Tomasz Jarmużek

Tableau Methods for Propositional Logic and Term Logic

Polish Contemporary Philosophy and Philosophical Humanities

Edited by Jan Hartman

Volume 20

PETER LANG

The book aims to formalise tableau methods for the logics of propositions and names. The methods described are based on Set Theory. The tableau rule was reduced to an ordered n-tuple of sets of expressions where the first element is a set of premises, and the following elements are its supersets.

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Bibliographic Information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available on the internet at http://dnb.d-nb.de.

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

The Publication is funded by Ministry of Science and Higher Education of the Republic of Poland as a part of the National Programme for the Development of the Humanities for the years 2018-2021,
Project No: 0257/NPRH6/H21/85/2017.

This publication reflects the views only of the author, and the Ministry cannot be held responsible for any use which may be made of the information contained therein.

NATIONAL PROGRAMME
FOR THE DEVELOPMENT OF HUMANITIES

ISSN 2191-1878
ISBN 978-3-631-83372-8 (Print)
E-ISBN 978-3-631-84653-7 (E-PDF)
E-ISBN 978-3-631-84666-7 (EPUB)
E-ISBN 978-3-631-84667-4 (MOBI)
DOI 10.3726/b18008

Peter Lang – Berlin · Bern · Bruxelles · New York ·
Oxford · Warszawa · Wien

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Preface

The aim of the book — according to its title — is to formalise the tableau methods for the propositional logic and term logic. By tableau methods we mean both ways of defining tableau systems and concepts which enable us to determine the occurrence of logical relationships within these systems.

In the book, we look into the problem of formal definition of concepts that are typical for such logical systems in which the occurrence of the logical relationships is examined by means of constructing the so-called tableau or proof tree.

Our considerations only apply to those systems that:

- firstly, are built for logics defined by bivalent semantics — the interpretation of formulas assigns to each formula either the value of truth or the value of false
- secondly, are systems of propositional logic or term logic.

Both conditions, as we will see later, have an important influence on the nature of the defined general concepts for the tableau methods. We write about establishing the overlapping relations of logical consequences in the tableau system because, in the presented approach, the starting point of the tableau system structure is the semantically defined logic for which the tableau system is constructed. We want to construct the tableau system in such a way that the relation of derivability determined by this system coincides with the relation of the semantic consequence for which the system was determined. In other words, we want the defined tableau system to be sound and complete to the initial semantics.

In many cases, it is of course possible to take the opposite approach — we can first define a tableau system and then determine the appropriate semantics for it. The methods presented in the book also make it possible to achieve this goal.

The formalisation of tableau methods described in this paper is based on the set theory — all concepts important for the theory of tableau methods are therefore defined as sets. The book analyses, among other things, the concepts of tableau rule, branch and tableau, offering their general and purely formal view. For example, the concept of a tableau rule is reduced to an ordered n-tuple of sets of expressions where the first element is a set of premises, and the following elements are the supersets of the set of premises — ways of drawing conclusions from it. At the same time, however, branches are defined as sequences of sets in which subsequent elements form the result of the application of the tableau rule. Finally, tableaux are sets of branches that meet certain additional conditions.

Still, the presented general and formal concepts do not interfere with the tableau concepts which are intuitive, standard and used in didactics. As we will show at the end of the book, the latter can be regarded as the application of formal concepts. The advantage of the formal approach is the possible generalization of conditions whose fulfilment by the tableau system is sufficient for its completeness and adequacy to the adopted semantics.

In Chapter One, the Reader will have the opportunity to get acquainted with the adopted strategy of formalisation of tableau concepts and an intuitive approach which is developed in the book.

In the following three chapters, three different cases are discussed in detail — different in terms of both syntax of the language in which we carry out the tableau proof and semantics. Contained in these chapters, structure of formal tableau systems for Classical Propositional Logic, Term Logic and modal logic **S5** is the starting point for the generalization of tableau methods which is done in the next chapter.

In Chapter Five, we describe the general theory of the structure of tableau systems and tableau concepts. The result of that chapter considerations is a theorem formulating sufficient conditions for the tableau system constructed with the given method to be adequate in relation to the initial semantics.

The final chapter describes the applications of the theory presented in Chapter Five. There, we will find the theory application to the construction of Modal Term Logic in the interpretation *de re* and the application to the theory of tableau systems for modal logics determined by the semantics of possible worlds. Another issue that we discuss in that chapter is the concept of the tableau system itself and the concepts that should be defined during its construction, as well as the resulting possibilities of examining the relations between the tableau systems. In the chapter, we also present a description of the transition from the abstractly understood concepts of tableau and branches to the standard, informal concepts of branches and tableau. This transition can also be considered as the application of general tableau concepts.

The author of the book would like to send greatest appreciation for the editorial reviewers of this book: prof. Andrzej Pietruszczak (Nicolaus Copernicus University in Toruń) and prof. Marcin Tkaczyk (The John Paul II Catholic University of Lublin). Moreover, the author would also like to express his gratitude to dr. Mateusz Klonowski (Nicolaus Copernicus University in Toruń), for the discussions and valuable remarks.

1 Introduction

1.1 Tableau methods

In the book, we look into the tableau methods. These methods are used to define logical systems in which through proofs — called tableau proofs — it is possible to show the occurrence of logical relations between sets of premises and conclusions.

Tableau methods in many ways constitute an interesting alternative to other methods of constructing logical systems. They are also interesting because the tableau systems have many advantages over other types of systems. Unfortunately, they also have their drawbacks. The aim of the book is to define tableau methods in such a way that for a certain class of tableau systems these drawbacks are minimized or, where possible, completely eliminated.

Let us briefly expose some features of tableau systems, comparing them with axiomatic systems.

One of the advantages of tableau systems is a fairly intuitive and simple mechanism of theorem proving. In most cases, knowing the tableau rules and the way they work, we can mechanically search for answers to the question whether a given formula is a logical consequence of a given set of premises. Such action does not require any particular ingenuity. Another advantage is the fact that if the answer is negative, most often — also intuitively — on the basis of an unsuccessful tableau proof we can build so-called countermodel, i.e. a model in which the premises are true, but the conclusion is false.

Unfortunately, the disadvantages of tableau systems are the complications that arise when trying to construct them precisely — there are many complex theoretical concepts. Obviously, we can use intuitive concepts of branch/track of proof, tableau/tree, open or closed tableau, complete tableau, etc. However, firstly, we are not dealing with a formal system, but with a preformal one, and thus potentially more or less burdened with serious logical errors. Secondly, it is difficult, if at all possible, to generalize our results by looking for metalogical dependencies between different tableau systems, as well as dependencies between classes of systems as for such actions we need general concepts whose specific cases occur within the construction of specific tableau systems.

In turn, axiomatic systems feature precisely defined basic concepts and these concepts can be generalized. For example, it can be assumed that each axiomatic system consists of a decidable set of formulas of a certain language For and a set of rules of proving $\mathbf{R}$. Since the system axioms can be described as zero-premiss rules, the general concept of *rule of proving* in the axiomatic system can be, for a given set of formulas For, defined as follows:

$R = \{\langle X,A\rangle : X \text{ is a subset of For}, A \text{ is a formula}\}$.

An axiomatic system is most often a ordered pair ⟨For, **R**⟩, where For is a decidable set of formulas, whereas **R** is non-empty set of rules of argumentation. When a rule is an axiomatic rule, it contains pairs $\langle X,A\rangle$, where X is an empty set, whereas A is a formula being introduced to the proof without premises.

With an axiomatic system ⟨For, **R**⟩ we can now define the general concept of proof of formula A on the basis of set of premises X. We shall state that A is *provable* on the basis of premises X iff there exists finite sequence of formulas $B_1, \ldots, B_n$ such that:

1. $B_n = A$
2. for any $1 \leq i \leq n$ at least one of two cases occurs:
   - $B_i \in X$
   - there exist: rule $R \in \mathbf{R}$ and such pair $\langle Y,C\rangle \in R$ that
     - $C = B_i$
     - Y is an empty set or for certain $m > 0$ and certain $0 < k_1, \ldots, k_m < i$, $Y = \{B_{k_1}, \ldots, B_{k_m}\}$.

With a defined axiomatic system and the concept of provability, we can proceed to the argumentation. However, unlike the tableau systems, in most cases it is not easy as it requires intuition, ingenuity and other skills. Moreover, it is usually difficult to move from an unsuccessful proof to a countermodel construction.

The above comparison could be summarised by saying that:

- axiomatic systems are simpler to define (which does not imply that a system with the required properties is easy to define, but the general definition of system and proof is simple) and can be defined precisely, however producing proofs on their ground and finding a countermodel is usually difficult
- tableau systems are more complicated to define and are usually not defined with sufficient precision (which makes it impossible to generalize tableau concepts and create metatheories of whole classes of tableau systems), but producing proofs in tableau systems is most often simple and in many cases enables finding a countermodel.

One of the primary goals of this study is to define the tableau concepts — concepts that seem necessary — that occur when defining various tableau systems precisely, and then try to generalize them within the scope determined by the use of tableau methods to construct propositional logic and term logic specified by bivalent semantics.

Therefore, we would like the tableau methods — similarly to axiomatic methods — to be based on certain constants and general concepts, and the construction of a given tableau system boiled down only to specifying those components that distinguish this tableau system from other tableau systems.

Such an approach will result not only in general and precise tableau concepts that can be used in the construction of different systems but also in proving metatheorems not of individual systems, but of entire classes of tableau systems. These metatheorems will apply to those properties of tableau systems that are independent of their specific features and concern all systems in a given class, constructed on the basis of general, developed tableau concepts.

By a tableau system — by analogy to the axiomatic system — we can understand a certain ordered triple of sets $\langle \mathsf{For}, \mathsf{Te}, \mathbf{Rt} \rangle$ where:

- For is a set of formulas of a given language
- Te is a set of tableau expressions — proof expressions of a tableau system being defined
- $\mathbf{Rt}$ is a set of tableau rules designed to produce tableau proofs.[1]

As distinct from the axiomatic system, the tableau system therefore features an additional element — a set of tableau expressions Te. In the specific case — such a case will be described in the next chapter — it may be so that $\mathsf{Te} = \mathsf{For}$. Most often, however, these sets are different because although we look for a logical relationship between a set of formulas and a given formula, we carry out a tableau proof based on tableau expressions Te for which a set of formulas alone is often insufficient.

Every axiomatic system $\langle \mathsf{For}, \mathbf{R} \rangle$ unambiguously determines a relation of derivability $\vdash_{\mathbf{R}}$ which occurs between such sets of premises and such formulas that on the grounds of this system the latter are derivable from the former. The axiomatic system can therefore be understood as pair $\langle \mathsf{For}, \vdash_{\mathbf{R}} \rangle$.

It should be similar for any tableau system $\langle \mathsf{For}, \mathsf{Te}, \mathbf{Rt} \rangle$, and thus it should unambiguously determine the tableau derivability relationship $\rhd_{\mathbf{Rt}}$ which contains such pairs of premises and conclusions $\langle X, A \rangle$ that in the system on grounds X we can tableau prove A. We call relationship $\rhd$ a tableau consequence, and more

1 It is an initial comprehension of a tableau system. We will not refer to it in further considerations. The proposed definition of a tableau rule will show that set Te is unnecessary for the characteristics of a tableau system since argumentation in that system does not go beyond the application of the tableau rules defined on the set of tableau expressions. In the last chapter, using the previously developed concepts, we will enunciate the concept of a tableau system, which will be more appropriate to the theory described in the book.

often *branch consequence*, showing that its occurrence can be identified with the existence of a tableau with specific properties.

We should therefore aim at such definitions of tableau concepts so that we can understand a tableau system as pair $\langle \mathsf{For}, \rhd_{\mathbf{Rt}} \rangle$. We have deliberately left aside the set of tableau expressions here Te as it will only serve as a domain for defining tableau rules, and as a consequence of creating proofs — since the tableau expressions are included in tableau rules.

Therefore, the problem of defining a tableau system should be reduced to defining a set of formulas, defining a set of tableau rules and the relationship between formulas and rules. The remaining tableau concepts — as in the case of the concept of provability in the axiomatic system — should be special cases of more general tableau concepts. All the more so as the concept of tableau proof is not a simple one, but — as we will show — dependent on many simpler concepts.

Hence, we want to define tableau concepts in such a way that tableau systems are simpler to define and at the same time more precise (which will enable the generalization of tableau concepts and the work on metatheories of entire classes of tableau systems), while maintaining the intuitiveness and automaticity present in producing proofs in tableau systems and, where possible, retaining the property of finding a countermodel.

The logic can also be perceived from the semantic side, related to the interpretation of expressions. Axiomatic systems are customarily treated as an expression of a syntactic approach to logic. Without referring to the meaning of the inscriptions used, axiomatic systems specify how to decide the occurrence of relation $\vdash$ by transforming expressions solely in terms of their structure and shape.

In the book, we approach the problem of tableau systems in an analogous manner, but the starting point is usually (although not necessarily so) a logic defined in a purely semantic way. It can be understood as pair $\langle \mathsf{For}, \models \rangle$ where For is a set of formulas of given language, and $\models$ is the relation between sets of formulas and individual formulas, defined in a standard way based on the interpretation of set For.

Thus, having such a semantically understood logic $\langle \mathsf{For}, \models \rangle$, we want to define an tableau system (often for the same purpose we define an axiomatic system) $\langle \mathsf{For}, \rhd \rangle$[2] which, by transforming the tableau rules themselves, would allow us — where possible — to determine the occurrence for a given pair of relationships $\models$. To demonstrate that relations $\models$ and $\rhd$ coincide (i.e. $\rhd = \models$) would thus be not

2 In the designation, we skip set of rules **Rt** since definition $\rhd$ will depend on a set of tableau rules.

only a correctness measure for the constructed tableau system, but it also is the purpose of defining the tableau system.

Our approach to tableau systems is therefore syntactic. We treat them as systems defining the code of transforming expressions without direct reference to the semantic concepts. This may seem controversial, especially in those cases (i.e. in the vast majority) where $\mathsf{For} \neq \mathsf{Te}$; hence, when we use expressions that are different from the formulas in the proofs, by conjecture they encode some semantic data/privileges. This controversy can be dismissed by referring to two arguments.

The first one is hinges upon the following fact. When defining, in some tableau system, the relation of branch consequence $\rhd$, with semantically determined logic $\langle \mathsf{For}, \models \rangle$, it is not at all self-evident that these approaches coincide. This requires additional proofs for two theorems:

- completeness theorem $\models \subseteq \rhd$
- soundness theorem $\rhd \subseteq \models$.

So despite different definitions of logic and the tableau system and mutually independent definitions of both relations, we are seeking a proof that these relations are extensionally equal.

Finally, it is also possible to define a tableau system in which relation $\rhd$ does not coincide with the output semantic relation $\models$. Relation $\rhd$ and, consequently, the tableau system that determines it, are therefore something different than the logic and relations of semantically defined consequences. We can then try to improve the defined tableau system, or look for a different semantics — both cases show that we are dealing with something different than the initial, semantically defined system.

Let us put forward the second argument proving that the fact that tableau expressions Te encode certain semantic properties does not necessarily lead to the conclusion that the tableau systems are not syntactic ones. After all, in the case of axiomatic systems, the axioms and rules also encode various semantic properties of functors in a sense, even though it is not directly visible. Of course, we can say that in many cases there exist various axiomatic systems (defined by disjoint sets of rules) corresponding to the same semantic relation $\models$, which means that the semantic intuitions present in the rules and axioms do not have to be identical. But it is also the case — as we will show — in many cases of the tableau systems. There may be more than one tableau systems that are equivalent to the same semantically determined logic, which would have to imply that also in this case the semantic intuitions present in their construction are not identical.

Regardless of how we evaluate tableau systems, we define here the relations of tableau provability in a different way than is the case in the semantic determination of consequence relations. For we do it in an autonomous way, although we undoubtedly transfer some semantic intuitions from semantically defined logic.

In this study, we will proof the theorems of completeness and soudness, leaning towards the view that we are dealing with a syntactic approach. However, a different orientation in this matter will not change the fact that we are considering something different than a purely semantically defined logic, nor will it invalidate the formal concepts presented in the paper.

1.2 Terminology and problems in the book

In each chapter of the study, we consider a case of a tableau system or make generalisations. The selected systems have different properties in some aspects which allows us to seek sufficiently general definitions of the tableau concepts so that all the systems described and all tableau systems similar to them — the systems described represent, due to their unique features, the classes of similar systems — are special cases of these definitions.

1.2.1 Study schedule and goals

In this study, we deal with the formalization of tableau methods. The result is a certain method of tableau systems construction, which corresponds to an intuitive approach, but apart from the precision we have already mentioned, it gives us additional benefits.

In Chapter Two, we will define the tableau system for Classical Propositional Logic as the basic case. As in work we focus on the economy principle when applying technical measures, we shall define this system based on set of tableau expressions Te equal to set of Classical Propositional Logic formulas For, although the tableau system for the Classical Propositional Logic could be defined based on set $\mathsf{Te} \neq \mathsf{For}$. Formally, this would be of course a different tableau system.

Hence, the case of a tableau system for the Classical Propositional Logic described in that chapter seems to be a borderline case. The system also features the property that any branch that begins with a finite set of tableau expressions can be extended to a branch that can no longer be governed by tableau rules and which is a branch of a finite length. We will call this feature a finite branch property. However, it is not a feature specific to all the tableau systems described in the book.

At the end of the chapter, we show that we actually have defined a tableau system for the Classical Propositional Logic because the tableau provability, which it determines, coincides with the consequence relation determined by the valuation of formulas.

In Chapter Three, we consider the tableau system for the simplest term logic (which we call Term Logic), i.e. the logic of classical and non-modal categorical propositions, allowing empty names. In the case of this tableau system for Term Logic, the set of formulas is actually contained in the set of tableau expressions. The proof language is therefore more complicated than in the previous case.

The issue of tableau concepts looks analogous, especially the key for proofs — as we will see — concept of the maximal branch. Also in this case, a maximal branch is always obtainable from a finite set of tableau expressions.

At the end of Chapter Three, we show the completeness and consistency of the tableau system with the normally and semantically defined consequence relations in Term Logic.

Chapter Four deals with another borderline case — but this time it is a borderline case on the opposite side to the case of Classical Propositional Logic. This is because we consider modal logic **S5**, by defining a set of tableau expressions in such a way that it neither coincides with the set of formulas of logic **S5** nor is it its proper superset — both sets are therefore disjoint. Also for this reason, the case is the most general one as the previously considered sets of expressions could also be defined, somewhat artificially, in such a way that the set of formulas would be disjoint from the set of tableau expressions. In the tableau system for logic **S5** those two sets are naturally disjoint.

We put *also* because in Chapter Four there emerges a problem with infinite branches. The systems described previously featured the property of a finite branch, which is not the case for the presented tableau system for **S5**. Therefore, it may happen that when constructing a branch and consequently a tableau which start with a finite set of expressions, it is not possible to finish them as some sequences of application of the rules become cyclical.

The lack of a finite branch property forces changes in some tableau concepts. This applies in particular to the concept of maximal branch and derived concepts. So, the tableau concepts defined in previous chapters become special cases of tableau concepts for systems that do not feature the property of a finite branch. The leading change is the generalisation of the concept of maximal branch. Intuitively, the maximal branch is the one to which we can no longer apply any rules. In previous cases, however, this meant that the maximal branch was of a finite length. It does not have to be the case this time. A maximal branch can be infinite, although not every infinite branch is a maximal one.

Despite the fact that, in our study, we do not focus on the issue of decidability in the tableau system, it is worth noting that the systems that feature the property of a finite branch, are decidible. For theoretically, always in a finite number of steps it is possible to construct a complete tableau for them — closed or open one, thus answering the question whether a given formula is or is not tableau provable on the grounds of given premises.

In the case of tableau systems that do not feature a finite branch property, these systems may not be decidable. So, although we prove that they are complete and sound in relation to the initial, semantically defined consequence relation, there does not have to exist a way of constructing an infinite tableau. Hence, starting from branches and tableaux as finite and potentially always constructible objects, and generalizing the tableau concepts, we come to theoretical branches or tableaux that are likely to exist, despite the fact that as setwise infinite objects, they cannot be defined by calculating their elements. However, as we said, in our study we do not deal directly with the problem of decidability in tableau systems, although examination of this problem is partly conditioned by the precision of the tableau concepts that we are working on.

In Chapter Five, we summarize the considerations of the previous chapters, defining general tableau concepts for propositional logic and term logic which cover all previous cases. We also show certain properties which, by virtue of these concepts, occur for the tableau systems defined by the presented method.

Finally, in Chapter Six, we apply general tableau concepts to a new case, showing how much their use simplifies the construction of a tableau system compared to previous cases where the tableau systems were defined from the simplest concepts. In this chapter we consider some modal term logic with the interpretation of modality *de re*. Since we already have tableau concepts for this type of systems, the construction of the tableau system and demonstrating that it is complete and consistent in relation to the semantics presented boils down to the statement of a few basic facts. We also present the application of general concepts to the case of modal logics determined by the semantics of possible worlds, showing how tableau concepts allow us to express conditions sufficient for a given tableau system for some modal logic to be sound and complete in relation to its semantics.

The formalisation of tableau concepts, which we propose, and its effects on the construction of tableau systems produce the main result of our study.

1.2.2 Terminology and issues in the book

Every time, before we move on to the construction of a tableau system, we start from a semantically defined logic, i.e. from ordered pair $\langle \mathsf{For}, \models \rangle$, where For is a set of formulas, and $\models$ is a relation of semantic consequence, defined in a standard way on the basis of set of all interpretations of set For.

For such understood logic, we construct a tableau system which — as mentioned before — by analogy to the axiomatic system — can be understood as a ordered triple of sets $\langle \mathsf{For}, \mathsf{Te}, \mathbf{Rt} \rangle$.

Such a triple, through the application of the definitions proposed by us further, unambiguously determines ordered pair $\langle \mathsf{For}, \rhd \rangle$, where $\rhd$ is a tableau provability/branch consequence relation, defined by a general definition, but in any case based on a predefined set of tableau rules **Rt**.

Now, let us notice that having initial pair $\langle \mathsf{For}, \models \rangle$ and observing the rules of defining a tableau system, which we will present below, we can usually define at least several different tableau systems which differ in terms of set of expressions Te or set of rules **Rt**, or both sets at the same time. Despite the fact that these are formally different systems, they can define the relations of tableau provability identical in scope to relation $\models$.

The above remark explains why we will be writing e.g. a tableau system for Classical Propositional Logic rather than a tableau system of Classical Propositional Logic. Potentially, there are many tableau systems for Classical Propositional Logic, whose relations $\rhd$ coincide with consequence relation $\models$ of Classical Propositional Logic.

So, in each case, we write about a tableau system for given logic rather than a tableau system of given logic, taking account of the multitude of possible but equivalent tableau systems.

Although in the following chapters the starting point will always be some semantically defined logic $\langle \mathsf{For}, \models \rangle$, the construction principles for the tableau system presented by us can be used for the construction of a tableau system regardless of any pair $\langle \mathsf{For}, \models \rangle$. For a tableau system defined in this way, it is only then possible to search for the appropriate semantics.

Within the said method, a set of formulas For is an essential initial ingredient for the construction of a tableau system.

With a fixed set For, we define a set of tableau expressions Te, that is a set of expressions on the basis of which we carry out the tableau proofs. Of course, to each formula corresponds at least one tableau expression that represents it in the tableau proof. In the chapters where we present a full description of tableau systems for selected logics, we define set Te based on the previously mentioned principle of economy — we choose the option that seems to be the simplest. Then we define the concept of a tableau inconsistent set of expressions, i.e. a set which should be searched for in each track of proof/branch of the tableau proof.

Next, on at least two-element Cartesian products of set of all subsets of the set of expressions, we define the tableau rules. The tableau rules of a given system are therefore sets of two or more element, ordered n-tuples which comprise the subsets of set Te.

In this approach, we apply tableau rules to sets, and consequently we get one or possibly more sets, if the application of rule to a given set is possible at all. Already in Chapter Two we present the mechanisms that we generally impose on the rules to avoid redundant — for several reasons — and unproductive applications of the tableau rules. We consider these mechanisms to be an important feature of the formalisation adopted.

The effect of defining tableau rules is always set of tableau rules **Rt**. By analogy to axiomatic systems, we treat determination of set **Rt** as an axiomatization of some logic by means of tableau rules. On given set Te it is often possible — as we mentioned before — to define different sets of rules that produce the same effects in terms of consequences of the tableau system. In this study we will provide examples of different axiomatizations by means of tableau rules of the same semantically determined logic. However, since the proof that two axiomatizations are equivalent without reference to the metatheory of tableau systems seems complicated, we examine only one axiomatization — the one that we consider natural and the simplest.

Although the tableau rules are intended to determine the tableau system unambiguously, without further tableau concepts it is either not clear how the rules determine further concepts or it is only intuitive. Therefore, in three subsequent chapters we define all tableau concepts, and in the next one we define general tableau concepts, which in combination with specific tableau rules sets automatically determine the tableau system. If the tableau rules have certain properties, the tableau system they determine is complete and sound with respect to the initial and semantically defined relationship of consequence $\models$.

In the presented approach, a set of tableau rules is a starting point in the process of defining more complex concepts of the tableau system. From a heuristic point of view, a set of rules is the most important component of the tableau system, but in order to precisely describe the concept of the tableau proof, we need further concepts that are to be general and, in individual cases, to define the initial set of tableau rules.

By applying the rules, we create proof tracks/branches. In the book, we use the term of *branch*. A branch is such a sequence of sets that each set is basically contained in its successor (except the last one — if it exists). Branches are created by using tableau rules that allow the last set belonging to a branch to be expanded in at least one way, so each branch element contains all the expressions present in the earlier elements of the branch.

Branches the last element of which contains a tableau inconsistent set are called *closed branches*. Such branches can no longer be extended because in our construction there are no rules whose premises would contain tableau inconsistent sets. Normally, branches that are not closed are called *open branches*.

There is one more, important from the viewpoint of the theory being developed, type of branches — maximal branches. Intuitively, the *maximal branch* is the one to which no more tableau rule can be adapted. Closed branches form a special type of maximal branches. However, maximal branches can also be open, even infinitely long, although, as we will see, the infinity of branches is neither a necessary nor a sufficient condition for its maximality.

A tableau proof is made up of branches that begin with the same set of expressions. We want to check whether formula A in a given tableau system is a consequence of set of formulas X. To this end, we construct branches which begin with expressions that are tableau equivalents of the formulas from set X and with the tableau equivalent of a formula contrary to formula A.

Hence, when constructing branches, we start from an assumption which is an indirect assumption. We develop branches in any way we want since the rules do not allow us to expand the initial sets in a way that would make the sequence of expressions cease to be a branch. Each non-maximal branch, which begins with a finite set of tableau expressions, can be extended — at least theoretically in the case of infinite branches — to a maximal branch, and each maximal branch is closed or open. In systems that do not have the property of a finite branch, we show that each branch beginning with a finite set of expressions can be extended to a maximal branch. In general, we also indicate conditions which, when met by set of tableau rules **Rt**, are sufficient to always obtain a maximal branch from a finite set of expressions. If now each maximal branch is closed, we conclude that formula A is *a branch consequence* (or *a tableau consequence*) of set of formulas X — for short $X \triangleright A$.

Sometimes we may be interested in the occurrence of branch consequence relation between infinite set of formulas Y and formula A. Then, an intuitive component appears in the proof — we have to show that there is such finite subset X of set Y that $X \triangleright A$. Although choosing the right set X is a non-formalised activity, we want the demonstration that $X \triangleright A$, where the property of a finite branch occurs, to be mechanical. This also distinguishes our approach to the tableau systems from other approaches. We do not restrict the branch consequences only to the finite sets.

It may even be practically impossible to check if there is a branch consequence relation in a given tableau system and in a given case, due to the number and complexity of branches. In order to reduce this complexity and make checking more practical, we introduce the concept of tableau/tree proof. In the study, we will use term *tableau*, although our concept of a tableau is far from the original one, i.e. a tableau with rows and columns. However, as it is customary to use this term, we will call the tableau proof a tableau.

As we have stated, the concept of tableau is intended to simplify checking whether there is a branch consequence in a given case. *Tableau* is always such a set of branches beginning with the same set of expressions that the existence of several different branches in a tableau implies the existence of tableau rules that led to the creation of these different branches.

Among tableaux we can distinguish *complete tableau* that, to put it intuitively, contain everything we need. Each branch belonging to such a tableau is a maximal one, and what is more, adding another branch to the complete tableau causes the obtained object not to be a tableau. We are also considering the issue of so-called *redundant branch variants*, i.e. branches that may or may not belong to a complete tableau.

A tableau that demonstrates the occurrence of branch consequence is a *closed tableau*, i.e. a complete tableau in which each branch is closed. It is the construction of a *closed tableau* that we consider to be a tableau proof, while the equivalence of a closed tableau existence with the occurrence of a branch consequence can be considered as a test of a good definition of a tableau system.

The concept of branch consequence is based on the inclusion of set of maximal branches in the set of closed branches. Most often the construction of a single closed tableau means the selection of a sufficient and appropriate subset of maximal branches that are closed. Theoretically, therefore, it is possible to construct many closed tableaux, but when constructing a tableau system, we should strive to show that one such tableau will suffice to recognise the occurrence of the tableau consequence. This postulate is our guiding principle in further studies on tableau systems.

On the other hand, it may happen that we will construct a complete tableau, which at the same time will be *open*, i.e. contain at least one maximal and open branch. The existence of such a tableau is a proof that the tableau consequence does not occur. The description of a complete and open tableau forms the basis for the construction of a counter-model, i.e. it allows to proceed to purely semantic concepts.

In the description of the method of construction of tableau systems we do not take account of the issue of effectiveness or length of proof. We do not aspire to create systems in which the proofs will be as short as possible and whose computer implementations will be as effective as possible. Our main goal is to formalise the tableau methods, and consequently to precisely define the tableau concepts. Despite this, both in the way we formulate tableau rules, consider the redundant branch variants or treat the existence of an appropriate tableau as a measure of the branch consequence occurrence, we are guided by a certain need for economy and efficiency. However, its implementation is contained in the very tableau concepts, rather that in the strategy of applying tableau rules or the strategy of carrying out a proof — in these aspects we leave room for discretion.

Our construction of the tableau systems is fully related to the set theory:

- the tableau rules are sets of ordered n-tuples, which comprise the sets of tableau expressions
- the branches are sequences of sets that are monotonic due to the inclusion
- tableaux are sets of branches.

When in the subsequent chapters considering three different cases of logic, for which we define tableau systems with this method, we aim to determine general tableau concepts which would reduce the construction of a tableau system to define a set of tableau rules according to a given scheme and to specify several concepts in more detail. The other concepts we have described would be constant and general.

We also want to simplify the construction of the tableau system, which would be complete and sound in relation to the semantically established logic, to check if there occur several system-specific facts.

Thus, in the book we present the formalisation of tableau methods, which not only brings a certain level of precision, but also leads to generalizations which allow to simplify the construction of the tableau system for activities that require intuition. On the other hand, in the case of determining the completeness and soundness with respect to semantics, they require only those facts that are specific to the system under examination.

Due to the fact that we usually deal with tableau methods which are applied more intuitively, further considerations contained in the book may raise the question whether the formalism presented in the book fits into the tableau methods, or whether it is something different, but similar.

Since the transition from intuitively understood tableaux to formalism proposed in the book and, in some cases, the transition from the concept of tableau in the proposed approach to the standard tableau is quite straightforward, the approach described in the book seems to be no less general than the standard approach. Apart from its already mentioned advantages, it contains all these intuitive ways of constructing tableau systems for propositional logic and term logic based on bivalent semantics.

1.3 Notations and concepts of the set theory

In this book we define all tableau concepts using the concepts of the set theory. We use standard denotations on the relations between sets or their elements: $\in$, $\subseteq$, $\supseteq$, $\subset$, $\supset$, $=$, $\neq$, etc. Also in a standard way we denote operations on sets $\cap$, $\bigcap$, $\cup$, $\bigcup$, $\setminus$, and $P(X)$, where X is a set, while $P(X)$ is a set of all subsets contained in set X. An empty set is denoted as $\varnothing$.

Sometimes, in metalanguage, we also use quantifiers $\exists$, $\forall$ and classical constants: $\Rightarrow$, $\Leftrightarrow$, &, etc.

With established set X, we shall state that its subset $Y \subseteq X$, which meets certain established condition (a), is a *maximal* set among subsets X which meet condition (a) iff there is no subset $Z \subseteq X$ such that Z meets condition (a) and $Y \subset Z$. On the other hand, with established set X, we shall state that its subset $Y \subseteq X$, which meets certain established condition (a), is a *minimal* set among subsets X which meet condition (a) iff there is no subset $Z \subseteq X$ such that Z meets condition (a) and $Z \subset Y$.

In the study we will very often use numbers from the set of natural numbers as well as the set of natural numbers itself $\mathbb{N}$ (without zero). Sometimes we will refer to the concept of set cardinality, i.e. its cardinal number, defining it with the use of notation $|X|$, where X is set. However, the cardinalities of the sets under consideration will never be greater than $|\mathbb{N}|$ cardinality of the set of natural numbers. When saying that given set X is *finite*, we will mean that $|X| < |\mathbb{N}|$.

We also often define multifold Cartesian products $X_1 \times \cdots \times X_n$, where $n \in \mathbb{N}$, $n \geq 2$ and $X_1, \ldots, X_n$ are sets. Elements of product $X_1 \times \cdots \times X_n$ are ordered n-tuples $\langle a_1, \ldots, a_n \rangle$, where $a_1 \in X_1, \ldots, a_n \in X_n$.

We also use the concept of function, using different letters or symbols, depending on the needs, to denote a function, e.g. $f : X \longrightarrow Y$.

We shall state that function f is *injective* or it is *injection* iff for any $x, y \in X$ it is the case that if $x \neq y$, then $f(x) \neq f(y)$. In turn, f is function *onto* iff for each $y \in Y$ there exists such $x \in X$ that $f(x) = y$. A function which is injective and *onto*, shall be called *bijection*.

We know that when f is a bijection, then there exists precisely one *inverse function* $g : Y \longrightarrow X$, i.e. such that for any $x \in X$, $g(f(x)) = x$. Function g will be denoted f^{-1}.

One of the fundamental concepts defined and used in the study is the concept of branch. With an established set of tableau expressions Te, a branch will take the form of function $\phi : K \longrightarrow P(\mathsf{Te})$, where $K = \{1, 2, 3, \ldots, n\}$ or $K = \mathbb{N}$, and ϕ meets certain additional conditions that are specified in the tableau theory, discussed further. Assuming that the cardinality of set K equals n, for certain $n \in \mathbb{N}$, we shall state that branch ϕ has *length of n* or is *n long*. In turn, if $|K| = |\mathbb{N}|$, we shall state that branch ϕ is *infinite* or is *infinitely long*.

Hence, branches are special types of sequences that to each natural number belonging to its domain assign a certain set of tableau expressions. Branches can be denoted in all possible ways in which we denote sequences, so we can list all the elements of branches with indices, describe a branch as a ordered set or an ordered n-tuple, if $|K| = n$, where $n \in \mathbb{N}$.

Branches are also sets and therefore we can define different relations on them, similarly as on sets. Assume we have two branches $\phi : K \longrightarrow P(\mathsf{Te})$ and $\psi : M \longrightarrow P(\mathsf{Te})$. Thus $\phi \subseteq \psi$ iff $K \subseteq M$ and for any number $i \in K$, $\phi(i) = \psi(i)$. In turn, $\phi = \psi$ iff $\phi \subseteq \psi$ and $\psi \subseteq \phi$. And finally, $\phi \subset \psi$ iff $\phi \subseteq \psi$ and $\phi \neq \psi$.

2 Tableau system for Classical Propositional Logic

2.1 Introductory remarks

In this chapter, we will define the tableau system for Classical Propositional Logic (for short **CPL**)[1], treating this case as a basic one.

This system can be considered basic because in the definition of tableau system for **CPL** we will use the propositional logic formulas themselves as tableau expressions. In the case of **CPL** it is possible, in other cases it may not be possible.

The defined system also features the property that any branch that begins with a finite set of tableau expressions can be extended to a branch of finite length to which the tableau rule can no longer be applied. We will call this feature *a finite branch property*. However, it is not a feature specific to all the tableau systems described in the book. This feature also makes the presented system for **CPL** a basic case, although for many logics the tableau systems feature the same property.

At the end of the chapter, we show that we actually have defined a tableau system for the **CPL**, because the relation of tableau derivability it determines coincides with the consequence relation determined by the valuation of formulas.

Throughout this chapter, we establish certain conventions of notation and order of defining tableau concepts and proving facts, which conventions and order will guide us within the book.

2.2 Language and semantics

The construction of a tableau system for the classical logic will start with the basic definitions. First, we will take up the language of **CPL**.

Definition 2.1 (Alphabet of **CPL**). *Alphabet of the Classical Propositional Logic* is the union of the following sets:

- set of logical constants: $\mathsf{Lc} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$
- set of propositional letters: $\mathsf{Var} = \{p_1, q_1, r_1, p_2, q_2, r_2, \dots\}$
- set of brackets: $\{\,),(\,\}$.

1 In the chapter we develop the approach that appeared for the first time in an English-language article [6]. In that study, both the concepts of tableau rules and the branches arising from the application of the rules were defined in a rather complicated way. In the meantime, these concepts have been simplified, so we present them here in an improved version. Of course, these changes affect the construction of the entire system.

Although the set of propositional letters is infinite and includes indexed propositional letters, in practice we will use a finite number of the following letters: p, q, r, s.

Definition 2.2 (Formula **CPL**). *Set of formulas* of **CPL** is the smallest set X which meets conditions:

1. $\mathsf{Var} \subseteq X$
2. if $A, B \in X$, then
   a. $\neg A \in X$
   b. $(A \wedge B) \in X$
   c. $(A \vee B) \in X$
   d. $(A \to B) \in X$
   e. $(A \leftrightarrow B) \in X$.

We specify this set as $\mathsf{For}_{\mathbf{CPL}}$, and its elements will be called *formulas*.

Remark 2.3. When considering different languages, we will use separate denotations for the sets of their formulas — though in a given context, we will use formulas of only one logic — so there will be no risk of mistake. However, in the chapter devoted to the generalisation of our considerations, we will use a denotation with no index. Additionally, sometimes, if possible, we will omit external brackets.

Let us proceed now to the semantics of classical logic. *Valuation of propositional letters* Var is any function $v : \mathsf{Var} \longrightarrow \{1,0\}$, that to each propositional letter assigns a value of truth or false. With function v, we can define valuation of formulas of **CPL**.

Definition 2.4 (Valuation of formulas of **CPL**). *Valuation of formulas* is function $V : \mathsf{For}_{\mathbf{CPL}} \longrightarrow \{1,0\}$, which for any $A, B \in \mathsf{For}_{\mathbf{CPL}}$ meets the following conditions:

1. $V(\neg A) = 1$ iff $V(A) = 0$
2. $V(A \wedge B) = 1$ iff $V(A) = 1$ and $V(B) = 1$
3. $V(A \vee B) = 1$ iff $V(A) = 1$ or $V(B) = 1$
4. $V(A \to B) = 1$ iff $V(A) = 0$ or $V(B) = 1$
5. $V(A \leftrightarrow B) = 1$ iff $V(A) = V(B)$.

We also call function V a *valuation of formulas of* **CPL** or shortly *valuation*.

Each function v, through conditions of definition 2.4, is unambiguously extendible to the valuation of formulas, i.e. function $V : \mathsf{For}_{\mathbf{CPL}} \longrightarrow \{1,0\}$ that to each formula assigns a value of truth or false.

Denotation 2.5. Let us adopt some abbreviations. Let '$V(X) = 1$' mean $V(X) \subseteq \{1\}$ (i.e. if $X \neq \varnothing$, then $V(X) = \{1\}$), while '$V(X) \neq 1$' means $V(X) \nsubseteq \{1\}$, (i.e. if $X \neq \varnothing$, then $V(X) \neq \{1\}$) for any set of formulas X and for any valuation V.

Using the concept of valuation of formulas, we can now define the semantic consequence of **CPL**.

Definition 2.6 (Semantic consequence of **CPL**). Let set $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$. Formula A *follows from* set X (for short: $X \vDash A$) iff for any valuation V, if $V(X) = 1$, then $V(A) = 1$. Relation $\vDash$ will be called *relation of classical semantic consequence* or shortly *relation of semantic consequence*.

Thus, classical semantic consequence $\vDash$ is such a relation that $\vDash \subseteq P(\mathsf{For}_{\mathbf{CPL}}) \times \mathsf{For}_{\mathbf{CPL}}$.

Remark 2.7. When considering different relations of consequences in the following chapters, we will not use separate denotations for them. In a given context, we will only examine one relation, so there will be no risk of mistake.

Denotation 2.8. For any set of formulas X and any formula A notation $X \nvDash A$ will mean that it is not the case that $X \vDash A$.

The last concept related to **CPL**, we will apply when constructing the tableau system is the concept of contradictory set of formulas.

Definition 2.9 (Contradictory set of formulas). Let $X \subseteq \mathsf{For}_{\mathbf{CPL}}$. Set X will be called *contradictory* iff there is no valuation V such that $V(X) = 1$. Set X will be called *non-contradictory* iff it is not contradictory.

2.3 Basic concepts of the tableau system for CPL

One of the basic concepts used to describe a tableau system, due to the nature of tableau proofs, is the concept of a tableau inconsistent set of proof expressions. In the case of a defined system for **CPL**, the proof expressions are the formulas themselves, so a set of tableau inconsistent expressions boils down to a certain set of formulas.

Definition 2.10 (Tableau inconsistent set of formulas). Set $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ will be called *tableau inconsistent* (for short: *t-inconsistent*) iff there exists such formula $A \in \mathsf{For}_{\mathbf{CPL}}$ that $A, \neg A \in X$. Set X will be called *tableau consistent* (for short: *t-consistent*) iff it is not t-inconsistent.

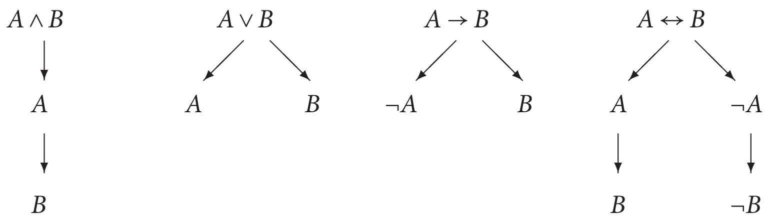
Corollary 2.11. *For any $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, if X is t-inconsistent, then X is contradictory.*

Proof. By definition 2.10, 2.4, 2.9. □

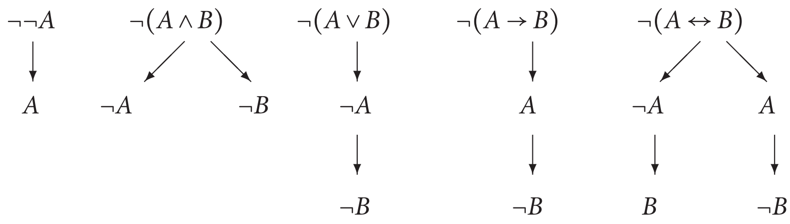
2.3.1 Tableau rules for CPL

We will now proceed to defining the proof tools. The central concept is tableau rules. Usually, in an intuitive approach, tableau rules are presented as graphs illustrating the way in which proof expressions are distributed. Rules for **CPL** are most often presented in the following way.[2]

First, we present positive rules — for formulas that do not begin with the negation functor.



The negative rules, on the other hand, for formulas preceded by negation, are presented in the following drawings.



In all of these rules, the arrows show the formulas we obtain by applying given rule, while some rules provide alternative formulas. Such a concept of rule is figurative and very intuitive, but at the same time not very formal. As a result, it is difficult to base on it to define the concept of tableau or tableau proof more formally than the concept of rule itself.

The starting point for the construction of a tableau system can therefore be a precise definition of the concept of tableau rule. Let us start with the general concept of rule.

2 See e.g. Priest G., [23], p. 5-6.

Definition 2.12 (Rule)**.** Let $P(\mathsf{For}_{\mathbf{CPL}})$ be the set of all subsets of set of formulas. Let $P(\mathsf{For}_{\mathbf{CPL}})^n$ be n-ry Cartesian product $\underbrace{P(\mathsf{For}_{\mathbf{CPL}}) \times \ldots \times P(\mathsf{For}_{\mathbf{CPL}})}_{n}$, for some $n \in \mathbb{N}$.

- By a *rule* we understand any such subset $R \subseteq P(\mathsf{For}_{\mathbf{CPL}})^n$ that if $\langle X_1, \ldots, X_n \rangle \in R$, then $X_1 \subset X_i$, for each $1 < i \leq n$.
- If $n \geq 2$, each element R will be called *ordered n-tuple* (pair, triple, etc., respectively).
- The first element of each n-tuple will be called an *input set* (*set of premises*) while the other elements *output sets* (*sets of conclusions*).[3]

Thus, according to the above definition, each rule is for some $n \in \mathbb{N}$ a set of n-tuples in the form of $\langle X_1, \ldots, X_n \rangle$, where $X_1, \ldots, X_n \subseteq \mathsf{For}_{\mathbf{CPL}}$. Not every rule is of course a tableau rule for **CPL**. A set of tableau rules for our tableau system for **CPL** shall be introduced by means of the following definition.

Definition 2.13 (Tableau rules for **CPL**)**.** *Tableau rules for* **CPL** are the following rules:

$R_\wedge = \{\langle X \cup \{(A \wedge B)\}, X \cup \{(A \wedge B), A, B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{(A \wedge B)\}$ is t-consistent$\}$

$R_\vee = \{\langle X \cup \{(A \vee B)\}, X \cup \{(A \vee B), A\}, X \cup \{(A \vee B), B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{(A \vee B)\}$ is t-consistent$\}$

$R_\rightarrow = \{\langle X \cup \{(A \rightarrow B)\}, X \cup \{(A \rightarrow B), \neg A\}, X \cup \{(A \rightarrow B), B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{(A \rightarrow B)\}$ is t-consistent$\}$

$R_\leftrightarrow = \{\langle X \cup \{(A \leftrightarrow B)\}, X \cup \{(A \leftrightarrow B), A, B\}, X \cup \{(A \leftrightarrow B), \neg A, \neg B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{(A \leftrightarrow B)\}$ is t-consistent$\}$

$R_{\neg\neg} = \{\langle X \cup \{\neg\neg A\}, X \cup \{\neg\neg A, A\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{\neg\neg A\}$ is t-consistent$\}$

$R_{\neg\wedge} = \{\langle X \cup \{\neg(A \wedge B)\}, X \cup \{\neg(A \wedge B), \neg A\}, X \cup \{\neg(A \wedge B), \neg B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{\neg(A \wedge B)\}$ is t-consistent$\}$

3 The property of rule that the first set is contained properly in every subsequent set seems to correspond well to the name proposed in the literature *expansion rule* (p. 60, [4]).

$R_{\neg\vee} = \{\langle X \cup \{\neg(A \vee B)\}, X \cup \{\neg(A \vee B), \neg A, \neg B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{\neg(A \vee B)\} \text{ is t-consistent}\}$

$R_{\neg\to} = \{\langle X \cup \{\neg(A \to B)\}, X \cup \{\neg(A \to B), A, \neg B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{\neg(A \to B)\} \text{ is t-consistent}\}$

$R_{\neg\leftrightarrow} = \{\langle X \cup \{\neg(A \leftrightarrow B)\}, X \cup \{\neg(A \leftrightarrow B), \neg A, B\}, X \cup \{\neg(A \leftrightarrow B), A, \neg B\}\rangle : X \subseteq \mathsf{For}_{\mathbf{CPL}}, A, B \in \mathsf{For}_{\mathbf{CPL}}, X \cup \{\neg(A \leftrightarrow B)\} \text{ is t-consistent}\}$.

Set of tableau rules for **CPL** will be denoted as $\mathbf{R}_{\mathbf{CPL}}$.

By definition 2.13 it follows that we have nine tableau rules for the tableau system being defined for **CPL**. Let us consider one of them, e.g. $R_{\neg\wedge}$. Take any x and assume that $x \in R_{\neg\wedge}$. By definitions 2.12 and 2.13, we get $\exists_{X \subseteq \mathsf{For}_{\mathbf{CPL}}}, \exists_{A,B} \in \mathsf{For}_{\mathbf{CPL}}$ that:

1. $x = \langle X \cup \{\neg(A \wedge B)\}, X \cup \{\neg(A \wedge B), \neg A\}, X \cup \{\neg(A \wedge B), \neg B\}\rangle$
2. $X \cup \{\neg(A \wedge B)\} \subset X \cup \{\neg(A \wedge B), \neg A\}$
3. $X \cup \{\neg(A \wedge B)\} \subset X \cup \{\neg(A \wedge B), \neg B\}$
4. $X \cup \{\neg(A \wedge B)\}$ is t-consistent.

Therefore, there is such a set and such formulas that the rule contains the ordered triple of sets which was constructed on them and has the following properties:

(a) the first element — the input set — is a proper subset of both the second and third element — the output sets
(b) the first element is a t-consistent set.

Both these properties are important for the method of constructing tableau systems presented in the book. They also form an element which distinguishes this method from other ones. But before we proceed to a more detailed discussion of the properties, we will look at the notation of the rules itself. The provided method of notation is quite precise, but at the same time complicated. Thus, despite the fact that the case of **CPL** is the simplest of those considered in the book, the description of sets — rules is somewhat complicated.

In order to simplify the notation, we will propose a fractional one. With rule $R = \{\langle X_1, \ldots, X_n\rangle : X_1 \text{ is t-consistent}\}$, where $n \geq 2$, we will write:

$$\frac{X_1}{X_2 | \ldots | X_n}$$

Heuristically speaking, the fraction bar in the above notation will tell us that from the input set X_1 we can proceed to one of the output sets $X_2, \ldots, X_n$, described under the bar. In the fractional notation, the individual sets will also be described structurally, i.e. the fraction will be a scheme of infinitely many n-tuples, which are elements of rule R.

Remark 2.14. Further in the book, we will capture specific tableau rules with fractional notation only, but we will have introduced a general definition of the rule for a given type of tableau systems beforehand.

Now, using the above method of notation, we will once again present set of rules $\mathbf{R_{CPL}}$. Set of rules $\mathbf{R_{CPL}}$ contains only rules defined by the following schemes in which the input sets are t-consistent:

$$R_{\wedge}: \frac{X \cup \{(A \wedge B)\}}{X \cup \{(A \wedge B), A, B\}} \qquad R_{\vee}: \frac{X \cup \{(A \vee B)\}}{X \cup \{(A \vee B), A\} | X \cup \{(A \vee B), B\}}$$

$$R_{\rightarrow}: \frac{X \cup \{(A \rightarrow B)\}}{X \cup \{(A \rightarrow B), \neg A\} | X \cup \{(A \rightarrow B), B\}}$$

$$R_{\leftrightarrow}: \frac{X \cup \{(A \leftrightarrow B)\}}{X \cup \{(A \leftrightarrow B), A, B\} | X \cup \{(A \leftrightarrow B), \neg A, \neg B\}}$$

$$R_{\neg\neg}: \frac{X \cup \{\neg\neg A\}}{X \cup \{\neg\neg A, A\}}$$

$$R_{\neg\wedge}: \frac{X \cup \{\neg(A \wedge B)\}}{X \cup \{\neg(A \wedge B), \neg A\} | X \cup \{\neg(A \wedge B), \neg B\}}$$

$$R_{\neg\vee}: \frac{X \cup \{\neg(A \vee B)\}}{X \cup \{\neg(A \vee B), \neg A, \neg B\}} \qquad R_{\neg\rightarrow}: \frac{X \cup \{\neg(A \rightarrow B)\}}{X \cup \{\neg(A \rightarrow B), A, \neg B\}}$$

$$R_{\neg\leftrightarrow}: \frac{X \cup \{\neg(A \leftrightarrow B)\}}{X \cup \{\neg(A \leftrightarrow B), \neg A, B\} | X \cup \{\neg(A \leftrightarrow B), A, \neg B\}}$$

Remark 2.15. The method of tableau rules defining we have presented — no matter what notation method is adopted — carries some benefits. The first is elimination of redundant applications of rule that may occur in the case of intuitive formulation of a tableau system. An redundant application of rule takes place when application of rule does not cause a new expression to appear on every branch it generates. In extreme cases, theoretically even intuitive rules can be applied without limitations, never receiving anything new. This is illustrated by the following example 2.16.

The rules defined according to our recipe exclude this type of situation, and the responsible factor is the condition that requires the input set to be a proper subset of each of the output sets. As per definition 2.12, for each rule R and $n \in \mathbb{N}$, if $\langle X_1, \ldots, X_n \rangle \in R$, then $X_1 \subset X_i$, for each $1 < i \leq n$. Therefore, no rule $R \in \mathbf{R_{CPL}}$ can be applied idly.

Example 2.16. Take formulas $(p \wedge q)$, $\neg\neg r$, p, q. Using the intuitive rules described at the beginning of this subchapter 2.3.1, we can apply to these formulas many times the rule for conjunction to formula $(p \wedge q)$, although it does not bring anything new — in the meantime we have applied the rule for double negation to formula $\neg\neg r$.

$$
\begin{array}{c}
(p \wedge q), p, q, \neg\neg r \\
| \\
p \\
| \\
q \\
| \\
r \\
| \\
p \\
| \\
q
\end{array}
$$

It is difficult to formally limit such redundant use, since the rule is illustrated in the formula scheme, whereas the proof consists of many elements. Therefore, it is necessary to refer to all previous elements in a limitation. However, some of these elements may have appeared after the earlier application of other rules. Hence, it is unclear at which level in this approach to formally exclude such application of the tableau rules.

Remark 2.17. Another benefit of the way the rules are defined is the exclusion of their applicability to the sets that are t-inconsistent. Within the classic approach to rules, there is nothing to prevent us from continuing to apply the rule, even if there is an explicit tableau inconsistency in the proof tree. Of course, one way to avoid this problem is to add a blocking rule to the set of tableau rules, which introduces a new element which closes the branch. Again, however, nothing formally excludes the application of further rules, even if there is a blocking element in the proof.

We illustrate this problem with example 2.18. In our approach, the informal "no use" directive makes technical sense: it is simply not possible — in the case of a t-inconsistent set, there are no rules that can be applied. Again, this is because of the rule definition. By definition 2.13 for each rule R and $n \in \mathbb{N}$, if $\langle X_1, \ldots, X_n \rangle \in R$, then X_1 is t-consistent, thus X_1 contains no pair of formulas A, $\neg A$, by definition 2.10. Consequently, no rule $R \in \mathbf{R_{CPL}}$ can be applied to a tableau inconsistent set.

Example 2.18. Take formulas $(p \wedge q)$, $\neg\neg\neg r$, p, r. Using the intuitive rules described at the beginning of this subchapter 2.3.1, we can apply the rule for conjunction to formula $(p \wedge q)$, even though previously we applied the rule for double negation to formula $\neg\neg\neg r$, which eventually produced a t-inconsistent set as we already had formula r.

$(p \wedge q), \neg\neg\neg r, p, r$
|
$\neg r$
|
p
|
q

Again, it is difficult to limit such application of rules if we do not have a precise concept of proof and rule.

Remark 2.19. Instead of set $\mathbf{R_{CPL}}$, we could use a different axiomatization to construct a tableau system. For instance, instead of rule $R_{\neg\wedge}$, we could consider the following tableau rule:

$$R'_{\neg\wedge} : \frac{X \cup \{\neg(A \wedge B)\}}{X \cup \{\neg(A \wedge B), (\neg A \vee \neg B)\}}$$

Similar proposals could be made for some other rules from set $\mathbf{R_{CPL}}$. Among many ideas, one of the simpler ones seems to be the replacement of rule $R_{\vee}$ e.g., the following one:

$$R'_{\vee} : \frac{X \cup \{(A \vee B)\}}{X \cup \{(A \vee B), (A \wedge B)\} | X \cup \{(A \vee B), A\} | X \cup \{(A \vee B), B\}}$$

Of course, the new rules — according to definition of tableau rule 2.13 — would also have t-consistent input sets, plus the input set would be contained properly in the output sets. Various axiomatizations can be examined for dependencies between sets of tableau rules, lengths of proofs and intuitiveness of rules.

For three reasons, however, we will not be researching such alternative tableau axiomatizations.

First of all, we want to describe the method of constructing tableau systems on the example of selected cases and then try to generalize this method on different cases. So in each case of the tableau systems considered in the book, it is sufficient to describe a single case, e.g. the most typical one. If we were to investigate an alternative axiomatization for a tableau system for **CPL**, e.g. using set $(\mathbf{R_{CPL}} \setminus \{R_{\neg\wedge}\}) \cup \{R'_{\neg\wedge}\}$, then in order to formally demonstrate that both systems yield the same logical consequences, it would require defining both systems and demonstrating that the sets of their consequences are identical. Meanwhile, we want to define a sample system for set of rules $\mathbf{R_{CPL}}$ and sample systems for logics specified in other languages, and then look for a general pattern for the construction of a tableau system that would simplify the construction of tableau systems and also help to study the equivalence of different sets of rules that axiomatize the same logic. So instead of describing consecutive tableau systems for the same semantically determined logic, we will try to seek common attributes of tableau systems for various semantically determined logic.

Secondly, the specified set of tableau rules $\mathbf{R_{CPL}}$ corresponds to the intuitive and normally adopted tableau rules for **CPL**. Thus, they are typical, because in the simplest way they correspond to the semantic content related to the interpretation of the classical connectives.

Thirdly, rules such as $R'_{\neg\wedge}$ do not seem good candidates for axiomatization. For example, the adoption of set $(\mathbf{R_{CPL}} \setminus \{R_{\neg\wedge}\}) \cup \{R'_{\neg\wedge}\}$, which is to replace rule $R_{\neg\wedge}$ with rule $R'_{\neg\wedge}$ would extend some of proofs. Because rule $R_{\neg\wedge}$ allows to proceed from formula $\neg(A \wedge B)$ to formulas $\neg A$ or $\neg B$. Meanwhile, in the case of rule $R'_{\neg\wedge}$ we proceed from formula $\neg(A \wedge B)$ to formula $(\neg A \vee \neg B)$. Transition to formulas $\neg A$ or $\neg B$ still requires the use of rule $R_{\vee}$. Thus, introduction of rule $R'_{\neg\wedge}$ in lieu of $R_{\neg\wedge}$ seems unnatural. It is not certain, however, that for the cases considered in the book there are no alternative rules that are not unnatural, or maybe even in some way they are better than the ones considered. In the next chapter, when describing the tableau system for term logic, we will show an example of alternative rules that seem at least equally good as the ones we will finally investigate there (see note 3.20).

However, among the reasons mentioned, the first one is conclusive. Therefore, although we will not examine alternative axiomatizations of tableau systems, it is worth stressing that the introduction of the general concept of rule 2.12 for a given set of tableau expressions Te is meaningful in overall. Most often it is possible to define different sets of tableau rules that define tableau systems corresponding to the same semantically determined logic.

2.3.2 Branches for CPL

Another thing in our theory that needs discussion is the concept of branch. It is a concept that depends on the tableau rule because branches are created by applying rules. Although we are not currently introducing any formal concept of *rule application*, intuitively the point is that a given rule R is applied to a given set Y when among the elements contained in R there is such n-tuple $\langle X_1, \ldots, X_n \rangle$ that $X_1 = Y$. The result of application of R is some set X_i, where $1 < i \leq n$, and consequently also sequence $\langle X_1, X_i \rangle$, which forms precisely a branch. Branches are therefore setwise objects consisting of sets. Let us now proceed to the formal definition of branch in the tableau system for **CPL**.

Definition 2.20 (Branch)**.** Let $K = \mathbb{N}$ or $K = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Let X be any set of formulas. *A branch* (or *a branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$: if $i+1 \in K$, then there exists such rule $R \in \mathbf{R}_{\mathbf{CPL}}$ and such n-tuple $\langle Y_1, \ldots, Y_n \rangle \in R$ that $\phi(i) = Y_1$ and $\phi(i+1) = Y_k$, for some $1 < k \leq n$.

Having two branches ϕ, ψ such that $\phi \subset \psi$ we shall state that:

- ϕ is a *sub-branch* of ψ
- ψ is a *super-branch* of ϕ.

Denotation 2.21. From now on — when speaking of branches — for convenience, we will use the following notations or designations:

1. $X_1, \ldots, X_n$, where $n \geq 1$
2. $\langle X_1, \ldots, X_n \rangle$, where $n \geq 1$
3. abbreviations: ϕ_M (where M is a domain ϕ, i.e. $\phi : M \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$)
4. or — to denote branches — small Greek letters: ϕ, ψ, etc.

The sets of branches, in turn, we shall denote with capital Greek letters: Φ, Ψ, etc. Furthermore, the domain cardinality of a given branch K we shall sometimes call a *length* of that branch.

Remark 2.22. As we can see, the concept of branch depends on some set of rules. In the case under consideration, the branch structure is based on the rules from set $\mathbf{R}_{\mathbf{CPL}}$. Further described complex tableau concepts will also depend on some sets of rules. Because in this chapter we are studying tableau system for **CPL**, based on rules from set $\mathbf{R}_{\mathbf{CPL}}$, so we are not going to make it any more complicated. In practice, however, the tableau concepts of systems constructed according to the presented idea always hinge upon some set of rules. In one of the further chapters, in the general description of the construction method itself, the set of rules will be a certain variable. In this chapter, it is specified as: $\mathbf{R_{CPL}}$, and the complex tableau concepts defined here depend on it.

By definition of rules 2.12, through the fact that the rules are defined by proper containing of the output set in each of the output sets, in any n-tuple, there is a conclusion.

Corollary 2.23. *Each branch is an injective sequence.*

We will now look at the issue of branch length. To investigate this problem, we need a function to measure the formula complexity. But, this is not about a syntactic complexity, but branch complexity. In practice, the branch complexity of the formula boils down to the maximal length of the branch, which is obtained by applying the tableau rules to a single formula only. First, we will give a definition of the function that measures this property and show that the function has been well defined, and then we will present an example. Assume that for any two natural numbers n, m, $max\{n, m\} = n$ iff $n \geq m$.

Definition 2.24 (Measure of the branch complexity of formula of **CPL**). *The measure of the branch complexity* is function $* : \mathsf{For}_{\mathbf{CPL}} \longrightarrow \mathbb{N}$, defined for each $x \in \mathsf{Var}$ and $A, B \in \mathsf{For}_{\mathbf{CPL}}$ with the below conditions:

1. $*(x) = 1$
2. $*(\neg x) = 1$
3. $*((A \wedge B)) = *(A) + *(B)$
4. $*((A \vee B)) = max\{*(A), *(B)\} + 1$
5. $*((A \rightarrow B)) = max\{*(\neg A), *(B)\} + 1$
6. $*((A \leftrightarrow B)) = max\{*(A) + *(B), *(\neg(A)) + *(\neg(B))\}$
7. $*(\neg\neg A) = *(A) + 1$
8. $*(\neg(A \wedge B)) = max\{*(\neg A), *(\neg B)\} + 1$
9. $*(\neg(A \vee B)) = *(\neg A) + *(\neg B)$
10. $*(\neg(A \rightarrow B)) = *(A) + *(\neg B)$
11. $*(\neg(A \leftrightarrow B)) = max\{*(\neg A) + *(B), *(A) + *(\neg B)\}$.

Function $*$, tells us how at most long a branch can be obtained by applying the rules from $\mathbf{R_{CPL}}$ to a given formula and its components. The following few facts clarify the problem. From the above definition and the definition of formula 2.2, it follows that:

Corollary 2.25. *For any $A \in \mathsf{For}_{\mathbf{CPL}}$: $*(A) \in \mathbb{N}$.*

We will now formulate a fact that describes the relationship between the branches and the measure of the branch complexity of formula.

Proposition 2.26. *Let $A \in \mathsf{For}_{\mathbf{CPL}}$. Let $X_1 = X \cup \{A\}, X_2, \ldots, X_n$ be such a branch that $Y_1, \ldots, Y_n$, where:*

- $Y_1 = \{A\}$
- $Y_2 = X_2 \setminus X$
- ...
- $Y_n = X_n \setminus X$

is a branch. Then $n \leq *(A)$.

Before we proceed to the proof of fact, let us try to explain its assumption. For each $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$ branch $X_1 = X \cup \{A\}, X_2, \ldots, X_n$ is constructed so that for each $1 < i \leq n$, $X_i = X \cup Y_i$. This means that X_2 was created by applying a certain rule to formula A, X_3 was created by applying a certain rule to formula from $X_2 \setminus X$, whereas X_{i+1} was created by applying a certain rule to formula from $X_i \setminus X$. So, to put it figuratively, the branch under consideration was created by applying the rules to formula A and the consequences of applying the rules to what was created previously. It is of course possible that at some point in time a certain rule may have been applied to a element of set X, but the effect must have been the same as applying the rule to A or its consequences.

Therefore, fact 2.26 tells us that a branch constructed through the application of rules to a single formula A and further effects of application of rules is not longer than $*(A)$, thus the measure of the branch complexity is properly defined.

Let us now move on to the proof of fact. It has an inductive nature and is based on the branch complexity of the formulas found in branches. For the proof, we will use the definition of branch 2.20 the definition of measure of branch complexity 2.24.

Proof. Take any branch $X_1 = X \cup \{A\}, X_2, \ldots, X_n$ that meets the theorem assumptions.

Initial step. Let $A = x$, for some $x \in \mathsf{Var}$. There is then no rule to apply to X_1, thus $n = 1$. Since $*(x) = 1$, so $n \leq *(x)$. If $A = \neg(x)$, for some $x \in \mathsf{Var}$, then there is either no rule to apply to X_1, so $n = 1$. Since $*(\neg x) = 1$, thus $n \leq *(\neg x)$.

Induction step. Assume that a fact thesis holds for each formula B such that $*(A) > *(B)$. So, for each branch beginning with $\{B\}$, m long, it is the case that $m \leq *(B)$. The branches will be called *B-branches*, *C-branches*, etc., depending on the formula they start with.

