffices to find for every i an element $\gamma_i \in \Gamma$ with

$$\varphi(\gamma_i) = e_i = (0, \dots, 0, 1, 0, \dots, 0) \in C^n.$$

Since C is cyclic of prime order, every non-zero element is a generator, and it suffices to choose $\gamma_i \in (\bigcap_{j \neq i} \Gamma_{g_j}) \setminus \Gamma_{g_i}$. This is possible since the reasoning in the first paragraph of this proof shows that the latter set is non-empty. In the end, we find a sequence

$$1 \to C^n \to \Gamma_0 / \Gamma_N \to G \to 1$$

where G acts on C^n by permuting the factors like it permutes the cosets of H_1, and this finishes the proof. □

Remark 7.4.2 In our setup, the universality property says the following: if we search for the "easiest possible" twisted Laplace operator on M_1, meaning associated to the Laplace operator on some prime order cyclic cover of M_1, we necessarily arrive at a diagram of the form (5.6).

Project

Assuming condition ($**$), one can now give the following alternative construction of diagram (5.6) used in the main Theorem 6.4.1: perform the above two step construction of M'_{new} and *define* an action of $\widetilde{G}$ on M'_{new} using the right hand side of Eqs. (7.13) and (7.14). Prove directly that this manifold satisfies the required properties.

Chapter 8
Homological Wideness

In this short chapter, we introduce a new topological notion: the action of a finite group G on a manifold is called *K-homologically wide* if the first homology group with coefficients in K contains the regular representation of G; this is in some sense complementary to the notion of homological triviality that is well studied in algebraic topology. We study how homological wideness behaves under reduction modulo primes. We also relate homological wideness to the previous conditions ($*$) and ($**$) for realisability of certain wreath products as covering groups of manifolds.

8.1 The Notion of Homological Wideness

Definition 8.1.1 Suppose G is a finite group acting (freely or not) on a closed connected (topological) manifold M. Let K denote a field. We say the action of G is *K-homologically wide* if the *K-homology representation* $h_K = h$ of G, given by the induced action on the first homology group

$$h_K : G \to \mathrm{Aut}(\mathrm{H}_1(M, K)) \tag{8.1}$$

contains the regular representation of G.

Recall that, for a ring R, an R-module M is called *cyclic* if there exists a *cyclic vector* $m \in M$, i.e., a vector such that $Rm = M$. The regular representation is a cyclic G-module (in this case, *any* element $g \in G$ is a cyclic vector, since the vector space span of the orbit $G \cdot g$ spans the entire representation space), hence another way to formulate homological wideness is as follows: *the action of G on M is homologically wide if and only if there exists a class $\omega \in \mathrm{H}_1(M, K)$ such that the orbit $G \cdot \omega$ spans a vector space of dimension $|G|$ inside* $\mathrm{H}_1(M, K)$. Indeed, if the regular representation is contained in the homology representation, just take a

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cyclic vector for that subrepresentation. Conversely, if such ω exists, then all $g \cdot \omega$ for $g \in G$ are linearly independent, and hence span a copy of the regular representation.

Lemma 8.1.2 *If the action of G on M is $\mathbf{F}_\ell$-homologically wide, then condition $(**)$, and hence condition $(*)$ for ℓ coprime to $|G|$, holds for any subgroup H_1 in G.*

Proof It suffices to show that for any $H_1 \leq G$, $\mathrm{Ind}_{H_1}^G \mathbf{1}$ is a subrepresentation of the regular representation. For that, it suffices to prove that the multiplicity of any irreducible G-representation ρ in $\mathrm{Ind}_{H_1}^G \mathbf{1}$ is less than or equal to $\dim \rho$, the multiplicity of ρ in the regular representation. By Frobenius reciprocity, we compute that the multiplicity of ρ in $\mathrm{Ind}_{H_1}^G \mathbf{1}$ is

$$\langle \rho, \mathrm{Ind}_{H_1}^G \mathbf{1} \rangle = \langle \mathrm{Res}_{H_1}^G \rho, \mathbf{1} \rangle = \frac{1}{|H_1|} \sum_{h_1 \in H_1} \mathrm{tr}(\rho(h_1)) \leq \frac{1}{|H_1|} |H_1| \dim \rho,$$

since the trace of $\rho(h_1)$ is a sum of $\dim \rho$ roots of unity (as $\rho(h_1)$ is of finite order). □

8.2 The Notion of Q-Homological Wideness

We first relate $\mathbf{Q}$-homological wideness (that has a transparent geometric meaning in terms of cycles) to $\mathbf{F}_\ell$-homological wideness (that is used in the proof of the main result). If the action of G on M is $\mathbf{Q}$-homologically wide, there exists a non-torsion homology class $\omega \in \mathrm{H}_1(M, \mathbf{Q}) = \mathrm{H}_1(M, \mathbf{Z}) \otimes \mathbf{Q}$ such that $\{g\omega\}$ is linearly independent over $\mathbf{Q}$. Fix an integer N such that $\omega' := N\omega \in \mathrm{H}_1(M, \mathbf{Z})$; then $\{g\omega'\}$ is a set of $\mathbf{Z}$-independent non-torsion homology classes for M. These classes will remain linearly independent modulo infinitely many ℓ. In fact, we can use representation theory to say more.

Lemma 8.2.1 *If the action of G on M is $\mathbf{Q}$-homologically wide, then it is $\mathbf{F}_\ell$-homologically wide for all ℓ coprime to $|G|$.*

Proof This follows from the basic theory of modular representations in "good" characteristics. In this proof, we write "R–mod" for the category of finitely generated modules over a ring R.

If $\mathscr{M}$ is a $\mathbf{Q}[G]$-module, then it is a $\mathbf{Q}_\ell[G]$-module (with $\mathbf{Q}_\ell$ the field of ℓ-adic numbers). We let $K \supseteq \mathbf{Q}_\ell$ denote a splitting field for all irreducible representations of G; it suffices to assume that K contains all m-th roots of unity where m runs over all orders of elements of G. Let $R \supseteq \mathbf{Z}_\ell$ denote the ring of integers of K and $\mathfrak{m}$ its maximal ideal with residue field $k := R/\mathfrak{m}$.

Fixing any $R[G]$-lattice $\mathscr{M}'$ in $\mathscr{M}$, we have a reduction map modulo $\mathfrak{m}$, producing a $k[G]$-module $\overline{\mathscr{M}'} = \mathscr{M}' \otimes k = \mathscr{M}'/\mathfrak{m}\mathscr{M}'$. The *decomposition map*

$$d\colon K_0(K[G]\text{–mod}) \to K_0(k[G]\text{–mod})\colon [\mathscr{M}] \to [\overline{\mathscr{M}'}]$$

is an isomorphism for ℓ coprime to $|G|$ and an effective map (i.e., positive integral combinations map to positive combinations) (see, e.g., [88, §15.5]; another formulation says that if ℓ is coprime to $|G|$, the Brauer character of the reduction of a G-representation modulo $\mathfrak{m}$ equals the character of the original representation, see, e.g., [54, Thm. 15.8]). By assumption, any irreducible (K-)representation of G occurs as direct summand in $\mathrm{H}_1(M, \mathbf{Z}) \otimes K$ (with multiplicity its dimension), and hence also every k-irreducible representation occurs as direct summand in $\mathrm{H}_1(M, \mathbf{Z}) \otimes k$ (with the same multiplicity). Since the regular representation $\mathbf{Q}[G]$ is defined over $\mathbf{Q}$, we also find the regular representation $\mathbf{F}_\ell[G]$ as direct summand in $\mathrm{H}_1(M, \mathbf{Z}) \otimes \mathbf{F}_\ell$. □

Example 8.2.2 For $G = \mathbf{Z}/2\mathbf{Z} = \langle \left(\begin{smallmatrix} 1 & 0 \\ 3 & -1 \end{smallmatrix}\right) \rangle$ acting on $\mathscr{M} = \mathbf{Z}^2$, $(1, 0)$ is a cyclic vector over $\mathbf{Q}$ but not over $\mathbf{F}_3$ (so $\ell = 3$ is excluded by the reasoning before the lemma). The proof of the lemma does not imply that an integral cyclic vector is a cyclic vector modulo ℓ; just that if one exists, then one exists modulo ℓ, as long as ℓ is coprime to $|G|$. In the example, $(1, 1)$ is a cyclic vector over both $\mathbf{Q}$ and $\mathbf{F}_3$. ◊

Chapter 9
Examples of Homologically Wide Actions

An action of a finite group G on a manifold M is homologically wide if the first homology of the manifold contains the regular representation of the group. In this chapter, we study this notion independently of the rest of this monograph. We first study the case where M is a surface and G acts freely, using the Lefschetz fixed point formula. We then recall a result of Broughton on the homology representation that allows us to deal with the case of a general action on a Riemann surface. After this, we switch to higher dimensional manifolds. We first recall Curtis's theory of the virtual Lefschetz characters, and use results of Cooper and Long to construct examples and counterexamples to homological wideness in all dimensions ≥ 3. Finally, we study locally symmetric spaces. Property (T) implies that only the case of rank 1 is interesting, where we make some remarks on the relation to automorphic representations and Mostow rigidity. Finally, we study an example of using torsion homology: Mednykh's explicit computation of the homology representation of the Seifert-Weber dodecahedral space, which we identify with a known modular representation through Brauer characters.

9.1 Surfaces

In dimension 2, the situation for a fixed-point free action is clear because the homology representation can be computed using the Lefschetz fixed point theorem [48, Theorem 2C.3].

