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The spread of Hindu-Arabic numerals among practitioners in Italy and England (13th-16th c.): two moments of a European innovation cycle?

Introduction

As part of the debate on the 'great divergence', a growing stream of literature has stressed the role of culture and of 'useful knowledge' in the advent of modern economic growth (Mokyr 2005; 2011; 2017; Clark 2007; Pomeranz 2000). In parallel, a recent debate on the long-run determinants of the industrial revolution has brought forward the idea of a 'little divergence' of the economies of the Low Countries and England in the pre-modern period. This literature shows that these northern European areas experienced a persistent growth in terms of real wages, GDP per capita, urbanisation and agricultural productivity since the late middle ages, setting them on a trajectory that culminated with the industrial revolution (Malinowski and van Zanden 2017; Broadberry et al. 2015; Álvarez-Nogal and Escosura 2013; van Zanden and van Leeuwen 2012; Allen 2000; 2001; van Zanden 1999; De Vries 1984). A debate ensued investigating the determinants of economic growth in the pre-modern period. A number of hypotheses emerged, with scholars first highlighting a relevant role of structural factors, such as international trade and institutional change (Acemoglu, Johnson, and Robinson 2005; Allen 2003) and, more recently, the role of 'human capital' and of access to information (De Pleijt and van Zanden 2016; Dittmar 2011; Van Zanden 2009). This paper contributes to this literature by focussing on the adoption of Hindu-Arabic numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) in European commercial practices. By providing a mathematical tool which made it possible to handle rational numbers more effectively than Roman numerals, Hindu-Arabic numerals arguably played a key role in the development of the innovations of the commercial revolution of the 13th century, and in their diffusion.

It has been argued that it is possible to identify a continuous spread of Hindu-Arabic numerals from the commercial revolution of the 13th century to the little divergence of England and the Low Countries (Danna 2022; 2021). In this paper, I expand on this hypothesis by investigating the diffusion of advanced commercial practices and of other forms of practical knowledge in the initial and final contexts of this process. As can be seen from Table 1, data on the publication of practical arithmetic manuals show that late medieval Italy and early modern England were, respectively, the early mover and the late adopter of Hindu-Arabic numerals in vernacular contexts. In this paper, I expand on the complex social and economic fac-

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tors that help to explain this remarkable distance in time between the adoption of this mathematics in Italy and England. Moreover, this analysis makes it possible to identify a number of significant parallels between the two contexts, showing how technical change (in this case, change in commercial practices) was associated with social and cultural changes.

I will first focus on the innovations of the commercial revolution of the 13th century, which led to the emergence of new markets and of new forms of enterprise. These structural changes corresponded with the emergence of new institutions, which were developed to bring to scale the innovations of the commercial revolution. Thanks to its increasing social diffusion, the practical knowledge underpinning these new economic practices started to spill over into fields beyond commerce, contributing to knock-on effects in new social and institutional contexts, and thereby contributing to influencing a broader set of fields and practices. As we will see, it is possible to identify a similar sequence of events in both late medieval Italy and in early modern England. By focussing on the initial and final phases of the diffusion of Hindu-Arabic numerals in vernacular contexts, it is possible to observe the continuities that characterised the European spread of this practical mathematics.

Tab. 1. **Publication of practical arithmetic manuals in Italy, England, and Europe**

	Italy	England	Europe TOTAL	% Italy/Total	% England/Total
1250-1300	2		2	100	
1301-1350	19		22	86	
1351-1400	24		27	89	
1401-1450	60		63	95	
1451-1500	133		163	82	
1501-1550	119	9	366	33	2
1551-1600	126	46	645	20	7

Source: Danna 2022.

In highlighting the central importance of commercial practices, I am borrowing a methodology that has been used extensively by social historians of science. Literature in the history of early modern science has in fact demonstrated the key importance of practice as the site where a new collaboration between practical knowledge and academic natural philosophy became possible. Thanks to this exchange between theoretical and practical knowledge, the use of instruments, of mathematical tools, and of experiments gained a central role in European science (Zilsel 1942; Taylor 1954; Shapin and Schaffer 1985; Johnston 1991; Shapin 2010; Cormack, Walton, and Schuster 2017). The historian interested in the economic impact of knowledge can find interesting perspectives in this literature, which has extensively investigated the social circulation of knowledge and its potential to bring about social and cultural change. In this paper, I am framing the question of the economic role of socially distributed knowledge as a question in between economic history and socio-intellectual history. In doing this, I am following Mokyr's own call for a closer integration between economic and intellectual history (Mokyr 2017, xiii-xiv), to which I am adding the perspective and methodologies of the 'social turn'. If we are interested in the economic impact of knowledge, in fact, we should not only ask what kind of knowledge was developed at different moments of time, but we should also – and more importantly – ask who was acting on the basis of such knowledge, in which social and institutional context, and with what institutional and economic outcomes.

In this paper, I use this methodological framework to investigate the adoption of Hindu-Arabic numerals in commercial practices in Italy and England. As the positional numeral system provided a mathematical foundation to the innovations of the commercial revolution, studying its diffusion provides a proxy to trace the useful knowledge associated with advanced commercial practices. Relying on a number of accounting documents, manuals, diaries, and other sources, I use Hindu-Arabic numerals as a yardstick to reconstruct the development, transmission, and spillover effects of applied knowledge in Italy and England. In both cases, it is possible to identify a process of change which started off informally in commercial practices, was subsequently institutionalised, and eventually spilled over to other fields. This case study makes it possible to both investigate the role of practical knowledge in pre-modern economic development, and to observe a number of analogies between changes in practices, institutions, and the broader society that linked the commercial revolution to the little divergence. The evidence provided in this paper raises the hypothesis of a continuity in the transmission of practical knowledge from the commercial revolution to the onset of the little divergence, in what can be thought of as a coherent innovation cycle.

The paper is divided into three sections. Section 1 discusses the adoption of Hindu-Arabic numerals in late medieval and early modern Italy. Section 2 discusses the same phenomena in late medieval and early modern England in comparative perspective. Section 3 wraps up and concludes.