Cases:

i) Let $A := \neg\neg B$. Then, by definition of function $*$, $*(A) > *(B)$. Under the induction hypothesis $*(B) \geq m$, where m is the length of any B–branch. Since $n = m_1 + 1$, for some $m_1 \leq *(B)$, by definition of branch 2.20, so $*(A) = *(B) + 1 \geq m_1 + 1 = n$, by definition 2.24.

ii) Let $A := (B \wedge C)$. Then, by definition of function $*$, $*(A) > *(B), *(C)$. Under the induction hypothesis, $*(B) \geq m$, where m is the length of any B-branch, and $*(C) \geq k$, where k is the length of any C-branch. Consequently, since $n \leq m_1 + k_1$, for some $m_1 \leq *(B)$, $k_1 \leq *(C)$, by definition of branch 2.20, so $*(A) = *((B \wedge C)) = *(B) + *(C) \geq m_1 + k_1 \geq n$, by definition 2.24.

iii) Let $A := (B \vee C)$. Then, by definition of function $*$, $*(A) > *(B)$, $*(C)$. Under the induction hypothesis, $*(B) \geq m$, where m is the length of any B-branch, and $*(C) \geq k$, where k is the length of any C-branch. Consequently, since $n = m_1 + 1$ or $n = k_1 + 1$, for some $m_1 \leq *(B)$, $k_1 \leq *(C)$, by definition of branch 2.20, so:

1. if $max\{*(B), *(C)\} = *(B)$, then $*(A) = *((B \vee C)) = *(B) + 1 \geq m_1 + 1 = n$, by definition 2.24
2. if $max\{*(B), *(C)\} = *(C)$, then $*(A) = *((B \vee C)) = *(C) + 1 \geq k_1 + 1 = n$, by definition 2.24.

iv) Let $A := (B \to C)$. Then, by definition of function $*$, $*(A) > *(\neg B), *(C)$. Under the induction hypothesis, $*(\neg B) \geq m$, where m is the length of any $\neg B$-branch, and $*(C) \geq k$, where k is the length of any C-branch. Consequently, since $n = m_1 + 1$ or $n = k_1 + 1$, for some $m_1 \leq *(\neg(B))$, $k_1 \leq *(C)$, by definition of branch 2.20, so:

1. if $max\{*(\neg B), *(C)\} = *(\neg B)$, then $*(A) = *((B \to C)) = *(\neg(B)) + 1 \geq m_1 + 1 = n$, by definition 2.24
2. if $max\{*(\neg B), *(C)\} = *(C)$, then $*(A) = *((B \to C)) = *(C) + 1 \geq k_1 + 1 = n$, by definition 2.24.

v) Let $A := (B \leftrightarrow C)$. Then, by definition of function $*$, $*(A) > *(B)$, $*(C)$, $*(\neg B), *(\neg C)$. Under the induction hypothesis, $*(B) \geq m$, $*(C) \geq k$, $*(\neg B) \geq l$, $*(\neg C) \geq o$, where m, k, l, o are respectively lengths of any B, C, $\neg B$ and $\neg C$-branch. Consequently, since $n \leq m_1 + k_1$ or $n \leq l_1 + o_1$, for some $m_1 \leq *(B)$, $k_1 \leq *(C)$, $l_1 \leq *(\neg B)$, $o_1 \leq *(\neg C)$, by definition of branch 2.20, so:

1. if $max\{*(B) + *(C), *(\neg B) + *(\neg C)\} = *(B) + *(C)$, then $*(A) = *((B \leftrightarrow C)) = *(B) + *(C) \geq m_1 + k_1 \geq n$, by definition 2.24

2. if $max\{*(B)+*(C),*(\neg B)+*(\neg C)\} = *(\neg B)+*(\neg C)$, then $*(A) = *(B \leftrightarrow C) = *(\neg B)+*(\neg C) \geq l_1 + o_1 \geq n$, by definition 2.24.

vi) Let $A := \neg(B \wedge C)$. Then, by definition of function $*$, $*(A) > *(\neg B), *(\neg C)$. Under the induction hypothesis, $*(\neg B) \geq m$, where m is the length of any $\neg B$-branch, and $*(\neg C) \geq k$, where k is the length of any $\neg C$-branch. Consequently, since $n = m_1 + 1$ or $n = k_1 + 1$, for some $m_1 \leq *(\neg B)$, $k_1 \leq *(\neg C)$, by definition of branch 2.20, so:

1. if $max\{*(\neg B),*(\neg C)\} = *(\neg B)$, then $*(A) = *(\neg(B \wedge C)) = *(\neg B)+1 \geq m_1 + 1 = n$, by definition 2.24
2. if $max\{*(\neg B), *(\neg C)\} = *(\neg C)$, then $*(A) = *(\neg(B \wedge C)) = *(\neg C)+1 \geq k_1 + 1 = n$, by definition 2.24.

vii) Let $A := \neg(B \vee C)$. Then, by definition of function $*$, $*(A) > *(\neg B), *(\neg C)$. Under the induction hypothesis, $*(\neg B) \geq m$, where m is the length of any $\neg B$-branch, and $*(\neg C) \geq k$, where k is the length of any $\neg C$-branch. Consequently, since $n \leq m_1 + k_1$, for some $m_1 \leq *(\neg B)$, $k_1 \leq *(\neg C)$, by definition of branch 2.20, so $*(A) = *(\neg(B \vee C)) = *(\neg B)+*(\neg C) \geq m_1 + k_1 \geq n$, by definition 2.24.

viii) Let $A := \neg(B \rightarrow C)$. Then, by definition of function $*$, $*(A) > *(B), *(\neg C)$. Under the induction hypothesis, $*(B) \geq m$, where m is the length of any B-branch, and $*(\neg C) \geq k$, where k is the length of any $\neg C$-branch. Consequently, since $n \leq m_1 + k_1$, for some $m_1 \leq *(B)$, $k_1 \leq *(\neg C)$, by definition of branch 2.20, so $*(A) = *(\neg(B \rightarrow C)) = *(B)+*(\neg C) \geq m_1 + k_1 \geq n$, by definition 2.24.

ix) Let $A := \neg(B \leftrightarrow C)$. Then, by definition of function $*$, $*(A) > *(B), *(C), *(\neg B), *(\neg C)$. Under the induction hypothesis, $*(B) \geq m$, $*(C) \geq k$, $*(\neg B) \geq l$, $*(\neg C) \geq o$, where m, k, l, o are respectively lengths of any B, C, $\neg B$ and $\neg C$-branch. Consequently, since $n \leq m_1 + o_1$ or $n \leq l_1 + k_1$, for some $m_1 \leq *(B)$, $k_1 \leq *(C)$, $l_1 \leq *(\neg B)$, $o_1 \leq *(\neg C)$, by definition of branch 2.20, so:

1. if $max\{*(\neg B)+*(C),*(B)+*(\neg C)\} = *(\neg B)+*(C)$, then $*(A) = *(\neg(B \leftrightarrow C)) = *(\neg B)+*(C) \geq l_1 + k_1 \geq n$, by definition 2.24
2. if $max\{*(\neg B)+*(C),*(B)+*(\neg C)\} = *(B)+*(\neg C)$, then $*(A) = *(\neg(B \leftrightarrow C)) = *(B)+*(\neg C) \geq m_1 + o_1 \geq n$, by definition 2.24. □

So the fact 2.26 tells us that if we create a branch based on a single A and make use of tableau rules in any order, then each produced branch will at most have a length of $*(A)$. See the example below.

Example 2.27. Let us consider set of formulas $Y \cup \{\neg(p \leftrightarrow \neg q)\}$. We decompose the highlighted formula using rule $R_{\neg\leftrightarrow}$ to this set. Due to two initial sets in the rule, we get two branches:

1. $X_1' = Y \cup \{\neg(p \leftrightarrow \neg q)\}, X_2' = Y \cup \{\neg(p \leftrightarrow \neg q), \neg p, \neg q\}$
2. $X_1'' = Y \cup \{\neg(p \leftrightarrow \neg q)\}, X_2'' = Y \cup \{\neg(p \leftrightarrow \neg q), p, \neg\neg q)\}$.

In the first case, we can no longer decompose the components of the initial formula, but in the case of X_2'' we can still apply rule $R_{\neg\neg}$, which produces: $X_3'' = Y \cup \{\neg(p \leftrightarrow \neg q), p, \neg\neg q, q\}$. (Of course, we assume that X_1'', X_2'', are t-consistent. Otherwise, we could not use the tableau rules.) The last sequence — X_1'', X_2'', X_3'' — is the longest among the sequences based on the decomposition of formula $\neg(p \leftrightarrow \neg q)$. At most, is has length of $*(\neg(p \leftrightarrow \neg q))$. Let us calculate: $*(p) = *(\neg q) = *(\neg p) = 1$, $*(\neg\neg q) = 1 + *(q) = 2$. Thus $max\{*(\neg p) + *(\neg q), *(p) + *(\neg\neg q)\} = max\{2,3\}$. Hence $*(\neg(p \leftrightarrow \neg q)) = 3$.

Another fact generalizes our observations on the relationship between the length of the branch and the branch complexity of the formula on the finite sets of formulas.

Proposition 2.28. *If X is a finite set of formulas, then each such branch $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ that $\phi(1) = X$ is finite.*

We will carry out an inductive proof with respect to the number of elements X by applying the previous fact 2.26.

Proof. Let X be a finite set of formulas, and $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ any such branch that $\phi(1) = X$.

Initial step. Assume that $|X| = 1$, so some formula $A \in X$. From the previous fact 2.26, we know that each branch based on decomposition of A has at most the length of $*(A)$. Hence, $|K| \leq *(A)$, so branch ϕ is finite, under 2.25.

Induction step. Assume that the fact thesis is true for each such set of formulas Y that $|Y| = n$. Let us consider a situation where $|X| = n+1$. So $X = Y' \cup \{A\}$ for certain set of formulas Y' such that $|Y'| = n$, and some new formula A.

From fact 2.26, we know that each branch based on the decomposition of A has at most the length of $*(A)$, which means it is finite. From the inductive assumption, we also know that each branch beginning with set of formulas Y' is finite.

Consequently, each branch ϕ beginning with $X = Y' \cup \{A\}$ is finite as it is composed of elements of some branch which begins with A and elements of some branch which begins with Y', by definition of branch 2.20, hence $|K| \leq n + *(A)$.

□

By virtue of the last fact, we know that there are no infinite branches beginning with finite sets of formulas, which is expressed by the following conclusion.

Corollary 2.29. *Let X be a finite set of formulas. Then there is no such branch* $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ *that:*

- $K = \mathbb{N}$
- $\phi(1) = X$.

Proof. From the previous fact 2.28 and by definition of branch 2.20. □

2.3.3 Maximal branches

Another important concept in the construction of a tableau system is the concept of a maximal branch. Intuitively, the maximal branch is a branch to which no rule can be applied anymore, extending it to some super-branch. The definition of the maximal branch for the currently defined system is as follows.

Definition 2.30 (Maximal branch)**.** Let $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ be a branch. We shall state that ϕ is *maximal* iff

1. $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$
2. there is no branch ψ such that $\phi \subset \psi$.

Before we discuss this definition, let us consider an example.

Example 2.31. Consider a branch beginning with set $X_1 = \{\underbrace{\neg \ldots \neg}_{n} r : n = m \cdot 3,$ where $m \in \mathbb{N}\}$. Set X_1 contains an infinite number of formulas in the form of $\underbrace{\neg \ldots \neg}_{n=m\cdot 3} r$, i.e. an infinite number of instances of propositional letter r, preceded in each instance by such number of instances of negation functor that is a multiple of 3.

We now define a branch based on set X_1 and rule $R_{\neg\neg}$, according to the following algorithm: for any $m \in \mathbb{N}$,

$$X_{m+1} = X_m \cup \{\underbrace{\neg \ldots \neg}_{m\cdot 3-2} r\}.$$

So, transition from set X_i to its superset X_{i+1} is an effect of addition through rule $R_{\neg\neg}$ to set X_i of formula $\underbrace{\neg \ldots \neg}_{i\cdot 3-2} r$.

The branch can be illustrated as follows:

$$
\begin{array}{c}
R_{\neg\neg} X_1 \\
| \\
R_{\neg\neg} X_2 = X_1 \cup \{\neg r\} \\
| \\
R_{\neg\neg} X_3 = X_2 \cup \{\neg\neg\neg\neg r\} \\
| \\
R_{\neg\neg} X_4 = X_3 \cup \{\neg\neg\neg\neg\neg\neg\neg\neg r\} \\
| \\
R_{\neg\neg} \ldots
\end{array}
$$

The defined branch is infinite. There is no super-branch to contain it. However, there is still an infinite number of formulas to which we might apply rule $R_{\neg\neg}$. As a matter of fact, already in the area of set X_4 we could obtain t-contradictory set, thereby closing the door on further extension of branches, defining $X_4 = X_3 \cup \{\neg\neg r\}$, with respect to $R_{\neg\neg}$, since $\neg r \in X_2 \subseteq X_3$.

Seemingly, we could limit the definition of maximal branch 2.30 to the second condition, i.e. to the non-existence of super-branch. However, we would then allow branches that contain non-decomposed expressions to which the tableau rules can still be applied.

Leaving out the first condition of the definition of maximal branch 2.30, would not change anything with respect to the cases of finite sets, but example 2.31 proves that we would allow cases of infinite branch that:

- begin with an infinite set
- meet the second condition of definition because they are infinite branches
- but not all expressions contained were decomposed, in particular those responsible for the emergence of t-inconistent subset.

Remark 2.32. The definition of maximal branch 2.30 is suitable for those tableau systems where only finite branches are obtained from finite sets of expressions. So it is i.a. good for the tableau system we construct for **CPL**.

For other systems, including modal logics, the definition is too narrow, because it does not include cases of branches that are not finite even though they start with a finite set of expressions.

We have now deliberately adopted definition 2.30, as sufficient for **CPL**. When we move on to defining the system for modal logic, we will generalize this definition. So it is going to describe special cases of infinite maximal branches which appear in the construction of tableau systems using the described method.

Extending the concept of branch onto infinite sets is problematic for many reasons, and what is more, it is unnecessary in practice, because important and sufficient concepts for our metatheory — the concept of branch consequences and tableau — we apply in practice to cases of finite sets.

Besides, we showed that a finite sets of expressions, using the rules from set of rules $\mathbf{R_{CPL}}$ we get branches of finite length (conclusion 2.29), so for the case of **CPL** the definition of maximal branch 2.30 is good enough.

The definition of maximal branch 2.30 in practice says that branch $X_1, \ldots, X_n$, for some $n \geq 1$, we shall call *maximal* iff there is no branch $X_1, \ldots, X_n, X_{n+1}$.

In the subsequent fact, we state that each finite set of formulas is the first element of some maximal branch.

Proposition 2.33. *Let X be a finite set of formulas. Then, there exists such maximal branch $X_1, \ldots, X_n$ that $X_1 = X$ and $n \geq 1$.*

Proof. Take any finite set of formulas X, and then indirectly assume that there is no maximal branch $X_1, \ldots, X_n$, where $X_1 = X$ and $n \geq 1$.

However, by definition of branch 2.20 we know that there exists at least one branch that starts with set X, i.e. $\langle X_1 \rangle$, where $X_1 = X$. From the indirect assumption and from the definition of maximal branch 2.30 it follows, however, that $\langle X_1 \rangle$ is not a maximal branch and it has some super-branch, so by definition of branch 2.20 there is some branch: X_1, X_2.

Let us now consider branch $X_1, \ldots, X_m$ m long, for some $m \in \mathbb{N}$. From the indirect assumption and from the definition of maximal branch 2.30 it follows, however, that $X_1, \ldots, X_m$ is not a maximal branch and it has some super-branch, so by definition of branch 2.20 there is some branch: $X_1, \ldots, X_m, X_{m+1}$.

So, from the indirect assumption and from the definition of maximal branch 2.30 results in a conclusion that ($\dagger$) for any $n \in \mathbb{N}$ there exists branch $X_1, \ldots, X_n$, X_{n+1} such that $X_1 = X$ and $n \geq 1$.

Let Φ be such a minimal set of branches that:

1. $\langle X_1 \rangle \in \Phi$
2. if $\langle X_1, \ldots, X_n \rangle \in \Phi$, then $\langle X_1, \ldots, X_n, X_{n+1} \rangle \in \Phi$, for any $n \in \mathbb{N}$.

Since Φ is a minimal set that meets conditions 1 and 2, then from ($\dagger$) it follows that for any $n \in \mathbb{N}$, there exists precisely one branch $\langle X_1, \ldots, X_n \rangle \in \Phi$, where $X_1 = X$.

K_i will now denote a domain of such branch ϕ_K contained in Φ that $|K| = i \in \mathbb{N}$. From the above considerations it follows that for each $i \in \mathbb{N}$ there exists precisely one set K_i that constitutes a domain of some branch which belongs to Φ. Then let $\overline{K} = \bigcup\{K_i : \phi_K \in \Phi\}$. Since $1 \in \overline{K}$ and $n + 1 \in \overline{K}$, if $n \in \overline{K}$, so $\overline{K} = \mathbb{N}$.

Let us now define sequence $\psi : \overline{K} \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ such that for any $i \in K$: $\psi(i) = \phi_{K_i}(i)$. From the definition of branch 2.20, that sequence is an infinite branch beginning with a finite set of formulas $X_1 = X$, which contradicts conclusion 2.29. □

Using the above fact, we can prove another fact important for our theory.

Proposition 2.34. *If X is a finite set of formulas, then for each branch $Y_1, \ldots, Y_n$ such that $Y_1 = X$ and $n \geq 1$, there exists maximal branch $Z_1, \ldots, Z_n, \ldots, Z_{n+m}$, where for any $i \leq n$ $Y_i = Z_i$ and $m \geq 0$.*

Proof. Let X be a finite set of formulas. Now, take any branch $Y_1, \ldots, Y_n$ such that $Y_1 = X$ and $n \geq 1$. From the definition of branch 2.20 and definition of tableau rules for **CPL** 2.13, we know that Y_n is a finite set. So, due to the last fact 2.33, for some $m \geq 0$ there exists such maximal branch $Z_n^1, \ldots, Z_{n+m}^{1+m}$ that $Z_n^1 = Y_n$, hence by virtue of definition of branch 2.20 there also exists maximal branch $Z_1, \ldots, Z_n, \ldots, Z_{n+m}$, where for any $i \leq n$ $Y_i = Z_i$ and $m \geq 0$. □

This fact tells us that any branch beginning with a finite set of formulas can be extended to the maximal branch in which it is included as a sequence.

For further consideration, the concept of branch which is maximal in a given set of branches, will be useful.

Definition 2.35 (Maximal branch in the set of branches)**.** Let Φ be a set of branches and let branch $\psi \in \Phi$. Branch ψ will be called *maximal in* Φ (or Φ–*maximal*) iff there is no such branch $\phi \in \Phi$ that $\psi \subset \phi$. Having some set of formulas X, $B(X)$ will denote the set of all branches $X_1, \ldots, X_n$ such that $X_1 = X$ and $n \geq 1$, while $MB(X)$ will denote the set of all maximal branches in $B(X)$.

Remark 2.36. The above concept of a maximal branch in a certain set of branches could be a starting point instead of the concept described in the definition of maximal branch 2.30. The latter is its special case in a situation when set Φ is identical to set $B(X)$, for some finite set of formulas X, considering fact 2.33.

However, we deliberately separated these concepts, because in one of the subsequent chapters we will change the definition of maximal branch in such a way that its scope will not be identical to the scope of definition 2.35 in relation to finite sets. This new concept of a maximal branch will also include cases where branches can be infinitely long, even though they start with a finite set of expressions. So, consequently, it can be the case that for a given branch ϕ there is no such branch ψ that $\phi \subset \psi$, and yet ϕ it will not be considered maximal for other reasons. Although we will continue to use the concept of a maximal branch in a given set, we will eventually extend the concept of a maximal branch.

Corollary 2.37. *Let X be a finite set of formulas. Then $B(X)$ contains a non-empty subset $MB(X)$.*

Proof. Take any finite set of formulas X. Due to fact 2.34, there exists maximal branch ψ beginning with X. Of course, branch ψ belongs to $MB(X)$, so set $MB(X)$ is non-empty. Since each branch contained in $MB(X)$ belongs to $B(X)$, so non-empty set $MB(X)$ is contained in $B(X)$. □

2.3.4 Closed and open branches

Another concept important for the tableau theory is the concept of a closed branch and an open branch. Intuitively, a branch is closed when we have reached a t-inconsistency set by decomposing formulas.

Definition 2.38 (Closed/open branch)**.** Branch $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ will be called *closed* iff $\phi(i)$ is a t-inconsistent set, for some $i \in K$. A branch will be called *open* iff it is not closed.

From the above definition, the definition of tableau rules for **CPL** 2.13 and the definition of branch 2.20, the following conclusion follows.

Corollary 2.39. *If branch $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ is closed, then $|K| \in \mathbb{N}$.*

Proof. Let branch $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ be closed. From the definition of closed branch 2.38 we know that there exists such $i \in K$ that $\phi(i)$ is a t-inconsistent set. From the definition of tableau rules for **CPL** it follows that there is no branch element $\phi(i+1)$, since the rules do not contain n-tuples with t-inconsistent input sets, hence none of the rules can be applied to set $\phi(i)$. So, from the definition of branch 2.20 we get: $K = \{1, 2, 3, \ldots i\}$, which means that $|K| \in \mathbb{N}$. □

In the case of a closed branch, the t-inconsistent element of sequence is therefore the last element. It is so because no more rule can be applied for branch extensions, because the tableau rules are defined in such a way that they cannot be applied to t-inconsistent sets. Therefore, from the definition of maximal branch 2.30 another conclusion follows.

Corollary 2.40. *If branch $\phi : K \longrightarrow P(\mathsf{For}_{\mathbf{CPL}})$ is closed, then it is maximal.*

2.3.5 Branch consequence relation

We will now proceed to the metalogical concept which occurs in each of the systems defined in the book. This concept seems to be a novelty, seemingly so far not defined in the studies on tableau systems.

As we wrote, the tableau method is treated and defined in the book in a purely syntactic manner, i.e. as a method of transforming the notations of a given language in order to answer the question whether the considered inference is correct.

A concept that corresponds to this question is the concept of branch consequence relation, specified with another definition.

Definition 2.41 (Branch consequence relation of **CPL**)**.** Let $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$. We shall state that A is a *branch consequence* of X (or short: $X \rhd A$) iff there exists such finite set $Y \subseteq X$ that each maximal branch beginning with $Y \cup \{\neg A\}$ is closed. Relation $\rhd$ will be called *branch consequence relation* of **CPL**.

Remark 2.42. When considering different branch consequence relations in the following chapters, we will not use separate denotations for them. In a given context, we will only examine one relation, so there will be no risk of mistake.

Denotation 2.43. For any set of formulas X and any formula A, notation $X \not\rhd A$ shall mean that it is not the case that $X \rhd A$.

In order to explain what is meant by definition 2.41, we will refer to an example.

Example 2.44. Let us consider the following example of occurrence of the relation of branch consequence. Take set $\{p\}$ and formula $(p \vee q)$. Relation $\{p\} \rhd (p \vee q)$ occurs since each maximal branch in the form $X_1 = \{p, \neg(p \vee q)\}, \ldots, X_n$ is closed. In fact, there is only one maximal branch in this form. It is branch: $X_1 = \{p, \neg(p \vee q)\}$, $X_2 = \{p, \neg(p \vee q), \neg p, \neg q\}$, which originates through rule $R_{\neg\vee}$. It is closed because $p \in X_2$ and $\neg p \in X_2$.

The above example shows the mechanism of the relation of branch consequence. It is based on the impossibility of finding a branch that would be both maximal and open. However, this example is somewhat misleading. We deliberately chose a set of formulas and a formula so that the branches beginning with this set and negation of the formula are neither too long nor too many. In practice, none of these features may actualize and most often they do not.

2.4 Relations of semantic consequence and branch consequence

Before we proceed to the issue of how to reduce as much as possible the number of branches, which suffice to consider to show the occurrence of the relation of branch consequence, we will first show that the concept of the branch consequence relation defines the same set of objects as the concept of the semantic consequence relation — it is thus the branch, syntactic counterpart of the semantic consequence relation.

2.4.1 Soundness theorem

In this subchapter, we will show that the relation of branch consequence is contained in the relation of semantic consequence, hence for any set of formulas X and any formula A it is the case that if $X \rhd A$, then $X \vDash A$.

We will use abbreviations concerning the valuations of the formulas adopted in denotation 2.5.

We will start with a lemma that tells us about the relationship between valuations of formulas and tableau rules.

Lemma 2.45. *Let X be a set of formulas and V be any valuation such that $V(X) = 1$. For any rule $R \in \mathbf{R_{CPL}}$:*

- *if $\langle X, Y \rangle \in R$, then $V(Y) = 1$*
- *if $\langle X, Y, Z \rangle \in R$, then $V(Y) = 1$ or $V(Z) = 1$.*

Proof. We carry out the proof by checking the rules one by one. We will use the definition of valuation of **CPL** formulas, 2.4.

Take any set of formulas X and any valuation V such that $V(X) = 1$. Take any rule R from set $\mathbf{R_{CPL}}$. If set X is an input set of some $n-tuple$ contained in R, there must exist: certain set of formulas X' and formulas A, B such that there occurs at least on of the below cases.

- $X = X' \cup \{(A \wedge B)\}$, $V(X' \cup \{(A \wedge B)\}) = 1$, $R = R_{\wedge}$ and $\langle X' \cup \{(A \wedge B)\}, X' \cup \{(A \wedge B), A, B\}\rangle \in R_{\wedge}$. Since $V((A \wedge B)) = 1$, so $V(A) = V(B) = 1$. Thus $V(X' \cup \{(A \wedge B), A, B\}) = 1$.
- $X = X' \cup \{(A \vee B)\}$, $V(X' \cup \{(A \vee B)\}) = 1$, $R = R_{\vee}$ and $\langle X' \cup \{(A \vee B)\}, X' \cup \{(A \vee B), A\}, X' \cup \{(A \vee B), B\}\rangle \in R_{\vee}$. Since $V((A \vee B)) = 1$, so $V(A) = 1$ or $V(B) = 1$. Thus $V(X' \cup \{(A \vee B), A\}) = 1$ or $V(X' \cup \{(A \vee B), A\}) = 1$.
- $X = X' \cup \{(A \to B)\}$, $V(X' \cup \{(A \to B)\}) = 1$, $R = R_{\to}$ and $\langle X' \cup \{(A \to B)\}, X' \cup \{(A \to B), \neg A\}, X' \cup \{(A \to B), B\}\rangle \in R_{\to}$. Since $V((A \to B)) = 1$, so $V(A) = 0$ or $V(B) = 1$. Hence $V(\neg A) = 1$ or $V(B) = 1$. Thus $V(X' \cup \{(A \to B), \neg A\}) = 1$ or $V(X' \cup \{(A \to B), B\}) = 1$.
- $X = X' \cup \{(A \leftrightarrow B)\}$, $V(X' \cup \{(A \leftrightarrow B)\}) = 1$, $R = R_{\leftrightarrow}$ and $\langle X' \cup \{(A \leftrightarrow B)\}, X' \cup \{(A \leftrightarrow B), A, B\}, X' \cup \{(A \leftrightarrow B), \neg A, \neg B\}\rangle \in R_{\leftrightarrow}$. Since $V((A \leftrightarrow B)) = 1$, so $V(A) = V(B)$. Hence $V(A) = V(B) = 1$ or $V(A) = V(B) = 0$. Thus $V(A) = V(B) = 1$ or $V(\neg A) = V(\neg B) = 1$. Consequently $V(X' \cup \{(A \leftrightarrow B), A, B\}) = 1$ or $V(X' \cup \{(A \leftrightarrow B), \neg A, \neg B\}) = 1$.
- $X = X' \cup \{\neg\neg A\}$, $V(X' \cup \{\neg\neg A\}) = 1$, $R = R_{\neg\neg}$ and $\langle X' \cup \{\neg\neg A\}, X' \cup \{\neg\neg A, A\}\rangle \in R_{\neg\neg}$. Since $V(\neg\neg A) = 1$, so $V(\neg A) = 0$, and $V(A) = 1$. Thus $V(X' \cup \{\neg\neg A, A\}) = 1$.
- $X = X' \cup \{\neg(A \wedge B)\}$, $V(X' \cup \{\neg(A \wedge B)\}) = 1$, $R = R_{\neg\wedge}$ and $\langle X' \cup \{\neg(A \wedge B)\}, X' \cup \{\neg(A \wedge B), \neg A\}, X' \cup \{\neg(A \wedge B), \neg B\}\rangle \in R_{\neg\wedge}$. Since $V(\neg(A \wedge B)) = 1$, so $V((A \wedge B)) = 0$. Hence $V(A) = 0$ or $V(B) = 0$, and consequently $V(\neg A) = 1$ or $V(\neg B) = 1$. Thus $V(X' \cup \{(A \wedge B), \neg A\}) = 1$ or $V(X' \cup \{(A \wedge B), \neg B\}) = 1$.
- $X = X' \cup \{\neg(A \vee B)\}$, $V(X' \cup \{\neg(A \vee B)\}) = 1$, $R = R_{\neg\vee}$ and $\langle X' \cup \{\neg(A \vee B)\}, X' \cup \{\neg(A \vee B), \neg A, \neg B\}\rangle \in R_{\neg\vee}$. Since $V(\neg(A \vee B)) = 1$, so $V((A \vee B)) = 0$, thus $V(A) = V(B) = 0$, and consequently $V(\neg A) = V(\neg B) = 1$. Thus $V(X' \cup \{\neg(A \vee B), \neg A, \neg B\}) = 1$.

- $X = X' \cup \{\neg(A \to B)\}$, $V(X' \cup \{\neg(A \to B)\}) = 1$, $R = R_{\neg\to}$ and $\langle X' \cup \{\neg(A \to B)\}, X' \cup \{\neg(A \to B), A, \neg B\}\rangle \in R_{\neg\to}$. Since $V(\neg(A \to B)) = 1$, so $V((A \to B)) = 0$, so $V(A) = 1$ and $V(B) = 0$. Thus $V(A) = 1$ and $V(\neg B) = 1$. Consequently $V(X' \cup \{\neg(A \to B), A, \neg B\}) = 1$.
- $X = X' \cup \{\neg(A \leftrightarrow B)\}$, $V(X' \cup \{\neg(A \leftrightarrow B)\}) = 1$, $R = R_{\neg\leftrightarrow}$ and $\langle X' \cup \{\neg(A \leftrightarrow B)\}, X' \cup \{\neg(A \leftrightarrow B), \neg A, B\}, X' \cup \{\neg(A \leftrightarrow B), A, \neg B\}\rangle \in R_{\neg\leftrightarrow}$. Since $V(\neg(A \leftrightarrow B)) = 1$, so $V(A) \neq V(B)$. Hence $V(A) = 0$ and $V(B) = 1$ or $V(A) = 1$ and $V(B) = 0$. Thus $V(\neg A) = V(B) = 1$ or $V(A) = V(\neg B) = 1$. Consequently, $V(X' \cup \{(A \leftrightarrow B), \neg A, B\}) = 1$ or $V(X' \cup \{(A \leftrightarrow B), A, \neg B\}) = 1$. □

Another lemma describes the relationship between finite, non-contradictory sets of formulas and the branches that originate from them. In this lemma we state that for a finite and non-contradictory set of formulas there is always at least one branch, beginning with this set, which is open and maximal.

Lemma 2.46 (On the existence of maximal and open branch)**.** *Let X be a finite set of formulas, and V be a valuation. If $V(X) = 1$, then there exists at least one maximal and open branch $X_1, \ldots, X_n$ such that $X_1 = X$ and $n \geq 1$.*

Proof. Take any finite set of formulas X and valuation V such that $V(X) = 1$. Based on conclusion 2.37, we know that set of all maximal branches $MB(X)$ is non-empty. Indirectly assume that none of the branches contained in $MB(X)$ is open.

Now, consider the branches beginning with set X, taking accounts of their lengths. Through inductive proof, we will show that for any $n \in \mathbb{N}$ there exists such open branch $X_1, \ldots, X_n$ that $X_1 = X$ and there exists such set of formulas Y that sequence $X_1, \ldots, X_n, X_{n+1}$, where $X_{n+1} = Y$, is also an open branch.

Initial step. There exists an open branch with the length of 1 beginning with set X. It is branch $X_1 = X$. Since $V(X) = 1$, by definition 2.9, set X is not contradictory, while by virtue of conclusion 2.11, set X is not t-inconsistent. So, by definition of open branch 2.38 branch X_1 is open. Hence, by virtue of the indirect assumption, X_1 is not a maximal branch. Thus, by definition of maximal branch 2.30 and lemma 2.45, there exists branch X_1, X_2 such that $V(X_2) = 1$. By definition 2.9, set X_2 is not contradictory, while by virtue of conclusion 2.11, set X_2 is not t-inconsistent. So, by definition of open branch 2.38, branch X_1, X_2 is open.

Induction step. Assume that for some $n \in \mathbb{N}$, there exists such open branch $X_1, \ldots, X_n$ that $X_1 = X$, $V(X_n) = 1$ and $n \geq 2$. Hence, by the indirect assumption, $X_1, \ldots, X_n$ is not a maximal branch.

Since $V(X_n) = 1$, by definition 2.9, set X_n is not contradictory, while by virtue of conclusion 2.11, set X_n is not t-inconsistent. Thus, by definition of maximal branch

2.30 and lemma 2.45, there exists branch $X_1, \ldots, X_n, X_{n+1}$ such that $V(X_{n+1}) = 1$. By definition 2.9, set X_{n+1} is not contradictory, while by virtue of conclusion 2.11, set X_{n+1} is not t-inconsistent. So, by definition of open branch 2.38, branch $X_1, \ldots, X_n, X_{n+1}$ is open.

Consequently, for any $n \in \mathbb{N}$, there exists such open branch $X_1, \ldots, X_n$ that $X_1 = X$ and there exists such set of formulas Y that sequence $X_1, \ldots, X_n, X_{n+1}$, where $X_{n+1} = Y$, is also an open branch and is not a maximal branch.

From this it follows that there exists an infinite branch $Y_1, Y_2, \ldots$ such that

1. $Y_1 = X$
2. $Y_{n+1} = X_{n+1}$, where X_{n+1} is such set that sequence $X_1, \ldots, X_n, X_{n+1}$, is also an open branch.

But, since X is a finite set of formulas, so the fact of existence of infinite branch $Y_1, Y_2, \ldots$ contradicts fact 2.28. So, the indirect assumption that each branch contained in $MB(X)$ is closed, is false. So there exists a branch beginning with X, which is maximal and open. □

With a lemma on the existence of maximal and open branch, we can now prove the soundness theorem.

Theorem 2.47 (Soundness). *For any $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, $A \in \mathsf{For}_{\mathbf{CPL}}$, if $X \rhd A$, then $X \vDash A$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, $A \in \mathsf{For}_{\mathbf{CPL}}$ and assume that $X \nvDash A$. So by definition 2.6 there exists such valuation V that $V(X \cup \{\neg A\}) = 1$. Hence for any finite set $Y \subseteq X$, $V(Y \cup \{\neg A\}) = 1$. From the previous lemma 2.46 it follows that for any finite $Y \cup \{\neg A\}$ there exists at least one maximal and open branch $X_1, \ldots, X_n$ such that $X_1 = Y \cup \{\neg A\}$ i $n \geq 1$. So there is no such finite set $Z \subseteq X$ that each maximal branch beginning with $Z \cup \{\neg A\}$ is closed. Hence by definition 2.41 $X \not\rhd A$. □

2.4.2 Completeness theorem

In this section, we will show that the relation of semantic consequence is contained in the relation of branch consequence, hence for any set of formulas X and any formula A it is the case that if $X \vDash A$, then $X \rhd A$.

Also here, we will use abbreviations concerning the valuations of the formulas adopted in denotation 2.5.

In our proofs, we will use the fact that the classical consequence relation $\vDash$ is compact. However, let us first recall the general definition of compactness of a binary relation, because the concept of a compact relation will be useful in the next chapter.

Definition 2.48 (Compact relation)**.** Take any set of objects X and binary relation R, specified as follows $R \subseteq P(X) \times X$. We shall state that relation R is *compact* iff for any $Y \subseteq X$ and for any $x \in X$: $YRx \Leftrightarrow$ there exists a finite set Y' such that $Y' \subseteq Y$ and $Y'Rx$.

The well-known fact of the compactness of semantic consequence relation of **CPL** is expressed in the next fact.

Proposition 2.49 (Compactness)**.** *For any* $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, $A \in \mathsf{For}_{\mathbf{CPL}}$, $X \vDash A$ *iff there exists such finite set* $Y \subseteq X$ *that* $Y \vDash A$.

We will start with a lemma which says about the relationship between the valuation of propositional letters and the negation of propositional letters in a maximal branch and the valuation of formulas from some initial set.

Lemma 2.50. *Let* $X_1, \ldots, X_n$, *for some* $n \geq 1$, *be a maximal and open branch. Let* $L(X_n) = \{x \in X_n : x = y$ *or* $x = \neg y$, *for some* $y \in \mathsf{Var}\}$. *Then for any valuation* V, *if* $V(L(X_n)) = 1$, *then* $V(X_n) = 1$.

Proof. We will carry out an inductive proof, taking account of the complexity of formulas in a branch.

Take any maximal and open branch $X_1, \ldots, X_n$, for some $n \geq 1$, and valuation V such that $V(L(X_n)) = 1$, where $L(X_n) = \{x \in X_n : x = y$ or $x = \neg y$, for some $y \in \mathsf{Var}\}$. Let $A \in X_n$.

Initial step. If $A = y$ or $A = \neg y$, for some $y \in \mathsf{Var}$, then $V(A) = 1$, since $A \in L(X_n)$.

Induction step. Assume that for all formulas B, $C \in X_n$ that have the lesser complexity than A, $V(B) = V(C) = 1$. Our proof will be based on possible cases of constructing formula A and on the initial assumption that $X_1, \ldots, X_n$ is a maximal branch. Since the branch is maximal, so if A could be decomposed by some of the tableau rules, then its component (components) are already in X_n (as for all $i \leq n$, $X_i \subseteq X_n$).

Now, for some formulas B, C occurs one of the cases:

- $A := (B \wedge C)$, and so $B, C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(B) = V(C) = 1 = V((B \wedge C))$
- $A := (B \vee C)$, and so B or $C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(B) = 1$ or $V(C) = 1$, therefore $V((B \vee C)) = 1$
- $A := (B \rightarrow C)$, and so $\neg B$ or $C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(\neg B) = 1$ or $V(C) = 1$, therefore $V((B \rightarrow C)) = 1$

- $A := (B \leftrightarrow C)$, and so B, C or $\neg B$, $\neg C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(B) = 1 = V(C)$ or $V(\neg B) = 1 = V(\neg C)$, therefore $V((B \leftrightarrow C)) = 1$
- $A := \neg\neg B$, and so $B \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(B) = 1 = V(\neg\neg B)$
- $A := \neg(B \wedge C)$, and so $\neg B$ or $\neg C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(\neg B) = 1$ or $V(\neg C) = 1$, therefore $V(\neg(B \wedge C)) = 1$
- $A := \neg(B \vee C)$, and so $\neg B, \neg C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(\neg B) = V(\neg C) = 1 = V(\neg(B \vee C))$
- $A := \neg(B \rightarrow C)$, and so $B, \neg C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(B) = V(\neg C) = 1 = V(\neg(B \rightarrow C))$
- $A := \neg(B \leftrightarrow C)$, and so $\neg B$, C or B, $\neg C \in X_n$, on the inductive assumption and by definition of valuation of formulas 2.4 we then get that $V(\neg B) = 1 = V(C)$ or $V(B) = 1 = V(\neg C)$, therefore $V(\neg(B \leftrightarrow C)) = 1$.

Consequently, $V(X_n) = 1$, since $V(A) = 1$, for all $A \in X_n$. □

The next lemma will show that a maximal and open branch allows to define a valuation that assigns the value of truth to each formula contained in the first element of branch.

Lemma 2.51 (Lemma on the existence of valuation)**.** *Let $X_1, \ldots, X_n$, for some $n \geq 1$, be a maximal and open branch. Then, there exists valuation V such that $V(X_1) = 1$.*

Proof. Take any maximal and open branch $X_1, \ldots, X_n$, for some $n \geq 1$. Now, we define $L(X_n) = \{x \in X_n : x = y \text{ or } x = \neg y, \text{ for some } y \in \mathsf{Var}\}$. Since $L(X_n)$ is t-consistent, we define valuation $v : \mathsf{Var} \longrightarrow \{1, 0\}$ such that for any $x \in \mathsf{Var}$:

- $v(x) = 1$, if $x \in L(X_n)$
- $v(x) = 0$, if $x \notin L(X_n)$.

We see that both $v(L(X_n)) = 1$ and $V(L(X_n)) = 1$, where V is an extension of function v to the set of all formulas.

Now, using lemma 2.50, we get thesis $V(X_1) = 1$, since $X_1 \subseteq X_n$. □

The lemma on the existence of valuation allows us to prove the completeness theorem.

Theorem 2.52 (Completeness). *For any $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, $A \in \mathsf{For}_{\mathbf{CPL}}$, if $X \vDash A$, then $X \rhd A$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, $A \in \mathsf{For}_{\mathbf{CPL}}$ and assume that $X \ntriangleright A$. Thus, by definition of branch consequence relation 2.41 for each finite subset $Y \subseteq X$ there exists a maximal branch beginning with $Y \cup \{\neg A\}$ which is open. By virtue of the lemma on the existence of valuation 2.51 for each finite set $Y \subseteq X$ there exists valuation V such that $V(Y \cup \{\neg A\}) = 1$. Thus, by definition of valuation of formulas 2.4 and consequence classical relation 2.6, for each finite set $Y \subseteq X$, $Y \nvDash A$. Hence and by virtue of the compactness property of relation of consequence $\vDash$, fact 2.49, we get thesis $X \nvDash A$. □

2.5 Tableaux for CPL vs. semantic consequence relation

So we can see that the concepts of relation of semantic consequence and relation of branch consequence denote exactly the same set of pairs $\langle X, A \rangle$, where X is a set of formulas, while A is a formula. In practice, however, it is not easy to determine whether a pair belongs to relation $\rhd$. According to definition 2.41, in order to achieve this, we need to select from X its finite subset Y such that each maximal branch beginning with set $Y \cup \{\neg A\}$ is closed. The first stage of this activity, i.e. the selection of an appropriate set, is difficult to study in general, since further we will deal with various different logics, and the relation of branch consequence has been defined for any sets, particularly infinite ones. However, if the set of premises is a finite set, we can proceed straight to the second stage, i.e. reviewing all maximal branches and checking if they are closed branches.

Unfortunately, although their number in the case of a finite set of premises will also be finite, it may be so large that their construction and examination are only theoretically possible. Therefore, we need a method that allows to select and study a sufficiently small number of maximal branches, the closure of which guarantees the occurrence of branch consequence relation.

In our theory, this method is based on the concept of a tableau. By tableau we mean a finite and minimal set of branches beginning with the same set. Of course, such a concept of tableau is far from its graphic presentation or one based on graphs. Due to its formal nature, it specifies the standard concept of tableau, plus its scope at least includes a set of those objects which are traditionally called semantic or analytical trees or tableaux. The problem of this relation will be dealt with in the last chapter of the book.

Let us now proceed to the definition of tableau.

Definition 2.53 (Tableau)**.** Let $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, $A \in \mathsf{For}_{\mathbf{CPL}}$, while Φ will be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called a *tableau for* $\langle X, A \rangle$ (or shortly *tableau*) iff the below conditions are met:

1. Φ is a non-empty subset of set of branches beginning with $X \cup \{\neg A\}$ (i.e. if $\psi \in \Phi$, then $\psi(1) = X \cup \{\neg A\}$)
2. each branch contained in Φ is Φ-maximal
3. for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
   - i and $i+1$ belong to the domains of functions $\psi_1, \ldots, \psi_n$
   - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

   then there exists such rule $R \in \mathbf{R}_{\mathbf{CPL}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$, that for any $1 \leq k \leq n$:
   - $\psi_k(i) = Y_1$
   - and there exists such $1 < l \leq m$ that $\psi_k(i+1) = Y_l$.

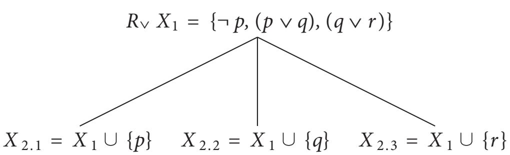
Remark 2.54. The first two conditions in the tableau definition are standard ones. Each branch in the tableau begins with a set which is the sum of the set of premises and the negation of the presumed conclusion. In addition, the tableau contains only Φ-maximal branches, so it does not include sub-branches of the branches belonging to Φ.

However, the third condition is particularly worth discussing. This condition says that branching can occur in a tableau only if there is a suitable rule and its ordered triple which contains two output sets corresponding to the branching.

We have deliberately reduced number m to range $\{2, 3\}$, since the rules in set $\mathbf{R}_{\mathbf{CPL}}$ have at least two elements each, but not more than three. In the case of tableau systems for other logics, however, it may happen that m will be any number greater than 1 which will allow for several branchings at a given stage of the tableau construction. In general, the upper range of number m minus 1 means the number of branchings that can appear in the tableau at a given stage of construction.

Therefore, definition of tableau 2.53 excludes such sets of branches from being considered tableaux as in example 2.55.

Example 2.55. We consider set $\{\neg p, (p \vee q), (q \vee r)\}$ and create three branches.



Although there exists a rule — it is rule $R_\vee$ — which contains ordered triples $\langle X_1, X_{2.1}, X_{2.2}\rangle$ and $\langle X_1, X_{2.2}, X_{2.3}\rangle$, set $\mathbf{R_{CPL}}$ includes no rule to comprise a ordered quadruple $\langle X_1, X_{2.1}, X_{2.2}, X_{2.3}\rangle$, therefore the presented set of branches $\{\langle X_1, X_{2.1}\rangle, \langle X_1, X_{2.2}\rangle, \langle X_1, X_{2.3}\rangle\}$ does not meet the third condition of the definition of tableau 2.53.

As we can see, the concept of tableaux has been defined in such a way that tableaux can also begin with infinite sets.

In practice, the construction of tableau is to show that a given formula is a branch consequence of a given finite set of premises. To this end, we must construct tableaux that have all the elements necessary to solve the problem. Tableaux with these properties are called complete tableaux. But, before we proceed to the definition of complete tableau, we will consider one more issue.

When constructing a tableau, it may happen that branchings and branches are formed which are redundant variants of the already existing branches. Let us consider the following two examples.

Example 2.56. Consider set $\{p \vee q, \neg\neg p\}$ and create a branch using a rule $R_{\neg\neg}$ to this set. We get the following branch.

$$\begin{array}{c} R_{\neg\neg}\ X_1 = \{p \vee q, \neg\neg p\} \\ | \\ X_2 = X_1 \cup \{p\} \end{array}$$

Branches X_1, X_2 cannot be any more extended since ordered triple $\langle X_1 \cup \{p\}, X_1 \cup \{p\}, X_1 \cup \{p, q\}\rangle$ does not belong to rule $R_\vee$ due to the fact that $X_1 \cup \{p\} \not\subset X_1 \cup \{p\}$, whereas the input set should be contained in each output set.

We can, however, starting from set $\{p \vee q, \neg\neg p\}$, through the use of rule $R_\vee$, produce two branches, and than apply rule $R_{\neg\neg}$ to set $X_{2.2}$. Then we get the following branches.

$$\begin{array}{ccc} & R_\vee\ X_1 = \{(p \vee q), \neg\neg p\} & \\ \swarrow & & \searrow \\ X_{2.1} = X_1 \cup \{p\} & & R_{\neg\neg}\ X_{2.2} = X_1 \cup \{q\} \\ & & | \\ & & X_3 = X_{2.2} \cup \{p\} \end{array}$$

In the light of definition of tableau 2.53, the set of these two branches is, obviously, a tableau. However, from the viewpoint of a tableau complexity and information it provides, the branch on the right seems unnecessary. This is because if we fail to get t-inconsistent set in the right-hand branch, we will also fail to get t-inconsistent set in the left-hand branch. On each set of formulas Y we generally know from the definition of tableau inconsistent set of formulas 2.10 that Y is not t-inconsistent iff each of its subsets is not t-inconsistent, and the point of constructing a tableau is precisely finding a t-inconsistency. Therefore, the branch on the right seems superfluous, and since it brings nothing important, it can be called redundant.

Such branchings are not a formal obstacle, hence they can be accepted. However, due to the economy of a tableau construction, they can be disregarded. Further concepts will be defined so that the tableau with or without redundant branches can be considered a complete tableau. Practically, we know that when we try to write or draw out a tableau proof, we endeavour to take account of all possibilities. So there is no reason to further restrict this process — and we allow both options. Let us now proceed to the formal concept of a redundant variant of branch.

Definition 2.57 (Redundant variant of branch)**.** Let ϕ and ψ be such branches that if there exists number i that i and $i+1$ belong to their domains, then for any $j \leq i$, $\phi(j) = \psi(j)$, but $\phi(i+1) \neq \psi(i+1)$. We shall state that branch ψ is a *redundant variant* of branch ϕ iff:

1. there exist such rule $R \in \mathbf{R_{CPL}}$ and such pair $\langle X, Y \rangle \in R$ that $X = \phi(i)$ and $Y = \phi(i+1)$
2. there exist such rule $R \in \mathbf{R_{CPL}}$ and such triple $\langle X, W, Z \rangle \in R$ that $X = \phi(i)$ and:
   a. $W = \phi(i+1)$ and $Z = \psi(i+1)$
   or
   b. $Z = \phi(i+1)$ and $W = \psi(i+1)$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. We shall state that Ψ is an *redundant superset* of Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ.

Let us consider another example of tableau with redundant branches.

Example 2.58. Take set $\{\neg\neg p, \neg\neg q, p \vee q\}$. By applying rules $R_\vee$ and $R_{\neg\neg}$ successively, we will get an interesting case of tableau.

$R_\vee \; X_1 = \{p \vee q, \neg\neg p, \neg\neg q\}$

$R_{\neg\neg} \; X_{2.1} = X_1 \cup \{p\}$ $\qquad$ $R_{\neg\neg} \; X_{2.2} = X_1 \cup \{q\}$

$X_{3.1} = X_{2.1} \cup \{q\}$ $\qquad$ $X_{3.2} = X_{2.2} \cup \{p\}$

Sets $X_{3.1}$ and $X_{3.2}$ are identical. So, we face an interesting situation where both branches "diverged" (sets $X_{2.1}$ and $X_{2.2}$ are different), and then "converged" (sets $X_{3.1}$ and $X_{3.2}$ are identical). Due to the definition of redundant variant of branch 2.57, each of those two branches is a redundant variant of the other.

Remark 2.59. The concept of a redundant variant of branch can be extended onto other cases of branchings where one of branches brings no expressions into the tableau that are necessary to obtain the answer to the question whether given inference is correct. Nevertheless, we will not expand this concept. After all, we are not interested in the economy of tableau proof. We introduced this concept in order to show that the adopted concepts of tableau and branch allows us to distinguish certain types of branches, and consequently, to distinguish certain sets of tableaux that are less complex.

Now, we will move on to the definition of complete tableau.

Definition 2.60 (Complete tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff:

1. each branch contained in Φ is maximal
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$
   b. $\langle X, A, \Psi \rangle$ is a tableau

   is a redundant superset of Φ.

A tableau is *incomplete* iff it is not complete.

In a complete tableau, all branches are maximal, not only the maximal ones in a given set. In addition, a complete tableau is such set of branches that adding a new branch to it at most gives us a redundant superset or the set ceases to be a tableau. In other words, a complete tableau is such set of maximal branches that any of its supersets ceases to be a tableau or is a redundant superset.

When constructing a complete tableau, we can come across a situation in which all the branches are closed, meaning each branch ends with a t-inconsistent set. Such a tableau is called closed tableau. Let us first define a closed/open tableau, and then discuss the definition.

Definition 2.61 (Closed/open tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff the below conditions are met:

1. each branch contained in Φ is closed
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$

b. $\langle X, A, \Psi \rangle$ is a tableau

is a redundant superset of Φ.

A branch is *open* iff it is not closed.

As we have said, by the above definitions — the definition of complete tableau 2.60 and the definition of closed tableau 2.61 — and the conclusion 2.40, we have another conclusion.

Corollary 2.62. *Each closed tableau is a complete tableau.*

Moreover, among complete tableaux, those tableaux that only contain sets of closed branches are closed tableaux. So we have another conclusion.

Corollary 2.63. *Each complete tableau in which the set of branches only contains closed branches is a closed tableau.*

Making use of conclusions 2.62 and 2.63, we can, therefore, simplify the definition of closed/open tableau by formulating another conclusion.

Corollary 2.64 (Closed/open tableau)**.** *Let $\langle X, A, \Phi \rangle$ be a tableau. Tableau $\langle X, A, \Phi \rangle$ is closed iff the below conditions are met:*

1. *$\langle X, A, \Phi \rangle$ is a complete tableau*
2. *each branch contained in Φ is closed.*

A branch is open iff it is not closed.

Conclusion 2.64 can be adopted as an equivalent version of the definition of closed/open tableau.

Further, we will show that the concept of tableau is significantly helpful in determining the occurrence of relation $\triangleright$.

Now, we will focus on the fact that the initial, finite set of formulas allows to construct a complete tableau.

Proposition 2.65. *Let X be a finite subset of set For_{CPL} and let $A \in \mathsf{For}$. Then, there exists at least one complete tableau $\langle X, A, \Phi \rangle$.*

Proof. Consider a set of all branches $B(X \cup \{\neg A\})$. We know that the set of maximal branches $MB(X \cup \{\neg A\})$ is non-empty (fact 2.37). In addition, for each branch $\phi \in B(X \cup \{\neg A\})$ there exists branch $\psi \in MB(X \cup \{\neg A\})$ such that $\phi \subseteq \psi$, by 2.34.

Let us now define such set $\Phi \subseteq MB(X \cup \{\neg A\})$ that Φ is a maximal set among the sets that meet the following condition:

- for any $n, i \in \mathbb{N}$ and any branches $\phi_1, \ldots, \phi_n \in \Phi$, if:
  - i and $i+1$ belong to domains of functions $\phi_1, \ldots, \phi_n$
  - for any $1 < k \leq n$ and any $o \leq i$, $\phi_1(o) = \phi_k(o)$,

  then there exists such rule $R \in \mathbf{R_{CPL}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$, that for any $1 \leq k \leq n$:
  - $\phi_k(i) = Y_1$
  - and there exists such $1 < l \leq m$ that $\phi_k(i+1) = Y_l$.

By definition of tableau 2.53, $\langle X, A, \Phi \rangle$ is a tableau. What is more, since Φ is a maximal set among those meeting the given condition, then by the definition of complete tableau 2.60, $\langle X, A, \Phi \rangle$ is a complete tableau. Since $MB(X \cup \{\neg A\})$ is non-empty, there exists at least one Φ that meets the given condition. □

We will now move on to a lemma that defines the relationship between the existence of a closed tableau and complete tableaux.

Lemma 2.66. *Let X be a finite subset of set $\mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$. Then, the following two statements are equivalent:*

1. *there exists closed tableau $\langle X, A, \Phi' \rangle$*
2. *each complete tableau $\langle X, A, \Phi'' \rangle$ is closed.*

Proof. Take any finite subset X of set $\mathsf{For}_{\mathbf{CPL}}$ and any formula $A \in \mathsf{For}_{\mathbf{CPL}}$.

First, we will take up the proof of implication (1) $\Rightarrow$ (2). Indirectly assume that there exists closed tableau $\langle X, A, \Phi' \rangle$ and not every complete tableau $\langle X, A, \Phi'' \rangle$ is closed. Hence, there exists complete tableau $\langle X, A, \Phi'' \rangle$ which is not closed. By definition of open tableau 2.61, $\langle X, A, \Phi'' \rangle$ is an open tableau. And since $\langle X, A, \Phi'' \rangle$ is a complete tableau, then each branch contained in Φ'' is maximal. Hence, there exists branch $\psi \in \Phi''$ such that:

1. ψ begins with $X \cup \{\neg A\}$
2. ψ is a maximal branch
3. ψ is an open branch.

Since ψ is an open branch, then by definition of open branch 2.38, there is no such formula A that A and $\neg A$ belong to the union of all the elements of branch $\bigcup \psi$.

Note that since branch ψ is maximal and open, then it is also closed under tableau rules in this regard that for any $R \in \mathbf{R_{CPL}}$ and any n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, where $n > 1$, if $X_1 \subseteq \bigcup \psi$, then for some $1 < j \leq n$, $X_j \subseteq \bigcup \psi$. For if that was not the case, there would exist rule $R \in \mathbf{R_{CPL}}$ and such n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, where $n > 1$, that $X_1 \subseteq \bigcup \psi$, but for none $1 < j \leq n$, $X_j \nsubseteq \bigcup \psi$. And it would mean that branch ψ

is not maximal, because $\bigcup\psi$ is identical to the last element of open and maximal branch ψ, by definition of branch 2.20.

We assumed that there existed closed tableau $\langle X, A, \Phi'\rangle$. By virtue of conclusion 2.64, $\langle X, A, \Phi'\rangle$ is a complete tableau. Hence, each branch contained in Φ' is maximal and Φ' contains all such branches without which ordered triple $\langle X, A, \Phi'\rangle$ would not be a complete tableau. Let us adopt designation $X \cup \{\neg A\} = Y_1$.

We will carry out an inductive proof with respect to n-th elements of branches contained in Φ', showing that they enable construction of an infinite branch beginning with set Y_1.

Initial step. Consider the first element of each branch contained in Φ'. It is set Y_1. Since $Y_1 \subseteq \bigcup\psi$, and ψ is an open branch, then there must exist a rule $R \in \mathbf{R_{CPL}}$ such that $\langle Y_1, Z_1, \ldots, Z_n\rangle \in R$, where $n \geq 1$, and — since $\langle X, A, \Phi'\rangle$ is a complete tableau — for each $1 \leq j \leq n$ there exists a branch in Φ' such that it contains Z_j. Since branch ψ is also closed under tableau rules, then certain $Z_j \subseteq \bigcup\psi$. So, Φ' includes such branch that its first element is Y_1, and second $Y_2 = Z_j$, and $Y_2 \subseteq \bigcup\psi$

Induction step. Assume that set Φ' includes such branch that its first n elements $Y_1, \ldots, Y_n$ are contained in set $\bigcup\psi$. Since $Y_n \subseteq \bigcup\psi$, and ψ is an open branch, then there must exist a rule $R \in \mathbf{R_{CPL}}$ such that $\langle Y_n, Z_1, \ldots, Z_k\rangle \in R$, where $k \geq 1$, and — since $\langle X, A, \Phi'\rangle$ is a complete tableau — for each $1 \leq j \leq k$ there exists a branch in Φ' such that it contains Z_j. Since branch ψ is also closed under tableau rules, then certain $Z_j \subseteq \bigcup\psi$. So, Φ' includes such branch that its first $n+1$ elements $Y_1, \ldots, Y_n, Y_{n+1}$ are contained in set $\bigcup\psi$.

So, for each branch in Φ', the first element equals Y_1 and $Y_1 \subseteq \bigcup\psi$ and for each $n \in \mathbb{N}$ if there exists in Φ' such branch that its first n elements $Y_1, \ldots, Y_n$ are contained in set $\bigcup\psi$, then also in Φ' there exists such branch that its first n elements equal $Y_1, \ldots, Y_n$, and its $n+1$-element Y_{n+1} is contained in set $\bigcup\psi$.

Now, we take all elements Y_i, where $i \in \mathbb{N}$ and we put them in increasing sequence $Y_1, Y_2, Y_3, \ldots$. This sequence is a branch, because for any Y_j there exists a rule $R \in \mathbf{R_{CPL}}$ such that $\langle Y_j, Z_1, \ldots, Z_k\rangle \in R$, where $k \geq 1$, and $Y_{j+1} = Z_l$, for some $1 \leq l \leq k$.

There exists therefore an infinitely long branch $Y_1, \ldots, Y_n, \ldots$ such that $Y_1 = X \cup \{\neg A\}$. But, since set $X \cup \{\neg A\}$, by assumption, is finite, then it contradicts fact 2.28.

Let us now move on to the proof of implication (2) ⇒ (1). Assume that each complete tableau $\langle X, A, \Phi''\rangle$ is closed. From fact 2.65, we know that for finite set of formulas X there exists at least one complete tableau $\langle X, A, \Phi\rangle$. Therefore, there exists closed tableau $\langle X, A, \Phi'\rangle$. □

We will now consider the relationship between complete and closed tableaux and the occurrence of branch consequence relation.

Lemma 2.67 (On relation between complete tableaux and branch consequence). *Let $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$. Then, the two below statements are equivalent:*

1. *there exists such finite set $Y \subseteq X$ that each complete tableau $\langle Y, A, \Phi \rangle$ is closed*
2. $X \triangleright A$.

Proof. The proof of the above equivalence is based on the definition of the notion tableau 2.60, notion of a closed tableau 2.61 and notion of a relation of branch consequence $\triangleright$.

Take $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$, and assume there exists such finite set $Y \subseteq X$ that each complete tableau $\langle Y, A, \Phi \rangle$ is closed. Therefore, each maximal branch that begins with set $Y \cup \{\neg A\}$ is closed. So, there exists such finite set $Y \subseteq X$ that each maximal branch that begins with set $Y \cup \{\neg A\}$ is closed. Hence, $X \triangleright A$.

On the other hand, if $X \triangleright A$, then there exists such finite set $Y \subseteq X$ that each maximal branch that begins with set $Y \cup \{\neg A\}$ is closed. Hence, there exists such finite set $Y \subseteq X$ that each complete tableau $\langle Y, A, \Phi \rangle$ is closed. □

We now proceed to the most important theorem of this subchapter. The occurrence of this theorem, or at least of its part "from the left to the right," may be a criterion for the correctness of tableau system construction. When constructing a tableau emerging from a given set of formulas, we can usually do it in many ways. In practice, however, we construct one tableau, checking whether each branch ends with a contradictory set. Usually we also assume that it is sufficient to state that given formula belongs to the set of correct conclusions from the initial set of formulas.

Intuitively, however, it seems doubtful. Why should one closed tableau be a proof if it is potentially possible to construct more tableaux? After all, usually, we have to rule out all cases in indirect proofs, and this is what the construction of a tableau is. Another theorem removes this doubt by saying that in order to determine the occurrence of relation of branch consequence, it is sufficient to construct one closed tableaux.

Theorem 2.68 (Theorem on the tableau arbitrariness). *Let $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$. Then, the two below statements are equivalent:*

1. *there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$*
2. $X \triangleright A$.

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$ and assume there exists such finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$. By virtue of lemma 2.66, we get the conclusion that there exists such finite set $Y \subseteq X$ that each complete tableau $\langle Y, A, \Phi' \rangle$ is closed. And from the above, and from lemma 2.67, it follows that $X \rhd A$.

Now, assume that $X \rhd A$. By virtue of lemma 2.67, there exists such finite set $Y \subseteq X$ that each complete tableau $\langle Y, A, \Phi' \rangle$ is closed. So, from lemma 2.66, it follows that there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$. □

The construction of one closed tableau which begins with a finite subset of set of premises X and negation of formula A is, therefore, equivalent to the fact that $X \rhd A$. Because previously, through the theorems on soudness and completeness, we proved that relations $\vDash$ and $\rhd$ are identical, i.e. they define the same set of pairs, now we can put in words the semantic form of the theorem on the tableau arbitrariness.

Theorem 2.69 (Theorem on the tableau arbitrariness — semantic form)**.** *Let $X \subseteq \mathsf{For}_{\mathbf{CPL}}$ and $A \in \mathsf{For}_{\mathbf{CPL}}$. Then, the two below statements are equivalent:*

1. *there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$*
2. *$X \vDash A$.*

This theorem says that the logical relation occurs between the premises and conclusion if and only if it is possible to select a finite set of premises, and then by attaching to it the negation of conclusion, construct a closed tableau emerging from that set.

2.6 Summary

In this chapter we presented a theory for the construction of a tableau system for **CPL**, using the method of defining tableau rules as rules that extend sets.

For the purposes of presentation, we separately showed the relationships between the semantic consequence relation, the branch consequence relation and the existence of a closed tableau, proving the theorems on the equivalence of these concepts. In practice, however, it is easier to prove a theorem equivalent to the conjunction of the above theorems, i.e. the following theorem.

Theorem 2.70 (Theorem on the completeness of tableau system of **CPL**)**.** *For any $X \subseteq \mathsf{For}_{\mathbf{CPL}}$, $A \in \mathsf{For}_{\mathbf{CPL}}$, the below statements are equivalent.*

- *$X \vDash A$*
- *$X \rhd A$*
- *there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$.*

Although this theorem is equivalent to the conjunction of earlier theorems — as we can see — its demonstration requires a proof of three implications. Certain proof transitions may therefore be omitted.

Theorem 2.70, was named a theorem on the completeness of tableau system of **CPL**, referring to the usual name to define the relationship between the semantic characteristics of given logic and its deductive definition. Often, when speaking on the completeness of a given deductive system, we mean not only a one-way relationship, but a relationship occurring two ways, that is, both soundness and completeness in the strict sense.

In the presentation of other tableau systems constructed using the presented method, we will always strive to demonstrate that there occurs a relevant theorem on the completeness of the tableau system, formulated in an analogous way as theorem 2.70. The proof of such theorem will be a positive criterion for the good formulation of the tableau system.

3 Tableau system for Term Logic

3.1 Introductory remarks

Now, we will describe the tableau system for Term Logic[1] (for short: **TL**).[2] By Term Logic we mean a logic in which both premises and conclusions have the form of classical categorical propositions:[3]

- Each *P* is *Q*.
- Some *P* is *Q*.
- No *P* is *Q*.
- Some *P* is not *Q*.

Moreover, we do not assume that in Term Logic the names appearing in categorical propositions are not empty. So we consider the most general approach — the simplest language and semantics.

The tableau system we will describe can be treated as a stand-alone system. But, this is not the purpose of its construction. We intend to indicate an example of the use of analogous tableau concepts, such as those we defined for the tableau system for **CPL**. Although the defined system will also feature the property of a finite branch, there will be new features that will be mentioned soon.

We will redefine the concepts of rule, various types of branches and tableaux in an almost identical way to the tableau concepts defined in the previous chapter. We mean *almost identical* because, after all, we will face a different language of tableau proof and a different set of tableau rules than in the case of the tableau

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1 Because we are going to name systems of various types of reasoning about the relationships between names, we will use rather term 'Term Logic' than 'Syllogistic logic'.

2 The considerations contained in this chapter are based on the English language study [9]. In that paper, we presented an outline of the tableau system for **TL** without a thorough analysis of details. Since the production of this article, we have also generalized the concept of tableau rules and modified other tableau concepts which has affected other concepts and the very nature of the tableau system. Some other variants of tableaux for syllogistic are presented in [15], [22].

3 We write about classic categorical propositions because categorical propositions can also take non-classical forms. They can be e.g. any numerical propositions: *At least five P are Q*, or e.g. modal propositions *de re*: *Each P is by necessity Q*. Besides, in the last chapter we will construct a tableau system for the logic of categorical modal propositions *de re*. In addition, there are many other possibilities to enrich classical categorical propositions.

system for **CPL**. However, these differences will not affect the formal nature of concepts themselves, so although, for example, a branch for **TL** will be built from different sets of expressions than a branch for **CPL**, the structure of presentation itself will be identical — because we aim at presenting a general scheme of the tableau system construction, a synthesis consisting in abstracting from tableau concepts those properties that are not specific, and therefore do not depend on the characteristics of exemplary systems which we define in detail in the initial chapters of the book.