Proposition 9.1.1 *A fixed-point free action of a non-trivial finite group G on a closed orientable surface M is* $\mathbf{Q}$*-homologically wide if and only if M is hyperbolic (i.e., has negative Euler characteristic). In particular, the property is independent of G.*

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Proof We will compute the character χ_h of the rational homology representation

$$h = h_{\mathbf{Q}} \colon G \to \mathrm{H}_1(M, \mathbf{Q}).$$

First of all, since we assume any $g \neq e$ has no fixed points, the map $g \colon M \to M$ has Lefschetz number 0, i.e.,

$$\mathrm{tr}(g_* | \mathrm{H}_0(M, \mathbf{Q})) - \mathrm{tr}(g_* | \mathrm{H}_1(M, \mathbf{Q})) + \mathrm{tr}(g_* | \mathrm{H}_2(M, \mathbf{Q})) = 0.$$

Since the action of g_* on $\mathrm{H}_0(M, \mathbf{Q}) \cong \mathbf{Q}$ and $\mathrm{H}_2(M, \mathbf{Q}) \cong \mathbf{Q}$ is trivial (induced by the action of g on the space of connected components of M, respectively the 2-cells, i.e., the one-element sets), both outer terms in this expression are 1, and the middle term is $\chi_h(g)$ by definition, so we find that $\chi_h(g) = 2$ for $g \neq e$. For $g = e$, on the other hand, we get directly from the definition of the character that $\chi_h(e) = \mathrm{tr}(e_* | \mathrm{H}_1(M, \mathbf{Q})) = \dim \mathrm{H}_1(M, \mathbf{Q}) = b_1(M) = 2 - \chi_M$. We conclude that

$$\chi_h(g) = \begin{cases} 2 & \text{if } g \neq e, \\ 2 - \chi_M & \text{if } g = e. \end{cases}$$

On the other hand, the character of the regular representation is

$$\chi_{G,\mathrm{reg}} = \begin{cases} 0 & \text{if } g \neq e, \\ |G| & \text{if } g = e. \end{cases}$$

We can match these expressions, and since representations are isomorphic if and only if their characters are equal, we find

$$h = 2 \cdot \mathbf{1}_G - \frac{\chi_M}{|G|} \cdot \rho_{G,\mathrm{reg}},$$

and hence the regular representation of a non-trivial group G occurs inside h if and only if $\chi_M < 0$. □

The above proposition has no (constant) curvature assumption. In constant curvature but with more general actions, we have the following.

Proposition 9.1.2 *Any action of a finite group G by* (*not necessarily fixed-point free*) *conformal automorphisms on a closed Riemann surface M is* $\mathbf{Q}$*-homologically wide if* $\chi_{G\backslash M} < 0$.

Proof We rely on the computation of the character of h in this branched setting by Broughton in [20, Prop. 2(iii)], using in addition (taking into account the holomorphic structure) the Eichler trace formula. Suppose that the G-cover is branched above t points. For each branch point, choose a lift to the cover and let $C_i \leq G$ denote the (cyclic) stabiliser of that lift (the stabilisers of any lift of a given point are conjugate in G). Then

$$h = 2 \cdot \mathbf{1} - \chi_{G\backslash M} \cdot \rho_{G,\mathrm{reg}} + \sum_{i=1}^{t} (\rho_{G,\mathrm{reg}} - \mathrm{Ind}_{C_i}^{G} \mathbf{1})$$
$$= 2 \cdot \mathbf{1} - \chi_{G\backslash M} \cdot \rho_{G,\mathrm{reg}} + \sum_{i=1}^{t} \mathrm{Ind}_{C_i}^{G} (\rho_{C_i,\mathrm{reg}} - \mathbf{1}),$$

where we have used that the induced representation of the regular representation of C_i to G is the regular representation of G. Since $\mathbf{1}$ occurs in $\rho_{C_i,\mathrm{reg}}$, the representations occurring in the sum are not virtual (i.e., every irreducible representation of G occurs in it with non-negative multiplicity) and therefore h contains $\rho_{G,\mathrm{reg}}$ as soon as $\chi_{G\backslash M} < 0$. □

Remark 9.1.3 In the "non-orbifold quotient" setting of Proposition 9.1.1, by the Riemann-Hurwitz formula, $\chi_M < 0$ if and only if $\chi_{G\backslash M} < 0$. This is no longer true in the setting of Proposition 9.1.2, when we only have $\chi_{G\backslash M} < 0 \Rightarrow \chi_M < 0$ but not the other way around.

The following is a detailed version of Corollary 1.2.3.

Corollary 9.1.4 *Let M_1, M_2 be two commensurable non-arithmetic closed Riemann surfaces. Then they admit a diagram (1.1) and, assuming the corresponding orbifold M_0 satisfies $\chi_{M_0} < 0$, isometry of M_1 and M_2 can be checked by computing the multiplicity of zero in at most*

$$4((\chi_{M_1}\chi_{M_2}/(\chi_{M_0}^{\mathrm{orb}})^2)!)^2$$

twisted Laplace spectra, where $\chi_{M_0}^{\mathrm{orb}}$ is the orbifold Euler characteristic given by

$$\chi_{M_0}^{\mathrm{orb}} := \chi_{M_0} - \sum (1 - 1/n_i), \tag{9.1}$$

with n_i the order of the stabiliser group at the orbifold points.

Proof By Proposition 2.5.2, hyperbolic non-arithmetic commensurable closed Riemann surfaces automatically admit a diagram of the form (1.1), and by Proposition 9.1.2, every group action is $\mathbf{Q}$-homologically wide since we assume $\chi_{M_0} < 0$. Therefore, Theorem 1.2.1 applies. To find a prime number ℓ coprime to $|G|$, we can always choose $\ell > |G|$, and by Bertrand's postulate [47, Thm. 418], we can find such a prime $\ell \leq 2|G|$. Hence we can make the bound in Theorem 1.2.1 weaker by $2\ell|H_2^{\mathrm{ab}}| \leq 4|G|^2$. To express this entirely in terms of the original diagram, we set d_i to be the degree of $M_i \to M_0$. Notice that the compositum $M_1 \bullet_{M_0} M_2$ is of degree at most $d_1 d_2$ over M_0, and the degree of the normal closure of the compositum is of degree at most $(d_1 d_2)!$ over M_0 (see (2.3)). Hence $|G| \leq (d_1 d_2)!$. Now $d_i = \chi_{M_i}/\chi_{M_0}^{\mathrm{orb}}$ where $\chi_{M_0}^{\mathrm{orb}}$ is the orbifold Euler characteristic given by $\chi_{M_0} - \sum(1 - 1/n_i)$ for n_i the order of the stabiliser group at the orbifold points [25, 5.1.3]. We find an upper bound of at most

$$4((d_1d_2)!)^2 \le 4((\chi_{M_1}\chi_{M_2}/(\chi_{M_0}^{\mathrm{orb}})^2)!)^2$$

for the number of equalities of multiplicities that needs to be checked. □

Remark 9.1.5 If M_1 and M_2 as in Corollary 9.1.4 are isospectral, by Weyl's law, they have the same volume. Since $d_i = \mathrm{vol}(M_i)/\mathrm{vol}(M_0)$, we can then assume that $d_1 = d_2$ and $\chi_{M_1} = \chi_{M_2}$.

In Chap. 11, one finds some detailed examples of surfaces with less crude bounds on the required number of equalities.

9.2 Using the Virtual Lefschetz Character

The above arguments in dimension 2 are based on very precise information given by fixed point formulæ. These admit a generalisation to a setup as in diagram (1.2) with M of arbitrary dimension, where they can sometimes be used to deduce some information about condition $(**)$ from Sect. 7.1; more specifically, whether $\mathrm{Ind}_H^G \mathbf{1}$ is a subrepresentation of the homology representation (over $\mathbf{Q}$). For this, we use that our manifolds are closed, and thus admit a regular triangulation, which allows us to apply the work of Curtis [32]. Consider the virtual Lefschetz character of G given as

$$\Lambda(g) = -h_{\mathbf{Q}}(g) + \sum_{i \neq 1} (-1)^i \operatorname{tr}(g_* |\, \mathrm{H}_i(M, \mathbf{Q})).$$

Then by [32, Prop. 1.6], we have

$$\langle \mathrm{Ind}_H^G \mathbf{1}, \Lambda \rangle = \chi(M_1), \tag{9.2}$$

the Euler characteristic of M_1. The formula provides no information for 3-dimensional manifolds, since then $\chi(M_1) = 0$ and $\Lambda = 0$. The next proposition provides an example of a result that can be deduced from such methods.

Proposition 9.2.1 *if M is of dimension 4 and $\chi(M_1) \le 0$, then $\mathrm{Ind}_H^G \mathbf{1}$ and $h_{\mathbf{Q}}$ have at least one irreducible representation in common.*

Proof By Poincaré duality, $\mathrm{H}_3(M, \mathbf{Q}) = \mathrm{Hom}(\mathrm{H}_1(M, \mathbf{Q}), \mathbf{Q}) \cong \mathrm{H}_1(M, \mathbf{Q})$, hence

$$\Lambda = 2 \cdot \mathbf{1} + \mathrm{H}_2(M, \mathbf{Q}) - 2h_{\mathbf{Q}},$$

and we conclude from (9.2) that

$$2\langle \mathrm{Ind}_H^G \mathbf{1}, h_{\mathbf{Q}} \rangle = 2 + \langle \mathrm{Ind}_H^G \mathbf{1}, \mathrm{H}_2(M, \mathbf{Q}) \rangle - \chi(M_1) \ge 2 - \chi(M_1) > 0.$$

Such results do not suffice to completely verify whether condition $(**)$ holds under general topological conditions in higher dimension (i.e., only referring to the vector space structure of $\mathrm{H}_1(M, \mathbf{Q})$, and not to its $\mathbf{Q}[G]$-module structure), and indeed, in the next few sections we will see examples showing that this is not possible.

9.3 Manifolds of Dimension ≥ 3

In case of 3-manifolds, the picture can vary widely: it is possible to construct a class of closed 3-manifolds with $\mathbf{Q}$-homologically wide group actions, but also hyperbolic 3-manifolds with large isometry group for which only the trivial group action is $\mathbf{Q}$-homologically wide. This upgrades to similar results in higher dimensions. The results are direct consequences of the work of Cooper and Long [27] for topological manifolds.

Proposition 9.3.1 *Suppose N is a smooth compact connected 3-manifold with a free smooth action by a finite group G, and let γ denote a smooth simple closed curve in N such that the orbit $G \cdot \gamma$ consists of $|G|$ disjoint smooth simple closed curves. Let*

$$X = N - \mathscr{N}(G \cdot \gamma)$$

denote the open manifold N with an open regular neighbourhood $\mathscr{N}(G \cdot \gamma)$ of the G-orbit of γ removed, and let M denote the double of the manifold X (i.e., two copies of X glued together along their boundaries). Assume that the embedding $\iota \colon \partial X \to X$ of the boundary induces a surjective map on first homology groups $\iota_* \colon \mathrm{H}_1(\partial X, \mathbf{Q}) \to \mathrm{H}_1(X, \mathbf{Q})$. Define a Riemannian metric on M as the pullback of any Riemannian metric on the quotient manifold $G\backslash M$; then if M' is any $(n-3)$-dimensional closed smooth connected Riemannian manifold with trivial G-action, $M \times M'$ is an n-dimensional closed smooth connected Riemannian manifold with a free isometric G-action that is $\mathbf{Q}$-homologically wide.

