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# An Introduction to Linear Algebra

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*To our families*

# Preface

Linear algebra is everywhere in the world of science and engineering. See [1, 4, 7, 10, 12, 14-19, 21-23, 25]. The present book is meant as a text for a course on linear algebra at the first-year undergraduate level. It is self-contained. The purpose of the book is to provide a solid foundation for further study of advanced mathematics.

At the beginning of the book, we introduce linear systems over the real field, solutions of linear systems by Gauss-Jordan elimination, and basic terminology of matrix. Especially we study elementary matrices to explain the processes of Gauss-Jordan elimination in matrix form.

We introduce determinant functions in Chapter 2 in order to study Cramer's rule which is an explicit representation for a linear system that has a unique solution. We discuss fundamental properties of determinants and the way to evaluate determinants through cofactor expansions.

As a fundamental example of vector spaces, we first introduce the Euclidean vector spaces in Chapter 3. We study the Cauchy-Schwarz inequality and linear transformations between two Euclidean vector spaces. The most important properties of the Euclidean vector spaces will be used to develop the concept of general vector spaces later.

In Chapter 4, we begin with the definition of general vector spaces over the real field. We mainly study subspaces, linearly independent sets, and bases for vector spaces. As important examples, we discuss four fundamental matrix spaces and study their properties. The dimension theorem for subspaces, the dimension theorem for matrices, and consistency theorems are also included.

As a superstructure of vector spaces, we introduce an inner product on general vector spaces in Chapter 5. By using the inner product, we can define notions of length, distance, angle, and orthogonality in general vector spaces. These notions are the foundation of subsequent studies on the Gram-Schmidt process for orthogonal bases and least squares problems. Besides, we also discuss the problem of change of basis in the last section of this chapter.

Chapter 6 presents one of the most important topics in linear algebra: eigenvalues and eigenvectors of square matrices. With these concepts and their related theorems, we study how to diagonalize a diagonalizable matrix, especially a symmetric matrix. Finally, the Jordan decomposition theorem is briefly mentioned.

In Chapter 7, we introduce general linear transformations between two general vector spaces and study their related properties which involve kernel, range, rank, nullity, inverse, and so on. We also discuss matrices of general linear transformations and show that a general linear transformation between two general vector spaces can be regarded as a matrix transformation between two Euclidean vector spaces.

In the last chapter, we develop several important topics in linear algebra, including quadratic forms, complex inner product spaces, Hermitian matrices and unitary matrices. A well-known fact in linear algebra is that the matrix product is not commutative, i.e., in general,

$$XY \neq YX,$$

where $X$ and $Y$ are square matrices. Böttcher and Wenzel proposed the following conjecture in 2005:

$$\|XY - YX\|_F \leqslant \sqrt{2}\,\|X\|_F\|Y\|_F,$$

where $\|\cdot\|_F$ is the Frobenius norm. In the last part of the book, we give an elementary proof of the Böttcher-Wenzel conjecture, where only several classical theorems studied in the book are used.

In writing the present book, many friends have offered us helps, advice, comments, and encouragement. First, we would like to express gratitude to the following people: Professors Raymond H.F. Chan, Hong-Kun Xu, Jin-Yun Yuan, Fu-Zhen Zhang, Zhao-Liang Xu, Chong Li, Dan-Fu Han, Wen Li, Jian-Long Chen, Qing-Biao Wu, Man-Chung Yeung, Yi-Min Wei, Che-Man Cheng, Michael K.P. Ng, Wai-Ki Ching, Fu-Rong Lin, Hai-Wei Sun, Hao-Min Zhou, Zheng-Jian Bai, Jian-Feng Cai, Vai-Kuong Sin, Gang Wu, Matthew M.H. Lin, Jin-Hua Wang, Wei-Ping Shen, Seak-Weng Vong, Siu-Long Lei, Kit-Ian Kou, Zhi-Gang Jia, Xiao-Shan Chen, Xiao-Fei Peng, Rong Huang, Juan Zhang, Ying-Ying Zhang, Hong-Kui Pang, Qing-Jiang Meng, Ze-Jia Xie, and Teng-Teng Yao. Special thanks go to one of the greatest mathematicians around the world, Professor Shing-Tung Yau from Department of Mathematics, Harvard University, for providing us valuable words at the beginning of Chapter 5. Of course, we are particularly grateful to Mr. Xiang Zhao for creating the cover painting for the book. Finally, we appreciate the most important institution in authors' life: University of Macau, for supplying such a wonderful intellectual atmosphere for writing the book. The book is dedicated to the 40th anniversary of University of Macau (1981–2021).

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# Contents

# Chapter 1

# Linear Systems and Matrices

"*No beginner's course in mathematics can do without linear algebra.*"

— Lars Gårding

"*Matrices act. They don't just sit there.*"

— Gilbert Strang

Solving linear systems (a system of linear equations) is the most important problem of linear algebra and possibly of applied mathematics as well. Usually, information in a linear system is often arranged into a rectangular array, called a "matrix". The matrix is particularly important in developing computer programs to solve linear systems with huge sizes because computers are suitable to manage numerical data in arrays. Moreover, matrices are not only a simple tool for solving linear systems but also mathematical objects in their own right. In fact, matrix theory has a variety of applications in science, engineering, and mathematics. Therefore, we begin our study on linear systems and matrices in the first chapter.

## 1.1 Introduction to Linear Systems and Matrices

Let $\mathbb{R}$ denote the set of real numbers. We now introduce linear equations, linear systems, and matrices.

### 1.1.1 Linear equations and linear systems

We consider

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ are coefficients, $x_i$ $(i = 1, 2, \ldots, n)$ are variables (unknowns), $n$ is a positive integer, and $b \in \mathbb{R}$ is a constant. An equation of this form is called a ***linear equation***, in which all variables occur to the first power. When $b = 0$, the linear equation is called a ***homogeneous linear equation***. A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a solution of the equation if $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ such that

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n = b.$$

The set of all solutions of the equation is called the ***solution set*** of the equation.

In the book, we always use example(s) to make our points clear.

**Example** We consider the following linear equations:

(a) $x + y = 1$.

(b) $x + y + z = 1$.

It is easy to see that the solution set of (a) is a line in $xy$-plane and the solution set of (b) is a plane in $xyz$-space.

We next consider the following $m$ linear equations in $n$ variables:

$$\begin{cases} a_{11}x_1 \ + \ a_{12}x_2 \ + \cdots + \ a_{1n}x_n \ = b_1 \\ a_{21}x_1 \ + \ a_{22}x_2 \ + \cdots + \ a_{2n}x_n \ = b_2 \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases} \tag{1.1}$$

where $a_{ij} \in \mathbb{R}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are coefficients, $x_j$ $(j = 1, 2, \ldots, n)$ are variables, and $b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m)$ are constants. A system of linear equations in this form is called a ***linear system***. A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a solution of the system if $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is a solution of each equation in the system. A linear system is said to be consistent if it has at least one solution. Otherwise, a linear system is said to be inconsistent if it has no solution.

**Example** Consider the following linear system

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2. \end{cases}$$

The graphs of these equations are lines called $l_1$ and $l_2$. We have three possible cases of lines $l_1$ and $l_2$ in $xy$-plane. See Figure 1.1.

- When $l_1$ and $l_2$ are parallel, there is no solution of the system.
- When $l_1$ and $l_2$ intersect at only one point, there is exactly one solution of the system.