The importance of the presented tableau system **TL** for our considerations consists in the fact that the set of formulas for Term Logic is a proper subset of the set of proof expressions, and moreover, there are no branchings in the tableaux. This case is located between borderline cases because of the relation of the set of logic formulas to the set of tableau expressions. However, in some respects it has a borderline character itself, because the tableaux do not host any branchings, so the tableau proofs boil down to the construction of a single maximal branch. General tableau concepts could be therefore simplified in the tableau system for **TL** although they still provide more detail on the general concepts that we describe in the book.

3.2 Language and semantics

The construction of tableau system for Term Logic requires, as usual, the presentation of basic concepts. Let us start with the alphabet of language of **TL**.

Definition 3.1 (Alphabet of **TL**). The alphabet of Term Logic is the sum of the following sets:

- set of logical constants: $\mathsf{Lc} = \{\mathbf{a}, \mathbf{i}, \mathbf{e}, \mathbf{o}\}$
- set of name letters: $\mathsf{Ln} = \{P^1, Q^1, R^1, P^2, Q^2, R^2, \ldots\}$.

Although the set of name letters is infinite and includes indexed letters, in practice we will use a finite number of the following letters: P, Q, R, S, T, U, treating them as metavariables ranging over set Ln.

Let us now proceed to the definition of formula of **TL**.

Definition 3.2 (Formula of **TL**). Set of formulas **TL** is the smallest set containing the following expressions:

- $P\mathbf{a}Q$
- $P\mathbf{i}Q$
- $P\mathbf{e}Q$
- $P\mathbf{o}Q$

where $P, Q \in \mathsf{Ln}$.
We specify this set as $\mathsf{For}_{\mathbf{TL}}$, and its elements will be called *formulas*.

Another basic concept is the concept of model for **TL**, and then the concept of truth in the model.

Definition 3.3 (Model for language of **TL**). *Model* $\mathfrak{M}_{\mathbf{TL}}$ *for language of* **TL** will be called such ordered pair $\langle D, d \rangle$ that:

- D is any set
- d is a function from set Ln in set $P(D)$ of all subsets of set D, i.e. $d : \mathsf{Ln} \longrightarrow P(D)$.

Definition 3.4 (Truth in model). Let $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$ be a model and $A \in \mathsf{For}_{\mathbf{TL}}$. We shall state that formula A is *true in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short $\mathfrak{M}_{\mathbf{TL}} \vDash A$) iff for some name letters P, $Q \in \mathsf{Ln}$, one of the below conditions is met:

1. $A := P\mathbf{a}Q$ and $d(P) \subseteq d(Q)$
2. $A := P\mathbf{i}Q$ and $d(P) \cap d(Q) \neq \varnothing$
3. $A := P\mathbf{e}Q$ and $d(P) \cap d(Q) = \varnothing$
4. $A := P\mathbf{o}Q$ and $d(P) \nsubseteq d(Q)$.

Formula A is *false in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short $\mathfrak{M}_{\mathbf{TL}} \nvDash A$) if for any name letters P, $Q \in \mathsf{Ln}$ none of the conditions is met.

Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$. Set X is *true in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short: $\mathfrak{M}_{\mathbf{TL}} \vDash X$) iff for any formula $A \in X$, $\mathfrak{M}_{\mathbf{TL}} \vDash A$. Set of formulas X is *false in model* $\mathfrak{M}_{\mathbf{TL}}$ (for short: $\mathfrak{M}_{\mathbf{TL}} \nvDash X$) iff it is not the case that for any formula $A \in X$, $\mathfrak{M}_{\mathbf{TL}} \vDash A$.

Making use of the concept of model, we can now define the concept of entailment or otherwise semantic consequence relation in **TL**.

Definition 3.5 (Semantic consequence of **TL**). Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and $A \in \mathsf{For}_{\mathbf{TL}}$. From set X *follows* formula A (for short: $X \vDash A$) iff for any model $\mathfrak{M}_{\mathbf{TL}}$, if $\mathfrak{M}_{\mathbf{TL}} \vDash X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash A$. Relation $\vDash$ will be also called *semantic consequence relation* of Term Logic, or shortly *semantic consequence*.

Denotation 3.6. For any set of formulas X and any formula A, $X \nvDash A$ will mean that it is not the case that $X \vDash A$.

We will now take up the issue of the compactness of semantic consequence relation. According to definition 2.48 compactness of relation $\vDash$ is expressed by the following definition.

Definition 3.7 (Compactness of semantic consequence of **TL**). Relation of semantic consequence $\vDash$ is *compact* iff for any set of formulas X and any formula A it is the case that: $X \vDash A \Leftrightarrow$ there exists such finite set $Y \subseteq X$ that $Y \vDash A$.

Compactness of relation $\models$ seems pretty obvious, so we are just going to outline the proof consisting in embedding **TL** into Monadic Logic of Predicates of which the relation of consequence is compact.

Before we proceed to the verbalization and proof of the relevant theorem, let us recall a few concepts concerning the Monadic Logic of Predicates (for short **MLP**):

- the alphabet of **MLP** contains:
  - classical, Boolean constants: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and quantifiers $\exists$, $\forall$
  - unary predicate letters $\mathsf{Lp} = \{p_1, q_1, r_1, p_2, q_2, r_2, \ldots\}$ (in practice, we will use a finite number of the following letters: p, q, r, s, treating them as metavariables ranging over set Lp)
  - set of individual constants Ci and individual variables Vi as well as auxiliary symbols:), (.
- **MLP** formulas will be constructed in a standard way and these are atomic expressions of type $p(x)$, where $p \in \mathsf{Lp}$, whereas $x \in \mathsf{Ci} \cup \mathsf{Vi}$, and more complex expressions using quantifiers, Boolean constants and brackets; set of formulas **MLP** we shall denote as $\mathsf{For}_{\mathbf{MLP}}$
- models for formulas from set $\mathsf{For}_{\mathbf{MLP}}$ are ordered triples $\mathfrak{M}_{\mathbf{MLP}} = \langle D, d_{\mathsf{Lp}}, d_{\mathsf{Ci}} \rangle$, where:
  - D is a non-empty set of any objects which is called a domain
  - d_{Lp} is a function from the set of predicate letters in the set of all subsets of D, i.e. in $P(D)$
  - d_{Ci} is a function from the set of individual constants in set D
- both truth conditions for **MLP** formulas and relation of semantic consequence $\models_{\mathbf{MLP}}$ are defined in a standard way
- relation $\models_{\mathbf{MLP}}$ is compact.

Since set of name letters Ln set of predicate letters Lp are countable, then there exists bijection: $\pi : \mathsf{Ln} \longrightarrow \mathsf{Lp}$, where $\pi(X) = x$, for all letters. Obviously, for π there exists the inverse function π^{-1}.

Next, we define function g from set of formulas $\mathsf{For}_{\mathbf{TL}}$ in set of formulas $\mathsf{For}_{\mathbf{MLP}}$ with the following conditions, for any name letters $P, Q \in \mathsf{Ln}$:

1. $g(P\mathbf{a}Q) = \forall_x(\pi(P)(x) \rightarrow \pi(Q)(x))$
2. $g(P\mathbf{i}Q) = \exists_x(\pi(P)(x) \wedge \pi(Q)(x))$
3. $g(P\mathbf{e}Q) = \forall_x(\pi(P)(x) \rightarrow \neg\pi(Q)(x))$
4. $g(P\mathbf{o}Q) = \exists_x(\pi(P)(x) \wedge \neg\pi(Q)(x))$,

where x is any, but fixed variable from set Vi.

In turn, having function g, we define transformation $\textsc{Tr}\colon \mathsf{For}_{\mathbf{TL}} \longrightarrow g(\mathsf{For}_{\mathbf{TL}})$ in such way that $\textsc{Tr}(y) = g(y)$, for any $y \in \mathsf{For}_{\mathbf{TL}}$. Note that function $\textsc{Tr}$ is a bijection because:

(a) for each $y \in g(\mathsf{For}_{\mathbf{TL}})$ there exists such $z \in \mathsf{For}_{\mathbf{TL}}$ that $\textsc{Tr}(z) = y$

(b) for any $y, z \in \mathsf{For}_{\mathbf{TL}}$, if y and z are various formulas, then $\textsc{Tr}(y) \neq \textsc{Tr}(z)$, by definition of function g.

Hence, there exists the inverse function to $\textsc{Tr}$, function $\textsc{Tr}^{-1}\colon g(\mathsf{For}_{\mathbf{TL}}) \longrightarrow \mathsf{For}_{\mathbf{TL}}$, so such function that for any $y \in \mathsf{For}_{\mathbf{TL}}$, $\textsc{Tr}^{-1}(\textsc{Tr}(y)) = y$.

Now, we will show that the following fact holds.

Proposition 3.8. *For any set of formulas $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and any formula A: $X \vDash A \Leftrightarrow \textsc{Tr}(X) \vDash_{\mathbf{MLP}} \textsc{Tr}(A)$.*

Proof. Take any set of formulas $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and formula A.

First, let us consider implication '$\Rightarrow$', assuming that $X \vDash A$. Take any model $\mathfrak{M}_{\mathbf{MLP}} = \langle D, d_{\mathsf{Lp}}, d_{\mathsf{Ci}} \rangle$ such that $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \textsc{Tr}(X)$. Based on model $\mathfrak{M}_{\mathbf{MLP}}$ we will define model $\mathfrak{M}_{\mathbf{TL}} = \langle D', d \rangle$ as follows:

- $D' = D$
- for any $P \in \mathsf{Ln}$, $d(P) = d_{\mathsf{Lp}}(\pi(P))$.

We will show that $\mathfrak{M}_{\mathbf{TL}} \vDash X$. We will now consider cases of formulas that can belong to set X. Take any name letters $P, Q \in \mathsf{Ln}$ and assume that some of the cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $P\mathbf{a}Q \in X$, then $\textsc{Tr}(P\mathbf{a}Q) = \forall_x(p(x) \to q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to q(x))$, consequently, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \subseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$
2. $P\mathbf{i}Q \in X$, then $\textsc{Tr}(P\mathbf{i}Q) = \exists_x(p(x) \wedge q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, hence there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$
3. $P\mathbf{e}Q \in X$, then $\textsc{Tr}(P\mathbf{e}Q) = \forall_x(p(x) \to \neg q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to \neg q(x))$, hence for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$

4. $P\mathbf{o}Q \in X$, then $\text{Tr}(P\mathbf{o}Q) = \exists_x(p(x) \wedge \neg q(x))$ and by assumption $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, hence there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, $d(P) \nsubseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$.

Hence, $\mathfrak{M}_{\mathbf{TL}} \vDash X$. In turn, from definition of relation $\vDash$ it follows that $\mathfrak{M}_{\mathbf{TL}} \vDash A$.

Now, we will show that $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(A)$. We will consider cases of formulas that can be identical to formula A. Take any name letters P, $Q \in \mathsf{Ln}$ and assume that some of the cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $A = P\mathbf{a}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$, so by definition of truth in model 3.4, $d(P) \subseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{a}Q)$
2. $A = P\mathbf{i}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) \neq \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{i}Q)$
3. $A = P\mathbf{e}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) = \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, so $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{e}Q)$
4. $A = P\mathbf{o}Q$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$, so by definition of truth in model 3.4, $d(P) \nsubseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{TL}}$, there exists such denotation of variable x, that this denotation belongs to set $d_{\mathsf{Lp}}(p)$, but it does not belong to set $d_{\mathsf{Lp}}(q)$, so $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{o}Q)$.

Hence, $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(A)$. Whereas from the arbitrariness of model $\mathfrak{M}_{\mathbf{MLP}}$ we have $\text{Tr}(X) \vDash_{\mathbf{MLP}} \text{Tr}(A)$.

Next, let us consider implication '$\Leftarrow$', assuming that $\text{Tr}(X) \vDash_{\mathbf{MLP}} \text{Tr}(A)$. Take any model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$ such that $\mathfrak{M}_{\mathbf{TL}} \vDash X$. Based on model $\mathfrak{M}_{\mathbf{TL}}$, we will define model $\mathfrak{M}_{\mathbf{MLP}} = \langle D', d_{\mathsf{Lp}}, d_{\mathsf{Ci}} \rangle$ as follows:

- $D' = D$, if $D \neq \varnothing$; otherwise $D' = D''$, for some whichever, fixed $D'' \neq \varnothing$
- for any $p \in \mathsf{Lp}$, $d_{\mathsf{Lp}}(p) = d(\pi^{-1}(p))$
- for any $x \in \mathsf{Ci}$, $d_{\mathsf{Ci}}(x) \in D'$.

We will show that $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(X)$. We will now consider cases of formulas that can belong to set X, and consequently, their images under function Tr belong to $\text{Tr}(X)$. Take any name letters $P, Q \in \mathsf{Ln}$ and assume that some of the below cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $P\mathbf{a}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$, so by definition of truth in model 3.4, $d(P) \subseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{a}Q)$
2. $P\mathbf{i}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) \neq \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, there exists such denotation of variable x, that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{i}Q)$
3. $P\mathbf{e}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$, so by definition of truth in model 3.4, $d(P) \cap d(Q) = \varnothing$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{e}Q)$
4. $P\mathbf{o}Q \in X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$, so by definition of truth in model 3.4, $d(P) \nsubseteq d(Q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$, but it does not belong to set $d_{\mathsf{Lp}}(q)$, hence $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, thus $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(P\mathbf{o}Q)$.

Hence, $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(X)$. From definition of relation $\vDash_{\mathbf{MLP}}$ it follows that $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \text{Tr}(A)$.

We will now show that $\mathfrak{M}_{\mathbf{TL}} \vDash A$. We will consider cases of formulas that can be identical to formula A. Take any name letters $P, Q \in \mathsf{Ln}$ and assume that some of the below cases occurs. We know that for some $p, q \in \mathsf{Lp}$, $\pi(P) = p$ and $\pi(Q) = q$.

1. $A := P\mathbf{a}Q$, then $\text{Tr}(P\mathbf{a}Q) = \forall_x(p(x) \to q(x))$, and since $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to q(x))$, consequently, for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it belongs to set $d_{\mathsf{Lp}}(q)$, hence by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \subseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$
2. $A := P\mathbf{i}Q$, then $\text{Tr}(P\mathbf{i}Q) = \exists_x(p(x) \wedge q(x))$, and because $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \exists_x(p(x) \wedge q(x))$, hence there exists such denotation of variable x that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$
3. $A := P\mathbf{e}Q$, then $\text{Tr}(P\mathbf{e}Q) = \forall_x(p(x) \to \neg q(x))$, and because $\mathfrak{M}_{\mathbf{MLP}} \vDash_{\mathbf{MLP}} \forall_x(p(x) \to \neg q(x))$, hence for each denotation of variable x, if denotation x belongs to set $d_{\mathsf{Lp}}(p)$, then it does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$

4. $A := P\mathbf{o}Q$, then $\textsc{Tr}(P\mathbf{o}Q) = \exists_x(p(x) \wedge \neg q(x))$, and because $\mathfrak{M}_{\mathbf{MLP}} \models_{\mathbf{MLP}} \exists_x(p(x) \wedge \neg q(x))$, hence there exists such denotation of variable x, that this denotation belongs to set $d_{\mathsf{Lp}}(p)$ and does not belong to set $d_{\mathsf{Lp}}(q)$, thus by definition of model $\mathfrak{M}_{\mathbf{MLP}}$, $d(P) \not\subseteq d(Q)$, so by definition of truth in model 3.4, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{o}Q$.

Hence, $\mathfrak{M}_{\mathbf{TL}} \models A$. Whereas from the arbitrariness of model $\mathfrak{M}_{\mathbf{TL}}$ we have $X \models A$. □

Let us now proceed to the very fact concerning the compactness of relation $\models$.

Proposition 3.9. *Relation of semantic consequence $\models$ is compact.*

Proof. Now, take any set $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and any formula $A \in \mathsf{For}_{\mathbf{TL}}$ and assume that $X \models A$. By virtue of fact 3.8 we know that $\textsc{Tr}(X) \models_{\mathbf{MLP}} \textsc{Tr}(A)$. And since relation $\models_{\mathbf{MLP}}$ is compact, then there exists such finite subset $Y' \subseteq \textsc{Tr}(X)$ that $Y' \models_{\mathbf{MLP}} \textsc{Tr}(A)$.

Due to definition of function $\textsc{Tr}$ and fact 3.8, there exists such finite subset $Y \subseteq X$ that $\textsc{Tr}^{-1}(Y') = Y$ and $Y \models A$.

On the other hand, let us assume there exists such finite subset $Y \subseteq X$ that $Y \models A$. Then, however, due to definition of relation of semantic consequence of **TL** 3.5, $X \models A$.

Thus, relation of semantic consequence $\models$ of **TL** is compact. □

3.3 Basic concepts of the tableau system for TL

Unlike the tableau system for **CPL** in the case of tableau system for **TL** the tableau proofs will be carried out in a more rich language than the set of formulas. The elements of expressions of this language will be simply called *tableau expressions*.

Definition 3.10 (Tableau expressions of **TL**). *Set of tableau expressions* is the union of the three following sets:

- $\{P_{+i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\{P_{-i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\mathsf{For}_{\mathbf{TL}}$.

We specify this set as $\mathsf{Te}_{\mathbf{TL}}$, and its elements will be called *expressions* or *tableau expressions*. Numbers that exist in expressions with + or − sign will be called *indices*.

Remark 3.11. In case of **TL** set $\mathsf{Te}_{\mathbf{TL}}$, i.e. set of proof expressions of which subsets will be used for construction of tableaux, is composed of formulas of **TL** and additional expressions which play a role worth explaining. We mean expressions P_{+i}, P_{-i}, where $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$. Although our approach to the tableau system is syntactic — we treat tableau proof as a transformation of sets of symbols without reference to their meaning — we can point to the semantic intuitions behind such kind of expressions. The natural numbers appearing in the expressions, in the construction of the model will denote the objects in the universe under consideration, while symbols +/− will mean being or not being the designator of a given name denoted by letter P. According to known literature, it seems that this sort of use of an additional language to describe whether or not given objects belong to the ranges of names in the context of tableau proofs has not yet been fully developed.

We will now define an auxiliary function that is to attribute formulas to formulas that contradict them. This function, among other things, will be used to begin tableau proofs, so it will play a similar role as negation in the tableau system from the previous chapter.

Definition 3.12. Function $\circ : \mathsf{For}_{\mathbf{TL}} \longrightarrow \mathsf{For}_{\mathbf{TL}}$, for any $P, Q \in \mathsf{Ln}$, is defined with the following conditions:

1. $\circ(P\mathbf{a}Q) = P\mathbf{o}Q$
2. $\circ(P\mathbf{i}Q) = P\mathbf{e}Q$
3. $\circ(P\mathbf{e}Q) = P\mathbf{i}Q$
4. $\circ(P\mathbf{o}Q) = P\mathbf{a}Q$.

Notice that by virtue of definition 3.12 and definition of truth in model 3.4, the following fact occurs.

Proposition 3.13. *For any formula A and any model $\mathfrak{M}_{TL}$: $\mathfrak{M}_{TL} \vDash A$ iff $\mathfrak{M}_{TL} \nvDash \circ(A)$.*

As we wrote, one of the basic concepts used to describe a tableau system, due to the nature of tableau proofs, is the concept of a tableau inconsistent set of proof expressions. Let it be reminded that in case of the defined system for **TL**, the proof expressions are the proper superset of the set of formulas, so the concept of t-incosistent set of formulas will also cover additional expressions.

Definition 3.14 (Tableau inconsistent set of expressions)**.** Set $X \subseteq \mathsf{Te}_{\mathbf{TL}}$ will be calles *tableau inconsistent* (for short: t-inconsistent) iff one of the below conditions is met:

1. there exists such formula $A \in \mathsf{For}_{\mathbf{TL}}$ that $A \in X$ and $\circ(A) \in X$
2. there exists such name letter $P \in \mathsf{Ln}$ and such number $i \in \mathbb{N}$ that $P_{+i} \in X$ and $P_{-i} \in X$.

Set X will be called *t-consistent* iff it is not t-inconsistent.

Remark 3.15. From the definition of tableau inconsistent set of expressions 3.14 we might remove the first condition and require the t-inconsistency to emerge in the set of "pure" tableau expressions. However, we will leave this condition to find t-inconsistency faster wherever possible, without the need for further application of tableau rules.

Let us now introduce the concept of model appropriate for the set of expressions. It is a generalisation of the concept of truth in model in the entire set $\mathsf{Te}_{\mathbf{TL}}$.

Definition 3.16 (Model appropriate for the set of expressions)**.** Let X be a set of tableau expressions, while $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ be a model. Model $\mathfrak{M}_{\mathbf{TL}}$ is *appropriate* for set X iff the below conditions are met:

1. $\mathfrak{M}_{\mathbf{TL}} \vDash X \cap \mathsf{For}_{\mathbf{TL}}$
2. there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $P \in \mathsf{Ln}$ and each $i \in \mathbb{N}$:
   a. if $P_{+i} \in X$, then $\gamma(i) \in d(P)$
   b. if $P_{-i} \in X$, then $\gamma(i) \notin d(P)$.

From the two above definitions, an important conclusion for metatheory follows, namely the relationship between the inconsistent sets of expressions and the appropriateness of models.

Corollary 3.17. *For any $X \subseteq \mathsf{Te}_{TL}$, if X is t-inconsistent, then there exists no model $\mathfrak{M}_{TL}$ appropriate for X.*

Proof. Take any tableau inconsistent set of expressions X and any model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$. From the definition of tableau inconsistent set of expressions 3.14 it follows that:

1. there exists such formula $A \in \mathsf{For}_{\mathbf{TL}}$ that $A \in X$ and $\circ(A) \in X$,
   or
2. there exists such name letter $P \in \mathsf{Ln}$ and such number $i \in \mathbb{N}$ that $P_{+i} \in X$ and $P_{-i} \in X$.

If the first case occurs, then from fact 3.13 we know that $\mathfrak{M}_{\mathbf{TL}} \nvDash A$ or $\mathfrak{M}_{\mathbf{TL}} \nvDash \circ(A)$. If the second case occurs, then from definition of model 3.3 we know that there exists no such function $\gamma : \mathbb{N} \longrightarrow D$ that for each $j \in \mathbb{N}$:

1. if $P_{+j} \in X$, then $\gamma(j) \in d(P)$
2. if $P_{-j} \in X$, then $\gamma(j) \notin d(P)$.

Since then $\gamma(i) \in d(P)$ and $\gamma(i) \notin d(P)$. Hence, from the definition of model appropriate for the set of expressions 3.16 it follows that $\mathfrak{M}_{\mathbf{TL}}$ is not a model appropriate for set of expressions X. Whereas from the arbitrariness of model $\mathfrak{M}_{\mathbf{TL}}$ it follows that there does not exist model $\mathfrak{M}_{\mathbf{TL}}$ appropriate for X. □

3.3.1 Tableau rules for TL

The starting point for the construction of a tableau system for **TL** should be a precise definition of the concept of tableau rule. Before we proceed to the general concept of rule, we will introduce a certain auxiliary function $* : \mathsf{Te}_{\mathbf{TL}} \setminus \mathsf{For}_{\mathbf{TL}} \longrightarrow \mathbb{N}$ such that for any $P \in \mathsf{Ln}$ and any $i \in \mathbb{N}$:

- $*(P_{+i}) = i$
- $*(P_{-i}) = i.$

To each expression not being a formula, meaning a name letter with an index, function $*$ attributes an index which is found in it.

Similar to the case of **CPL**, we will first provide the general concept of rule. Not only because it allows to provide the general conditions that a tableau rule must meet. In the case of **TL** we will also provide alternative sets of tableau rules that are suitable for construction of a tableau system for **TL**. This means that within the below general concept of a tableau rule, we can define different sets of tableau rules that define various, however equivalent in terms of scope for correct inferences, tableau systems for **TL** (see note 3.20).

Definition 3.18 (Rule). Let $P(\mathsf{Te}_{\mathbf{TL}})$ be a power set of the set of tableau expressions. Let $P(\mathsf{Te})^n$ be n–ry Cartesian product $\underbrace{P(\mathsf{Te}_{\mathbf{TL}}) \times \cdots \times P(\mathsf{Te}_{\mathbf{TL}})}_{n}$, for some $n \in \mathbb{N}$.

- By a *rule* we understand any subset $R \subseteq P(\mathsf{Te}_{\mathbf{TL}})^n$ such that if $\langle X_1, \ldots, X_n \rangle \in R$, then:
  1. X_1 is t-consistent
  2. $X_1 \subset X_i$, for each $1 < i \leq n$.
- If $n \geq 2$, then each element R will be called *ordered n-tuple* (pair, triple, etc., respectively).
- The first element of each n-tuples will be called an *input set* (*set of premises*), while its remaining elements *output sets* (*sets of conclusions*).

Definition of rule for **TL** differs from the definition of rule for **CPL** 2.12 among other things[4] in having introduced to the definition of rule for **TL** a condition of t-consistency of the input sets. We will no longer put down the rules in the form of sets, but immediately in the schematic, fractional form. Thus, from the general definition of rule itself, it follows that the input sets of tableau rules will be t-consistent.

A set of tableau rules designed for the defined tableau system for **TL** shall be introduced by means of the following definition.

Definition 3.19 (Tableau rules for **TL**). *Tableau rules for* **TL** are the following rules:

$$R\mathbf{a}_{+} : \frac{X \cup \{P\mathbf{a}Q, P_{+j}\}}{X \cup \{P\mathbf{a}Q, P_{+j}, Q_{+j}\}}$$

$$R\mathbf{e}_{-} : \frac{X \cup \{P\mathbf{e}Q, P_{+j}\}}{X \cup \{P\mathbf{e}Q, P_{+j}, Q_{-j}\}}$$

$$R\mathbf{i} : \frac{X \cup \{P\mathbf{i}Q\}}{X \cup \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{TL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}\} \nsubseteq X$.

$$R\mathbf{o} : \frac{X \cup \{P\mathbf{o}Q\}}{X \cup \{P\mathbf{o}Q, P_{+j}, Q_{-j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{TL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{-k}\} \nsubseteq X$.

Set of tableau rules for **TL** will be defined as $\mathbf{R}_{\mathbf{TL}}$.

According to the general definition of rule 3.18, the input sets of each rule are t-consistent. In addition, in each tableau rule, the input set is basically contained in the output set. The notations provided are schemes of pairs belonging to the rules, so for each of the rules *X* is any set of expressions, *P*, *Q* are any name letters and *j* is any index such that all these elements satisfy the conditions imposed on the rule. One novelty is that set $\mathbf{R}_{\mathbf{TL}}$ contains rules that must include at least two premises, e.g. rule $R\mathbf{a}_{+}$.

---

4 We write *among other things* since this definition above all differs in the set on which we define rules. Nevertheless, all the other conditions are nearly identical to those of the concept of rule for **CPL**.

Another new and important aspect of the tableau rules in this system for **TL** are conditions in rules $R\mathbf{i}$, $R\mathbf{o}$. In both rules, we basically have the same conditions, so through one example we will discuss them collectively.

These conditions must be met if a given set is to be an output set with an assumed input set. So, the notation adopted in both rules says, for example, that if pair $\langle X \cup \{P\mathbf{i}Q\}, X \cup \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}\rangle$ belongs to rule $R\mathbf{i}$, then:

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{TL}})$ and
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}\} \nsubseteq X$.

These conditions are therefore necessary conditions for a given pair to belong to the rule.

Condition 1 requires the index which is entered to be new, meaning not appearing in any expression belonging to the output set. The semantic intuitions behind this procedure require the object denoted by new index to be new as well and not to remain in any positive relationship with the other names that appear in a given proof sequence.

In turn, condition 2 requires the input of a pair of expressions (in this example pair P_{+j}, Q_{+j}) to take place only when a similar pair does not already belong to the output set. In practice, this condition makes it impossible to enter unnecessary expressions in the proof, as is the case in example 3.21.

Analogous conditions will be considered in the next chapter which will be devoted to modal logic. There, we are going to provide an example which will illustrate the problem of infinite branches. In that case, even applying conditions blocking the unnecessary use of rules will not prevent the emergence of infinite branches. It will, however, prevent the creation of infinite branches in situations where this is not a consequence of logic itself, but of the wrong definition of the tableau system.

Remark 3.20. We can consider alternative sets of rules for the construction of a tableau system for **TL**. The following rule would help.

$$R\mathbf{e}'_{-} : \frac{X \cup \{P\mathbf{e}Q, Q_{+j}\}}{X \cup \{P\mathbf{e}Q, Q_{+j}, P_{-j}\}}$$

Rule $R\mathbf{e}'_{-}$ allows to proceed from premises $P\mathbf{e}Q$, Q_{+j} to conclusion P_{-j}. The semantic intuition contained in this rule says that if a name from the predicate in a general contradicting proposition has a subject j as the designator, then this object is not the designator of the subject of this proposition.

Making use of rule $R\mathbf{e}'_{-}$, we can define the following sets of tableau rules, different from set $\mathbf{R_{TL}}$:

(a) $\mathbf{R_{TL}} \cup \{R\mathbf{e}'_{-}\}$

(b) $(\mathbf{R_{TL}} \setminus \{R\mathbf{e}_{-}\}) \cup \{R\mathbf{e}'_{-}\}$.

Likewise, we could consider another rule that says that if we have a proposition $P\mathbf{a}Q$ and expression Q_{-j} saying — intuitively — that an object denoted by j is not a designator of name Q, then this object is not a designator of name P either.

$$R\mathbf{a}_{-} : \frac{X \cup \{P\mathbf{a}Q, Q_{-j}\}}{X \cup \{P\mathbf{a}Q, Q_{-j}, P_{-j}\}}$$

Making use of rule $R\mathbf{a}_{-}$, we can define the following sets of tableau rules, different from set $\mathbf{R_{TL}}$:

(a) $\mathbf{R_{TL}} \cup \{R\mathbf{a}_{-}\}$

(b) $(\mathbf{R_{TL}} \setminus \{R\mathbf{a}_{+}\}) \cup \{R\mathbf{a}_{-}\}$.

Furthermore, extending the language of logic of **TL** to the language which includes expressions denoting statements that a particular object is or is not the designator of a given name, we could consider another rule that allows "reversing" general contradicting propositions:

$$R\mathbf{e} : \frac{X \cup \{P\mathbf{e}Q\}}{X \cup \{Q\mathbf{e}P\}}$$

It is worth noting that the tableau system for **TL** we are now describing, there are no branching rules (all tableau rules are sets of pairs). So there will be no branchings in the tableaux. However, we might consider such variant of rule $R\mathbf{a}_{+}$ which would allow branchings:

$$R\mathbf{a}'_{+} : \frac{X \cup \{P\mathbf{a}Q\}}{X \cup \{P\mathbf{a}Q, P_{-i}\} | X \cup \{P\mathbf{a}Q, Q_{+i}\}}, \text{ where } i \in *(X)$$

However, due to the economy of tableau proof, it is better — as far as possible — to introduce the fewest possible number of tableau rules that are not sets of pairs.

Although all these rules and sets of rules seem to be interesting, we will not consider them all — neither for the language of **TL**, nor for extended languages — for the reasons described in the previous chapter (note 2.19). Instead, we will focus on the tableau system determined by set of rules $\mathbf{R_{TL}}$.

Nonetheless, let us stress once again that the examples provided indicate that the general definition of rule 3.18 makes sense. There may exist many sets of tableau rules that potentially — we mean (potentially) because to determine this it always requires a proof — they define the same logic.

Example 3.21. Starting from set $X \cup \{P\mathbf{i}Q\}$ and using rule $R\mathbf{i}$ without condition 2, we could get an infinite sequence in which we enter a new index in each of the elements:

$$\begin{array}{rcl}
R\mathrm{i}\ X_1 & = & X \cup \{P\,\mathrm{i}Q\} \\
 & | & \\
R\mathrm{i}\ X_2 & = & X_1 \cup \{P_{+\mathrm{i}}, Q_{+\mathrm{i}}\} \\
 & | & \\
R\mathrm{i}\ X_3 & = & X_2 \cup \{P_{+\mathrm{j}}, Q_{+\mathrm{j}}\} \\
 & | & \\
R\mathrm{i}\ X_4 & = & X_3 \cup \{P_{+\mathrm{k}}, Q_{+\mathrm{k}}\} \\
 & | & \\
 & \dots &
\end{array}$$

Such a sequence is infinite not because of the properties of the logic itself, but because of the unnecessary acceptance of continuous application to the same elements of the rule already applied once.

The adopted set of tableau rules — in our case set $\mathbf{R_{TL}}$ — determines the content of the range of successive concepts of the tableau system. Although the formal concepts that we will describe will be analogous to those from the previous chapter, each of them will depend on set $\mathbf{R_{TL}}$.

3.3.2 Branches for TL

With a fixed set of tableau rules, we can proceed to the concept of branch. As we already know, branches are such sequences of sets that each two adjacent elements are in turn: an input set and an output set of some n-tuple that belongs to the set of tableau rules. Branches are therefore setwise objects consisting of sets. Below, we present the formal definition of branch in the tableau system for **TL**.

Definition 3.22 (Branch)**.** Let $K = \mathbb{N}$ or $K = \{1, 2, \dots, n\}$, where $n \in \mathbb{N}$. Let X be any set of expressions. *A branch* (or *a branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$: if $i + 1 \in K$, then there exists such rule $R \in \mathbf{R_{TL}}$ and such pair $\langle Y_1, Y_2 \rangle \in R$ that $\phi(X_i) = Y_1$ i $\phi(X_{i+1}) = Y_2$.

Having two branches ϕ, ψ such that $\phi \subset \psi$ we shall state that:

- ϕ is a sub-branch of ψ
- ψ is a super-branch of ϕ.

Denotation 3.23. From now on — when speaking of branches for **TL** — for convenience, we will use the following notations or designations:

1. $X_1, \ldots, X_n$, where $n \geq 1$
2. $\langle X_1, \ldots, X_n \rangle$, where $n \geq 1$
3. abbreviations: ϕ_M (where M is a domain ϕ, i.e. $\phi : M \longrightarrow P(\mathsf{Te}_{\mathbf{TL}}))$
4. or — to denote branches — small Greek letters: ϕ, ψ, etc.

The sets of branches, in turn, we shall denote with capital Greek letters: Φ, Ψ, etc. Furthermore, the domain cardinality of a given branch K we shall sometimes call a *length* of that branch.

Remark 3.24. We will repeat here in part the remark from the previous chapter. As we can see, the concept of branch depends on some set of tableau rules. In the case under consideration, the branch structure is based on the rules from set $\mathbf{R_{TL}}$. Further described complex tableau concepts will also depend on some sets of rules. Because in this chapter we are studying tableau system for **TL** based on rules from set $\mathbf{R_{TL}}$, so we are not going to make it any more complicated.

In practice, however, the tableau concepts of systems constructed according to the presented idea always base on some set of rules. In one of the subsequent chapters, at the general description of the construction way itself, the set of rules will be a variable. In this chapter it is specified as: $\mathbf{R_{TL}}$ and the complex tableau concepts defined here depend on it. And since set $\mathbf{R_{TL}}$ only includes such rules that constitute sets of ordered pairs, so in the definition of branch for **TL** we specified that it is about pairs, contrary to the definition of tableau system in the previous chapter, where in the definition of branch 2.20, we wrote about the existence of an appropriate n-tuple.

This remark applies to the rest of the study.

By definition of rule 3.18, through the fact that the rules are defined by proper containing of the input set in each of the output sets, in any n-tuple, there is a conclusion.

Corollary 3.25. *Each branch is an injective sequence.*

3.3.3 Maximal branches

Among the branches constructed through applying the rules from set $\mathbf{R_{TL}}$, we will distinguish such branches to which no more rules from set $\mathbf{R_{TL}}$ can be applied, expanding them into some super-branches. As we already know, such branches are called maximal branches. The definition of maximal branch is the same as in the previous chapter, except that of course the maximal branches here are branches for **TL**.

Definition 3.26 (Maximal branch). Let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ be a branch. We shall state that ϕ is *maximal* iff

1. $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$
2. there is no branch ψ such that $\phi \subset \psi$.

As in the case described in the previous chapter, definition 3.26 could be shortened to the second condition. This would not change anything with regard to the cases of finite sets, but we would allow cases of infinite branches that:

- begin with an infinite set
- meet the second condition of definition because they are infinite branches
- do not, however, "resolve" all expressions.

The following example 3.27 illustrates a situation where without the first condition in the definition of maximal branch 3.26, we would consider a presented branch to be maximal. This branch would be infinite and, at the same time, would still contain expressions that were not used in the proof.

Example 3.27. Let us consider the following branch, beginning from set $\{P^1\mathbf{i}Q^1\} \cup \{P^n\mathbf{i}Q^n : n > 1\}$.

$$
\begin{array}{rcl}
R\mathrm{i}\ X_1 & = & \{P^1\mathrm{i}Q^1\} \cup \{P^n\mathrm{i}Q^n : n > 1\} \\
 & | & \\
R\mathrm{i}\ X_2 & = & X_1 \cup \{P^2_{+1}, Q^2_{+1}\} \\
 & | & \\
R\mathrm{i}\ X_3 & = & X_2 \cup \{P^3_{+2}, Q^3_{+2}\} \\
 & | & \\
R\mathrm{i}\ X_4 & = & X_3 \cup \{P^4_{+3}, Q^4_{+3}\} \\
 & | & \\
 & \ldots &
\end{array}
$$

This branch is infinite, although rule $R\mathbf{i}$ has never been applied to any of the sets in such a way as to draw conclusions from formula $P^1\mathbf{i}Q^1$.

Thus, the definition of maximal branch 3.26 says that if sequence $X_1, \ldots, X_n$, for some $n \geq 1$, is a branch, then we call it a *maximal* branch iff there does not exist branch $X_1, \ldots, X_n, X_{n+1}$.

Extending the concept of branch onto infinite sets is unnecessary in practice, because important concepts for our metatheory — the concept of branch consequence and tableau — we apply to cases of finite sets. At this stage of the book, the finite sets of tableau expressions do not begin the branch of infinite length.

However, the definition of maximal branch will change in the study. To the above concept of branch applicable are all reservations from note 2.32, concerning the maximal branch for the tableau system for **CPL**. Definition 3.26 is suitable for those tableau systems where only finite branches are obtained from finite sets of expressions. For other systems this definition is too narrow. It does not include cases of branches which, even though they begin with a finite set of expressions, are not finite.

In the following chapter, we will proceed to defining the system for modal logic, and we will generalize this definition. Hence, the above definition 3.26, and especially definition 2.20 will describe special cases of maximal branches which appear in the construction by the described method of such tableau systems as system for **CPL** or **TL**. Besides, also in the case of **TL** we will show that from finite sets of expressions we always get branches of finite length (fact 3.32).

3.3.4 Closed and open branches

Among the maximal branches, the closed branches deserve special attention. In addition, set of open branches complements the set of closed branches. As we remember, intuitively a branch is closed when we get a t-inconsistent set having decomposed tableau expressions.

Definition 3.28 (Closed/open branch). A branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ will be called *closed* iff $\phi(i)$ is a t-inconsistent set, for some $i \in K$. A branch will be called *open* iff it is not closed.

From the above definition, the definition of tableau rules for **TL** 3.19 and the definition of branch 3.22 the following conclusion follows.

Corollary 3.29. *If branch $\phi : K \longrightarrow P(\mathsf{Te}_{TL})$ is closed, then $|K| \in \mathbb{N}$ and $\phi(|K|)$ is a t-inconsistent set.*

In the case of a closed branch, the t-inconsistent sequence element is the last element because no rule can be applied to it anymore to extend the branch. For the rules are defined in such a way that they cannot be applied to t-inconsistent sets. Therefore, from the definition of maximal 3.26 another conclusion follows.

Corollary 3.30. *If branch $f : K \longrightarrow P(\mathsf{Te}_{TL})$ is closed, then it is maximal.*

We are now going to show two facts that are needed for further proofs. The first one says that a branch that begins with a finite set of expressions is also finite in length, not greater than a certain number.

Proposition 3.31. *Let X be a finite set of expressions. Let $\phi : K \longrightarrow P(\mathsf{Te}_{TL})$ be any such branch that $\phi(1) = X$. Then, there exists number $n \in \mathbb{N}$ such that $|K| \leq n$.*

Proof. Take any finite set of expressions X and such branch $\phi : K \to P(\mathsf{Te}_{\mathbf{TL}})$ that $\phi(1) = X$. We will carry out an inductive proof due to the cardinality of the first element of the branch.

Initial step. Assume that $|X| = 1$. We have six types of cases that can take place. There exist name letters P, $Q \in \mathsf{Ln}$ and index $i \in \mathbb{N}$ such that one of the following cases occurs:

1. $P_{+i} \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
2. $P_{-i} \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
3. $P\mathbf{a}Q \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
4. $P\mathbf{i}Q \in X$, then, however, $|K| \leq 2 \in \mathbb{N}$, by definition of branch 3.22, since there only exists one rule $R \in \mathbf{R_{TL}}$, rule $R\mathbf{i}$ that allows to extend branch $\langle X \rangle$ with set $Y = \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}$, for some $j \in \mathbb{N}$, whereas there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X, Y \rangle$
5. $P\mathbf{e}Q \in X$, then, however, $|K| = 1 \in \mathbb{N}$, by definition of branch 3.22, since there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X \rangle$
6. $P\mathbf{o}Q \in X$, then, however, $|K| \leq 2 \in \mathbb{N}$, by definition of branch 3.22, since there only exists one rule $R \in \mathbf{R_{TL}}$, rule $R\mathbf{o}$ that allows to extend branch $\langle X \rangle$ with set $Y = \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}$, for some $j \in \mathbb{N}$, whereas there does not exist such rule $R \in \mathbf{R_{TL}}$ that would allow to extend branch $\langle X, Y \rangle$.

Thus, when $|X| = 1$, then $|K| \leq 2$. So, if $|X| = 1$, then there exists such number $n \in \mathbb{N}$ that $|K| \leq n$.

Induction step. Assume that the theorem thesis holds for each such set of expressions Y that $|Y| = m$. Thus, for any branch $\psi : M \longrightarrow P(\mathsf{Te_{TL}})$ such that $\psi(1) = Y$, there exists number l such that $|M| \leq l$.

We will show that the theorem thesis also occurs for $|X| = m+1$. Take any set of expressions Y such that $Y \subseteq X$ and $|Y| = m$. Thus, for any branch $\chi : N \longrightarrow P(\mathsf{Te_{TL}})$ such that $\chi(1) = Y$ there exists number l such that $|N| \leq l$.

We have six types of cases that can take place. There exist name letters P, $Q \in \mathsf{Ln}$ and index $i \in \mathbb{N}$ such that one of the following cases occurs:

1. $X = Y \cup \{P_{+i}\}$, then, however, $|K| \leq l + k \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{a}_+$ and $R\mathbf{e}_-$ which allow to extend each branch containing additional expression P_{+i} at most with k elements
2. $X = Y \cup \{P_{-i}\}$, then, however, $|K| \leq l \in \mathbb{N}$, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ does not comprise any rule which would allow to extend a branch containing additional expression P_{-i}
3. $X = Y \cup \{P\mathbf{a}Q\}$, then, however, $|K| \leq l + ((k+1) \cdot o) \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, while o is the number

of particular propositions that belong to set Y and in the subject or predicate have name letter P, and expressions P_{+j}, for some index j, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{a}_+$ and $R\mathbf{e}_-$ which allow to extend each branch containing additional expression $P\mathbf{a}Q$ at most with $(k+1)\cdot o$ elements

4. $X = Y \cup \{P\mathbf{i}Q\}$, then, however, $|K| \leq l + k + 1 \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{i}$, $R\mathbf{a}_+$ and $R\mathbf{e}_-$, which allow to extend each branch containing additional expression $P\mathbf{i}Q$ at most with $k+1$ elements
5. $X = Y \cup \{P\mathbf{e}Q\}$, then, however, $|K| \leq l + k \in \mathbb{N}$, where k is the number of particular propositions that belong to set Y and in the subject or predicate have name letter P, and expressions P_{+j}, for some index j, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rule $R\mathbf{e}_-$, which allows to extend each branch containing additional expression $P\mathbf{e}Q$ at most with k elements
6. $X = Y \cup \{P\mathbf{o}Q\}$, then, however, $|K| \leq l + k + 1 \in \mathbb{N}$, where k is the number of propositions in form $R\mathbf{a}S$ and $T\mathbf{e}U$ that belong to set Y, by definition of branch 3.22, since set $\mathbf{R_{TL}}$ contains rules $R\mathbf{o}$, $R\mathbf{a}_+$ and $R\mathbf{e}_-$, which allow to extend each branch containing additional expression $P\mathbf{o}Q$ at most with $k+1$ elements.

So, if $|X| = m + 1$, then there exists such number $n \in \mathbb{N}$ that $|K| \leq n$. Then, there exists such number $n \in \mathbb{N}$ that $|K| \leq n$. □

The second fact says that for each finite set of expressions, there exists a maximal branch which begins with this set.

Proposition 3.32. *Let $X \subseteq \mathsf{Te}_{\mathbf{TL}}$ be a finite set of expressions. Then, there exists maximal branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ such that $\phi(1) = X$.*

Proof. Let $X \subseteq \mathsf{Te}_{\mathbf{TL}}$ be a finite set of expressions. Then, by fact 3.31, for each branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{TL}})$ such that $\phi(1) = X$, there exists such number $n \in \mathbb{N}$ that $|K| \leq n$.

Indirectly assume that no branch ϕ beginning with set X is a maximal branch. By an inductive proof through the branch length we will show that this assumption leads to a contradiction.

Initial step. We know there exists at least one branch. This is the branch that begins with set X and that a length of 1. However, by indirect assumption, it is not a maximal branch, so by definition of maximal branch 3.26, there exists a branch that begins with X, with length of 2.

Induction step. Now, let us take a branch beginning with X, of length m, and assume there exists a branch of length $m+1$ beginning with set X. Again, by indirect assumption, it is not a maximal branch, so by definition of maximal branch 3.26, there exists a branch that begins with X of length $m+2$.

Thus for any $m \in \mathbb{N}$, there exists a super-branch of length $m+1$ beginning with finite set of expressions X. This, however, contradicts the fact that for any branch beginning with set X there exists such number $n \in \mathbb{N}$ that bounds the length of that branch. □

3.3.5 Relation of branch consequence

We will now define the concept of branch consequence using the concepts of branch, maximal branch and closed branch.

Definition 3.33 (Branch consequence of **TL**). Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and $A \in \mathsf{For}_{\mathbf{TL}}$. Formula A is a *branch consequence* of X (for short: $X \triangleright A$) iff there exists such finite set $Y \subseteq X$ that each maximal branch beginning with set $Y \cup \{\circ(A)\}$ is closed.

Denotation 3.34. For any set of formulas X and any formula A notation $X \not\triangleright A$ means that it is not the case that $X \triangleright A$.

In the definition of branch consequence, we refer to the function specified by definition 3.12. Function $\circ$ to each formula assigns a formula which contradicts it 3.13. The above concept of branch consequence differs from the analogous concept for **CPL** in the fact that (apart from defining on a different language — set of expressions) in the previous case negation was used, and here we use a contradictory formula. In fact, however, it can be assumed that in both cases the point is to start with a formula contradictory to the formula which could potentially be a branch consequence of a certain set of premises, but in the case of **CPL** a contradictory formula could be obtained directly by preceding the formula with a negation.

Also in the present case, we will refer to the example of the branch consequence occurrence.

Example 3.35. Let us consider an example — Barbara syllogism — premises $\{P\mathbf{a}Q, Q\mathbf{a}R\}$, conclusion $P\mathbf{a}R$[5]. We want to answer the question whether $\{P\mathbf{a}Q, Q\mathbf{a}R\} \triangleright P\mathbf{a}R$?

The first set of each branch we need to examine is set $\{P\mathbf{a}Q, Q\mathbf{a}R, \circ(P\mathbf{a}R)\}$, by definition of function $\circ$, equal to set $\{P\mathbf{a}Q,\ Q\mathbf{a}R,\ P\mathbf{o}R\}$. On the left side, we put the branch elements, on the right side, we put the rules we use to transform the sets. There exists one type of maximal branches beginning with this last set:

1. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R\} \subset$ $R\mathbf{o}$, where $i \in \mathbb{N}$

5 Note that we are considering a pattern of inference rather than a specific inference as we use metavariables and letters instead of indices. It is obvious, however, that further considerations would be identical if we used name letters and digits.

2. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R, P_{+i}, R_{-i}\} \subset$ $R\mathbf{a}_{+}$
3. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R, P_{+i}, R_{-i}, Q_{+i}\} \subset$ $R\mathbf{a}_{+}$
4. $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R, P_{+i}, R_{-i}, Q_{+i}, R_{+i}\}$ t-inconsistency R_{-i}, R_{+i}

We said one *type* since i is any natural number. So, actually there exists an infinite number of branches beginning with set $\{P\mathbf{a}Q, Q\mathbf{a}R, P\mathbf{o}R\}$.[6] The described branches are maximal ones because their last element is t-inconsistent as it contains expressions R_{-i}, R_{+i}, always for some $i \in \mathbb{N}$.

So we showed that all maximal branches beginning with set $\{P\mathbf{a}Q, Q\mathbf{a}R, \circ(P\mathbf{a}R)\}$ are closed. Thus $\{P\mathbf{a}Q, Q\mathbf{a}R\} \rhd P\mathbf{a}R$.

The tableau concepts we have described for **TL** are very similar to their equivalents for **CPL**, despite the fact that we have dealt with a different language of proof and with indices.

3.4 Tableaux for TL

As in the case of the tableau system for **CPL**, the practical examining of the branch consequence occurrence should boil down to building of an appropriate tableau.

Nevertheless, we are going to have a discussion on the definition of the concepts of tableau, complete tableau, closed/open tableau of a tableau system for **TL**. We will define all these concepts in two variants.

The first variant will make a direct reference to the fact that among the rules belonging to set $\mathbf{R_{TL}}$ there are no rules that contain an ordered n-tuples longer than two, and as a consequence there never occur any branchings. The second variant of the concepts will be based on the relevant definitions already provided in the tableau system for **CPL**, however related to the set of branches of the tableau system for **TL**.

Although in each case the first definition variant will appear to be a special and simplified case of the second variant, in fact the two variants will prove to be equivalent, as we will demonstrate. In this way we will also show that the book considerations aim to describe certain general concepts for the tableau systems, independent of many specific properties of a given system, and thus to the general theory of tableau systems.

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6 Of course, we might define equivalents of rules $R\mathbf{i}$ and $R\mathbf{o}$ in such a way that the introduced expressions as indices only had for instance the least number that does not appear in the set to which we apply the rule. Then, the number of branches would amount to one. There are many ways to define these rules similarly, but from the formal point of view each time we would then examine a different axiomatization than the assumed set of rules $\mathbf{R_{TL}}$.

Therefore, the first variant of each definition will always be a specific version, defined according to the needs of the current system, while the second one will be a definition analogous to the one used in the previous chapter. As usual, we will start with the definition of tableau.

Definition 3.36 (Tableau — variant one)**.** Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called a *tableau for* $\langle X, A \rangle$ (or for short *tableau*) iff Φ is an one-element subset of the set of branches beginning with set $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$).

Remark 3.37. According to the definition of tableau 3.36, a tableau for pair $\langle X, A \rangle$ is ordered triple $\langle X, A, \Phi \rangle$ in which Φ is such an one-element set of branches that any branch in this set begins with set $X \cup \{\circ(A)\}$. This definition could be technically simplified.

Such simplification would consist in the fact that in an ordered triple, instead of set of branches Φ, we would simply place a branch belonging to set Φ. However, as we strive for a general theory of tableaux, so we use a general notation of a tableau as a triple in which we distinguish a set of premises, a potential conclusion and a set of branches that meet the conditions from the definition. The case of the system described in the present chapter is in terms of the number of branches — as we wrote at the beginning of the chapter — a borderline case, so we will not alter the convention for it.

Let us now define the concept of tableau for **TL** in a similar way the concept of tableau for **CPL**. Similar, since this definition is analogous to the corresponding definition for the tableau system for **CPL**. However, we cannot say that it is identical, because it pertains to the objects that were constructed from other sets, in spite of similarity of the same objects, i.e. branches. This definition uses — just like the definition of tableau 2.53 in the tableau system for **CPL** — the concept of maximal branch in the set of branches which we will adapt to the current situation. Here, the analogous remarks apply as in the case of definition 2.36 from Chapter Two — so we will not reiterate those here.

Definition 3.38 (Maximal branch in the set of branches)**.** Let Φ be a set of branches and let branch $\psi \in \Phi$. Branch ψ will be called *maximal in* Φ (or Φ–*maximal*) iff there is no such branch $\phi \in \Phi$ that $\psi \subset \phi$.

With the concept of maximal branch in the set of branches, we can proceed to the second variant of the tableau definition.

Definition 3.39 (Tableau — variant two)**.** Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called *a tableau for* $\langle X, A \rangle$ (or for short *tableau*) iff the below conditions are met:

1. Φ is a non-empty subset of set of branches beginning with $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$)
2. each branch contained in Φ is Φ-maximal
3. for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
   - i and $i+1$ belong to domains of functions $\psi_1, \ldots, \psi_n$
   - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

   then there exists such rule $R \in \mathbf{R_{TL}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$ that for any $1 \leq k \leq n$:
   - $\psi_k(i) = Y_1$
   - and there exists such $1 < l \leq m$ that $\psi_k(i+1) = Y_l$.

The above tableau definition for **TL** seems too complex. It is not too broad because — as we will see — it does not cover setwise constructions that we would not consider as tableaux for the tableau system for **TL**. However, condition 3 of this definition seems to be overly expanded. Note the conclusion that follows from the definition of tableau in variant two and the definition of set of tableau rules $\mathbf{R_{TL}}$.

Corollary 3.40. *Let $\langle X, A, \Phi \rangle$ be a tableau in variant two 3.39. Then, set Φ contains precisely one branch.*

Proof. Let $\langle X, A, \Phi \rangle$ be a tableau. In the proof, we will make use of definition of tableau 3.39. Assume that set Φ contains two branches ϕ_1 and ϕ_2. Point 1. of the tableau definition implies that $\phi_1(1) = \phi_2(1) = X \cup \{\circ(A)\}$. While point 2. and definition of maximal branch in the set of branches 3.38 imply that $\phi_1 \notin \phi_2$ and $\phi_2 \notin \phi_1$ since each branch in set Φ is Φ-maximal.

Now, take any such number $i \in \mathbb{N}$ that for any $o \leq i$, $\phi_1(o) = \phi_2(o)$, plus $i+1$ belong to domains of functions ϕ_1 and ϕ_2. From point 3. it follows that there exists such rule $R \in \mathbf{R_{TL}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$ that for any $1 \leq k \leq 2$:

- $\phi_k(i) = Y_1$
- there exists such $1 < l \leq m$ that $\phi_k(i+1) = Y_l$.

And yet, since for each rule $R \in \mathbf{R_{TL}}$, for each ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m \leq 3$, $m = 2$, so $\phi_1(i+1) = Y_l = \phi_2(i+1)$. Hence $\phi_1 = \phi_2$. □

This conclusion shows that the concept of tableau in the second variant for **TL** system could be simpler. Nonetheless, it is not too broad since both definitions of tableau for the tableau system for **TL** are equivalent, which gets proven by the below fact.

Proposition 3.41. *Let* $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ *and* Φ *be a set of branches.* $\langle X, A, \Phi \rangle$ *is a tableau in variant one 3.36 iff* $\langle X, A, \Phi \rangle$ *is a tableau in variant two 3.39.*

Proof. Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches.

Assume that $\langle X, A, \Phi \rangle$ is a tableau in variant one 3.36. In view of this definition, Φ is a non-empty subset of set of branches beginning with $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$), so the first condition from definition 3.39 is satisfied. Definiens of definition 3.36 also claims that Φ contains precisely one branch. So, each branch that belongs to Φ is Φ-maximal by virtue of definition 3.38, which is the second condition of definition 3.39. Finally, also the third condition of definition 3.39 is met as Φ contains precisely one branch ψ. Thus, $\langle X, A, \Phi \rangle$ is a tableau in variant two.

Now, assume that $\langle X, A, \Phi \rangle$ is a tableau in variant two 3.39. Due to the first condition of this definition, Φ is a non-empty subset of set of branches beginning with $X \cup \{\circ(A)\}$ (i.e. if $\phi \in \Phi$, then $\phi(1) = X \cup \{\circ(A)\}$). While due to conclusion 3.40, Φ contains precisely one branch. Thus, as per definition 3.36 $\langle X, A, \Phi \rangle$ is a tableau in variant one. □

Therefore, any of the definitions of tableau in the tableau system considered here can be adopted for **TL**, although due to the economy of phrasing, definition 3.36 seems better. Its scope covers exactly the same objects as in the definition of tableau that was created by applying analogies to the definition of tableau in the system for **CPL**.

We will now take up the issue of the complete tableau, also considering simpler and more complex variants, modelled on the system for **CPL**. Here is variant one.

Definition 3.42 (Complete tableau — variant one)**.** Let triple $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff a branch contained in Φ is maximal. A tableau is *incomplete* iff it is not complete.

In a complete tableau, a branch that belongs to it is maximal, so it cannot be extended. In variant two, which in a moment will be adapted to the context of **TL**, in a complete tableau all branches are maximal, not only those maximal ones in a given set. In addition, a complete tableau contains such set of branches that it is no longer possible to add any new branches to it, without causing the set to cease to be a tableau. More specifically, a complete tableau contains such set of maximal branches that any non-redundant superset of it ceases to co-create the tableau, similar to definition 2.60.

Consideration of the second variant, based on the tableau definition from the previous chapter, requires the introduction of an auxiliary definition, including the definition of a redundant superset.

Definition 3.43 (Redundant variant of branch)**.** Let ϕ and ϕ' be such branches that for some numbers i and $i+1$ that belong to their domains, it is the case that for any $j \leq i$, $\phi(j) = \phi'(j)$, but $\phi(i+1) \neq \phi'(i+1)$. We shall state that branch ϕ' is an *redundant variant* of branch ϕ iff:

- there exists such rule $R \in \mathbf{R_{TL}}$ and such pair $\langle X, Y \rangle \in R$ that $X = \phi(i)$ and $Y = \phi(i+1)$
- there exists such rule $R \in \mathbf{R_{TL}}$ and such triple $\langle X, W, Z \rangle \in R$ that $X = \phi(i)$ and:
  1. $W = \phi(i+1)$ and $Z = \phi'(i+1)$
     or
  2. $Z = \phi(i+1)$ and $W = \phi'(i+1)$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. We shall state that Ψ is an *redundant superset* Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ.

However, in the case of our system, the concept of redundant variant of branch is empty. It is covered by another conclusion.

Corollary 3.44. *For no branch ϕ there exists such branch ψ that ψ is a redundant variant of branch ϕ.*

Proof. Let us consider any branch ϕ and indirectly assume that there exists such branch ψ that ψ is a redundant variant of branch ϕ. According to definition of redundant variant of branch 3.43, for some numbers i and $i+1$ that belong to domains ϕ and ψ, it is the case that $\phi(i) = \psi(i)$, but $\phi(i+1) \neq \psi(i+1)$, and there exists such rule $R \in \mathbf{R_{TL}}$ and such triple $\langle X, Y, Z \rangle \in R$ that $X = \phi(i)$ and $Y = \phi(i+1)$, $Z = \psi(i+1)$ or $Y = \psi(i+1)$, $Z = \phi(i+1)$. However, this leads to a contradiction to the fact that there exists no rule $R \in \mathbf{R_{TL}}$ to comprise any triple $\langle X, Y, Z \rangle$. □

And since there exist no redundant variants of branches, there are neither redundant nor proper supersets of sets of branches, which is confirmed by another conclusion.

Corollary 3.45. *Let Φ be a set of branches. There exists no set of branches Ψ such that $\Phi \subset \Psi$ and Ψ is a redundant superset of Φ.*

Proof. Take any set of branches Φ and indirectly assume that there exists set of branches Ψ such that $\Phi \subset \Psi$ and Ψ is a redundant superset of Φ. Thus — according to definition of redundant variant of branch 3.43 — for any branch $\psi \in \Psi \setminus \Phi$, there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ. Since $\Phi \subset \Psi$, so there exists some branch $\psi \in \Psi \setminus \Phi$. Assume that set Φ is non-empty. Consequently, there exists such branch $\phi \in \Phi$ that some ψ is a redundant variant of ϕ, which contradicts the previous conclusion 3.44. □

We have shown that both concepts — of a redundant variant of branch and a redundant superset, in the phrasing appropriate for our tableau system for **TL**, are empty. After considering the redundant variants of branches and redundant supersets of branches, we can proceed to the definition of complete tableau in variant two.

Definition 3.46 (Complete tableau — variant two)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff:

1. each branch contained in Φ is maximal
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$
   b. $\langle X, A, \Psi \rangle$ is a tableau

   is a redundant superset of Φ.

A tableau is *incomplete* iff it is not complete.

Let us note that, once again, the two variants of the concept are equivalent. This is covered by another fact.

Proposition 3.47. *Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. $\langle X, A, \Phi \rangle$ is a complete tableau in variant one 3.36 iff $\langle X, A, \Phi \rangle$ is a complete tableau in variant two 3.46.*

Proof. Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches.

Assume that $\langle X, A, \Phi \rangle$ is a complete tableau in variant one 3.42. Since by virtue of the definition of tableau 3.36, set Φ contains precisely one branch and according to definition 3.42, that branch is maximal, so each branch contained in set Φ is maximal, which constitutes condition 1 of being a complete tableau in variant two of the definition of complete tableau 3.46. Let us now check if condition 2 of the definition in variant two holds. It claims that for any set of branches Ψ such that:

(a) $\Phi \subset \Psi$

(b) $\langle X, A, \Psi \rangle$ is a tableau

Ψ is a redundant superset of Φ.

So, take any such set of branches Ψ that $\Phi \subset \Psi$. However, by definition of tableau 3.36, triple $\langle X, A, \Psi \rangle$ is not a tableau since Ψ is a set of at least two elements, whereas in a tableau, a set of branches includes precisely one branch. Thus, condition 2 of the definition of complete tableau in variant two 3.46 is satisfied in the empty way.

Now, assume that $\langle X, A, \Phi \rangle$ is a complete tableau in variant two 3.46. According to definition 3.46, each branch contained in Φ is maximal, and what is more, by virtue of definition of tableau 3.36, Φ contains precisely one branch. Thus, a branch contained in Φ is a maximal branch, and consequently, tableau $\langle X, A, \Phi \rangle$ is complete according to the definition of complete tableau in variant one 3.42. □

Evidently, both concepts of a redundant variant of branch and a redundant superset, and their conclusions for the tableau system for **TL** play no role whatsoever in the equivalence proof.

When constructing a complete tableau, we may face a situation where a branch not only becomes maximal, but also finishes with a t-inconsistent set. Such a tableau is called a closed tableau. Let us, again, consider two variants. Variant one to begin with.

Definition 3.48 (Closed/open tableau — variant one)**.** Assume that $\langle X, A, \Phi \rangle$ is a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff a branch contained in Φ is closed. A tableau is *open* iff it is not closed.

Variant two, in turn, corresponding to the definition aiming at the general definition of a closed tableau, has the following form.

Definition 3.49 (Closed/open tableau — variant two)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff the below conditions are met:

1. $\langle X, A, \Phi \rangle$ is a complete tableau
2. each branch contained in Φ is closed.

A tableau is *open* iff it is not closed.

Proposition 3.50. *Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches. $\langle X, A, \Phi \rangle$ is a closed tableau in variant one 3.48 iff $\langle X, A, \Phi \rangle$ is a closed tableau in variant two 3.49.*

Proof. Let $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$ and Φ be a set of branches.

Assume that $\langle X, A, \Phi \rangle$ is a closed tableau in variant one 3.48. Thus, in the light of the definition of complete tableau 3.42, $\langle X, A, \Phi \rangle$ is a complete tableau — since branch which is contained in it is, by conclusion 3.30, maximal — which constitutes condition 1 of being a closed tableau in variant two of the definition of closed tableau 3.49. Since by virtue of the definition of tableau 3.36, set Φ contains precisely one branch and according to definition 3.48, that branch is closed, so each branch contained in set Φ is closed, which constitutes condition 2 of being a closed tableau in variant two of the definition of closed tableau 3.49.

Now, assume that $\langle X, A, \Phi \rangle$ is a closed tableau in variant two 3.49. Thus, by condition 2 of definition 3.49, each branch contained in Φ is closed, and, what is more, by virtue of definition of tableau 3.36, Φ contains precisely one branch, which constitutes a condition of being a closed tableau in variant one of the definition of closed tableau 3.48. □

To sum up, we have shown that specific tableau concepts of the tableau system for **TL** are special cases of more general concepts applied earlier (*modulo* set of tableau rules $\mathbf{R_{TL}}$ and resulting sets of branches), and in the considered cases are equivalent to them.

By virtue of the definition of closed tableau and the definition of complete tableau, and conclusion 3.30, we get another conclusion.

Corollary 3.51. *Each closed tableau is a complete tableau.*

Further, we will show that the concept of tableau is significantly helpful in determining the occurrence of relation $\triangleright$, while this concept, in terms of range, is equal to the concept of implication $\vDash$.

3.5 Completeness theorem for the tableau system for TL

Let us begin with the definition of model generated by a branch.

Definition 3.52 (Model generated by branch)**.** Let ϕ be any branch. We define the following function $At(\phi) = \bigcup \phi \cap (\mathsf{Te}_{\mathbf{TL}} \setminus \mathsf{For}_{\mathbf{TL}})$.

We shall state that model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$ is *generated by branch* ϕ iff:

- $D = \{x \in \mathbb{N} : x \in *(At(\phi))\}$
- for any name latter $P \in \mathsf{Ln}$, $x \in d(P)$ iff $P_{+x} \in At(\phi)$.

From this definition, we get the following conclusion.

Corollary 3.53. *Let ϕ be an open branch. Then, there exists a model generated by ϕ.*

Proof. By definition of open branch 3.28, definition of model generated by branch 3.52 and definition of model 3.3. □

Lemma 3.54 (On generation of model)**.** *Let ϕ be an open and maximal branch. Then, there exists model $\mathfrak{M}_{\mathbf{TL}}$ generated by ϕ such that $\mathfrak{M}_{\mathbf{TL}} \vDash \bigcup \phi \cap \mathsf{For}_{\mathbf{TL}}$.*

Proof. Take any open and maximal branch ϕ. Since branch ϕ is open, so by previous conclusion 3.53, there exists model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d \rangle$ generated by ϕ.

We will now show that for any formula A contained in $\bigcup\phi$, it is the case that $\mathfrak{M}_{\mathbf{TL}} \vDash A$, i.e. $\mathfrak{M}_{\mathbf{TL}} \vDash \bigcup\phi \cap \mathsf{For}_{\mathbf{TL}}$. The proof will be carried out with consideration of all the possible cases of construction of formula A. Now, assume that $A \in \bigcup\phi$. By definition of formula, for some name letters $P, Q \in \mathsf{Ln}$, there must occur one of the following cases.