Proof Since [27] concerns topological manifolds, we start by observing that the double of a *smooth* manifold has a smooth structure compatible with the embedding of the original manifold, but composed with a diffeomorphism on one of the copies, see, e.g., [58, VI.5]. Hence M is a smooth compact connected manifold on which the finite group G acts smoothly (and properly) without fixed points. Therefore, the quotient $G\backslash M$ is a smooth compact connected manifold, too, and the quotient map $M \to G\backslash M$ is a smooth covering map. Choose a Riemannian structure on the quotient $G\backslash M$ such that it becomes a closed Riemannian manifold, and make M into a closed Riemannian manifold by giving it the pullback Riemannian structure. Now G acts on M by fixed-point free Riemannian isometries.

Cooper and Long [27, Lemma 2.3] have proven that, by the assumption that ι_* is surjective,

$$\mathrm{H}_1(M, \mathbf{Q}) = \rho_{G,\mathrm{reg}} \oplus (\rho_{G,\mathrm{reg}} - \mathbf{1})$$

as G-modules. Hence, in particular, $\mathrm{H}_1(M, \mathbf{Q})$ contains the regular representation.

Now G acts trivially on M', so G acts by isometries on the cartesian product $M \times M'$. By the Künneth formula, $M \times M'$ has first homology group

$$\mathrm{H}_1(M \times M', \mathbf{Q}) = \mathrm{H}_1(M, \mathbf{Q}) \oplus \mathrm{H}_1(M', \mathbf{Q}),$$

so $\rho_{G,\mathrm{reg}}$ is also a subrepresentation of $\mathrm{H}_1(M \times M', \mathbf{Q})$, and the action of G on $M \times M'$ is $\mathbf{Q}$-homologically wide. □

In the other direction, Cooper and Long have also shown that through Dehn surgeries, it is possible to "remove" the canonical $\mathbf{Q}[G]$-modules $\rho_{G,\mathrm{reg}}$ and also $\rho_{G,\mathrm{reg}} - \mathbf{1}$ from the homology representation to arrive at a rational homology 3-sphere with a G-action. Further surgery along an embedded hyperbolic knot allows one to construct such a *hyperbolic* (i.e., constant -1 curvature) manifold [27, Theorem 2.6].

Proposition 9.3.2 *For any finite non-trivial group G, there exists a hyperbolic rational homology 3-sphere M with a free action of G by isometries on M; in particular, the action of G on M is not $\mathbf{Q}$-homologically wide.*

We conclude that in dimension 3 homological wideness is unrelated to hyperbolicity (in marked contrast to the case of dimension 2).

Corollary 9.3.3 *For any finite non-trivial group G, and any dimension $n \geq 3$, there exists an n-dimensional closed connected Riemannian manifold M' with a free action of G by isometries on M' for which the action of G on M' is not $\mathbf{Q}$-homologically wide.*

Proof Let M be as in Proposition 9.3.2, and let G act trivially on the $(n-3)$-dimensional sphere S^{n-3}. Setting $M' = M \times S^{n-3}$, by the Künneth formula, we have $\mathrm{H}_1(M', \mathbf{Q}) = 0$ for $n \neq 4$ and $\mathrm{H}_1(M', \mathbf{Q}) = \mathbf{Q}$ for $n = 4$, so it is impossible for non-trivial G to act homologically wide on M'. □

Remark 9.3.4 Bartel and Page have shown that there exists a closed hyperbolic 3-manifold M with a free action of any given finite group G by isometries on M such that additionally, $\mathrm{H}_1(M, \mathbf{Q})$ is any given $\mathbf{Q}[G]$-module [7].

9.4 Locally Symmetric Spaces of Rank ≥ 2

Let $\mathbf{G}$ denote a connected semisimple Lie group with trivial center, $\mathbf{K}$ a maximal compact subgroup of $\mathbf{G}$, and Γ a discrete subgroup of $\mathbf{G}$ such that $\Gamma\backslash\mathbf{G}$ is compact. Consider the locally symmetric Riemannian manifold $M := \Gamma\backslash\mathbf{G}/\mathbf{K}$. If all factors of $\mathbf{G}$ have real rank ≥ 2, then Γ has Kazhdan's property (T), and hence $\mathrm{H}_1(M, \mathbf{Q}) = \Gamma^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q} = \{0\}$ (see, e.g., [8, Cor. 1.3.6]). This shows the following.

Proposition 9.4.1 *Only the trivial group can have a* $\mathbf{Q}$*-homologically wide action on a locally symmetric space of rank* ≥ 2.

9.5 Locally Symmetric Spaces of Rank 1

On the other hand (keeping the notations of Sect. 9.4), if $\mathbf{G}$ has rank 1, the first Betti number of M can be expressed in terms of representation theory via a formula of Matsushima's [67]; more precisely, a sum of multiplicities of specific representation occurring in the representation R_Γ of $\mathbf{G}$ by right multiplication on $L^2(\Gamma\backslash\mathbf{G})$. If $\mathbf{G} = \mathrm{SO}(n,1)$ for $n \geq 3$, there is a unique representation J_1 in that sum and $b_1(M)$ equals the multiplicity of the representation J_1 in R_Γ. Here, J_1 is the unique unitary irreducible representation with non-zero Lie algebra cohomology. Except for $n = 3$, J_1 is not in the discrete or principal series ([34, Thm. V.5; Rem. V.8; Prop. V.6]; [50, Lemma 4.4] or [14, VII.4.9]). For $n = 3$, J_1 is the principal series representation of $\mathrm{PSL}(2,\mathbf{C})$ on $L^2(\mathbf{C})$ with Gelfand-Graev-Vilenkin parameters $(2,0)$, given explicitly as

$$J_1\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(f)(z) := (cz+d)^2 f\left(\frac{az+b}{cz+d}\right).$$

Proposition 9.5.1 *Let* $M = \Gamma\backslash\mathbb{H}^n$ *denote a closed hyperbolic* n*-manifold* ($n \geq 3$)*, corresponding to a cocompact discrete subgroup* Γ *in* $\mathrm{SO}(n,1)$. *If* G *is a finite group acting* $\mathbf{Q}$*-homologically widely on* M*, then*

$$|G| \leq \langle J_1, R_\Gamma \rangle,$$

the multiplicity of the representation J_1 *described above in the* $\mathrm{SO}(n,1)$*-representation* R_Γ *given by right multiplication on* $L^2(\Gamma\backslash\mathrm{SO}(n,1))$.

9.6 Hyperbolic Manifolds; Formulation in Terms of Uniform Lattices

If $M = \Gamma\backslash\mathbb{H}^n$ is a compact connected hyperbolic manifold of dimension $n \geq 3$ with finite full isometry group $\mathrm{Isom}(M)$, Mostow rigidity implies that $\mathrm{Isom}(M) \cong \mathrm{Out}(\Gamma)$, the outer automorphism group of Γ; indeed, M is an Eilenberg-MacLane $K(\Gamma,1)$, and hence $\mathrm{Out}(\Gamma)$ is isomorphic to the group of homotopy self-equivalences up to free homotopy; but by Mostow rigidity, every homotopy equivalence is homotopic to an isometry [72, Thm. 24.1']. Hence homological wideness of the action of a subgroup $G \hookrightarrow \mathrm{Isom}(M)$ on M can be formulated in purely group theoretical terms.

Proposition 9.6.1 *The action of a finite group G of isometries on a compact connected hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^n$ of dimension $n \geq 3$ is K-homologically wide if and only if the representation*

$$G \hookrightarrow \mathrm{Out}(\Gamma) \to \mathrm{Aut}(\Gamma^{\mathrm{ab}} \otimes_{\mathbf{Z}} K) \cong \mathrm{GL}(b_1(\Gamma), K)$$

contains the regular representation.

Recall again that Proposition 9.3.2 gives an example where this representation is trivial for $K = \mathbf{Q}$. Belolipetsky and Lubotzky [9] have shown that, given any finite group G, there exist infinitely many compact connected hyperbolic manifolds with G as isometry group.

Remark 9.6.2 Besson et al. [12, Théorème 9.1] have proven that *homeomorphic* oriented hyperbolic manifolds of the same dimension $n \geq 3$ with the same volume are isometric. In our situation, we start from a "correspondence" as in diagram (1.1) and a homeomorphism is not given.

9.7 Using Torsion Homology for Homological Wideness

Instead of using $\mathbf{Q}$-homological wideness, one may try to find suitable ℓ for which the action of G on $\mathrm{H}_1(M)$ is $\mathbf{F}_\ell$-homologically wide for specific ℓ. For example, $\mathrm{H}_1(M)$ might be torsion, so that no non-trivial group acts $\mathbf{Q}$-homologically wide, but nevertheless, $\mathrm{H}_1(M, \mathbf{F}_\ell)$ can contain $\mathbf{F}_\ell[G]$. We content ourselves with commenting on one example.

Example 9.7.1 Let M denote the Seifert–Weber dodecahedral space, a hyperbolic 3-manifold with first Betti number zero; cf. [98]. By Mostow rigidity, M is uniquely described by its fundamental group

$$\langle a_1, \ldots, a_6 | a_3^{-1} a_6 a_4^{-1} a_5 a_2, a_2^{-1} a_6 a_3^{-1} a_4 a_1, a_6 a_2^{-1} a_3 a_5 a_1^{-1}, \\ a_2 a_4 a_5^{-1} a_6 a_1^{-1}, a_3 a_4^{-1} a_6 a_5^{-1} a_1, a_4 a_2 a_5 a_3 a_1 \rangle.$$

For the following facts, especially the computation of the homology representation, we refer to Mednykh [68]:

- $\mathrm{H}_1(M, \mathbf{Z}) = \mathbf{F}_5^3$, admitting non-trivial maps to a cyclic group $\mathbf{Z}/\ell\mathbf{Z}$ for $\ell = 5$.
- The full isometry group of M is isomorphic to S_5. If we write generators as $r = (12)$ and $c = (12345)$, there is a faithful action on $\mathrm{H}_1(M, \mathbf{F}_\ell)$ through matrices in $\mathrm{GL}(3, \mathbf{F}_5)$ given as

$$r = \begin{pmatrix} 4 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

The isometry group $G = \langle r \rangle \cong \mathbf{Z}/2\mathbf{Z}$ of M has a cyclic vector $(1, 1, 0)$ in $\mathrm{H}_1(M, \mathbf{F}_\ell)$, so the action of G on M is $\mathbf{F}_5$-homologically wide (but not $\mathbf{Q}$-homologically wide). Similarly, the isometry group $G = \langle crc^{-1}r \rangle \cong \mathbf{Z}/3\mathbf{Z}$ of M has cyclic vector $(1, 0, 0)$. ◊

We can identify this homology representation on the nose, as follows. Let

$$\mathrm{sgn}\colon S_5 \to \mathbf{F}_5^*$$

denote the linear character given by the sign of a permutation (modulo 5), and let

$$\psi\colon S_5 \to \mathrm{GL}(3, \mathbf{F}_5)$$

denote the 3-dimensional irreducible representation constructed as follows as composition factor of the standard permutation representation of S_5. The group S_5 acts on $V := \mathbf{F}_5^5$ by permuting the standard basis vectors; consider the quotient $W = V/L$ by the S_5-invariant line $L := \mathbf{F}_5 \cdot (1, 1, 1, 1, 1)$ spanned by the all-one vector, and consider the S_5-invariant hyperplane

$$U := \{[w = (w_1, \ldots, w_5)] \in W\colon \sum w_i = 0\}$$

in W (this makes sense since we work modulo 5). Then the natural induced action of S_5 on $U \cong \mathbf{F}_5^3$ is the 3-dimensional faithful mod-5 representation ψ, that turns out to be irreducible.