- When $l_1$ and $l_2$ coincide, there are infinitely many solutions of the system.

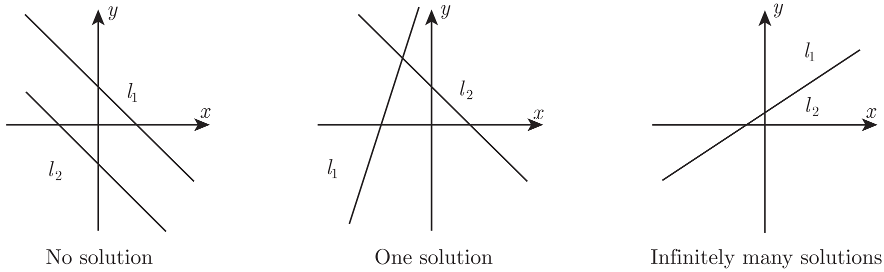



Figure 1.1

### 1.1.2 Matrices

The term matrix was first introduced by a British mathematician James Sylvester in the 19th century. Another British mathematician Arthur Cayley developed basic algebraic operations on matrices in the 1850s. Up to now, matrices have become the language to know.

**Definition** *A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.*

**Remark** The size of a matrix is described in terms of the number of rows and columns it contains. Usually, a matrix with $m$ rows and $n$ columns is called an $m \times n$ matrix. If $A$ is an $m \times n$ matrix, then we denote the entry in row $i$ and column $j$ of $A$ by the symbol $(A)_{ij} = a_{ij}$. Moreover, a matrix with real entries will be called a real matrix and the set of all $m \times n$ real matrices will be denoted by the symbol $\mathbb{R}^{m \times n}$. For instance, a matrix $A$ in $\mathbb{R}^{m \times n}$ can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $a_{ij} \in \mathbb{R}$ for any $i$ and $j$. When compactness of notation is desired, the preceding matrix can be written as

$$A = [a_{ij}] .$$

In particular, if $A \in \mathbb{R}^{1\times 1}$, then $A = a_{11} \in \mathbb{R}$.

We now introduce some important matrices with special sizes. A ***row matrix*** is a general $1 \times n$ matrix $\mathbf{a}$ given by

$$\mathbf{a} = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{1\times n}.$$

A ***column matrix*** is a general $m \times 1$ matrix $\mathbf{b}$ given by

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{m\times 1}.$$

A ***square matrix*** is an $n \times n$ matrix $A$ given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{1.2}$$

The ***main diagonal*** of the square matrix $A$ is the set of entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (1.2).

For linear system (1.1), we can write it briefly as the following matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix},$$

which is called the ***augmented matrix*** of (1.1).

**Remark** When we construct an augmented matrix associated with a given linear system, the unknowns must be written in the same order in each equation and the constants must be on the right.

### 1.1.3 Elementary row operations

In order to solve a linear system efficiently, we replace the given system with its augmented matrix and then solve the same system by operating on the rows of the augmented matrix. There are three elementary row operations on matrices defined as follows:

(1) Interchange two rows.

(2) Multiply a row by a nonzero number.

(3) Add a multiple of one row to another row.

By using elementary row operations, we can always reduce the augmented matrix of a given system to a simpler augmented matrix from which the solution of the system is evident. See the following example.

**Example** Consider the following system

$$\begin{cases} x + \phantom{4}y + \phantom{3}z = 6 \\ 2x + 4y - 3z = 1 \\ 3x + 2y - 2z = 1. \end{cases}$$

The augmented matrix of the system is given by

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 4 & -3 & 1 \\ 3 & 2 & -2 & 1 \end{bmatrix}.$$

By using elementary row operations, actually one can transform the augmented matrix of the system to a simpler form,

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 4 & -3 & 1 \\ 3 & 2 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Then from the simpler form, we immediately have

$$x = 1, \quad y = 2, \quad z = 3,$$

which is obviously the solution of the original system. See next section for details.

## 1.2 Gauss-Jordan Elimination

In this section, we develop a method called ***Gauss-Jordan elimination*** [1] for solving linear systems. In fact, Gauss-Jordan elimination is the most frequently used algorithm in scientific computing.

### 1.2.1 Reduced row-echelon form

In the example of Subsection 1.1.3, we solved the given linear system by reducing the augmented matrix to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

from which the solution of the system was evident. This is an example of a matrix that is in ***reduced row-echelon form***. We therefore give the following definition.

**Definition** *For any matrix in reduced row-echelon form, it must satisfy the following conditions.*

(i) *For the rows that consist entirely of zeros, they are grouped together at the bottom of the matrix. The rows that consist entirely of zeros will be called zero rows.*

(ii) *If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a* ***leading 1****.*

(iii) *For two successive rows that both contain leading 1's, the leading 1 in the higher row occurs farther to the left than the leading 1 in the lower row.*

(iv) *Each column that contains a leading 1 has zeros in all its other entries.*

**Remark** A matrix having properties (i), (ii), (iii), but not necessarily (iv), is said to be in ***row-echelon form***. The following example is in row-echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

### 1.2.2 Gauss-Jordan elimination

Gauss-Jordan elimination is a standard technique for solving linear systems. Actually Gauss-Jordan elimination is a step-by-step elimination procedure which reduces an augmented matrix of a given linear system to reduced row-echelon form. Then the solution set of the system can be found by just inspection. We illustrate the idea by the following example.

**Example** We solve the following system

$$\begin{cases} \quad\;\; - \;3x_2 \qquad\qquad\qquad\quad +7x_5 = 15 \\ 2x_1 + \;6x_2 + 6x_3 + 4x_4 + 2x_5 = 28 \\ 2x_1 + 11x_2 + 6x_3 + 4x_4 - 9x_5 = \;\;5. \end{cases}$$

The augmented matrix of the system is given by

$$\begin{bmatrix} 0 & -3 & 0 & 0 & 7 & 15 \\ 2 & 6 & 6 & 4 & 2 & 28 \\ 2 & 11 & 6 & 4 & -9 & 5 \end{bmatrix}.$$

Now, by using the elementary row operations, we are going to reduce the matrix to reduced row-echelon form.

**Step 1.** Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the leftmost column that does not consist entirely of zeros:

$$\xrightarrow{\text{Interchange 1st row and 2nd row}} \begin{bmatrix} 2 & 6 & 6 & 4 & 2 & 28 \\ 0 & -3 & 0 & 0 & 7 & 15 \\ 2 & 11 & 6 & 4 & -9 & 5 \end{bmatrix}.$$

**Step 2.** If the entry that is now at top of the column found in **Step 1** is $a \neq 0$, multiply the first row by $1/a$ in order to introduce the leading 1:

$$\xrightarrow{1/2 \times \text{1st row}} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 14 \\ 0 & -3 & 0 & 0 & 7 & 15 \\ 2 & 11 & 6 & 4 & -9 & 5 \end{bmatrix}.$$

**Step 3.** Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros:

$$\xrightarrow{\text{3rd row} + (-2) \times \text{1st row}} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 14 \\ 0 & -3 & 0 & 0 & 7 & 15 \\ 0 & 5 & 0 & 0 & -11 & -23 \end{bmatrix}.$$

**Step 4.** Now cover the top row in the matrix and begin again with **Step 1** applied to the submatrix remained. Continue in this way until the entire matrix is in row-echelon form:

$$\xrightarrow{(-1/3) \times \text{2nd row}} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 14 \\ 0 & 1 & 0 & 0 & -\dfrac{7}{3} & -5 \\ 0 & 5 & 0 & 0 & -11 & -23 \end{bmatrix}$$

$$\xrightarrow{\text{3rd row} + (-5) \times \text{2nd row}} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 14 \\ 0 & 1 & 0 & 0 & -\dfrac{7}{3} & -5 \\ 0 & 0 & 0 & 0 & \dfrac{2}{3} & 2 \end{bmatrix}$$

$$\xrightarrow{3/2 \times \text{3rd row}} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 14 \\ 0 & 1 & 0 & 0 & -\dfrac{7}{3} & -5 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

The entire matrix is now in row-echelon form. To find the reduced row-echelon form we need the following additional step.