1. The adoption of Hindu-Arabic numerals and the culture of Italian abacus mathematics

The 'commercial revolution of the 13th century' was a watershed moment in European economic history (De Roover 1953, 80).[1] This profound change in commercial practices started in the Italian city states of the 13th and 14th centuries, whose merchants introduced a wave of key innovations that led to the emergence

[1] «By a commercial revolution I understand a complete or drastic change in the methods of doing business or in the organization of business enterprise just as an industrial revolution means a complete change in the methods of production […]. The commercial revolution marks the beginning of mercantile or commercial capitalism, while the industrial revolution marks the end of it».

of new markets and to new forms of enterprise.[2] Tuscan merchant-bankers developed the first European international commercial-banking companies, with branches located in the main hubs of European commerce (Tognetti 2015, 704; Spufford 1988, 253). The corporate structure of these firms was often based on an organisational innovation, i.e. the so-called *compagnia* (or 'partnership') contract (Goldthwaite 2009, 64-66; Padgett and McLean 2006).[3] In the early 14th century, Tuscan *compagnie* became the most important commercial enterprises in Europe, and Tuscan merchant-bankers consolidated a monopoly on European financial markets that lasted over a century (Tognetti 2015; Kaeuper 1973; Sapori 1926; 1944). The complex structure of these companies promoted a strong experimentation in accounting techniques, which led to the invention of double-entry bookkeeping (Padgett and Powell 2012; Parker and Yamey 1994; Nobes 1984; Lee 1977; De Roover 1956; Yamey and Littleton 1956). The international nature of these firms, moreover, led to the development of financial innovations, the two most important of which were the bill of exchange and insurance contracts (Ceccarelli 2012; Melis and Dini 1975). Bypassing the theological ban on interest rates, the bill of exchange made it possible to extend credit at interest, and it thereby revolutionised European financial markets (Adrian R. Bell, Brooks, and Moore 2017; Rubin 2010; Kohn 1999; Mueller 1997, 293-307; Spufford 1988; De Roover 1944; 1967).

Influences from the east had a key role in these developments (Abulafia 1997). The literature on the commercial revolution often overlooks the fact that the merchant-bankers of the commercial revolution were the first European economic agents to adopt Arabic arithmetic in their commercial practices. Before the adoption of the positional numeral system, Europeans used a number of reckoning technologies, such as tallies and the counter abacus (or casting counter) (Day 2013; Baxter 1989; Barnard 1916). Thanks to the principle of positionality and to the symbol for zero, Hindu-Arabic numerals make it possible to handle the fundamental mathematical operations relying on only ten figures (Ifrah 2000, 679). Moreover, the positional numeral system was adopted together with the symbolism for fractions. This was an important breakthrough, as Roman numerals allow to handle only a limited set of rational numbers. Roman 'fractions' were in fact limited to duodecimal fractions, based on the ancient subdivision of Roman units (Yeldham 1927; Maher and Makowski 2001). A system in which each fraction is represented with a different symbol is structurally limited in its handling of rational numbers. On the contrary, the positional numeral system provided a systematic way to write

[2] The concept of the 'commercial revolution' has also been used to identify a series of wider socio-economic phenomena occurring in this period (Lopez 1976). Following De Roover's original definition, however, the concept of the 'commercial revolution' used in this paper is limited to business methods.

[3] This new corporate structure made it possible to both make mid-term business strategies and to provide the necessary flexibility to exploit unforeseen business opportunities. In Florence, where it was non-notarised, the *compagnia* contract also made it possible to considerably reduce transaction costs. It is interesting to note that the mediation of notaries remained in use for a long time for transactions involving actors who were not part of the tightly-knit network of international merchants (Tognetti 2018).

fractions using only ten symbols, and made it potentially possible to handle *any* rational number.

These characteristics are key to understand the adoption of Hindu-Arabic numerals among merchants of the commercial revolution. A first source attesting the exchanges between the Arabic civilisation and the Italian maritime republics is the *Liber abaci* by Leonardo Fibonacci (c. 1228). This is an extensive *summa* of the mathematical knowledge of the time, combining the Arabic, Greek, and Latin traditions (Giusti and D'Alessandro 2020; Folkerts 2004). While previous Latin translations of Arabic mathematical works were not concerned with the practical applications of mathematics, Fibonacci's work showed explicitly how Arabic arithmetic could be applied to solving commercial and financial problems (Franci 2002; Giusti and Petti 2002; Goetzmann 2003; Biggs 2009).

Solving mathematical calculations related to exchange was of key importance in the development of the innovations of the commercial revolution. For example, the Italian monetary system of the time generated a need to handle these calculations. As the monetary system was trimetallic and the relative value of coins was continuously subject to change, Italian late medieval accounts are kept in the so-called moneys of account, or 'ghost money' (Lane and Mueller 1985, 3-13; Cipolla 1956, 38). This means that every transaction occurring in cash required that the accountant calculate the exchange rate between the actual currency and the unit of account used in the ledger. As the relative price of these moneys could have taken any value, calculating ratios between these units implied calculating with almost any possible rational number. Similar calculations concerned both merchant-bankers active in international trade and local merchants since moneys of account were used also by small shopkeepers (Goldthwaite 2009, 355). Moreover, problems of exchange were essential in the development of the bill of exchange, which was based on the calculation of exchange rates. One can wonder to what extent it would have been possible to develop these innovations relying on a numeral system – the Roman one – that was structurally limited in handling rational numbers.

Innovations such as double-entry bookkeeping, the use of a symbolic numerical notation, and the bill of exchange were shared among a community of users, as each party needs to recognise and trust the meaning and validity of their conventions, in a sort of network effect. In other words, a large subset of the mercantile community had to adopt the innovations of the commercial revolution for them to achieve their full potential. As a consequence, following the first experimental period in which practical arithmetic was transmitted informally, Italian city states developed an institutional setting that aimed at making a wide social diffusion of these practices possible, thanks to the creation of a specialised schooling system. While in the early middle ages basic education had been provided by church schools, between the late 13th and the 14th century Italian city states revolutionised the educational sector by founding schools that employed lay teachers, provided teaching in the vernacular, offered intensive curricula, and were often publicly funded (Black 2007, 186-87). As it was not possible to master the new commercial practices without being literate and numerate, these schools taught basic grammar and practical arithmetic to prospective merchants, artisans, and to members of urban society more generally. By providing a training in the fundamentals of the techniques of the commercial revolution, this schooling system made possible a wide adoption of the innovations of the commercial revolution in urban societies.

In most cities, pupils were taught practical arithmetic in specialised schools, the so-called 'scuole d'abaco' (Grendler 1989, 44-77; Arrighi 1966; Goldthwaite 1972).[4] Despite what their name might suggest, abacus schools taught both how to use traditional reckoning techniques (such as the counter abacus) and arithmetic with Hindu-Arabic numerals.[5] By the 15th century, abacus schools were founded in most central-northern Italian cities, and there is evidence of their presence in some centres of the south (Ulivi 2008). As they were often publicly funded, these schools were established as a deliberate policy to promote local useful knowledge. In some cases, this intention is explicitly attested in local archives. For example, in the 14th century the abacus school of Pistoia was founded on the premises that without practical arithmetic local merchants and artisans could not have exercised their activities appropriately («Sine scientia abaci mercatores et artifices utiliter et bene se exercere non possunt») (Grendler 1989, 22). In major cities, such as Florence and Venice, the demand for this kind of training was sufficiently strong that practical arithmetic schools did not need public support, as the fees paid by their students covered their financing (Black 2007, 545-611; Ulivi 2002b). At the same time, these fees were not particularly high, as abacus-school pupils came from a wide social spectrum, which comprised both families of the mercantile élite and those of small shopkeepers (Van Egmond 1977; Goldthwaite 1972).