1. $A = P\mathbf{a}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 3.52, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{a}_{+}$, $\bigcup\phi$ also contains tableau expression Q_{+i}. By definition of model generated 3.52, $i \in d(Q)$. Hence, $d(P) \subseteq d(Q)$, and by definition of truth in model 3.4, we thus get that $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$. In turn, if there exists no such $i \in D$ that $i \in d(P)$, then $\varnothing = d(P) \subseteq d(Q)$, so by definition of truth in model 3.4, we get $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{a}Q$.
2. $A = P\mathbf{i}Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{i}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{+i}, for some $i \in \mathbb{N}$. By definition of model generated 3.52, $i \in d(P)$ and $i \in d(Q)$. Since $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 3.4, we get that $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{i}Q$.
3. $A = P\mathbf{e}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 3.52, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{e}_{-}$, $\bigcup\phi$ also contains tableau expression Q_{-i}. Since branch ϕ is open, so expression $Q_{+i} \notin \bigcup\phi$, and consequently, by definition of model generated 3.52, $i \notin d(Q)$. Thus, $d(P) \cap d(Q) = \varnothing$ and by definition of truth in model 3.4, we get $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$. In turn, if there exists no object $i \in D$ such that $i \in d(P)$, then $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 3.4, we get that $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{e}Q$.
4. $A = P\mathbf{o}Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{o}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{-i}, for some $i \in \mathbb{N}$. By definition of model generated 3.52, $i \in d(P)$ and — since branch ϕ is open and, consequently, expression $Q_{+i} \notin \bigcup\phi$ — $i \notin d(Q)$, so $d(P) \nsubseteq d(Q)$. Thus, by definition of truth in model 3.4, we get $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$. □

We will now move on to a lemma which claims that application of tableau rules for the extension of branches does not reach beyond the model which is appropriate. In other words, according to the definition of model appropriate for the set of expressions 3.16, if we have a set of expressions such that the formulas contained in this set of expressions are true in given model and expressions stating inclusion or non-inclusion of a denotation of given index in the scope of name are interpretable in the model (they do not contradict the state of affairs in the model), then the extension of that set with the use of rules produces a set that still has the above properties.

Lemma 3.55. *Let* $\mathfrak{M}_{TL}$ *be any model,* X, $Y \subseteq \mathsf{Te}_{TL}$, *and let* $R \in \mathbf{R}_{TL}$. *Then, if* $\langle X, Y\rangle \in R$ *and* $\mathfrak{M}_{TL}$ *is appropriate for set of expressions* X, *then* $\mathfrak{M}_{TL}$ *is appropriate for* Y.

Proof. In the proof, we will make use of definition of model appropriate for the set of expressions 3.16. Let $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ be any model and X, $Y \subseteq \mathsf{Te}_{\mathbf{TL}}$. We will consider all cases of rules $R \in \mathbf{R}_{\mathbf{TL}}$, assuming that $\langle X, Y\rangle \in R$ and $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, and showing that then $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for Y.

1. Let $R = R\mathbf{a}_+$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{a}Q, P_{+i}\}, Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{a}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{a}Q$, hence by definition of truth in model 3.4, $\gamma(i) \in d(Q)$, since $d(P) \subseteq d(Q)$; consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}$.
2. Let $R = R\mathbf{i}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{i}Q\}, Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{i}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{i}$ enriches set X with expressions P_{+i}, Q_{+i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{i}Q$, so by virtue of definition of truth in model 3.4, in the domain there exists certain object x such that $x \in d(P) \cap d(Q)$; so we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}$.
3. Let $R = R\mathbf{e}_-$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{e}Q, P_{+i}\}, Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{e}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{TL}} \models P\mathbf{e}Q$, hence by virtue of definition of truth in model 3.4 $\gamma(i) \notin d(Q)$, since $d(P) \cap d(Q) = \varnothing$; consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}$.

4. Let $R = R\mathbf{o}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{o}Q\}, Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{TL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions X, so by definition 3.16, $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{o}$ enriches set X with expressions P_{+i}, Q_{-i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{TL}} \vDash P\mathbf{o}Q$, so by virtue of definition of truth in model 3.4, in the domain there exists certain object x such that $x \in d(P)$, but $x \notin d(Q)$; so we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 3.16, model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}$. □

We will now proceed to the main theorem which synthesizes all so far covered facts and lemmas, stating the dependencies between the semantic consequence relation, branch consequence relation and the existence of closed tableau.

Theorem 3.56 (Theorem on the completeness of tableau system for **TL**). *For any $X \subseteq \mathsf{For}_{\mathbf{TL}}$, $A \in \mathsf{For}_{\mathbf{TL}}$, the below statements are equivalent.*

- $X \vDash A$
- $X \rhd A$
- *there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{TL}}$ and $A \in \mathsf{For}$. We will prove three implications.

(a) $X \vDash A \Rightarrow X \rhd A$

Assume that $X \not\rhd A$. We must show that $X \nvDash A$. From the assumption and definition $\rhd$ 3.33, we know that for each finite set $Y \subseteq X$, there exists a branch beginning with set $Y \cup \{\circ(A)\}$ which is maximal and open. Take any finite subset $Y' \subseteq X$. Thus, there exists branch ϕ beginning with set $Y' \cup \{\circ(A)\}$ which is maximal and open. Since branch ϕ is maximal and open, so by lemma on generation of model 3.54, there exists model $\mathfrak{M}_{\mathbf{TL}}$ such that $\mathfrak{M}_{\mathbf{TL}} \vDash Y' \cup \{\circ(A)\}$.

Due to the fact that set Y' is arbitrary, so for any Y, finite subset of X, there exists model $\mathfrak{M}_{\mathbf{TL}}$ such that $\mathfrak{M}_{\mathbf{TL}} \vDash Y \cup \{\circ(A)\}$. Thus, for any Y, finite subset of X, $Y \nvDash A$, due to the fact 3.13. While by fact 3.9, relation $\vDash$ is compact, so $X \nvDash A$.

(b) $X \rhd A \Rightarrow$ there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$

Assume that for any finite subset $Y \subseteq X$, each tableau $\langle Y, A, \Phi\rangle$ is open. Take any finite subset $Y \subseteq X$. By the assumption, each tableau $\langle Y, A, \Phi\rangle$ is open. By definition of tableau 3.36, each set Φ contains one branch beginning with set

$Y \cup \{\circ(A)\}$. Consequently, by definition of open tableau 3.48, each branch beginning with set $Y \cup \circ(A)$ is open. However, from fact 3.32, we know that since set $Y \cup \{\circ(A)\}$ is finite, then there exists branch beginning with $Y \cup \{\circ(A)\}$ that is maximal. Since that branch is open and set Y is a finite subset of X, by definition of relation of branch consequence 3.33 $X \not\triangleright A$,

(c) there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle \Rightarrow X \vDash A$

Assume that there exists finite subset $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$. Indirectly assume that $X \nvDash A$, thus, by definition of relation of semantic consequence 3.5, there exists such model $\mathfrak{M}_{\mathbf{TL}}$ that $\mathfrak{M}_{\mathbf{TL}} \vDash X$, but $\mathfrak{M}_{\mathbf{TL}} \nvDash A$. From fact 3.13, we know that $\mathfrak{M}_{\mathbf{TL}} \vDash \circ(A)$. Consequently, $\mathfrak{M}_{\mathbf{TL}} \vDash X \cup \{\circ(A)\}$, and thus $\mathfrak{M}_{\mathbf{TL}} \vDash Y \cup \{\circ(A)\}$. Since tableau $\langle Y, A, \Phi \rangle$ is closed, then by definition 3.48, Φ contains branch ψ beginning with set $Y \cup \{\circ(A)\}$ which is closed. So, branch ψ is maximal by conclusion 3.30 and has lengths n, for certain $n \in \mathbb{N}$. What is more, by conclusion 3.29, set $\psi(n)$ is t-inconsistent.

Since $\mathfrak{M}_{\mathbf{TL}} \vDash Y \cup \{\circ(A)\}$, so model $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set $Y \cup \{\circ(A)\}$. Now, applying lemma 3.55, $n-1$-times we get conclusion that $\mathfrak{M}_{\mathbf{TL}}$ is appropriate for set $\psi(n)$. However, due to the fact that $\psi(n)$ is t-inconsistent and conclusion 3.17, there exists no model appropriate for $\psi(n)$. □

3.5.1 Estimation of cardinality of model for TL

When applying tableau methods for **TL** another issue appears. With the tableau proof, we can estimate the upper limit of the cardinality of models that we only need to check in order to establish whether a given inference is correct[7]. While, in the study we do not take up this issue in general (as it is related to the issue of decidaility), but this outcome for **TL** we can virtually get directly from the theorem on completeness of tableau system for **TL** 3.56.[8]

By *existential formula*, we mean any formula in form $P\mathbf{i}Q$ or $P\mathbf{o}Q$, where P, Q, $\in \mathsf{Ln}$.

Now, we shall define function $\lambda : P(\mathsf{For}_{\mathbf{TL}}) \longrightarrow P(\mathsf{For}_{\mathbf{TL}})$ with the following condition: for any set $\Phi \in P(\mathsf{For}_{\mathbf{TL}})$, $\lambda(\Phi) = \{x \in \Phi : x \text{ is an existential formula}\}$. So, from each set of formulas, function λ "selects" all existential formulas that belong to a given set.

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7 Estimations of cardinality of model for syllogistic, for languages richer than the language of **TL**, have been examined in the studies by: A. Pietruszczak [20], [21], P. Kulicki [17], [18].

8 This outcome was originally described in article [9]. However, when defining the tableau system for **TL**, in that study, we applied another set of rules and non-formalised tableau concepts.

Now, in turn, we shall define function $\sigma : \{\Psi \in P(\mathsf{For}_{\mathbf{TL}}) : \Psi \text{ is a finite set}\} \longrightarrow \mathbb{N}$ with the following condition: for any finite set $\Psi \in P(\mathsf{For}_{\mathbf{TL}})$, $\sigma(\Psi) = |\lambda(\Psi)|$. So, function σ "counts" the number of existential formulas that are found in any finite set of formulas.

We have the theorem.

Theorem 3.57. *Let X be a finite set of formulas and let $A \in \mathsf{For}_{TL}$. Then:*
$\forall_{\mathfrak{M}_{TL}=\langle D,d\rangle} (|D| = \sigma(X \cup \{\circ(A)\}) \Rightarrow (\mathfrak{M}_{TL} \vDash X \Rightarrow \mathfrak{M}_{TL} \vDash A))$ *iff* $X \vDash A$.

Proof. Take any finite set of formulas X and any formula A.

The implication "from the right to the left" follows from the definition of relation of semantic consequence 3.5. Because if $X \vDash A$, then for any model $\mathfrak{M}_{\mathbf{TL}}$, if $\mathfrak{M}_{\mathbf{TL}} \vDash X$, then $\mathfrak{M}_{\mathbf{TL}} \vDash A$. Particularly, for such models $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ that $|D| = \sigma(X \cup \{\circ(A)\})$.

Now, assume that $X \nvDash A$. From the theorem on completeness of tableau system of **TL** 3.56, we get that $X \not\triangleright A$. By definition of relation of branch consequence 3.33, for any finite $Y \subseteq X$, there exists maximal and open branch beginning with set $Y \cup \{\circ(A)\}$. Since X is a finite set, so there exists maximal and open branch ϕ beginning with set $X \cup \{\circ(A)\}$. So, by lemma on generation of model 3.54, there exists such model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ that $\mathfrak{M}_{\mathbf{TL}} \vDash X \cup \{\circ(A)\}$, thus $\mathfrak{M}_{\mathbf{TL}} \vDash X$ and $\mathfrak{M}_{\mathbf{TL}} \nvDash A$, by fact 3.13.

By definition 3.52, domain $D = \{i \in \mathbb{N} : i \in *(At(\phi))\}$, and each object $i \in D$ emerged in some expression in $\bigcup\phi \cap (\mathsf{Te}_{\mathbf{TL}} \setminus \mathsf{For})$ by the application of rule R**i** or R**o** to some existential formula. As for each existential formula, we can only one time apply rule R**i** or R**o**, thus $|D| \leq \sigma(X \cup \{\circ(A)\})$.

Model $\mathfrak{M}_{\mathbf{TL}}$ was generated by an open and maximal branch, so rule R**i** or R**o** was applied to each existential formula. Thus, for any existential formula, there exists at least one object $i \in D$, due to the fact that rules R**i** and R**o** introduce expressions with new indices. Hence, $|D| \geq \sigma(X \cup \{\circ(A)\})$.

So, consequently, there exists such model $\mathfrak{M}_{\mathbf{TL}} = \langle D, d\rangle$ that $|D| = \sigma(X \cup \{\circ(A)\})$, $\mathfrak{M}_{\mathbf{TL}} \vDash X$ and $\mathfrak{M}_{\mathbf{TL}} \nvDash A$. $\square$

4 Tableau system for modal logic S5

4.1 Introductory remarks

Chapter Four will be devoted to a case extremely different than, presented in Chapter One, the case of tableau system for **CPL**. For we consider modal logic **S5** (for short: **S5**), defining the set of tableau expressions in such a way for it neither to concur with the set of formulas of logic **S5**, nor to be its proper superset — both sets are disjoint[1]. The proofs in the tableau system that we are going to construct will therefore be carried out in a language different from the one in which we want to determine whether or not a given consequence relation holds.

Another important difference between the previous systems and the one currently being defined is that the systems described previously featured the property of a finite branch, which is not the case for the presented tableau system for **S5**.

Therefore, it may happen that when constructing a branch and consequently a tableau which begins with a finite set of expressions, it is not possible to finish it as some sequences of application of the rules become cyclical. The lack of a finite branch property forces changes in a concept of maximal branch and in dependent concepts.

So, the tableau concepts defined in previous chapters will become special cases of tableau concepts for systems that do not feature the property of a finite branch. The leading change is the generalisation of the concept of maximal branch. Still, the maximal branch is a branch to which no rule can be applied anymore in order to extend it as it contains everything tableau rules are capable of introducing to it. Previously, however, for the cases of finite sets this meant that the maximal branch was of a finite length. It does not have to be the case this time. A maximal branch can be infinite even though the infinity of a branch does not imply its maximal nature. For there exist cases of infinite-length branches that are not maximal.

It is worth noting that the systems that feature the property of a finite branch are decidable. For theoretically, always in a finite number of steps it is possible to construct a complete tableau for them — closed or open one, thus answering the question whether a given formula is or is not tableau derivable on the grounds of given premises. In the case of tableau systems that do not feature a finite branch

1 Some of the findings we will present here were described in English-language article [8]. In particular, we took up the general definition of concepts for modal logics defined by the semantics of possible worlds, to which we will return in the final chapter of the book.

property, these systems may not be decidable. So, although we prove that they are complete and consistent in relation to the initial, semantically defined relation, there does not have to exist a general way of constructing a complete tableau even though such a tableau can exist.

So, we treat the case of a tableau system for **S5** logic as a model for two reasons:

- the set of tableau expressions is naturally different than the set of formulas
- branches beginning with finite sets of expressions can feature an infinite lengths.

Both reasons lead to the need of generalization of the so-far applied tableau concepts, affecting the formalisation of tableau methods offered in the book.

4.2 Language and semantics

As usual, for the start we will define the basic concepts for the logic **S5**. First, we will take up the language of **S5**.

Definition 4.1 (Alphabet of **S5**). *Alphabet of the modal logic* **S5** is the sum of the following sets:

- set of logical constants: $\mathsf{Lc} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Diamond, \Box\}$
- set of propositional letters: $\mathsf{Var} = \{p_1, q_1, r_1, p_2, q_2, r_2, \ldots\}$
- set of brackets: $\{)$, $(\}$.

Although the set of propositional letters is infinite and includes indexed letters, in practice we will use a finite number of the following letters: *p*, *q*, *r*, *s*.

Definition 4.2 (Formula **S5**). *Set of formulas of modal logic* **S5** is the smallest set *X* which meets conditions:

1. $\mathsf{Var} \subseteq X$
2. if $A, B \in X$, then
   a. $\neg A \in X$
   b. $\Box A \in X$
   c. $\Diamond A \in X$
   d. $(A \wedge B) \in X$
   e. $(A \vee B) \in X$
   f. $(A \rightarrow B) \in X$
   g. $(A \leftrightarrow B) \in X$.

We specify this set as $\mathsf{For}_{\mathbf{S5}}$, and its elements will be called *formulas*.

We will now proceed to the interpretation of set $\mathsf{For}_{\mathbf{S5}}$. To begin with, let us recall some properties of binary relations.

Definition 4.3. Let R be a binary relation defined on Cartesian product $X \times X$, for some set X. We shall state that:

1. R is a *universal* relation iff $\forall_{x,y \in X}\, xRy$
2. R is an *equivalence* relation iff
   a. R is a reflexive relation, i.e. $\forall_{x \in X}\, xRx$
   b. R is a symmetric relation, i.e. $\forall_{x,y \in X}\, (xRy \Rightarrow yRx)$
   c. R is a transitive relation, i.e. $\forall_{x,y,z \in X}\, (xRy \,\&\, yRz \Rightarrow xRz)$.

From the above definition 4.3, the following conclusion follows.

Corollary 4.4. *Each universal relation is an equivalence relation.*

We will now proceed to the concept of model for formulas **S5**.

Definition 4.5 (Model for language of **S5**). *Model* $\mathfrak{M}_{\mathbf{S5}}$ *for language of* ***S5*** will be called such ordered quadruple $\langle W, R, V, w \rangle$ that:

- W is a non-empty set
- R is a universal relation defined on Cartesian product $W \times W$, i.e. $R = W \times W$
- V is a function valuating propositional letters in the elements of set W, i.e. $V : \mathsf{Var} \times W \longrightarrow \{0,1\}$
- $w \in W$.

From definition of model 4.5 and conclusion 4.4 another conclusion results.

Corollary 4.6. *Each model with a universal relation is a model with an equivalence relation.*

Models with equivalence relation will be denoted as $\mathfrak{M}'_{\mathbf{S5}}$.

Now, we shall define truth in model. This definition also applies to models with equivalence relation $\mathfrak{M}'_{\mathbf{S5}}$.

Definition 4.7 (Truth in model). Let $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w \rangle$ be a model and let $A \in \mathsf{For}_{\mathbf{S5}}$. Formula A is *true* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \models A$) iff for any formulas $B, C \in \mathsf{For}_{\mathbf{S5}}$ the below conditions are met:

1. if $A \in \mathsf{Var}$, then $V(A, w) = 1$
2. if $A := \neg B$, then formula B is not true in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \not\models B$)
3. if $A := (B \wedge C)$, then $\mathfrak{M}_{\mathbf{S5}} \models B$ and $\mathfrak{M}_{\mathbf{S5}} \models C$
4. if $A := (B \vee C)$, then $\mathfrak{M}_{\mathbf{S5}} \models B$ or $\mathfrak{M}_{\mathbf{S5}} \models C$
5. if $A := (B \rightarrow C)$, then $\mathfrak{M}_{\mathbf{S5}} \not\models B$ or $\mathfrak{M}_{\mathbf{S5}} \models C$
6. if $A := (B \leftrightarrow C)$, then $\mathfrak{M}_{\mathbf{S5}} \models B$ iff $\mathfrak{M}_{\mathbf{S5}} \models C$

7. if $A := \Box B$, then $\forall_{u \in W}(wRu \Rightarrow \langle W, R, V, u\rangle \vDash B)$
8. if $A := \Diamond B$, then $\exists_{u \in W}(wRu \,\&\, \langle W, R, V, u\rangle \vDash B)$.

Formula A is *false* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \nvDash A$) iff it not true.

Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$. Set of formulas X is *true* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \vDash X$) iff for any formula $A \in X$, $\mathfrak{M}_{\mathbf{S5}} \vDash A$. Set of formulas X is *false* in model $\mathfrak{M}_{\mathbf{S5}}$ (for short: $\mathfrak{M}_{\mathbf{S5}} \nvDash X$) iff it is not the case that $\mathfrak{M}_{\mathbf{S5}} \vDash X$.

Now, we will show a fact that displays an connection between models with equivalence relation and those with universal relation.

Proposition 4.8. *For any model* $\mathfrak{M}'_{\mathbf{S5}} = \langle W', R', V', w'\rangle$, *there exists such model* $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ *that for any formula* $A \in \mathsf{For}_{\mathbf{S5}}$, $\mathfrak{M}'_{\mathbf{S5}} \vDash A$ *iff* $\mathfrak{M}_{\mathbf{S5}} \vDash A$.

Proof. Take any model with equivalence relation $\mathfrak{M}'_{\mathbf{S5}} = \langle W', R', V', w'\rangle$. Next, define model with universal relation $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ as follows:

- $W = \{x \in W' : w'Rx\}$
- $R = \{\langle x, y\rangle : x, y \in W\}$
- $V = V'_W$, where V'_W is a restriction of function V' to set $\mathsf{Var} \times W$
- $w = w'$.

R' is an equivalence relation, $R \subseteq R'$, $R = R'_W$, where R'_W constitutes a restriction of relation R' to set $W = \{x \in W' : w'Rx\}$, so R'_W is a universal relation. Since relation R' is reflexive, so obviously $w' \in W$, thus model $\mathfrak{M}_{\mathbf{S5}}$ is well defined.

Consider any propositional letter $q \in \mathsf{Var}$. By definition of model $\mathfrak{M}_{\mathbf{S5}}$, since $V = V'_W$, so for any $u \in W$ it is the case that $(*)$ $V'(q, u) = 1$ iff $V(q, u) = 1$.

Now, take any formula $A \in \mathsf{For}_{\mathbf{S5}}$. We will consider various construction possibilities for formula A, carrying out an inductive proof in respect of the complexity of formula A and showing that for any $u \in W$, the below thesis occurs:

$(**)$ $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$.

Initial step. Take any $u \in W$. Assume that $A \in \mathsf{Var}$. By $(*)$ and definition of truth in model 4.7, we get $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$.

Induction step. Take any formulas $B, C \in \mathsf{For}_{\mathbf{S5}}$ and assume that $(***)$ for B, C and for any $u \in W$, the following occurs:

$\langle W', R', V', u\rangle \vDash B$ iff $\langle W, R, V, u\rangle \vDash B$

$\langle W', R', V', u\rangle \vDash C$ iff $\langle W, R, V, u\rangle \vDash C$.

Take any $u \in W$ and consider the following cases:

1. $A = \neg B$, $A = (B \wedge C)$, $A = (B \vee C)$, $A = (B \rightarrow C)$ or $A = (B \leftrightarrow C)$; then by virtue of assumption $(***)$ and definition of truth in model 4.7, we get $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$

2. $A = \Box B$; then, by virtue of definition of truth in model 4.7, $\langle W', R', V', u\rangle \vDash \Box B$ iff $\forall_{z\in W'}(uR'z \Rightarrow \langle W', R', V', z\rangle \vDash B)$, by definition of set W and relation R and by assumption $(***)$, it is the case iff $\forall_{z\in W}(uRz \Rightarrow \langle W, R, V, z\rangle \vDash B)$, while by virtue of definition of truth in model 4.7, iff $\langle W, R, V, u\rangle \vDash \Box B$, thus $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$
3. $A = \Diamond B$; then, by virtue of definition of truth in model 4.7, $\langle W', R', V', u\rangle \vDash \Diamond B$ iff $\exists_{z\in W'}(uR'z \,\&\, \langle W', R', V', z\rangle \vDash B)$, by definition of set W and relation R and by assumption $(***)$, it is the case iff $\exists_{z\in W}(uRz \,\&\, \langle W, R, V, z\rangle \vDash B)$, while by virtue of definition of truth in model 4.7, iff $\langle W, R, V, u\rangle \vDash \Diamond B$, thus $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$.

Consequently, we get thesis:

for any $u \in W$, $\langle W', R', V', u\rangle \vDash A$ iff $\langle W, R, V, u\rangle \vDash A$. However, since $w = w'$ and $w \in W$, $\mathfrak{M}'_{\mathbf{S5}} \vDash A$ iff $\mathfrak{M}_{\mathbf{S5}} \vDash A$. □

Making use of the concept of model, we can now define the concept of semantic consequence relation in **S5**. For the entire class of models with universal relation, in a normal way on set $P(\mathsf{For}_{\mathbf{S5}}) \times \mathsf{For}_{\mathbf{S5}}$ we define the consequence relation.

Definition 4.9 (Semantic consequence of **S5**). Let $A \in \mathsf{For}_{\mathbf{S5}}$ and $X \subseteq \mathsf{For}_{\mathbf{S5}}$. We shall state that from set X *follows* formula A (for short: $X \vDash A$) iff for any model $\mathfrak{M}_{\mathbf{S5}}$, if $\mathfrak{M}_{\mathbf{S5}} \vDash X$, then $\mathfrak{M}_{\mathbf{S5}} \vDash A$. Relation $\vDash$ will also be called *semantic consequence relation of logic* **S5** or for short *semantic consequence relation*.

Denotation 4.10. For any set of formulas X and any formula A notation $X \nvDash A$ will mean that it is not the case that $X \vDash A$.

Let us remind that we defined models for **S5** with the use of universal relations. It is known, however, we could define models for **S5** using equivalence relations — since both classes of models determine the same semantic relation of consequence, which is expressed by another fact.

Proposition 4.11. *Let $A \in \mathsf{For}_{\mathbf{S5}}$ and $X \subseteq \mathsf{For}_{\mathbf{S5}}$. Then, $X \vDash A$ iff for any model with equivalence relation $\mathfrak{M}'$, if $\mathfrak{M}' \vDash X$, then $\mathfrak{M}' \vDash A$.*

Proof. Let $A \in \mathsf{For}_{\mathbf{S5}}$ and $X \subseteq \mathsf{For}_{\mathbf{S5}}$.

First, we will prove the implication "from the left to the right". Assume that $X \vDash A$ and take any model with equivalence relation $\mathfrak{M}'_{\mathbf{S5}} = \langle W, R, V, w\rangle$ such that $\mathfrak{M}'_{\mathbf{S5}} \vDash X$. From fact 4.8 we know that for certain model with universal relation $\mathfrak{M}_{\mathbf{S5}}$ it is the case that for any formula $B \in \mathsf{For}_{\mathbf{S5}}$, $\mathfrak{M}'_{\mathbf{S5}} \vDash B$ iff $\mathfrak{M}_{\mathbf{S5}} \vDash B$. Thus $\mathfrak{M}_{\mathbf{S5}} \vDash X$, and consequently, by assumption $\mathfrak{M}_{\mathbf{S5}} \vDash A$. Again, making use of fact 4.8, we get $\mathfrak{M}'_{\mathbf{S5}} \vDash A$.

We prove implication 'from the left to the right' with the use of conclusion 4.6. Assume that for any model with equivalence relation $\mathfrak{M}'$, if $\mathfrak{M}' \vDash X$, then $\mathfrak{M}' \vDash A$, and take any model $\mathfrak{M}_{\mathbf{S5}}$ such that $\mathfrak{M}_{\mathbf{S5}} \vDash X$. But, by conclusion 4.6, model $\mathfrak{M}_{\mathbf{S5}}$ is a model of equivalence relation, thus $\mathfrak{M}_{\mathbf{S5}} \vDash A$. □

The above issue will be discussed in subchapter devoted to the axiomatization with tableau rules as it directly suggests the possibility of two different axiomatizations.

Let us now go to the concept of contradictory set of formulas.

Definition 4.12. Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$. Set of formulas X is *contradictory* iff for any model $\mathfrak{M}_{\mathbf{S5}} \nvDash X$. Set X is *non-contradictory* iff X it is not contradictory.

Another conclusion follows from definition of contradictory set of formulas 4.12 and definition of truth in model 4.7.

Proposition 4.13. *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. If $\{A, \neg A\} \subseteq X$, then X is contradictory.*

4.3 Basic concepts of the tableau system for S5

Unlike the tableau system for **CPL**, in the case of tableau system for **TL** the tableau proofs were carried out in a more rich language than the set of formulas. In the case of tableau system for **S5**, the language of tableau proof has no common elements with the language of logic **S5** — thus in this respect, the described system constitutes a completely new case.

Let us now define the proof language for the tableau system for logic **S5**.

Definition 4.14 (Tableau expressions for **S5**)**.** *Set of tableau expressions* is the union of two following sets:

- Cartesian product: $\mathsf{For}_{\mathbf{S5}} \times \mathbb{N}$
- $\{irj : i, j \in \mathbb{N}\}$.

We specify this set as $\mathsf{Te}_{\mathbf{S5}}$, and its elements will be called *tableau expressions* or simply *expressions*. Numbers present in the expressions will be called *indices*.

In the event of tableau systems for the modal logic, sometimes there is a need of "selecting" indices from the tableau expressions. Thus, let us define an appropriate function to enable the above. Before that, however, we will introduce an auxiliary function $h : \mathsf{Te}_{\mathbf{S5}} \longrightarrow P(\mathbb{N})$ defined for any $A \in \mathsf{For}_{\mathbf{S5}}$ and $i, j \in \mathbb{N}$ with conditions:

- $h(\langle A, i\rangle) = \{i\}$
- $h(irj) = \{i, j\}$.

Definition 4.15 (Function selecting indices)**.** *Function selecting indices* will be called function $* : P(\mathsf{Te}_{\mathbf{S5}}) \longrightarrow P(\mathbb{N})$ defined for any $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ with condition:

- $*(X) = \bigcup\{h(y) : y \in X\}$.

For any subset $\mathsf{Te}_{\mathbf{S5}}$, function $*$ selects all indices present in the expressions from that set.

Let us now proceed to the concept of similar sets of expressions. Intuitively, two sets of expressions are similar iff their expressions contain exactly the same formulas and all expressions in both sets are structurally similar with respect to the indices. Formally:

Definition 4.16 (Similar set of expressions)**.** Let X, $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$. X is *similar* to Y iff there exists such bijection $g : *(X) \longrightarrow *(Y)$ that for any $A \in \mathsf{For}_{\mathbf{S5}}$ and indices $i, j \in \mathbb{N}$:

- $\langle A, i\rangle \in X$ iff $\langle A, g(i)\rangle \in Y$
- $irj \in X$ iff $g(i)rg(j) \in Y$.

Based on definition 4.16, we can draw the following conclusion.

Corollary 4.17. *The relation of similarity is symmetric, i.e. For any sets of expressions X, Y, if set X is similar to Y, then Y is similar to set X.*

Proof. By definition of similar sets of expressions 4.16, by the fact that function g is bijection and by the equivalences present in both conditions. □

Next, we introduce a definition of tableau inconsistent set.

Definition 4.18 (Tableau inconsistent set of expressions)**.** Set $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ will be called *tableau inconsistent* (for short: t-inconsistent) iff for some formula $A \in \mathsf{For}_{\mathbf{S5}}$ and some index $i \in \mathbb{N}$, $\langle A, i\rangle \in X$ and $\langle \neg A, i\rangle \in X$. Set X is *tableau consistent* (for short: t-consistent) iff X is not t-inconsistent.

From this definition, the following conclusion results.

Corollary 4.19. *Let X, $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$. If set X is similar to set Y, then X is t-consistent iff Y is t-consistent.*

Proof. Let X, $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$, and assume that X is similar to set Y. Also, assume that set X is t-inconsistent. Then, by definition of t-inconsistent set 4.18, set X contains expressions $\langle A, i\rangle$, $\langle \neg A, i\rangle$, for some $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. By definition of similar set 4.16, there exists bijection $g : *(X) \longrightarrow *(Y)$ and set Y contains expressions $\langle A, g(i)\rangle$, $\langle \neg A, g(i)\rangle$, thus by definition of t-inconsistent set 4.18, set Y

is t-inconsistent. On the other hand, since relation of set similarity is symmetric 4.17, then by assumption, set Y is similar to set X. Also, assume that set Y is t-inconsistent. Then, by definition of t-inconsistent set 4.18, set Y contains expressions $\langle A, i\rangle$, $\langle \neg A, i\rangle$, for some $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. By definition of similar set 4.16, there exists bijection $g' : *(Y) \longrightarrow *(X)$ and set X contains expressions $\langle A, g'(i)\rangle$, $\langle \neg A, g'(i)\rangle$, thus by definition of t-inconsistent set 4.18, set X is t-inconsistent. □

For further studies, we also need a concept that would combine models with the set of expressions.

Definition 4.20 (Model appropriate for set of expressions)**.** Let $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ be a model and let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$. We shall state that model $\mathfrak{M}_{\mathbf{S5}}$ is *appropriate* for X iff there exists such function $f : \mathbb{N} \longrightarrow W$, that for any $A \in \mathsf{For}_{\mathbf{S5}}$ and $i, j \in \mathbb{N}$:

- if $\langle A, i\rangle \in X$, then $\langle W, R, V, f(i)\rangle \models A$
- if $irj \in X$, then $f(i)Rf(j)$.

Another fact follows from the above definition.

Proposition 4.21. *Let X be a t-inconsistent set of expressions. Then, there exists no model $\mathfrak{M}_{\mathbf{S5}}$ appropriate for X.*

Proof. Take any set of expressions X and any model $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, w\rangle$ and assume that X is t-inconsistent. Then, by definition of tableau inconsistent set of expressions 4.18, for some formula $A \in \mathsf{For}_{\mathbf{S5}}$ and for some index $i \in \mathbb{N}$, $\langle A, i\rangle \in X$ and $\langle \neg A, i\rangle \in X$. If model $\mathfrak{M}_{\mathbf{S5}}$ were appropriate for set of expressions X, then by definition of model appropriate for the set of expressions 4.20, there would exist such function $f : \mathbb{N} \longrightarrow W$ that if $\langle A, i\rangle \in X$, then $\langle W, R, V, f(i)\rangle \models A$ and if $\langle \neg A, i\rangle \in X$, then $\langle W, R, V, f(i)\rangle \models \neg A$, and so $\langle W, R, V, f(i)\rangle \models A$ and $\langle W, R, V, f(i)\rangle \models \neg A$. However, from fact 4.13 and definition 4.12, it follows that there exists no such a model. Hence, model $\mathfrak{M}_{\mathbf{S5}}$ is not appropriate for set of expressions X. □

4.3.1 Tableau rules for S5

As in the previous cases of described tableau systems, we first provide the general concept of rule. Not only because it facilitates provision of the general features that a tableau rule must meet. In the case of system construction for **S5** we will provide — as in the case of **TL** — alternative sets of tableau rules that are suitable for construction of tableau system for logic **S5**. This means that within the below general concept of rule, we can define different sets of tableau rules that define various, however equivalent in terms of scope of branch consequence, tableau systems for **S5** (see note 2.19 and 3.20).

Definition 4.22 (Rule). Let $P(\mathsf{Te}_{\mathbf{S5}})$ be a power set of the set of tableau expressions. Let $P(\mathsf{Te}_{\mathbf{S5}})^n$ be n-ary Cartesian product $\underbrace{P(\mathsf{Te}_{\mathbf{S5}}) \times \cdots \times P(\mathsf{Te}_{\mathbf{S5}})}_{n}$, for some $n \in \mathbb{N}$.

- By a *rule* we understand any subset $R \subseteq P(\mathsf{Te}_{\mathbf{S5}})^n$ such that if $\langle X_1, \ldots, X_n \rangle \in R$, then:
  a. X_1 is t-inconsistent
  b. $X_1 \subset X_i$, for each $1 < i \leq n$.
- If $n \leq 2$, then each element R will be called an *ordered n-tuple* (pair, triple, etc., respectively).
- The first element of each n-tuple will be called an *input set* (*set of premises*), while its remaining elements *output sets* (*sets of conclusions).*

In the case of **S5**, the rule definition differs from the rule definition for **TL** (definition 3.18) only in that it is specified on a different set of expressions and in a different nature of t-inconsistency. Beyond that, these definitions are structurally similar, because we are seeking to formulate general concepts. A set of tableau rules for the tableau system for **CPL** we describe, shall be introduced by means of another definition.

Definition 4.23 (Tableau rules for **S5**). Tableau rules for **S5** are the following rules:

$$R_{\wedge}: \frac{X \cup \{\langle (A \wedge B), i \rangle\}}{X \cup \{\langle (A \wedge B), i \rangle, \langle A, i \rangle, \langle B, i \rangle\}}$$

$$R_{\vee}: \frac{X \cup \{\langle (A \vee B), i \rangle\}}{X \cup \{\langle (A \vee B), i \rangle, \langle A, i \rangle\} | X \cup \{\langle (A \vee B), i \rangle, \langle B, i \rangle\}}$$

$$R_{\rightarrow}: \frac{X \cup \{\langle (A \rightarrow B), i \rangle\}}{X \cup \{\langle (A \rightarrow B), i \rangle, \langle \neg A, i \rangle\} | X \cup \{\langle (A \rightarrow B), i \rangle, \langle B, i \rangle\}}$$

$$R_{\leftrightarrow}: \frac{X \cup \{\langle (A \leftrightarrow B), i \rangle\}}{X \cup \{\langle (A \leftrightarrow B), i \rangle, \langle A, i \rangle, \langle B, i \rangle\} | X \cup \{\langle (A \leftrightarrow B), i \rangle, \langle \neg A, i \rangle, \langle \neg B, i \rangle\}}$$

$$R_{\neg\neg}: \frac{X \cup \{\langle \neg\neg A, i \rangle\}}{X \cup \{\langle \neg\neg A, i \rangle, \langle A, i \rangle\}}$$

$$R_{\neg\wedge}: \frac{X \cup \{\langle \neg(A \wedge B), i \rangle\}}{X \cup \{\langle \neg(A \wedge B), i \rangle, \langle \neg A, i \rangle\} | X \cup \{\langle \neg(A \wedge B), i \rangle, \langle \neg B, i \rangle\}}$$

$$R_{\neg\vee}: \frac{X \cup \{\langle\neg(A \vee B), i\rangle\}}{X \cup \{\langle\neg(A \vee B), i\rangle, \langle\neg A, i\rangle, \langle\neg B, i\rangle\}}$$

$$R_{\neg\rightarrow}: \frac{X \cup \{\langle\neg(A \rightarrow B), i\rangle\}}{X \cup \{\langle\neg(A \rightarrow B), i\rangle, \langle A, i\rangle, \langle\neg B, i\rangle\}}$$

$$R_{\neg\leftrightarrow}: \frac{X \cup \{\langle\neg(A \leftrightarrow B), i\rangle\}}{X \cup \{\langle\neg(A \leftrightarrow B), i\rangle, \langle\neg A, i\rangle, \langle B, i\rangle\} \mid X \cup \{\langle\neg(A \leftrightarrow B), i\rangle, \langle A, i\rangle, \langle\neg B, i\rangle\}}$$

$$R_{\neg\Box}: \frac{X \cup \{\langle\neg\Box A, i\rangle\}}{X \cup \{\langle\neg\Box A, i\rangle, \langle\Diamond\neg A, i\rangle\}}$$

$$R_{\neg\Diamond}: \frac{X \cup \{\langle\neg\Diamond A, i\rangle\}}{X \cup \{\langle\neg\Diamond A, i\rangle, \langle\Box\neg A, i\rangle\}}$$

$$R_{\Box}: \frac{X \cup \{\langle\Box A, i\rangle, irj\}}{X \cup \{\langle\Box A, i\rangle, irj, \langle A, j\rangle\}}$$

$$R_{\Diamond}: \frac{X \cup \{\langle\Diamond A, i\rangle\}}{X \cup \{\langle\Diamond A, i\rangle, irj, \langle A, j\rangle\}}, \text{ where:}$$

1. $j \notin *(X \cup \{\langle\Diamond A, i\rangle\})$
2. for any $k \in \mathbb{N}$, $\{irk, \langle A, k\rangle\} \nsubseteq X$.

$$R_r: \frac{X}{X \cup \{irj\}}, \text{ where } i, j \in *(X).$$

Set of tableau rules for **S5** will be defined as $\mathbf{R_{S5}}$.

Let us now devote a few words to discussing the described rules. Although in many ways they resemble the rules from set $\mathbf{R_{TL}}$, there are also some differences.

According to the definition of rule 4.22, the input sets of each rule are t-consistent. In addition, in each rule, the input set is basically contained in each output set. Again, there appears the two-premise rule — rule $R_{\Box}$.

On the other hand, similar to the case of rules R**i** and R**o** for **TL**, we have a rule with limitations on introducing new expressions. We mean rule $R_{\Diamond}$. This rule says that the expressions introduced are to have a new index (condition 1), moreover, the input set must not contain expressions similar to the one entered, beginning with the index appearing in the expression with $\Diamond$ (condition 2).

The semantic intuitions on which condition 1 is based are as follows. The object denoted by a new index is also to be new and its relationship with other objects that are denoted by the other indices is to remain unresolved.

Condition 2 prevents unnecessary expressions from being entered in the proof. In practice, it also prevents the proof from being unnecessarily extended indefinitely — so it prevents the creation of infinite branches when this is not a consequence of logic itself, but of the wrong definition of tableau system (example 4.24).

The last rule is a rule that corresponds to universal relation in the models. With rule R_r, each two indices that appear in the expressions found in the proof may cause the addition to the proof of another expression that contains those indices. These indices do not have to be different, of course.

Example 4.24. Consider the following set of expressions $X \cup \{\langle \Diamond p, 1\rangle\}$ and rule $R_\Diamond$ without condition 2. The use of rule $R_\Diamond$ without condition 2 can result in infinite branches by entering expressions with new indices.

$$
\begin{array}{rl}
R_\Diamond \; X_1 = & X \cup \{\langle \Diamond p, 1\rangle\} \\
& | \\
R_\Diamond \; X_2 = & X_1 \cup \{ 1r2, \langle p, 2\rangle\} \\
& | \\
R_\Diamond \; X_3 = & X_2 \cup \{ 1r3, \langle p, 3\rangle\} \\
& | \\
R_\Diamond \; X_4 = & X_3 \cup \{ 1r4, \langle p, 4\rangle\} \\
& | \\
& \ldots
\end{array}
$$

The example presented above could be part of some kind of proof in which we could still apply rule $R_\Diamond$ to each new set without condition 2, utilizing the fact that it contains expression $\langle \Diamond A, 1\rangle$.

In a standard, informal and intuitive approach to the logic within a tableau approach[2], we can find an equivalent of rule $R_\Diamond$ — below, we denote it in a conventional form:

$$
\begin{array}{l}
\Diamond A, i \\
\quad \downarrow \\
irj \\
A, j, \text{ where } j \text{ is new in the branch.}
\end{array}
$$

However, this formulation of the rule causes various problems. The basic problem — a fundamental one, we can say — is that the rule refers to the concept of

2 For instance, in the study by G. Priest, *Introduction to Non-classical Logic* [23].

branch, although the concept of branch is connected with the concept of set of rules — in our approach a branch is determined by the application of rules from the output set of rules. So, this way of defining a rule has the features of circularity — we define a rule by reference to a branch, whereas the definition of branch requires the definition of set of rules.

In addition, this formulation of rule $R_{\Diamond}$ seems too weak. In literature, it is only the comments to the rule that prohibit its application to the same expression more than once[3].

In the view presented in the book, rule $R_{\Diamond}$ formally requires a new index when applied (condition 1 allows only such pairs of sets), but also blocks the creation of *catch-22 situation* of applications (due to condition 2). In the approach we offer, we formally subsume the fact that *index must be new* and *the rule can only be applied to the same formula once*. Besides, the latter formulation is not precise either; in some cases condition 2 does not allow us to apply rule $R_{\Diamond}$ to given formula even once (example 4.25).

Example 4.25. Consider the following set of expressions $X \cup \{\langle \Diamond p, 1\rangle, \langle \Diamond(p \wedge q), 1\rangle\}$ and rule $R_{\Diamond}$ with condition 2.

$$
\begin{array}{rl}
R_{\Diamond} & X_1 = X \cup \{\langle \Diamond p, 1\rangle, \langle \Diamond(p \wedge q), 1\rangle\} \\
& \quad | \\
R_{\wedge} & X_2 = X_1 \cup \{1r2, \langle (p \wedge q), 2\rangle\} \\
& \quad | \\
& X_3 = X_2 \cup \{\langle p, 2\rangle, \langle q, 2\rangle\}
\end{array}
$$

To set X_1 we applied rule $R_{\Diamond}$, drawing conclusions from expression $\langle \Diamond(p \wedge q), 1\rangle$. Next, to set X_2 we applied rule $R_{\wedge}$ by decomposing expression $\langle (p \wedge q), 2\rangle$. However, to set X_3 we cannot anymore apply rule $R_{\Diamond}$ in order to decompose expression $\langle \Diamond p, 1\rangle$, since X_3 contains expressions $1r2$ and $\langle p, 2\rangle$. So, to expression $\langle \Diamond p, 1\rangle$ we did not apply rule $R_{\Diamond}$ even once.

However, as noted before, in modal logic, even application of conditions blocking the unnecessary use of rules may not prevent the emergence of infinite branches, which will be proven later, having defined the concept of modal branch.

Remark 4.26. We can consider alternative sets of rules for the construction of a tableau system for **S5**. The following rules would help (on their right side we put the condition that fulfils the relation in the model).

---

3 This is why G. Priest notes: *in the rule for* $\diamond$ *(...) the number j must be new*, *not mentioning that this rule can be applied only once for one formula* ([23], s. 25).

(Reflexivity)

$$R_{ref}: \frac{X \cup \{\langle A,i \rangle\}}{X \cup \{\langle A,i \rangle, iri\}} \qquad \forall_{w_1 \in W}\, w_1 R w_1$$

(Symmetry)

$$R_{sym}: \frac{X \cup \{irj\}}{X \cup \{irj, jri\}} \qquad \forall_{w_1, w_2 \in W}\, (w_1 R w_2 \Rightarrow w_2 R w_1)$$

(Transitivity)

$$R_{trans}: \frac{X \cup \{irj, jrk\}}{X \cup \{irj, jrk, irk\}} \qquad \forall_{w_1, w_2, w_3 \in W}\, (w_1 R w_2 \,\&\, w_2 R w_3 \Rightarrow w_1 R w_3)$$

The above rules correspond to the properties of equivalence relations: reflexivity, transitivity and symmetry. Fact 4.11 states that a class of models with universal relation and a class of models with equivalence relation define exactly the same logic. Therefore, an alternative set of tableau rules could be defined through the above three rules: $(\mathbf{R_{S5}} \setminus R_r) \cup \{R_{ref}, R_{sym}, R_{trans}\}$.

Although both sets of rules seem to be interesting, for the reasons we have already described, we will not investigate the second one — for we intend to provide a generalisation further in the book which will also include this approach. So, now we will focus on the tableau system determined by set of rules $\mathbf{R_{S5}}$. The example provided indicates that the general definition of rule 4.22 makes sense. There may exist many sets of tableau rules that potentially — to determine this always requires a proof — define the same logic.

The adopted set of tableau rules — in our case set $\mathbf{R_{S5}}$ — specifies the content of the range of successive concepts of the tableau system. Formally, the concepts we will describe will be analogous to those from the previous chapter. However, each of them will depend on set of tableau rules $\mathbf{R_{S5}}$.

4.3.2 Branches for S5

With a fixed set of tableau rules, we can proceed to the concept of branch. Branches are such sequences of sets that each two adjacent elements constitute an input set and an output set of some n-tuple that belongs to the set of tableau rules. Branches are therefore setwise objects consisting of sets. Let us now proceed to the formal definition of branch in the tableau system for **S5** and derived concepts. The individual concepts will be similar to those from the previous chapters, and all of them will be defined on the currently adopted set $\mathsf{Te}_{\mathbf{S5}}$.

Definition 4.27 (Branch)**.** Let $K = \mathbb{N}$ or $K = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Let X be any set of expressions. *A branch* (or *a branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$, if $i+1 \in K$, then there exists such rule $R \in \mathbf{R_{S5}}$ and such n-tuple $\langle Y_1,\ldots,Y_n\rangle \in R$, that $\phi(i) = Y_1$ and $\phi(i+1) = Y_k$, for certain $1 < k \leq n$.

Having two branches ϕ, ψ such that $\phi \subset \psi$ we shall state that:

- ϕ is a sub-branch of ψ
- ψ is a super-branch of ϕ.

Denotation 4.28. From now on — when speaking of branches — for convenience, we will use the following notations or designations:

1. $X_1,\ldots,X_n$, where $n \geq 1$
2. $\langle X_1,\ldots,X_n\rangle$, where $n \geq 1$
3. abbreviations: ϕ_K, where K is a domain ϕ, i.e. $\phi : K \longrightarrow P(\mathsf{Te}_{S5})$
4. or — to denote branches — small Greek letters: ϕ, ψ, etc.

The sets of branches, in turn, we shall denote with capital Greek letters: Φ, Ψ, etc. Furthermore, the domain cardinality of a given branch K we shall sometimes call a *length* of that branch.

The concept of branch depends on the initial set of tableau rules. In the case under consideration, the branch structure is based on the rules from set $\mathbf{R_{S5}}$. Further described tableau concepts also depend on that set of rules. In harmony with the adopted convention, we will not entangle the notations since alls the subsequent concepts depend on set $\mathbf{R_{S5}}$.

By definition of rules 4.22, through the fact that they are defined by proper inclusion of the input set in each of the output sets in any n-tuple, we get a conclusion.

Corollary 4.29. *Each branch is an injective sequence.*

4.3.3 Closed and open branches

Unlike usual, we will first take up a certain type of maximal branches, and only then we will introduce the concept of maximal branch. The reason for this is that because of the emergence of infinite branches beginning with finite sets of expressions, we will have to extend the concept of maximal branch. The concept of a closed branch will be useful for the extension, so we start with it. In addition, the set of closed branches is completed by the set of open branches, so we will also introduce the concept of open branch.

As we remember, intuitively a branch is closed when we get a t-inconsistent set having decomposed expressions by rules.

Definition 4.30 (Closed/open branch)**.** Branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ will be called *closed* iff $\phi(i)$ is a t-inconsistent set for some $i \in K$. Branch ϕ will be called *open* iff it is not closed.

From the above definition, the definition of tableau rules for **S5** 4.23 and the definition of branch 4.27, we get the fact.

Proposition 4.31. *If branch $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ is closed, then $|K| \in \mathbb{N}$ and $\phi(|K|)$ is a t-inconsistent set.*

In the case of a closed branch, the t-inconsistent sequence element is the last element because no rule can be applied to it anymore to extend the branch. For the rules are defined in such a way that they cannot be applied to t-inconsistent sets.

4.3.4 Maximal branches

We will commence the issue of maximal branches for the tableau system for **S5** with an initial and non-proper concept of maximal branch. By analogy to the definitions of maximal branch in tableau systems for **CPL** and **TL** (definitions 2.30 and 3.26), we could adopt the following definition.

Definition 4.32 (Maximal branch — variant one)**.** Let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be a branch. We shall state that ϕ is *maximal* iff

1. $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$
2. there is no branch ψ such that $\phi \subset \psi$.

Unfortunately, there are cases where the first set is finite, but this does not guarantee the finiteness of branch. So, regardless of either we used all the rules from set $\mathbf{R}_{\mathbf{S5}}$ which could have been applied to some expressions appearing in the branch, or we still have expressions to which no rules were applied, the branch can be infinite.

Example 4.33. Consider an example of set $\{\langle \neg(\Diamond p \rightarrow \Diamond \Box p), 1\rangle\}$. It is a finite set. Below, we describe an infinite branch beginning with set X and constructed by applying the rules from set of tableau rules $\mathbf{R}_{\mathbf{S5}}$.

$$
\begin{array}{ll}
R_{\neg} & X_1 = \{\langle\neg(\Diamond p \rightarrow \Diamond\Box p), 1\rangle\} \\
& \quad | \\
R_{\Diamond} & X_2 = X_1 \cup \{\langle\Diamond p, 1\rangle, \langle\neg\Diamond\Box p, 1\rangle\} \\
& \quad | \\
R_{\neg\Diamond} & X_3 = X_2 \cup \{1r2, \langle p, 2\rangle\} \\
& \quad | \\
R_{\Box} & X_4 = X_3 \cup \{\langle\Box\neg\Box p, 1\rangle\} \\
& \quad | \\
R_{\neg\Box} & X_5 = X_4 \cup \{\langle\neg\Box p, 2\rangle\} \\
& \quad | \\
R_{\Diamond} & X_6 = X_5 \cup \{\langle\Diamond\neg p, 2\rangle\} \\
& \quad | \\
R_r & X_7 = X_6 \cup \{2r3, \langle\neg p, 3\rangle\} \\
& \quad | \\
R_{\Box} & X_8 = X_7 \cup \{1r3\} \\
& \quad | \\
R_{\neg\Box} & X_9 = X_8 \cup \{\langle\neg\Box p, 3\rangle\} \\
& \quad | \\
R_{\Diamond} & X_{10} = X_9 \cup \{\langle\Diamond\neg p, 3\rangle\} \\
& \quad | \\
& \quad \ldots
\end{array}
$$

The branch of infinite length is obtained by applying by turns the following rules $R_{\Diamond}$, R_r, $R_{\Box}$, $R_{\neg\Box}$ to set X_{10} and to each subsequent set that is created following the application of the given sequence of rules.

The definition of maximal branch in the given version specifies that the maximal branch is finite and there exists no super-branch for it. This definition is good for those systems where applying tableau rules to finite sets of expressions always produces finitely long branches. In these cases, using the tableau rules for a given system, we can decompose all the initial expressions from a finite set of tableau expressions in a finite number of steps. It is therefore not a good definition for modal logic because of the following fact.

Proposition 4.34. *There exists such finite set of expressions X that a branch beginning with X is infinite.*

Proof. Example 4.33. □

It would appear that removal of the first condition from definition 4.32 will improve the situation. Removal of the first condition would mean that the branch does not have to be of finite length. Its maximality would be based on the fact that there exists no super-branch of it. However, the lack of a super-branch for the branch beginning with a finite set does not mean that all the tableau rules that could be used to construct the branch were actually used. Even in example 4.33, in individual sequences, we did not make use of all the possibilities of using rule R_r.

Note that by adding to set $X = \{\langle\neg(\Diamond p \rightarrow \Diamond\Box p),1\rangle\}$ any new expression — e.g. $\langle(q \wedge \neg q),1\rangle$, we get a finite set again. For this set, there exists a branch of infinite length in which the rule for this new expression has not been applied in any of the subsets (example 4.35).

Therefore, even though the branch may satisfy the second condition from definition 4.32 — there is no branch to be contained in it — it does not, however, draw all the possible conclusions from the expressions that belong to the elements of this branch, including the initial set.

Example 4.35. Consider an example of set of expressions $X = \{\langle\neg(\Diamond p \rightarrow \Diamond\Box p),1\rangle, \langle(q \wedge \neg q),1\rangle\}$. It is a finite set. Below, we describe an infinite branch beginning with set X and constructed by applying the rules from set of tableau rules $\mathbf{R_{S5}}$. This branch is similar to the one in example 4.33, except that each of its elements contains expression $\langle(q \wedge \neg q),1\rangle$ from which in none of the branch elements conclusions have been drawn applying rule $R_{\wedge}$.

$$
\begin{array}{ll}
R_{\neg} & X_1 = \{\langle\neg(\Diamond p \rightarrow \Diamond\Box p), 1\rangle, \langle(q \wedge \neg q), 1\rangle\} \\
& \quad | \\
R_{\Diamond} & X_2 = X_1 \cup \{\langle\Diamond p, 1\rangle, \langle\neg\Diamond\Box p, 1\rangle\} \\
& \quad | \\
R_{\neg\Diamond} & X_3 = X_2 \cup \{1r2, \langle p, 2\rangle\} \\
& \quad | \\
R_{\Box} & X_4 = X_3 \cup \{\langle\Box\neg\Box p, 1\rangle\} \\
& \quad | \\
R_{\neg\Box} & X_5 = X_4 \cup \{\langle\neg\Box p, 2\rangle\} \\
& \quad | \\
R_{\Diamond} & X_6 = X_5 \cup \{\langle\Diamond\neg p, 2\rangle\} \\
& \quad | \\
R_{r} & X_7 = X_6 \cup \{2r3, \langle\neg p, 3\rangle\} \\
& \quad | \\
R_{\Box} & X_8 = X_7 \cup \{1r3\} \\
& \quad | \\
R_{\neg\Box} & X_9 = X_8 \cup \{\langle\neg\Box p, 3\rangle\} \\
& \quad | \\
R_{\Diamond} & X_{10} = X_9 \cup \{\langle\Diamond\neg p, 3\rangle\} \\
& \quad | \\
& \quad \dots
\end{array}
$$

A branch of infinite length is obtained by applying by turns the following rules $R_{\Diamond}$, R_r, $R_{\Box}$, $R_{\neg\Box}$ to set X_{10} and to each subsequent set that is created following the application of the given sequence of rules. Whereas, in no step we apply rule $R_{\wedge}$.

Before we can introduce a somewhat more general and ultimate definition of maximal branch, we still need a few auxiliary concepts.

Definition 4.36 (Core of rule)**.** Let rule $R \in \mathbf{R_{S5}}$ and $n \in \mathbb{N}$. Let $\langle X_1,\ldots,X_n\rangle \in R$ and $\langle Y_1,\ldots,Y_n\rangle \in R$. We shall state that set $\langle Y_1,\ldots,Y_n\rangle \in R$ is a *core of rule R in set* $\langle X_1.\ldots,X_n\rangle$ iff

1. $Y_1 \subseteq X_1$
2. there exists no such proper subset $U_1 \subset Y_1$ that for some n-tuple $\langle U_1,\ldots,U_n\rangle \in R$
3. for any $1 < i \leq n$, $Y_i = Y_1 \cup (X_i \setminus X_1)$.

Remark 4.37. Note that in the case of tableau rules from set $R \in \mathbf{R_{S5}}$:

- the cores of rule for given n-tuple are expressions that play important roles at given stage of the tableau proof — most often, the core input set is one-element set
- in a special case, the core input set for the rule for given n-tuple is two-element set of expressions — it is the case, for instance, in the event of rule $R_\Box$.

Even though the rules have been defined on sets, the concept of core of rule in set indicates an essential element or elements which enable drawing conclusions through the use of rule.

The above concepts result in the following conclusion.

Corollary 4.38. *Let rule $R \in \mathbf{R_{S5}}$, $n \in \mathbb{N}$, and let $\langle X_1, \ldots, X_n\rangle \in R$. Then, there exists such n-tuple $\langle Y_1,\ldots,Y_n\rangle \in R$ that $\langle Y_1,\ldots,Y_n\rangle$ is the core of rule R in set $\langle X_1,\ldots,X_n\rangle$.*

Proof. By definition of tableau rules 4.23 and by definition of core of rule 4.36. □

Now, we can proceed to the concept of strong similarity between the sets of expressions. It is a speciality of the concept of similarity (definition 4.16). We will need the concept of strong similarity for the definition of maximal branch.

Definition 4.39 (Strong similarity)**.** Let rule $R \in \mathbf{R_{S5}}$ and let $\langle X_1,\ldots,X_n\rangle \in R$, for some $n \in \mathbb{N}$. On any set of expressions $W \subseteq \mathsf{Te}_{S5}$, we will state that it is *strongly similar to* set X_i, where $1 < i \leq n$, iff

1. W is similar to X_i
2. for certain n-tuple $\langle Y_1,\ldots,Y_n\rangle$, which is the core of rule R in set $\langle X_1,\ldots,X_n\rangle$, and for certain $W' \subseteq W$, the following conditions are met:
   a. $Y_1 \subseteq W'$
   b. W' is similar to $Y_1 \cup (X_i \setminus X_1)$.

Having adopted the concept of strong similarity, we can proceed to the concept of maximal branch in the final version.

Definition 4.40 (Maximal branch)**.** Let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be a branch. We shall state that ϕ is *maximal* iff it meets one of the below conditions:

1. ϕ is closed
2. for any rule $R \in \mathbf{R}_{\mathbf{S5}}$, any $n \in \mathbb{N}$ and any n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, if $\phi(k) = X_1$, for certain $k \in K$, then for some $j \in K$, there exist $\phi(j)$ and such set of expressions $W \subseteq \mathsf{Te}_{\mathbf{S5}}$ that for some $1 < i \leq n$, W is strongly similar to X_i and $W \subseteq \phi(j)$.

Remark 4.41. According to the above definition, a maximal branch is closed or, in a sense, closed under application of rules (both conditions do not necessarily have to be mutually exclusive). Closure under rules means that if a branch is not closed and it was possible to apply some rule to one of its elements, then some of the branch elements includes a set strongly similar to the one that could have been a result of application of that rule. There are two things worth clarifying.

In the definition, we mention a strongly similar set W because the application of rule $R_{\Diamond}$, in subsequent stages, can result in different sets than the ones we would have gotten by applying this rule earlier.

Set W is to be contained in one of the elements of branch $\phi(j)$, and not necessarily be identical to it, since the rule could have been applied to set $\phi(j-1)$ which can be a proper superset of set X.

Therefore, maximal branches can be either finite or infinite. Of course, if a branch is maximal in terms of the first definition 4.32, that means it is maximal in terms of the definition we have adopted — 4.40.

Corollary 4.42. *Each branch which is maximal in terms of definition 4.32 is maximal in terms of definition 4.40.*

Proof. Take any branch ϕ maximal in terms of definition 4.32 and assume that ϕ is not closed. If it does not meet the second condition of definition 4.40, then since ϕ is finite — the first condition of definition 4.32, so there exists branch ψ such that $\phi \subset \psi$ which obviously contradicts the second condition of definition 4.32. □

We know, therefore, that the concept of maximal branch as used so far, expressed in definition 4.32 is a special case of the concept defined by the definition of maximal branch 4.40. The latter definition is therefore taken as a model definition for the theory of tableau systems.

Important cases for the theory of tableaux that meet the definition of maximal branch used in the chapters on the tableau system for **CPL** and **TL**, also meet definition 4.40. Important cases, both for the branch consequence relation and the tableau, are those where the initial set — a set of tableau expressions or formulas — is finite. Therefore, since we know that from finite sets in the tableau system for **CPL** and in the tableau system for **TL** only emerge finite branches (see fact 2.34 and fact 3.31), so we can state that definitions like 4.40 by virtue of conclusion 4.42 — we mean *like* since even though obviously, constructively the same idea stands behind them, in both cases we face different rules and different sets of expressions — are appropriate for the system with the property of a finite branch.

From definitions 4.30 and 4.40, we get a self-evident conclusion.

Corollary 4.43. *Each closed branch is maximal.*

Therefore, sometimes when constructing a tableau proof using tableau tools for logic **S5**, when we seek maximal branches, we can deal with infinite branches. It is understandable that those branches cannot be described with anything different than a scheme. In general, branches, as part of a tableau proof, are not sequences put on paper, but sequences of abstract objects that we can or cannot say are or are not maximal.

4.3.5 Relation of branch consequence

As in the case of previous tableau systems, we will also define the concept of branch consequence for the presently described system using the following terms: branch, maximal branch and closed branch.

Definition 4.44 (Branch consequence **S5**). Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Formula A is a *branch consequence* of X (for short: $X \rhd A$) iff there exists such finite set $Y \subseteq X$ and index $i \in \mathbb{N}$ that each maximal branch beginning with set $\{\langle B, i\rangle : B \in Y \cup \{\neg A\}\}$ is closed.

Denotation 4.45. For any set of formulas X and any formula A notation $X \not\rhd A$ will mean that it is not the case that $X \rhd A$.

The above concept of branch consequence relation differs from the analogous concept for tableau system for **CPL** and **TL** in the fact that (obviously, apart from defining on a different language — different set of expressions) when determining whether given pair belongs to the relation of branch consequence we can encounter a problem of branches that are maximal and infinite alike.

4.4 Tableaux for S5

It is usually inconvenient, or even infeasible, to investigate whether a pair belongs to the relation of branch consequence. As we remember, in the approach presented in the book, it is the construction of tableau that is supposed to solve this problem.

When defining a tableau for logic **S5**, we will readdress the auxiliary concept of maximality in a set of branches. It is necessary because, unlike the case of tableau system for **TL**, and like the case of **CPL**, a tableau can contain many branches. So for the sake of order, we must avoid a situation in which sub-branches belong to the same tableau as their super-branches.

Definition 4.46 (Maximal branch in the set of branches)**.** Let Φ be a set of branches and let branch $\psi \in \Phi$. We shall state that ψ is *maximal in set* Φ (for short: Φ–*maximal*) iff there is no such branch $\phi \in \Phi$ that $\psi \subset \phi$.

We can now move on to the concept of tableau. Tableau is a special and non-empty set of branches that *i*) begin with the same set of expressions and *ii*) each branching requires a tableau rule that allows for such branching, and what is more *iii*) each branch that belongs to the tableau must be maximal in this set.

Definition 4.47 (Tableau)**.** Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$ and Φ be a set of branches. Ordered triple $\langle X, A, \Phi\rangle$ will be called *a tableau for* $\langle X, A\rangle$ (or for short: *tableau*) iff the below conditions are met:

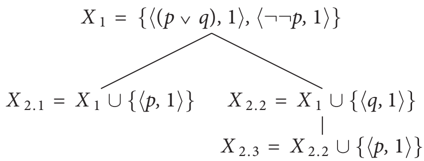
1. Φ is a non-empty subset of set of branches beginning with $\{\langle B, i\rangle : B \in X \cup \{\neg A\}\}$, for some index $i \in \mathbb{N}$ (i.e. if $\psi \in \Phi$, then $\psi(1) = \{\langle B, i\rangle : B \in X \cup \{\neg A\}\}$)
2. each branch contained in Φ is Φ-maximal
3. for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
   - i and $i+1$ belong to domains of functions $\psi_1, \ldots, \psi_n$
   - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

   then there exists such rule $R \in \mathbf{R}_{\mathbf{S5}}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m\rangle \in R$, where $1 < m \leq 3$ that for any $1 \leq k \leq n$:
   - $\psi_k(i) = Y_1$
   - and there exists such $1 < l \leq m$ that $\psi_k(i+1) = Y_l$.

Again, the concept of tableaux has been defined in such a way that tableaux can also begin with infinite sets. Practicably, however, the construction of tableau is to show that a given formula is a branch consequence of given finite set of premises — according to the definition of branch consequence 4.44. To this end, we must construct tableaux containing all elements that are sufficient to solve the problem. Such tableaux are called complete tableaux. Before that, we will consisder the problem of redundant branches.

The definition of redundant branches resembles a similar definition, worded for **CPL**, while the difference, as usual, boils down to the fact that its subject are branches composed of other objects. Let us consider an example analogous to the one we discussed in the chapter devoted to the classical logic (example 4.48).

Example 4.48. Consider set of expressions $\{\langle p \vee q, 1\rangle, \langle \neg\neg p, 1\rangle\}$. Through the use of rule $R_{\vee}$, we get two branches, and then to set $X_{2.2}$ we apply rule $R_{\neg\neg}$. Then we get the following branches.



In the light of definition of tableau 4.47, the set of these two branches is a tableau for pair $\langle \{p \vee q\}, \neg p\rangle$.

However, from the viewpoint of a tableau complexity, the branch on the right seems unnecessary. This is because if we fail to get t-inconsistent set in the right-hand branch, we will also fail to get t-inconsistent set in the left-hand branch. Therefore, the branch on the right seems superfluous, and since it brings nothing important, it can be conventionally called redundant.

Let us also repeat that such branchings do not form a formal obstacle. We suggest their bypassing merely for the sake of the economy of a construction. Therefore, further concepts will be defined in a similar way as in Chapter Two, i.e. so that the tableau with or without redundant branches can be considered a complete tableau. Practically, we know that when we try to write or draw out a tableau proof, we endeavour to take account of all possibilities. If required, however, the redundant branches can be bypassed. Let us now update the concept of redundant variant of branch for the current context.