**Step 5.** Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's:

$$\xrightarrow{\text{2nd row} + (7/3) \times \text{3rd row}} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 14 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{1st row} + (-1) \times \text{3rd row}} \begin{bmatrix} 1 & 3 & 3 & 2 & 0 & 11 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{1st row} + (-3) \times \text{2nd row}} \begin{bmatrix} 1 & 0 & 3 & 2 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

The last matrix is in reduced row-echelon form.

The corresponding system is

$$\begin{cases} x_1 \qquad + 3x_3 + 2x_4 \qquad = 5 \\ \quad x_2 \qquad\qquad\qquad\quad = 2 \\ \qquad\qquad\qquad\qquad x_5 = 3. \end{cases}$$

Since $x_1$, $x_2$, and $x_5$ correspond to leading 1's in reduced row-echelon form of the augmented matrix, we call them leading variables. The remaining variables $x_3$ and $x_4$ are called free variables. Solving the leading variables yields

$$\begin{cases} x_1 = -3x_3 - 2x_4 + 5 \\ x_2 = 2 \\ x_5 = 3. \end{cases}$$

Setting $x_3 = s$ and $x_4 = t$, we therefore obtain the solution set of the system,

$$\begin{cases} x_1 = -3s - 2t + 5 \\ x_2 = 2 \\ x_5 = 3, \end{cases}$$

where $s$ and $t$ can take arbitrary values.

### 1.2.3 Homogeneous linear systems

A linear system is called to be ***homogeneous*** if the constant terms are all zero. Consider

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0. \end{cases}$$

Obviously, $x_1 = x_2 = \cdots = x_n = 0$ is a solution of the system, which is called the trivial solution. Any nonzero solutions are called nontrivial solutions. For nontrivial solutions, we have the following theorem.

**Theorem 1.1** *A homogeneous linear system has infinitely many solutions if there are more variables than equations.*

*Proof* Let

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m + \cdots + a_{mn}x_n = 0, \end{cases}$$

where $m < n$. By using elementary row operations, one can obtain the reduced row-echelon form of the augmented matrix of the system. It follows from the reduced row-echelon form that the corresponding system has the following form

$$\begin{cases} x_{k_1} & & & + \sum(\ ) = 0 \\ & x_{k_2} & & + \sum(\ ) = 0 \\ & & \ddots & \qquad \vdots \\ & & x_{k_r} & + \sum(\ ) = 0, \end{cases}$$

where $k_1 < k_2 < \cdots < k_r$ are numbers in the set $\{1, 2, \ldots, m\}$ and $\sum(\ )$ denotes sums that involve the $n - r$ free variables. We remark that $r \leqslant m < n$ and usually

$k_1 = 1$. If $r = m$, then there is no zero row. Finally, we obtain

$$\begin{cases} x_{k_1} = -\sum(\ ) \\ x_{k_2} = -\sum(\ ) \\ \quad\vdots \\ x_{k_r} = -\sum(\ ), \end{cases}$$

which implies that the system has infinitely many solutions. $\square$

**Example** We consider the following linear system with 6 variables and 4 equations,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 + a_{16}x_6 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 + a_{26}x_6 = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + a_{35}x_5 + a_{36}x_6 = 0 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + a_{45}x_5 + a_{46}x_6 = 0. \end{cases}$$

The augmented matrix $A$ of the system is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 7}.$$

By using elementary row operations, if the reduced row-echelon form of $A$ is obtained as

$$\begin{bmatrix} 1 & b_{12} & 0 & 0 & 0 & b_{16} & 0 \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} & 0 \\ 0 & 0 & 0 & 0 & 1 & b_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then the corresponding system is

$$\begin{cases} x_1 + b_{12}x_2 \qquad\qquad\qquad\qquad + b_{16}x_6 = 0 \\ \qquad\qquad\qquad x_3 + b_{24}x_4 \qquad + b_{26}x_6 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad x_5 + b_{36}x_6 = 0. \end{cases}$$

We therefore have

$$\begin{cases} x_1 = -b_{12}x_2 - b_{16}x_6 \\ x_3 = -b_{24}x_4 - b_{26}x_6 \\ x_5 = -b_{36}x_6. \end{cases}$$

From the proof of Theorem 1.1, it follows that $r = 3$, $k_1 = 1$, $k_2 = 3$, and $k_3 = 5$. There are three free variables, say, $x_2$, $x_4$, and $x_6$. Thus, the system has infinitely many solutions.

## 1.3 Matrix Operations

Matrices appear in many contexts other than as augmented matrices for linear systems. In this section, we begin our study on matrix theory by giving some basic definitions of the subject. We also introduce some operations on matrices and discuss their fundamental properties.

### 1.3.1 Operations on matrices

Now, we develop an arithmetic of matrices which contains the sum, difference, product of matrices, and so on. We have the following definition of operations on matrices.

**Definition**

(i) *Equal of matrices*: *Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, denoted by $A = B$, if they have the same size and $a_{ij} = b_{ij}$ for all $i, j$.*

(ii) *Sum and difference*: *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size. Then $A + B$ is a matrix with the entries given by $(A + B)_{ij} := a_{ij} + b_{ij}$ for all $i, j$, and $A - B$ is a matrix with the entries given by $(A - B)_{ij} := a_{ij} - b_{ij}$ for all $i, j$.*

(iii) *Scalar multiplication*: *Let $A = [a_{ij}]$ and $c$ be any scalar. Then $cA$ is a matrix with the entries given by $(cA)_{ij} := ca_{ij}$ for all $i, j$.*

(iv) *Linear combination of matrices*: $\sum_{i=1}^{s} c_i A_{(i)}$, *where $A_{(i)}$ $(1 \leqslant i \leqslant s)$ are matrices of the same size and $c_i$ $(1 \leqslant i \leqslant s)$ are scalars.*

(v) *Matrix product*: *Let $A = [a_{ij}]$ be a general $m \times r$ matrix and $B = [b_{ij}]$ be a general $r \times n$ matrix. Then the product of $A$ and $B$ is an $m \times n$ matrix denoted* *by*

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ \hline a_{i1} & a_{i2} & \cdots & a_{ir} \\ \hline \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

*with entries* $(AB)_{ij}$ *defined by*

$$(AB)_{ij} := a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \sum_{k=1}^{r} a_{ik}b_{kj}$$

*for all* $1 \leqslant i \leqslant m$ *and* $1 \leqslant j \leqslant n$.

**Remark** In the definition of matrix product, the number of columns of the first factor $A$ must be the same as the number of rows of the second factor $B$ in order to form the product $AB$. If the condition is not satisfied, then the product is undefined. Even if $A$ and $B$ are both $n \times n$ matrices, we usually have $AB \neq BA$.