The teachers who worked in these schools show interesting characteristics as well. While in the 12th century most teachers were clergymen affiliated to church schools, by the end of the 13th century masters active in Italian city states were mostly laymen whose work was embedded in the society of the commune (Black 2007, 191). Apart from working as teachers of practical arithmetic, abacus masters were often employed by the commune as expert reckoners and as surveyors (Goldthwaite 1972, 428; Ulivi 2002a; Grendler 1989, 22-23). Abacus masters had a distinct professional identity. For example, they created a separate membership in mercantile guilds, as by 1316 it is possible to find an *ars magistrorum abaci* in Milan and in Venice (Van Egmond 1977, 13). Moreover, abacus masters were keenly competitive. They competed to secure higher numbers of pupils, and challenged each other to solve mathematical problems of increasing difficulty, both within and across cities (Goldthwaite 1972; Ulivi 2015). In terms of social status, abacus masters generally belonged to a urban middle class, as their tax reports place them in a comparable position to that of shopkeepers, artisans and small merchants (Martines 1963; Van Egmond 1977, 8-14; Tognetti 1995).

The main sources available to study abacus mathematics are the so-called abacus manuals. These texts are written in Italian vernaculars, and their entire tradition

[4] It seems that in the Venetian state pupils were taught grammar and practical arithmetic in parallel, in what Grendler called «vernacular combination schools» (Grendler 1989, 22-23). Despite these regional variations, the mathematical training provided by these schools was substantially the same.

[5] The name «abacus school» is derived from the vernacular 'scuola d'abaco», a name that can be misleading. In late medieval Italian, «abaco» was used to indicate not just the reckoning board, but any reckoning device.

can be considered as a vernacularisation of the mathematics that was transmitted from the Arabic world to southern Europe across the Mediterranean, as exemplified by Fibonacci's 13th-century *Liber abaci*. The importance of the tradition of abacus mathematics is shown by the number of extant texts, as we know over 280 manuscripts and over 150 printed editions of abacus manuals written between the late 13th century and 1600 (Van Egmond 1980). As most authors of these texts were abacus masters, it is possible to hypothesise that these texts mainly circulated in the context of abacus schools. In some cases, we also know that they were passed down across generations of masters (Arrighi 1966; Ulivi 2004, 70-71; Murano 2015; Danna 2019). Abacus manuals mainly consist of long lists of worked examples (called *ragioni*), and their theoretical sections are kept to a bare minimum. These *ragioni* are often ordered according to their domain of application (for example: problems of exchange, of conversion, of division of profits and losses, etc.). This structure suggests that these texts were used as repositories of the mathematical problems explained by abacus masters while teaching. This is particularly true for early manuals, as there is evidence that with time these texts increasingly circulated in the hands of other practitioners.

The oldest abacus manuals were written at the end of the 13th century. Starting from that period, the increasing number of extant documents suggests that there was an acceleration in the production of these texts in the 14th century. Let us consider a paradigmatic example of one of these manuals: Paolo dell'Abaco's *Trattato di tutta l'arte dell'abaco*. The author (c. 1300-c.1366) was a prominent abacus master in 14th-century Florence who worked in the abacus school of Santa Trinita and wrote different works. Judging from the number of extant copies, Paolo dell'Abaco's *Trattato di tutta l'arte dell'abaco* was the most successful abacus manual of the century, with ten manuscripts attesting the work (Van Egmond 1980; Murano 2015; Danna 2019, 259). The manuscript Fond. Princ. II. IX. 57 of the National Library of Florence is probably the oldest copy, and may have been written by Paolo himself (Van Egmond 1977, 12). It is a manuscript of 188 paper folios which has been drafted by a single main hand, but also shows a number of subsequent hands that extended the initial text.

The manuscript opens with miscellaneous folios that include a medical section. The main text of the *Trattato* starts at folio 17r, with a short incipit: «Al cominciamento del nostro trattato sarae scritta e provata tutta l'arte dell'abacho gieneralmente ciò che ddire se ne puote».[6] The same folio reports a table of contents, which is followed by equivalence tables, by an introduction of the positional numeral system (fol. 23r), and of the fundamental arithmetical operations, first with integers and then with fractions (fols. 25-42). Following this theoretical preface, the manuscript presents a long section on the practical applications of mathematics which include conversions, calendrical calculations, payments, monetary exchange, alloying, division of profits and losses, barter, interest rates (both simple and compound), recre-

[6] «At the beginning of our treatise will be written and proved all the art of the abacus in all its aspects, i.e., everything it is possible to say about it».

ational problems,[7] and a section on practical geometry that shows several applications for land and architectural surveying (fols. 43-142). Folios 142-156 include a further section on medicine where astronomical calculations are used to identify the most appropriate time to give medical remedies. Folios 157-181 contain a miscellany of mercantile problems, some of which are solved with algebraic methods. The last folios (182-188) provide a further section on astronomy/astrology. While it is possible to find a broadly coherent structure in this manuscript, it is an open document characterised by numerous stratifications, rather than a systematic text. It is possible that this manuscript was among the books that Paolo dell'Abaco mentioned in his testament, and which were eventually given to a younger abacus master, Antonio Mazzinghi da Peretola. If this was the case, this manuscript would be particularly significant, as the several hands found therein would correspond to different generations of masters who used and integrated the text. Moreover, this manuscript would also provide direct documentation of the kind of mathematics that circulated in abacus schools (Danna 2019).

The mathematics found in these texts is an inherently applied knowledge. As can be seen from the contents listed above, the first and most important field of application was the economy. The text presents sections explicitly dealing with how to handle interest rates – a topic which at the time was highly debated in other social and cultural contexts – and uses the most advanced mathematics of the time – algebra – to solve mercantile problems. Moreover, this manuscript also shows that around the middle of the 14th century abacus mathematics was starting to be applied in fields beyond commerce. Paolo himself was active as a mathematical practitioner, as he worked as a consultant for the commune, advised on military matters, probably worked as a physician, and did experiments in astronomy using instruments. His work in astronomy and measurement was influential, as it seems that Paolo dal Pozzo Toscanelli resorted to some of Paolo's calculations when working on the problem of calculating longitude at sea (Murano 2015; Danna 2019, 255-66). Paolo dell'Abaco's testament also illustrates his interests in these disciplines, as it mentions books and a number of instruments pertaining to astrology and medicine whose detailed description has recently been identified in a Florentine inventory (Ulivi 1996; Murano 2015).