Proposition 9.7.2 *The mod-5 homology representation of the isometry group S_5 of the Seifert–Weber dodecahedral space is irreducible, and can be identified with*

$$\rho \cong \mathrm{sgn} \otimes \psi. \tag{9.3}$$

Proof Let $\rho\colon S_5 \to \mathrm{GL}(3, \mathbf{F}_5)$ denote the explicit realisation of the representation, as in Example 9.7.1. To see the isomorphism in (9.3), we compute that the values of the Brauer character of ρ on the conjugacy classes of elements of order coprime to 5 (given by cycle type) are as in Table 9.1.

This equals the Brauer character of $\mathrm{sgn} \otimes \psi$, so ρ has the same semi-simplification, but since $\mathrm{sgn} \otimes \psi$ is irreducible, it is actually isomorphic to ρ, that is then also automatically irreducible. □

Table 9.1 Brauer character of the mod-5 homology representation for the Seifert–Weber dodecahedral space

Conjugacy class	$()$	(ab)	(abc)	$(abcd)$	$(ab)(cd)$	$(ab)(cde)$
Matrix element	1	r	$rc^{-1}rc$	rc	$c^{-1}rc^2rc$	$rc^{-1}rcrc^{-1}$
Character value	3	-1	0	1	-1	2

Remark 9.7.3 If M is a closed hyperbolic 3-manifold with large $\dim_{\mathbf{F}_\ell} \mathrm{H}_1(M, \mathbf{F}_\ell)$, there are sometimes explicit lower bounds on the volume of M, expounded in works of Culler and Shalen. For example, if for some prime ℓ, $\dim_{\mathbf{F}_\ell} \mathrm{H}_1(M, \mathbf{F}_\ell) \geq 5$ then $\mathrm{vol}(M) > 0.35$ [31].

Remark 9.7.4 The scope of the method of using ℓ-torsion in $\mathrm{H}_1(M)$ in establishing homological wideness (or the weaker conditions we outlined) is unclear, although some heuristics can be set up by considering the size of the (ℓ-)torsion subgroup. Bader et al. [5, Theorem 1.8] have constructed, for any $\alpha \geq 0$, sequences $\{M_m\}$ of (non-arithmetic) closed hyperbolic rational homology 3-spheres, converging in Benjamini–Schramm topology, for which

$$|\mathrm{H}_1(M_m)_{\mathrm{tors}}| \sim e^{\alpha\, \mathrm{vol}(M_m)}. \tag{9.4}$$

A conjecture of Bergeron and Venkatesh [10, Conjecture 1.3 for SO(3, 1), and bottom of page 122] states that if $M = \Gamma\backslash\mathbb{H}^3$ is closed hyperbolic *arithmetic* manifold and $M_n := \Gamma_m\backslash\mathbb{H}^3$ for a chain of congruence subgroups of Γ_m of Γ with trivial intersection, then the growth in (9.4) holds with $\alpha = 1/(6\pi)$.

Project

Explicitly determine the isometry group of rational homology 3-spheres and its representation on the torsion in the first homology. Which (modular) representations can occur? (cf. Remark 9.3.4 for the free part.)

Open Problem

In the situation of Remark 9.7.4, understand not just the size, but also (some of) the decomposition of $\mathrm{H}_1(M_m, \mathbf{F}_\ell)$ as a $\mathbf{Z}[\mathrm{Out}(\Gamma_m)]$-module. A subproblem is, given a hyperbolic manifold $\Gamma\backslash\mathbb{H}^n$ corresponding to a cocompact discrete group Γ of isometries of hyperbolic n-space $\mathbb{H}^n$ ($n \geq 3$), to determine the structure of its first homology Γ^{ab} as a $\mathbf{Z}[\mathrm{Out}(\Gamma)]$-module (compare Sect. 9.6).

Project

Develop other topological criteria on manifolds that imply homological wideness for large classes of group actions.

Chapter 10
Homological Wideness, "Class Field Theory" for Covers, and a Number Theoretical Analogue

In a previous chapter, we have constructed a particular Riemannian covering realising a wreath product. In this chapter, we first return to that example and use class field theory for Riemannian coverings (à la Sunada) to study the behaviour of geodesic in such covers. We then relate, in the general case, homological wideness of a group G acting on a manifold M (i.e., the question whether the first homology of M contains the regular representation of G) to the existence of geodesics with certain splitting behaviour. In exact analogy to an classical argument in analytic number theory, we use the Ruelle zeta function to show the existence of infinitely many totally split geodesics for a given covering in the negative curvature case. Finally, the analogy with class field theory allows us to study an analogue of homological wideness in the theory of extensions of number fields.

10.1 Abelian Class Field Theory Applied to the Cover $M' \to M$

We briefly revert to the setup of Chap. 7, now under the stronger assumption that the action of G on M is $\mathbf{F}_\ell$-homologically wide. In this situation, the intermediate covers M''_{g_i} can be characterised in terms of the splitting behaviour of certain geodesics, as we now explain.

First of all, fix $\omega \in \mathrm{H}_1(M, \mathbf{F}_\ell)$ to be a cyclic vector for the G-action, so that by assumption

$$\mathrm{H}_1(M, \mathbf{F}_\ell) = \mathbf{F}_\ell[G]\omega \oplus U$$

for some complementary $\mathbf{F}_\ell[G]$-module U. Recall also that we have chosen a set of representatives $\{g_1, \ldots, g_n\}$ for the left cosets G/H_1, so $\{g_1^{-1}, \ldots, g_n^{-1}\}$ is a

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set of representatives for the right cosets $H_1\backslash G$ (by Hall's marriage theorem, there even exists a set $\{g_j\}$ that simultaneously represents the left and right cosets, so alternatively we could use the same set of representatives, if chosen suitably.) This allows us to decompose $\mathrm{H}_1(M, \mathbf{F}_\ell) = \mathbf{F}_\ell[G]\omega \oplus U = U' \oplus U$ as direct sum of $\mathbf{F}_\ell[H_1]$-modules, where $U' = \bigoplus_{j=1}^n \mathbf{F}_\ell[H_1]g_j^{-1}\omega$ is, by the assumption of homological wideness, a free $\mathbf{F}_\ell[H_1]$-module with basis $\omega_j := g_j^{-1}\omega$. Thus, the quotient map to the H_1-coinvariants on U is as described in (7.2), and since this can be identified with the map q_{1*} by Lemma 7.2.1, we find an isomorphism of $\mathbf{F}_\ell$-vector spaces $\mathrm{H}_1(M_1, \mathbf{F}_\ell) = \bigoplus \mathbf{F}_\ell\omega'_j \oplus U_{H_1}$, where $\omega'_j := q_{1*}(\omega_j)$; in particular, the ω'_j are linearly independent. With these identifications, the map q_{1*} is given as in the following diagram.

$$\begin{tikzcd}
\mathrm{H}_1(M, \mathbf{F}_\ell) \arrow[r, "q_{1*}"] \arrow[d, "\cong"] & \mathrm{H}_1(M_1, \mathbf{F}_\ell) \arrow[d, "\cong"] \\
\bigoplus\limits_{j=1}^{n} \mathbf{F}_\ell[H_1]\omega_j \oplus U \arrow[r, "\underline{\mathrm{t}}"] & \bigoplus\limits_{j=1}^{n} \mathbf{F}_\ell\, \omega'_j \oplus U_{H_1}
\end{tikzcd}$$

$$\sum_{j=1}^{n} \sum_{h \in H_1} k_{j,h} h\omega_j + u \mapsto \sum_{j=1}^{n} \left(\sum_{h \in H_1} k_{j,h} \right) \omega'_j + \underline{\mathrm{t}}(u)$$

with $k_{i,h} \in \mathbf{F}_\ell, u \in U$. Since every Riemannian covering of M is (isomorphic to) a quotient of $\widetilde{M}$ by a subgroup of Γ, every *abelian* cover of M is (isomorphic to) a quotient of $[\Gamma, \Gamma]\backslash\widetilde{M}$, and hence Galois groups of abelian covers of M correspond to quotient groups of $\mathrm{H}_1(M) \cong \Gamma^{\mathrm{ab}}$. The coverings $\varpi_i : M''_{g_i} \to M$ are abelian, and they correspond, by construction, to the surjective maps

$$\varphi_i : \mathrm{H}_1(M) \xrightarrow{\otimes \mathbf{F}_\ell} \mathrm{H}_1(M, \mathbf{F}_\ell) \xrightarrow{q_{1*}=\underline{\mathrm{t}}} \mathrm{H}_1(M_1, \mathbf{F}_\ell) = \bigoplus_{j=1}^{n} \mathbf{F}_\ell\omega'_j \oplus U_{H_1} \to C \cong \mathbf{F}_\ell,$$

$$\sum_{j=1}^{n} k_j\omega'_j + u \mapsto k_i \bmod \ell$$

with $k_j \in \mathbf{F}_\ell, u \in U$. We let $\{\Omega_j\}$ denote a set of linearly independent elements of $\mathrm{H}_1(M)$ that map to $\{\omega_j\}$ in $\mathrm{H}_1(M, \mathbf{F}_\ell)$.