**Example 1** Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 0 & 4 \\ 0 & -3 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

Since $A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 4$ matrix, the product $AB$ is a $2 \times 4$ matrix given by

$$AB = \begin{bmatrix} 7 & 7 & 3 & 4 \\ 4 & 7 & 2 & 0 \end{bmatrix}.$$

But the product $BA$ is undefined.

**Example 2** Let

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 2 & -3 & 0 \\ 0 & 4 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -2 \\ 1 & 6 & 2 \end{bmatrix}.$$

The product of matrices $A$ and $B$ is given by

$$AB = \begin{bmatrix} 1 & 5 & 4 \\ 2 & -3 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -2 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 20 & 0 \\ 6 & 5 & 10 \\ 1 & 2 & -6 \end{bmatrix}$$

and the product of matrices $B$ and $A$ is given by

$$BA = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -2 \\ 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 4 \\ 2 & -3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 20 & 14 \\ -2 & -5 & -2 \\ 13 & -5 & 6 \end{bmatrix}.$$

But $AB \neq BA$.

### 1.3.2 Partition of matrices

A matrix can be partitioned into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For instance, below are three possible partitions of a general $4 \times 5$ matrix $A$. The first one is a partition of $A$ into four ***submatrices*** $A_{11}$, $A_{12}$, $A_{21}$, and $A_{22}$; the second one is a partition of $A$ into its row matrices $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$; the third one is a partition of $A$ into its column matrices $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$, and $\mathbf{c}_5$:

$$A = \left[\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};$$

$$A = \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hdashline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ \hdashline a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right] = \left[\begin{array}{c} \mathbf{r}_1 \\ \hdashline \mathbf{r}_2 \\ \hdashline \mathbf{r}_3 \\ \hdashline \mathbf{r}_4 \end{array}\right];$$

$$A = \left[\begin{array}{c:c:c:c:c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right] = \left[\begin{array}{c:c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 \end{array}\right].$$

### 1.3.3 Matrix product by columns and by rows

Sometimes it may be desirable to find a particular row or column of a matrix product $AB$ without computing the entire product. The following results are useful for that purpose. Let $A \in \mathbb{R}^{m \times r}$ and $B \in \mathbb{R}^{r \times n}$. Then

$$j\text{th column matrix of } AB = \begin{bmatrix} \displaystyle\sum_{k=1}^{r} a_{1k} b_{kj} \\ \vdots \\ \displaystyle\sum_{k=1}^{r} a_{mk} b_{kj} \end{bmatrix} = A[\, j\text{th column matrix of } B \,]$$

and

$$i\text{th row matrix of } AB = \left[\ \sum_{k=1}^{r} a_{ik}b_{k1}, \ldots, \sum_{k=1}^{r} a_{ik}b_{kn}\ \right] = [\ i\text{th row matrix of } A\ ]B.$$

**Remark** Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ denote the row matrices of $A$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ denote the column matrices of $B$. It follows from the formulas above that

$$AB = A\left[\begin{array}{c:c:c:c} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{array}\right] = \left[\begin{array}{c:c:c:c} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array}\right], \tag{1.3}$$

which shows that $AB$ can be computed column by column, and

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ -- \\ \mathbf{a}_2 \\ -- \\ \vdots \\ -- \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ --- \\ \mathbf{a}_2 B \\ --- \\ \vdots \\ --- \\ \mathbf{a}_m B \end{bmatrix}, \tag{1.4}$$

which shows that $AB$ can also be computed row by row.

### 1.3.4 Matrix product of partitioned matrices

From the remark in Subsection 1.3.3, we know that the computation of a matrix product can be completed by some special partitions of matrices. We now introduce the general case. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{bmatrix} \in \mathbb{R}^{m\times l}, \qquad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{st} \end{bmatrix} \in \mathbb{R}^{l\times n},$$

where the number of columns of submatrix $A_{ik}$ is equal to the number of rows of submatrix $B_{kj}$ for each $1 \leqslant i \leqslant r$, $1 \leqslant k \leqslant s$, and $1 \leqslant j \leqslant t$. Then we construct

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ \vdots & \vdots & & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rt} \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where each submatrix of $C$ is given by

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{is}B_{sj} = \sum_{k=1}^{s} A_{ik}B_{kj}$$

for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant t$. In fact, the partitioned matrix $C$ is nothing new but the product of matrices $A$ and $B$, i.e., $C = AB$. See the following example.

**Example** Let $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 2}$. Below are the partitions of $A$ and $B$:

$$A = \left[\begin{array}{cc:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hdashline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \left[\begin{array}{c:c} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \hdashline b_{31} & b_{32} \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$= \left[\begin{array}{c:c} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} b_{31} \end{bmatrix} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} b_{32} \end{bmatrix} \\ \hdashline \begin{bmatrix} a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + \begin{bmatrix} a_{33} \end{bmatrix} \begin{bmatrix} b_{31} \end{bmatrix} & \begin{bmatrix} a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} + \begin{bmatrix} a_{33} \end{bmatrix} \begin{bmatrix} b_{32} \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c:c} \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{bmatrix} + \begin{bmatrix} a_{13}b_{31} \\ a_{23}b_{31} \end{bmatrix} & \begin{bmatrix} a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{13}b_{32} \\ a_{23}b_{32} \end{bmatrix} \\ \hdashline \begin{bmatrix} a_{31}b_{11} + a_{32}b_{21} \end{bmatrix} + \begin{bmatrix} a_{33}b_{31} \end{bmatrix} & \begin{bmatrix} a_{31}b_{12} + a_{32}b_{22} \end{bmatrix} + \begin{bmatrix} a_{33}b_{32} \end{bmatrix} \end{array}\right]$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix} = AB.$$

### 1.3.5 Matrix form of a linear system

In fact, the matrix product has an important application in solving linear systems. Consider linear system (1.1) of $m$ linear equations in $n$ unknowns. We can replace the $m$ equations in this system with the single matrix equation

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

By using the product of matrices, it follows that

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then the original system has been replaced by the single matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

Here $A$ is called the ***coefficient matrix*** of the system. Thus, the augmented matrix for the system is obtained by adjoining $\mathbf{b}$ to $A$ as the last column, i.e., $[\ A \mid \mathbf{b}\ ]$.

**Remark** Note that by using matrix operations on the above linear system, it can also be written as follows:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

i.e.,

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}, \tag{1.5}$$

where $\mathbf{c}_j$ is the $j$th column matrix of $A$ for $1 \leqslant j \leqslant n$.

### 1.3.6 Transpose and trace of a matrix

**Definition** *The **transpose** of an $m \times n$ matrix $A = [a_{ij}]$, denoted by $A^T$, is defined to be the $n \times m$ matrix with entries given by*

$$(A^T)_{ij} = a_{ji}.$$

*The **trace** of an $n \times n$ matrix $A = [a_{ij}]$ is given by*

$$\mathrm{tr}(A) := \sum_{i=1}^{n} a_{ii}.$$

For instance,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad \mathrm{tr}(A) = a_{11} + a_{22} + a_{33}.$$

Some important properties of the transpose are listed in the following theorem.

**Theorem 1.2** *Let the sizes of matrices $A$ and $B$ be such that the stated operations can be performed. Then*

(a) $(A^T)^T = A$.

(b) $(A \pm B)^T = A^T \pm B^T$.

(c) $(kA)^T = kA^T$, *where $k$ is any scalar.*

(d) $(AB)^T = B^T A^T$.