In other words, Paolo dell'Abaco's *Trattato di tutta l'arte dell'abaco* shows that around the middle of the 14th century abacus mathematics was an essential knowledge for anyone engaging in market exchanges. Moreover, it also shows that this kind of knowledge was starting to spill over into other fields, such as surveying, medicine, and astronomy. With time, these spillovers increased, with practical arithmetic and Hindu-Arabic numerals reaching further afield. This can first be ob-

[7] The so-called 'recreational problems' are miscellaneous exercises in which the reader is required to identify the correct resolution method among the several formulae used in these texts. On the one hand, these problems can be thought of as a form of amusement (similar to contemporary mathematical puzzles). On the other hand, these sections also played an educational role. While most sections of abacus manuals are subject-specific – and therefore consistently apply the same resolution methods – recreational problems trained the reader to identify the correct resolution method: an essential skill once the learner left the abacus school and faced the breadth of problems stemming from commercial practices.

served from evidence about who owned practical arithmetic manuals. While early manuscripts - such as Paolo's *Trattato* - were mostly circulating in the hands of abacus masters, with time there is evidence of an increasing diversity of owners of abacus texts. By the second half of the fifteen century, it is possible to find manuscript manuals that belonged to a Florentine carpenter,[8] a medical practitioner from Cremona,[9] a Bolognese rug trader,[10] and several artists active across Italian cities.

Artists provide a particularly interesting case, as Hindu-Arabic numerals were used in the development of a key innovation of Italian renaissance painting, i.e., linear perspective. This can not only be observed from their preparatory sketches - where Hindu-Arabic numerals and fractions appear regularly[11] - but also from the fact that some protagonists of Italian linear perspective wrote texts of abacus mathematics. For example, Piero della Francesca was not only one of the most prominent artists of his time, but also a writer of geometry and mathematics, as one of his first written works is a full-fledged abacus manual (Arrighi 1970; Dalai Emiliani et al. 2012). Architects were influenced by abacus mathematics as well. In Vasari's *Le vite de' più eccellenti pittori, scultori e architettori*, Filippo Brunelleschi is the first artist who is said to have attended an abacus school during his youth (Bellosi and Rossi 1986, 294). In his *Libri della Famiglia*, Leon Battista Alberti recommended that youngsters attend abacus schools (Romano, Tenenti, and Furlan 1994, 73-74). Hindu-Arabic numerals were also employed in naval architecture, as shown by the sketches for the construction of galleys copied – probably from original documentation of the Venetian arsenal – by Michael of Rhodes in his 15th-century manuscript (Long et al. 2009). By the 16th century, abacus mathematics had become so widespread in Italian urban societies that Girolamo Cardano (1501-1576) could claim in the incipit of his *Artis arithmeticae tractatus de integris* that practical arithmetic was essential for the economic success of any professional:

> Itaque factum est ut Oratores, Poetae, Iurisconsulti, Medici, Agricolae, Architecti, Exercituum duces ac demum omnes quotquot sunt operibus suis aut aliqua ratione illustres nihil ab hac [numerandi scientia] efficere possint, cum plures etiam hac sola industria locupletes evadant atque inter hos mensarii, collibytae, mercatores, quibus nemo vel promptius vel facilius ditatur. Laborant Medici Rhetoresque tota vita, ut quater sestertium (nec id frequentius) lucrifaciant, at hi quam paucis annis centies sestertium congregant (Cardano 1663, 10; 117).[12]

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8 Florence, Biblioteca Marucelliana, ms. A. c. s. 47.

9 Florence, Biblioteca Mediceo-Laurenziana, ms. Ash 1128.

10 Bologna, Biblioteca Universitaria, ms. 1612.

11 This is evident, for example, in Leonardo da Vinci's *Codice Atlantico* or Piero della Francesca's *De prospectiva pingendi* (Marinoni 1973; Dalai Emiliani et al. 2017).

12 «We have come to the point that orators, poets, jurisprudents, physicians, farmers, architects, army leaders and actually all those who are somehow reputed for their works or for some other reason cannot accomplish anything without that [the science of numbers or arithmetic]. In fact, many become wealthy only thanks to that activity [of counting numbers], among them bankers, money exchangers and traders. None grows rich more rapidly

While Cardano's incipit represents a rhetorical exaggeration, it shows that, by the 16th century, abacus mathematics had become an integral part of Italian urban culture. It was a knowledge that was widely shared across social strata and across cities, and it was applied to a variety of fields, such as mercantile practices, shipbuilding, architecture, and painting. In his reconstruction of the social history of pictorial style, Baxandall discussed at length the distinctively quantitative characteristics of 15th-century Italian artistic taste. He argued that this sensibility was so widespread in 15th-century societies that it is possible to identify a quantitative 'period eye' shared across the public. The reconstruction of this quantitative culture is based on Baxandall's discussion of abacus mathematics and abacus schools (Baxandall 1972). More recently, Goldthwaite has made similar claims concerning the practice of accounting for renaissance Florence (Goldthwaite 2015). If it is possible to claim that practices that made use of abacus mathematics (such as linear perspective and accounting) can be considered as forms of culture in renaissance Italy, then it should be possible to consider the arithmetic on which these practices were founded as part of this urban culture as well.

2. The spread of practical arithmetic in England and the culture of the 'mathematicalls'

Following its origins in the 13th-century Mediterranean, practical arithmetic spread to the rest of the European continent. The evidence extant on the publication of practical arithmetic manuals shows that the spread of this mathematics followed a south-to-north axis, with northern European countries adopting this mathematical knowledge well into the 16th century (Danna 2021; 2022). This evidence suggests that there may also have been a continuity in the diffusion of advanced business methods between the commercial revolution and the little divergence. The case of England is particularly interesting, as it sheds light on the varying social circulation of this mathematical knowledge. England was in fact an early adopter of Hindu-Arabic numerals in scholarly contexts, but it was a late adopter of the same mathematics in practical contexts. Interestingly, once Hindu-Arabic numerals started to circulate in English vernacular contexts, it is possible to observe a series of changes that are akin to the ones outlined above for the Italian case.