The analogue of abelian class field theory for manifolds (described by Sunada in [93, §5], compare [90, §4]) allows us to distinguish the different covers ϖ_i, as follows.

We consider geodesics in M (smooth closed curves in M locally of minimal length) as oriented cycles, forgetting the parametrisation. A *prime geodesic* is a geodesic that is not a multiple of another geodesic. Let I_M denote the free abelian group generated by the prime geodesics of M, and let $I_M \to \mathrm{H}_1(M)$ denote the map that associates to a prime geodesic the homology class of the corresponding closed loop. We denote the kernel of this map by I_M^0, the subgroup of elements that are homologous to zero. The map is surjective: choose any lift of an element in $\mathrm{H}_1(M) \cong \Gamma^{\mathrm{ab}}$ to Γ, and consider the free homotopy class of free loops in M corresponding to its conjugacy class; by shortening, that free homotopy class contains a closed geodesic (E. Cartan's theorem, Note IV in his "Leçons sur la géométrie des espaces de Riemann"; see, e.g. [35, Ch. 12, Thm. 2.2]; as is written in that reference, the result of Cartan does not require negative curvature), and that geodesic maps to the given element in $\mathrm{H}_1(M)$. We conclude that there is an isomorphism $I_M/I_M^0 \cong \mathrm{H}_1(M)$.

If $\mathfrak{p}$ is a prime geodesic on M, let $(\mathfrak{p}|\varpi_i) \in \mathbf{F}_\ell$ denote a generator for the (cyclic) stabiliser of any lift of $\mathfrak{p}$ to M''_{g_i} (since the cover ϖ_i is abelian, this does not depend on the chosen lift: in general, the stabilisers of different lifts are conjugate). By the orbit-stabiliser theorem, the number of prime geodesics in M''_{g_i} above $\mathfrak{p}$ is given by $|\mathbf{F}_\ell|/\langle(\mathfrak{p}|\varpi_i)\rangle$. This number is either ℓ (the prime geodesic "splits", and $(\mathfrak{p}|\varpi_i) = 0$) or 1 (the prime geodesic is "inert", and $(\mathfrak{p}|\varpi_i) \neq 0$).

A main result in abelian class field theory for manifolds, [93, Prop. 7], says, since the cover ϖ_i is abelian, the group homomorphism $I_M \to \mathbf{F}_\ell$ given by $\mathfrak{p} \mapsto (\mathfrak{p}|\varpi_i)$ is surjective with kernel $I_M^0 \cdot \varpi_i(I_{M''_{g_i}})$, and we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi_i) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_1(M) \arrow[r, "\varphi_i"] \arrow[d, "\cong"] & \mathbf{F}_\ell \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \varpi_i(I_{M''_{g_i}}) \arrow[r] & I_M/I_M^0 \arrow[r, "(\cdot,\varpi_i)"] & \mathbf{F}_\ell \arrow[r] & 0
\end{tikzcd}$$

Since $\ker(\varphi_i)$ consists of the H_1-orbit of all homology classes spanned by both the classes Ω_j with $j \neq i$, as well as the classes in the complement U, we deduce from this diagram the following result.

Proposition 10.1.1 *The prime geodesics of M that are inert in the cover $\varpi_i : M''_{g_i} \to M$ are precisely the prime geodesics in the H_1-orbit of all geodesics whose homology class lies in the one-dimensional subspace $\langle\Omega_i\rangle$ of* $\mathrm{H}_1(M)$.

Since g_i represent different conjugacy classes of H_1 in G, the subspaces $\langle\Omega_i\rangle$ of $\mathrm{H}_1(M)$ are distinct, and hence so are the covers $\varpi_i : M''_{g_i} \to M$ in the fiber product (7.11).

10.2 Homological Wideness and Geodesics

The question whether the action of G on M is $\mathbf{Q}$-homologically wide may be approached by splitting it into two separate questions:

(a) Does there exist of a prime closed geodesic on M whose G-orbit consists of $|G|$ distinct geodesics?

(b) Do the loops corresponding to these geodesics become homologous in $\mathrm{H}_1(M, \mathbf{Q})$?

We have no general framework to deal with the second question (notice the example of the loop separating the two tori in the connected sum $T^2 \# T^2$; it is non-trivial in the fundamental group, but becomes trivial in the first homology group, since it bounds one of the tori). However, concerning the first question, we can say the following.

Proposition 10.2.1 *Suppose that M and M_0 are negatively curved closed Riemannian manifolds and $M \to M_0$ is a G-Galois Riemannian covering. Then there exists infinitely many closed prime geodesics in M_0 that lift to $|G|$ distinct closed prime geodesics in M, that hence form one G-orbit of such geodesics.*

Proof We use the following analytical argument (similar to the analytic argument that split primes in number fields exist). Let

$$Z_{M_0}(s) := \prod_{\mathfrak{p}} (1 - e^{-s\ell(\mathfrak{p})})^{-1}$$

where $\mathfrak{p}$ runs over closed prime geodesics $\mathfrak{p}$ in M_0 of length $\ell(\mathfrak{p})$ (i.e., the Ruelle zeta function for the geodesic flow on M).

Let $h > 0$ denote the volume entropy of the universal covering of M (which is also the volume entropy of M_0). Since M is negatively curved, the geodesic flow is weak mixing Anosov, so it follows that $Z_M(s)$ converges for $\mathrm{Re}(s) > h$ but has a pole at $s = h$ [75, Prop. 9].

Since the covering $M \twoheadrightarrow M_0$ is Galois, a prime geodesic $\mathfrak{p}$ splits into $r_{\mathfrak{p}}$ distinct prime geodesics in M, which are all of the same length, say, $f_{\mathfrak{p}}$ times the length of $\mathfrak{p}$. Then $r_{\mathfrak{p}} f_{\mathfrak{p}} = |G|$ (see [93, §5] or [90, §4]).

Suppose, by contradiction, that the set S of geodesics of M_0 that split completely into $|G|$ distinct prime geodesics in M is finite. Then $f_{\mathfrak{p}} \geq 2$ for all $\mathfrak{p} \notin S$. We find, with $\mathfrak{P}$ running over the closed prime geodesics of M, and for real $s > h$,

$$\begin{aligned} Z_M(s) &= \prod_{\mathfrak{P}} \left(1 - e^{-s\ell(\mathfrak{P})}\right)^{-1} \leq \prod_{\mathfrak{p}} \left(1 - e^{-s f_{\mathfrak{p}} \ell(\mathfrak{p})}\right)^{-r_{\mathfrak{p}}} \\ &\leq \left(Z_{M_0}(2s)\right)^{|G|} \prod_{\mathfrak{p} \in S} \left(1 + e^{-s\ell(\mathfrak{p})}\right)^{|G|}. \end{aligned} \tag{10.1}$$

Now $Z_{M_0}(2s)$ converges at $s = h$, and hence also the right hand side of the inequality (10.1) converges at $s = h$; this contradicts the fact that the left hand side has a pole at $s = h$, and hence shows that S is infinite. □

Remark 10.2.2 The result also follows from the more general Riemannian covering version of the Chebotarev density theorem due to Parry and Pollicott [76, Theorem 3] applied to the trivial conjugacy class in G.

Kojima [57, Prop. 2] gave a different proof for the case of closed orientable hyperbolic 3-manifolds admitting a totally geodesic embedding of a Riemann surface of genus ≥ 3, using projective laminations.

10.3 An Analogue of Homological Wideness for Number Fields

Suppose that G acts on M with (orbifold) quotient M_0, and denote, as usual, the fundamental group of M by Γ and that of M_0 by Γ_0. Then, as in Remark 6.2.2, G acts by outer automorphisms on Γ, and hence it acts by automorphisms on the abelianisation Γ^{ab}; in this interpretation, the homology representation is given by $h\colon G \to \mathrm{Aut}(\Gamma^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q})$. This has the following analogue in number theory: if $K/\mathbf{Q}$ is a finite Galois extension with Galois group G, and $G_K := \mathrm{Gal}(\overline{\mathbf{Q}}/K)$, $G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we have a short exact sequence $1 \to G_K \to G_{\mathbf{Q}} \to G \to 1$, and hence an action of G by outer automorphisms on G_K, given by conjugation by any lift of an element of G to $G_{\mathbf{Q}}$. This induces a group representation

$$\mathbf{h}\colon G \to \mathrm{Aut}(G_K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}).$$

The analogue of $\mathbf{Q}$-homological wideness in this context is the following.

Proposition 10.3.1 *The representation* $\mathbf{h}$ *contains the regular representation* $\mathbf{Q}[G]$.

Proof There exists a prime number p that is totally split in $K/\mathbf{Q}$ (which follows from Chebotarev's theorem, or easier manipulation with zeta functions much as in the proof of Proposition 10.2.1). Let $\mathfrak{p}$ denote any prime ideal of K above such p. Consider the reciprocity map from class field theory

$$\vartheta : \mathbf{A}_{K,f}^{*} \to G_K^{\mathrm{ab}}$$

from the finite idele group $\mathbf{A}_{K,f}^{*}$ of K, let $\pi_{\mathfrak{p}}$ denote any uniformiser in the $\mathfrak{p}$-completion of K, and let

$$F := \vartheta((1, \dots, 1, \pi_{\mathfrak{p}}, 1, \dots, 1))$$

denote a "Frobenius" of $\mathfrak{p}$; then for any $g \in G$,

$$g(F) = \vartheta((1, \dots, 1, \pi_{g(\mathfrak{p})}, 1, \dots, 1)).$$

By the assumption of total splitting, all $g(\mathfrak{p})$ for g running through G and for $\mathfrak{p}$ a fixed prime above the given p are distinct. Since the kernel of ϑ is the closure of the diagonally embedded K^* in $\mathbf{A}_{K,f}^{*}$, we see that $\{g(F) : g \in G\}$ are distinct commuting elements of infinite order in G_K^{ab}, and hence F is a cyclic vector for G. $\square$

Remark 10.3.2 Compared to the case of manifolds, in the number theory case, the group $G_K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}$ is of infinite rank (and captures all ramified abelian extensions), whereas $\mathrm{H}_1(M, \mathbf{Q})$ is always of finite rank (and captures topological abelian covers). For number fields, subproblem (a) as in Remark 10.2 is answered affirmatively by a similar splitting theorem as Proposition 10.2.1 for manifolds; and problem (b)—linear combinations of geodesics becoming homologous—does not occur at all, due to the specific nature of the reciprocity map.