*Proof* Parts (a), (b), and (c) are self-evident. We therefore only prove (d). Let

$$A = [a_{ij}] \in \mathbb{R}^{m \times r}, \qquad B = [b_{ij}] \in \mathbb{R}^{r \times n}.$$

Then the products $(AB)^T$ and $B^T A^T$ can both be formed and they have the same size. It only remains to show that corresponding entries of $(AB)^T$ and $B^T A^T$ are the same, i.e., for all $i, j$,

$$\left((AB)^T\right)_{ij} = (B^T A^T)_{ij}. \tag{1.6}$$

Applying the definition of transpose of a matrix to the left-hand side of (1.6) and then using the definition of matrix product, we obtain

$$\left((AB)^T\right)_{ij} = (AB)_{ji} = \sum_{k=1}^{r} a_{jk} b_{ki}.$$

To evaluate the right-hand side of (1.6), let $A^T = [a'_{ij}]$ and $B^T = [b'_{ij}]$, then

$$a'_{ij} = a_{ji}, \qquad b'_{ij} = b_{ji}.$$

Furthermore, we have for all $i$ and $j$,

$$(B^T A^T)_{ij} = \sum_{k=1}^{r} b'_{ik} a'_{kj} = \sum_{k=1}^{r} b_{ki} a_{jk} = \sum_{k=1}^{r} a_{jk} b_{ki} = \left((AB)^T\right)_{ij}.$$

Thus, (d) holds. □

Some important properties of the trace are included in the following theorem.

**Theorem 1.3** *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then*

(a) $\mathrm{tr}(A) = \mathrm{tr}(A^T)$.

(b) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(c) $\mathrm{tr}(\alpha A + \beta B) = \alpha \mathrm{tr}(A) + \beta \mathrm{tr}(B)$, *where $\alpha$ and $\beta$ are any scalars.*

(d) $\mathrm{tr}(AB - BA) = 0$.

(e) $\mathrm{tr}(B) = 0$ *if* $B^T = -B$.

*Proof* Part(a) is obvious. For (b), because addition is associative and commutative, we have

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} = \sum_{k=1}^{n} (BA)_{kk} = \mathrm{tr}(BA).$$

For (c), we have

$$\begin{aligned} \mathrm{tr}(\alpha A + \beta B) &= \sum_{i=1}^{n} (\alpha A + \beta B)_{ii} = \sum_{i=1}^{n} (\alpha \cdot a_{ii} + \beta \cdot b_{ii}) \\ &= \alpha \sum_{i=1}^{n} a_{ii} + \beta \sum_{i=1}^{n} b_{ii} = \alpha \mathrm{tr}(A) + \beta \mathrm{tr}(B). \end{aligned}$$

For (d), it follows from (c) and (b) that

$$\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(AB) - \mathrm{tr}(AB) = 0.$$

For (e), by using (a), the given condition $B^T = -B$, and (c), we deduce

$$\mathrm{tr}(B) = \mathrm{tr}(B^T) = \mathrm{tr}(-B) = -\mathrm{tr}(B).$$

Thus, $\mathrm{tr}(B) = 0$. □

## 1.4 Rules of Matrix Operations and Inverses

In this section, we study some basic properties of the arithmetic operations on matrices.

### 1.4.1 Basic properties of matrix operations

**Theorem 1.4** *Let $A$, $B$, and $C$ be matrices and the sizes of matrices be assumed such that the indicated operations can be performed. The following rules of matrix operations are valid.*

(a) $A + B = B + A$. (*Commutative law for addition*)

(b) $A + (B + C) = (A + B) + C$. (*Associative law for addition*)

(c) $(AB)C = A(BC)$. (*Associative law for product*)

(d) $A(B \pm C) = AB \pm AC$. (*Left distributive law*)

(e) $(B \pm C)A = BA \pm CA$. (*Right distributive law*)

(f) $a(B \pm C) = aB \pm aC$.

(g) $(a \pm b)C = aC \pm bC$.

(h) $a(bC) = (ab)C$.

(i) $a(BC) = (aB)C = B(aC)$.

*Here $a$ and $b$ are any scalars.*

*Proof* We only prove (c) of the associative law for matrix product. The other parts here are left as an exercise. Assume that

$$A = [a_{ij}] \in \mathbb{R}^{s \times n}, \quad B = [b_{jk}] \in \mathbb{R}^{n \times m}, \quad C = [c_{kl}] \in \mathbb{R}^{m \times r}.$$

We want to show

$$(AB)C = A(BC).$$

Let

$$V = AB = [v_{ik}] \in \mathbb{R}^{s \times m}, \qquad W = BC = [w_{jl}] \in \mathbb{R}^{n \times r}.$$

Then

$$v_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

for $1 \leqslant i \leqslant s$ and $1 \leqslant k \leqslant m$, and

$$w_{jl} = \sum_{k=1}^{m} b_{jk} c_{kl}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant l \leqslant r$. Since $(AB)C = VC$, the entry in row $i$ and column $l$ of matrix $VC$ is given as follows:

$$(VC)_{il} = \sum_{k=1}^{m} v_{ik}c_{kl} = \sum_{k=1}^{m} \left( \sum_{j=1}^{n} a_{ij}b_{jk} \right) c_{kl} = \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}b_{jk}c_{kl} \tag{1.7}$$

for $1 \leqslant i \leqslant s$ and $1 \leqslant l \leqslant r$. Since $A(BC) = AW$, the entry in row $i$ and column $l$ of matrix $AW$ is given as follows:

$$(AW)_{il} = \sum_{j=1}^{n} a_{ij}w_{jl} = \sum_{j=1}^{n} a_{ij} \left( \sum_{k=1}^{m} b_{jk}c_{kl} \right) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ij}b_{jk}c_{kl} \tag{1.8}$$

for $1 \leqslant i \leqslant s$ and $1 \leqslant l \leqslant r$. Because addition is associative and commutative, the results in (1.7) and (1.8) should be the same. Hence the proof is completed. $\square$

### 1.4.2 Identity matrix and zero matrix

We define the identity matrix and the zero matrix as follows:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

**Remark** Throughout the book, we use the symbol $I_n$ to denote the $n \times n$ identity matrix. If there is no confusion, we sometimes use $I$ to denote the identity matrix with an appropriate size. Besides, we also use $\mathbf{e}_i$ to denote the $i$th column matrix of $I_n$, i.e.,

$$\mathbf{e}_i = \begin{bmatrix} 0 & \cdots & 0 & \underset{\substack{\uparrow \\ i\text{th}}}{1} & 0 & \cdots & 0 \end{bmatrix}^T.$$

**Theorem 1.5** *Let the sizes of the matrices be such that the indicated operations can be performed. The following rules of matrix operations are valid.*

(a) $AI = A, \quad IA = A.$

(b) $A + \mathbf{0} = \mathbf{0} + A = A.$

(c) $A\mathbf{0} = \mathbf{0}, \quad \mathbf{0}A = \mathbf{0}.$

The proof of the theorem is trivial and we therefore omit it.

**Example** Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $AB = \mathbf{0}$ even if both $A$ and $B$ are nonzero matrices. Thus, if $AB = \mathbf{0}$ for $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times r}$, perhaps it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$.

As the following theorem shows, the identity matrix is useful in studying reduced row-echelon forms of square matrices. The proof of the theorem is left as an exercise.