Definition 4.49 (Redundant variant of branch)**.** Let ϕ and ψ be such branches that for some numbers i and $i+1$ that belong to their domains, it is the case that for any $j \leq i$, $\phi(j) = \psi(j)$, but $\phi(i+1) \neq \psi(i+1)$. We shall state that branch ψ is a *redundant variant* of branch ϕ iff:

- there exists such rule $R \in \mathbf{R_{S5}}$ and such pair $\langle X, Y\rangle \in R$ that $X = \phi(i)$ and $Y = \phi(i+1)$
- there exists such rule $R \in \mathbf{R_{S5}}$ and such triple $\langle X, W, Z\rangle \in R$ that $X = \phi(i)$ and:

1. $W = \phi(i+1)$ and $Z = \psi(i+1)$
   or
2. $Z = \phi(i+1)$ and $W = \psi(i+1)$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. We shall state that Ψ is a *redundant superset* Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ.

Having adopted the concept of redundant superset of branches, we can proceed to the definition of complete tableau.

Definition 4.50 (Complete tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff:

1. each branch contained in Φ is maximal
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$
   b. $\langle X, A, \Psi \rangle$ is a tableau

   is a redundant superset of Φ.

We shall state that a tableau is *incomplete* iff it is not complete.

A complete tableau contains such set of branches that it is no longer possible to add any new branches to it, without causing the ordered triple to cease to be a tableau, or there appears a redundant variant of some branch. As we already know, in a complete tableau, all branches are maximal, not only the maximal ones in a given set. Thus, a complete tableau contains such set of maximal branches that any of its supersets does not anymore produce a tableau, or it features at least one redundant variant of some branch that already earlier belonged to the tableau.

When constructing a complete tableau, we can face a situation in which all the branches are closed, meaning each branch ends with a set that is t-inconsistent. Such a tableau will be called a closed tableau.

Definition 4.51 (Closed/open tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff the below conditions are met:

1. $\langle X, A, \Phi \rangle$ is a complete tableau
2. each branch contained in Φ is closed.

We shall state that a tableau is *open* iff it is not closed.

By the above definition of closed tableau, we get another conclusion on the relation of closed tableaux with complete ones.

Corollary 4.52. *Each closed tableau is a complete tableau.*

4.5 Theorem on the completeness of the tableau system for S5

Further, we will show that the concept of tableau is significantly helpful in determining the occurrence of relation $\triangleright$, and that the existence of a closed tableau is equivalent to the occurrence of semantic consequence $\models$. But before we move on to these problems, we must introduce a few definitions and determine a few facts.

First, we will show that for any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if $X \models A$, then $X \triangleright A$. Let us begin with the definition of closure under tableau rules.

Definition 4.53 (Closure under tableau rules)**.** Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$. We shall state that set $Y \subseteq \mathsf{Te}_{\mathbf{S5}}$ is a *closure of set X under tableau rules* iff Y is a set that meets the following conditions:

- $X \subseteq Y$
- for any rule $R \in \mathbf{R}_{\mathbf{S5}}$ and any n-tuple $\langle Z_1, Z_2, \ldots, Z_n \rangle \in R$, where $n \in \mathbb{N}$, if $X \subseteq Z_1 \subseteq Y$, then $Z_j \subseteq Y$, for some $2 \leq j \leq n$.

If set Y is a closure of set X under tableau rules, then it will be denoted as X^Y. Sometimes, on set Y we will simply state that it is a *closure*.

Obviously, each set of expressions X has its closure Y such that $X \subseteq Y \subseteq \mathsf{Te}_{\mathbf{S5}}$. Some sets can have more that one closure. Note that by definition of branch 4.27, the following fact occurs.

Proposition 4.54. *Each closure under tableau rules is a branch of length one.*

Making use of the concept of closure, we will now show a relationship between the existence of maximal and open branches originating from finite subsets of some set of expressions and the existence of closure of that set which is an open and maximal branch.

Lemma 4.55 (On the existence of maximal and open branch)**.** *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. If for any finite subset $Y \subseteq X$ there exists an open and maximal branch beginning with set of expressions $Y^i = \{\langle A, i \rangle : A \in Y\}$, then there exists such closure Z of set of expressions $X^i = \{\langle A, i \rangle : A \in X\}$ under tableau rules that Z is an open and maximal branch.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. Next, assume that $(*)$ for any finite subset $Y \subseteq X$ there exists open and maximal branch beginning with set of expressions $Y^i = \{\langle A, i \rangle : A \in Y\}$.

Now, we take the set of all maximal and open branches beginning with set $Y^i = \{\langle A, i \rangle : A \in Y\}$, for any finite subset $Y \subseteq X$. The set will be denoted as $\mathbf{X}$.

Next, we will define set $\overline{\mathbf{X}}$ with the following conditions:

1. $\overline{\mathbf{X}} \subseteq \mathbf{X}$
2. for any two branches ϕ and ψ contained in $\mathbf{X}$, if there exist such two numbers $i, k \in \mathbb{N}$ that $\phi(i) \cup \psi(k)$ is a t-inconsistent set, then $\phi \notin \overline{\mathbf{X}}$ or $\psi \notin \overline{\mathbf{X}}$
3. $\overline{\mathbf{X}}$ is a maximal set among subsets $\mathbf{X}$ that meet conditions 1 and 2.

There exists at least one such set $\overline{\mathbf{X}}$ that $\overline{\mathbf{X}} \subseteq \mathbf{X}$. Take one such set $\overline{\mathbf{X}}$ and denote it as $\overline{X}$.

Consider set $\bigcup\{\phi(1) : \phi \in \overline{X}\}$. Note that $(**)\ X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$. If that was not the case, there would exist such $x \in X^i$ that $x \notin \bigcup\{\phi(1) : \phi \in \overline{X}\}$ and for each such branch $\psi \in \mathbf{X}$ that $x \in \psi(1)$, $\psi(1) \subseteq X^i$ and $\psi(1)$ is a finite set, it would be the case that $\psi \notin \overline{X}$. And then, for some finite subset $Y^i \subseteq X^i$ there would exist no maximal and open branch beginning with set $Y^i \cup \{x\}$ which contradicts assumption $(*)$.

Now, using condition:

$$U \in \overline{\overline{X}} \text{ iff there exists such branch } \phi \text{ that } \phi \in \overline{X} \text{ and } U = \bigcup \phi$$

we define set $\overline{\overline{X}}$. Further, we define set $Z = \bigcup \overline{\overline{X}}$.

We claim that set Z is a closure of set of expressions $X^i = \{\langle A, i\rangle : A \in X\}$ under tableau rules, according to definition 4.53, and that Z is an open and maximal branch.

First, we will show that Z is a closure of set of expressions $X^i = \{\langle A, i\rangle : A \in X\}$. Thus, we will show that Z meets conditions of closure, according to definition 4.53.

Note that $X^i \subseteq Z$, since by $(**)\ X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$, and by construction of set Z, $\bigcup\{\phi(1) : \phi \in \overline{X}\} \subseteq Z$.

Now, take any rule $R \in \mathbf{R_{S5}}$ and any n-tuple $\langle U_1, \ldots, U_n\rangle \in R$, for some $n \in \mathbb{N}$, and assume that $X^i \subseteq U_1 \subseteq Z$. From the definition of tableau rules for **S5** 4.23, it follows that there exists such n-tuple $\langle U'_1, \ldots, U'_n\rangle \in R$ that:

- for any $1 \leq j \leq n$, U'_j is a minimal finite set such that if U_j is not such a minimal finite set that $\langle U_1, \ldots, U_n\rangle \in R$, then $U'_j \subset U_j$
- for any $1 < j \leq n$, $U_j \setminus U_1 = U'_j \setminus U'_1$.

Therefore, assuming that $U'_1 \subseteq Z$, we must show that for some $1 < l \leq n$, $U'_l \subseteq Z$, since $U'_l \cup U_1 = U_l$. Since for finite set of expressions U'_1, it is the case that $U'_1 \subseteq Z$, there exists such finite number of branches $\phi_1, \phi_2, \ldots, \phi_o$ in set $\overline{X}$ that for some $k \in \mathbb{N}$, $U'_1 \subseteq \phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$. So, set $\overline{X}$ contains branch ψ such that $\psi(1) = \phi_1(1) \cup \phi_2(1) \cup \cdots \cup \phi_o(1)$ and $U'_1 \subseteq \psi(m)$, for certain $m \in \mathbb{N}$, since ψ is a maximal branch and $\phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$ is a t-consistent set. But, from the fact that ψ is a maximal branch and from definition of maximal branch 4.40, it follows that for certain $1 < l \leq n$, $U'_l \subseteq \bigcup \psi$, and thus $U'_l \subseteq Z$, since by construction $Z, \bigcup \psi \subseteq Z$.

We will now show that Z is an open and maximal branch.

By conclusion 4.54, set Z is a branch. By construction of set Z, Z is an open branch, that is none of the subsets of Z is t-inconsistent, by virtue of definition of set $\overline{X}$.

Let us now check if Z is a maximal branch. Making use of the definition of maximal branch 4.40, assume that there exists a tableau rule $R \in \mathbf{R_{S5}}$ and n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, for some $n \in \mathbb{N}$, such that $X_1 = Z$. By definition of tableau rules 4.23, there exists n-tuple $\langle X'_1 \ldots, X'_n \rangle \in R$ such that for any $1 < j \leq n$, $X_j \setminus X_1 = X'_j \setminus X'_1$ and $X^i \subseteq X'_1 \subseteq Z$. Since Z is a closure under tableau rules of set X^i, so $X'_j \subseteq Z$, for certain $1 < j \leq n$, by definition 4.53. Therefore, also $X_j \subseteq Z$, since $X_j = X_1 \cup X'_j$. But then $X_1 \not\subset X_j$, which by definition of tableau rules 4.23 is out of the question. Consequently, there exists no tableau rule and n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ such that $X_1 = Z$, for some $n \in \mathbb{N}$. Therefore, Z is a maximal branch, by definition of maximal branch 4.40 □

Let us define the concept of model generated by branch.

Definition 4.56 (Model generated by branch)**.** Let ϕ be a branch. Let X be a non-empty subset of set of formulas $\mathsf{For_{S5}}$ and $\{\langle A, k \rangle : A \in X\} \subseteq \bigcup \phi$, for some $k \in \mathbb{N}$. We define function $AT(\phi) \subseteq \mathsf{Te_{S5}}$ as follows, $x \in AT(\phi)$ iff one of the below conditions is met:

- $x \in \bigcup \phi \cap \{irj : i, j \in \mathbb{N}\}$
- $x \in \bigcup \phi \cap (\mathsf{Var} \times \mathbb{N})$.

We shall state that model $\mathfrak{M}_{S5} = \langle W, R, V, w \rangle$ is *generated by branch* ϕ iff:

- $W = \{i : i \in *(AT(\phi))\} \cup \{k\}$
- for any $i, j \in \mathbb{N}$, $\langle i, j \rangle \in R$ iff $irj \in AT(\phi)$
- $V(x, i) = 1$ iff $\langle x, i \rangle \in AT(\phi)$
- $w = k$.

Let model $\mathfrak{M}_{S5}$ be generated by ϕ. Then, we shall state that ϕ *generates the model.*

From the definition of generated model, another conclusion results.

Corollary 4.57. *Let ϕ be such an open branch that for certain non-empty set of formulas X and some index $i \in \mathbb{N}$, $\{\langle A, i \rangle : A \in X\} \subseteq \bigcup \phi$. Then, ϕ generates model $\mathfrak{M}_{S5} = \langle W, R, V, i \rangle$.*

Proof. By definition of open branch 4.30, definition of model 4.5 and definition of model generated by branch 4.56. □

Lemma 4.58 (On generation of model)**.** *Let ϕ be an open and maximal branch. Then, for any non-empty $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and any index i such that $\{\langle A,i\rangle : A \in X\} \subseteq \bigcup\phi$ there exists model $\mathfrak{M}_{\mathbf{S5}}$ such that for any formula A, if $A \in X$, then $\mathfrak{M}_{\mathbf{S5}} \vDash A$.*

Proof. Take any maximal and open branch ϕ such that for certain set $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and certain index i, $\{\langle A,i\rangle : A \in X\} \subseteq \bigcup\phi$. Since ϕ is open, then according to conclusion 4.57, there exists, generated by ϕ, model $\mathfrak{M}_{\mathbf{S5}} = \langle W,R,V,w\rangle$, specified in accordance with definition 4.56, where $w = i$.

We will carry out an inductive proof due to the construction of formulas, showing that for any formula E and any $k \in \mathbb{N}$, if $\langle E,k\rangle \in \bigcup\phi$, then $\mathfrak{M}_{\mathbf{S5}} = \langle W,R,V,k\rangle \vDash E$.

Initial step. Take any $x \in \mathsf{Var}$ and some index $j \in \mathbb{N}$.

If $\langle x,j\rangle \in \bigcup\phi$, then — according to the definition of generated model $\mathfrak{M}_{\mathbf{S5}}$ — $V(x,j) = 1$, thus by definition of truth in model 4.7, $\langle W,R,V,j\rangle \vDash x$.

If $\langle \neg x,j\rangle \in \bigcup\phi$, then since branch ϕ is open, $\langle x,j\rangle \notin \bigcup\phi$, and — according to the definition of generated model $\mathfrak{M}_{\mathbf{S5}}$ — $V(x,j) = 0$, thus by definition of truth in model 4.7, $\langle W,R,V,j\rangle \vDash \neg x$.

Induction step. (†) Take any formula $E \in \mathsf{For}_{\mathbf{S5}}$ and indices $j,k \in \mathbb{N}$ and assume that for each tableau expression $\langle D,n\rangle$, where $D \in \mathsf{For}_{\mathbf{S5}}$ and $n \in \mathbb{N}$ that belongs to set $\bigcup\phi$ as a result of application of some tableau rule to set $\{\langle E,j\rangle\} \subseteq \bigcup\phi$ or set $\{\langle E,j\rangle, jrk\} \subseteq \bigcup\phi$, it is the case that $\langle W,R,V,n\rangle \vDash D$.

Making use of the inductive assumption, let us consider all cases of construction of formula E. Take some index $j \in \mathbb{N}$.

1. Let $E = (B \wedge C)$ and $\langle (B \wedge C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\wedge}$, both $\langle B,j\rangle$ and $\langle C,j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W,R,V,j\rangle \vDash (B \wedge C)$.
2. Let $E = (B \vee C)$ and $\langle (B \vee C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\vee}$, $\langle B,j\rangle$ or $\langle C,j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W,R,V,j\rangle \vDash (B \vee C)$.
3. Let $E = (B \rightarrow C)$ and $\langle (B \rightarrow C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\rightarrow}$, $\langle \neg B,j\rangle$ or $\langle C,j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W,R,V,j\rangle \vDash (B \rightarrow C)$.
4. Let $E = (B \leftrightarrow C)$ and $\langle (B \leftrightarrow C),j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\leftrightarrow}$, $\langle B,j\rangle$, $\langle C,j\rangle$ belong to $\bigcup\phi$ or $\langle \neg B,j\rangle$, $\langle \neg C,j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W,R,V,j\rangle \vDash (B \leftrightarrow C)$.
5. Let $E = \neg\neg B$ and $\langle \neg\neg B,j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\neg}$, $\langle B,j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W,R,V,j\rangle \vDash \neg\neg B$.

6. Let $E = \neg(B \wedge C)$ and $\langle \neg(B \wedge C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\wedge}$, $\langle \neg B, j\rangle$ or $\langle \neg C, j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \models \neg(B \wedge C)$.
7. Let $E = \neg(B \vee C)$ and $\langle \neg(B \vee C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\vee}$, $\langle \neg B, j\rangle$ and $\langle \neg C, j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \models \neg(B \vee C)$.
8. Let $E = \neg(B \to C)$ and $\langle \neg(B \to C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\to}$, $\langle B, j\rangle$ and $\langle \neg C, j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \models \neg(B \to C)$.
9. Let $E = \neg(B \leftrightarrow C)$ and $\langle (B \leftrightarrow C), j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\leftrightarrow}$, $\langle \neg B, j\rangle$, $\langle C, j\rangle$ belong to $\bigcup\phi$ or $\langle B, j\rangle$, $\langle \neg C, j\rangle$ belong to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \models (B \leftrightarrow C)$.
10. Let $E = \neg\Box B$ and $\langle \neg\Box B, j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\Box}$, $\langle \Diamond\neg B, j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \models \neg\Box B$.
11. Let $E = \neg\Diamond B$ and $\langle \neg\Diamond B, j\rangle \in \bigcup\phi$. Then, since ϕ is a maximal branch, due to rule $R_{\neg\Box}$, $\langle \Box\neg B, j\rangle$ belongs to $\bigcup\phi$. From assumption (†) and definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \models \neg\Diamond B$.
12. Let $E = \Box B$ and $\langle \Box B, j\rangle \in \bigcup\phi$. We have theoretically two cases: either (i) for none $l \in \mathbb{N}$ expression jrl belongs to $\bigcup\phi$, or (ii) there exists such $l \in \mathbb{N}$ that expression jrl belongs to $\bigcup\phi$. However, case (i) does not hold, because by rule R_r, expression jrj belongs to $\bigcup\phi$ at least. In case (ii) we take set $\{l : jrl \in \bigcup\phi\}$ — by assumption, this set is non-empty. Since branch ϕ is maximal, by virtue of rule $R_\Box$ for any $m \in \{l : jrl \in \bigcup\phi\}$ set $\bigcup\phi$ contains expression $\langle B, m\rangle$. Whereas due to construction of model $\mathfrak{M}_{\mathbf{S5}}$ and (†), we know that $\langle W, R, V, m\rangle \models B$. Therefore, by definition of truth in model 4.7, we get that $\langle W, R, V, j\rangle \models \Box B$.
13. Let $E = \Diamond B$ and $\langle \Diamond B, j\rangle \in \bigcup\phi$. Since branch ϕ is maximal, due to rule $R_\Diamond$, there exists index $l \in \mathbb{N}$ such that expressions $\langle B, l\rangle$ and jrl belong to $\bigcup\phi$. From the construction of model $\mathfrak{M}_{\mathbf{S5}}$ and (†), we know that $l \in W$, $\langle j, l\rangle \in R$ and $\langle W, R, V, l\rangle \models B$. Therefore, by definition of truth in model 4.7, we get $\langle W, R, V, j\rangle \models \Diamond B$.

Thus, we have proven that for any formula E and any index j, if $\langle E, j\rangle \in \bigcup\phi$, then $\langle W, R, V, j\rangle \models E$. Therefore, there exists such model $\mathfrak{M}_{\mathbf{S5}}$ that for any formula A, if $A \in X$, then $\mathfrak{M}_{\mathbf{S5}} \models A$ since $\mathfrak{M}_{\mathbf{S5}} = \langle W, R, V, i\rangle$. □

The above concepts and facts allow us to demonstrate a partial relationship between the relation of semantic consequence and the relation of branch consequence in the tableau system for **S5**.

Lemma 4.59. *For any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if $X \models A$, then $X \rhd A$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Assume that $X \not\rhd A$. We must show that $X \not\models A$. From the assumption and definition of $\rhd$, 4.44, we know that there exists no such index $i \in \mathbb{N}$ and such finite set $Y \subseteq X$ that each maximal branch beginning with set $\{\langle B, i\rangle : B \in Y \cup \{\neg A\}\}$ is closed. Therefore, for each index $i \in \mathbb{N}$ and each finite set $Y \subseteq X$, there exists a maximal branch beginning with set $\{\langle B, i\rangle : B \in Y \cup \{\neg A\}\}$ which is open. By lemma 4.55, for some index $i \in \mathbb{N}$, there exists such closure Z of set of expressions $X^i = \{\langle B, i\rangle : B \in X \cup \{\neg A\}\}$ under tableau rules that Z is an open and maximal branch. Since $Z = \bigcup Z$ and $\{\langle B, i\rangle : B \in X \cup \{\neg A\}\} \subseteq Z$, from lemma 4.58, we know that there exists model $\mathfrak{M}_{\mathbf{S5}}$ such that $\mathfrak{M}_{\mathbf{S5}} \models X \cup \{\neg A\}$. Hence, by definition of $\models$, 4.9, $X \not\models A$. □

We will now proceed to the determination of relationship between the branch consequence relation and the existence of a closed tableau. However, this will require some introduction of another concepts.

Let us now define the concept of **R**-branch, that is such branch that originated by the application of rules exclusively from set $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$, for some **R**.

Definition 4.60 (**R**-branch)**.** Let $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$, let $K = \mathbb{N}$ or $K = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Moreover, let X be a set of expressions. **R**-*branch* (or **R**-*branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$, if $i + 1 \in K$, then there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle Y_1, \ldots, Y_n\rangle \in R$ that $\phi(i) = Y_1$ and $\phi(i+1) = Y_k$, for certain $1 < k \leq n$.

Having established set **R**, the resultant branch will be then called **R**-*branch*.

Definition of **R**-branch differs from definition of branch 4.27 in the fact that the applied rules come from a subset of set of tableau rules $\mathbf{R}_{\mathbf{S5}}$. In a special case when $\mathbf{R} = \mathbf{R}_{\mathbf{S5}}$, both definitions would be identical. But, since set **R** does not have to be identical with set $\mathbf{R}_{\mathbf{S5}}$, so we have a conclusion.

Corollary 4.61. *For any $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$, each **R**-branch is a branch.*

In a similar manner, we will define another auxiliary concept, namely the concept of quasi-maximal branch.

Definition 4.62 (Quasi-maximal branch)**.** Let $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$ and let $\phi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be a branch. We shall state that ϕ is a *quasi-maximal branch* iff it meets one of the below conditions:

1. ϕ is closed
2. for any rule $R \in \mathbf{R}$, any $n \in \mathbb{N}$ and any n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, if $\phi(k) = X_1$, for certain $k \in K$, then for some $j \in K$, there exist $\phi(j)$ and such set of expressions $W \subseteq \mathsf{Te}_{\mathbf{S5}}$ that for some $1 < i \leq n$, W is strongly similar to X_i (according to $\mathbf{R}$) and $W \subseteq \phi(j)$.

Having established set **R**, the resultant quasi-maximal branch will be called **R**-*quasi-maximal branch*.

The provided definition of quasi-maximal branch also resembles the definition of maximal branch 4.40, while in a special case, when $\mathbf{R} = \mathbf{R}_{\mathbf{S5}}$, both definitions would be identical. Again, the difference pertains to the reference to the set of rules which is some subset $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$, so possibly proper subset of tableau rules. Since a maximal branch must be a sequence closed under all rules, so the relationship that occurs between the quasi-maximal branches and maximal branches is one-directional. That relationship is expressed by another conclusion which follows from the definition of maximal branch 4.40 and definition of quasi-maximal branch 4.62.

Corollary 4.63. *Each maximal branch is* **R***-quasi-maximal branch, for any* $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$.

The next conclusion is consequential for further considerations. It follows directly from the definition of quasi-maximal branch 4.62. In the proofs of further facts, the content of that conclusion shall be deemed self-evident.

Corollary 4.64. *For any* $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$*, each* **R***-quasi-maximal branch is a branch.*

Let us now introduce the definition of addition of branches.

Definition 4.65 (Addition of branches)**.** Let $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ and $\psi : M \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ be branches, for some $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$, and let $\phi(n) = \psi(1)$. The results of the operation $\phi \oplus \psi$ is function $\varphi : K \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ defined as follows:

1. if $M = \mathbb{N}$, then $K = \mathbb{N}$
2. if $|M| \in \mathbb{N}$, then $K = \{1, \ldots, n, n+1, n+2, \ldots, n+|M|-1\}$
3. for each $i \in K$
   a. if $1 \leq i \leq n$, $\varphi(i) = \phi(i)$
   b. if $i > n$, then $\varphi(i) = \psi((i-n)+1)$.

From definition of addition of branches 4.65, definition of tableau rules 4.23 and definition of branches 4.27, another conclusion follows.

Corollary 4.66. *Let* $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ *and* $\psi : M \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ *be branches, for some* $n \in \mathbb{N}$ *and* $M \subseteq \mathbb{N}$*, and let* $\phi(n) = \psi(1)$*. Then* $\phi \oplus \psi$ *is also a branch.*

Now, we have several facts concerning the relationship between the quasi-maximal branches and the finite sets of expressions.

Proposition 4.67. *Let* $\mathbf{R}_{CPL} = \{R_\wedge, R_\vee, R_\to, R_\leftrightarrow, R_{\neg\neg}, R_{\neg\wedge}, R_{\neg\vee}, R_{\neg\to}, R_{\neg\leftrightarrow}\}$. *Let* $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ *be a finite set of tableau expressions. Then, there exists a* $\mathbf{R}_{CPL}$*-quasi-maximal branch beginning with set X.*

Proof. Take set of rules $\mathbf{R}_{CPL} = \{R_\wedge, R_\vee, R_\to, R_\leftrightarrow, R_{\neg\neg}, R_{\neg\wedge}, R_{\neg\vee}, R_{\neg\to}, R_{\neg\leftrightarrow}\}$ and set of tableau expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$.

If set X is t-inconsistent, then — by definition 4.62 — one-element sequence $\langle X\rangle$ is a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X. Then, assume that X is not t-inconsistent.

Since X is a finite set, then $*(X)$, that is a set of indices that appear in the expressions in set X (definition of function selecting indices 4.15), is also finite.

If $*(X)$ is an empty set, because $X = \varnothing$, then — by definition 4.62 — one-element sequence $\langle X\rangle$ is a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Assume that $*(X)$ is a non-empty set. By *quasi-modal* formula we will mean each such formula A of logic **S5** that A is different from each following formula $\Diamond B$, $\Box B$, $\neg\Diamond B$, $\neg\Box B$, where B is a formula of **S5**. The set of all quasi-modal formulas will be denoted with symbol $\mathsf{For}^Q_{\mathbf{S5}}$. We will divide set of propositional letters Var into two disjoint subsets Var_1 and Var_2 so that propositional letter $x \in \mathsf{Var}_1$ iff $x = p_i$, for some $i \in \mathbb{N}$. From definitions of set Var it follows that both set Var_1 and Var_2 are infinite sets, plus their union equals to set Var.

Since set $\mathsf{For}_{\mathbf{S5}} \setminus \mathsf{For}^Q_{\mathbf{S5}}$ and set of propositional letters Var_1 are infinite and countable sets, we can determine bijection $\bullet : \mathsf{For}_{\mathbf{S5}} \setminus \mathsf{For}^Q_{\mathbf{S5}} \longrightarrow \mathsf{Var}_1$ that assigns exactly one propositional letter to each formula which is not quasi-modal.

Now, for each index $i \in *(X)$, we define set $X^i = \{\langle A, i\rangle : \langle A, i\rangle \in X$ *and A is a quasi-modal formula*$\}$. Set X^i contains all and only those expressions that belong to set of expressions X which constitute an ordered pair: some quasi-modal formula A, i.e., in terms of structure, formula corresponding to some formula of **CPL**, and index i. Since set X is finite, so for any i, set X^i is also finite.

If for any $i \in *(X)$, set X^i is an empty set, then initial set X does not comprise any subset to which we could apply one of rules $\mathbf{R}_{CPL}$. Therefore — by definition 4.62 — one-element sequence $\langle X\rangle$ is a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Assume that it is not the case that for any $i \in *(X)$, set X^i is an empty set. Now, note that set of rules $\mathbf{R}_{CPL}$ includes analogons of tableau rules from the set of tableau rules for **CPL** — from set $\mathbf{R}_{\mathbf{CPL}}$. In other words, rules for **CPL** "split" formulas into subformulas, and at the same time rules $\mathbf{R}_{CPL}$ for the expressions constructed from quasi-modal formulas and index identically "split" formulas preserving the initial index in the new expression.

For rules $\mathbf{R_{CPL}}$, we have fact 2.34 stating that each finite set of formulas of the classical logic, that is non-modal formulas, is the first element of some such branch of a finite length ϕ that there does not exist super-branch $\phi \subset \psi$.

Now, for any formulas A, B and C, we will define substitution $e : \mathsf{For}_{\mathbf{S5}} \longrightarrow \mathsf{For}_{\mathbf{S5}}$ using the following conditions:

1. if $A \in \mathsf{Var}$, then for any $i \in \mathbb{N}$:
   a. if $A = p_i$ and $i = 1$, then $e(A) = q_1$
   b. if $A = p_i$ and $i \neq 1$, then $e(A) = q_j$, where j is the smallest odd number greater that index in $e(p_{i-1})$
   c. if $A = q_i$, then $e(A) = q_j$, where $j = i \cdot 2$
   d. if $A = r_i$, then $e(A) = r_i$
2. if $A = \neg B$, A is a quasi-modal formula and there is no such formula D that $B = \neg D$, then $e(A) = \neg e(B)$
3. in the other cases:
   a. if $A = \neg\neg B$, then $e(A) = \neg\neg e(B)$
   b. if $A = (B \wedge C)$, then $e(A) = (e(B) \wedge e(C))$
   c. if $A = (B \vee C)$, then $e(A) = (e(B) \vee e(C))$
   d. if $A = (B \rightarrow C)$, then $e(A) = (e(B) \rightarrow e(C))$
   e. if $A = (B \leftrightarrow C)$, then $e(A) = (e(B) \leftrightarrow e(C))$
   f. if $A = \Diamond B$, then $e(A) = \bullet(\Diamond B)$
   g. if $A = \Box B$, then $e(A) = \bullet(\Box B)$
   h. if $A = \neg \Diamond B$, then $e(A) = \bullet(\neg \Diamond B)$
   i. if $A = \neg \Box B$, then $e(A) = \bullet(\neg \Box B)$.

Note that for any formula A, its images under function e, i.e. $e(A)$ is a formula of **CPL**, that is $e(A) \in \mathsf{For}_{\mathbf{CPL}}$. Let us define function $e' : \mathsf{For}_{\mathbf{S5}} \longrightarrow \mathsf{For}_{\mathbf{CPL}}$ with condition: for any formula $A \in \mathsf{For}_{\mathbf{S5}}$, $e'(A) = e(A)$.

Due to the fact that function $\bullet$ is a bijection and that function e is injective, function e' is also a bijection. Hence, there exists inverse function e'^{-1} such that for any formula $A \in \mathsf{For}_{\mathbf{S5}}$, $e'^{-1}(e'(A)) = A$.

For any $i \in *(X)$, we now define set $\overline{X^i} = \{e'(A) : \langle A, i \rangle \in X^i\}$.

Obviously, $\overline{X^i}$ is, by virtue of the construction of set X^i, a finite subset of formulas **CPL**. So, from the mentioned fact 2.34, it follows that there exists such a finite branch of length n beginning with set $\overline{X^i}$ that it cannot be anymore extended using the tableau rules for tableau system for **CPL**:

(1) $\overline{X^i_1}, \overline{X^i_2}, \ldots, \overline{X^i_n}$.

Hence, there exists such finite $\mathbf{R}_{CPL}$-branch of length n, beginning with set X^i:

(2) $X_1^i, X_2^i, \ldots, X_n^i$

where for any $1 \leq j \leq n$, $X_j^i = \{\langle A, i\rangle : A \in e'^{-1}(\overline{X_j^i})\}$, that it cannot be anymore extended using the tableau rules from set $\mathbf{R}_{CPL}$. If branch (2) was extendible by some of rules from set $\mathbf{R}_{CPL}$, also branch (1) would be extendible by an equivalent of that rule from set $\mathbf{R_{CPL}}$. But this would contradict fact 2.34.

Note that branch:

$$X_1^i, X_2^i, \ldots, X_n^i$$

is 4.62 quasi-maximal as it was created using rules that belonged to set of rules $\mathbf{R}_{CPL}$ and no rule from set of rules $\mathbf{R}_{CPL}$ can be applied to set X_n^i.

Now, take initial set of expressions X and branch $X_1^i, X_2^i, \ldots, X_n^i$, and for any index $i \in *(X)$ define branch:

$$(\dagger)\ Y_1^i = X_1^i \cup X,\ Y_2^i = X_2^i \cup X, \ldots, Y_n^i = Y_n^i \cup X.$$

Set $Y_1^i = X_1^i \cup X$, by definition of set X_1^i, is equal to set X. So, the defined branch begins with set X.

If for some index $i \in *(X)$, a branch defined with scheme $(\dagger)$ ends with a t-inconsistent set, then according to definition of quasi-maximality 4.62, that branch is quasi-maximal, while since Y_1^i is equal to set X, so there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Let us assume, however, that for no index $i \in *(X)$, a branch defined with scheme $(\dagger)$ ends with a t-inconsistent set.

As we know, set of indices $i \in *(X)$ is finite — assume it contains m indices.

If set $i \in *(X)$ only includes one index, say index $j \in *(X)$, then according to definition of quasi-maximality 4.62 there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X, defined with scheme $(\dagger)$:

$$Y_1^j = X_1^j \cup X,\ Y_2^j = X_2^j \cup X, \ldots, Y_n^j = Y_n^j \cup X.$$

Let us assume, however, that number of indices m that belong to set $*(X)$, is greater than one. Let us arrange the indices that belong to set $i \in *(X)$ in sequence $i_1, i_2, \ldots, i_m$. For indices i_1, i_2, we take two branches, as per scheme $(\dagger)$:

(a) $Y_1^{i_1}, Y_2^{i_1}, \ldots, Y_n^{i_1}$

(b) $Y_1^{i_2}, Y_2^{i_2}, \ldots, Y_l^{i_2}$

and define the third branch, summing up each of sets of branches (b) and last element of branch (a), $Y_n^{i_1}$:

(c) $Y_n^{i_1} \cup Y_1^{i_2}, Y_n^{i_1} \cup Y_2^{i_2}, \ldots, Y_n^{i_1} \cup Y_l^{i_2}$.

Since $Y_1^{i_2} = X$, by definition of branch (†), $Y_n^{i_1} \cup Y_1^{i_2} = Y_n^{i_1}$, because $X \subseteq Y_n^{i_1}$ by definition of branch (†). Now, we will make use of fact on addition of branches 4.66 and define a branch by adding together branches (a) and (c):

(a)⊕(c) $Z_1 = Y_1^{i_1}, Z_2 = Y_2^{i_1}, \ldots, Z_n = Y_n^{i_1}, Z_{n+1} = Y_n^{i_1} \cup Y_2^{i_2}, \ldots, Z_{n+l-1} = Y_n^{i_1} \cup Y_l^{i_2}$.

Branch (a)⊕(c) will be called i_2-branch. Now, assume we have defined i_k-branch for $k < m$, of length $o \in \mathbb{N}$:

(d) $Y_1^{i_k}, Y_2^{i_k}, \ldots, Y_o^{i_k}$,

Next, we take a branch for index i_{k+1}, as per scheme (†):

(e) $Y_1^{i_{k+1}}, Y_2^{i_{k+1}}, \ldots, Y_n^{i_{k+1}}$

and define the third branch, summing up each of sets of branches (e) and last element of branch (d), $Y_o^{i_k}$:

(f) $Y_o^{i_k} \cup Y_1^{i_{k+1}}, Y_o^{i_k} \cup Y_2^{i_{k+1}}, \ldots, Y_o^{i_k} \cup Y_n^{i_{k+1}}$.

Since $Y_1^{i_{k+1}} = X$, by definition of branch (†), $Y_o^{i_k} \cup Y_1^{i_{k+1}} = Y_o^{i_k}$, because $X \subseteq Y_o^{i_k}$ by definition of branch (†) and construction of i_k-branch. Again, we make use of fact on addition of branches 4.66 and define a branch by adding branches (d) and (f):

(d)⊕(f) $Z_1 = Y_1^{i_k}, Z_2 = Y_2^{i_k}, \ldots, Z_o = Y_o^{i_k}, Z_{o+1} = Y_o^{i_k} \cup Y_2^{i_{k+1}}, \ldots, Z_{o+n-1} = Y_o^{i_k} \cup Y_n^{i_{k+1}}$.

Branch (d)⊕(f) will be called i_{k+1}-branch. Carrying out the above actions $m-1$ times, we get i_m-branch, of length n, for some $n \in \mathbb{N}$:

(g) $Y_1^{i_m}, Y_2^{i_m}, \ldots, Y_n^{i_m}$.

We claim that branch (g) is — according to definition of quasi-maximality 4.62 — quasi-maximal, and since $Y_1^{i_m}$ is equal to set X, there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X.

Assume that branch (g) is not closed. Take any rule $R \in \mathbf{R}_{CPL}$ and any such ordered l-tuple $\langle X_1, \ldots, X_l \rangle \in R$ that for some $1 \leq j \leq n$, $Y_j^{i_m} = X_1$. Therefore, according to definition of tableau rules 4.23 and set of rules $\mathbf{R}_{CPL}$, in set of tableau expressions $\mathsf{Te}_{\mathbf{S5}}$, there exist such expressions $A_1, \ldots, A_l$ that $X_2 \setminus X_1 = \{A_2\}, \ldots, X_l \setminus X_1 = \{A_l\}$ and $\langle \{A_1\}, \ldots, \{A_l\} \rangle \in R$. Since $R \in \mathbf{R}_{CPL}$, so expression A_1 is composed of some formula $B \in \mathsf{For}_{\mathbf{S5}}$ and some index $k \in \mathbb{N}$, thus it has a structure of $\langle B, k \rangle$.

We know that there exists $\mathbf{R}_{CPL}$-quasi-maximal branch:

(h) $Z_1 = \{\langle A, k \rangle : \langle A, k \rangle \in X\}, \ldots, Z_{n'}$

for certain $n' \in \mathbb{N}$. Since set Z_1 contains all expressions like $\langle A, k \rangle$ present in set X, so $\langle B, k \rangle$, due to definition of rules $\mathbf{R}_{CPL}$, belongs to some set Z_o, where $1 \leq o \leq n'$ such that Z_o is the first set in branch (h) comprising expression $\langle B, k \rangle$.

From the construction of branch (g), it follows that $Z_o \subseteq Y_j^{i_m}$. And since branch (h) is quasi-maximal, so some of expressions $A_2, \ldots, A_l$ belongs to some set $Z_{o+o'}$, where $o + o' \leq n'$. From the construction of branch (g), it follows that $Z_{o+o'} \subseteq Y_{j+o'}^{i_m}$. Due to the fact that $Y_j^{i_m} \subseteq Y_{j+o'}^{i_m}$, set $Y_j^{i_m} \cup \{A_{l'}\} \subseteq Y_{j+o'}^{i_m}$, where $2 \leq l' \leq l$. And since by definition of strong similarity of sets of expressions 4.39, set $Y_j^{i_m} \cup \{A_{l'}\}$ is strongly similar to set $X_{l'}$, so from definition of quasi-maximal branch 4.62, it follows that (g) is $\mathbf{R}_{CPL}$-quasi-maximal. □

Now, we expand the above fact into richer set of rules $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$.

Proposition 4.68. *Let $\mathbf{R}_{CPL\neg\Diamond\neg\Box} = \mathbf{R}_{CPL} \cup \{R_{\neg\Diamond}, R_{\neg\Box}\}$. Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be any finite set of tableau expressions. Then, there exists a $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch beginning with set X.*

Proof. Let $\mathbf{R}_{CPL\neg\Diamond\neg\Box} = \mathbf{R}_{CPL} \cup \{R_{\neg\Diamond}, R_{\neg\Box}\}$. Take any and finite set of expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$.

From fact 4.67, we know that there exists a $\mathbf{R}_{CPL}$-quasi-maximal branch beginning with set X:

(a) $Y_1, \ldots, Y_n$

where:

1. $\mathbf{R}_{CPL} = \{R_\wedge, R_\vee, R_\rightarrow, R_\leftrightarrow, R_{\neg\neg}, R_{\neg\wedge}, R_{\neg\vee}, R_{\neg\rightarrow}, R_{\neg\leftrightarrow}\}$
2. $n \in \mathbb{N}$
3. $Y_1 = X$.

If branch (a) is closed, then by definition 4.62, there exists $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch that begins with set X. Assume, however, that (a) is not a closed branch.

The last element of branch (a), set Y_n, contains a finite number of elements, due to the fact that branch (a) is finite and that for each $0 \le i < n$, set Y_{i+1} is also finite, by definition of rules $\mathbf{R}_{CPL}$.

Since set Y_n is finite, then for some $m \in \mathbb{N}$, it contains at most m of expressions like $\langle \neg \Diamond A, i \rangle$, where $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$.

Therefore, making use of rule $R_{\neg\Diamond}$, we can define a branch beginning with set Y_n of length at most $m+1$. Take any such branch of a maximal length:

(b) $Y_n^1, \ldots Y_n^o$

where $o \in \mathbb{N}$ and $o \le m+1$. Branch (b) is $R_{\neg\Diamond}$-quasi-maximal, by definition 4.62.

In view of the fact that the last element of branch (a) is the first element of branch (b), we can add both branches, by virtue of fact 4.66, to get branch (a)⊕(b):

(c) $Y_1, \ldots, Y_{n+o-1}$

of length $n+o-1$.

If branch (b) is closed, then also branch (c) is closed, and — by definition 4.62 — it is $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal, and moreover it begins with set X. Assume, however, that (c) is not a closed branch.

The last element of branch (c), set Y_{n+o-1}, features a finite number of elements, due to the fact that branch (c) is finite and that for each $0 \le i < n+o-1$, set Y_{i+1} is also finite, by definition of rules $\mathbf{R}_{CPL\neg\Diamond}$.

Since set Y_{n+o-1} is finite, then for some $k \in \mathbb{N}$, it contains at most k of expressions like $\langle \neg \Box A, i \rangle$, where $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$.

Therefore, making use of rule $R_{\neg\Box}$, we can define a branch beginning with set Y_{n+o-1} of length at most $k+1$. Take any such branch of a maximal length:

(d) $Y_{n+o-1}^1, \ldots Y_{n+o-1}^j$

where $j \in \mathbb{N}$ and $j \le k+1$. Branch (d) is $R_{\neg\Box}$-quasi-maximal, by definition 4.62.

In view of the fact that the last element of branch (c) is the first element of branch (d), we can add both branches, by virtue of fact 4.66, to get branch (c)⊕(d):

(e) $Y_1, \ldots, Y_{n+o+j-2}$

of length $n+o+j-2$.

If branch (d) is closed, then also branch (e) is closed, and — by definition 4.62 — it is $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal, and moreover it begins with set X. Assume, however, that (e) is not a closed branch.

Now, take any rule $R \in \mathbf{R}_{CPL\neg\Diamond\neg\Box}$ and any ordered m-tuple $\langle Z_1,\ldots,Z_m\rangle \in R$, for some $m \in \mathbb{N}$. Assume that set Z_1 is an element of branch (e). By the construction of branch (e), the sequence also includes such set U that for some $1 < i \leq m$, $Z_i \subseteq U$ and U is an element of branch (a), (b) or (d), thus by the construction of branch (e), it is also an element of branch (e). And since set Z_i by virtue of definition of strong similarity of sets of expressions 4.39 is strongly similar to Z_i, due to definition of quasi-maximal branch 4.62, branch (e) is $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch.

Since branch (e) begins with set of expressions X, so there exists a $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch beginning with set X. □

Again, we expand the above fact into richer set of rules $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{S5}}$.

Proposition 4.69. *Let $\mathbf{R}_r = \mathbf{R}_{CPL\neg\Diamond\neg\Box} \cup \{R_r\}$. Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be any finite set of tableau expressions. Then, there exists a $\mathbf{R}_r$-quasi-maximal branch beginning with set X.*

Proof. Let $\mathbf{R}_r = \mathbf{R}_{CPL\neg\Diamond\neg\Box} \cup \{R_r\}$. Take any and finite set of expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$.

From fact 4.68 we know that there exists a $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$-quasi-maximal branch beginning with set X:

(a) $Y_1, \ldots, Y_n$

where:

1. $\mathbf{R}_{CPL\neg\Diamond\neg\Box} = \{R_\wedge,\ R_\vee,\ R_\rightarrow,\ R_\leftrightarrow,\ R_{\neg\neg},\ R_{\neg\wedge},\ R_{\neg\vee},\ R_{\neg\rightarrow},\ R_{\neg\leftrightarrow} R_{\neg\Diamond}, R_{\neg\Box}, R_\Diamond, R_\Box\}$
2. $n \in \mathbb{N}$
3. $Y_1 = X$.

If branch (a) is closed, then by definition 4.62, there exists $\mathbf{R}_r$-quasi-maximal branch that begins with set X. Assume, however, that (a) is not a closed branch.

The last element of branch (a), set Y_n, contains a finite number of elements, due to the fact that branch (a) is finite and that for each $0 \leq i < n$, set Y_{i+1} is also finite, by definition of rules $\mathbf{R}_{CPL\neg\Diamond\neg\Box}$.

Since set Y_n is finite, then for some $m \in \mathbb{N}$, set $*(Y_n)$ contains at most m indices.

Therefore, making use of rule R_r, we can define a branch beginning with set Y_n of length at most of $(m \cdot m) + 1$, because by definition of rule R_r, if indices i, j

belong to given set $*(Z)$, then that rule makes it possible to add set $Z \cup \{irj\}$ in the branch, as long as $irj \notin Z$.

Take any branch maximal in length.

(b) $Y_n^1, \ldots Y_n^o$

where $o \in \mathbb{N}$ and $o \leq (m \cdot m) + 1$. Branch (b) is R_r-quasi-maximal, by definition 4.62.

In view of the fact that the last element of branch (a) is the first element of branch (b), we can add both branches, by virtue of fact 4.66, to get branch (a)⊕(b):

(c) $Y_1, \ldots, Y_{n+o-1}$

of length $n + o - 1$.

If branch (b) is closed, then also branch (c) is closed, and — by definition 4.62 — it is $\mathbf{R}_r$-quasi-maximal, and moreover it begins with set X. Assume, however, that (b) is not a closed branch.

Now, take any rule $R \in \mathbf{R}_r$ and any ordered m-tuple $\langle Z_1, \ldots, Z_m \rangle \in R$, for some $m \in \mathbb{N}$. Assume that set Z_1 is an element of branch (c). By the construction of branch (c), that sequence also includes such set U that for some $1 < i \leq m$, $Z_i \subseteq U$ and U is an element of branch (a) or (b), thus by the construction of branch (c), it is also an element of branch (c). And since set Z_i by virtue of by definition of strong similarity of sets of expressions 4.39 is strongly similar to Z_i, due to definition of quasi-maximal branch 4.62, branch (c) is $\mathbf{R}_r$-quasi-maximal branch.

Since branch (c) begins with set of expressions X, so there exists a $\mathbf{R}_r$-quasi-maximal branch beginning with set X. □

In order to extend the latter fact onto further rules from set $\mathbf{R}_{\mathbf{S5}}$, we need additional definitions. Prior to expressing the definition of cycle of rules $R_\Box$, we will first explain the idea of cyclical application of that rule.

Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Let the following branch (a) $X_1, \ldots, X_n$, where $n \in \mathbb{N}$ and $X_1 = X$, be $\mathbf{R}_r$-quasi-maximal. We specify two sets of expressions:

$$X' = \{irj : i,j \in \mathbb{N}, irj \in X_1\}$$

$$X'' = \{\langle \Box A, i\rangle : \Box A \in \mathsf{For}_{\mathbf{S5}}, i \in \mathbb{N}, \text{and } \langle \Box A, i\rangle \in X_1\}.$$

Union $X' \cup X''$ is a finite set because set X_1 is finite by assumption. Moreover, union $X' \cup X'' \subseteq X_n$ because $X_1 \subseteq X_n$.

Set $X' \cup X''$ contains a finite number of such two-element subsets $\{\langle \Box A, i\rangle, irj\}$ that if expression $\langle A,j\rangle$ does not belong to t-consistent set of tableau expressions Y, but set Y contains set $\{\langle \Box A, i\rangle, irj\}$, then pair $\langle Y, Y \cup \{\langle A,j\rangle\}\rangle \in R_{\Box}$. Thus, in particular, if expression $\langle A,j\rangle$ does not belong to set X_n and set X_n is t-consistent, then pair $\langle X_n, X_n \cup \{\langle A,j\rangle\}\rangle \in R_{\Box}$. Assume that the number of subsets $\{\langle \Box A, i\rangle, irj\} \subseteq X' \cup X''$ is l, for some $l \geq 0$.

Now, we extend branch (a) by rule $R_{\Box}$, taking account of all l sets $\{\langle \Box A, i\rangle, irj\} \subseteq X' \cup X''$, in an arbitrary order. Thereby, we get a branch of length at most $m \leq n + l$:

(b) $X_1, \ldots, X_n, X_{n+1} = X_n \cup \{\langle A_{n+1}, i_{n+1}\rangle\}, \ldots, X_m = X_{m-1} \cup \{\langle A_m, i_m\rangle\}$

where for any $n < j \leq m$, if set X_j belongs to branch (b), then set X_{j-1} is t-consistent and expression $\langle A_j, i_j\rangle$ does not belong to set X_{j-1}.

If set $\{\langle \Box A, k\rangle, kro\} \subseteq X' \cup X''$ does not exist, for some indices k, o, such that pair $\langle X_m, X_{m+1} \cup \{A, o\}\rangle \in R_{\Box}$, then branch (b) cannot be anymore extended using rule $R_{\Box}$ by applying it to some pair from set $\{\{x,y\} : x \in X', y \in X''\} \subseteq X_1$. This does not mean, of course, that there no new pairs appeared in the branch, in the subsequent elements of the branch to which we could apply rule $R_{\Box}$. Nevertheless, we have exhausted all the initial possibilities, closing a certain stage the result of which will be called the cycle of rule $R_{\Box}$. Let us now proceed to a formal definition.

Definition 4.70 (Cycle of rule $R_{\Box}$). Let X be a finite set of expressions. Let branch ϕ: $X_1, \ldots, X_n$ be such branch that $X_1 = X$ and $n \in \mathbb{N}$. Branch ϕ will be called a *cycle of rule* $R_{\Box}$ iff the below conditions are met:

1. for certain $m \leq n$, branch $X_1, \ldots, X_m$ is $\mathbf{R}_r$-quasi-maximal
2. for each $m < l \leq n$ there exist such indices $o, k \in \mathbb{N}$ and formula A that $X_l = X_{l-1} \cup \{\langle A, o\rangle\}$ and $\{\langle \Box A, k\rangle, kro\} \subseteq X_1$
3. there is no set $\{\langle \Box A, k\rangle, kro\} \subseteq X_1$, for some indices k, o and formula A, such that pair $\langle X_n \cup \{\langle \Box A, k\rangle, kro\}, X_n \cup \{\langle A, o\rangle\}\rangle \in R_{\Box}$.

Another fact follows from fact 4.69 and definition 4.70.

Proposition 4.71. *Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Then, there exists cycle of rule $R_{\Box}$.*

Now, we expand the concept of cycle onto rule $R_{\Diamond}$, which will make our considerations to cover all the rules from set of tableau rules $\mathbf{R}_{\mathbf{S5}}$. Prior to that, however, let us look into the issue in a similar way as the expansion onto rule $R_{\Box}$.

Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Let branch (a) $X_1, \ldots, X_n$, where $n \in \mathbb{N}$ and $X_1 = X$, be a cycle of rule $R_{\Box}$.

We specify set $X' = \{\langle \Diamond A, i\rangle : A \in \mathsf{For}_{\mathsf{S5}}, i \in \mathbb{N}$ and $\langle \Diamond A, i\rangle \in X_1\}$.

Set X' is a finite set because set X_1 is finite by assumption. Moreover, $X' \subseteq X_n$ because $X_1 \subseteq X_n$.

Set X' contains a finite number of such expressions $\langle \Diamond A, k\rangle$ that if subset $\{\langle A, j\rangle, krj\}$ is not contained in t-consistent set of tableau expressions Y and $j \notin *(Y)$, but $\langle \Diamond A, k\rangle \in Y$, then pair $\langle Y, Y \cup \{\langle A, j\rangle, krj\}\rangle \in R_{\Diamond}$. Thus, in particular, if for any index j subset $\{\langle A, j\rangle, krj\}$ is not contained in set X_n, X_n is t-consistent and $j \notin *(X_n)$, then pair $\langle X_n, X_n \cup \{\langle A, j\rangle, krj\}\rangle \in R_{\Diamond}$. Assume that the number of such expressions in set X' is $l \geq 0$.

Now, we extend branch (a) by rule $R_{\Diamond}$, taking account of all l expressions $\langle \Diamond A, k\rangle \in X'$, in an arbitrary order. Thereby, we get a branch of length at most $m \leq n + l$:

(b) $X_1, \ldots, X_n, X_{n+1} = X_n \cup \{\langle A_{n+1}, i_{n+1}\rangle, k_n r i_{n+1}\}, \ldots, X_m = X_{m-1} \cup \{\langle A_m, i_m\rangle, k_{m-1} r i_m\}$

where for any $n < j \leq m$, if set X_j belongs to branch (b), then X_{j-1} is t-consistent, $\{\langle \Diamond A_j, k_{j-1}\rangle\} \in X_{j-1}$, set $\{\langle A_j, o\rangle, k_{j-1} r o\}$ is not contained in set X_{j-1} for any $o \in \mathbb{N}$, and $o \notin *(X_{j-1})$.

If there is no tableau expression $\langle \Diamond A, k\rangle \in X'$, for some index k, such that pair $\langle X_m, X_{m+1} \cup \{\langle A, o\rangle, kro\}\rangle \in R_{\Diamond}$, for some index o, then branch (b) cannot be anymore extended using rule $R_{\Diamond}$ by applying it to the expressions from set X_1. Again, this does not mean, of course, that there no new pairs appeared in the branch, in the subsequent elements of the branch to which we could apply rule $R_{\Diamond}$. However, we have exhausted all the initial possibilities, closing a certain stage the result of which will be called the cycle, precisely. Let us now proceed to a formal definition of cycle.

Definition 4.72 (Cycle)**.** Let X be a finite set of expressions. Let branch ϕ: $X_1, \ldots, X_n$ be such branch that $X_1 = X$ and $n \in \mathbb{N}$. Branch ϕ will be called a *cycle* iff the below conditions are met:

1. for certain $m \leq n$, branch $X_1, \ldots, X_m$ is a cycle of rule $R_{\Box}$
2. for each $m < l \leq n$ there exist such indices $o, k \in \mathbb{N}$ and formula A that $X_l = X_{l-1} \cup \{\langle A, o\rangle, kro\}$ and $\{\langle \Diamond A, k\rangle\} \subseteq X_1$
3. there is no set $\{\langle \Diamond A, k\rangle\} \subseteq X_1$, for some index k and formula A, such that for some index o pair $\langle X_n \cup \{\langle \Diamond A, k\rangle\}, X_n \cup \{\langle \Diamond A, k\rangle, kro, \langle A, o\rangle\}\rangle \in R_{\Diamond}$.

Another fact follows from fact 4.71 and definition 4.72.

Proposition 4.73. *Let $X \subseteq \mathsf{Te}_{\mathsf{S5}}$ be a finite set of tableau expressions. Then, there exists a cycle.*

Now, we will proceed to the principal fact from among those concerning quasi-maximal branches. We will show that for a finite set of expressions, there exists **R**-quasi-maximal branch that begins with that set, for $\mathbf{R} = \mathbf{R_{S5}}$, thus by definition of maximal branch 4.40 and definition of quasi-maximal branch 4.62, a maximal branch.

Proposition 4.74. *Let $X \subseteq \mathsf{Te}_{\mathbf{S5}}$ be a finite set of tableau expressions. Then, there exists a maximal branch beginning with set X.*

Proof. Take any finite set of expressions $X \subseteq \mathsf{Te}_{\mathbf{S5}}$. From fact 4.73, it follows that there exists cycle:
(1) $X_1^1, \ldots, X_o^1$
where:

1. $o \in \mathbb{N}$
2. $X_1^1 = X$
3. all elements of the sequence have originated by the rules applicable for set X_1^1 from set of tableau rules $\mathbf{R_{S5}}$.

(†) For any $n > 1$, we now define the following cycle:

$(n)\ X_1^n, \ldots, X_m^n$

where:

1. $m \in \mathbb{N}$
2. $X_1^n = X_k^{n-1}$ and X_k^{n-1} is the last element of cycle $(n-1)$ which is k long, for certain $k \in \mathbb{N}$
3. all elements of the sequence have originated through the rules applicable for set X_1^n from set of tableau rules $\mathbf{R_{S5}}$.

There may exist an infinite number of cycles such that the last element of cycle (n) is the first element of cycle $(n+1)$, and there exists at least one cycle like (1), that is such that the first element of that cycle is set of tableau expressions X. From the set of all cycles, we select the minimal set of cycles **C** such that:

1. precisely one cycle like (1) belongs to **C**
2. if cycle (n) belongs to **C** and set Z is the last element of cycle (n), then set **C** contains cycle $(n+1)$ with set Z as the first element.

So, in set **C** for each (n) there exists precisely one cycle.

Now, assume that in set **C** there exists one-element cycle. Let it be cycle (k). Therefore by definition of cycle 4.72 and definition (†), for each $i \in \mathbb{N}$, each cycle $(k+i)$ has one element, what is more, cycle (k) is identical to cycle $(k+i)$.

In such case, on cycles from (1) to (k), making use $k-1$ times of conclusion concerning addition of branches 4.66, we define branch:

$$(\ddagger)\ (\ldots(\underbrace{X_1^1,\ldots,X_n^1}_{(1)} \oplus \underbrace{X_1^2,\ldots,X_m^2}_{(2)}) \oplus \ldots) \oplus \underbrace{X_1^k}_{(k)}$$

where $n, m, \ldots, o$ are the lengths of individual cycles.

Assume that branch $(\ddagger)$ is not closed. We claim that branch $(\ddagger)$ is maximal. Because, take any rule $R \in \mathbf{R}_{\mathbf{S5}}$ and any j-tuple $\langle Z_1, \ldots, Z_j \rangle \in R$ such that set Z_1 is an element of branch $(\ddagger)$. If there is no such element W of branch $(\ddagger)$ that some set $U \subseteq W$ is strongly similar, within the meaning of definition of strong similarity 4.39, to set of expressions Z_i, for some $1 < i \leq j$, then from definition of tableau rules 4.23, it follows that rule R contains j-tuple $\langle Z_1 \cup X_1^k, \ldots, Z_j' \cup X_1^k \rangle$, where for any $1 < i \leq j$, $Z_i' \cup X_1^k$ is a set of expressions similar to set $Z_i \cup X_1^k$. What follows further, either (k) is not a one-element cycle or it is not a cycle. This, however, contradicts the construction of set of cycles $\mathbf{C}$.

Now, assume that in set $\mathbf{C}$ there does not exist one-element cycle. Then, each cycle contained in $\mathbf{C}$ has at least two elements. By definition of set $\mathbf{C}$ — as it is the minimal set of cycles — for each cycle (k), cycle $(k+1)$ differs from it since at least its second element is a superset of the last element of cycle (k).

Now, we will arrange all cycles from set $\mathbf{C}$ as per their numbers in the following sequence of sequences:

$$(\ddagger\ddagger)\ \underbrace{X_1^1,\ldots,X_n^1}_{(1)}, \underbrace{X_1^2,\ldots,X_m^2}_{(2)}, \ \ldots, \underbrace{X_1^k,\ldots,X_o^k}_{(k)}, \ldots$$

where $n, m, \ldots, o, \ldots$ are the lengths of individual cycles.

Next, for any cycle (i), where $1 < i$, we remove set X_1^i — this element is identical to the last element of cycle $(i-1)$ — to get sequence:

$$(\ddagger\ddagger\ddagger)\ \underbrace{X_1^1,\ldots,X_n^1}_{(1)}, \underbrace{X_2^2,\ldots,X_m^2}_{(2)}, \ \ldots, \underbrace{X_2^k,\ldots,X_o^k}_{(k)}, \ldots$$

where $n, m, \ldots, o, \ldots$ are the lengths of individual cycles.

That sequence is an infinite branch. It can also be defined as follows. Take function $\phi : \mathbb{N} \longrightarrow P(\mathsf{Te}_{\mathbf{S5}})$ specified by the below conditions:

1. $\phi(1) = X_1^1$
2. for any $i, j, o \in \mathbb{N}$, if $\phi(i) = X_j^o$, then:

a. $\phi(i+1) = X^{o}_{j+1}$, if element X^{o}_{j+1} belongs to (‡‡‡)
b. $\phi(i+1) = X^{o+1}_{2}$, otherwise.

By definition of sequence (‡‡‡) and definition of branch 4.27, sequence ϕ is a branch since for any $i \in \mathbb{N}$, there exists such rule $R \in \mathbf{R_{S5}}$ and there exists such l-tuple $\langle Y_1, \ldots, Y_l \rangle$ that $\phi(i) = X_1$ i $\phi(i+1) = Y_{l_1}$, where $1 < l_1 \leq l$.

As branch ϕ is infinite, so it is not closed, by fact 4.31.

We claim that branch ϕ is maximal. Because, take any rule $R \in \mathbf{R_{S5}}$ and any n-tuple $\langle Z_1, \ldots, Z_n \rangle \in R$ such that set Z_1 is an element of branch ϕ. So, by definition of branch ϕ, set Z_1 is an element of some cycle (k).

If there is no such element W of cycle (k) that for some set $U \subseteq W$, set U is strongly similar, within the meaning of definition of strong similarity 4.39, to set of expressions Z_i, for some $1 < i \leq n$, then from definition of cycles (†) and definition of tableau rules 4.23 it follows that rule R contains n-tuple $\langle Z_1 \cup X^{k+1}_{2}, \ldots, Z'_n \cup X^{k+1}_{2} \rangle$, where for any $1 < i \leq n$, $Z'_i \cup X^{k+1}_{2}$ is a set of expressions similar to set $Z_i \cup X^{k+1}_{2}$. Thus, by definition of cycle 4.72, it follows that cycle $(k+1)$ contains set of expressions W such that for some set $U \subseteq W$, set U is strongly similar, within the meaning of definition of strong similarity 4.39, to set of expressions Z_i, for some $1 < i \leq n$.

From the arbitrariness of rule R and set Z_1, it follows that branch ϕ is a maximal branch. □

The above concepts and facts let us show a partial relationship between the branch consequence relation and the existence of a closed tableau in the tableau system for **S5**.

Lemma 4.75. *For any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if $X \rhd A$, then there exists finite subset $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Assume that $X \rhd A$. Therefore, by definition of $\rhd$, there exists such finite set $Y \subseteq X$ and such index $i \in \mathbb{N}$ that each maximal branch beginning with set $\{\langle B, i \rangle : B \in Y \cup \{\neg A\}\}$ is closed. Thus, by fact 4.74 and by definition of complete tableau 4.50, there exists such non-empty subset Φ of set of branches beginning with set $\{\langle B, i \rangle : B \in Y \cup \{\neg A\}\}$ that $\langle Y, A, \Phi \rangle$ is a closed tableau. □

We will now proceed to the description of dependencies between the existence of a closed tableau and the semantic consequence in the **S5**. However, this will require a determination of several more fundamental facts.

Lemma 4.76. *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ be a finite set of formulas, $A \in \mathsf{For}_{\mathbf{S5}}$ and $i \in \mathbb{N}$. If there exists a maximal and open branch beginning with set $\{\langle B, i\rangle : B \in X \cup \{\neg A\}\}$, then each complete tableau $\langle X, A, \Phi\rangle$ is open.*

Proof. Take finite set $X \subseteq \mathsf{For}_{\mathbf{S5}}$, any formula $\neg A \in \mathsf{For}_{\mathbf{S5}}$ and index $i \in \mathbb{N}$ such that there exists a maximal and open branch beginning with set $X^i = \{\langle B, i\rangle : B \in X \cup \{\neg A\}\}$. We will denote that branch with letter ϕ.

$(*)$ Since branch ϕ is open, so no element ϕ is a t-inconsistent set, by definition 4.30.

$(**)$ Since branch ϕ is maximal and open, so for any rule $R \in \mathbf{R_{S5}}$, any $n \in \mathbb{N}$ and any element $Y \in \phi$, if $\langle Y, Y_1, \ldots, Y_n\rangle \in R$, then there exists some element $Z \in \phi$ such that some subsist $W \subseteq Z$ is a set strongly similar to set Y_i, for certain $1 \leq i \leq n$, by definition of maximal branch 4.40.

Now, we indirectly assume that there exists complete and closed tableau $\langle X, A, \Psi\rangle$.

Since tableau $\langle X, A, \Psi\rangle$ is complete, so Ψ is such a minimal subset of set of all maximal branches that $\langle X, A, \Psi\rangle$ is a complete tableau, by definition of complete tableau 4.50.

Since tableau $\langle X, A, \Psi\rangle$ is closed, so each branch that belongs to Ψ, is closed, by definition of closed tableau 4.51. For certain $k \in \mathbb{N}$, each of these branches:

- begins with set $X^k = \{\langle B, k\rangle : B \in X \cup \{\neg A\}\}$, by definition of tableau 4.47,
- and its last element is a t-inconsistent set of expressions, by definition of closed tableau 4.51.

We intend to show that there exists some branch χ such that $\chi \notin \Psi$ and $\langle X, A, \Psi \cup \{\chi\}\rangle$ is a tableau, which contradicts the assumption that $\langle X, A, \Psi\rangle$ is a complete tableau if χ is not a redundant variant of any branch which belongs to Φ.

To this end, we will apply the induction through the branch length in order to construct infinite branches beginning with set X^k. The construction method for such branches will be denoted as $(\dagger)$.

Consider the first element of each branch contained in set of branches Ψ. It is set $X_1 = X^k = \{\langle B, k\rangle : B \in X \cup \{\neg A\}\}$. X_1 is a similar set of expressions — within the meaning of definition of similarity 4.16 — to set $X^i = \{\langle B, i\rangle : B \in X \cup \{\neg A\}\}$. Since $X^i \in \phi$ and branch ϕ is open, so X^i and X_1 are t-consistent, by conclusion 4.19.

Nevertheless, due to the fact that Ψ is a set of closed branches and the considered tableau $\langle X, A, \Psi\rangle$ is complete, there must exist a tableau rule $R \in \mathbf{R_{S5}}$ such that $\langle X_1, Z_2, \ldots, Z_l\rangle \in R$, where $1 < l$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 4.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. As due to definition of tableau rules 4.23, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — in the sense of definition of similarity 4.16 — to some set $W_m \subseteq Y_m$ and it is t-consistent since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}_{\mathbf{S5}}$, by the fact that ϕ is an $(*)$ open and $(**)$ maximal branch. Set Z_m will be denoted as X_2, while element W_m as X_2^*.

Therefore, for number 1 there exist such branches $\psi_1, \psi_2 \in \Psi$ that:

- $X_1 \in \psi_1$
- set X_2 originated by the application of certain rule $R \in \mathbf{R_{S5}}$ to set X_1, ultimately producing a second element of branch $\psi_2 \in \Psi$
- $X_2 \in \psi_2$
- X_2 is a t-consistent set
- $X_1 \subset X_2$
- for some $j \in \mathbb{N}$, set $X_2^* \subseteq X_j \in \phi$, moreover set X_2^* is similar, within the meaning of definition of similarity 4.16 — to set X_2.