The first Latin translations of Arabic arithmetic works were carried out in 12th-century al-Andalus by scholars such as Robert of Chester and Gerard of Cremona (Ambrosetti 2008; Djebbar 2003; Folkerts 2001; 2003; Folkerts and Kunitzsch 1997). These translations were at the origin of the tradition of the Latin *algorismi*, i.e. brief mathematical primers introducing Hindu-Arabic numerals and the methods to

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and more easily than they! Physicians and orators work their whole life to spare a quarter sesterce (and they do not often succeed), whereas those people can amass hundreds of sesterces in very few years». English translation from (Omodeo 2017, 324). The *Artis arithmeticae tractatus de integris* was published as part of Cardano's *Opera omnia* in 1663, but its remarks concern the 16th century, as Cardano died in 1576.

use them to calculate the four fundamental operations. These texts spread in European monasteries and universities, as Hindu-Arabic numerals were used in fields where advanced calculations were needed, such as astronomy (Nothaft 2014). English scholars were quick to adopt these techniques, as a few 12th-century *algorismi* from England are extant. The catalogue of Medieval Libraries of Great Britain – which aims to record evidence on all extant medieval books that belonged to a British library in the middle ages – includes a number of references to *algorismi*.[13] Among these is one of the oldest copies of the so-called *Dixit Algorizmi*, which was probably written in the monastery of Bury St. Edmunds (Karpinski 1921; Vogel 1963; Crossley and Henry 1990).[14]

However, the circulation of these techniques remained limited to the learned classes. An interesting example of this is provided by the so-called *Dialogus de scaccario*. This is essentially a government memo, in which royal treasurer Richard FitzNeal explains how the English Exchequer was run. Fiscal revenues were recorded using a large exchequer to cast calculations with counters. When explaining how this reckoning tool was used, the author makes a revealing reference to Arabic arithmetic, excluding its use in this context:

> Quid ad calculatorem
> Huius autem hec est ratio secundum consuetum cursum scaccarii non legibus arismeticis. Memoriter, ut credo, dixisse me retines scaccario superponi pannum uirgis distinctum in cuius intersticiis numerales acerui collocantur. Porro calculator in medio lateris residet ut pateat omnibus et ut liberum habeat ministra mannus excursum (Amt 2007, 36-37).[15]

As a learned churchman, Richard FitzNeal was probably aware of the Arabic methods of calculation. However, to carry out calculations that involved laymen – like sheriffs, who were for the most part illiterate – he preferred to use a counter abacus (C. Johnson 1950, xxiv-xxxvi).

This hesitation to adopt Hindu-Arabic numerals in English practical contexts persisted for a long time, and it was not facilitated by the tense relationships between Italian and English merchants. For centuries, Italian merchants were the only practitioners active in England who mastered the techniques of the commercial revolution. By 1277, these merchants were active in English trade and finance, holding a dominant position in the export of wool, advancing loans to the crown, and administering the church's tenth in the kingdom. This is the period in which Edward I granted to the community of Italian merchants the area in east London

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13 This resource is accessible online at http://mlgb3.bodleian.ox.ac.uk

14 Cambridge University Library, ms. Ii.6.5. On the library of the abbey of Bury St. Edmunds in this period, see (Gransden 2007; 2015).

15 «But the system of this is according to the usual course of the Exchequer, not by the rules of Arabian arithmetic. You remember my saying, I imagine, that a cloth is laid on the Exchequer table ruled with lines, and that the coins used as counters are placed in the spaces between them. The Accountant sits in the middle of his side of the table, so that everybody can see him, and so that his hand can move freely at its work» (C. Johnson 1950, 24-25).

that is still known as Lombard Street. While royal and church officials relied on the financial services of the Italians, they did not know how their techniques worked. For example, in 1291 a dispute involving bills of exchange was heard by Exchequer officials. As – by their own admission – these officials were not competent in these practices, they preferred to appoint arbitrators from the community of Italian merchants to settle the case (Rawcliffe 1991, 100-101). This penetration of Italian finance in England reached its peak in the late 1330s, when the financial difficulties of the English crown determined the bankruptcy of some major Tuscan merchant-banking companies (Sapori 1934; 1955; Bell, Brooks, and Moore 2009; Bell, Brooks, and Moore 2009; Tognetti 2014).

Relations between Italian merchants, the English crown, and the community of English merchants remained difficult for a long time. Following the bankruptcies of the 1330s, the second half of the century was possibly a period in which the presence of the Italian community was reshaped, but by the 15th century Italian galleys were a key link for English international trade (Holmes 1960; Bradley 1992). Moreover, Italians constantly held a dominant role – if not a monopoly – on international exchange throughout the period. This created tensions which surfaced explicitly in the 15th-century *Libelle of Englyshe Polycye* (1436). The author of this pamphlet, probably an English merchant active in international trade, argued that Italian merchants bought in England on credit, sold for cash, and invested this money in bills of exchange purchased by English merchants (Tanzini and Tognetti 2012, 132). In his own words, they would «wipe our nose with our owne selve» (Holmes 1961, 201). The perception of a general economic downturn around the middle of the 15th century led to the introduction of stricter rules for 'alien' merchants, whose hosts were required to provide official reports of their activities (H. Bradley 2012). In 1456, a riot that broke out in London was triggered by a series of attacks against the Italian mercantile community (Bolton 1986). Similar tensions were still at play in the following century, for example with the Merchant Adventurers calling on the monarchy to adopt policies specifically aimed at penalising Italian merchants, as documented in the *Declaration of the misdemeanours of Italians* of 1561 (Ramsay 1973, 31).

While these tensions did not favour knowledge exchange between the two communities, there were also opportunities for – or necessity of – local cooperation, especially far from the rivalries of the court. An interesting example is provided by Southampton, which in the 15th century emerged as the main hub for Italian shipping in England, and had a comparatively large community of Italians merchants and sailors. Everyday life between the Italian and local English communities was characterised by a continuous negotiation between factional tensions and the benefits of collaboration (Ruddock 1951; 1946). Upon his arrival in Southampton in 1429-30, Luca di Maso degli Albizzi, captain of the Florentine galleys, reported in his diary that he was welcomed by William Soper, collector of the customs of the port and keeper of the king's ships. Soper hosted Luca di Maso in his country estate and showed him the main ships of the royal navy. A quick observation about the accounting methods used by Southampton customs officials opens the possibility that Luca di Maso exchanged some knowledge concerning bookkeeping with them (Mallett 1967, 259). It may be as a consequence of this kind of local knowledge exchanges that Southampton collectors were among the first English officials to use bills of exchange drawn on Italian companies in London to transfer tax revenues to the Exchequer (Rose 1982, 16-17). Moreover, the account books of Southampton show the use of Hindu-Arabic numerals by the 15th century, while English public accounts tend to employ Hindu-Arabic numerals at a much later date, starting from as late as the 17th century (Jenkinson 1926). For example, Hindu-Arabic numerals appear in the port accounts by Robert Florys of 1435-36, and in the Southampton steward's accounts dated 1457-58 and 1492-93 (Foster 1963; Thick 1995; 1999). Interestingly, the account book written under the direction of William Soper himself reports Hindu-Arabic numerals, providing what may be among the earliest appearances of the new notation in English public accounts.[16]