**Theorem 1.6** *Let $R$ be the reduced row-echelon form of a square matrix $A$. Then either $R$ has a row of zeros or $R = I$.*

### 1.4.3 Inverse of a matrix

**Definition** *Let $A$ and $B$ be square matrices of the same size such that*

$$AB = BA = I.$$

*Then $B$ is called an* ***inverse*** *of $A$, denoted by $B = A^{-1}$, and $A$ is said to be invertible. If no such $B$ exists, then $A$ is said to be not invertible.*

**Example** Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

One can verify that $B$ is an inverse of $A$ since

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

The next theorem shows that an invertible matrix has exactly one inverse.

**Theorem 1.7** *Let $B$ and $C$ be both inverses of the matrix $A$. Then $B = C$.*

*Proof* Since $B$ and $C$ are both inverses of $A$, we have

$$AB = BA = I, \qquad AC = CA = I.$$

From $BA = I$, multiplying both sides on the right by $C$ yields

$$(BA)C = IC = C.$$

However, we obtain by Theorem 1.4 (c),

$$(BA)C = B(AC) = BI = B.$$

Thus, $C = B$. $\square$

For a $2 \times 2$ invertible matrix, the following theorem gives a formula for constructing the inverse.

**Theorem 1.8** *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*is invertible if $ad - bc \neq 0$. The inverse is given by the formula*

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

*Proof* Verify that $AA^{-1} = I_2$ and $A^{-1}A = I_2$. $\square$

The following theorem is concerned with the invertibility of the product of invertible matrices.

**Theorem 1.9** *Let $A$ and $B$ be $n \times n$ invertible matrices. Then $AB$ is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

*In general,*

$$\left(A_{(1)}A_{(2)}\cdots A_{(p)}\right)^{-1} = A_{(p)}^{-1}\cdots A_{(2)}^{-1}A_{(1)}^{-1}, \tag{1.9}$$

*where $A_{(i)}$ $(1 \leqslant i \leqslant p)$ are $n \times n$ invertible matrices and $p$ is any positive integer.*

*Proof* Since $A$ and $B$ are invertible, we obtain by Theorem 1.4 (c),

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I.$$

Thus,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

We are now going to prove the general case of (1.9) by using induction. When $p = 2$, we have just proved that (1.9) is true. We assume that (1.9) is true for $p = k - 1$, i.e.,

$$\big(A_{(1)}A_{(2)} \cdots A_{(k-1)}\big)^{-1} = A_{(k-1)}^{-1} \cdots A_{(2)}^{-1}A_{(1)}^{-1}. \tag{1.10}$$

Considering the case of $p = k$, it follows from the case of $p = 2$ and (1.10) that

$$\begin{aligned}
&\big(A_{(1)}A_{(2)} \cdots A_{(k-1)}A_{(k)}\big)^{-1} = \big[\big(A_{(1)}A_{(2)} \cdots A_{(k-1)}\big)A_{(k)}\big]^{-1} \\
&= A_{(k)}^{-1}\big(A_{(1)}A_{(2)} \cdots A_{(k-1)}\big)^{-1} = A_{(k)}^{-1}\big(A_{(k-1)}^{-1} \cdots A_{(2)}^{-1}A_{(1)}^{-1}\big) \\
&= A_{(k)}^{-1}A_{(k-1)}^{-1} \cdots A_{(2)}^{-1}A_{(1)}^{-1}.
\end{aligned}$$

Therefore, (1.9) is true for any positive integer $p$. $\square$

The following theorem gives a relationship between the inverse of an invertible matrix $A$ and the inverse of $A^T$.

**Theorem 1.10** *If $A$ is invertible, then $A^T$ is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T.$$

*Proof* We have by using Theorem 1.2 (d),

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^TA^T = (AA^{-1})^T = I^T = I.$$

Thus, $(A^T)^{-1} = (A^{-1})^T$. $\square$

### 1.4.4 Powers of a matrix

**Definition** *If $A$ is square, then we define the powers of $A$ for any integer $n \geqslant 0$,*

$$A^0 := I, \qquad A^n := \underbrace{AA \cdots A}_{n}.$$

*Moreover, if $A$ is invertible, then*

$$A^{-n} := (A^{-1})^n = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{n}.$$

If $r$ and $s$ are integers, then it follows from the definition of powers that

$$A^r A^s = A^{r+s}, \qquad (A^r)^s = A^{rs}.$$

**Theorem 1.11** *If $A$ is invertible, then*

(a) *$A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.*

(b) *$A^n$ is invertible and $(A^n)^{-1} = (A^{-1})^n$ for any integer $n \geqslant 0$.*

(c) *For any scalar $k \neq 0$, the matrix $kA$ is invertible and $(kA)^{-1} = \dfrac{1}{k} A^{-1}$.*

The proof of the theorem is straightforward and we therefore omit it.

## 1.5 Elementary Matrices and a Method for Finding $A^{-1}$

We develop an algorithm for finding the inverse of an invertible matrix in this section.

### 1.5.1 Elementary matrices and their properties

**Definition** *An $n \times n$ **elementary matrix** can be obtained by performing a single elementary row operation on $I_n$. The following are three types of elementary matrices.*

(i) *Interchange rows $i$ and $j$ of $I_n$:*

$$E(i,j) = \begin{bmatrix}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 && \cdots && 1 &&& \\
&&&& 1 &&&&&& \\
&&& \vdots && \ddots && \vdots &&& \\
&&&&&& 1 &&&& \\
&&& 1 && \cdots && 0 &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{bmatrix}
\begin{array}{l}
\\ \\ \\ \textit{row } i \\ \\ \\ \\ \textit{row } j \\ \\ \\ \\
\end{array}$$

(ii) *Multiply row $i$ of $I_n$ by $c$ ($c \neq 0$):*

$$E\big(i(c)\big) = \begin{bmatrix} 1 &&&&& \\ & \ddots &&&& \\ && c &&& \\ &&& 1 && \\ &&&& \ddots & \\ &&&&& 1 \end{bmatrix} \begin{matrix} \\ \\ \textit{row } i \\ \\ \\ \\ \end{matrix}$$

(iii) *Add $k$ times row $j$ to row $i$ of $I_n$:*

$$E\big(i,j(k)\big) = \begin{bmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 & \cdots & k && \\ &&& \ddots & \vdots && \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{bmatrix} \begin{matrix} \\ \\ \textit{row } i \\ \\ \textit{row } j \\ \\ \\ \end{matrix}$$

**Remark** A square matrix is called a ***permutation matrix*** if it can be written as a product of elementary matrices of type (i).

When a matrix $A$ is multiplied on the left by an elementary matrix $E$, the result is to perform an elementary row operation on $A$. More precisely, we have the following theorem.

**Theorem 1.12** *Let $A$ be an $m \times n$ matrix. If the elementary matrix $E$ results from performing a certain row operation on $I_m$, then the product $EA$ is the matrix that results when this same row operation is performed on $A$.*

*Proof* We only prove the statement concerned with the elementary matrix $E(i,j)$. One can prove the statements concerned with $E\big(i(c)\big)$ and $E\big(i,j(k)\big)$ easily by using the same trick. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ denote the row matrices of $A$ and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ denote the column matrices of $I_m$. It follows that

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_m \end{bmatrix}, \qquad E(i,j) = \begin{bmatrix} \mathbf{e}_1^T \\ \vdots \\ \mathbf{e}_j^T \\ \vdots \\ \mathbf{e}_i^T \\ \vdots \\ \mathbf{e}_m^T \end{bmatrix}.$$

Since $\mathbf{e}_k^T A = \mathbf{r}_k$ for $1 \leqslant k \leqslant m$, we have by (1.4),

$$E(i,j)A = \begin{bmatrix} \mathbf{e}_1^T \\ \vdots \\ \mathbf{e}_j^T \\ \vdots \\ \mathbf{e}_i^T \\ \vdots \\ \mathbf{e}_m^T \end{bmatrix} A = \begin{bmatrix} \mathbf{e}_1^T A \\ \vdots \\ \mathbf{e}_j^T A \\ \vdots \\ \mathbf{e}_i^T A \\ \vdots \\ \mathbf{e}_m^T A \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_m \end{bmatrix}.$$

Hence $E(i,j)A$ is a matrix that results when rows $i$ and $j$ of $A$ are interchanged. □

The following theorem is concerned with the invertibility of elementary matrices. The proof of the theorem is left as an exercise.