Now, assume that for certain $n \in \mathbb{N}$ there exist such branches $\psi_1, \ldots, \psi_n \in \Psi$ that:

- for any $1 < j \leq n$, set X_j originated through the application of certain rule $R \in \mathbf{R_{S5}}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$
- $X_n \in \psi_n$
- X_n is a t-consistent set
- $X_1 \subset X_2 \subset \ldots \subset X_n$
- for some $i \in \mathbb{N}$, set $X_n^* \subseteq X_i \in \phi$, moreover X_n^* is similar, within the meaning of definition of similarity 4.16 — to set X_n.

Nevertheless, due to the fact that Ψ is a set of closed branches, the considered tableau $\langle X, A, \Psi \rangle$ is complete and set X_n is a t-consistent set, there must exist a tableau rule $R \in \mathbf{R_{S5}}$ such that $\langle X_n, Z_2, \ldots, Z_l \rangle \in R$, where $1 < l$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 4.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. As due to definition of tableau rules 4.23, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — within the meaning of definition of similarity 4.16 — to some set $W_m \subseteq Y_m$ and it is t-consistent since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}_{\mathbf{S5}}$, by virtue of the fact that ϕ is an $(*)$ open and $(**)$ maximal branch. Set Z_m will be denoted as X_{n+1}, while element W_m as X_{n+1}^*.

Thus, fo any $n \in \mathbb{N}$, there exist such branches $\psi_1, \ldots, \psi_n, \psi_{n+1} \in \Psi$ that:

- for any $1 < j \leq n+1$, set X_j originated through the application of certain rule $R \in \mathbf{R_{S5}}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$
- $X_{n+1} \in \psi_{n+1}$
- X_{n+1} is a t-consistent set
- $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1}$
- for some $i \in \mathbb{N}$, set $X^*_{n+1} \subseteq X_i \in \phi$, moreover X^*_{n+1} is similar, within the meaning of definition of similarity 4.16 — to set X_{n+1}.

Set of all sets that originate this way $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$ will be denoted as $\mathbf{X}$. Set $\mathbf{X}$ contains at least one branch ψ such that for any $i \in \mathbb{N}$, if $X_i \in \psi$, then there exists set $X_i \in \mathbf{X}$.

Branch ψ can be defined through the specification of such minimal subset of $\mathbf{X}$, set $\mathbf{X}'$ that:

1. $X_1 \in \mathbf{X}'$
2. for any $i \in \mathbb{N}$, if $X_i \in \mathbf{X}'$, then exactly one $X_{i+1} \in \mathbf{X}'$.

Branch ψ is infinite, and as a consequence of conclusion 4.31 it is an open branch.

Since set X_1, the first element of branch ψ, is equal to set X^k, and moreover for any element $X_i \in \psi$, where $i > 1$, there exists such rule $R \in \mathbf{R_{S5}}$ and such n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi' \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi'$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\psi\} \rangle$ by virtue of definition of tableau 4.47 is a tableau for pair $\langle X, A \rangle$.

However, branch ψ does not belong to set Ψ because tableau $\langle X, A, \Psi \rangle$, contrary to the assumption, would not be a closed tableau.

Let us now consider the question whether or not set $\Psi \cup \{\psi\}$ is a redundant superset of set Ψ, in the light of definition of redundant variant of branch 4.49. Let us now carry out the following argument.

($\dagger\dagger$) Assume that branch ψ is a redundant variant of some branch $\psi' \in \Psi$. Thus, for certain minimal $1 \leq i \in \mathbb{N}$ there exist two such rules $R, R' \in \mathbf{R_{S5}}$, ordered couple $\langle U_1, U_2 \rangle \in R$ and ordered triple $\langle W_1, W_2, W_3 \rangle \in R'$, such that:

- $U_1 = W_1$
- $U_1 = X_i \in \psi$ and $U_1 = Y_i \in \psi'$
- $U_2 = W_2$ or $U_2 = W_3$

- (a) if $U_2 = W_2$, then $U_2 = Y_{i+1}$ and $X_{i+1} = W_3$
- (b) if $U_2 = W_3$, then $U_2 = Y_{i+1}$ and $X_{i+1} = W_2$.

No matter which case occurs, (a) or (b), since branch ϕ is open and maximal (assumptions $(*)$ and $(**)$), so also element Y_{i+1} that belongs to ψ' is t-consistent because it is similar to certain set of expressions W included in certain element of branch ϕ.

Therefore, we can construct another infinite and open branch $Z_1, \ldots, Z_{i+1}, \ldots$, making use of construction $(\dagger)$ for which, by virtue of reasoning analogous to $(\dagger\dagger)$, there exists such subsequent element U_{i+2} that $Z_1, \ldots Z_{i+1}, U_{i+2}$ is a t-consistent branch and is not a redundant variant of any sub-branch of any branch from set Ψ.

So, by application of inductive reasoning and steps $(\dagger)$ and $(\dagger\dagger)$ we get an infinite branch — call it χ — and, consequently, open which is not a redundant variant of any branch that belongs to set of branches Ψ and begins with set X_1.

Since Ψ, by assumption, contains closed branches, $\chi \notin \Psi$. Since set X_1, the first element of branch χ, is equal to set X^k, and moreover for any element $X_i \in \chi$, where $i > 1$, there exists such rule $R \in \mathbf{R_{S5}}$ and such n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi' \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\chi\} \rangle$ by virtue of definition of tableau 4.47 is a tableau for pair $\langle X, A \rangle$.

Thus, $\langle X, A, \Psi \rangle$ is not a complete tableau which contradicts the initial assumption. □

In the next fact, we will take up the relationship between the rules from set $\mathbf{R_{S5}}$ and models. That fact states that rules have the following property: for any model, if the model is appropriate for the input set of the rule, then it is also appropriate for at least one output set of that rule.

Proposition 4.77. *Let $X \subseteq \mathsf{Te}_{S5}$, $\mathfrak{M}_{S5}$ be a model and rule $R \in \boldsymbol{R}_{S5}$. If $\langle X, X_1, \ldots, X_n \rangle \in R$, where $1 \leq n \leq 2$, and model $\mathfrak{M}_{S5}$ is appropriate for set X, then $\mathfrak{M}_{S5}$ is appropriate for some set X_i, where $1 \leq i \leq n$.*

Proof. Take any: $X \subseteq \mathsf{Te}_{S5}$, model $\mathfrak{M}_{S5} = \langle W, Q, V, w \rangle$, rule $R \in \mathbf{R_{S5}}$, and $n+1$-tuple $\langle X, X_1, \ldots, X_n \rangle \in R$, for some $1 \leq n \leq 2$.

From definition of model appropriate for the set of expressions 4.20, we know that model $\mathfrak{M}_{S5}$ is appropriate for X iff there exists such function $f : \mathbb{N} \longrightarrow W$ that for any $A \in \mathsf{For}_{S5}$ and $i, j \in \mathbb{N}$:

- if $\langle A,i\rangle \in X$, then $\langle W,Q,V,f(i)\rangle \vDash A$
- if $irj \in X$, then $f(i)Qf(j)$.

Let us consider all possible cases of rule R, and $n+1$-tuples $\langle X, X_1, \dots, X_n\rangle \in R$, where $1 < n \leq 2$. Take any two formulas $A, B \in \mathsf{For}_{\mathbf{S5}}$, any indices $i,j \in \mathbb{N}$ and any set of expressions $Y \subset X$.

1. Assume that $R = R_{\wedge}$ and $X = Y \cup \{\langle (A \wedge B),i\rangle\}$, $X_1 = X \cup \{\langle A,i\rangle, \langle B,i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash (A \wedge B)$. Therefore, by definition of truth in model 4.7, $\langle W,Q,V,f(i)\rangle \vDash A$ and $\langle W,Q,V,f(i)\rangle \vDash B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
2. Assume that $R = R_{\vee}$ and $X = Y \cup \{\langle (A \vee B),i\rangle\}$, $X_1 = X \cup \{\langle A,i\rangle\}$ and $X_2 = X \cup \{\langle B,i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash (A \vee B)$. Therefore, by definition of truth in model 4.7, $\langle W,Q,V,f(i)\rangle \vDash A$ or $\langle W,Q,V,f(i)\rangle \vDash B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
3. Assume that $R = R_{\rightarrow}$ and $X = Y \cup \{\langle (A \rightarrow B),i\rangle\}$, $X_1 = X \cup \{\langle \neg A,i\rangle\}$ and $X_2 = X \cup \{\langle B,i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash (A \rightarrow B)$. Therefore, by definition of truth in model 4.7, $\langle W,Q,V,f(i)\rangle \vDash \neg A$ or $\langle W,Q,V,f(i)\rangle \vDash B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
4. Assume that $R = R_{\leftrightarrow}$ and $X = Y \cup \{\langle (A \leftrightarrow B),i\rangle\}$, $X_1 = X \cup \{\langle A,i\rangle, \langle B,i\rangle\}$ and $X_2 = X \cup \{\langle \neg A,i\rangle, \langle \neg B,i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash (A \leftrightarrow B)$. Therefore, by definition of truth in model 4.7, $\langle W,Q,V,f(i)\rangle \vDash A$ and $\langle W,Q,V,f(i)\rangle \vDash B$ or $\langle W,Q,V,f(i)\rangle \vDash \neg A$ and $\langle W,Q,V,f(i)\rangle \vDash \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
5. Assume that $R = R_{\neg\neg}$ and $X = Y \cup \{\langle \neg\neg A,i\rangle\}$, $X_1 = X \cup \{\langle A,i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash \neg\neg A$. Therefore, by definition of truth in model 4.7, $\langle W,Q,V,f(i)\rangle \nvDash \neg A$, and consequently $\langle W,Q,V,f(i)\rangle \vDash A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
6. Assume that $R = R_{\neg\wedge}$ and $X = Y \cup \{\langle \neg(A \wedge B),i\rangle\}$, $X_1 = X \cup \{\langle \neg A,i\rangle\}$ and $X_2 = X \cup \{\langle \neg B,i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since

model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f:\mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \models \neg(A \wedge B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \not\models A$ or $\langle W, Q, V, f(i)\rangle \not\models B$, so $\langle W, Q, V, f(i)\rangle \models \neg A$ or $\langle W, Q, V, f(i)\rangle \models \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.

7. Assume that $R = R_{\neg\vee}$ and $X = Y \cup \{\langle\neg(A \vee B), i\rangle\}$, $X_1 = X \cup \{\langle\neg A, i\rangle, \langle\neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f:\mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \models \neg(A \vee B)$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \not\models A$ and $\langle W, Q, V, f(i)\rangle \not\models B$, and consequently $\langle W, Q, V, f(i)\rangle \models \neg A$ and $\langle W, Q, V, f(i)\rangle \models \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
8. Assume that $R = R_{\neg\rightarrow}$ and $X = Y \cup \{\langle\neg(A \rightarrow B), i\rangle\}$, $X_1 = X \cup \{\langle A, i\rangle, \langle\neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f:\mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \models \neg(A \rightarrow B)$. Therefore, by definition of truth in model 4.7, we get: $\langle W, Q, V, f(i)\rangle \not\models (A \rightarrow B)$, and thus $\langle W, Q, V, f(i)\rangle \models A$ and $\langle W, Q, V, f(i)\rangle \not\models B$, hence $\langle W, Q, V, f(i)\rangle \models \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
9. Assume that $R = R_{\neg\leftrightarrow}$ and $X = Y \cup \{\langle\neg(A \leftrightarrow B), i\rangle\}$, $X_1 = X \cup \{\langle\neg A, i\rangle, \langle B, i\rangle\}$ and $X_2 = X \cup \{\langle A, i\rangle, \langle\neg B, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f:\mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \models \neg(A \leftrightarrow B)$. Therefore, by definition of truth in model 4.7, either $\langle W, Q, V, f(i)\rangle \not\models A$ and $\langle W, Q, V, f(i)\rangle \models B$, or $\langle W, Q, V, f(i)\rangle \models A$ and $\langle W, Q, V, f(i)\rangle \not\models B$, and thus $\langle W, Q, V, f(i)\rangle \models \neg A$ and $\langle W, Q, V, f(i)\rangle \models B$ or $\langle W, Q, V, f(i)\rangle \models A$ and $\langle W, Q, V, f(i)\rangle \models \neg B$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1 or X_2, it is appropriate for some set X_i, where $1 \leq i \leq n$.
10. Assume that $R = R_{\neg\Box}$ and $X = Y \cup \{\langle\neg\Box A, i\rangle\}$, $X_1 = X \cup \{\langle\Diamond\neg A, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f:\mathbb{N} \longrightarrow W$ that $\langle W, Q, V, f(i)\rangle \models \neg\Box A$. Therefore, by definition of truth in model 4.7, $\langle W, Q, V, f(i)\rangle \not\models \Box A$, and consequently there exists such $u \in W$ that $f(i)Qu$ & $\langle W, Q, V, u\rangle \not\models A$, so $f(i)Qu \,\&\, \langle W, Q, V, u\rangle \models \neg A$, and thus $\langle W, Q, V, f(i)\rangle \models \Diamond\neg A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is appropriate for some set X_i, where $1 \leq i \leq n$.
11. Assume that $R = R_{\neg\Diamond}$ and $X = Y \cup \{\langle\neg\Diamond A, i\rangle\}$, $X_1 = X \cup \{\langle\Box\neg A, i\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$ is appropriate for set X, there exists such function $f:\mathbb{N} \longrightarrow W$

that $\langle W,Q,V,f(i)\rangle \vDash \neg \Diamond A$. Therefore, by definition of truth in model 4.7, $\langle W,Q,V,f(i)\rangle \nvDash \Diamond A$, and consequently there is no such $u \in W$ that $f(i)Qu \& \langle W,Q,V,u\rangle \vDash A$, so for any $u \in W$, if $f(i)Qu$, then $\langle W,Q,V, u\rangle \nvDash A$, that is for any $u \in W$, if $f(i)Qu$, then $\langle W,Q,V,u\rangle \vDash \neg A$, and thus $\langle W,Q,V,f(i)\rangle \vDash \Box\neg A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is also appropriate for some set X_i, where $1 \leq i \leq n$.

12. Assume that $R = R_{\Box}$ and $X = Y \cup \{\langle \Box A,i\rangle, irj\}$, $X_1 = X \cup \{\langle A,j\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash \Box A$ and $f(i)Qf(j)$. Therefore, by definition of truth in model 4.7, for any $u \in W$, if $f(i)Qu$, then $\langle W,Q,V,u\rangle \vDash A$, so $\langle W,Q,V,f(j)\rangle \vDash A$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, thus it is appropriate for some set X_i, where $1 \leq i \leq n$.
13. Assume that $R = R_{\Diamond}$ and $X = Y \cup \{\langle \Diamond A,i\rangle\}$, $X_1 = X \cup \{irj, \langle A,j\rangle\}$. Assume that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X. Since model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, there exists such function $f : \mathbb{N} \longrightarrow W$ that $\langle W,Q,V,f(i)\rangle \vDash \Diamond A$. Therefore, by definition of truth in model 4.7, there exists such $u \in W$ that $f(i)Qu \& \langle W,Q,V,u\rangle \vDash A$. By definition of rule $R_{\Diamond}$, index $j \notin *(X)$. So, we define new function $f' : \mathbb{N} \longrightarrow W$, so that for any $k \neq j$, $f'(k) = f(k)$, whereas $f'(j) = u$. Then, for any $k \neq j$ and $l \neq j$, if $\langle A,k\rangle \in X_1$, then $\langle W,Q,V,f'(k)\rangle \vDash A$, and if $krl \in X_1$, then $f'(k)Qf'(l)$. Moreover, $f'(i)Qf'(j) \& \langle W,Q,V,f'(j)\rangle \vDash A$, since $f'(j) = u$. Hence, from the above and from definition of model appropriate for the set of expressions 4.20, model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X_1. While since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, it is also appropriate for some set X_i, where $1 \leq i \leq n$.
14. Assume that $R = R_r$ and $X_1 = X \cup \{irj\}$, where $i, j \in *(X)$. Assume that model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is appropriate for set X, thus there exists function $f : \mathbb{N} \longrightarrow W$ that meets the conditions from definition of appropriate model 4.20 and $f(i), f(j) \in W$. Since relation R in model $\mathfrak{M}_{\mathbf{S5}} = \langle W,Q,V,w\rangle$ is universal, $f(i)Qf(j)$. Thus model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for X_1, thus it is also appropriate for some set X_i, where $1 \leq i \leq n$.

□

The above fact 4.77 will be used for the proof of another lemma. This lemma determines the relationship between the finite sets of formulas and the existence of maximal and open branches.

Lemma 4.78. *Let $X \subseteq \mathsf{For}_{\mathbf{S5}}$ be a finite set of formulas, $i \in \mathbb{N}$ and let $\mathfrak{M}_{\mathbf{S5}}$ be a model. If $\mathfrak{M}_{\mathbf{S5}} \vDash X$, then there exists a maximal and open branch beginning with set $\{\langle A,i\rangle : A \in X\}$.*

Proof. Take any finite set of formulas $X \subseteq \mathsf{For}_{\mathbf{S5}}$, any index $i \in \mathbb{N}$ and any model $\mathfrak{M}_{\mathbf{S5}} = \langle W, Q, V, w\rangle$, and then assume that $\mathfrak{M}_{\mathbf{S5}} \vDash X$. Let us define set $\{\langle A, i\rangle : A \in X\}$. Set $\{\langle A, i\rangle : A \in X\}$ will be denoted as X^i.

Now, we take any function $f : \mathbb{N} \longrightarrow W$ such that $f(i) = w$. By definition of model appropriate for the set of expressions 4.20, model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set X^i, since for any formula $A \in \mathsf{For}_{\mathbf{S5}}$ and any index $j \in \mathbb{N}$, if $\langle A, j\rangle \in X^i$, then $j = i$. While due to the fact that $f(i) = w$ and the assumption that $\mathfrak{M}_{\mathbf{S5}} \vDash X$, we get a constitution that if $\langle A, i\rangle \in X^i$, then $\langle W, Q, V, f(i)\rangle \vDash A$. Moreover no expression krl, where $k, l \in \mathbb{N}$, belongs to set X^i, so there also holds the second condition of definition of model appropriate for the set of expressions 4.20.

Now, indirectly assume that each maximal branch beginning with set $X^i = \{\langle A, i\rangle : A \in X\}$ is closed.

As $\Phi(X^i)$ we will denote the set of all maximal branches beginning with set X^i. From fact 4.74, we know that for each finite set of tableau expressions Y there exists a maximal branch beginning with set Y. Thus, set $\Phi(X^i)$ is non-empty.

Since set $\Phi(X^i)$ is a set of all maximal branches beginning with set X^i, so it also has the following property. Assume that it contains branch χ. Let for certain $n \in \mathbb{N}$ exist such rule $R \in \mathbf{R}_{\mathbf{S5}}$ and such triple $\langle Z_1, Z_2, Z_3\rangle \in R$ that $\chi(n) = Z_1$ and $\chi(n+1) = Z_2$ or $\chi(n+1) = Z_3$. Note that then both Z_2 and Z_3 are finite sets of expressions because each rule extends set to add at most two tableau expressions (by definition of tableau rules 4.23), branch χ begins with finite set X^i and we consider its n-th element. Thus, from fact 4.74 we know that:

- there exists maximal branch ϕ beginning with set X^i such that $\phi(n+1) = Z_2$
- there exists maximal branch ψ beginning with set X^i such that $\psi(n+1) = Z_3$.

$(*)$ Thus, for any $n \in \mathbb{N}$, if there exist: such rule $R \in \mathbf{R}_{\mathbf{S5}}$, and such l-tuple $\langle Z_1, \ldots, Z_l\rangle \in R$, where $1 < l \leq 3$, branch $\chi \in \Phi(X^i)$ such that $\chi(n) = Z_1$ and $\chi(n+1) = Z_2$ or $\chi(n+1) = Z_3$, then there exist branches $\psi \in \Phi(X^i)$ and $\phi \in \Phi(X^i)$ such that $\psi(n) = Z_1$, $\psi(n+1) = Z_2$ and $\phi(n) = Z_1$, $\phi(n+1) = Z_3$, if $l = 3$.

$(**)$ By assumption, each branch that belongs to set $\Phi(X^i)$ is closed, thus by fact 4.31, each branch that belongs to set $\Phi(X^i)$ has a finite length of m, for some $m \in \mathbb{N}$.

From the initial assumption, we know that each of branches in set $\Phi(X^i)$ begins with set X^i.

Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set of expressions X^i, by virtue of fact 4.21, which says that for a t-inconsistent set of expressions there is no appropriate model, set X^i is not t-inconsistent. Hence, we have a conclusion that in set $\Phi(X^i)$ there is no branch of length of one as a one long branch would be open.

Due to fact 4.77 which states that for any rule $R \in \mathbf{R_{S5}}$ any l-tuple — where $2 \leq l \leq 3$ — $\langle Z_1, \ldots Z_l \rangle \in R$, if model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set Z_1, then it is appropriate for some set Z_i, where $2 \leq i \leq 3$, and $(*)$, there exists branch $\chi \in \Phi(X^i)$ such that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set $\chi(2)$ and $\chi(1) = X^i$. The set of those branches that belong to $\Phi(X^i)$, and at the same time model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for their k-th element, will be denoted as $\Phi(X^i)_k$. So, we have $\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \neq \varnothing$.

Now, assume that for some $n \in \mathbb{N}$, where $n > 1$, set $\Phi(X^i)_{n-1} \supseteq \Phi(X^i)_n \neq \varnothing$. Since set $\Phi(X^i)_n$ is non-empty, so take some branch $\psi \in \Phi(X^i)_n$.

By assumption, model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set of expressions $\psi(n)$. Since model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set of expressions $\psi(n)$, so by virtue of fact 4.21, which says that for a t-inconsistent set of expressions there is no appropriate model, set of expressions $\psi(n)$ is not t-inconsistent. Thus, branch ψ is longer that n because otherwise it would be an open branch. Due to fact 4.77 which states that for any rule $R \in \mathbf{R_{S5}}$ any l-tuple — where $2 \leq l \leq 3$ — $\langle Z_1, \ldots Z_l \rangle \in R$, if model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set Z_1, then it is appropriate for some set Z_i, where $2 \leq i \leq 3$, and $(*)$, there exists branch $\phi \in \Phi(X^i)_1$ such that model $\mathfrak{M}_{\mathbf{S5}}$ is appropriate for set $\phi(n+1)$ and $\phi \in \Phi(X^i)_n$. Thus, $\Phi(X^i)_n \supseteq \Phi(X^i)_{n+1}$ and $\Phi(X^i)_{n+1} \neq \varnothing$.

Therefore, for each $k \in \mathbb{N}$, $\Phi(X^i)_k \neq \varnothing$ and

$$\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \supseteq \cdots \supseteq \Phi(X^i)_k \supseteq \ldots$$

Next, we take the intersection of all those sets $\Phi(X^i)_k$, where $k \in \mathbb{N}$. Intersection $\bigcap\{\Phi(X^i)_k : k \in \mathbb{N}\} = \Phi$ is non-empty as for each k, subset $\Phi(X^i)_k$ is also non-empty. So, set Φ includes at least one branch χ. That branch is maximal and begins with set X^i since $\Phi \subseteq \Phi(X^i)$.

But, branch χ is infinite which contradicts conclusion $(**)$. □

We can now move on to the last lemma of this chapter.

Lemma 4.79. *For any $X \subseteq \mathsf{For}_{\mathbf{S5}}$, $A \in \mathsf{For}_{\mathbf{S5}}$, if there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$, then $X \models A$.*

Proof. Take any $X \subseteq \mathsf{For}_{\mathbf{S5}}$ and $A \in \mathsf{For}_{\mathbf{S5}}$. Assume that $X \not\models A$. So, by definition of relation of semantic consequence 4.9, there exists such model $\mathfrak{M}_{\mathbf{S5}}$ that $\mathfrak{M}_{\mathbf{S5}} \models X$ and $\mathfrak{M}_{\mathbf{S5}} \not\models A$. Therefore, by definition of truth in model 4.7, we have $\mathfrak{M}_{\mathbf{S5}} \models \neg A$, and consequently $\mathfrak{M}_{\mathbf{S5}} \models X \cup \{\neg A\}$. Thus for any finite set $Y \subseteq X$, also $\mathfrak{M}_{\mathbf{S5}} \models Y \cup \{\neg A\}$.

Take any finite subset $Y' \subseteq X$. From lemma 4.78 we get a conclusion that for any $i \in \mathbb{N}$ there exists maximal and open branch beginning with set $\{\langle B, i \rangle : B \in Y' \cup \{\neg A\}\}$. And from the above, and from lemma 4.76 we know that each complete tableau $\langle Y', A, \Phi\rangle$ is open. Since Y' was an arbitrary finite subset of set of formulas X, so there is no finite set $Y \subseteq X$ and closed tableau $\langle Y', A, \Phi\rangle$. □

To sum up the lemmas we have presented so far, we move on to the theorem on completeness for the tableau system we have discussed.

Theorem 4.80 (Theorem on the completeness of tableau system of **S5**). *For any $X \subseteq \mathsf{For}_{S5}$, $A \in \mathsf{For}_{S5}$, the below statements are equivalent.*

- $X \vDash A$
- $X \rhd A$
- *there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$*

Proof. Take any $X \subseteq \mathsf{For}_{S5}$ and $A \in \mathsf{For}_{S5}$. For the theorem proof, it is sufficient to show the occurrence of three implications:

(a) $X \vDash A \Rightarrow X \rhd A$
(b) $X \rhd A \Rightarrow$ there exists finite $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle$
(c) there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi\rangle \Rightarrow X \vDash A$.

Implication (a) results from lemma 4.59. Implication (b) results from lemma 4.75. Implication (c) results from lemma 4.79. □

5 Metatheory of tableau systems for propositional logic and term logic

5.1 Introductory remarks

In this chapter, we establish general tableau concepts for systems constructed using the method described in the book. These concepts make it possible to utter and justify a number of basic facts, the specific cases of which we have been proving in the previous chapters.

Using the further defined general concepts of the tableau systems, we will be able to utter and prove a general theorem on the relationship between the tableau systems and the semantics adopted for them. The construction of the tableau system, which is adequate in terms of the adopted semantics, will boil down to defining the basic concepts of this system in such a way that they are special cases of the general concepts and meet certain general conditions which we will define further. In this way, we will shorten the definition of the tableau systems to a minimal — in comparison to the previous chapters — number of procedures that describe the features of the considered system that distinguish it from other tableau systems.

5.2 Language and semantics

By set For we mean any set of formulas of some language. We call its elements *formulas*.

Remark 5.1. For our considerations, we adopt any but fixed such set of formulas For that $|\mathsf{For}|$ is an even number or For is an infinite set. Set For will remain unchanged until the end of this chapter.

We will now look at the issue of interpretation of formulas. In the cases presented in the previous chapters, the interpretations were the valuations of formulas or models. However, we will deliberately use the concept of *interpretation* in order to cover all those cases. We intend to describe in general the concept of interpretation which will be applicable in the presented metatheory of the construction of tableau systems that will allow us to draw conclusions on the general relationships between the semantic form of a given logic and its tableau approach.

In our considerations, we will make use of the fact that we only look into such logics whose interpretations assign exactly one of the two logical values to each formula. So, a given interpretation divides a set of formulas in an unambiguous, disjoint and exhaustive way into a subset of true formulas and a subset of false formulas under this interpretation. This division coincides with some division of the set of formulas into mutually contradictory formulas, because for any formula, a formula is true in a given interpretation if and only if the formula contradicting it is false in that interpretation.

The interpretation of a set of formulas can therefore be identified with the segment of the logical division of a set of formulas in terms of the contradiction of formulas which contains exactly all the formulas that are true in this interpretation. The starting point, however, will be a function that assigns a contradictory formula to each formula, thus always dividing the set of formulas into two segments of the logical division. At least one of those segments may correspond to some interpretation of a set of formulas in which all and only those formulas that belong to this segment of the division are true. We will illustrate this with an example.

Example 5.2. Let us consider the case of **TL** discussed in Chapter Three. Function $\circ$ (definition 3.12) assigns a contradictory formula to each formula from set $\mathsf{For}_{\mathbf{TL}}$ (fact 3.13). Let us now divide set of formulas $\mathsf{For}_{\mathbf{TL}}$ into the following pairs of sets: X' and complement of X' to set of formulas $\mathsf{For}_{\mathbf{TL}}$, i.e. $X'' = \mathsf{For}_{\mathbf{TL}} \setminus X'$, as per the principle of division, for each formula $A \in \mathsf{For}_{\mathbf{TL}}$:

$$(\dagger)\ A \in X' \text{ iff } \circ(A) \notin X'.$$

The division of set of formulas $\mathsf{For}_{\mathbf{TL}}$ into sets X' and X'' is a logical division as $X' \cap X'' = \varnothing$ and $X' \cup X'' = \mathsf{For}_{\mathbf{TL}}$, and what is more, sets X' and X'' are non-empty.

Let us denote the set of all models $\mathfrak{M}_{\mathbf{TL}}$ for **TL** as $\mathbf{M}_{\mathbf{TL}}$, while the set of all such segments of division by function $\circ$ that meet equivalence $(\dagger)$ as $\mathbf{X}_{\circ}$.

Let $\mathfrak{M}_{\mathbf{TL}}$ be any model. We define subset of formulas $\mathfrak{M}'_{\mathbf{TL}} = \{A \in \mathsf{For}_{\mathbf{TL}} : \mathfrak{M}_{\mathbf{TL}} \models A\}$. A complement of set $\mathfrak{M}'_{\mathbf{TL}}$ to set $\mathsf{For}_{\mathbf{TL}}$ is set $\mathsf{For}_{\mathbf{TL}} \setminus \mathfrak{M}'_{\mathbf{TL}}$, i.e. set $\mathfrak{M}''_{\mathbf{TL}} = \{A \in \mathsf{For}_{\mathbf{TL}} : \mathfrak{M}_{\mathbf{TL}} \not\models A\}$.

Function $\circ$ assigns a contradictory formula to each formula from set $\mathsf{For}_{\mathbf{TL}}$ (fact 3.13), since for any model $\mathfrak{M}_{\mathbf{TL}}$ for **TL** and for any formula $A \in \mathsf{For}_{\mathbf{TL}}$ it is the case that:

$$(\dagger\dagger)\ \mathfrak{M}_{\mathbf{TL}} \models A \text{ iff } \mathfrak{M}_{\mathbf{TL}} \not\models \circ(A).$$

Since sets $\mathfrak{M}'_{\mathbf{TL}}$ and $\mathfrak{M}''_{\mathbf{TL}}$ are disjoint, exhaustive and non-empty, so they make up one of many logical divisions of set $\mathsf{For}_{\mathbf{TL}}$ by function $\circ$ according to principle $(\dagger)$, for any formula $A \in \mathsf{For}_{\mathbf{TL}}$:

$$(\dagger\dagger\dagger)\ A \in \mathfrak{M}'_{\mathbf{TL}} \text{ iff } \circ(A) \notin \mathfrak{M}'_{\mathbf{TL}}.$$

and belong to set $\mathbf{X}_{\circ}$.

So with function ∘ we can unambiguously identify each model $\mathfrak{M}_{\mathbf{TL}}$ (and more precisely the set of all formulas that are true in this model) with some segment of the logical division of the set of formulas by the contradiction of formulas, determined by function ∘ with equivalence (††).

For each $\mathfrak{M}_{\mathbf{TL}} \in \mathbf{M}_{\mathbf{TL}}$, there exists precisely one set $X \in \mathbf{X}_{\circ}$ such that $\mathfrak{M}'_{\mathbf{TL}} = X$. Therefore, we can identify set of all models $\mathbf{M}_{\mathbf{TL}}$ with some subset of set $\mathbf{X}_{\circ}$, that is with some subset of set of all subsets of set of formulas $P(\mathsf{For}_{\mathbf{TL}})$.

However, the opposite dependence does not occur. The segments of logical division that belong to $\mathbf{X}_{\circ}$ may correspond to many models that vary in terms of the domain cardinality, but not formulas which are true in them.

In some cases, a segment of logical division that belongs to $\mathbf{X}_{\circ}$ may not be determined by any models. If, for instance, we take such segments of division Y' and Y'' that belong to $\mathbf{X}_{\circ}$ that for certain name letters P, $Q \in \mathsf{Ln}$, $P\mathbf{e}P \in Y'$ and $P\mathbf{i}Q \in Y'$, then there is no such model $\mathfrak{M}_{\mathbf{TL}} \in \mathbf{M}_{\mathbf{TL}}$ that $\mathfrak{M}_{\mathbf{TL}} \models Y'$, since the truth of set of formulas Y' would require the denotation of letter P to be an empty set and non-empty one at the same time. Further, we will also provide an example for **CPL** (example 5.6).

Therefore, models/valuations, that is the set of interpretations should be identified with a certain subset of the set of logical divisions of the set of formulas determined by a certain function that assigns a contradictory formula to each formula. Whether or not it is a proper subset, may vary from one case to another (see example 5.7).

In the definition of interpretation of formulas, we will employ a reference to the sets of formulas that are true in a given model or valuation, because in the general approach to the tableau metatheory we cannot penetrate the structure of particular types of interpretations for different logics. At the same time, we must retain the general aspects, in particular those that correspond to the tableau proof, i.e. adopting in the proof a formula that is contradictory to the formula we are proving.

We will now propose the general definition of set of divisions of set of formulas For. There is a function in this definition that implicitly assigns a contradictory formula to each formula. In many cases, however, even if this function is well established, we will identify a set of initial valuations/models with a proper subset of some sort of a set of all divisions.

Definition 5.3. Let $f : \mathsf{For} \longrightarrow \mathsf{For}$ be an injective function and $X \subseteq \mathsf{For}$. We shall state that X is a *division* of For iff for any formula $A \in \mathsf{For}$ the following condition is met: $A \in X$ iff $f(A) \notin X$.

By set $\mathbf{X}_f$ we will mean a set of all divisions of set of formulas For determined by the established function f.

Remark 5.4. In some cases, function f may result in the non-existence of any division $X \subseteq \mathsf{For}$, and then, consequently, set of all divisions $\mathbf{X}_f$ is empty. It is the case for instance when for some formula $A \in \mathsf{For}$ $f(A) = A$. Thus, not each function f is suitable for the definition of set $\mathbf{X}_f$.

In some cases, such function may fail to determine divisions corresponding to all models/valuations (example 5.7). For these cases, the specified method cannot be used to define a tableau system.

Function f is determined by default by the way of adopting an equivalent of formula contradictory to the formula being proved. So, function f can correspond to negation — it does in the case of **CPL** and **S5**[1]. Moreover, it may not correspond to any functor from the language — as it does not in the event of the tableau system for **TL**, where it could be identified with function $\circ$ 3.13 (example 5.2) that assigns a contradictory formula to each formula. Generally speaking, function f, to each formula, assigns such formula that for any division X it is the case that exactly one of those formulas belongs to X.

Remark 5.5. For our considerations, we adopt any but fixed and non-empty set of all divisions $\mathbf{X}_f$, for some function $f : \mathsf{For} \longrightarrow \mathsf{For}$ that meets the conditions of the last definition 5.3. The set will remain unchanged until the end of this chapter. We will, on the other hand, refer to function f.

As stated in example 5.2, for a given set of all valuations/models and given function f, it does not have to be the case that each element of set $\mathbf{X}_f$ corresponds to some valuation/model. Below, we provide an example for **CPL** (example 5.6).

Example 5.6. Take set of formulas of **CPL** $\mathsf{For}_{\mathbf{CPL}}$. We will define function f as follows $f(A) = \neg A$, for any formula $A \in \mathsf{For}_{\mathbf{CPL}}$. Function f meets the condition from definition of division of set of formulas 5.8, i.e. it is an injective function. Now, let $\mathbf{X}_f$ be a set of all divisions of set of formulas determined by function f.

The set contains such division X that $(p \wedge q) \in X$, $\neg p \in X$ and $\neg q \in X$. Obviously, no valuation of formulas V corresponds to set X, since for each V, if $V((p \wedge q)) = 1$, then $V(\neg p) = 0$. Therefore, there does not exist such valuation V that $V' = \{A \in \mathsf{For} : V(A) = 1\}$ and $V' = X$.

Moreover, no valuation corresponds to set $\mathsf{For}_{\mathbf{CPL}} \setminus X$, since for each V, if $V(\neg(p \wedge q)) = 1$, then $V(\neg\neg p) = 0$ or $V(\neg\neg q) = 0$, and $V(p) = 0$ or $V(q) = 0$. Therefore, there does not exist such valuation V that $V' = \{A \in \mathsf{For} : V(A) = 1\}$ and $V' = \mathsf{For}_{\mathbf{CPL}} \setminus X$.

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1 However, function f does not have to imply the addition of negation to the formula. For we could, for instance, assign a contradictory formula to each formula of **CPL** which would not be a negation of the initial formula. In the same way, we could also define a complete tableau system.

On the other hand, however, for each valuation V there exists such set $V' = \{A \in \mathsf{For} : V(A) = 1\}$ that for some division $Y \in \mathbf{X}_f$, $V' = Y$, because for each formula $A \in \mathsf{For}_{\mathbf{CPL}}$ it is the case that $A \in V'$ iff $\neg A \notin V'$.

By virtue of the above, we can identify the set of all valuations of formulas with a proper subset of set of all divisions $\mathbf{X}_f$ by function f.

In some cases and for some functions f it may be the case that the set of divisions corresponding to all models/valuations is identical to given set $\mathbf{X}_f$ (example 5.7).

Example 5.7. Consider two similar cases. In the first one, function f cannot be established in such a way that each model has a corresponding division that determines f. In the second case, for each division by function f there exists at least one corresponding model.

Take such subset X of set of all formulas $\mathsf{For}_{\mathbf{TL}}$ that $X = \{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\}$. Consider set of all models $\mathbf{M}_X$ for a new, more sparing language. They will be analogous to the models for **TL**, but their denotation sets will only be assigned to two name letters that occur in a new, more sparing language — letters P^1 and Q^1.

Next, in a usual manner we will define the relation of semantic consequence determined on set $P(X) \times X$ (in a sense truncating the relation of semantic consequence of **TL** to set X) and denote it as $\models'$. Only four arguments are correct in such logic:

$$\{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\} \models' P^1\mathbf{a}Q^1$$

$$\{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\} \models' P^1\mathbf{e}Q^1$$

$$\{P^1\mathbf{a}Q^1\} \models' P^1\mathbf{a}Q^1$$

$$\{P^1\mathbf{e}Q^1\} \models' P^1\mathbf{e}Q^1$$

since:

$$\{P^1\mathbf{a}Q^1\} \not\models' P^1\mathbf{e}Q^1$$

$$\{P^1\mathbf{e}Q^1\} \not\models' P^1\mathbf{a}Q^1$$

due to the fact that in $\mathbf{M}_X$ there exist models in which the denotation of letters P^1 is a non-empty set. In the event when that set is contained in the set of denotations Q^1 proposition $P^1\mathbf{a}Q^1$ is true, while proposition $P^1\mathbf{e}Q^1$ is not true. Instead, in the event when the set of denotations of name letter P^1 is disjoint with the set of denotations of letter Q^1, proposition $P^1\mathbf{e}Q^1$ is true, while proposition $P^1\mathbf{a}Q^1$ is not true.

In turn, due to the fact that in $\mathbf{M}_X$ there exist models in which the denotation of letter P^1 is an empty set, set $X = \{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\}$ is not a contradictory set. Therefore, we are unable to establish function f in such a way that each model has a division of set X determined by function f. For there exists only one injective function $f : X \longrightarrow X$ such that for any $y \in X$ $f(y) \neq y$. It is defined as follows:

$$f(P^1\mathbf{a}Q^1) = P^1\mathbf{e}Q^1$$

$$f(P^1\mathbf{e}Q^1) = P^1\mathbf{a}Q^1.$$

Now, take any division Y of set X by function f. It meets condition: $A \in Y$ iff $f(A) \notin Y$, for any $A \in X$. If $P^1\mathbf{a}Q^1 \in Y$, then $f(P^1\mathbf{a}Q^1) \notin Y$, i.e. $P^1\mathbf{e}Q^1 \notin Y$, thus $P^1\mathbf{e}Q^1 \in X \setminus Y$. If, in turn, $P^1\mathbf{a}Q^1 \notin Y$, then $f(P^1\mathbf{a}Q^1) \in Y$, i.e. $P^1\mathbf{e}Q^1 \in Y$, thus $P^1\mathbf{e}Q^1 \notin X \setminus Y$. An analogous sequence of implications occurs for the other formula. Thus, for model in which set of formulas $X = \{P^1\mathbf{a}Q^1, P^1\mathbf{e}Q^1\}$ is true, there does not exist any corresponding division of set X by function f. So, for such relation of consequence like $\models'$, we will not construct a tableau system using the provided method[2].

On the other hand, take such subset Y of set of all formulas $\mathsf{For}_{\mathbf{TL}}$, that $Y = \{P^1\mathbf{a}Q^1, P^1\mathbf{o}Q^1\}$, and set of all models $\mathbf{M}_Y$ (which is identical to set $\mathbf{M}_X$, as we still have the same letters in the alphabet), and then define in a usual manner the relation of semantic consequence on set $P(Y) \times Y$ (in a sense truncating the relation of semantic consequence of **TL** to set Y) and denote it as $\models''$. In such logic, there occur analogous implications like for relation $\models'$. However, in $\mathbf{M}_Y$ there does not exist model such that set $Y = \{P^1\mathbf{a}Q^1, P^1\mathbf{o}Q^1\}$ is true in it, because formulas $P^1\mathbf{a}Q^1$, $P^1\mathbf{o}Q^1$ are contradictory. We establish function $f : Y \longrightarrow Y$ as follows: $f(P^1\mathbf{a}Q^1) = P^1\mathbf{o}Q^1$ and $f(P^1\mathbf{o}Q^1) = P^1\mathbf{a}Q^1$. We only get two divisions of set Y by function f, $Y' = \{P^1\mathbf{a}Q^1\}$ and $Y'' = \{P^1\mathbf{o}Q^1\}$ [3].

Note that for each model $\mathfrak{M}_Y$ there exists division Z of set Y by function f such that $\mathfrak{M}'_Y = \{A \in Y : \mathfrak{M}_Y \models'' A\} = Z$. On the other hand, for each division Z of set Y by function f there exists model $\mathfrak{M}_Y$ such that $\mathfrak{M}'_Y = \{A \in Y : \mathfrak{M}_Y \models'' A\} = Z$.

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2 Perhaps for relation $\models'$ it is impossible at all to define a tableau system in a standard way, as it is not the case that for each formula there exists a contradictory formula. We can, however, expand the provided method, requiring the adoption at the beginning of proof of suitable auxiliary expressions instead of the contradictory formula, when it does not exist. In the discussed case, for instance for formula $P^1\mathbf{a}Q^1$ it would be set $\{P^1_{+i}, Q^1_{-i}\}$, for some $i \in \mathbb{N}$.

3 The example shown is interesting insomuch that in the tableau system for it, the set of tableau rules could be empty. For each branch and tableau would be one-step, because having adopted a set of premises in the very first step, we would receive a t-inconsistent set, with the natural assumption that $\{A, f(A)\}$ is a t-inconsistent set.

For the reasons we mentioned, in further semantic considerations we will refer to some established set $\mathbf{I} \subseteq \mathbf{X}_f$. So, let us proceed to the definition.

Definition 5.8 (Interpretation of formulas)**.** *A set of interpretations determined by function f* (for short: *interpretations*) is each subset $\mathbf{I} \subseteq \mathbf{X}_f$. The elements of set $\mathbf{I}$ will be called *interpretations of formulas* (or for short: *interpretations*) and denoted by letter $\mathfrak{I}$, possibly with indices.

Denotation 5.9. Let $\mathbf{I} \subseteq \mathbf{X}_f$. Let $\mathfrak{I} \in \mathbf{I}$ be any interpretation of formulas. Let us adopt the following denotations:

- for any formula $A \in \mathsf{For}$, the fact that $A \in \mathfrak{I}$ will be put as $\mathfrak{I} \models A$, whereas the fact that $A \notin \mathfrak{I}$ will be put as $\mathfrak{I} \not\models A$
- for any subset of formulas $X \subseteq \mathsf{For}$, $\mathfrak{I} \models X$ iff for any formula $A \in X$, $\mathfrak{I} \models A$; while $\mathfrak{I} \not\models X$ iff it is not the case that $\mathfrak{I} \models X$.

We will now proceed to the concept of relation of semantic consequence.

Definition 5.10 (Semantic consequence)**.** Let $\mathbf{I} \subseteq \mathbf{X}_f$ be a set of interpretations. Let $X \subseteq \mathsf{For}$ and $A \in \mathsf{For}$.

- Formula *A follows from X under* $\mathbf{I}$ (for short: $X \models_{\mathbf{I}} A$) iff for any interpretation $\mathfrak{I} \in \mathbf{I}$, if $\mathfrak{I} \models X$, then $\mathfrak{I} \models A$. Whereas formula *A does not follow from X under* $\mathbf{I}$ (for short: $X \not\models_{\mathbf{I}} A$) iff it is not the case that $X \models_{\mathbf{I}} A$.
- When set $\mathbf{I}$ is fixed, we apply notation $X \models A$ and respectively $X \not\models A$.
- Relation $\models$ will be called a *relation of semantic consequence* (*defined by set of interpretations* $\mathbf{I}$).

With an established relation $\models$ we can proceed to the concept of semantically defined logic.

Definition 5.11. Let set $\mathbf{I} \subseteq \mathbf{X}_f$ be a set of interpretations of formulas For. Pair $\langle \mathsf{For}, \models_{\mathbf{I}} \rangle$ will be called a *semantically defined logic*.

Another fact says that two relations of consequence defined on set of formulas For are identical iff they are defined with the same set of interpretations $\mathbf{I} \subseteq \mathbf{X}_f$, so when speaking of relation $\models$, we do not have to refer to set $\mathbf{I}$, as it is unambiguously determined by relation $\models$.

Proposition 5.12. *Let $\mathbf{I}_1 \subseteq \mathbf{X}_f$ and $\mathbf{I}_2 \subseteq \mathbf{X}_f$ be sets of interpretations of formulas* For. *Let pairs $\langle \mathsf{For}, \models_{\mathbf{I}_1} \rangle$ and $\langle \mathsf{For}, \models_{\mathbf{I}_2} \rangle$ be semantically defined logics. In such case $\models_{\mathbf{I}_1} = \models_{\mathbf{I}_2}$ iff $\mathbf{I}_1 = \mathbf{I}_2$.*

Proof. Take any sets of interpretations of formulas $\mathbf{I}_1 \subseteq \mathbf{X}_f$, $\mathbf{I}_2 \subseteq \mathbf{X}_f$, and logics $\langle \mathsf{For}, \models_{\mathbf{I}_1} \rangle$ and $\langle \mathsf{For}, \models_{\mathbf{I}_2} \rangle$ they semantically defined. Assume that $\mathbf{I}_1 \neq \mathbf{I}_2$. We have two cases:

(1) there exists such interpretation $\mathfrak{I} \in \mathbf{I}_1$ that $\mathfrak{I} \notin \mathbf{I}_2$

or

(2) there exists such interpretation $\mathfrak{I} \in \mathbf{I}_2$ that $\mathfrak{I} \notin \mathbf{I}_1$.

Let us only consider case (1) as case (2) is analogous.

Now, take any such interpretation $\mathfrak{I} \in \mathbf{I}_1$ that $\mathfrak{I} \notin \mathbf{I}_2$.

Assume that $\mathbf{I}_2$ is an empty set. By definition of relation of semantic consequence 5.10, $\models_{\mathbf{I}_2} = P(\mathsf{For}) \times \mathsf{For}$. On the other hand, since $\mathfrak{I} \in \mathbf{I}_1$, so by definition 5.3, for certain formula A, $\mathfrak{I} \models A$ and $\mathfrak{I} \not\models f(A)$, so by definition of relation of semantic consequence 5.10, $\{A\} \not\models_{\mathbf{I}_1} f(A)$, thus $\models_{\mathbf{I}_1} \neq \models_{\mathbf{I}_2}$.

$(*)$ Now, assume that $\mathbf{I}_2$ is not an empty set.

Since $\mathfrak{I} \notin \mathbf{I}_2$, then by definition 5.3 for each interpretation $\mathfrak{I}' \in \mathbf{I}_2$ there exists such formula $B \in \mathsf{For}$ that:

(a) $\mathfrak{I}' \models B$ and $\mathfrak{I} \not\models B$

or

(b) $\mathfrak{I}' \not\models B$ and $\mathfrak{I} \models B$.

Take any interpretation $\mathfrak{I}' \in \mathbf{I}_2$ and such formula $B \in \mathsf{For}$ that there occurs case (a) $\mathfrak{I}' \models B$ and $\mathfrak{I} \not\models B$. It is the case when and only when — by definition 5.3 — (a)' $\mathfrak{I}' \not\models f(B)$ and $\mathfrak{I} \models f(B)$, thus there exists such formula $C \in \mathsf{For}$ that $f(B) = C$ and (a)” $\mathfrak{I}' \not\models C$ and $\mathfrak{I} \models C$.

Now, we define set of formulas $X \subseteq \mathsf{For}$ as follows: for any formula A, $A \in X$ iff there exists such interpretation $\mathfrak{I}' \in \mathbf{I}_2$ that $\mathfrak{I}' \not\models A$ and $\mathfrak{I} \models A$.

From $(*)$, (a)”, (b) it follows that set X is non-empty and for each interpretation $\mathfrak{I}' \in \mathbf{I}_2$, there exists such formula $D \in X$ that $\mathfrak{I}' \not\models D$. What is more, for each formula $D \in X$, $\mathfrak{I} \models D$, so $\mathfrak{I} \models X$.

Therefore, there does not exist such interpretation $\mathfrak{I}' \in \mathbf{I}_2$ that $\mathfrak{I}' \models X$. From the above and from definition of relation of semantic consequence 5.10, we get that $X \models_{\mathbf{I}_2} D$, for any formula $D \in \mathsf{For}$, since $\mathfrak{I}' \not\models X$, for any interpretation of formulas $\mathfrak{I}' \in \mathbf{I}_2$.

While since $\mathfrak{I} \models X$, so $\mathfrak{I} \not\models f(E)$, for each $E \in X$, by definition 5.3. From the above and from definition of semantic consequence 5.10, $X \not\models_{\mathbf{I}_1} f(E)$, for certain $E \in X$. And at the same time $X \models_{\mathbf{I}_2} f(E)$, since $X \models_{\mathbf{I}_2} D$, for any formula $D \in \mathsf{For}$, thus $\models_{\mathbf{I}_1} \neq \models_{\mathbf{I}_2}$

Now, assume that $\mathbf{I}_1 = \mathbf{I}_2$. Then obviously $\models_{\mathbf{I}_1} = \models_{\mathbf{I}_2}$, since $\models_{\mathbf{I}_1} = \models_{\mathbf{I}_1}$. □

Remark 5.13. For our considerations, we adopt any, but fixed set of interpretations $\mathbf{I} \subseteq \mathbf{X}_f$. The set will remain unchanged until the end of this chapter. We will, on the other hand, refer to function f.

Remark 5.14. For our considerations, we adopt any, but fixed, semantically defined logic $\langle \mathsf{For}, \models \rangle$. Due to fact 5.12, since relation of consequence $\models$ determines exactly one set of interpretations of formulas $\mathbf{I} \subseteq \mathbf{X}_f$, we do not have to include a reference to set $\mathbf{I}$ in the notation. Logic $\langle \mathsf{For}, \models \rangle$ will remain unchanged until the end of this chapter. On the other hand, we will from time to time refer to the set of interpretations of formulas $\mathbf{I}$ determined by $\models$.

5.3 Basic concepts of the tableau system

Now, we will define the set of conditions that should be satisfied by the tableau expressions.

Definition 5.15 (Tableau expressions)**.** *The set of tableau expressions* will be called any set Te that meets the following conditions:

1. there exists such injective function $g : \mathsf{For} \longrightarrow P(\mathsf{Te})$ that for any formula $A \in \mathsf{For}$, $g(A)$ is a countable subset of set Te and for any formula $B \in \mathsf{For}$, if $A \neq B$, then $g(A) \cap g(B) = \varnothing$
2. there exists at least one distinguished and finite set $\mathsf{Te}' \subseteq \mathsf{Te}$ such that for any subset $X \subseteq \mathsf{For}$, if there exists interpretation of formulas $\mathfrak{I} \in \mathbf{I}$ such that $\mathfrak{I} \models X$, then $\mathsf{Te}' \nsubseteq \bigcup\{g(A) : A \in X\}$ (the set of all such distinguished sets will be specified as Te^{in}).

The elements of set Te will be called *tableau expressions* or simply *expressions*.

Making use of condition 2 of definition of set of tableau expressions 5.15, we will now introduce the general concept of t-inconsistent set (and concept of t-consistent set).

Definition 5.16. Let Te be a set of tableau expressions and let $\mathsf{Te}'' \subseteq \mathsf{Te}$. Set Te'' will be called *tableau inconsistent* (for short: *t-inconsistent*) iff there exists such $\mathsf{Te}' \in \mathsf{Te}^{in}$ that $\mathsf{Te}' \subseteq \mathsf{Te}''$. Set Te'' will be called *tableau consistent* (for short: *t-consistent*) iff Te'' is not t-inconsistent.

Remark 5.17. For any set of expressions Te we assume that the values of function $g : \mathsf{For} \longrightarrow P(\mathsf{Te})$ are sets of indexed elements of set Te, i.e. for any formula $A \in \mathsf{For}$, there exists such countable set $\{x^1, x^2, x^3, \dots\} \subseteq \mathsf{Te}$ that $g(A) = \{x^1, x^2, x^3, \dots\}$. At the same time, we do not assume that for any x^i, $x^j \in g(A)$, if $i \neq j$, then $x^i \neq x^j$. The numbers visible in the indices of tableau expressions $x^1, x^2, x^3, \dots$ will be called *indices*.

Remark 5.18. As we remember, in the case for **CPL** considered in the book, the set of expressions was identical to the set of formulas. In this case, we would adopt

$\mathsf{Te} = \{A^i : A \in \mathsf{For}_{\mathbf{CPL}}, i \in \mathbb{N}\}$[4]. In a more complex case, i.e. **TL**, set $\{A^i : A \in \mathsf{For}_{\mathbf{TL}}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$. In both those cases, we associate a little artificially with each formula an infinite set of expressions that correspond to it, whereas for each formula A, if $g(A) = \{x^1, x^2, x^3, \dots\}$, then for any indices i, j we would have: $x^i = x^j$.

In the case of **S5** such association is more natural, as the elements of set along with indices $\{A^i : A \in \mathsf{For}_{\mathbf{S5}}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$ would correspond to the ordered couples $\langle A, i\rangle$ we use in the proofs.

In both latter cases (**TL**, **S5**), in order to obtain entire set Te we would have to still define additional auxiliary expressions. Our definition does not specify what these expressions would be, but in order to get a complete tableau system, they would have to meet further conditions that we will provide. This reservation also applies to the concept of an inconsistent set of expressions, which in some cases could even contain one element. This is the case in those systems where the branch containing certain expressions (considered to be inconsistent) is closed with a special sign by means of an additional rule or rules, e.g. × (refer for example [23]) and only then the branch is considered closed.

Ultimately, however, the role of these expressions which, through function g represent formulas in a tableau proof, could be fulfilled by any other symbols, not structurally (graphically) related to formulas, yet meeting the conditions from definition of set of tableau expressions 5.15.

New denotations will be useful for further work.

Denotation 5.19. Let us adopt the following denotations:

- for any formula $A \in \mathsf{For}$ and any $i \in \mathbb{N}$, $A^i = x^i$ iff $x^i \in g(A)$
- for any subset $X \subseteq \mathsf{For}$, $X^i = \{A^i : A \in X\}$.

Remark 5.20. For our considerations, we adopt any, but fixed set of tableau expressions Te and included in Te at least one tableau contradictory set Te'. These arrangements will remain unchanged until the end of this chapter.

Remark 5.21. We also assume the option of inclusion in set Te of auxiliary expressions which correspond to expressions such as irj in case of the described tableau system for **S5**, or such as expressions P_{-i} and P_{+i} in case of the described tableau system for **TL**, for any $i, j \in \mathbb{N}$ and any name letter $P \in \mathsf{Ln}$. The auxiliary expressions also feature indices.

Such expressions will be specified by means of set TeA, subset of Cartesian product $\{\alpha_1, \alpha_2, \alpha_3, \dots\} \times \{\beta_1, \beta_2, \beta_3, \dots\}$, where for any $i, j \in \mathbb{N}$, if $\langle \alpha_i, \beta_j, \rangle \in \mathsf{TeA}$,

4 In Chapter Two, we adopted the simplest variant. We could have, however, adopted the set of tableau expressions for **CPL** different from the set of formulas.

then β_j is an ordered n-tuple $\langle k_1, k_2, \ldots, k_n \rangle$ of indices present in that sequence in expression α_i, for some $n, k_1, k_2, \ldots, k_n \in \mathbb{N}$.

Set $\mathsf{TeA} \subseteq \mathsf{Te}$, moreover TeA may be empty, because the construction of tableau proof for a given logic may not require at all a richer set than what is required by definition 5.15, or its definition uses a different set of auxiliary expressions than TeA.

Definition 5.22 (Function selecting indices). *Function selecting indices* will be called function $* : P(\mathsf{Te}) \longrightarrow P(\mathbb{N})$ defined for any $i, j \in \mathbb{N}$ with conditions:

- for any formula $A \in \mathsf{For}$, $*(\{A^i\}) = \{i\}$
- for any $\langle \alpha_i, \beta_j \rangle \in \mathsf{TeA}$, $*(\{\langle \alpha_i, \beta_j \rangle\}) = \{k : k$ is an element of $\beta_j\}$
- for any $X \subseteq \mathsf{Te}$, if $|X| > 1$, then $*(X) = \bigcup \{*(\{y\}) : y \in X\}$.

Therefore, for any subset of set $\{A^i : A \in \mathsf{For}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$ function $*$ selects all indices present in the expressions from that set.

We will now proceed to the general conditions that specify the relation of similarity between the sets of expressions, regardless of whether or not set TeA is non-empty and set Te contains any other specific auxiliary expressions. Therefore, in the definition of similarity we assume the condition of having the same cardinality.

Definition 5.23 (Similar set of expressions). Let X, $Y \subseteq \mathsf{Te}$. We shall state that X is *similar* to Y iff:

- X is t-consistent iff Y is t-consistent
- X and Y have the same cardinality
- there exists such bijection $h : *(X) \longrightarrow *(Y)$ that:
  - for any formula $A \in \mathsf{For}$ and $i \in \mathbb{N}$, $A^i \in X$ iff $A^{h(i)} \in Y$.
  - for any $i, n, k_1, k_2, \ldots, k_n \in \mathbb{N}$, there exist such $j \in \mathbb{N}$, $\langle \alpha_i, \beta_j \rangle \in \mathsf{TeA}$ that $\langle \alpha_i, \beta_j \rangle \in X$ and $\beta_j = \langle k_1, k_2, \ldots, k_n \rangle$ iff there exist such $l \in \mathbb{N}$, $\langle \alpha_i, \beta_l \rangle \in \mathsf{TeA}$ that $\langle \alpha_i, \beta_l \rangle \in Y$ and $\beta_l = \langle h(k_1), h(k_2), \ldots, h(k_n) \rangle$.

From definition of similarity 5.23 the following conclusion results.

Corollary 5.24. *Let $X, Y \subseteq \mathsf{Te}$. If X is similar to Y, then Y is similar to X.*

For further work, we need a definition of relation that occurs between interpretation of formulas $\mathfrak{I}$ and subset of tableau expressions $\{A^i : A \in \mathsf{For}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$.

Definition 5.25. Let $i \in \mathbb{N}$. By $\Vdash_i$ we mean any such relation specified on Cartesian product $\mathbf{I} \times \mathsf{For}$ that for any formula $A \in \mathsf{For}$ and any interpretation $\mathfrak{I} \in \mathbf{I}$:

- if $\mathfrak{I} \models A$, then $\mathfrak{I} \Vdash_i A$.
- $\mathfrak{I} \Vdash_i A$ iff it is not the case that $\mathfrak{I} \Vdash_i f(A)$ (for short: $\mathfrak{I} \nVdash_i A$).
- if for certain $j \in \mathbb{N}$, $\mathfrak{I} \Vdash_j A$, then $\mathfrak{I} \Vdash_i A$.

By $\Vdash$ we will mean set $\{\langle \mathfrak{I}, A\rangle : \mathfrak{I} \in \mathbf{I}, A \in \mathsf{For}$ and for some $i \in \mathbb{N}\ \mathfrak{I} \Vdash_i A\}$.

Remark 5.26. For our considerations, we adopt any, but fixed relation $\Vdash$. It will remain unchanged until the end of this chapter.

Remark 5.27. Intuitively, relation $\Vdash$ is in a sense expansion of relation $\models$ onto set of tableau expressions $\{A^i : A \in \mathsf{For}, i \in \mathbb{N}\} \subseteq \mathsf{Te}$. We used “in a sense”, because in the cases of **CPL** and **TL** we can identify it with relation $\models$, as for each formula A, of one of these logics, $A^i = A$, for any $i \in \mathbb{N}$. On the other hand, in the event of modal logic (e.g. **S5**), notation $\mathfrak{I} \Vdash_i A$ may indicate that if based on model $\mathfrak{M}$, that we identify with interpretation $\mathfrak{I}$, we construct model $\mathfrak{M}'$, where the distinguished world will be corresponded by index i, then $\mathfrak{M}' \models A$.

We will now specify the general conditions describing the interpretation appropriate for set of expressions.

Definition 5.28 (Interpretation appropriate for set of expressions)**.** Let $\mathfrak{I} \in \mathbf{I}$ be an interpretation, and let $X \subseteq \mathsf{Te}$ be a set of tableau expressions. We shall state that interpretation $\mathfrak{I}$ is *appropriate* for set X iff:

1. X is t-consistent
2. for any $i \in \mathbb{N}$ and any $A \in \mathsf{For}$, if $A^i \in X$, then $\mathfrak{I} \Vdash_i A$.

5.4 Tableau rules

Now, we will proceed to the conditions that should be satisfied by the tableau rules. So, let us define the general concept of rule.

Definition 5.29 (Rule)**.** Assume that $P(\mathsf{Te})$ is a set of all subsets of set Te. Let $P(\mathsf{Te})^n$ be n-element of Cartesian product $\underbrace{P(\mathsf{Te}) \times \cdots \times P(\mathsf{Te})}_{n}$, for some $n \in \mathbb{N}$, whereas $\bigcup_{n \in \mathbb{N}} P(\mathsf{Te})^n$ be a union of all n-ary products such that $n \geq 2$.

- *Rule* will be called any subset $R \subseteq \bigcup_{n \in \mathbb{N}} P(\mathsf{Te})^n$ such that if $\langle X_1, \ldots, X_n\rangle \in R$, then the following conditions are satisfied:
  - $X_1 \subset X_i$, for any $1 < i \leq n$
  - X_1 is a t-consistent set
  - if $k \neq l$, then $X_k \neq X_l$, for any $1 < k, l \leq n$
  - (Closure under similarity) for any set of expressions Y_1 such that Y_1 is similar to X_1, there exist such sets of expressions $Y_2, \ldots, Y_n$ that $\langle Y_1, \ldots, Y_n\rangle \in R$ and for any $1 < i \leq n$, Y_i is similar to X_i

- (Existence of core of rule) for some finite set $Y \subseteq X_1$ there exists ordered n-tuple $\langle Z_1, \dots, Z_n \rangle \in R$ such that:
  1. $Z_1 = Y$
  2. for any $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$
  3. there does not exist proper subset $U_1 \subset Y$ and n-tuple $\langle U_1, \dots, U_n \rangle \in R$
- (Closure under expansion) for any t-consistent set of expressions Z_1, such that $X_1 \subset Z_1$ and for each $1 < i \leq n$, X_i is not similar to any subset Z_1 containing X_1, there exist such set of expressions $Z_2, \dots, Z_n$ that $\langle Z_1, \dots, Z_n \rangle \in R$ and for any $1 < i \leq n$, X_i is similar to $X_1 \cup (Z_i \setminus Z_1)$
- (Closure under finite sets) if X_1 is a finite set, then for each $1 < i \leq n$, X_i is a finite set
- (Closure under subsets) for any subset $X' \subseteq X_1$, if for certain set $Y \subseteq X_1$ there exists such n-tuple $\langle W_1, \dots, W_n \rangle \in R$ that:
  1. $W_1 = Y$
  2. for any $1 < i \leq n$, $W_i = W_1 \cup (X_i \setminus X_1)$
  3. $W_1 \subseteq X'$

  then there exist such sets of expressions $Z_1, \dots, Z_n$ that:
  1. $\langle Z_1, \dots, Z_n \rangle \in R$
  2. $Z_1 = X'$
  3. for any $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$.

- We shall state that rule R has been *applied* to set X_1 iff for certain $1 < i \leq n$, exactly one set X_i was selected from certain n-tuple $\langle X_1, \dots, X_n \rangle \in R$.

Remark 5.30. Note that the general concept of rule we have introduced with definition 5.29, is in a way less general than the general concepts of rules used in the previous chapters — since we have added several additional conditions that were not present before. For we will not use a specific set of rules in further proofs, but we will define another conditions that should be jointly met by the set of tableau rules selected for axiomatization of the tableau system.

However, in some respects, definition 5.29 is more general than the definitions of rules in the previous chapters. In those cases, the rule was to be a subset of Cartesian product $R \subseteq P(\mathsf{Te})^n$, where $n \geq 2$. And here, the rule is defined as a subset of the union under Cartesian products $R \subseteq \bigcup_{n \in \mathbb{N}} P(\mathsf{Te})^n$, where $n \geq 2$. Of course, the rules that meet the first condition also meet the second one. The more general condition was provided for a number of reasons.

First of all, even in the case of structural definition of tableau rules — as we have done so far — there may be a situation in which ordered pairs of different numbers of elements belong to a single rule. An example is a logic with relating connectives, in which we sometimes have to describe relationships between propositional letters when distributing expressions in tableaux. In this case, the number of elements in a given n-tuple may depend on the number of propositional letters in a given formula.[5]

Secondly, even if we define the tableau rules structurally, it is also possible to define them for some system in a more complex way. We could, for example, sum up some (or even all) of the rules described in the chapter on the tableau system for **CPL**, to get for instance rule $R' = R_{\neg\neg} \cup R_{\vee}$. That rule would consist of two types of n-tuples: ordered pairs and ordered triples.

Finally, we can build tableau systems for non-classical reasoning (inferences), in which we draw conclusions in a non-deductive way. In such systems, there may occur a need for unstructured definition of tableau rules, e.g. in such a way that from a given structurally described set of premises it is possible to move in the tableau to one or more branches, depending on which subformulas the premises are composed of. Excluding certain conditions from definition 5.29, that definition may be useful in such cases.

We will now frame a general definition of the core of rule in a given set.