Open Problem

Continuing along the lines of Remark 10.3.2, in [29] it was shown that isomorphism of number fields is equivalent to topological conjugacy of associated dynamical systems built from the reciprocity map. The analogous question for manifolds becomes: associate to a manifold M the dynamical system given by the monoid generated by prime geodesics acting on $\mathrm{H}_1(M, \mathbf{Z})$, where a prime geodesic acts by adding the homology class of the corresponding closed loop; then under what conditions is isometry of two manifolds M_1 and M_2 equivalent to the topological conjugacy of the corresponding dynamical systems, where one additionally assumes that the identification of the prime geodesics is length-preserving? (In number fields, the analogue of the additional assumption would be that the map of prime ideals preserves the norm map, but it turns out that this is automatic in that case, given the other assumptions.)

Open Problem

Study (in examples; or find a criterion) whether and how distinct geodesics in a G-orbit become homologous.

Chapter 11
Examples Concerning the Main Result

We study whether it is possible to apply twisted Laplace spectra (and, if so, how many are necessary) to deduce isometry of some well-known examples of isospectral manifolds in the literature, due to Schüth (simply connected manifolds), Ikeda (lens spaces), Vignéras/Linowitz and Voight (arithmetic surfaces), Milnor (lattice examples), Doyle and Rossetti (Tetra and Didi), Sunada (based on group-theoretical examples from Gerst, Gaßmann and Komatsu), Brooks and Tse (surfaces of small genus, Riemann surfaces of small genus), Barden and Kang (surfaces of genus two), and Miatello and Rossetti (flat manifolds isospectral for all twists by linear characters).

11.1 Examples Where Theorem 1.2.1 Does not Apply

We first give some examples where the conditions of Theorem 1.2.1 are not met.

Example 11.1.1 Schüth constructed isospectral, non-isometric simply connected manifolds M_1 and M_2 [87]: these are non-isometric manifolds that are indistinguishable by any twisted spectrum on functions, simply because there is nothing to twist by.

From the perspective of our two conditions, this example already violates the first: if a diagram like (1.1) would exist, then by simple connectedness, M_1 and M_2 would both be isometric to $\widetilde{M}_0$, and hence isometric, which they are not.

Sutton [94] and An et al. [3] constructed further examples of isospectral simply connected manifolds by using an extension of Sunada's method valid for continuous groups, rather than finite groups, as we are considering in this book. $\lozenge$

Example 11.1.2 The isospectral compact Riemann surfaces constructed by Vignéras [97] are described by commensurable arithmetic lattices and by [24, Proposition 3] there does not exist a diagram (1.1), so our first condition is

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violated. The original examples have Euler characteristic -201600 (corrected value from [62]) but Linowitz and Voight have constructed similar examples of Euler characteristic -10 and proven that this is the maximal Euler characteristic that can occur for the class of "unitive" torsion free Fuchsian groups; cf. [62]. $\diamondsuit$

Example 11.1.3 Ikeda's isospectral non-isometric lens spaces from Example 4.2.1 have $M = S^5$, and $M_1 = L(11; 1, 2, 3)$ and $M_2 = L(11, 1, 2, 4)$. In this example, $H_i \cong \mathbf{Z}/11\mathbf{Z}$. The action is not $\mathbf{Q}$-homologically wide; actually, the only cyclic cover of M_1 is M itself. $\diamondsuit$

Example 11.1.4 We consider Milnor's example [71] where $M_i = \Gamma_i \backslash \mathbf{R}^{16}$ with $\Gamma_1 = E_8 \oplus E_8$ and $\Gamma_2 = E_{16}$. The group $\langle \Gamma_1, \Gamma_2 \rangle = \Gamma_1 + \Gamma_2$ is a lattice, and, as Chen [24, §3] observed, there is a diagram of the form (1.2) with $M := (\Gamma_1 \cap \Gamma_2) \backslash \mathbf{R}^{16}$, $G = (\Gamma_1 + \Gamma_2)/(\Gamma_1 \cap \Gamma_2)$ and $H_i = \Gamma_i/(\Gamma_1 \cap \Gamma_2)$. One easily checks, by writing down explicit lattice bases that $G = H_1 \times H_2$ is the Klein four-group with $H_i \cong \mathbf{Z}/2\mathbf{Z}$. This can be done, for example, by using the MAGMA [15] code below.

MAGMA Program Code for Computing G

```
L0:=LatticeWithBasis(8,
[2,0,0,0,0,0,0,0,
-1,1,0,0,0,0,0,0,
0,-1,1,0,0,0,0,0,
0,0,-1,1,0,0,0,0,
0,0,0,-1,1,0,0,0,
0,0,0,0,-1,1,0,0,
0,0,0,0,0,01,1,0,
1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2]);
L1:=DirectSum(L0,L0);
L2:=LatticeWithBasis(16,
[2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,-1,1,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,-1,1,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,-1,1,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,-1,1,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,-1,1,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,-1,1,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,-1,1,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,-1,1,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,-1,1,0,
1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2,1/2]);
L3:=L1 meet L2;
Index(L2,L3);
Index(L1,L3);
G:=(L1+L2)/L3;
G;
```

Since G is abelian, the subgroups H_1 and H_2 cannot be weakly conjugate (since they are not equal as subgroups of G). A cover realising the wreath product cannot exist: the group G acts non-trivially on the coset space $G/H_1 = H_2$, so the wreath product $C^2 \rtimes G$ is non-commutative, and hence cannot occur as subgroup of the (abelian) fundamental group of M.

Chen also observed that for these M_1, M_2, there exists another diagram of the form (1.2) in which the corresponding groups *are* weakly conjugate, see [24, Prop. 1]. In that example, the group G contains an element involving a non-trivial translation. One easily computes the corresponding lattices bases (e.g., again using the MAGMA [15] code below) to see that in that case, $H_i \cong (\mathbf{Z}/2\mathbf{Z})^{12}$.

MAGMA Program Code for Computing H_1 and H_2

```
Ls:=LatticeWithBasis(4,
[2,0,0,0,
0,2,0,0,
0,0,0,2,
1,1,1,1]);
Lt:=DirectSum(Ls,Ls);
L:=DirectSum(Lt,Lt);
H1:=L1/L;
H2:=L2/L;
H1;
H2;
```

Again, the wreath product cannot be realised for the same reason as above (the corresponding wreath product is non-commutative but the fundamental group of M is commutative). $\diamondsuit$

Example 11.1.5 Doyle and Rossetti [36] constructed two isospectral, non-isometric closed flat 3-manifolds, called "Tetra" and "Didi". These are commensurable, given as quotients of $\mathbf{R}^3/(\mathbf{Z}^2 \times 2\mathbf{Z})$ by the action of the two non-isomorphic groups of order 4, but there is no diagram (1.1) (if so, they would be strongly isospectral, but they cannot be isospectral on 1-forms, since they have different first Betti numbers). $\diamondsuit$

11.2 Examples from Sunada's Construction

If M_1 and M_2 are isospectral via the Sunada construction, then a diagram of the form (1.2) exists by default (possibly with orbifold M_0). We discuss some "small" examples of closed surfaces (where a group G is realised by choosing a compact hyperbolic Riemann surface M_0 whose genus is larger than or equal to the number of

Table 11.1 Examples of Sunada triples (with p an odd prime number) from [90, Example 1], [6, 19] and [45, Example 4.1]. We indicate the dimension of the representations involved in Theorem 1.2.1, as well as an upper bound on the number of equalities of spectra that need to be checked. In the last line, we choose $\ell = 2$ and use the bound in Remark 5.3.3

Name	$\lvert G\rvert$	$\lvert H_i\rvert$	Smallest ℓ	Dim. of reps.	# equalities $\leq$
"Gerst"	32	4	3	8	24
"Gaßmann"	720	4	7	180	56
"Komatsu"	$(p^3)!$	p^3	$\leq 2p^3 - 3$	$(p^3 - 1)!$	$2p^2(2p^3 - 3)$
E.g., $p = 3$	$\approx 10^{28}$	27	29	$\approx 4 \cdot 10^{26}$	522
Brooks-Tse	168	24	5	7	20
Barden-Kang	96	8	5	12	80
Guralnick	p^5	p^2	2	p^3	$4p^2 + 2$

generators of G, so that there is a surjection $\pi_1(M_0) \twoheadrightarrow G$). These examples satisfy the requirements of Theorem 1.2.1 and illustrate numerically that, whereas our auxiliary construction involves manifolds and groups of relatively large order and negative Euler characteristic, the dimension of the representations and the number of required twists can be rather small (dictated by the degrees of the corresponding coverings $M \to M_i$).

Example 11.2.1 Sunada lists three examples in [90, §1, Example 1–3], for which we indicate in the first three rows of Table 11.1 the dimensions of the representations by which one needs to twist, as well as how many equalities of multiplicities of zero suffice in Theorem 1.2.1. The examples are

- "Gerst": $G = (\mathbf{Z}/8\mathbf{Z})^* \rtimes \mathbf{Z}/8\mathbf{Z}$, $H_1 = \{(1,0), (3,0), (5,0), (7,0)\}$ and $H_2 = \{(1,0), (3,4), (5,4), (7,0)\}$ (both isomorphic to the Klein four group, but not conjugate in G).
- "Gaßmann": Example 1.2.2 from the introduction.
- "Komatsu": $G = S_{p^3}$, $H_1 = (\mathbf{Z}/p\mathbf{Z})^3$ and H_2 the Heisenberg group modulo p (i.e., the group $\{\left(\begin{smallmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{smallmatrix}\right) : a, b, c \in \mathbf{F}_p\}$), of order p^3 and exponent p. Both embed in G by the action of left multiplication on themselves, and H_1 is commutative, whereas H_2 is not.

The computations of the data in the table in these cases is straightforward, except for the last case, named "Komatsu"; here, we need to find a prime ℓ coprime to $|G| = (p^3)!$. If we observe that all prime divisors of $|G|$ are $< p^3$ and use Bertrands postulate/Chebyshev's theorem that there is a prime between p^3 and $2p^3 - 2$ [47, Thm. 418], we can certainly find $\ell \leq 2p^3 - 3$. We also use that $|H_2^{\mathrm{ab}}| \leq p^2$; for this, note the commutator identities

$$\left[\left(\begin{smallmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{smallmatrix}\right),$$
$$\left[\left(\begin{smallmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{smallmatrix}\right)\right] = \left[\left(\begin{smallmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix}\right),$$

which imply that the abelianisation of the group of upper triangular matrices in $\mathrm{GL}(3, \mathbf{F}_p)$ with all diagonal entries equal to 1 is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^2$. $\diamondsuit$

We now discuss in some more detail follow-up examples of Brooks–Tse and Barden–Kang that have the smallest known genus (in variable or constant curvature). The results are summarised in the fifth and sixth row of Table 11.1.