These documents suggest a pattern of adoption of Hindu-Arabic numerals in English vernacular contexts similar to what we have seen for Italy. They suggest both that this adoption was driven by the mercantile classes, and that the early adopters of the new notation were also among the early adopters of the techniques of the commercial revolution. This is confirmed by subsequent accounts kept by English merchants. For example, the letters of the Celys (1472-1488), a family active in the international wool trade, show that these merchants were acquainted with the new notation. Moreover, it seems that they were also among the first English merchants to use bills of exchange (Hanham 1975; 1985, 165). In the first half of the 16th century, the accounts of the Kytson family – active in trade with the Low Countries – show that these merchants used Hindu-Arabic numerals not just to report quantities, but also to calculate.[17] One of the first English account books to show an influence of double-entry bookkeeping – the ledger by Thomas Howell (1517-28), also active in international trade – reports Hindu-Arabic numerals as well (Connell-Smith 1951; Winjum 1971).[18] Around 1530, Richard Hill, a grocer from London active in the wool trade with the Low Countires, wrote a miscellaneous memorandum-book that includes a section on practical arithmetic and conversion tables for wool prices across the Channel.[19] Interestingly, very similar patterns of adoption of Hindu-Arabic numerals in accounting sources can be found in the Low Countries. Also in this context, it is possible to observe a piecemeal adoption of Hindu-Arabic numerals in accounting sources, with private accounts playing the leading role, and public accounts following at a distance of at least a number of decades.[20]

This chronology of the adoption of practical arithmetic suggests that the first circulation of Hindu-Arabic numerals in England was driven by merchants active in international trade, who were exposed to the influence of foreign commercial techniques and faced operations of exchange in their business practices. The first circu-

[16] Greenwich, National Maritime Museum (ms. PLA/18), f. 41v.

[17] Cambridge University Library, Hengrave Hall Mss Collection, Hengrave 78/1-4. The accounts belonging to Thomas Kytson (1529-1540) have recently been published in (Brett 2020).

[18] The ledger of Thomas Howell is kept at the Drapers Company in London.

[19] Oxford, Balliol College, ms. 354, fols. 186r-189v.

[20] I am grateful to Erik Aerts for sharing with me the early drafts of his study on the use of Hindu-Arabic numerals in accounting sources of the Low Countries, which is forthcoming in the *Journal of European Economic History*.

lation of this knowledge was most likely informal, but it generated an incipient demand for more structured training in practical arithmetic. As far as the available evidence makes it possible to reconstruct, this incipient demand that emerged among international English merchants is similar to the initial circulation of practical arithmetic among Tuscan merchant-bankers during the 13th century. As in the Italian case, this informal circulation of practical arithmetic progressively grew until it required some form of formal training in practical arithmetic. Once this demand was sufficiently strong, it was met by the foundation of the first vernacular schools of practical arithmetic in England.

England was the latecomer in this innovation cycle that started from the Mediterranean. Interestingly, practical arithmetic spread progressively from the south to the north of Europe not following the main routes of international trade, but through an inland network based on proximity. The key channels for this transmission were master-pupil relationships and the migration of skilled people (Danna 2021; 2022). The foundation of practical arithmetic schools also followed this path. Starting from the middle of the 15th century, we have the first evidence of practical arithmetic schools in the south of France and Upper Germany. Among the protagonists of this early phase were Nicolas Chuquet, who was active in Lyon in the 1480s, and Ulrich Wagner, who ran a school of practical arithmetic in Nuremberg in the second half of the 15th century (Benoit 1988; Flegg 1988; Gärtner 2000, 189). From these first cities north of the Alps, practical arithmetic schools spread to other centres. The first evidence of practical arithmetic schools in the Low Countries is from the early 16th century, with Antwerp as the hub for this kind of training (Kool 1988; Meskens 2013). From the north of the continent, practical arithmetic schools eventually crossed the Channel towards England.

The first masters of practical arithmetic active in London were in fact immigrants from the continent. In the preface to his *The welspring of sciences* (1574), Humfrey Baker complained that continental masters active in London in the 1570s still argued that their knowledge of practical arithmetic was superior to that of English masters:

> For when I perceaued the importunitye of certayne straungers not borne within this lande, at this present, and of later daies so farre proceadinge, that they aduanced and extoiled them selues in open talke and writinges, that they had attained such knowledge and perfection in Arithmeticke, as no English man the like: Truly me thought that the same report not onely tended to the dispraise our Countreymen in generall: But touched especially some others & me, that had travailed & written publiquely in the same facultye. For unto this same effecte they have of late painted the corners and postes in every place within this Citie with their peevishe billes, makinge promise, and bearinge men in hande that they coulde teache the summe of that Science in breife Methode and compendious rules suche as before their arrivall have not bene taught within this realme.[21]

[21] Humfrey Baker (1574), *The welspring of sciences*, fols. 6r-v.

This evidence suggests, therefore, that the practical arithmetic schools of England were distant descendants of the model of Italian vernacular schools, and close descendants of the first practical arithmetic schools founded in the north of the European continent.

This hypothesis is confirmed by the first manuals of practical arithmetic published in England, as these texts show strong influences from the continent. The first evidence of such a manual written in English is preserved in a single leaf kept in the British Library.[22] This is the frontpage of a manual titled *Art and science of arithmetic*, printed in 1526 by Richard Fakes, and the colophon introduces the work as an English translation of a manual written in French. Moreover, the printer himself was an immigrant from the continent, as he was a Norman by birth and his original name was «Faques» (Williams 2012, 166). The second English practical arithmetic is the anonymous *An introduction for to lerne to recken*, printed in 1536/37 and in 1539. This manual is not presented as a translation of a work written on the continent, but its language shows strong influences from French and Netherlandish. Also in this case, the biography of the printer is significant, as John Herford was a Fleming who started to operate as a printer in England in the 1530s. This evidence confirms the influence of sources from the north of the European continent for the origins of the English tradition of practical arithmetic (Bockstaele 1960; Richeson 1947; Williams 2012).

These early anonymous works were followed by manuals written by the first generation of English practical arithmeticians. The most influential among these was Robert Recorde (c. 1512-1558). Born into a family of Welsh middle-class merchants, Recorde studied medicine in Oxford and Cambridge, and worked as a physician and as a tutor of practical arithmetic. He was the author of a series of practical texts and was also active as an early mathematical practitioner, serving as an official in royal mints. Recorde died prematurely in 1558 in a debtors' prison, but the influence of his works lasted long after his death (Roberts 2016; Roberts and Smith 2012; Williams 2011). In 1543, Recorde published his *The Ground of Artes Teachyng the Worke and Practise of Arithmetike*. As suggested by its title, this was a foundational work which covered the essentials of practical arithmetic with Hindu-Arabic numerals. The text introduced the new numeral system, showed how to make calculations with integer numbers, and included a final section on calculations with the counter abacus and finger reckoning. In his preface to the reader, Recorde lamented the low consideration given to arithmetic in English society, and addressed his work to vernacular readers, who could use the manual to self-teach in case they could not afford a formal education:

> Yet am I bolde to put my selfe in presse with such abylyte as God hath lent me, though not with so greate conynge as many men, yet with as greate affection as any man to helpe my countre men, I wyll not cease dayly (as much as my small abylyte wyl suffre me) to endyre some suche thynge that shalbe

[22] London , British Library, Bagford (formerly Harleian 5919), item 178.