Definition 5.31 (Core of rule). Let R be a rule and $n \in \mathbb{N}$. Let $\langle X_1, \ldots, X_n \rangle \in R$ and $\langle Z_1, \ldots, Z_n \rangle \in R$. We shall state that $\langle Z_1, \ldots, Z_n \rangle \in R$ is a *core of rule R in set* $\langle X_1, \ldots, X_n \rangle$ iff

1. $Z_1 \subseteq X_1$
2. there do not exist proper subset $U_1 \subset Z_1$ and n-tuple $\langle U_1, \ldots, U_n \rangle$ such that $\langle U_1, \ldots, U_n \rangle \in R$ and $U_i \setminus U_1 = Z_i \setminus Z_1$, for any $1 < i \leq n$
3. for any $1 < i \leq n$, $Z_i = Z_1 \cup (X_i \setminus X_1)$.

By definition of rule 5.29 (Existence of core of rule), we get the following conclusion.

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5 Although the book does not directly consider logics with relating connectives, the results of this chapter also apply to the tableau systems for such logics. An introduction to relating logics was presented inter alia in [12], [7] and [11]. The simplest example of a tableau system of logic with relating connectives is described in paper [10]. In turn, the issue of rules containing various n-tuples appears in those logics with relating connectives for which the semantic structure is constrained by various conditions motivated by philosophically oriented interpretation of connectives. Such an approach to tableau methods for logics with relating connectives in the context of causality can be found in [11] as well as connexivity in [12], [13]. It is worth noting that in the case of [13] in a sense relating logics were combined with modal ones and two tableau approaches were joined.

Corollary 5.32. *Let R be a rule and $n \in \mathbb{N}$. If n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, then there exists n-tuple $\langle Y_1,\ldots,Y_n \rangle \in R$ such that $\langle Y_1,\ldots,Y_n \rangle$ is the core of rule R in set $\langle X_1,\ldots,X_n \rangle$.*

We will now define two additional technical concepts that will allow us to frame a general definition of a set of tableau rules.

Let $X \subseteq \mathsf{Te}$ be a set of tableau expressions and let $\mathbf{R}$ be a set of rules. By $\mathbf{R}_X$ we will mean a set of all and only such rules from set $\mathbf{R}$ that are applicable to set X. Formally, $R \in \mathbf{R}_X$ iff $R \in \mathbf{R}$ and there exists such n-tuple $\langle Y_1,\ldots,Y_n \rangle \in R$ that $Y_1 = X$.

Let $R \in \mathbf{R}_X$, by R_X we will mean a set of all and only such n-tuples from R that their first element equals X, and if some of the remaining elements of two n-tuples that belong to R_X differ, then these two n-tuples have different input sets of the core of rule. Formally, for any $n \in \mathbb{N}$, $\langle Y_1,\ldots,Y_n \rangle \in R_X$ iff:

- $\langle Y_1,\ldots,Y_n \rangle \in R$ and $Y_1 = X$
- for any set of expressions $Z_1, \ldots, Z_n, Z'_1, \ldots, Z'_n, Y'_1, \ldots, Y'_2 \subseteq \mathsf{Te}$, if:
  - $\langle Z_1,\ldots,Z_n \rangle \in R_X$
  - $\langle Y_1,\ldots,Y_n \rangle \neq \langle Z_1,\ldots,Z_n \rangle$
  - $\langle Y'_1,\ldots,Y'_2 \rangle$ is the core of rule R in set $\langle Y_1,\ldots,Y_n \rangle$
  - $\langle Z'_1,\ldots,Z'_n \rangle$ is the core of rule R in set $\langle Z_1,\ldots,Z_n \rangle$

  then $Y'_1 \neq Z'_1$.

The limitations in the definition of set R_X are the results of the fact that some rules, e.g. $R_\Diamond$ in the described tableau system for **S5**, may introduce completely new expressions that are absent in any form in the previous portions of the proof.

In the case of rule $R_\Diamond$, we can introduce to the branch some set $\{\langle A,i \rangle, irj\}$, where $A \in \mathsf{For}_{\mathbf{S5}}$ and $i,j \in \mathbb{N}$, selected from among many such sets. Usually, there exist multiple such n-tuples that belong to $R_\Diamond$ which are applicable to set X, if $\langle \Diamond A, i \rangle \in X$, after even one application we cannot apply rule $R_\Diamond$ to set X anymore due to expression $\langle \Diamond A, i \rangle \in X$, because of the limitations in the application of this rule (see example 4.24). So, we want set R_X to include only one such n-tuple, since we can only use one.

With the above concepts, we can proceed to the definition of set of tableau rules.

Definition 5.33 (Tableau rules)**.** Let $\mathbf{R}$ be a set of rules. We shall state that $\mathbf{R}$ is a *set of tableau rules* iff

1. $\mathbf{R}$ is a finite set
2. for any $X \subseteq \mathsf{Te}$, if X is a finite set, then for any rule $R \in \mathbf{R}_X$, each set R_X is a finite set.

So, set of tableau rules **R** must include a finite number of rules, and what is more, for any finite set of expressions, each of rules R that belong to **R** can be applied a finite number of times — taking account of the set of n-tuples that belong to R_X.

Remark 5.34. We adopt any, but fixed set of tableau rules **R**. That set will remain unchanged until the end of this chapter. It is worth noting again that all further tableau concepts: branches and tableaux of different types will depend on set **R**.

5.4.1 Branches

Conventionally, another concept in our theory that will be discussed is the concept of branch. It is a concept that depends on the notion of the tableau rule because branches are created by applying rules. Branches — as mentioned before — are setwise objects consisting of sets. The below definition corresponds to all the definitions of branch used so far, only that it depends on the general notion of set of tableau rules **R**.

Definition 5.35 (Branch)**.** Let $K = \mathbb{N}$ or $K = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Let X be any set of expressions. *A branch* (or *a branch beginning with* X) will be called any sequence $\phi : K \longrightarrow P(\mathsf{Te})$ that meets the following conditions:

1. $\phi(1) = X$
2. for any $i \in K$: if $i+1 \in K$, then there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ that $\phi(i) = Y_1$ and $\phi(i+1) = Y_k$, for some $1 < k \leq n$.

Having two branches ϕ, ψ such that $\phi \subset \psi$ we shall state that:

- ϕ is a sub-branch of ψ
- ψ is a super-branch of ϕ.

Denotation 5.36. From now on — when speaking of the branches constructed by the application of rules from set **R** — for the sake of convenience, we will use the following notations or denotations:

1. $X_1, \ldots, X_n$, where $n \geq 1$
2. $\langle X_1, \ldots, X_n \rangle$, where $n \geq 1$
3. abbreviations: ϕ_M (where M is a domain ϕ, i.e. $\phi : M \longrightarrow P(\mathsf{Te})$)
4. or — to denote branches — small Greek letters: ϕ, ψ, etc.

The sets of branches, in turn, we shall denote with capital Greek letters: Φ, Ψ, etc. Furthermore, the domain cardinality of a given branch K we shall sometimes call a *length* of that branch.

All the so far considered branches have been specific cases of the above concept, assuming an appropriate set of rules **R**.

Let us now introduce the general definition of addition of branches.

Definition 5.37 (Addition of branches)**.** Let $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te})$ and $\psi : M \longrightarrow P(\mathsf{Te})$ be branches, for some $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$, and let $\phi(n) = \psi(1)$. The results of the operation $\phi \oplus \psi$ is function $\varphi : K \longrightarrow P(\mathsf{Te})$ defined as follows:

1. if $M = \mathbb{N}$, then $K = \mathbb{N}$
2. if $|M| \in \mathbb{N}$, then $K = \{1, \ldots, n, n+1, n+2, \ldots, n+|M|-1\}$
3. for each $i \in K$
   a. if $1 \leq i \leq n$, $\varphi(i) = \phi(i)$
   b. if $i > n$, then $\varphi(i) = \psi((i-n)+1)$.

From definition of branch 5.35 and definition of addition of branches 5.37, follows an analogous conclusion as in the case of the tableau system for logic **S5**.

Corollary 5.38. *Let $\phi : \{1, \ldots, n\} \longrightarrow P(\mathsf{Te})$ and $\psi : M \longrightarrow P(\mathsf{Te})$ be branches, for some $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$, and let $\phi(n) = \psi(1)$. Then $\phi \oplus \psi$ is also a branch.*

5.4.2 Closed and open branches

An important classification of branches is the division into closed and open branches. A branch is closed when, applying the rules in subsequent steps, we have reached a t-inconsistent set. Below, we present the definition which is directly based on set of tableau rules **R**, as it refers to definition of branch 5.35.

Definition 5.39 (Closed/open branch)**.** Branch $\phi : K \longrightarrow P(\mathsf{Te})$ will be called *closed* iff $\phi(i)$ is a t-inconsistent set for some $i \in K$. A branch will be called *open* iff it is not closed.

From the above definition 5.39, definition of tableau rules 5.33 and definition of branch 5.35, the following conclusion results.

Corollary 5.40. *If branch $\phi : K \longrightarrow P(\mathsf{Te})$ is closed, then $|K| \in \mathbb{N}$.*

Again, in the case of a closed branch, the t-inconsistent sequence element is the last element and no rule can be applied to it anymore to extend the branch. For the tableau rules have been defined in such a way that they cannot be applied to t-inconsistent sets.

5.4.3 Maximal branches

One more important concept in the construction of a tableau system is the concept of a maximal branch. The definition of maximal branch is based on the concept of strong similarity. As we already know from the previous chapter, the concept of strong similarity of sets of expressions is a special case of the similarity of sets. Below, we provide its version that is generalized to the context of rules from set **R**.

Definition 5.41 (Strong similarity)**.** Let rule $R \in \mathbf{R}$ and let $\langle X_1, \ldots, X_n \rangle \in R$, for some $n \in \mathbb{N}$. On any set of expressions $W \subseteq \mathsf{Te}$ we will state that it is *strongly similar to* set X_i, where $1 < i \leq n$, iff

1. W is similar to X_i
2. for certain n-tuple $\langle Y_1, \ldots, Y_n \rangle$, which is the core of rule R in set $\langle X_1, \ldots, X_n \rangle$, the following conditions are satisfied:
   a. for certain $W' \subseteq W$, $Y_1 \subseteq W'$
   b. W' is similar to $Y_1 \cup (X_i \setminus X_1)$.

Having adopted the concept of strong similarity, we can proceed to the concept of maximal branch in the general version, also referred to set of tableau rules **R**.

Definition 5.42 (Maximal branch)**.** Let $\phi : K \longrightarrow P(\mathsf{Te})$ be a branch. We shall state that ϕ is *maximal* iff it meets one of the below conditions:

1. ϕ is closed
2. for any rule $R \in \mathbf{R}$, any $n \in \mathbb{N}$ and any n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, if $\phi(k) = X_1$, for certain $k \in K$, then for some $j \in K$, there exist $\phi(j)$ and such set of expressions $W \subseteq \mathsf{Te}$ that for some $1 < i \leq n$, W is strongly similar to X_i and $W \subseteq \phi(j)$.

Remark 5.43. We will repeat here the remark from the previous chapter. According to the above definition, a maximal branch is closed or, in a sense, closed under effect of rules (both conditions do not necessarily have to be mutually exclusive). Closure under rules means that if a branch is not closed and it was possible to apply some rule to one of its elements, then some of the branch elements includes a set strongly similar to the one that could have been a result of application of that rule. Set W is to be contained in one of the elements of branch $\phi(j)$, and not necessarily be identical to it, since we applied the rule to set $\phi(j-1)$, which can be a proper superset of set X, and consequently we obtained more expressions, and what is more, those elements could have been obtained as a result of application of another rule.

Maximal branches as defined by the definition 5.42 can be either finite or infinite. There occurs an analogous case here as in the considerations on logic **S5**: if the branch is finite and there does not exist super-branch, then it is also a maximal branch.

Corollary 5.44. *If branch ϕ is finite in length, and there does not exist such branch ψ that $\phi \subset \psi$, then ϕ is a maximal branch.*

Proof. Take any branch ϕ which is finite and there does not exist such branch ψ that $\phi \subset \psi$. Now, assume that ϕ is not closed. If it does not meet the second conditions of definition 5.44, then since ϕ is finite, so there exists branch ψ such that $\phi \subset \psi$, which obviously contradicts the assumption. □

So we have a general definition of maximal branch, which includes both finite cases — especially in systems that feature the finite branch property, and infinite cases, such as those that occur, for example, in modal logics.

The described concepts, therefore, apply to systems in which, by building a tableau proof using tableau tools and looking for maximal branches, we may be dealing with infinite branches. This is the case when a branch cannot be closed or certain rule application sequences are repeated.

From definitions 5.39 and 5.42 we get the following conclusion again.

Corollary 5.45. *Each closed branch is maximal.*

5.4.4 Branch consequence relation

We will now move on to the general concept of branch consequence which we will define using the following concepts: branch, maximal branch, closed branch, and denotation 5.19.

Definition 5.46 (Branch consequence)**.** Let $X \subseteq \mathsf{For}$ and $A \in \mathsf{For}$. We shall state that A is *branch consequence of* X (for short: $X \rhd A$) iff there exists such finite set $Y \subseteq X$ and such index $i \in \mathbb{N}$ that each maximal branch beginning with set $Y^i \cup \{f(A)^i\}$ is closed. By $X \not\rhd A$, we mean that A is not branch consequence X.

The general concept of branch consequence relation corresponds to the so far defined concepts of branch consequence relation, taking account of remark 5.18. This remark is valid for all tableau concepts defined hereafter. So, when constructing a branch or tableau for some set of formulas $X \cup \{A\}$, we begin with set $X^i \cup \{f(A)^i\}$, for some index $i \in \mathbb{N}$.

5.5 Tableaux

In this subchapter, we will move on to the general definition of tableau and various variants of tableaux. However, we will start with an auxiliary concept of maximality in set of branches which we already have used in the previous chapters.

Definition 5.47 (Maximal branch in the set of branches)**.** Let Φ be a set of branches and let branch $\psi \in \Phi$. We shall state that ψ is *maximal in set* Φ (for short: Φ–*maximal*) iff there is no such branch $\phi \in \Phi$ that $\psi \subset \phi$.

We can now move on to the general concept of tableau.

Definition 5.48 (Tableau)**.** Let $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$ a Φ be a set of branches. Ordered triple $\langle X, A, \Phi \rangle$ will be called *a tableau for* $\langle X, A \rangle$ (or for short: *tableau*) iff the below conditions are met:

1. Φ is a non-empty subset of set of branches beginning with set $X^i \cup \{f(A)^i\}$, for some index $i \in \mathbb{N}$ (i.e. if $\psi \in \Phi$, then $\psi(1) = X^i \cup \{f(A)^i\}$)
2. each branch contained in Φ is Φ-maximal
3. for any $n, i \in \mathbb{N}$ and any branches $\psi_1, \ldots, \psi_n \in \Phi$, if:
   - i and $i+1$ belong to domains of functions $\psi_1, \ldots, \psi_n$
   - for any $1 < k \leq n$ and any $o \leq i$, $\psi_1(o) = \psi_k(o)$

   then there exists such rule $R \in \mathbf{R}$ and such ordered m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $1 < m$, that for any $1 \leq k \leq n$:
   - $\psi_k(i) = Y_1$
   - and there exists such $1 < l \leq m$ that $\psi_k(i+1) = Y_l$.

The above concept of tableau covers all notions of tableau considered so far in a book, with a properly defined set of tableau rules $\mathbf{R}$.

When considering the tableaux in general, we can also generalize the concept of redundant branch which is useful for the definition of complete tableau.

Definition 5.49 (Redundant variant of branch)**.** Let ϕ and ψ be such branches that for some numbers i and $i+1$ that belong to their domains, it is the case that for any $j \leq i$, $\phi(j) = \psi(j)$, but $\phi(i+1) \neq \psi(i+1)$. We shall state that branch ψ is a *redundant variant* of branch ϕ iff:

- there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ that $\phi(i) = X_1$ and $\phi(i+1) = X_j$, for certain $1 < j \leq n$
- there exists rule $R \in \mathbf{R}$ and such m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R$, where $m > n$, that $X_1 = \psi(i) = Y_1$ and:
  1. $\psi(i+1) = Y_k$, for certain $1 < k \leq m$
  2. for any $1 < l \leq n$ there exists such $1 < o \leq m$ that $o \neq k$ and $X_l = Y_o$.

Let Φ, Ψ be sets of branches and $\Phi \subset \Psi$. We shall state that Ψ is a *redundant superset* Φ iff for any branch $\psi \in \Psi \setminus \Phi$ there exists such branch $\phi \in \Phi$ that ψ is a redundant variant of ϕ.

Making use of the general concept of redundant superset of branches we can define the general concept of complete tableau.

Definition 5.50 (Complete tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *complete* iff:

1. each branch contained in Φ is maximal
2. any set of branches Ψ such that:
   a. $\Phi \subset \Psi$
   b. $\langle X, A, \Psi \rangle$ is a tableau

   is a redundant superset of Φ.

A tableau is *incomplete* iff the tableau is not complete.

Now we can define the general concept of closed and open tableaux.

Definition 5.51 (Closed/open tableau)**.** Let $\langle X, A, \Phi \rangle$ be a tableau. We shall state that $\langle X, A, \Phi \rangle$ is *closed* iff the below conditions are met:

1. $\langle X, A, \Phi \rangle$ is a complete tableau
2. each branch contained in Φ is closed.

A tableau is *open* iff the tableau is not closed.

By virtue of the above definitions of closed tableau and complete tableau, we get a conclusion which makes up a generalization of the analogous conclusions from the preceding chapters.

Corollary 5.52. *Each closed tableau is a complete tableau.*

5.6 Completeness theorem

In this chapter, we will define several general concepts and establish facts that will allow us to prove a claim from which we can deduce the theorem on completeness for the tableau system that meets the conditions given below.

In the first place, we will address the concepts used to demonstrate the relationship between relation $\models$ and $\triangleright$. We will begin with the definition of branch generating interpretation.

Definition 5.53 (Branch generating interpretation)**.** Let Φ be a set of all open and maximal branches that contain some tableau equivalents of formulas and let $\mathfrak{I}$ be an interpretation of formulas. We shall state that branch $\phi \in \Phi$ *generates interpretation* $\mathfrak{I}$ iff there exists such function $\succcurlyeq : \Phi \longrightarrow \mathbf{I}$ that $\succcurlyeq(\phi) = \mathfrak{I}$.

Remark 5.54. The general definition of generating interpretation of formulas by branch is purely auxiliary and redundant in nature. For it is difficult to generally establish a definition of function $\succcurlyeq$. However, in the case of specific logics or whole classes of logics, this concept takes on a very specific meaning — we can then describe the transition from the open and maximal branch to the construction of interpretation. We will present this issue in the next chapter, describing examples of application.

Another important concept is the concept of set of interpretations good for rules.

Definition 5.55 (Interpretations good for rules)**.** Let

- **R** be a set of tableau rules
- ϕ be an open and maximal branch
- $X^i \subseteq \bigcup \phi$, for some non-empty $X \subseteq \mathsf{For}$ and some $i \in \mathbb{N}$
- **I** be a set of interpretations of formulas.

We shall state that set **I** is *good* for set of rules **R** iff branch ϕ generates such interpretation $\mathfrak{I}$ that:

- $\mathfrak{I} \in \mathbf{I}$
- $\mathfrak{I} \models X$.

We will now define the general concept of closure under rules.

Definition 5.56 (Closure under rules)**.** Let $X \subseteq \mathsf{Te}$. We shall state that $Y \subseteq \mathsf{Te}$ is a *closure* of set X under rules **R** iff Y is a set that meets the following conditions:

- $X \subseteq Y$
- for any rule $R \in \mathbf{R}$ and any n-tuple $\langle Z_1, Z_2, \ldots, Z_n \rangle \in R$, where $n \in \mathbb{N}$, if $X \subseteq Z_1 \subseteq Y$, then $Z_j \subseteq Y$, for some $2 \leq j \leq n$.

On set Y we will also state that is a *closure*.

For any set of expressions, there exists at least one closure, at times there may exist more closures.

Using the above concept of closure, we can move on to the verbalization and proof of the following lemma.

Lemma 5.57 (On the existence of open and maximal branch)**.** *Let $X \subseteq \mathsf{For}$ and $i \in \mathbb{N}$. If for each finite $Y \subseteq X$, there exists a maximal and open branch beginning with Y^i, then there exists a closure of set X^i under rules **R** which is an open and maximal branch.*

Proof. Take any $X \subseteq \mathsf{For}$, $i \in \mathbb{N}$, and assume that $(*)$ for each finite $Y \subseteq X$ there exists an open and maximal branch beginning with set Y^i.

Next, we specify the set of all maximal and open branches that begin with set Y^i, for some finite $Y \subseteq X$ — we will denote that set as $\mathbf{X}$.

We define set $\overline{\mathbf{X}}$, through the following conditions:

1. $\overline{\mathbf{X}} \subseteq \mathbf{X}$
2. for any two branches ϕ and ψ contained in $\mathbf{X}$, if there exist such $i, k \in \mathbb{N}$ that $\phi(i) \cup \psi(k)$ is a t-inconsistent set, then $\phi \notin \overline{\mathbf{X}}$ or $\psi \notin \overline{\mathbf{X}}$
3. $\overline{\mathbf{X}}$ is a maximal set among those subsets $\mathbf{X}$ that meet conditions 1 and 2.

There exists at least one set $\overline{\mathbf{X}}$ such that $\overline{\mathbf{X}} \subseteq \mathbf{X}$. We take one of such sets $\overline{\mathbf{X}}$ and denote it as $\overline{X}$.

Consider set $\bigcup\{\phi(1) : \phi \in \overline{X}\}$. Note that $(**)$ $X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$. For when $X^i \nsubseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$, there would exist such $x \in X^i$ that $x \notin \bigcup\{\phi(1) : \phi \in \overline{X}\}$ and, consequently, for any such branch $\psi \in \mathbf{X}$ that $x \in \psi(1)$, $\psi(1) \subseteq X^i$ and $\psi(1)$ is a finite set, it would be the case that $\psi \notin \overline{X}$. Then, however, for some finite set $Y^i \subseteq X^i$ there would exist no maximal and open branch beginning with set $Y^i \cup \{x\}$ which would contradict assumption $(*)$.

We define the condition that specifies new set $\overline{\overline{X}}$:
$U \in \overline{\overline{X}}$ iff there exists such branch ϕ that $\phi \in \overline{X}$ and $U = \bigcup \phi$
Now, we can define set $Z = \bigcup \overline{\overline{X}}$.

We claim that Z is a closure of set X^i under tableau rules $\mathbf{R}$ (definition 5.56), and that Z is an open and maximal branch.

First, we will show that Z is a closure of set X^i, thus that it meets conditions of definition of closure 5.56.

Note that $X^i \subseteq Z$, since $(**)$ $X^i \subseteq \bigcup\{\phi(1) : \phi \in \overline{X}\}$, and by definition of set Z, $\bigcup\{\phi(1) : \phi \in \overline{X}\} \subseteq Z$.

Now, take any rule $R \in \mathbf{R}$ and any n-tuple $\langle U_1, \ldots, U_n \rangle \in R$, for some $n \in \mathbb{N}$, and assume that $X^i \subseteq U_1 \subseteq Z$. From definition 5.33, it follows that there exists such n-tuple $\langle U_1', \ldots, U_n' \rangle \in R$ that:

- for each $1 \leq j \leq n$, U_j' is such a minimal and finite set that if U_j is not such a minimal and finite set such that $\langle U_1, \ldots, U_n \rangle \in R$, then $U_j' \subset U_j$
- for any $1 < j \leq n$, $U_j \setminus U_1 = U_j' \setminus U_1'$.

Consequently, assuming that $U_1' \subseteq Z$, we must show that for certain $1 < l \leq n$, $U_l' \subseteq Z$, since $U_l' \cup U_1 = U_l$. Since $U_1' \subseteq Z$ and U_1' is a finite set, thus there exists a finite number of such branches $\phi_1, \phi_2, \ldots, \phi_o$ in set $\overline{X}$ that for certain $k \in \mathbb{N}$, $U_1' \subseteq \phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$. So, set $\overline{X}$ contains such branch ψ that $\psi(1) = \phi_1(1) \cup \phi_2(1) \cup \cdots \cup \phi_o(1)$ and $U_1' \subseteq \psi(m)$, for certain $m \in \mathbb{N}$, and since $\phi_1(k) \cup \phi_2(k) \cup \cdots \cup \phi_o(k)$ is a t-consistent set, so set $\overline{X}$ contains such maximal branch ψ' that — by definition 5.42 — for certain $1 < l \leq n$, $U_l' \subseteq \bigcup \psi'$. Finally, $U_l' \subseteq Z$, since by construction Z, $\bigcup \psi \subseteq Z$.

We will now move on to showing that Z is an open and maximal branch.

From the definition of branch — 5.35 — it follows that Z is a branch.

Whereas by construction of Z, Z is an open branch, i.e. no subset Z is t-inconsistent, by definition $\overline{X}$.

Let us now check if Z is a maximal branch. According to definition 5.42, we assume that there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle X_1, \ldots, X_n \rangle \in R$, for some $n \in \mathbb{N}$ that $X_1 = Z$. By definition of tableau rules 5.33 (Existence if core) and (Closure under subsets), there exists such n-tuple $\langle X_1' \ldots, X_n' \rangle \in R$ that for any $1 < j \leq n$, $X_j \setminus X_1 = X_j' \setminus X_1'$ and $X^i \subseteq X_1' \subseteq Z$. Since Z is a closure X^i, so $X_j' \subseteq Z$, for certain $1 < j \leq n$, by definition 5.56. Therefore, $X_j \subseteq Z$, since $X_j = X_1 \cup X_j'$. But then $X_1 \notin X_j$, which by definition 5.33 is out of the question. Consequently, there exists no such tableau rule R and n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ that $X_1 = Z$, for some $n \in \mathbb{N}$. Therefore, Z is a maximal branch, by definition 5.42. □

Now, we will verbalize and prove a fact that is needed to demonstrate the relationship between relation $\triangleright$ and the existence of a closed tableau.

Proposition 5.58. *Let $X \subseteq \mathsf{Te}$. If X is a finite set, then there exists such maximal branch ϕ that $\phi(1) = X$.*

Proof. Take any subset $X \subseteq \mathsf{Te}$. The set of all branches beginning with set of expressions X will be denoted as $\mathbf{X}$. Set $\mathbf{X}$ is non-empty as by definition of branch 5.35, such mapping $\psi : \{1\} \longrightarrow P(\mathsf{Te})$ that $\psi(1) = X$, is a branch.

We have two options: (1) there exists a closed branch in set $\mathbf{X}$, or (2) there does not exist any closed branch in set $\mathbf{X}$.

If case (1) occurs, then by definition of maximal branch 5.42, there exists such maximal branch ϕ that $\phi(1) = X$.

Assume that case (1) does not occur.

Let $Y \subseteq \mathsf{Te}$ be any finite set of expressions. By definition 5.33, the number of tableau rules that belong to set $\mathbf{R}_Y$ is finite and different from zero. Now, assume there is j of them, for some $j \in \mathbb{N}$. We assign to each number from range $1 \leq i \leq j$ exactly one of rules that belong to set $\mathbf{R}_Y$, to obtain the sequence of all rules from set $\mathbf{R}_Y$: $R^1, \ldots, R^j$.

By definition of tableau rules 5.33, for each rule $R^i \in \mathbf{R}_Y$, there exists a finite number of n-tuples $\langle Y, X_1, \ldots, X_{n-1} \rangle$ in set R^i_Y. Therefore, each set R^i_Y, where $R^i \in \mathbf{R}_Y$, is finite — contains at most k of ordered n-tuples, for some $k \geq 1$. We take account of some sets $R^1_Y, \ldots, R^j_Y$, one for each $R^i \in \mathbf{R}_Y$.

We assign to each number from range $1 \leq i \leq k$ exactly one n-tuple from set R^i_Y, and denote given n-tuple as r_i to obtain the sequence of all n-tuples from set R^i_Y: $r_1, \ldots, r_k$.

Consequently, in any R^i_Y there exists a finite number of ordered n-tuples $\langle Y, X_1, \ldots, X_{n-1}\rangle$ that we can arrange in sequence: $r^i_1, \ldots, r^i_k$, for certain $k \geq 1$.

Next, we define a list of all n-tuples r^i_l from each R^i_Y, imposing a kind of lexicographical order on that list:

$$\underbrace{r^1_1, \ldots, r^1_m}_{R^1{}_Y}, \underbrace{r^2_1, \ldots, r^2_n}_{R^2{}_Y}, \ldots, \underbrace{r^j_1, \ldots, r^j_o}_{R^j{}_Y}, \text{ where } 1 \leq m, n, \ldots, o.$$

Such defined list of ordered n-tuples from set of expressions Y will be called *Y-list* and denoted as L_Y. Of course, there may exist multiple Y-lists. Still, there exists at least one Y-list that can be empty.

Let L_Y be Y-list and let $r_i \in L_Y$. We know that $r_i \in R^k{}_Y \subseteq R^k$, for some $k \leq j$. Let $r_i = \langle X_1, \ldots, X_n\rangle$. We shall state that ordered n-tuple $\langle Z_1, \ldots, Z_n\rangle$ is an *expansion* of r_i iff:

- $\langle Z_1, \ldots, Z_n\rangle \in R^k$
- for each $1 \leq l \leq n$ the following conditions are satisfied:
  1. $X_l \subset Z_l$.
  2. X_l is a set that is similar to $X_1 \cup (Z_l \setminus Z_1)$.

If $\langle Z_1, \ldots, Z_n\rangle$ is the considered expansion r_i, instead of $\langle Z_1, \ldots, Z_n\rangle$ we will write r'_i.

$(*)$ From definition of rules 5.29 (Closure under expansion), (Existence of core of rule), we know that for any $r_i = \langle X_1, \ldots, X_n\rangle$ that belongs to rule R and for any t-consistent set of expressions Z_1, such that $X_1 \subset Z_1$ and for each $1 < i \leq n$, X_i is not similar to any subset Z_1 that contains X_1, there exists such $r_j \in R$ that r_j is an expansion of r_i, where $r_j = \langle Z_1, \ldots, Z_n\rangle$, for some $Z_2, \ldots Z_n \subseteq \mathsf{Te}$.

Let L_Y be certain Y-list. By induction we define the *closure* of set Y under L_Y. $L_Y(Y)$ is a maximally long sequence of sets of expressions $Z_1, \ldots, Z_o$, such that for some $o \in \mathbb{N}$ and for any $1 \leq n \leq o$:

1. if $n = 1$, then $Z_n = Y$
2. if $n = 2$, then $Z_n = X_j$, where:
   a. r_1 is the first n-tuple in L_Y
   b. $r_1 = \langle Y, X_1, \ldots, X_n\rangle$, for $n \geq 1$
   c. $X_j = X_1$
3. if $n > 2$, then
   a. Z_{n-1} belongs to sequence $L_Y(Y)$
   b. Z_{n-1} is a consequence of expansion of certain m-tuple $r_l \in L_Y$ applied to Z_{n-2}, thus $r'_l = \langle Z_{n-2}, W_1, \ldots, W_m\rangle$, for $m \geq 1$, and $Z_{n-1} = W_1$

   and $Z_n = X_j$, where:

a. there exists r_{l+m}, for some $m \geq 1$, and it is the first element after r_l in L_Y such that:
b. r'_{l+m} is an expansion of r_{l+m}
c. $r'_{l+m} = \langle Z_{n-1}, X_1, \ldots, X_i \rangle$, for $i \geq 1$
d. $X_j = X_1$.

By definition of branch 5.35, each closure $L_Y(Y) := Z_1, \ldots, Z_n$, for some $n \in \mathbb{N}$, is a branch.

Now, let us investigate the initial set of expressions X. By virtue of the previous findings, we conclude:

- X is a finite set, so we have such branch $L^1_X(X) := X_1, \ldots, X_k$, for some X-list and some $k \in \mathbb{N}$ that:
- X_k is a finite set of expressions as set of tableau rules **R** is closed under finite sets 5.29 (Closure under finite sets).

Let us investigate a sequence of closures under some number of lists L^j — where $j \in \mathbb{N}$ — and assume that the last set from the last closure X_o is a finite set:

$$
\begin{array}{ll}
L^1_X(X) = X_1, \ldots, X_k & \text{for some } k \geq 1, \text{ where } k \in \mathbb{N} \\
L^2_{X_k}(X_k) = X_k, \ldots, X_l & \text{for some } l \geq k, \text{ where } k \in \mathbb{N} \\
. & \\
. & \\
. & \\
L^{j-1}_{X_{l+m}}(X_{l+m}) = X_{l+m}, \ldots, X_n & \text{for some } n \text{ and } m \in \mathbb{N}, \text{ where } n \geq l+m \\
L^j_{X_n}(X_n) = X_n, \ldots, X_o & \text{for some } o \geq n, \text{ where } o \in \mathbb{N}.
\end{array}
$$

Since set of expressions X_o is finite, so we can define another branch $L^{j+1}_{X_o}(X_o) := X_o, \ldots, X_r$, for some X_o-list and certain $r \in \mathbb{N}$ such that:

1. by definition of addition of branches 5.37 and conclusion on addition of branches 5.38, $(((\ldots(L^1_X(X) \oplus L^2_{X_k}(X_k)) \oplus \ldots) \oplus L^{j-1}_{X_{l+m}}(X_{l+m})) \oplus L^j_{X_n}(X_n)) \oplus L^{j+1}_{X_o}(X_o)$ is a branch.
2. X_r is a finite set of expressions as set of tableau rules **R** is closed under finite sets by definition of tableau rules 5.29 (Closure under finite sets).

Consequently, for any $j \in \mathbb{N}$ there exists such branch $L^{j+1}_{X_m}(X_m)$ that $L^j_{X_l}(X_l) = X_l, \ldots, X_m$, $L^1_X(X) = X_1, \ldots, X_k$, and for some $k \leq \cdots \leq l \leq m \in \mathbb{N}$, $X_1 \subseteq X_k \subseteq \cdots \subseteq X_l \subseteq X_m$. We can extract all those branches:

$$\underbrace{X_1 = X, \ldots, X_k}_{L^1_X(X)}, \underbrace{X_k, \ldots, X_l}_{L^2_{X_k}(X_k)}, \underbrace{X_l, \ldots, X_m}_{L^3_{X_l}(X_l)}, \ldots$$

After removal of all duplicates of elements, we get a branch — let us call it χ — which begins with initial set X. We claim that branch χ is a maximal branch. There exist two options:

1. branch χ is finite in length
2. branch χ is infinite.

If the first option occurs, then from the construction of branch χ we know that there does not exist super-branch χ. And from conclusion 5.44 we deduce that branch χ is maximal.

Assume that the second case occurs — so branch χ is infinite in length. Let us investigate if χ is a maximal branch. Taking account of definition of maximal branch 5.42, assume that there exists such tableau rule $R \in \mathbf{R}$ and such sets of expressions $Y_1, \ldots Y_n \subseteq \mathsf{Te}$, for some $1 < n \in \mathbb{N}$, that:

- $\langle Y_1, \ldots, Y_n \rangle \in R$
- for some $1 \leq i$, $X_i = Y_1$ and $X_i \in \chi$.

We must demonstrate that there exists such index $j \in \mathbb{N}$ that for certain $1 < k \leq n$, some subset W of element of branch $X_j \in \chi$ is strongly similar to set Y_k.

From the construction of branch χ we know that $X_i \in L^k_{X_m}(X_m)$, for some $k \geq 1$ and $m \leq i$.

By definition of tableau rules 5.33 and construction of branch χ two cases are possible:

(a) $i = m$ and $R \in \mathbf{R}_{X_m}$

(b) $i > m$ and there exist:

1. some $l \in \mathbb{N}$, where $m < l$
2. subsequent sequence $L^{k+1}_{X_l}(X_l)$ and $R \in \mathbf{R}_{X_l}$.

Assume the first case, meaning $i = m$ and $R \in \mathbf{R}_{X_m}$. By virtue of construction of branch χ we have three options:

1. $X_{i+1} = Y_k$, for some $1 < k \leq n$
2. there exist: n-tuple $\langle W_1, \ldots, W_n \rangle \in R$ which is an expansion of the initial n-tuple $\langle Y_1, \ldots, Y_n \rangle$, and such element of branch χ $X_{i+o} = W_1$, for some $o \geq 1$ that $X_{i+o+1} = W_k$ and additionally certain subset W_k is strongly similar to set of expressions Y_k, for some $1 < k \leq n$

3. there exist: set of expressions $X_{i+o} \in \chi$, for some $o \geq 1$, and such rule $R' \in \mathbf{R}_{X_m}$ that certain ordered n-tuple $\langle X_{i+o-1}, Y_2, \ldots, Y_n \rangle \in R'$, for some $n \in \mathbb{N}$, $X_{i+o} = Y_{n_1}$, for some $1 < n_1 \leq n$, and certain subset X_{i+o} is strongly similar to Y_k, for some $1 < k \leq n$.

Case (b) consists of similar options, only that we consider such expansion of n-tuple $\langle Y_1, \ldots, Y_n \rangle \in R$ that the first set of that expansion contains set X_l.

Therefore, χ is a maximal branch. □

We still need a few additional concepts for the proof of the theorem on completeness. We will utilize them for demonstration of relationship between the existence of closed tableau an relation $\models$.

Now, we will define a successive concept relevant for the general theorem on completeness.

Definition 5.59 (Rules good for interpretations)**.** We shall state that set of rules **R** is *good* for set of interpretations **I** iff for any sets $X_1, \ldots, X_i \subseteq \mathsf{Te}$ (where $1 < i$), any interpretation $\mathfrak{I} \in \mathbf{I}$ and any rule $R \in \mathbf{R}$, if:

- $\langle X_1, \ldots, X_i \rangle \in R$
- $\mathfrak{I}$ is appropriate for X_1,

then $\mathfrak{I}$ is appropriate for X_j, for some $1 < j \leq i$.

We will use the above definition 5.59 for the proof of another lemma, assuming the property it defines. This lemma determines the relationship between the finite sets of formulas and the existence of maximal and open branches.

Lemma 5.60. *Let $X \subseteq \mathsf{For}$ be a finite set of formulas, $i \in \mathbb{N}$ and let $\mathfrak{I} \in \boldsymbol{I}$ be an interpretation of formulas. If set of rules* **R** *is good for set of interpretations* $\boldsymbol{I}$ *and $\mathfrak{I} \models X$, then there exists a maximal and open branch beginning with set $\{A^i : A \in X\}$.*

Proof. Take any finite set of formulas $X \subseteq \mathsf{For}$, any index $i \in \mathbb{N}$ and any interpretation of formulas $\mathfrak{I}$, and then assume that set of rules **R** is good for set of interpretations **I** and $\mathfrak{I} \models X$. We define the following set $\{A^i : A \in X\}$. Set $\{A^i : A \in X\}$ will be denoted as X^i.

Since $\mathfrak{I} \models X$, so from definition of relation $\Vdash_i$ 5.25 it follows that $\mathfrak{I} \Vdash X$. Moreover, by definition of tableau expressions 5.15, set X is t-consistent.

Consequently, due to definition 5.28, interpretation $\mathfrak{I}$ is appropriate for set X^i.

Now, indirectly assume that each maximal branch beginning with set X^i is closed.

As $\Phi(X^i)$ we will denote the set of all maximal branches beginning with set X^i. From fact 5.58, we know that for each finite set of tableau expressions Y there exists a maximal branch beginning with set Y. Thus, set $\Phi(X^i)$ is non-empty.

Since set $\Phi(X^i)$ is a set of all maximal branches beginning with set X^i, so it has the following property.

Now, assume that for some branch $\chi \in \Phi(X^i)$. Let for certain $n \in \mathbb{N}$ exist such rule $R \in \mathbf{R}$ and such m-tuple $\langle Z_1, \ldots, Z_m \rangle \in R$ that $\chi(n) = Z_1$ and $\chi(n+1) = Z_j$, for some $1 < j \leq m$.

Note that each set Z_j is a finite set of expressions since each rule expands the finite input set to the finite output set (by definition of tableau rules 5.29 (Closure under finite sets)), branch χ begins with finite set X^i, and we investigate its n-th element. Thus, from fact 5.58 we know that:

- for each set Z_j there exists maximal branch ϕ_j beginning with set X^i such that $\phi_i(n+1) = Z_j$.

$(*)$ Consequently, set $\Phi(X^i)$ contains such branches $\chi, \phi_2, \ldots, \phi_m$ that $\chi = \phi_j$, for some $1 < j \leq m$.

Thus for any $n \in \mathbb{N}$, if there exist: such rule $R \in \mathbf{R}$, such m-tuple $\langle Z_1, \ldots, Z_m \rangle \in R$, and branch $\chi \in \Phi(X^i)$ such that $\chi(n) = Z_1$ and $\chi(n+1) = Z_j$, then there exists branch $\psi \in \Phi(X^i)$ such that $\psi(n+1) = Z_j$ and for any $k \leq n+1$, $\chi(k) = \psi(k)$.

$(**)$ By assumption, each branch that belongs to set $\Phi(X^i)$ is closed, thus by virtue of fact 5.40, each branch that belongs to set $\Phi(X^i)$ has a finite length of m, for some $m \in \mathbb{N}$.

From the initial assumption, we know that each of branches in set $\Phi(X^i)$ begins with set X^i.

Since interpretation $\mathfrak{I}$ is appropriate for set of expressions X^i, so due to the definition of interpretation appropriate for set of expressions 5.28, set X^i is not t-inconsistent. Hence, we get a conclusion that there are no branches of length one in set $\Phi(X^i)$.

Due to the assumption that set of rules $\mathbf{R}$ is good for set of interpretations $\mathbf{I}$ and definition 5.59, for any rule $R \in \mathbf{R}$ and any l-tuple $\langle Z_1, \ldots Z_l \rangle \in R$, if interpretation of formulas $\mathfrak{I}$ is appropriate for set Z_1, then it is also appropriate for some set Z_j, where $1 < j \leq l$, and by virtue of $(*)$, there exists branch $\chi \in \Phi(X^i)$ such that interpretation of formulas $\mathfrak{I}$ is appropriate for set $\chi(2)$ and $\chi(1) = X^i$.

The set of those branches that belong to $\Phi(X^i)$, and simultaneously interpretation of formulas $\mathfrak{I}$ is appropriate for their k-th element, will be denoted as $\Phi(X^i)_k$.

So, we have $\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \neq \varnothing$.

Now, assume that for some $n \in \mathbb{N}$, where $n > 1$, set $\Phi(X^i)_{n-1} \supseteq \Phi(X^i)_n \neq \varnothing$. Since set $\Phi(X^i)_n$ is non-empty, so take some branch $\psi \in \Phi(X^i)_n$.

By assumption, interpretation of formulas $\mathfrak{I}$ is appropriate for set of expressions $\psi(n)$, so due to the definition of interpretation appropriate for set of expressions 5.28, set $\psi(n)$ is not t-inconsistent.

Due to the assumption that set of rules **R** is good for set of interpretations **I** and definition 5.59 which claims that for any rule $R \in \mathbf{R}$ and any l-tuple $\langle Z_1, \dots Z_l \rangle \in R$, if interpretation of formulas $\mathfrak{I}$ is appropriate for set Z_1, then it is also appropriate for some set Z_j, where $1 < j \leq l$, and $(*)$, there exists branch $\phi \in \Phi(X^i)_{n+1}$ such that interpretation $\mathfrak{I}$ is appropriate for set $\phi(n+1)$ and $\phi \in \Phi(X^i)_n$. Thus, $\Phi(X^i)_n \supseteq \Phi(X^i)_{n+1}$ and $\Phi(X^i)_{n+1} \neq \varnothing$.

Therefore, for each $k \in \mathbb{N}$:

$$\Phi(X^i) = \Phi(X^i)_1 \supseteq \Phi(X^i)_2 \supseteq \dots \supseteq \Phi(X^i)_k \supseteq \dots$$

We take the intersection of all those sets $\Phi(X^i)_k$, where $k \in \mathbb{N}$. Intersection $\bigcap\{\Phi(X^i)_k : k \in \mathbb{N}\} = \Phi$ is non-empty as for each k, subset $\Phi(X^i)_k$ is also non-empty. So, set Φ includes at least one branch χ. That branch is maximal and begins with set X^i since $\Phi \subseteq \Phi(X^i)$.

But, branch χ is infinite which contradicts conclusion $(**)$. □

We will now move on to the final, auxiliary relationship between the tableau concepts.

Lemma 5.61. *Let $X \subseteq \mathsf{For}$ be a finite set of formulas, $A \in \mathsf{For}$ and $i \in \mathbb{N}$. If there exists a maximal and open branch beginning with set $\{B^i : B \in X \cup \{f(A)\}\}$, then each complete tableau $\langle X, A, \Phi \rangle$ is open.*

Proof. Take finite set $X \subseteq \mathsf{For}$, any formula $A \in \mathsf{For}$ and index $i \in \mathbb{N}$ such that there exists a maximal and open branch beginning with set $\{B^i : B \in X \cup \{f(A)\}\}$. We will denote that branch by letter ϕ, and set $\{B^i : B \in X \cup \{f(A)\}\}$, for simplicity, will be denoted by X^i.

$(*)$ Since branch ϕ is open, so no element ϕ is a t-inconsistent set, by definition 5.39.

$(**)$ Since branch ϕ is maximal and open, so for any rule $R \in \mathbf{R}$, any $n \in \mathbb{N}$ and any element $Y \in \phi$, if $\langle Y, Y_1, \dots, Y_n \rangle \in R$, then there exists some element $Z \in \phi$ such that some subset $W \subseteq Z$ is a set strongly similar to set Y_i, for certain $1 \leq i \leq n$, by definition of maximal branch 5.42.

Now, we indirectly assume that there exists complete and closed tableau $\langle X, A, \Psi \rangle$.

Since tableau $\langle X, A, \Psi \rangle$ is complete, so Ψ is such subset of set of all maximal branches that $\langle X, A, \Psi \rangle$ is a complete tableau, by definition of complete tableau 5.50.

Since tableau $\langle X, A, \Psi \rangle$ is closed, so each branch that belongs to Ψ, is closed, by definition of closed tableau 5.51. For certain $k \in \mathbb{N}$, each of these branches:

- begins with set $X^k = \{B^k : B \in X \cup \{f(A)\}\}$, by definition of tableau 5.48
- and its last element is a t-inconsistent set of expressions, by definition of closed tableau 5.51.

We intend to show that there exists some open branch ψ in set of branches Ψ, which contradicts the assumption that $\langle X, A, \Psi \rangle$ is a closed tableau. To this end, we will apply the induction through the branch length in order to construct infinite branches beginning with set X^k. The construction method for such branches will be denoted as $(\dagger)$.

Consider the first element of each branch contained in set of branches Ψ. It is set $X_1 = X^k = \{B^k : B \in X \cup \{f(A)\}\}$. X_1 is a set of expressions similar — within the meaning of definition of similarity 5.23 — to set $X^i = \{B^i : B \in X \cup \{f(A)\}\}$. Since $X^i \in \phi$ and branch ϕ is open, so X^i and X_1 are t-consistent, by definition 5.23.

Nevertheless, due to the fact that Ψ is a set of closed branches and the considered tableau $\langle X, A, \Psi \rangle$ is complete, there must exist a tableau rule $R \in \mathbf{R}$ such that $\langle X_1, Z_2, \ldots, Z_l \rangle \in R$, where $l < 1$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 5.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. Because due to definition of tableau rules 5.29, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — within the meaning of definition of similarity 5.23 — to some set $W_m \subseteq Y_m$ and it is t-consistent, since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}$, by virtue of the fact that ϕ is an open, by $(*)$, and maximal branch, by $(**)$. Set Z_m will be denoted as X_2, while element W_m as X_2^*.

Therefore, for number 1 there exist such branches $\psi_1, \psi_2 \in \Psi$ that:

- $X_1 \in \psi_1$
- set X_2 originated by the application of certain rule $R \in \mathbf{R}$ to set X_1, ultimately producing a second element of branch $\psi_2 \in \Psi$
- $X_2 \in \psi_2$
- X_2 is a t-consistent set
- $X_1 \subset X_2$
- for some $j \in \mathbb{N}$, set $X_2^* \subseteq X_j \in \phi$, where set X_2^* is similar, in the sense of definition of similarity 4.16, to set X_2.

Now, assume that for certain $n \in \mathbb{N}$ there exist such branches $\psi_1, \ldots, \psi_n \in \Psi$ that:

- for any $1 < j \leq n$, set X_j originated by the application of certain rule $R \in \mathbf{R}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$

- $X_n \in \psi_n$
- X_n is a t-consistent set
- $X_1 \subset X_2 \subset \ldots \subset X_n$
- for some $i \in \mathbb{N}$, set $X_n^* \subseteq X_i \in \phi$, where X_n^* is similar, in the sense of definition of similarity 4.16, to set X_n.

Nevertheless, due to the fact that Ψ is a set of closed branches, the considered tableau $\langle X, A, \Psi \rangle$ is complete and set X_n is a t-consistent set, there must exist a tableau rule $R \in \mathbf{R}$ such that $\langle X_n, Z_2, \ldots, Z_l \rangle \in R$, where $l > 1$, and for each $1 < j \leq l$ there exists such branch in set Ψ that Z_j belongs to that branch, by definition of complete tableau 5.50.

Nonetheless, certain set Z_m — for $1 < m \leq l$ — must be t-consistent. Because due to definition of tableau rules 5.29, there exists such l-tuple that $\langle Y_1, \ldots, Y_l \rangle \in R$, where Z_m is a similar set — within the meaning of definition of similarity 5.23 — to some set $W_m \subseteq Y_m$ and it is t-consistent since $Y_m \subseteq U \in \phi$, for certain $U \subseteq \mathsf{Te}$, by virtue of the fact that ϕ is an open $(*)$ and maximal branch $(**)$. Set Z_m will be denoted as X_{n+1}, while element W_m as X_{n+1}^*.

Thus, for any $n \in \mathbb{N}$, there exist such branches $\psi_1, \ldots, \psi_n, \psi_{n+1} \in \Psi$ that:

1. for any $1 < j \leq n+1$, set X_j originated by the application of certain rule $R \in \mathbf{R}$ to set X_{j-1}, ultimately producing j-th element of branch $\psi_j \in \Psi$
2. $X_{n+1} \in \psi_{n+1}$
3. X_{n+1} is a t-consistent set
4. $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1}$
5. for some $i \in \mathbb{N}$, set $X_{n+1}^* \subseteq X_i \in \phi$, where X_{n+1}^* is similar, within the meaning of definition of similarity 5.23 — to set X_{n+1}.

Set of all sets that originate this way $X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$ will be denoted as $\mathbf{X}$. Set $\mathbf{X}$ contains at least one branch ψ such that for any $i \in \mathbb{N}$, if $X_i \in \psi$, then there exists set $X_i \in \mathbf{X}$.

Branch ψ can be defined through the specification of such minimal subset of $\mathbf{X}$, set $\mathbf{X}'$ that:

- $X_1 \in \mathbf{X}'$
- for any $i \in \mathbb{N}$, if $X_i \in \mathbf{X}'$, then exactly one $X_{i+1} \in \mathbf{X}'$.

Branch ψ is infinite, and as a consequence of conclusion 5.40 it is an open branch.

Since set X_1, the first element of branch ψ, is equal to set X^k, and moreover for any element $X_i \in \psi$, where $i > 1$, there exist rule $R \in \mathbf{R_{S5}}$ and n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ such that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi' \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi'$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\psi\} \rangle$ by definition of tableau 5.48 is a tableau for pair $\langle X, A \rangle$.

However, branch ψ does not belong to set Ψ because tableau $\langle X, A, \Psi \rangle$, contrary to the assumption, would not be a closed tableau.

Let us now consider the question whether or not set $\Psi \cup \{\psi\}$ is a redundant superset of set Ψ, in the light of definition of redundant variant of branch 5.49. Let us now carry out the following argument.

(††) Assume that branch ψ is a redundant variant of some branch $\psi' \in \Psi$ different from ψ. For a certain minimal $1 \leq i \in \mathbb{N}$:

- there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle X_1, \ldots, X_n \rangle \in R$ that $\psi'(i) = X_1$ and $\psi'(i+1) = X_j$, for certain $1 < j \leq n$
- there exists rule $R \in \mathbf{R}$ and such m-tuple $\langle Y_1, \ldots, Y_m \rangle \in R'$, where $n < m$, that $X_1 = \psi(i) = Y_1$ and:
  1. $\psi(i+1) = Y_k$, for certain $1 < k \leq m$
  2. for any $1 < l \leq n$ there exists such $1 < o \leq m$ that $o \neq k$ and $X_l = Y_o$ and there exists such branch $\psi'' \in \Psi$ that $\psi''(i+1) = Y_o$.

But since branch ϕ is open and maximal (assumptions $(*)$ and $(**)$), so also some element $X_l = \psi''(i+1)$ for some branch $\psi'' \in \Phi$ is t-consistent because it is similar to some set of expressions W included in some element of branch ϕ.

Therefore, we can construct yet another infinite and open branch $X_1, \ldots, X_{j+1}$, making use of construction (†) which again, for at least successive element, i.e. $X_{j+1} = X_l$, by virtue of reasoning analogous to (††) is t-consistent and it is not a redundant variant of any sub-branch of any branch from set Ψ.

So, by application of induction and steps (†) and (††) we get an infinite branch — call it χ — and, consequently, open which is not a redundant variant of any branch that belongs to set of branches Ψ and begins with set X_1.

Since Ψ, by assumption, contains closed branches, so $\chi \notin \Psi$. Since set X_1, the first element of branch χ, is equal to set X^k, and moreover for any element $X_i \in \chi$, where $i > 1$, there exists such rule $R \in \mathbf{R}$ and such n-tuple $\langle Y_1, \ldots Y_n \rangle \in R$ that:

- $Y_1 = X_{i-1}$
- $X_i = Y_k$, for certain $1 < k \leq n$
- for each $1 < j \leq n$, if $j \neq k$, then there exists branch $\psi \in \Psi$ such that for some Z_l, where $1 \leq l$, $Z_l \in \psi$, $Z_l = Y_1$ and $Z_{l+1} = Y_j$,

so $\langle X, A, \Psi \cup \{\chi\} \rangle$ by virtue of definition of tableau 5.48 is a tableau for pair $\langle X, A \rangle$.

Thus, $\langle X, A, \Psi \rangle$ is not a complete tableau which contradicts the initial assumption. □

Summing up the definitions, lemmas and facts that we have presented so far, we move on to the general theorem on completeness for the tableau systems constructed using the method presented in the book.

Theorem 5.62 (General theorem on completeness). *If:*

1. *set of interpretations* $\mathbf{I}$ *is good for set of tableau rules* $\mathbf{R}$
2. *set of tableau rules* $\mathbf{R}$ *is good for set of interpretations* $\mathbf{I}$

then for any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$ *the below statements are equivalent:*

- $X \models A$
- $X \rhd A$
- *there exists finite subset* $Y \subseteq X$ *and closed tableau* $\langle Y, A, \Phi \rangle$.

Proof. We assume 1., 2. and take any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$. We must prove three implications.

(a) $X \models A \Longrightarrow X \rhd A$.

Assume that $X \not\rhd A$. Hence, for any finite $Y \subseteq X$ there exists an open and maximal branch beginning with set $Y^i \cup f(A)^i$ — for some $i \in \mathbb{N}$ — by definition of branch consequence 5.46. By lemma 5.57 (On the existence of open and maximal branch), there exists a closure of set $X^i \cup f(A)^i$ under rules $\mathbf{R}$ which constitutes a maximal and open branch ψ.

By assumption 1, we know that there exists interpretation $\mathfrak{I} \in \mathbf{I}$ generated by ψ and $\mathfrak{I} \models X \cup \{f(A)\}$. Therefore $\mathfrak{I} \models X$ and $\mathfrak{I} \not\models A$, by definition 5.8. Consequently $X \not\models A$.

(b) $X \rhd A \Longrightarrow$ there exists finite subset $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$.

$(*)$ Assume that for each finite subset $Y \subseteq X$ all tableaux $\langle Y, A, \Phi \rangle$ are open.

Take any finite subset $Y \subseteq X$. Note that from fact 5.58 $(**)$ it follows that for any finite set of tableau expressions there exists a maximal branch which begins with that set. Therefore for any index $i \in \mathbb{N}$ there exists maximal branch beginning with set $Y^i \cup \{f(A)^i\}$.

Take any index $i \in \mathbb{N}$. So, set of maximal branches Φ^i beginning with set $Y^i \cup \{f(A)^i\}$ is non-empty. What is more, by $(**)$ set Φ^i contains at least one such subset Φ that ordered triple $\langle Y, A, \Phi \rangle$ is a complete tableau.

Due to assumption $(*)$ tableau $\langle Y, A, \Phi \rangle$ is open.

Since $\langle Y, A, \Phi \rangle$ is open and complete, so Φ contains maximal and open branch ϕ which begins with set $Y^i \cup \{f(A)^i\}$.

Since Y is any finite subset X and i is any index, so for any finite subset $Y \subseteq X$ and any index $i \in \mathbb{N}$, there exists some maximal and open branch ψ beginning with set $Y^i \cup \{f(A)^i\}$.

Consequently, there does not exist such finite subset $Y \subseteq X$ and such index $i \in \mathbb{N}$ that each maximal branch beginning with $Y^i \cup \{f(A)^i\}$ is closed. Therefore — by definition 5.46 — $X \not\triangleright A$.

(c) there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle \Longrightarrow X \models A$.

Assume that $X \not\models A$. So, by definition of relation of semantic consequence 5.10, there exists such interpretation of formulas $\mathfrak{I}$ that $\mathfrak{I} \models X$ and $\mathfrak{I} \not\models A$. Thus $\mathfrak{I} \models f(A)$, and consequently $\mathfrak{I} \models X \cup \{f(A)\}$.

Hence, for any finite subset $Y \subseteq X$, also $\mathfrak{I} \models Y \cup \{f(A)\}$.

Take any finite subset $Y' \subseteq X$. From lemma 5.60 and assumption 2 (set of tableau rules **R** is good for set of interpretations **I**), we get a conclusion that for any $i \in \mathbb{N}$ there exists maximal and open branch beginning with set $\{B^i : B \in Y' \cup \{f(A)\}\}$.

And from lemma 5.61 we know that each complete tableau $\langle Y', A, \Phi \rangle$ is open. Since Y' was an arbitrary finite subset of set of formulas X, so there is no finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$. □

6 Examples of applications

6.1 Introductory remarks

In this chapter, we will show exemplary applications of the general tableau concepts we defined in Chapter Five. Thanks to the general concepts we have at our command, and their interrelationships we have demonstrated, we can significantly shorten the construction of a complete tableau system.

We already have the general concepts:

- set of tableau rules
- branches
- closed/open branch
- relation of branch consequence
- tableau
- open/closed tableau.

We also know that there exists a general connection between a properly defined set of tableau rules **R** and properly defined semantics. Further work on the construction of the complete tableau system must therefore focus solely on defining the detailed concepts of tableau system in such a way that the general conditions are met — if that is the case, then we get a tableau system that is complete in terms of the initial semantics.

In the next four subchapters, we will describe three different applications. The first one will be of a detailed nature. We will consider an example of the logic of categorical propositions with modalities *de re*. We will define the basic concepts and then show that they meet the sufficient conditions for the general theorem on completeness 5.62, which will allow us to reach the conclusion that the defined tableau system is adequate to the initial semantics. This kind of application of the general theorem on completeness can be considered paradigmatic, because the theorem is intended primarily to shorten the construction of complete tableau systems when we want to construct a tableau system for a logic that is already semantically defined.

The second application, described in subsequent subchapter, has a general character. In this subchapter, we describe how to apply the general tableau concepts and general tableau theorem to the entire class of logics defined with the same type of semantics. Using an example of modal logics specified with the semantics of possible worlds, we will show how to obtain a less general theorem on completeness, specified for this class of logics. It allows even simpler proof of the completeness of specific tableau systems, because some more general properties are fulfilled by the entire class of modal logics specified with the semantics of possible worlds. Similar general applications can occur in all cases where we consider the classes of logic defined with a common type of semantics.

The next subchapter is devoted to the concept of a tableau system. We will try to define the general concept of tableau system and show what benefits to the investigation of dependencies between tableau systems brings the way the book describes tableau systems.

In the last subchapter, we will outline the transition between the formalised tableaux and standard tableaux/trees. Therefore, we will try to show that the approach presented in the book corresponds to the standard approach, as to the practical construction of proof itself, while at the same time emphasizing the general nature of concepts that the standard approach does not bear.

6.2 Tableau system for Modal Term Logic *de re*

We will now turn to the logic of categorical propositions *de re*.[1] It is an extension of logic **TL** with new categorical propositions with modalities in the interpretation *de re*. This logic will be called Modal Term Logic *de re*, for short **MTL**.

When defining set of formulas of **MTL** on the right hand side we will provide schemes of propositions in English which correspond to particular formulas and may occur in the reasonings described by **MTL**.

6.2.1 Language

Let us begin with the alphabet of **MTL**.

Definition 6.1 (Alphabet of **MTL**). Alphabet of Modal Term Logic is made up by the sum of the following sets:

- set of logical constants $\mathsf{Lc} = \{\mathbf{a}, \mathbf{i}, \mathbf{e}, \mathbf{o}, \mathbf{a}^{\Diamond}, \mathbf{i}^{\Diamond}, \mathbf{e}^{\Diamond}, \mathbf{o}^{\Diamond}, \mathbf{a}^{\Box}, \mathbf{i}^{\Box}, \mathbf{e}^{\Box}, \mathbf{o}^{\Box}\}$
- set of name letters $\mathsf{Ln} = \{P^1, Q^1, R^1, P^2, Q^2, R^2, \dots\}$.

Even though the set of name letters is infinite and contains indexed letters, practicably we will use a finite number of the following letters: *P*, *Q*, *R*, *S*, *T*, *U*, treating them as metavariables ranging over set Ln.

Let us now proceed to the definition of formula of **MTL**. Modal Term Logic is defined on the following set of formulas.

1 The idea for the system presented here was offered first in [14]. A simplified and better developed in some respects version of this material is presented in [22]. Also other variants of syllogistic are studied there.

Definition 6.2 (Formula of **MTL**)**.** Set of formulas of **MTL** is the smallest set containing the following expressions.

- $P\mathbf{a}Q$ — *Each P is Q.*
- $P\mathbf{i}Q$ — *Some P is Q.*
- $P\mathbf{e}Q$ — *No P is Q.*
- $P\mathbf{o}Q$ — *Some P is not Q.*
- $P\mathbf{a}^{\Box}Q$ — *Each P must be Q.* *Each P is necessarily Q.*
- $P\mathbf{i}^{\Box}Q$ — *Some P must be Q.* *Some P is necessarily Q.*
- $P\mathbf{e}^{\Box}Q$ — *No P may be Q.* *No P is possibly Q.*
- $P\mathbf{o}^{\Box}Q$ — *Some P must not be Q.* *Some P is not possibly Q.*
- $P\mathbf{a}^{\Diamond}Q$ — *Each P may be Q.* *Each P is possibly Q.*
- $P\mathbf{i}^{\Diamond}Q$ — *Some P may be Q.* *Some P is possibly Q.*
- $P\mathbf{e}^{\Diamond}Q$ — *No P must be Q.* *No P is necessarily Q.*
- $P\mathbf{o}^{\Diamond}Q$ — *Some P may not be Q.* *Some P is not necessarily Q.*

where $P, Q \in \mathsf{Ln}$.
We specify set of formulas as $\mathsf{For}_{\mathbf{MTL}}$, and its elements will be called *formulas*.

6.2.2 Semantics

Let us define the concept of model. We will use semantics without possible worlds.

Definition 6.3 (Model for language of **MTL**)**.** *Model* will be called ordered quadruple $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$, where:

1. D is a set
2. $d^{\Box}$, d, $d^{\Diamond}$ are such functions from set of name letters Ln in set $P(D)$ that for any name letter $P \in \mathsf{Ln}$, $d^{\Box}(P) \subseteq d(P) \subseteq d^{\Diamond}(P)$.

Remark 6.4. The proposed concept of model expresses the following intuitions. Functions $d^{\Box}, d, d^{\Diamond}$ assign to each name letter P those sets of objects that are — respectively — necessarily P-s, are P-s, and can be P-s. The objects that are necessarily P-s, are also simply P-s and are contained in the set of those objects that can be P-s. Hence, we have a chain of inclusions: $d^{\Box}(P) \subseteq d(P) \subseteq d^{\Diamond}(P)$.

Now, we proceed to the concept of truth in model.

Definition 6.5 (Truth in model)**.** Let $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$ be a model and let $A \in \mathsf{For}_{\mathbf{MTL}}$. We shall state that formula A is *true in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short $\mathfrak{M}_{\mathbf{MTL}} \models A$) iff for some name letters P, $Q \in \mathsf{Ln}$, one of the below conditions is met:

1. $A = P\mathbf{a}Q$ and $d(P) \subseteq d(Q)$
2. $A = P\mathbf{i}Q$ and $d(P) \cap d(Q) \neq \varnothing$
3. $A = P\mathbf{e}Q$ and $d(P) \cap d(Q) = \varnothing$
4. $A = P\mathbf{o}Q$ and $d(P) \nsubseteq d(Q)$
5. $A = P\mathbf{a}^{\Box}Q$ and $d(P) \subseteq d^{\Box}(Q)$
6. $A = P\mathbf{i}^{\Box}Q$ and $d(P) \cap d^{\Box}(Q) \neq \varnothing$
7. $A = P\mathbf{e}^{\Box}Q$ and $d(P) \cap d^{\Diamond}(Q) = \varnothing$
8. $A = P\mathbf{o}^{\Box}Q$ and $d(P) \nsubseteq d^{\Diamond}(Q)$
9. $A = P\mathbf{a}^{\Diamond}Q$ and $d(P) \subseteq d^{\Diamond}(Q)$
10. $A = P\mathbf{i}^{\Diamond}Q$ and $d(P) \cap d^{\Diamond}(Q) \neq \varnothing$
11. $A = P\mathbf{e}^{\Diamond}Q$ and $d(P) \cap d^{\Box}(Q) = \varnothing$
12. $A = P\mathbf{o}^{\Diamond}Q$ and $d(P) \nsubseteq d^{\Box}(Q)$.

If for any propositional letters P, $Q \in \mathsf{Ln}$ none of the conditions is met, then we shall state that formula A is *false in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short $\mathfrak{M}_{\mathbf{MTL}} \not\models A$).
Let $X \subseteq \mathsf{For}_{\mathbf{MTL}}$. We shall state that set of formulas X is *true in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short: $\mathfrak{M}_{\mathbf{MTL}} \models X$) iff for any formula $A \in X$, $\mathfrak{M}_{\mathbf{MTL}} \models A$. We shall state that set of formulas X is *false in model* $\mathfrak{M}_{\mathbf{MTL}}$ (for short: $\mathfrak{M}_{\mathbf{MTL}} \not\models X$) iff it is not the case that $\mathfrak{M}_{\mathbf{MTL}} \models X$.