Example 11.2.2 Brooks and Tse (see [19] and [17]) constructed closed surfaces of Euler characteristic -4 (genus 3) that are isospectral but not isometric for well-chosen metrics whose curvature is not constant. Here, $G = \mathrm{PSL}(2, 7) = (\mathrm{P})\mathrm{SL}(3, 2)$ is the unique simple group of order $168 = 2^3 \cdot 3 \cdot 7$ [51, 6.14(4) & 6.15], the automorphism group of the Fano plane $\mathbf{P}^2(\mathbf{F}_2)$, and H_1, H_2 are index 7 subgroups given as 3×3 matrices in $\mathrm{SL}(3, 2)$ in which the first column, respectively the first row, is $(1, 0, 0)$ (stabilisers of a point and a dual hyperplane in $\mathbf{P}^2(\mathbf{F}_2)$).

In this case, M_0 is an orbifold sphere with 3 singular points of order 7, and $\chi_M = -2^5 \cdot 3$. The smallest possible ℓ that can be chosen is $\ell = 5$, and then $\widetilde{G}$ is of order $5^7 \cdot 168 \approx 13 \cdot 10^6$ and $\chi_{M'} = -2^5 \cdot 3 \cdot 5^7$. The dimension of the required representations in Proposition 5.3.1, on the other hand, is only 7. Recall that S_4 has a unique normal subgroup isomorphic to the Klein four-group (generated by products of two two-cycles) and quotient isomorphic to $S_3 \cong \mathrm{GL}(2, 2)$ [51, 5.2]. Since $H_i \cong (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \rtimes \mathrm{GL}(2, 2) \cong S_4$ has commutator subgroup A_4, the number of spectral equalities that needs to be checked is just $2 \cdot 5 \cdot |H_2^{\mathrm{ab}}| = 20$ (cf. Proposition 5.3.2). $\diamondsuit$

Example 11.2.3 In the same reference, Brooks and Tse (see [19] and [17]) used a different representation of the same group G and the same subgroups H_i to construct isospectral non-isometric compact Riemann surfaces (surfaces of *constant negative curvature* -1) of Euler characteristic -6 (genus 4). Here, M_0 is an orbifold torus with a single singular point of order 7. Then $\chi_{M'} = -2^4 \cdot 3^2 \cdot 5^7$, but one needs to check only 20 equalities of spectral multiplicities of 7-dimensional representations. $\diamondsuit$

Example 11.2.4 A similar construction of such surfaces (not of constant curvature) of Euler characteristic -2 (genus 2) by Barden and Kang [6] has the largest currently known Euler characteristic. In their case, G has order 96, and H_1 and H_2 are of index 12 in G, both isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$. We can choose $\ell = 5$ and $\widetilde{G}$ of order $5^{12} \cdot 96 \approx 2 \cdot 10^{10}$. With $\chi_{M_i} = -2$ for $i = 1, 2$, we have $\chi_M = -16$, $\chi_{M'} = -2^4 \cdot 5^{12}$, and the dimension of the required representations is 12. In this case, one needs to check 80 equalities of multiplicities of zero in various spectra. $\diamondsuit$

Example 11.2.5 We consider an example where the order of $|G|$ is odd (the situation can be realised using Riemann surfaces of sufficiently high genus, as above). Note that G is forcedly solvable, by the Feit–Thompson theorem. Let p denote an odd prime number. As in Guralnick [45, Example 4.1], consider the group G of order p^5 given as the semidirect product

$$G = A \rtimes H, \text{ where } A = (\mathbf{Z}/p^2\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}) \text{ and } H = (\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z})$$

with the action $a \mapsto a^h$ of $h \in H$ on $a \in A$ determined by

$$(1,0)^{(1,0)} = (1,1),\ (0,1)^{(1,0)} = (p,1),\ (1,0)^{(0,1)} = (p+1,0),\ (0,1)^{(0,1)} = (0,1).$$

Now G has two subgroups $H_i \cong \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$ that are not weakly conjugate; more precisely, in the above description,

$$H_1 = H = \langle((0,0),(1,0)),((0,0),(1,0))\rangle \text{ and } H_2 = \langle((0,0),(1,0)),((p,0),(0,1))\rangle.$$

Using the smallest prime $\ell \geq 3$ coprime to $|G|$ in the main theorem, we need to check the following number of equalities: (a) if $p = 3$, 90 equalities, using $\ell = 5$; (b) if $p \neq 3$, $6p^2$ equalities, using $\ell = 3$. In this case, a better result is possible using Pintonello's method from Remark 5.3.3, where one can choose $\ell = 2$ at the cost of adding extra identities, leading to $4p^2 + 2$ equalities to be checked (this number equals $38 < 90$ for $p = 3$ and is always smaller than $6p^2$). Note that an earlier similar example with $|G| = p^6$, $|H_i| = p^2$ can be found in [2, Ch. IV, Ex. 2, p. 63]. $\diamondsuit$

11.3 Flat Manifolds Isospectral for All Twists by Linear Characters

Example 11.3.1 Miatello and Rossetti constructed non-isometric closed flat manifolds M_1 and M_2 that admit non-trivial twists but are twisted isospectral on functions and forms for all twists by linear characters (and hence also any representation that decomposes as a direct sum of such) [70, 4.5]. Twisted Laplacians of such linear characters act on sections of flat line bundles.

The manifolds are constructed as follows (our notations differ from [70]). Consider the group of affine transformations $\mathbf{R}^4 \rtimes \mathrm{O}(4)$ of $\mathbf{R}^4$. Let τ_v denote the translation by $v \in \mathbf{R}^4$, let

$$\Lambda := \{\tau_v : v \in \mathbf{Z}^4\} \cong \mathbf{Z}^4,$$

and denote the standard basis vectors as $e_1, \ldots, e_4$. Consider the two orthogonal matrices

$$A := \mathrm{diag}(1,1,-1,-1) \text{ and } A' := \mathrm{diag}(1,-1,1,-1)$$

and the vectors

$$b_1 = (e_2 + e_4)/2,\ b_1' = e_3/2,\ b_2 = e_2/2,\ b_2' = e_1/2.$$

The manifolds are $M_i = \Gamma_i\backslash\mathbf{R}^4$ with $\Gamma_i = \langle A\tau_{b_i}, A'\tau_{b'_i}, \Lambda\rangle$. The Γ_i are Bieberbach groups fitting into an exact sequence

$$1 \to \Lambda \cong \mathbf{Z}^4 \to \Gamma_i \to \langle A, A'\rangle \cong (\mathbf{Z}/2\mathbf{Z})^2 \to 1.$$

Concerning our two conditions, the situation in this example is as follows. If $A \in \mathrm{O}(4)$ is of order two and $b \in 1/2\mathbf{Z}^4$, then $(A\tau_b)^2 = \tau_{b+Ab} \in \Lambda$, since $b + Ab \in \mathbf{Z}^4$; hence for any $\gamma \in \Gamma_i$, we have $\gamma^2 \in \Lambda \leq \Gamma_1 \cap \Gamma_2$. Therefore, with $\Gamma_0 := \langle\Gamma_1, \Gamma_2\rangle$ we have $\Gamma_0/\Gamma_i \cong (\mathbf{Z}/2\mathbf{Z})^2$, and a diagram such as (1.1) exists with $M_0 := \Gamma_0\backslash\mathbf{R}^4$. Then we can set $M := \Gamma\backslash\mathbf{R}^4$ with $\Gamma := \Gamma_1 \cap \Gamma_2$ to get a diagram (1.2). From the presentation of Γ_i, one may compute the intersection to be $\Gamma = \Lambda \cong \mathbf{Z}^4$, so M is a 4-torus. Since $G = \Gamma_0/\Gamma$ is a group of exponent 2 and order 16, we find $G = H_1 \times H_2 \cong (\mathbf{Z}/2\mathbf{Z})^4$ with $H_i = (\mathbf{Z}/2\mathbf{Z})^2$.

We now look at the second condition. Since $G/H_1 \cong H_2$ with left G-action induced by the multiplication in H, it follows that $\mathrm{Ind}_{H_1}^G \mathbf{1} \cong \mathbf{Z}[H_2]$ is the (4-dimensional) regular representation of H_2, i.e., the direct sum of the four linear characters of the Klein four-group. Note that, similarly, $\mathrm{Ind}_{H_2}^G \mathbf{1} \cong \mathbf{Z}[H_1]$, and hence $\mathrm{Ind}_{H_1}^G \mathbf{1} \cong \mathrm{Ind}_{H_2}^G \mathbf{1}$, meaning that H_1 and H_2 are weakly conjugate subgroups of G.

To analyse the homology representation, we note that $\mathrm{H}_1(M) \cong \Gamma^{\mathrm{ab}} \cong \mathbf{Z}^4$. The group G acts on this through conjugation by (outer) automorphisms, i.e., if $A\tau_b$ represents an element of G (with $A \in \mathrm{O}(4)$ of order two, $b \in 1/2\mathbf{Z}^2$), then it acts by $\tau_{-b}A\tau_v A\tau_b = \tau_{Av}$. We conclude that the representation of G on $\mathrm{H}_1(M)$ factors through the representation of $\langle A, A'\rangle \cong (\mathbf{Z}/2\mathbf{Z})^2$ in $\mathrm{GL}(4, \mathbf{Z})$. Looking at characters, this is the regular representation of the Klein four-group. We conclude that

$$\mathrm{H}_1(M) \cong \mathrm{Ind}_{H_1}^G \mathbf{1}$$

as G-modules.

We find that the two conditions of our main result are satisfied; although the action of G is not homologically wide (G is of order 16 but the first Betti number of M is only 4), the above calculation shows that condition $(*)$ does hold. Using $\ell = 2$ as in Remark 5.3.3, we find that the manifolds M_1 and M_2 can be distinguished by 18 equalities of twisted spectra for 4-dimensional representations. These representation are then forcedly not all direct sums of linear characters.