**Theorem 1.13** *For three types of elementary matrices, we have*

$$E(i,j)^{-1} = E(i,j), \qquad E\big(i(c)\big)^{-1} = E\big(i(c^{-1})\big), \qquad E\big(i,j(k)\big)^{-1} = E\big(i,j(-k)\big).$$

**Remark** It follows from Theorem 1.13 that the inverse of any elementary matrix is still an elementary matrix.

### 1.5.2 Main theorem of invertibility

The next theorem establishes some equivalent statements of the invertibility of a matrix. These results are extremely important and will be used many times later.

**Theorem 1.14** *Let $A$ be an $n \times n$ matrix. Then the following statements are equivalent, i.e., all true or all false.*

(a) *$A$ is invertible.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced row-echelon form of $A$ is $I_n$.*

(d) *$A$ is expressible as a product of elementary matrices.*

*Proof* It is sufficient to prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Let $\mathbf{x}_0$ be any solution of $A\mathbf{x} = \mathbf{0}$, i.e., $A\mathbf{x}_0 = \mathbf{0}$. Since $A$ is invertible, multiplying both sides of $A\mathbf{x}_0 = \mathbf{0}$ by $A^{-1}$, we have $A^{-1}A\mathbf{x}_0 = A^{-1}\mathbf{0}$, which implies $I_n\mathbf{x}_0 = \mathbf{0}$. Thus, $\mathbf{x}_0 = \mathbf{0}$. Therefore, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b) $\Rightarrow$ (c): Let $R$ be the reduced row-echelon form of $A$. Then by Theorem 1.6, $R = I_n$ or $R$ has a zero row. If $R$ has a zero row, then it follows from Theorem 1.1 that $R\mathbf{x} = \mathbf{0}$ has infinitely many solutions, i.e., nontrivial solutions. Therefore, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, which contradicts (b). Thus, $R = I_n$.

(c) $\Rightarrow$ (d): If the reduced row-echelon form of $A$ is $I_n$, then there exist some elementary matrices $E_{(1)}, E_{(2)}, \ldots, E_{(k)}$ such that

$$E_{(k)} \cdots E_{(2)} E_{(1)} A = I_n. \tag{1.11}$$

By Theorem 1.13, we know that every elementary matrix is invertible and the inverse of an elementary matrix is still an elementary matrix. It follows from (1.11) and Theorem 1.9 that

$$A = E_{(1)}^{-1} E_{(2)}^{-1} \cdots E_{(k)}^{-1}.$$

Thus, (d) holds.

(d) $\Rightarrow$ (a): It is obtained directly from Theorems 1.13 and 1.9. $\square$

### 1.5.3 A method for finding $A^{-1}$

In the following, we establish a method for constructing the inverse of an $n \times n$ invertible matrix $A$. Multiplying both sides of (1.11) on the right by $A^{-1}$ yields

$$A^{-1} = E_{(k)} \cdots E_{(2)} E_{(1)},$$

where $E_{(1)}, E_{(2)}, \ldots, E_{(k)}$ are elementary matrices. We next construct an $n \times 2n$ matrix $\left[\begin{array}{c:c} A & I \end{array}\right]$. We have by using (1.3),

$$\begin{aligned} E_{(k)} \cdots E_{(2)} E_{(1)} \left[\begin{array}{c:c} A & I \end{array}\right] &= \left[\begin{array}{c:c} E_{(k)} \cdots E_{(2)} E_{(1)} A & E_{(k)} \cdots E_{(2)} E_{(1)} I \end{array}\right] \\ &= \left[\begin{array}{c:c} I & E_{(k)} \cdots E_{(2)} E_{(1)} \end{array}\right] \\ &= \left[\begin{array}{c:c} I & A^{-1} \end{array}\right]. \end{aligned}$$

Thus, the sequence of elementary row operations that reduces $A$ to $I$ actually converts $I$ to $A^{-1}$ simultaneously.

**Example** Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 6 \\ 2 & 7 & 3 \end{bmatrix}.$$

*Solution* The computations are as follows:

$$\left[\begin{array}{c:c} A & I_3 \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 3 & 1 & 6 & 0 & 1 & 0 \\ 2 & 7 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{3rd row} + (-2) \times \text{1st row}]{\text{2nd row} + (-3) \times \text{1st row}} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 7 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{3rd row} + (-7) \times \text{2nd row}} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & 19 & -7 & 1 \end{array}\right]$$

$$\xrightarrow{(-1) \times \text{3rd row}} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -19 & 7 & -1 \end{array}\right]$$

$$\xrightarrow{\text{1st row} + (-2) \times \text{3rd row}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 39 & -14 & 2 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -19 & 7 & -1 \end{array}\right] = \left[\begin{array}{c:c} I_3 & A^{-1} \end{array}\right].$$

Thus,

$$A^{-1} = \begin{bmatrix} 39 & -14 & 2 \\ -3 & 1 & 0 \\ -19 & 7 & -1 \end{bmatrix}. \qquad \square$$

## 1.6 Further Results on Systems and Invertibility

We develop more results concerned with linear systems and invertibility of matrices in this section.

### 1.6.1 A basic theorem

**Theorem 1.15** *Every linear system has either no solution, exactly one solution, or infinitely many solutions.*

*Proof* If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, then exactly one of the following is true:

(a) the system has no solution;

(b) the system has exactly one solution;

(c) the system has more than one solution.

The proof will be completed if we can show that the system has infinitely many solutions in case (c). Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Therefore, $\mathbf{x}_0$ is nonzero. Moreover,

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Let $k$ be any scalar. Then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + A(k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) = \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

i.e., $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{x}_0 \neq \mathbf{0}$ and there are infinitely many choices for $k$, we conclude that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. □

### 1.6.2 Properties of invertible matrices

From the definition of the inverse of an invertible matrix $A$, it is necessary to find a square matrix $B$ such that

$$AB = I, \qquad BA = I.$$

The next theorem shows that if $B$ satisfies either condition, then the other condition holds automatically.