> to the enstruction, though not of learned men, yet at the leaste of the vulgare sorte […] but some wyll lyke this my boke above any other Englysche Arithmetike hetherto wryten, and namely suche as shall lacke estructers, for whole sake I have so playnely set forthe the examples, as no boke (that I have sene) hath done hertherto, which thyng shall be great ease to ye rude reader.[23]

The Ground of Artes had remarkable success, as it was reprinted over 40 times until 1699. Starting from the 1552 edition, Recorde included in the text calculations with fractions, the 'golden rule' of three, and a new set of practical problems. New editions appeared after Recorde's death, further updated by two prominent mathematical practitioners – John Dee (1561) and John Mellis (1582) – who also added a section on bookkeeping. Along these updates, the parts on calculations on the abacus and finger reckoning were dropped.

While *The Ground of Artes* was Recorde's most popular text, he also wrote other works. *The Urinal of Physike*, a work on medicine, was published in London in 1547. In 1551, he published *The Pathway to Knowledge*, the first English exposition of practical geometry. In 1556, he published *The Castle of Knowledge*, the first introduction to astronomy written in English, which also mentions Copernicus' new heliocentric theory. *The Whetstone of Witte* appeared in 1557, and, building on continental sources, provided the first English exposition of algebra. In order to make equations easier to handle algorithmically, in this work Recorde put forward the idea of using two parallel lines as a sign for equality ('='). This is the first known appearance of the equals sign, making *The Whetstone of Witte* a key text in the history of mathematical notation (Cajori 1928). As stated explicitly in the prefaces to his texts, Recorde sought to provide a foundation for a new English practical science. His works covered practical arithmetic (1543), medicine (1547), practical geometry (1551), astronomy (1556), and algebra (1557), often for the first time in the English language.

There is evidence that Recorde's works circulated among English practitioners. John Mellis, who was one the first writers of bookkeeping in English, wrote in the preface to his 1596 edition of *The Ground of Arts* that he had been introduced to practical arithmetic through Recorde's manual. He subsequently attended a vernacular school, became a practical arithmetic master himself, and eventually agreed to publish a new edition of Recorde's text following a growing demand for this kind of books. The text circulated among mariners. It is quoted in almanacs covering celestial navigation in the 1570s and in manuals of navigation with the compass in the 1580s (Williams 2011, 218-19). *The Castle of Knowledge* was included in the small library of the first ships that sailed in search of the Northwest passage. John Dee, who edited an edition of *The Ground of Artes*, himself spent time on board these ships, instructing mariners in practical arithmetic, geometry, and astronomy (Patterson 1951, 209-10). The majority of surviving copies of English practical arithmetic manuals presents marginalia, showing that these texts were actually used to learn

[23] Recorde (1543), *The ground of artes teachyng the worke and practise of arithmetike*, fol. iv v, fol. vii r.

arithmetic, often passing down across generations of the same family and sometimes also reaching women (Otis 2017).

Starting from the 1550s, a growing number of schools of practical arithmetic were founded. As we have seen, in the preface to his *The Welspring of Sciences* dedicated to the Merchant Adventurers (1574), Humfrey Baker complained about the unfair competition that English masters suffered from continental masters active in London. A surviving broadside advertisement from 1590 attests that Baker had a reckoning school in London, where pupils were instructed in practical arithmetic, bookkeeping, and international exchange. Starting from the late 16th century, arithmetic increasingly appears in schoolmasters' licences, and the number of schools teaching practical mathematics grew throughout England in the following century (Otis 2017, 472-76).

This increasing demand for training in practical arithmetic was met by the growing publication of practical arithmetic manuals. Apart from Humfrey Baker's *The Welspring of Sciences* – printed at least 20 times since 1562 – it is worth mentioning Dionis Gray's *The Store-House of Brevitie in Woorkes of Arithemetike* (1577) – which included rules and definitions in rhyming verses –, Thomas Masterson's *First Booke of Arithmeticke* (1592), and Thomas Hylles' *The Arte of Vulgar Arithmeticke* (1600). The influence of the continental tradition of practical arithmetic continued during this period, as signalled by a number of translations of foreign works. Among these are François Flory's *The Practice of Cifering* (1593) – an English translation of a French work published in Antwerp in 1577 –, Claes Pietersz van Deventer's *The Pathway to Knowledge* (1596) – a translation of the author's *Practicque, om te leeren rekenen, cypheren ende boeckhouwen* –, and Christian Wurstisen's *The Elements of Arithmeticke most Methodically Delivered* (1596) – translation of the author's *Elementa Arithmeticae* (1579).

Moreover, in this period practical arithmetic started to spill over to new fields, such as navigation – as we have seen – and to military matters. In 1579, Leonard Digges published his *An Arithmeticall Militare Treatise*, the first practical arithmetic by an English author to include a section on the military applications of arithmetic. Thomas Bedwell, another mathematical practitioner, was active in surveying and military engineering (Johnston 1991, 321-30). Following the attack of the Spanish armada in 1588, the interest in these applications spiked. Cyprian Lucar translated into English Tartaglia's treatise on gunnery, which was printed with the title *Three Books of Colloquies Concerning the Art of Shooting in Great and Small Pieces of Artillery* just a few months after the attack (1588). Moreover, a group of merchants and city officials sponsored the first series of public lectures on arithmetic held in London. The lecturer, Thomas Hood, addressed his classes first of all to navy officials, covering military and maritime applications of arithmetic (Johnson 1942). The lecture series was discontinued in 1594, but in 1598 Gresham College was founded, whose faculty promoted an alliance between scholars and practitioners (Johnson 1942, 95).

This growing number of publications concerning arithmetic and its applications, together with the diffusion of mathematical practitioners in a number of levels of English society, determined the rise of a 'culture of the mathematicalls', characterised by a shared 'perception of mathematics as a vernacular, practical, accessible, and worldly activity', that was often coupled by an extensive use of instruments (Johnston 1991, 341). These mathematical practitioners held a middling position between the world of the arts and crafts and that of the universities, and had an important role in making English natural philosophy turn towards mathematisation, instruments, and experimentation (Taylor 1954; Johnston 1991, 341). A striking example of this mathematics moving upwards in the social ladder can be found in Samuel Pepys' *Diary*. Educated according to the traditional curriculum of the liberal arts, Pepys was not trained in arithmetic during his studies. Therefore, when he was appointed clerk of the acts to the navy board in 1662, he started to study practical arithmetic taking lessons from a mariner of the HMS Royal Charles:

> 4th Up by five o'clock, and after my journall put in order, to my office about my business, which I am resolved to follow, for every day I see what ground I get by it. By and by comes Mr. Cooper, mate of the Royall Charles, of whom I intend to learn mathematiques, and do begin with him to-day, he being a very able man, and no great matter, I suppose, will content him. After an hour's being with him at arithmetique (my first attempt being to learn the multiplication-table); then we parted till to-morrow (Latham and Matthews 1996).