In a different direction, Gordon, Ouyang and Schüth [41] [86] have shown how to distinguish the manifolds in this example using a *non-flat* Hermitian *line* bundle. ♢

11.4 An Example that Does not Arise from Sunada's Construction

Example 11.4.1 We consider a finite group G with two non-weakly conjugate subgroups H_1 and H_2. Let M_0 denote a compact hyperbolic Riemann surface of genus larger or equal to the number of generators of G, and fix a surjective morphism $\pi_1(M_0) \twoheadrightarrow G$. This leads to a diagram of the form (1.2), and homological wideness is satisfied by Proposition 9.1.1. For example, set $G = S_4$ with $H_1 = \langle(1234)\rangle \cong \mathbf{Z}/4\mathbf{Z}$ and $H_2 = \langle(12)(34), (13)(24)\rangle \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Since the cycle types in H_1 and H_2 are different, they are not weakly conjugate. We choose M_0 to be a genus two compact Riemann surface and let $\ell = 5$; now we can distinguish the genus 7 Riemann surfaces M_1 and M_2 using 40 equalities of multiplicities of zero in the spectra of Laplacians twisted by 6-dimensional representations. $\diamondsuit$

Project

For the "Komatsu" example, find representations of smaller dimension twisting by which implies isometry.

Project

Study the use of spectra of operators on non-flat line bundles to detect isometry.

Chapter 12
Length Spectrum

In this chapter, we prove a version of the main result using the length spectrum instead of the Laplace spectrum, in case the manifold is negatively curved. We introduce the L-series corresponding to a unitary representation of the fundamental group, recall its convergence properties in relation to volume entropy, and its behaviour in finite covers.

12.1 L-Series

We assume that M is a connected negatively curved oriented closed Riemannian manifold. Then the following properties hold:

1. each free homotopy class $[\gamma]$ contains a unique closed geodesic [56, Thm. 3.8.14]. We write $\ell(\gamma)$ for the length of that geodesic.
2. the geodesic flow is of Anosov type, and the topological entropy equals the *volume entropy*, defined as $h_M := \lim_{R\to+\infty} (\log \operatorname{vol}(B(x, R)))/R$, where $B(x, R)$ is a geodesic ball of radius R in the universal cover $\widetilde{M}$ of M centered at some point $x \in \widetilde{M}$ [65].

Recall some elements of the theory of prime geodesics (for general M but an abelian cover, we treated this in Sect. 10.1). We call $[\gamma]$ *prime* if the associated geodesic is not a multiple of another geodesic (in the sense of oriented cycles). Let $\varpi : M' \to M$ denote a G-cover for a finite group G. Notice that $h_{M'} = h_M$ by definition. Above each fixed prime geodesic of M lie finitely many prime geodesics of M', the set of which carries a transitive G-action. For a fixed γ' mapping to γ, we let $(\varpi|\gamma)$ denote any element of G that generates the stabiliser of γ'. All such elements are conjugate in G (in the abelian case, the element does not depend on the choice of γ', cf. Sect. 10.1).

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Let $\rho\colon G \to \mathrm{U}(N, \mathbf{C})$ denote a representation. Since the determinant takes the same value on conjugate matrices, we have a well-defined associated *L-series*

$$L_M(\rho, s) := \prod_{[\gamma]} \det(1_N - \rho((\varpi|\gamma))e^{-s\ell(\gamma)})^{-1},$$

where 1_N is the identity matrix of size $N \times N$. If ρ_1 and ρ_2 are two representations of G, then [76, p, 135]

$$L_M(\rho_1 \oplus \rho_2) = L_M(\rho_1)L_M(\rho_2). \tag{12.1}$$

Choosing $\rho = \mathbf{1}$, $L_M(\mathbf{1}, s)$ is related to analogues of the Selberg zeta function. Standard theory of Dirichlet series (applying Möbius inversion to the coefficients of the logarithmic derivative) implies that knowledge of $L_M(\mathbf{1}, s)$ is equivalent to knowledge of the multiset of lengths $\{\ell(\gamma)\}$. Parry and Pollicott and Adachi and Sunada have proven:

Lemma 12.1.1 ([76, Thm. 1 and 2] and [1, Thm. A]) *The function $L_M(\rho, s)$ converges absolutely for $\mathrm{Re}(s) > h_M$ and can be analytically continued to an open set D that contains the half-plane $\mathrm{Re(s)} \geq h_M$. For an irreducible representation ρ, $L(\rho, s)$ is holomorphic in D unless $\rho = \mathbf{1}$. Furthermore, $L_M(\mathbf{1}, s)$ has a simple pole at $s = h_M$.*

12.2 Main Result for the Length Spectrum

In this setup, the analogue of Lemma 3.9.1 is the following:

Lemma 12.2.1 *Let G be a finite group acting by fixed-point free isometries on a negatively curved Riemannian manifold M' with quotient $M = G\backslash M'$. Set $h := h_M = h_{M'}$. If ρ is any unitary representation of G, then the multiplicity $\langle \rho, \mathbf{1}\rangle$ of the trivial representation in the decomposition of ρ into irreducibles equals $-\mathrm{ord}_{s=h} L_M(\rho, s)$, the order of the pole of $L_M(\rho, s)$ at $s = h$.*

Proof Let D denote the extended region of convergence as in Lemma 12.1.1. Decompose $\rho = \bigoplus_{i=1}^{N} \langle \rho_i, \rho\rangle \rho_i$ as a sum over irreducible representations ρ_i. Then by formula (12.1), we have

$$L_M(\rho, s) = \prod_{i=1}^{N} L(\rho_i, s)^{\langle \rho_i, \rho\rangle},$$

and set $\rho_1 = \mathbf{1}$ for convenience. Applying Lemma 12.1.1, we find from this product decomposition that $\mathrm{ord}_{s=h} L_M(\rho, s) = \langle \mathbf{1}, \rho\rangle$. □

We have the following further two analogues of results previously shown for the Laplace spectra. The analogue of Lemma 3.8.1 for L-series is the following.

Lemma 12.2.2 ([91, Remark 2 After Lemma 1]) *If $M \to M_1 \to M_0$ is a tower of finite Riemannian coverings and $M \to M_0$ is Galois with group G, $M \to M_1$ with group H, and $\rho\colon H \to \mathrm{U}(N, \mathbf{C})$ a representation, then*

$$L_{M_0}(\mathrm{Ind}_H^G \rho, s) = L_{M_1}(\rho, s).$$

The analogue of Proposition 4.1.1 for L-series is the following.

Lemma 12.2.3 *Suppose that we have diagram (1.2) and set $h := h_{M_1} = h_{M_2} = h_M$; then, for two linear characters $\chi_1 \in \check{H}_1$ and $\chi_2 \in \check{H}_2$, a representation isomorphism $\mathrm{Ind}_{H_1}^G \chi_1 \cong \mathrm{Ind}_{H_2}^G \chi_2$ is equivalent to*

$$\mathrm{ord}_{s=h} L_{M_i}(\overline{\chi}_i \otimes \mathrm{Res}_{H_i}^G \mathrm{Ind}_{H_j}^G \chi_j, s) \tag{12.2}$$

being the same for the pairs (i, j) given by $(1, 1)$, $(2, 1)$ *and* $(1, 2)$, $(2, 2)$.

Proof The proof is essentially the same as that of Proposition 4.1.1, but now using Lemma 12.2.1 instead of Lemma 3.9.1. □

We can adopt the reasoning in the proof of the main theorem to find:

Theorem 12.2.4 *Suppose we have a diagram (1.1) of negatively curved Riemannian manifolds with (common) volume entropy h, and suppose that the action of G on M in the extended diagram (1.2) is homologically wide. Then M_1 and M_2 are equivalent Riemannian covers of M_0 if and only if the pole orders at $s = h$ of a finite number of specific L-series of representations on M_1 and M_2 is equal.*

More specifically, using the notation of Theorem 6.4.1, M_1 and M_2 are equivalent Riemannian covers of M_0 if and only if there exists a linear character $\chi : \widetilde{H}_2 \to \mathbf{C}^*$ such that

$$\mathrm{ord}_{s=h} L_{M_1}(\overline{\Xi} \otimes \mathrm{Res}_{\widetilde{H}_1}^{\widetilde{G}} \mathrm{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi) = \mathrm{ord}_{s=h} L_{M_2}(\overline{\chi} \otimes \mathrm{Res}_{\widetilde{H}_2}^{\widetilde{G}} \mathrm{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \Xi)$$

and

$$\mathrm{ord}_{s=h} L_{M_1}(\overline{\Xi} \otimes \mathrm{Res}_{\widetilde{H}_1}^{\widetilde{G}} \mathrm{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \chi) = \mathrm{ord}_{s=h} L_{M_2}(\overline{\chi} \otimes \mathrm{Res}_{\widetilde{H}_2}^{\widetilde{G}} \mathrm{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \chi).$$

There are $\ell|H_2^{\mathrm{ab}}|$ linear characters χ on $\widetilde{H}_2$, and the dimension of the representations involved is the index $[G : H_2]$.

Remark 12.2.5 In case of a surface of constant curvature -1, Theorems 1.2.1 and 12.2.4 are equivalent using a twisted version of the Selberg trace formula as in [49, III.4.10].

Open Problem

In [28, Theorem 3.1], it was proven that two global function fields (corresponding to smooth projective algebraic curves over finite fields) are isomorphic if and only if there is a group isomorphism between their abelianised absolute Galois groups such that the corresponding L-series are equal. In [13], it is shown that one may restrict to unramified characters by admitting extensions of the ground field.

For negatively curved (e.g., hyperbolic) manifolds M_1 and M_2, the corresponding question is whether they are isometric if and only if there is an isomorphism $\psi\colon \mathrm{H}_1(M_1, \mathbf{Z}) \cong \mathrm{H}_1(M_2, \mathbf{Z})$ such that

$$L_{M_2}(\chi, s) = L_{M_1}(\chi \circ \psi, s)$$

for all $\chi \in \mathrm{Hom}(\pi_1(M_2), \mathbf{C}^*) = \mathrm{Hom}(\mathrm{H}_1(M_2, \mathbf{Z}), \mathbf{C}^*)$.

Open Problem

The geodesic length function defines the *marked length spectrum* $[\gamma] \mapsto \ell(\gamma)$ as a function from conjugacy classes in $\pi_1(M)$ to $\mathbf{R}_{>0}$. Croke and Otal [30] [74] showed that this characterises the isometry class of M in dimension two (in arbitrary dimension, this is an open conjecture of Burns and Katok, see [22, Problem 3.1]; a local version was proven in [44]). This motivates the question: what is the relation between the marked length spectrum and the information encoded in the L-series $L_M(\rho, s)$ where ρ runs over all unitary representations of $\pi_1(M)$ (this information might be called the "twisted length spectrum")?

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