Remark 6.6. The semantics for modal syllogistic we offer in this study refers to the semantics presented in studies of F. Johnson [16] and S. K. Thomason [26] i [27].[2]

In [16] Johnson introduced model in form $\langle W, V^e, V^a, V^a_c, V^e_c \rangle$ where V^e, V^a, V^a_c, and V^e_c are functions that assign subsets of set W to name letters, where W is treated as a set of all objects ("world"); $V^e(S)$ is a set of objects that essentially are S-s; $V^a(S)$ is a set of objects that accidentally are S-s; $V^e_c(S)$ is a set of objects that essentially are non-S-s; $V^a_c(S)$ is a set of objects that accidentally are non-S-s. Furthermore, an auxiliary function V was adopted as $V(S) = V^e(S) \cup V^a(S)$, that is $V(S)$ is a set of all S-s. Therefore, our D, d, and $d^{\Box}$ are Johnson's W, V, and V^e

2 The application of this type of semantics was launched in [14].

respectively. Moreover, our function $d^{\Diamond}$ can be defined with formula $d^{\Diamond}(S) := W \smallsetminus V^e_c(S)$; set of all those objects that are not essentially non-S-s, i.e. can be S-s.[3]

Our interpretation of proposition $S\mathbf{a}^{\Box}P$ corresponds to the interpretation of Johnson: $V(S) \subseteq V^e(P)$; and for $S\mathbf{e}^{\Box}P$ we have: $d(S) \cap d^{\Diamond}(P) = \varnothing$, i.e. $V(S) \cap (W \smallsetminus V^e_c(P)) = \varnothing$ iff $V(S) \subseteq V^e_c(P)$. For the other propositions [16] adopted different interpretation that ours, in order to reproduce the modal syllogistic of Aristotle.[4]

In [26] Thomason used a semantics based on the ordered quadruples in form $\langle W, \mathrm{Ext}, \mathrm{Ext}^+, \mathrm{Ext}^- \rangle$, where W is a set of objects and Ext, Ext^+, and Ext^- are functions that assign subsets of set W to name letters and meet the following conditions: $\varnothing \neq \mathrm{Ext}^+(x) \subseteq \mathrm{Ext}(x)$ and $\mathrm{Ext}^-(x) \cap \mathrm{Ext}(x) \neq \varnothing$, for each letter x. Thomason's W, Ext and Ext^+ correspond to our D, d and $d^{\Box}$. Furthermore, his $\mathrm{Ext}^-(S)$ is a set of objects that cannot be S-s. We can express that set through our $D \smallsetminus d^{\Diamond}(S)$. We have $\mathrm{Ext}(S) \subseteq (W \smallsetminus \mathrm{Ext}^-(S))$, that is $d(S) \subseteq d^{\Diamond}(S)$. Obviously, Ext, Ext^+ and Ext^- correspond to functions V, V^e, and V^e_c from [16].

Denotation 6.7. Let $\bullet \in \{\Box, \Diamond\}$. Let us adopt denotation: if $\bullet = \Box$, then $\bullet' = \Diamond$, and if $\bullet = \Diamond$, then $\bullet' = \Box$.

Now, we will define a function that assigns a contradictory formula to each formula.

Definition 6.8 (Contradictory formulas)**.** Let $\circ\colon \mathsf{For}_{\mathbf{MTL}} \longrightarrow \mathsf{For}_{\mathbf{MTL}}$ be a function specified for any $P, Q \in L\mathsf{n}$ and $\bullet \in \{\Box, \Diamond\}$ with the following conditions:

1. $\circ(P\mathbf{a}Q) = P\mathbf{o}Q$
2. $\circ(P\mathbf{i}Q) = P\mathbf{e}Q$
3. $\circ(P\mathbf{e}Q) = P\mathbf{i}Q$
4. $\circ(P\mathbf{o}Q) = P\mathbf{a}Q$
5. $\circ(P\mathbf{a}^{\bullet}Q) = P\mathbf{o}^{\bullet'}Q$
6. $\circ(P\mathbf{i}^{\bullet}Q) = P\mathbf{e}^{\bullet'}Q$
7. $\circ(P\mathbf{e}^{\bullet}Q) = P\mathbf{i}^{\bullet'}Q$
8. $\circ(P\mathbf{o}^{\bullet}Q) = P\mathbf{a}^{\bullet'}Q$.

Directly from definition of truth in model 6.5 and definition of function $\circ$ 6.8 we get a conclusion.

Corollary 6.9. *For any model $\mathfrak{M}_{\mathbf{MTL}}$ and any formula A, $\mathfrak{M}_{\mathbf{MTL}} \vDash A$ iff $\mathfrak{M}_{\mathbf{MTL}} \nvDash \circ(A)$*

3 Function V^a_c is superfluous in this semantics. This is also evident in studies of Thomason [26] and [27] who disregards that function.

4 For instance, propositions $S\mathbf{i}^{\Box}P$ and $P\mathbf{i}^{\Box}S$ are to be equivalent in this semantics.

Note that each model $\mathfrak{M}_{\mathbf{MTL}}$ can be identified with interpretation $\mathfrak{I}$ — in accordance with definition of general interpretation of formulas 5.8. Take any model $\mathfrak{M}_{\mathbf{MTL}}$ and define set of formulas $X_{\mathfrak{M}_{\mathbf{MTL}}} = \{A \in \mathsf{For}_{\mathbf{MTL}} : \mathfrak{M}_{\mathbf{MTL}} \vDash A\}$. Function $\circ$ is injective and for any formula B, $B \in X_{\mathfrak{M}_{\mathbf{MTL}}}$ iff $\circ(B) \notin X_{\mathfrak{M}_{\mathbf{MTL}}}$, by conclusion 6.9.

We define conventionally the relation of semantic consequence.

Definition 6.10 (Semantic consequence relation)**.** Let set $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and $A \in \mathsf{For}_{\mathbf{MTL}}$. We shall state that formula A *follows from* set of formulas X (for short: $X \vDash A$) iff for any model $\mathfrak{M}_{\mathbf{MTL}}$, if $\mathfrak{M}_{\mathbf{MTL}} \vDash X$, then $\mathfrak{M}_{\mathbf{MTL}} \vDash A$. We shall state that from set of formulas X *does not follow* formula A (for short: $X \nvDash A$) iff it is not the case that $X \vDash A$.

Pair $\langle \mathsf{For}_{\mathbf{MTL}}, \vDash \rangle$ is a semantically defined logic, in accordance with general definition 5.11. This implies that relation of semantic consequence $\vDash$ both is unambiguously determined by set of all models $\mathfrak{M}_{\mathbf{MTL}}$ for $\mathsf{For}_{\mathbf{MTL}}$ of set of formulas **MTL**, and unambiguously determines set of all models $\mathfrak{M}_{\mathbf{MTL}}$ for $\mathsf{For}_{\mathbf{MTL}}$ of set of formulas **MTL**, by fact 5.12.

6.2.3 Tableau expression

Before we move on to the definition of Te — set of tableau expression for **MTL** in accordance with general definition of tableau expression 5.15 — let us define several auxiliary concepts.

Definition 6.11 (Expressions)**.** Set of expressions Ex is the union of the following sets.

- $\{A^i : A \in \mathsf{For}_{\mathbf{MTL}}, i \in \mathbb{N}\}$
- $\{P_{+i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\{P_{-i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$
- $\{P^{\bullet}_{+i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$, where $\bullet \in \{\Box, \Diamond\}$
- $\{P^{\bullet}_{-i} : P \in \mathsf{Ln}, i \in \mathbb{N}\}$, where $\bullet \in \{\Box, \Diamond\}$.

By virtue of definition 6.11, the following conclusion occurs

Corollary 6.12. *There exists function $g : \mathsf{For}_{\mathbf{MTL}} \longrightarrow P(\mathsf{Ex})$, defined with condition: for any formula $A \in \mathsf{For}_{\mathbf{MTL}}$, $g(A) = \{A^1, A^2, A^3, \dots\}$, where $\{A^1, A^2, A^3, \dots\} \subseteq \mathsf{Ex}$.*

Remark 6.13. Practically, in the case of **MTL**, each $A^i \in g(A)$ will be simply identified with formula A, since in the tableau proof each formula will be self-represented.

Next, based on set Ex we define the concept of inconsistent set of expressions that ultimately will correspond to the concept of tableau inconsistent set of expressions.

Definition 6.14 (Inconsistent set of expressions)**.** Let $X \subseteq \mathsf{Ex}$. We shall state that set X is an *inconsistent set of expressions* iff X meets one of the following conditions:

1. $A \in X$ and $\circ(A) \in X$, for some $A \in \mathsf{For}_{\mathbf{MTL}}$
2. $P_{+i} \in X$ and $P_{-i} \in X$, for some $P \in \mathsf{Ln}$ and some $i \in \mathbb{N}$
3. $P^{\bullet}_{+i} \in X$ and $P^{\bullet}_{-i} \in X$, for some $P \in \mathsf{Ln}$, some $i \in \mathbb{N}$ and $\bullet \in \{\Diamond, \Box\}$.

We shall state that set X is a *consistent set of expressions* iff X is not an inconsistent set of expressions.

Based on definition of model 6.3, conclusion 6.9 and definition 6.14, we get another conclusion.

Corollary 6.15. *Let $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and $i \in \mathbb{N}$. If there exists model $\mathfrak{M}_{\mathbf{MTL}}$ such that $\mathfrak{M}_{\mathbf{MTL}} \models X$, then set $\{x^i : x \in g(A), A \in X\}$ is a consistent set of expressions.*

Based on definition of set of expressions 6.11, conclusion 6.12, definition of inconsistent set of expressions 6.14 and conclusion 6.15 we get the following fact.

Proposition 6.16. *Set of expressions Ex meets the conditions of general definition of tableau expressions 5.15.*

Due to the above fact, set Ex will be denoted as $\mathsf{Te}_{\mathbf{MTL}}$, while its elements will be called *expressions* or *tableau expressions*. In turn, the inconsistent sets of expressions will be called *tableau inconsistent* (*t-inconsistent*), while the consistent sets of expressions will be called *tableau consistent* (*t-consistent*).

Now, we define the function selecting indices.

Definition 6.17 (Function selecting indices)**.** Let $* : \mathsf{Te}_{\mathbf{MTL}} \longrightarrow \mathbb{N}$ be such function that for any name letter $P \in \mathsf{Ln}$, any $i \in \mathbb{N}$, any $\bullet \in \{\Box, \Diamond\}$ and any formula $A \in \mathsf{For}_{\mathbf{MTL}}$:

1. $*(A^i) = i$
2. $*(P_{+i}) = i$
3. $*(P_{-i}) = i$
4. $*(P^{\bullet}_{+i}) = i$
5. $*(P^{\bullet}_{-i}) = i$.

Denotation 6.18. Let $x \in \mathsf{Te}_{\mathbf{MTL}}$ and $*(x) = i$. We adopt denotation x^i.

Now, we will define binary relation $\equiv$ specified on the Cartesian product $P(\mathsf{Te}_{\mathbf{MTL}}) \times P(\mathsf{Te}_{\mathbf{MTL}})$ that will correspond to the general definition of similarity of sets of expressions (definition 5.23).

Definition 6.19. Let X, $Y \subseteq \mathsf{Te}_{\mathbf{MTL}}$. We define relation $\equiv$ with condition: $X \equiv Y$ iff there exists bijection $h : *(X) \longrightarrow *(Y)$ such that for any expression $x^i \in \mathsf{Te}_{\mathbf{MTL}}$: $x^i \in X$ iff $x^{h(i)} \in Y$.

From definition 6.19 results the following conclusion.

Corollary 6.20. *Let X, $Y \subseteq \mathsf{Te}_{\mathbf{MTL}}$. If $X \equiv Y$, then:*

- *X is t-consistent iff Y is t-consistent*
- *sets X and Y have the same cardinality.*

By conclusion 6.20 and definition of relation $\equiv$ 6.19, we claim that relation $\equiv$ meets the conditions of general definition of similarity of sets of expressions 5.23.

Proposition 6.21. *Relation $\equiv$ is the relation of similarity of sets in accordance with the general definition of similarity of sets of expressions 5.23.*

We will now define the concept that establishes the relation between the models and sets of expressions.

Definition 6.22. Let X be a set of expressions, while $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$ be a model. We shall state that $\mathfrak{M}_{\mathbf{MTL}}$ is *appropriate* for set X iff the below conditions are met:

1. $\mathfrak{M}_{\mathbf{MTL}} \models X \cap \mathsf{For}_{\mathbf{MTL}}$
2. there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $P \in \mathsf{Ln}$, each $i \in \mathbb{N}$ and for any $\bullet \in \{\Box, \Diamond\}$:
   a. if $P_{+i} \in X$, then $\gamma(i) \in d(P)$
   b. if $P_{-i} \in X$, then $\gamma(j) \notin d(P)$
   c. if $P^{\bullet}_{+i} \in X$, then $\gamma(i) \in d^{\bullet}(P)$
   d. if $P^{\bullet}_{-i} \in X$, then $\gamma(j) \notin d^{\bullet}(P)$.

From definition of inconsistent set of expressions 6.14 and definition of model appropriate for the set of expressions 6.22 follows a condition concerning the relationship between the inconsistent sets of expressions and the appropriateness of models.

Corollary 6.23. *For any $X \subseteq \mathsf{Te}_{\mathbf{MTL}}$, if X is t-inconsistent, then there exists no model $\mathfrak{M}_{\mathbf{MTL}}$ appropriate for X.*

Next, note that relation $\vDash$ for **MTL** meets the conditions of relation $\Vdash$ (definition 5.25). Thus, by conclusion 6.23 and definition of appropriate model 6.22 we have the fact.

Proposition 6.24. *The notion of model appropriate for set of expressions meets the general conditions of interpretation appropriate for the set of expressions described in definition 5.28.*

Thus, we have demonstrated that the presented concepts for the tableau system for **MTL** are special cases of the general concepts described in the previous chapter.

6.2.4 Rules for the tableau system for logic MTL

We can proceed to the rules. On each rule, we conventionally assume that each of its input sets is t-consistent and that each input set is basically contained in the output set.

Definition 6.25 (Tableau rules for **MTL**). *Tableau rules for system* **MTL** are the following rules:

Classical rules

$$R\mathbf{a}_{+} : \frac{X \cup \{P\mathbf{a}Q, P_{+j}\}}{X \cup \{P\mathbf{a}Q, P_{+j}, Q_{+j}\}}$$

$$R\mathbf{e}_{-} : \frac{X \cup \{P\mathbf{e}Q, P_{+j}\}}{X \cup \{P\mathbf{e}Q, P_{+j}, Q_{-j}\}}$$

$$R\mathbf{i} : \frac{X \cup \{P\mathbf{i}Q\}}{X \cup \{P\mathbf{i}Q, P_{+j}, Q_{+j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}\} \nsubseteq X$.

$$R\mathbf{o} : \frac{X \cup \{P\mathbf{o}Q\}}{X \cup \{P\mathbf{o}Q, P_{+j}, Q_{-j}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{-k}\} \nsubseteq X$.

Rules for $\Diamond$

$$R\mathbf{a}^{\Diamond}_{+} : \frac{X \cup \{P\mathbf{a}^{\Diamond}Q, P_{+j}\}}{X \cup \{P\mathbf{a}^{\Diamond}Q, P_{+j}, Q^{\Diamond}_{+j}\}}$$

$$Re_{-}^{\Diamond}: \frac{X \cup \{P\mathbf{e}^{\Diamond}Q, P_{+j}\}}{X \cup \{P\mathbf{e}^{\Diamond}Q, P_{+j}, Q_{-j}^{\Box}\}}$$

$$R\mathbf{i}^{\Diamond}: \frac{X \cup \{P\mathbf{i}^{\Diamond}Q\}}{X \cup \{P\mathbf{i}^{\Diamond}Q, P_{+j}, Q_{+j}^{\Diamond}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}^{\Diamond}\} \nsubseteq X$.

$$R\mathbf{o}^{\Diamond}: \frac{X \cup \{P\mathbf{o}^{\Diamond}Q\}}{X \cup \{P\mathbf{o}^{\Diamond}Q, P_{+j}, Q_{-j}^{\Box}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{-k}^{\Box}\} \nsubseteq X$.

Rules for □

$$R\mathbf{a}_{+}^{\Box}: \frac{X \cup \{P\mathbf{a}^{\Box}Q, P_{+j}\}}{X \cup \{P\mathbf{a}^{\Box}Q, P_{+j}, Q_{+j}^{\Box}\}}$$

$$R\mathbf{e}_{-}^{\Box}: \frac{X \cup \{P\mathbf{e}^{\Box}Q, P_{+j}\}}{X \cup \{P\mathbf{e}^{\Box}Q, P_{+j}, Q_{-j}^{\Diamond}\}}$$

$$R\mathbf{i}^{\Box}: \frac{X \cup \{P\mathbf{i}^{\Box}Q\}}{X \cup \{P\mathbf{i}^{\Box}Q, P_{+j}, Q_{+j}^{\Box}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{+k}^{\Box}\} \nsubseteq X$.

$$R\mathbf{o}^{\Box}: \frac{X \cup \{P\mathbf{o}^{\Box}Q\}}{X \cup \{P\mathbf{o}^{\Box}Q, P_{+j}, Q_{-j}^{\Diamond}\}}, \text{ where:}$$

1. $j \notin *(X \setminus \mathsf{For}_{\mathbf{MTL}})$
2. for any $k \in \mathbb{N}$, $\{P_{+k}, Q_{-k}^{\Diamond}\} \nsubseteq X$.

Bridging rules

$$R_{+}^{\Box}: \frac{X \cup \{P_{+j}^{\Box}\}}{X \cup \{P_{+j}^{\Box}, P_{+j}\}}$$

$$R_{+}: \frac{X \cup \{P_{+j}\}}{X \cup \{P_{+j}, P_{+j}^{\Diamond}\}}$$

$$R_{-}^{\Diamond}: \frac{X \cup \{P_{-j}^{\Diamond}\}}{X \cup \{P_{-j}^{\Diamond}, P_{-j}\}}$$

$$R_{-}: \frac{X \cup \{P_{-j}\}}{X \cup \{P_{-j}, P_{-j}^{\Box}\}}$$

Set of rules for **MTL** will be denoted as $\mathbf{R_{MTL}}$.

Note that set $\mathbf{R_{MTL}}$ meets both the general conditions of rule (definition 5.29), and the general conditions of set of tableau rules 5.33 — the cores of each rule from $\mathbf{R_{MTL}}$ (definition 6.25) are any ordered pairs in which set X is empty.

6.2.5 Branches and tableaux for MTL

We accept all general definitions from the previous chapter:

- branch 5.35
- closed/open branch 5.39
- maximal branch 5.42
- tableau 5.48
- complete tableau 5.50
- closed/open tableau 5.51
- branch consequence 5.46

assuming that these concepts are dependent on set of tableau rules $\mathbf{R_{MTL}}$.

6.2.6 Theorem on the completeness of the tableau system for MTL

In order to demonstrate that for the tableau system for **MTL** the theorem on completeness holds, we must show that the assumptions of general theorem 5.62 are met. So, we must demonstrate that:

- set of models for language of **MTL** is good for set of tableau rules $\mathbf{R_{MTL}}$ (according to general definition 5.55)
- set of tableau rules $\mathbf{R_{MTL}}$ is good for model for language of **MTL** (according to general definition 5.59).

Let us first define the concept of model defined by a branch.

Definition 6.26 (Model generated by branch)**.** Let ϕ be any branch. We define the following function $AT(\phi) = \bigcup \phi \cap (\mathsf{Te}_{\mathbf{MTL}} \setminus \mathsf{For}_{\mathbf{MTL}})$.

We shall state that model $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$ is *generated by branch* ϕ iff:

1. $D = \{x \in \mathbb{N} : x \in *(At(\phi))\}$
2. for any name letter $P \in \mathsf{Ln}$:

a. $x \in d(P)$ iff $P_{+x} \in At(\phi)$
b. for any $\bullet \in \{\Box, \Diamond\}$, $x \in d^{\bullet}(P)$ iff $P^{\bullet}_{+x} \in At(\phi)$.

From this definition, we get the following conclusion.

Corollary 6.27. *Let ϕ be an open and maximal branch. Then, there exists a model generated by ϕ.*

Proof. Take any open and maximal branch ϕ. From definition of open branch 5.39 and definition of model generated by branch 6.26 we get ordered quadruple $\langle D, d^{\Box}, d, d^{\Diamond} \rangle$.

We must investigate if for any name letter $P \in \mathsf{Ln}$ occurs $d^{\Box}(P) \subseteq d(P) \subseteq d^{\Diamond}(P)$. Since branch ϕ is maximal and open, so by virtue of bridging rules from definition of tableau rules $\mathbf{R_{MTL}}$ (definition 6.25), for any $i \in \mathbb{N}$ and for any name letter $P \in \mathsf{Ln}$:

- if $i \in d^{\Box}(P)$, then $i \in d(P)$, due to rule $R^{\Box}_{+}$
- if $i \in d(P)$, then $i \in d^{\Diamond}(P)$, due to rule R_{+}.

Thus, from definition of model 6.3 we get a conclusion that $\langle D, d^{\Box}, d, d^{\Diamond} \rangle$ is a model. □

As we can see, the general definition of branch generating model 5.53 gains content in the context of the tableau system of **MTL**. We might define a function assigning models to open and maximal branches, but we will refrain from doing so and instead directly use conclusion 6.27. While on model $\mathfrak{M}_{\mathbf{MTL}}$ generated by branch ϕ we shall state that branch ϕ *generates* model $\mathfrak{M}_{\mathbf{MTL}}$.

We will now show that set of models of **MTL** is good for tableau rules $\mathbf{R_{MTL}}$ (according to general definition 5.55). The following lemma will be useful for that.

Lemma 6.28. *Let ϕ be an open and maximal branch. Let $X^i \subseteq \bigcup \phi$, for some $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and $i \in \mathbb{N}$. Then branch ϕ generates such model $\mathfrak{M}_{\mathbf{MTL}}$ that $\mathfrak{M}_{\mathbf{MTL}} \vDash X$.*

Proof. Take any open and maximal branch ϕ. Take any set $X^i \subseteq \bigcup \phi$, for some $X \subseteq \mathsf{For}_{\mathbf{MTL}}$ and $i \in \mathbb{N}$. Note that in the case of **MTL** set X^i is simply a set of formulas (remark 6.13).

Since branch ϕ is open and maximal, so by virtue of the previous conclusion 6.27 there exists model $\mathfrak{M}_{\mathbf{MTL}} = \langle d^{\Box}, d, d^{\Diamond} \rangle$ generated by ϕ.

We will now show that for any formula A contained in $\bigcup \phi$, it is the case that $\mathfrak{M}_{\mathbf{MTL}} \vDash A$, i.e. $\mathfrak{M}_{\mathbf{MTL}} \vDash \bigcup \phi \cap \mathsf{For}_{\mathbf{MTL}}$. This implies that $\mathfrak{M}_{\mathbf{MTL}} \vDash X$.

The proof will be carried out by consideration of all the possible cases of construction of formula A. Assume that $A \in \bigcup \phi$. By definition of formula, for some name letters P, $Q \in \mathsf{Ln}$, there must occur one of the following cases.

1. $A = P\mathbf{a}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of generated model 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{a}_+$, $\bigcup\phi$ also contains tableau expression Q_{+i}. By definition of model generated 6.26, $i \in d(Q)$. Hence, $d(P) \subseteq d(Q)$, and by definition of truth in model 6.5, we thus get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$. In turn, if there exists no such $i \in D$ that $i \in d(P)$, then $\varnothing = d(P) \subseteq d(Q)$, so by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$.
2. $A = P\mathbf{i}Q$. Since ϕ is a maximal and open branch, so by tableau rule $R\mathbf{i}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{+i}, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and $i \in d(Q)$. Since $d(P) \cap d(Q) \neq \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}Q$.
3. $A = P\mathbf{e}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{e}_-$, $\bigcup\phi$ also contains tableau expression Q_{-i}. Since branch ϕ is open, so expression $Q_{+i} \notin \bigcup\phi$, and consequently, by definition of model generated 6.26, $i \notin d(Q)$. Thus, $d(P) \cap d(Q) = \varnothing$ and by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$. In turn, if there exists no object $i \in D$ such that $i \in d(P)$, then $d(P) \cap d(Q) = \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$.
4. $A = P\mathbf{o}Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{o}$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, Q_{-i}, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and — since branch ϕ is open and, consequently, expression $Q_{+i} \notin \bigcup\phi$ — $i \notin d(Q)$, so $d(P) \nsubseteq d(Q)$. Thus, by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{o}Q$.

The remaining eight cases of the possible construction of formula A will be reduced to four cases as both rules that we apply to them and conditions of truth that define their truth are analogous. So, take $\bullet \in \{\Box, \Diamond\}$. We have four cases.

1. $A = P\mathbf{a}^{\bullet}Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{a}^{\bullet}_{+}$, $\bigcup\phi$ also contains tableau expression $Q^{\bullet}_{+i}$. By definition of model generated 6.26, $i \in d^{\bullet}(Q)$. Hence, $d(P) \subseteq d^{\bullet}(Q)$, and by definition of truth in model 6.5, we thus get that $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}^{\bullet}Q$. In turn, if there exists no such $i \in D$ that $i \in d(P)$, then $\varnothing = d(P) \subseteq d^{\bullet}(Q)$, so by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}^{\bullet}Q$.

2. $A = P\mathbf{i}^\bullet Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{i}^\bullet$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, $Q^\bullet_{+i}$, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and $i \in d^\bullet(Q)$. Since $d(P) \cap d^\bullet(Q) \neq \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{i}^\bullet Q$.
3. $A = P\mathbf{e}^\bullet Q$. Take any object $i \in D$ such that $i \in d(P)$. By definition of model generated 6.26, set $\bigcup\phi$ contains tableau expression P_{+i}. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{e}^\bullet_-$, $\bigcup\phi$ also contains tableau expression $Q^{\bullet'}_{-i}$. Since branch ϕ is open, so expression $Q^{\bullet'}_{+i} \notin \bigcup\phi$, and consequently, by definition of model generated 6.26, $i \notin d(Q)^{\bullet'}$. Thus, $d(P) \cap d^{\bullet'}(Q) = \varnothing$ and by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{e}Q$. In turn, if there exists no object $i \in D$ such that $i \in d(P)$, then $d(P) \cap d^{\bullet'}(Q) = \varnothing$, so by definition of truth in model 6.5, we get that $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{e}^\bullet Q$.
4. $A = P\mathbf{o}^\bullet Q$. Since ϕ is a maximal and open branch, so by virtue of tableau rule $R\mathbf{o}^\bullet$, set $\bigcup\phi$ also contains tableau expressions P_{+i}, $Q^{\bullet'}_{-i}$, for some $i \in \mathbb{N}$. By definition of model generated 6.26, $i \in d(P)$ and — since branch ϕ is open and, consequently, expression $Q^{\bullet'}_{+i} \notin \bigcup\phi$ — $i \notin d^{\bullet'}(Q)$, so $d(P) \nsubseteq d^{\bullet'}(Q)$. Thus, by definition of truth in model 6.5, we get $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{o}^\bullet Q$. □

From lemma 6.28 and definition 5.55 applied to the set of models for the language of **MTL** we get a conclusion.

Corollary 6.29. *Set of models for language of* **MTL** *is good for set of tableau rules* $\boldsymbol{R_{MTL}}$.

We will now proceed to the demonstration of an opposite dependence between rules and models.

Lemma 6.30. *Let* $\mathfrak{M}_{MTL}$ *be any model,* $X, Y \subseteq \mathsf{Te}_{MTL}$*, and let* $R \in \boldsymbol{R_{MTL}}$*. Then, if* $\langle X, Y\rangle \in R$ *and* $\mathfrak{M}_{MTL}$ *is appropriate for set of expressions* X*, then* $\mathfrak{M}_{MTL}$ *is appropriate for* Y*.*

Proof. In the proof, we will make use of definition of model appropriate for the set of expressions 6.22. Let $\mathfrak{M}_{\mathbf{MTL}} = \langle D, d^\square, d, d^\diamond\rangle$ be any model and $X, Y \subseteq \mathsf{Te}_{\mathbf{MTL}}$. We will consider all cases of rules $R \in \mathbf{R_{MTL}}$, assuming that $\langle X, Y\rangle \in R$ and $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, and showing that then $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for Y.

We have four cases of rules for the classical functors.

1. Let $R = R\mathbf{a}_+$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{a}Q, P_{+i}\}, Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X,

so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}Q$, hence by definition of truth in model 6.5, $\gamma(i) \in d(Q)$, since $d(P) \subseteq d(Q)$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{a}Q, P_{+i}, Q_{+i}\}$.

2. Let $R = R\mathbf{i}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{i}Q\}, Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{i}$ enriches set X with expressions P_{+i}, Q_{+i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P) \cap d(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{i}Q, P_{+i}, Q_{+i}\}$.
3. Let $R = R\mathbf{e}_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{e}Q, P_{+i}\}, Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}Q$, hence by definition of truth in model 6.5 $\gamma(i) \notin d(Q)$, since $d(P) \cap d(Q) = \varnothing$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{e}Q, P_{+i}, Q_{-i}\}$.
4. Let $R = R\mathbf{o}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{o}Q\}, Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P, Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{o}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S_{-j} \in X$, then $\gamma(j) \notin d(S)$; however, rule $R\mathbf{o}$ enriches set X with expressions P_{+i}, Q_{-i} and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{o}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P)$, but $x \notin d(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{o}Q, P_{+i}, Q_{-i}\}$.

We have eight rules for formulas with modal functors. However, we will restrict our considerations to four cases as they are analogous in terms of application. So, take $\bullet \in \{\Box, \Diamond\}$.

1. Let $R = R\mathbf{a}^{\bullet}_{+}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{a}^{\bullet}Q, P_{+i}\}, Z \cup \{P\mathbf{a}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ i.a. such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet}_{+j} \in X$, then $\gamma(j) \in d^{\bullet}(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{a}^{\bullet}Q$, hence by definition of truth in model 6.5, $\gamma(i) \in d^{\bullet}(Q)$, since $d(P) \subseteq d^{\bullet}(Q)$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{a}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}$.
2. Let $R = R\mathbf{i}^{\bullet}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{i}^{\bullet}Q\}, Z \cup \{P\mathbf{i}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ i.a. such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet}_{+j} \in X$, then $\gamma(j) \in d^{\bullet}(S)$; however, rule $R\mathbf{i}^{\bullet}$ enriches set X with expressions $P_{+i}, Q^{\bullet}_{+i}$ and index i is new, it has not occurred in any of expressions from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{i}^{\bullet}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P) \cap d^{\bullet}(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{i}^{\bullet}Q, P_{+i}, Q^{\bullet}_{+i}\}$.
3. Let $R = R\mathbf{e}^{\bullet}_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{e}^{\bullet}Q, P_{+i}\}, Z \cup \{P\mathbf{e}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that i.a. for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet'}_{-j} \in X$, then $\gamma(j) \notin d^{\bullet'}(S)$; due to the fact that $P_{+i} \in X$, also $\gamma(i) \in d(P)$, while since $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{e}^{\bullet}Q$, hence by definition of truth in model 6.5 $\gamma(i) \notin d^{\bullet'}(Q)$, since $d(P) \cap d^{\bullet'}(Q) = \varnothing$; consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{e}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}$.
4. Let $R = R\mathbf{o}^{\bullet}$, then $\langle X, Y\rangle = \langle Z \cup \{P\mathbf{o}^{\bullet}Q\}, Z \cup \{P\mathbf{o}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, P, $Q \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, $\mathfrak{M}_{\mathbf{MTL}} \vDash P\mathbf{o}^{\bullet}Q$ and there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that for each name letter $S \in \mathsf{Ln}$ and each $j \in \mathbb{N}$: if $S_{+j} \in X$, then $\gamma(j) \in d(S)$ and if $S^{\bullet'}_{-j} \in X$, then $\gamma(j) \notin d^{\bullet'}(S)$; however, rule $R\mathbf{o}^{\bullet}$ enriches set X with expressions

$P_{+i}, Q^{\bullet'}_{-i}$ and index i is new, it has not occurred in any expression from set X, while since $\mathfrak{M}_{\mathbf{MTL}} \models P\mathbf{o}^{\bullet}Q$, so by definition of truth in model 6.5, in the domain there exists certain object x such that $x \in d(P)$, but $x \notin d^{\bullet'}(Q)$; we define function $\gamma' : \mathbb{N} \longrightarrow D$ such that for any $k \in \mathbb{N}$, if $k \neq i$, then $\gamma'(k) = \gamma(k)$ and $\gamma'(i) = x$, consequently, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P\mathbf{o}^{\bullet}Q, P_{+i}, Q^{\bullet'}_{-i}\}$.

We have four cases for the bridging rules.

1. Let $R = R^{\Box}_{+}$, then $\langle X, Y\rangle = \langle Z \cup \{P^{\Box}_{+i}\}, Z \cup \{P^{\Box}_{+i}, P_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \in d^{\Box}(P)$; furthermore $d^{\Box}(P) \subseteq d(P)$, by definition of model 6.3, so $\gamma(i) \in d(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P^{\Box}_{+i}, P_{+i}\}$.
2. Let $R = R_{+}$, then $\langle X, Y\rangle = \langle Z \cup \{P_{+i}\}, Z \cup \{P_{+i}, P^{\Diamond}_{+i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \in d(P)$; furthermore $d(P) \subseteq d^{\Diamond}(P)$, by definition of model 6.3, so $\gamma(i) \in d^{\Diamond}(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P_{+i}, P^{\Diamond}_{+i}\}$.
3. Let $R = R^{\Diamond}_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P^{\Diamond}_{-i}\}, Z \cup \{P^{\Diamond}_{-i}, P_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \notin d^{\Diamond}(P)$; furthermore $d(P) \subseteq d^{\Diamond}(P)$, by definition of model 6.3, so $\gamma(i) \notin d(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P^{\Diamond}_{-i}, P_{-i}\}$.
4. Let $R = R_{-}$, then $\langle X, Y\rangle = \langle Z \cup \{P_{-i}\}, Z \cup \{P_{-i}, P^{\Box}_{-i}\}\rangle$, for some $Z \subseteq \mathsf{Te}_{\mathbf{MTL}}$, $P \in \mathsf{Ln}$ and $i \in \mathbb{N}$; since model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions X, so by definition 6.22, there exists function $\gamma : \mathbb{N} \longrightarrow D$ such that $\gamma(i) \notin d(P)$; furthermore $d^{\Box}(P) \subseteq d(P)$, by definition of model 6.3, so $\gamma(i) \notin d^{\Box}(P)$; thus, by definition of model appropriate for the set of expressions 6.22, model $\mathfrak{M}_{\mathbf{MTL}}$ is appropriate for set of expressions $Y = Z \cup \{P_{-i}, P^{\Box}_{-i}\}$. □

From lemma 6.30 and definition 5.59 applied to set of tableau rules $\mathbf{R}_{\mathbf{MTL}}$ we get a conclusion.

Corollary 6.31. *Set of tableau rules $\boldsymbol{R_{MTL}}$ is good for set of models for the language of **MTL**.*

Applying both conclusions: 6.29 and 6.31, concepts defined in this chapter and general tableau theorem 5.62, from the previous chapter, we get the completeness theorem for system of **MTL**.

Theorem 6.32 (Completeness theorem for **MTL**). *For any set $X \subseteq \mathsf{For}_{\boldsymbol{MTL}}$ and any formula $A \in \mathsf{For}_{\boldsymbol{MTL}}$ the following statements are equivalent:*

- $X \vDash A$
- $X \rhd A$
- *there exists finite set $Y \subseteq X$ and closed tableau $\langle Y, A, \Phi \rangle$*

Thus, we have shown how — with application of the general tableau concepts — we can shorten the proof of completeness theorem to the appropriate definition of specific concepts and application of the general theorem.

6.2.7 Estimation of cardinality of model for MTL

When applying the tableau methods to **TL** we received a possibility of estimation of the limitation of cardinality of models that can be countermodels for the considered inference (theorem 3.57). The situation is similar in the case of system for **MTL**. We can get an identical outcome nearly directly from the completeness theorem of the tableau system for **MTL**, carrying out a proof analogous to the one for theorem 3.57.

By *existential formula*, we mean any formula in form $P\mathbf{i}Q$, $P\mathbf{o}Q$, $P\mathbf{i}^{\bullet}Q$, $P\mathbf{o}^{\bullet}Q$, where $P, Q, \in \mathsf{Ln}$ and $\bullet \in \{\Box, \Diamond\}$.

Next, we define function $\lambda' : P(\mathsf{For}_{\mathbf{TL}}) \longrightarrow P(\mathsf{For}_{\mathbf{TL}})$ with the following condition: for any set $\Phi \in P(\mathsf{For}_{\mathbf{TL}})$, $\lambda'(\Phi) = \{x \in \Phi : x \text{ is an existential formula}\}$. So, from each set of rules, function λ' "selects" all existential formulas that belong to a given set.

Now, in turn, we shall specify function $\sigma : \{\Psi \in P(\mathsf{For}_{\mathbf{TL}}) : \Psi \text{ is a finite set}\} \longrightarrow \mathbb{N}$ with the following condition: for any finite set $\Psi \in P(\mathsf{For}_{\mathbf{TL}})$, $\sigma'(\Psi) = |\lambda'(\Psi)|$. So, function σ' "counts" the number of existential formulas that are found in any finite set of formulas.

With the use of the defined functions, we can frame the following theorem.

Theorem 6.33. *Let X be a finite set of formulas and let $A \in \mathsf{For}_{\boldsymbol{MTL}}$. Then, the below statements are equivalent:*

- *for any model $\mathfrak{M}_{\boldsymbol{MTL}} = \langle D, d^{\Box}, d, d^{\Diamond} \rangle$, if $|D| = \sigma'(X \cup \{\circ(A)\})$ and $\mathfrak{M}_{\boldsymbol{MTL}} \vDash X$, then $\mathfrak{M}_{\boldsymbol{MTL}} \vDash A$*
- $X \vDash A$.

6.3 Tableau systems for modal logics

In this subchapter, we will investigate the application of the theory of tableau systems we covered in Chapter Five to the general case. We will show how to apply the general concepts described previously to construct tableau systems for modal logics determined by models with possible worlds.

By using general concepts, we will provide the conditions whose occurrence demonstration is sufficient to get a complete tableau system for a given modal logic.[5]

6.3.1 Language and semantics

We adopt the set of formulas for modal logic defined in one of the previous chapters with definition 4.2 for logic **S5**. We will denote that set as $\mathsf{For}_{\mathbf{ML}}$.

Next, we adopt the general concept of model with possible worlds $\mathfrak{M}_{\mathbf{ML}}$, in accordance with definition of model 4.5 for logic **S5** — with any relation of accessibility. We will denote the set of all such models as **M**.

We define the concept of truth (and falsehood) of formula in model, analogously to definition 4.7.

According to definition 5.8, each model $\mathfrak{M}_{\mathbf{ML}} \in \mathbf{M}$ can be identified with interpretation of formulas as for function $f : \mathsf{For}_{\mathbf{ML}} \longrightarrow \mathsf{For}_{\mathbf{ML}}$ defined with condition $f(A) = \neg A$, for any $A \in \mathsf{For}_{\mathbf{ML}}$ it is the case that $A \in \{B \in \mathsf{For}_{\mathbf{ML}} : \mathfrak{M}_{\mathbf{ML}} \vDash B\}$ iff $f(A) \notin \{B \in \mathsf{For}_{\mathbf{ML}} : \mathfrak{M}_{\mathbf{ML}} \vDash B\}$.

Taking any subset $\mathbf{M'} \subseteq \mathbf{M}$, we conventionally define relation of semantic consequence $\vDash_{\mathbf{M'}}$. Based on that relation, we semantically define modal logic $\langle \mathsf{For}_{\mathbf{ML}}, \vDash_{\mathbf{M'}} \rangle$.

6.3.2 Tableau expressions

Now, we will proceed to the issue of expressions representing formulas and other properties in the tableau proof.

First, we define the set of expressions. Next, we will show that it meets the general conditions imposed on the set of tableau expressions (definition 5.15).

Definition 6.34 (Expressions)**.** Set of expressions Ex is a set that is the union of the below sets:

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5 The issues described in this subchapter were partially presented in article [8]. However, the general tableau theorem 5.62 was not used there. In addition, some concepts were defined differently in that paper.

- $\mathsf{For}_{\mathbf{ML}} \times \mathbb{N}$
- $\{irj : i,j \in \mathbb{N}\}$
- $\{\sim irj : i,j \in \mathbb{N}\}$
- $\{i = j : i,j \in \mathbb{N}\}$
- $\{\sim i = j : i,j \in \mathbb{N}\}$.

The elements of set $\mathbb{N}$ are called *indices*.

Remark 6.35. New types of expressions appeared in the set of expressions, which in the proof correspond to the negation of relation occurrence, the identity and negation of identity. They are not needed in all constructed systems, but in some systems they will be used by the tableau rules.

We can define the following function $g : \mathsf{For}_{\mathbf{ML}} \longrightarrow P(\mathsf{Ex})$, for any $A \in \mathsf{For}_{\mathbf{ML}}$, $g(A) = \{\langle A, i\rangle : i \in \mathbb{N}\}$. Function g has important properties, for any two formulas A, B: $A \neq B$ iff $g(A) \cap g(B) = \varnothing$, and $g(A)$ is a countable subset of set Ex.

Note, furthermore, that each expression $\langle A, i\rangle$ can be identified with expression A^i, which corresponds to function g from general definition of tableau expressions 5.15.

Definition 6.36. Let $X \subseteq \mathsf{Ex}$. We shall state that X is *tableau inonsistent* (for short: t-inconsistent) iff for some $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$ at least one of the below conditions is met:

1. $\langle A, i\rangle, \langle \neg A, i\rangle \in X$
2. $irj, \sim irj \in X$
3. $i = j, \sim i = j \in X$.

We shall state that X is *tableau consistent* (for short: t-consistent) iff it is not tableau inconsistent.

From definition 6.36 results the following conclusion.

Corollary 6.37. *Let $X \subseteq \mathsf{For}_{ML}$ and $\mathfrak{M}_{ML} \in \boldsymbol{M}$. If $\mathfrak{M}_{ML} \vDash X$, then $\{x^i : x^i \in g(A), A \in X\}$ is a t-consistent set.*

By virtue of definition of expressions 6.34, definition 6.36, existence of function g, conclusion 6.37 and general definition of tableau expressions 5.15, we can assume that set of expressions Ex is a set of tableau expressions. The set will be denoted as $\mathsf{Te}_{\mathbf{ML}}$.

Now, we will define more auxiliary concepts.

Definition 6.38 (Function selecting indices)**.** Function $* : P(\mathsf{Te}_{\mathbf{ML}}) \longrightarrow P(\mathbb{N})$ is a *function selecting indices* iff for any $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$ and $X \subseteq \mathsf{Te}_{\mathbf{ML}}$ the below conditions are met:

- $*(\{\langle A,i\rangle\}) = \{i\}$
- $*(\{irj\}) = \{i,j\}$
- $*(\{\sim irj\}) = \{i,j\}$
- $*(\{i=j\}) = \{i,j\}$
- $*(\{\sim i=j\}) = \{i,j\}$
- if $|X|>1$, then $*(X) = \bigcup\{*(\{x\}) : x \in X\}$.

For any subset Y of set of expressions $\mathsf{Te}_{\mathbf{ML}}$ function $*$ selects all indices present in Y.

Now, we will define binary relation $\equiv$ specified on Cartesian product $P(\mathsf{Te}_{\mathbf{ML}}) \times P(\mathsf{Te}_{\mathbf{ML}})$, that will correspond to the general definition of similarity of sets of expressions (5.23).

Definition 6.39. Let X, $Y \subseteq \mathsf{Te}_{\mathbf{ML}}$. We define relation $\equiv$ with condition: $X \equiv Y$ iff there exists bijection $h : *(X) \longrightarrow *(Y)$ (where $*(X)$, $*(Y)$ are sets of indices present in the expressions from X and from Y) such that for any $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$:

- $\langle A,i\rangle \in X$ iff $\langle A,h(i)\rangle \in Y$
- $irj \in X$ iff $h(i)rh(j) \in Y$
- $\sim irj \in X$ iff $\sim h(i)rh(j) \in Y$
- $i=j \in X$ iff $h(i)=h(j) \in Y$
- $\sim i=j \in X$ iff $\sim h(i)=h(j) \in Y$.

From definition 6.39 results the following conclusion.

Corollary 6.40. *Let X, $Y \subseteq \mathsf{Te}_{\mathbf{ML}}$. If $X \equiv Y$, then:*

- *X is t-consistent iff Y is t-consistent*
- *sets X and Y have the same cardinality.*

By virtue of conclusion 6.40 and definition of relation $\equiv$ 6.39 we claim that relation $\equiv$ meets the conditions of general definition of similarity of sets of expressions 5.23.

We will now define the concept that describe the relation between the models and sets of expressions.

Definition 6.41. Let $\mathfrak{M}_{\mathbf{ML}} = \langle W,Q,V,w\rangle$ and $X \subseteq \mathsf{Te}_{\mathbf{ML}}$. We shall state that model $\mathfrak{M}_{\mathbf{ML}}$ is *appropriate* for set of expressions X iff there exists such function $\gamma : \mathbb{N} \longrightarrow W$, that for any $A \in \mathsf{For}_{\mathbf{ML}}$, $i,j \in \mathbb{N}$ the following conditions occur:

- if $\langle A,i\rangle \in X$, then $\langle W,Q,V,\gamma(i)\rangle \vDash A$
- if $irj \in X$, then $\gamma(i)Q\gamma(j)$
- if $\sim irj \in X$, then it is not the case that $\gamma(i)Q\gamma(j)$

- if $i = j \in X$, then $\gamma(i)$ is identical to $\gamma(j)$
- if $\sim i = j \in X$, then $\gamma(i)$ is different from $\gamma(j)$.

From definition of tableau inconsistent set of expressions 6.36 and definition of model appropriate for set of expressions 6.41 the following conclusion results.

Corollary 6.42. *For any set of expressions $X \subseteq \mathsf{Te}_{ML}$, if X is t-inconsistent, then there exists no model $\mathfrak{M}_{ML}$ appropriate for X.*

Note that relation $\vDash$ defined by any subset $\mathbf{M}' \subseteq \mathbf{M}$ meets the conditions of relation $\Vdash$ (definition 5.25). Thus, by conclusion 6.42 and definition of appropriate model 6.41 we get a fact.

Proposition 6.43. *The notion of model appropriate for set of expressions meets the general conditions of interpretation appropriate for set of expressions described in definition 5.25.*

Thus, we have demonstrated that the presented concepts for the modal logics, determined with the semantics of possible worlds, are special cases of the general concepts described in the previous chapter.

6.3.3 Rules, branches and tableaux for modal logics

First, we will adopt the concept of tableau rule, originating by the application of concept of tableau expression $\mathsf{Te}_{\mathbf{ML}}$ and other presented concepts for modal logics to the general concept of rule 5.29 and general concept of tableau rule 5.33. Set of tableau rules for a given modal logic will be denoted as $\mathbf{R}_{\mathbf{ML}}$.

Furthermore, we adopt all general definitions from the previous chapter:

- branch 5.35
- closed/open branch 5.39
- maximal branch 5.42
- tableau 5.48
- complete tableau 5.50
- closed/open tableau 5.51
- branch consequence 5.46

assuming that these concepts are always dependent on some fixed set of tableau rules $\mathbf{R}_{\mathbf{ML}}$ for a given modal logic.

6.3.4 Generating of model

As we mentioned in the previous chapter, in remark 5.54, it is difficult to establish a general method for transition from the maximal and open branch to the generated model. For we do not know how the model is constructed and what types of expressions are used in the proof which leads to the maximal and open branch. Nonetheless, for a single logic it is possible — we did so in the case of a tableau system for **MTL**.

We can try to do the same in the case discussed, i.e. in relation to certain class of logics which in many respects are similar. The definition provided below is quite broad and on its basis we can define many types of models for modal logics determined by the semantics of possible worlds. For we have the general concept of model and the concept of set of tableau expressions $\mathsf{Te}_{\mathbf{ML}}$ which determines the range and elements used to define the model.

Definition 6.44 (Branch generating model)**.** Let $\mathbf{R_{ML}}$ be a set of tableau rules and ϕ be $\mathbf{R_{ML}}$-branch. Let $X = \{\langle A,k\rangle : A \in Y\} \subseteq \bigcup\phi$, for some $k \in \mathbb{N}$ and non-empty subset $Y \subseteq \mathsf{For}_{\mathbf{ML}}$. We define set $AT(\phi)$ as follows: $x \in AT(\phi)$ iff one of the below conditions occurs

- $x \in \bigcup\phi \cap (\{irj : i,j \in \mathbb{N}\} \cup \{i = j : i,j \in \mathbb{N}\})$
- $x \in \bigcup\phi \cap (\mathsf{Var} \times \mathbb{N})$.

We shall state that branch ϕ *generates* model $\mathfrak{M}_{\mathbf{ML}} = \langle W, Q, V, w\rangle$ iff

- W is a maximal subset of set $\{i : i \in *(AT(\phi))\}$ such that:
  a. for any $i,j \in \mathbb{N}$, if $i = j \in AT(\phi)$, then $i \notin W$ or $j \notin W$
  b. $k \in W$
- for any $i,j \in \mathbb{N}$
  a. $\langle i,j\rangle \in Q$ iff $irj \in AT(\phi)$ and $i,j \in W$
  b. $V(x,i) = 1$ iff $\langle x,i\rangle \in AT(\phi)$ and $i \in W$
- $w = k$.

Remark 6.45. In definition of branch generating model 6.44, the domain of model W was specified i.a. as follows: W is a maximal subset of set $\{i : i \in *(AT(\phi))\}$ such that for any $i,j \in \mathbb{N}$, if $i = j \in AT(\phi)$, then $i \notin W$ or $j \notin W$. Model generated by an open and maximal branch consists of indices included in the tableau expressions. In the event when the branch contains expression $i = j$, for some $i,j \in \mathbb{N}$, we must select one of the indices that belong to $*(\{i = j\})$, since expression $i = j$ is expected to state that it is about the same object in the domain.

Definition 6.44 obviously meets the general conditions of the definition of branch generating interpretation, since by virtue of the next conclusion, each open and maximal branch can be assigned a model.

Corollary 6.46. *Let $\boldsymbol{R_{ML}}$ be a set of tableau rules. Let ϕ be an open and maximal $\boldsymbol{R_{ML}}$-branch and let $X = \{\langle A,k\rangle : A \in Y\} \subseteq \bigcup\phi$, for some $k \in \mathbb{N}$ and non-empty subset $Y \subseteq \mathsf{For}_{ML}$. Then there exists model $\mathfrak{M}_{ML}$ such that branch ϕ generates $\mathfrak{M}_{ML}$.*

Proof. We get that conclusion from definition of open branch 5.39 applied to the modal tableau rules and definition of branch generating model 6.46. □

So, checking for a given set of tableau rules $\mathbf{R_{ML}}$ and given class of models $\mathbf{M}' \subseteq \mathbf{M}$, if set $\mathbf{M}'$ is good for set of rules $\mathbf{R_{ML}}$, does not require demonstration of the existence of model. We only have to — in accordance with definition 5.55 — demonstrate that in the generated model true are those formulas whose equivalents belonged to the branch and were used for the definition of model and that this model belongs to set $\mathbf{M}'$.

6.3.5 Completeness theorem of tableau systems for modal logics

We will now verbalize a general theorem on completeness of tableau systems for modal logics defined with the semantics of possible worlds.

Applying the general definitions of set of interpretations good for set of rules 5.55 and set of rules good for set of interpretations 5.59, the concepts defined in this chapter and general tableau theorem 5.62 we proved in the previous chapter, we obtain a general theorem on completeness for modal logics of defined with the semantics of possible worlds.

In order to frame the theorem, let us assume that for any set of models $\mathbf{M}' \subseteq \mathbf{M}$ and any set of tableau rules $\mathbf{R_{ML}}$ notation $X \vDash_{\mathbf{M}'} A$ means relation $\vDash$ defined by set of models $\mathbf{M}'$, whereas notation $X \rhd_{\mathbf{R_{ML}}} A$ means relation $\rhd$ defined by set of tableau rules $\mathbf{R_{ML}}$. Similarly, when writing $\mathbf{R_{ML}}$-tableau $\langle Y, A, \Phi\rangle$, we mean that tableau $\langle Y, A, \Phi\rangle$ and branches that belong to set of branches Φ originated merely by the application of rules from set $\mathbf{R_{ML}}$ to the expressions from set $\mathsf{Te}_{\mathbf{ML}}$.

Let us proceed to the theorem.

Theorem 6.47. *Let $\mathbf{M}' \subseteq \mathbf{M}$ be a set of models. Let $\mathbf{R_{ML}}$ be a set of tableau rules. If:*

- *set $\mathbf{M}'$ is good for set of rules $\mathbf{R_{ML}}$*
- *set $\mathbf{R_{ML}}$ is good for set of models $\mathbf{M}'$,*

then for any set $X \subseteq \mathsf{For}_{\mathbf{ML}}$ and any formula $A \in \mathsf{For}_{\mathbf{ML}}$ the following statements are equivalent:

- $X \vDash_{\mathbf{M}'} A$
- $X \rhd_{\mathbf{R_{ML}}} A$
- *there exists such finite set Y that $Y \subseteq X$ and closed $\mathbf{R_{ML}}$-tableau $\langle Y, A, \Phi\rangle$.*

Theorem 6.47 reduces the constitution of complete tableau system for modal logic defined by the semantics of possible worlds to the demonstration of two facts that form the assumptions of this theorem. So, with the established set of models $\mathbf{M}'$ we must define the set of tableau rules so $\mathbf{R_{ML}}$ as to these two facts occur. Analogously, we may start from set of tableau rules $\mathbf{R_{ML}}$ and specify subset $\mathbf{M'} \subseteq \mathbf{M}$ of all models with possible worlds so as to the theorem assumptions occur. In both cases we will get a complete tableau system.

6.4 Tableau system

In this study, we have often used term *tableau system*, even though the emphasis was on the concept of tableau and branch consequence. Ultimately, however, it is the tableau system that constitutes the tableau recognition of given logic. Therefore, we will now try to clarify the concept of tableau system.

By a tableau system we may mean pair $\langle \mathsf{For}, \rhd_{\mathbf{R}} \rangle$, where For is a set of formulas of given logic, $\mathbf{R}$ set of tableau rules, whereas $\rhd$ is a relation of branch consequence, defined based on the set of all maximal $\mathbf{R}$-branches.

Such approach of the tableau system indirectly contains all concepts we have defined when constructing relation $\rhd$. To define the relation of branch consequence, it takes i.a. the concept of tableau expressions, t-inconsistent set, concept of tableau rules, description of relationship between the formulas and tableau expressions as well as the concepts of open/closed and maximal branches.

The concept of tableau and its various variants (open/closed tableau, complete/incomplete tableau) can be, in turn, regarded as an apprehension of the method for choosing a relatively small subset of set of branches that allows to determine the occurrence of relation of branch consequence $\rhd$.

The concepts presented here enable the investigation of the relationships between different tableau systems defined by the method described, e.g. related to the questions about the dependency of sets of tableau rules or to the economics of sets system construction in general. For example, using the modification of the proof of the general completeness theorem 5.62 (for one set of tableau rules, considering the implication (a), and for another, considering the implications (b) and (c)), we can frame a theorem on the relationships between the tableau systems.

Theorem 6.48. *Let $\langle \mathsf{For}, \vDash_{\mathbf{M}} \rangle$ be a logic semantically defined with set of interpretations $\mathbf{M}$. Let $\langle \mathsf{For}, \rhd_{\mathbf{R}} \rangle$ and $\langle \mathsf{For}, \rhd_{\mathbf{R'}} \rangle$ be tableau systems. If:*

1. *set $\mathbf{M}$ is good for set of rules $\mathbf{R'}$*
2. *set $\mathbf{R}$ is good for set of interpretations $\mathbf{M}$*

then for any set of formulas $X \subseteq \mathsf{For}$ and any formula $A \in \mathsf{For}$, if $X \rhd_{\mathbf{R}} A$, then $X \rhd_{\mathbf{R'}} A$.

Note that sets of tableau rules **R** and **R'** can be defined on completely different sets of tableau expressions, using different auxiliary concepts, so that the rules they include can be directly incomparable. But comparing their deductive power can be done by checking the dependencies between the sets of tableau rules and the set of interpretations **M**.

6.5 Transition from the formalized tableaux to the standard tableaux

The last issue we will be dealing with is the problem of the relationship between the concepts presented in the book — the concept of branch and the concept of tableau — and the conventionally comprehended tableaux.

We discuss this problem in the part devoted to the application of the theory of tableaux, guided by the belief that standard tableaux are a practical application of abstract concepts.[6]

However, since in most cases tableaux and tableau systems are presented primarily in graphic form, our goal was to create a formalization, and thus the theory of tableaux, independent of the tableaux comprehended in this way. The theory described here is precisely an abstract approach to the tableau methods for which the standard tableaux can be considered as applications.

Take any tableau system $\langle \mathsf{For}, \triangleright_{\mathbf{R}} \rangle$, where For is a set of formulas, whereas **R** is a set of tableau rules. From general definition of tableau rules 5.33, it follows that the set of rules **R** is associated to a minimal set of tableau expressions Te which was used for the definition of rules from set **R**.

Consider any branch created by the application of rules from set **R**. In accordance with general definition of branch 5.35, that branch is a certain injective function $\phi : K \longrightarrow P(\mathsf{Te})$, where $K = \mathbb{N}$ or $K = \{1, 2, 3, \ldots, n\}$, for some $n \in \mathbb{N}$, while $P(\mathsf{Te})$ is a set of all subsets of the set of tableau expressions.

We will now define an intuitive branch that will correspond to branch ϕ. Let M be linearly ordered by relation $\leq_M$ set of points such that there exists bijection $\alpha : K \longrightarrow M$ that meets condition $\alpha(i) \leq_M \alpha(j)$, if $i \leq j$, for any $i, j \in K$.

Now, we define function $\phi' : M \longrightarrow P(\mathsf{Te})$ from the set of points into the set of all subsets of the set of tableau expressions by the following condition for any $i \in M$:

1. $\phi'(i) = \phi(1)$, if $\alpha^{-1}(i) = 1$
2. $\phi'(i) = \phi(n) \setminus \phi(n-1)$, if $\alpha^{-1}(i) = n$ and $n > 1$, for any $n \in K$.

Function ϕ' specifies a linearly ordered set of points in which each point has a certain subset Te assigned. Anyway, point $\alpha(1)$ has assigned entire set $\phi(1)$ from

6 So we share the remark of Melvin Fitting who wrote in *Handbook of Tableau Methods* that the proof trees (graphs), applied as proofs in the tableau systems, make up an application of a more abstract approach to logics (p. 5, [2]).

branch ϕ — the literature usually refers to that point as *root* — while the other points have assigned sets of elements that make up the difference between the elements of branch ϕ and their immediate predecessors.

In this way, we can assign to each branch ϕ at least one branch ϕ', with some established domain — set of points M linearly ordered by relation $\leq_M$.

Example 6.49. Take set of tableau expressions $\{\langle\Diamond p,1\rangle, \langle\Diamond(p \wedge q),1\rangle, \langle\neg\neg q,1\rangle\}$. Applying rules from $\mathbf{R_{S5}}$ in the order described on left hand side, we get the following branch which will be defined as ψ.

$$\begin{array}{rl}
R_\Diamond \;\; X_1 = & \{\langle\Diamond p, 1\rangle, \langle\Diamond(p \wedge q), 1\rangle, \langle\neg\neg q, 1\rangle\} \\
& | \\
R_\wedge \;\; X_2 = & X_1 \cup \{1r2, \langle(p \wedge q), 2\rangle\} \\
& | \\
R_{\neg\neg} \;\; X_3 = & X_2 \cup \{\langle p, 2\rangle, \langle q, 2\rangle\} \\
& | \\
X_4 = & X_3 \cup \{\langle q, 1\rangle\}
\end{array}$$

Transition from branch ϕ to function ϕ' will be called a *branch translation* or simply *translation*. If function $\phi' : M \longrightarrow P(\mathsf{Te})$ is a translation, then notation ϕ'_M indicates domain of ϕ'.

The result of certain translation ϕ' will be presented in the graphs below. Example 6.49 described branch ψ. Now, if we carry out the translation of branch ψ, then we can get the following graph presenting branch ψ'. This time — unlike so far — we will put the application of the rule on the right hand side and next to the points of branch that resulted from the application of given rule.

$$\begin{array}{ll}
1. & \{\langle\Diamond p, 1\rangle, \langle\Diamond(p \wedge q), 1\rangle, \langle\neg\neg q, 1\rangle\} \\
& | \\
2. & \{1r2, \langle(p \wedge q), 2\rangle\} \quad R_\Diamond \\
& | \\
3. & \{\langle p, 2\rangle, \langle q, 2\rangle\} \quad R_\wedge \\
& | \\
4 & \{\langle q, 1\rangle\} \quad R_{\neg\neg}
\end{array}$$

And if we remove the numbering, brackets of sets and brackets from the tableau expressions, and in lieu of numbers we will insert points •, then we will get a graph of branch ψ' corresponding to the graphs usually used to present branches.

$$\begin{array}{l}
\bullet\; \Diamond p, 1, \; \Diamond(p \wedge q), 1, \; \neg\neg q, 1 \\
| \\
\bullet\; 1r2, (p \wedge q), \; 2 \quad R_\Diamond \\
| \\
\bullet\; p, 2, \; q, 2 \quad R_\wedge \\
| \\
\bullet\; q, 1 \quad R_{\neg\neg}
\end{array}$$

So we can see that we can move from the branch defined in our theory — the formalized branch — to the standard branch, and the other way round — from the intuitive branch to the formalized branch. However, moving in the other direction requires a more precise determination of what a branch is, as well as determination of what the tableau rules are. Nonetheless, these issues are clarified by the theory that has just been presented — so it is easier to apply the general concepts in practice.

Since we already have a certain method of applying the general concept of branch in practice, we can now describe the transition from formalized tableaux to standard ones.

We are still considering an arbitrary tableau system $\langle \mathsf{For}, \rhd_{\mathbf{R}} \rangle$ — where For is a set of formulas, while **R** is a set of tableau rules — and set of tableau expressions Te on which the tableau rules from set **R** were defined.

Consider tableau $\langle X, A, \Phi \rangle$, where $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$, whereas Φ is a set of branches — in accordance with general definition of tableau 5.48.

Based on set Φ we can define sets that only include translations of branches from set Φ — one for each branch from Φ. We impose condition: $(*)$ Ψ is such set of translations of branches from Φ that for each branch $\phi \in \Phi$, Ψ contains precisely one translation of ϕ. Since for each branch $\phi \in \Phi$, Ψ contains precisely one translation of ϕ, so the translation of branch ϕ in given set Ψ will be denoted as ϕ'.

Among the sets that meet condition $(*)$ there is at least one such set Φ' that the distribution of points and assigned in the translations included in set Φ' sets of expressions corresponds to the distribution of differences between the consequents and antecedents in the relevant branches contained in Φ as well as the inclusion of sub-branches of some branches in another ones.

We can carry out the following action on set Φ'. Define set $\langle \Phi'', \leq_{\Phi''} \rangle$:

1. $\Phi'' = \bigcup \Phi'$
2. $\langle n, X \rangle \leq_{\Phi''} \langle k, Y \rangle$ iff there exists such translation $\phi'_M \in \Phi'$ that $n, k \in M$, $\phi'_M(n) = X$, $\phi'_M(k) = Y$ and $n \leq_M k$.

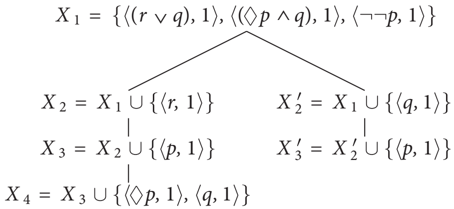
In set Φ'' there exists an element which is the smallest in terms of relation $\leq_{\Phi''}$ — that element is included in a set that is identical to the first element of each branch that belongs to set of branches Φ, i.e. the first step in the proof of fact that $X \rhd A$.

Set $\langle \Phi'', \leq_{\Phi''} \rangle$ is an application of the definition of abstract concept of tableau $\langle X, A, \Phi \rangle$. In this way, to each tableau $\langle X, A, \Phi \rangle$ we can assign at least one tableau $\langle \Phi'', \leq_{\Phi''} \rangle$.

The transition from tableau $\langle X, A, \Phi\rangle$ to tableau $\langle \Phi'', \leq_{\Phi''}\rangle$ will be called *tableau translation* or simply *translation*.

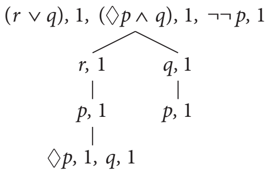
Here is an example. Again, consider set of tableau rules $\mathbf{R_{S5}}$ described in Chapter Four.

Example 6.50. Consider the following set of expressions $\{\langle(r \vee q),1\rangle, \langle\neg\neg p,1\rangle, \langle(\Diamond p \wedge q),1\rangle\}$. By the use of rule $R_\vee$, and then on left hand side — rules $R_{\neg\neg}$ and $R_\wedge$, and on the right have side — rules $R_{\neg\neg}$, we get the following branches.



In the light of definition of tableau 4.47, the set of these two branches is a tableau for pair $\langle\{(r \vee q), (\Diamond p \wedge q)\}, \neg p\rangle$, although it is not complete tableau.

And if we remove the brackets of sets and brackets from the tableau expressions, then we will get a graph of tableau corresponding to the graphs usually used to present tableaux.



This will be a result of some translation of set of branches Φ, i.e. some partially ordered set with the smallest element which is the root of the proof tree. We do not insert the denotations of points •. Instead of the denotations, there are individual lines with expressions.

Glossary

CPL	Classical Propositional Logic
Lc	logical constants
Var	propositional letters
For_{CPL}	formulas of **CPL**
v	valuation of propositional letters
V	valuation of formulas
t-inconsistent	tableau inconsistent set
t-consistent	tableau consistent set
$\boldsymbol{R}_{CPL}$	tableau rules for **CPL**
Ln	name letters
For_{TL}	formulas of **TL**
Ci	individual constants
Vi	individual variables
Lp	predicate letters
Tr	transformation
Te_{TL}	tableau expressions for *TL*
$\boldsymbol{R}_{TL}$	tableau rules for **TL**
For_{S5}	formulas *S*5
Te_{S5}	tableau expressions for *S*5
$\boldsymbol{R}_{S5}$	tableau rules for **S5**
For	formulas generally
$\langle \mathsf{For}, \vDash_{\mathfrak{J}} \rangle$	semantically defined logic
Te^{in}	tableau contradictory set
Te	tableau expressions
TeA	tableau auxiliary expressions
$\triangleright$	branch consequence relation
$\langle X, A, \Phi \rangle$	tableau
For_{MTL}	formulas *MTL*
Te_{MTL}	tableau expressions for *MTL*
For_{ML}	formulas for modal logic
Te_{ML}	tableau expressions for modal logic

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