John Wallis, one of the founders of English algebra, recorded in his autobiography that he first came to know about arithmetic in 1631 not during his studies in Cambridge, but from the textbooks of one of his brothers, who attended a vocational school that covered the standard curriculum of practical arithmetic: «the Practical part of Common Arithmetick in Numeration, Addition, Substraction, Multiplication, Division, The Rule of Three (Direct and Inverse,) the Rule of Fellowship (with and without Time), the Rule of False Position, Rules of Practise and Reduction of Coins and some other little things» (Scriba 1970, 26). Wallis explicitly recorded that this kind of mathematics was more common among practitioners than in university circles:

> For Mathematicks, (at that time, with us) were scarce looked upon as Accademical studies, but rather Mechanical; as the business of Traders, Merchants, Seamen, Carpenters, Surveyors of Lands, or the like; and perhaps some Almanak-makers in London. And amongst more than Two hundred Students (at that time) in our College, I do not know of any Two (perhaps not any) who had more of Mathematicks than I, (if so much) which was then but little; And but very few, in that whole University. For the Study of Mathematicks was at that time more cultivated in London than in the Universities (Scriba 1970, 27).

Wallis portrays a society where practical arithmetic had been widely adopted by urban practitioners, and where it was starting to influence the practices of the learned elites. This evidence shows that the 'dramatic transformation' in the levels of numeracy of English society that occurred between 1500 and 1700 (Thomas 1987) was the outcome of a bottom-up process. This bottom-up diffusion started with the adoption of practical arithmetic and of the business techniques of the commercial revolution among English merchants, and subsequently spilled over into new fields, progressively influencing larger subsets of English society.

3. **Conclusion**

We have seen that the first adoption of Hindu-Arabic numerals in both Italian and English vernacular contexts was driven by the emerging need to solve calculations stemming from commercial practices. Among these calculations, particularly important were those relating to practices of exchange, as they implied the need to handle rational numbers – operations which were facilitated by the positional numeral system. The first merchants who adopted these techniques transmitted their knowledge informally. This initial informal circulation of practical knowledge led to an emerging demand for training in practical arithmetic. When this demand became sufficiently strong to sustain a more structured training in practical arithmetic, practical arithmetic schools were founded. These first schools were staffed by specialised masters, who often moved to the new centre from a centre of previous adoption. In turn, practical arithmetic schools acted as accelerators of the social circulation of this practical knowledge, which produced a number of spillover effects in fields beyond commerce.

It is possible to identify these steps in both Italy and England, with England following at a distance of around two centuries. Italian and English schools of practical arithmetic show similar characteristics, and English schools were designed following the model of Netherlandish and French schools, which in turn were distant descendants of Italian abacus schools. The profiles of the mathematical practitioners active in these schools also show similar features. Both in Italy and in England, these mathematical practitioners were lay masters, who taught and wrote in the vernacular, and mainly addressed their works to practitioners. Even if in England quite a few of them were university-trained, neither Italian abacus masters nor English mathematical practitioners were full members of the academic world. These mathematical practitioners conceived mathematics as an inherently applied knowledge. They wrote manuals of practical arithmetic that shared the same fundamental structure and transmitted substantially the same mathematical knowledge.

These similarities are exemplified by a comparison between Paolo dell'Abaco and Robert Recorde. Even though these two figures lived two centuries apart, it is possible to identify a degree of 'kinship' between their works. As discussed in Section 1, Paolo dell'Abaco's *Trattato di tutta l'arte dell'abaco* can be considered as a particularly good source to study the mathematics taught in 14th-century Florentine abacus schools. It covers the fundamentals of practical arithmetic, a wide range of commercial and financial problems, practical geometry, astronomical and calendrical calculations, medicine, and algebra. Considered together, Robert Recorde's works sought to provide a foundation for a new vernacular science in England. Recorde's works covered practical arithmetic (*The Ground of Artes*, 1543), medicine (*The Urinal of Physicke*, 1547), practical geometry (*The Pathway to Knowledge*, 1551), astronomy (*The Castle of Knowledge*, 1556), and algebra (*The Whetstone of Witte*, 1557), often for the first time in the English language. There are clear parallels between the subject matters of these works.

Both Paolo dell'Abaco and Robert Recorde were active as masters of practical arithmetic and as mathematical practitioners. Paolo dell'Abaco advised the commune of Florence on military matters, and Robert Recorde worked in royal mints. Moreover, the works of both authors show the initial spillover of practical arithmetic in fields beyond commerce in their respective societies. Both authors wrote on medicine, astronomy, and measurement, and their research was influential for navigational calculations. While covering a variety of fields, their works consistently give practice a central role in their epistemologies, as both authors were primarily concerned with the practical applications of their knowledge. Despite working at a distance of two centuries, Paolo dell'Abaco's and Robert Recorde's works show a degree of familiarity, as they seem to appear in similar moments of development of their respective societies.

More generally, the evidence provided in this paper shows that it is possible to observe a number of analogies between the Italian and the English spread of practical arithmetic in vernacular contexts. While in Italy this process took off in the 13th century, in England it started in the late 15th century. In both cases, this adoption was driven by an emerging need to handle advanced calculations stemming from commercial practices. After their first adoption in commercial practices, practical arithmetic spilled over to other fields, generating changes in wider social contexts. In both cases, these spillovers occurred bottom-up, and became sufficiently widespread to generate considerable cultural and social changes.

The comparative perspective developed in this paper is significant because Italy and England represent, respectively, the early mover and the late adopter in the spread of the body of practical knowledge. While a distance of around two centuries between the adoption of Hindu-Arabic numerals in Italian and English vernacular contexts may seem surprising, this paper has highlighted the complexity of the economic and social phenomena that underpinned the adoption of this mathematics. As we have seen, this transmission occurred through a complex interaction of informal contacts, emerging demand for quantitative skills, migration of skilled people, wider social diffusion, and institutionalisation of knowledge. We have focussed on the early and the late stages in the transmission of this knowledge to highlight the parallels between these phenomena even at the two furthest remote moments of the spread of this practical knowledge.

As we have seen, the adoption of Hindu-Arabic numerals among practitioners opened new avenues for the application of mathematics to a variety of fields. From this perspective, the diffusion of this mathematics provides a proxy to reconstruct the spread of a wider body of practical knowledge that was incrementally transmitted from the Mediterranean context to the north of Europe. The parallels highlighted in this paper raise the hypothesis of conceiving the onset of the little divergence as the final phase of an innovation cycle. This innovation cycle started with the commercial revolution of the 13th century, and was incrementally transmitted to the rest of Europe. This, in turn, opens new questions about how the social circulation of this practical knowledge differed between European areas.

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