**Theorem 1.16** *Let $A$ be a square matrix.*

(a) *If $B$ is square and satisfies $BA = I$, then $B = A^{-1}$.*

(b) *If $B$ is square and satisfies $AB = I$, then $B = A^{-1}$.*

*Proof* We only prove (a) and the proof of (b) is similar. We consider the system $A\mathbf{x} = \mathbf{0}$ and show that this system only has the trivial solution. Let $\mathbf{x}_0$ be any solution of this system, i.e., $A\mathbf{x}_0 = \mathbf{0}$. Multiplying both sides by $B$ yields

$$BA\mathbf{x}_0 = B\mathbf{0}.$$

Since $BA = I$, we deduce

$$I\mathbf{x}_0 = \mathbf{0}, \quad \text{i.e.,} \quad \mathbf{x}_0 = \mathbf{0}.$$

Thus, the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. It follows from Theorem 1.14 that $A^{-1}$ exists. Multiplying both sides of $BA = I$ on the right by $A^{-1}$, we obtain

$$BA \cdot A^{-1} = I \cdot A^{-1} \quad \Longrightarrow \quad B = A^{-1}.$$ □

**Theorem 1.17** *Let $A$ and $B$ be square matrices of the same size. If $AB$ is invertible, then $A$ and $B$ must also be invertible.*

*Proof* Since $AB$ is invertible, there exists $(AB)^{-1}$ such that

$$(AB)(AB)^{-1} = I \quad \Longrightarrow \quad A[B(AB)^{-1}] = I,$$

and

$$(AB)^{-1}(AB) = I \quad \Longrightarrow \quad [(AB)^{-1}A]B = I.$$

By Theorem 1.16, both $A$ and $B$ are invertible. □

**Example** Let $A$ and $B$ be square matrices of the same size. Show that if $I - AB$ is invertible, then $I - BA$ is also invertible.

*Proof* By Theorem 1.16, we only need to find a matrix $X$ such that $(I-BA)X = I$. Actually,

$$\begin{aligned} I &= I - BA + BA = I - BA + BIA \\ &= I - BA + B(I - AB)(I - AB)^{-1}A \\ &= I - BA + (B - BAB)(I - AB)^{-1}A \\ &= I - BA + (I - BA)B(I - AB)^{-1}A \\ &= (I - BA)[I + B(I - AB)^{-1}A]. \end{aligned}$$

Thus, $I - BA$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$ □

The following theorem shows that we can solve a certain linear system by using the inverse of its coefficient matrix.

**Theorem 1.18** *Let $A$ be an invertible $n \times n$ matrix. Then for each $n \times 1$ matrix $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

*Proof* Since $A$ is invertible, there exists $A^{-1}$. For each $n \times 1$ matrix $\mathbf{b}$, let $\mathbf{x}_0$ be an arbitrary solution of $A\mathbf{x} = \mathbf{b}$, i.e., $A\mathbf{x}_0 = \mathbf{b}$. Multiplying both sides of $A\mathbf{x}_0 = \mathbf{b}$ by $A^{-1}$, we have

$$A^{-1}A\mathbf{x}_0 = A^{-1}\mathbf{b}.$$

Thus, $\mathbf{x}_0 = A^{-1}\mathbf{b}$ is the only solution of $A\mathbf{x} = \mathbf{b}$. □

Furthermore, we add two more equivalent statements into Theorem 1.14.

**Theorem 1.19** *Let $A$ be an $n \times n$ matrix. Then the following are equivalent.*

(a) *A is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row-echelon form of* $A$ *is* $I_n$.

(d) *A is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

*Proof* Since we know that (a), (b), (c), and (d) are equivalent, it is sufficient to prove that (a) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ (f): This was proved in Theorem 1.18.

(f) $\Rightarrow$ (e): This is self-evident.

(e) $\Rightarrow$ (a): If the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$, then in particular, the systems

$$A\mathbf{x} = \mathbf{e}_1, \quad A\mathbf{x} = \mathbf{e}_2, \quad \ldots, \quad A\mathbf{x} = \mathbf{e}_n$$

are consistent, where $\mathbf{e}_i$ denotes the $i$th column matrix of $I_n$ for $1 \leqslant i \leqslant n$. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be solutions of the respective systems, and let us form an $n \times n$ matrix $C$ having these solutions as columns:

$$C = \left[\begin{array}{c:c:c:c} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array}\right].$$

Then we have by (1.3),

$$AC = \left[\begin{array}{c:c:c:c} A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \end{array}\right] = \left[\begin{array}{c:c:c:c} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = I_n.$$

It follows from Theorem 1.16 (b) that $C = A^{-1}$. Thus, $A$ is invertible. $\square$

## 1.7 Some Special Matrices

Certain classes of matrices have special structures, which are useful in linear algebra and also have many applications in practice.

### 1.7.1 Diagonal and triangular matrices

The following square matrix is called a ***diagonal matrix***:

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix},$$

which is usually denoted by

$$D = \mathrm{diag}(d_{11}, d_{22}, \ldots, d_{nn}).$$

If $d_{ii} \neq 0$ for $1 \leqslant i \leqslant n$, then

$$D^{-1} = \begin{bmatrix} d_{11}^{-1} & 0 & \cdots & 0 \\ 0 & d_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^{-1} \end{bmatrix} = \mathrm{diag}(d_{11}^{-1}, d_{22}^{-1}, \ldots, d_{nn}^{-1}).$$

The following square matrices are called ***triangular matrices***: a lower triangular matrix

$$L = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \qquad (a_{ij} = 0,\ i < j)$$

and an upper triangular matrix

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad (a_{ij} = 0,\ i > j).$$

**Theorem 1.20** *We have*

(a) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*

(b) *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*

(c) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

(d) *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

*Proof* Part (a) is obvious. We defer the proof of (c) until the next chapter (after Theorem 2.8). Here we prove (b) and (d) only.

For (b), we will prove the result for lower triangular matrices. The proof for upper triangular matrices is similar. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be lower triangular $n \times n$ matrices, and let $C = AB = [c_{ij}]$. Obviously, $a_{ij} = b_{ij} = 0$ for $i < j$. We can prove that $C$ is lower triangular by showing that $c_{ij} = 0$ for $i < j$. If $i < j$, then the terms in the expression of $c_{ij}$ can be grouped as follows:

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i,j-1}b_{j-1,j}}_{\substack{\text{Terms in which the row}\\ \text{number of } b \text{ is less than}\\ \text{the column number of } b}} + \underbrace{a_{ij}b_{jj} + a_{i,j+1}b_{j+1,j} + \cdots + a_{in}b_{nj}}_{\substack{\text{Terms in which the row}\\ \text{number of } a \text{ is less than}\\ \text{the column number of } a}}.$$

In the first grouping all of the $b$ factors are zero since $b_{ij} = 0$ for $i < j$, and in the second grouping all of the $a$ factors are zero since $a_{ij} = 0$ for $i < j$. Thus, $c_{ij} = 0$ for $i < j$. It follows that $C$ is a lower triangular matrix.

For (d), we only prove the result for lower triangular matrices again. Let $A = [a_{ij}]$ be an invertible lower triangular $n \times n$ matrix, where $a_{ij} = 0$ for $i < j$. From (c), we know that $a_{ii} \neq 0$ for all $i$. Suppose that $B = [b_{ij}]$ is the inverse of $A$. Then

$$AB = I. \tag{1.12}$$

We now compare the entries in both sides of (1.12) row by row. Beginning with the first row, we have for $j > 1$,

$$0 = (I)_{1j} = a_{11}b_{1j} + a_{12}b_{2j} + \cdots + a_{1n}b_{nj} = a_{11}b_{1j} + 0 \cdot b_{2j} + \cdots + 0 \cdot b_{nj} = a_{11}b_{1j},$$

which implies $b_{1j} = 0$. For the second row, we have for $j > 2$,

$$0 = (I)_{2j} = \sum_{t=1}^{n} a_{2t}b_{tj} = a_{21}b_{1j} + a_{22}b_{2j} = a_{22}b_{2j} \quad \Longrightarrow \quad b_{2j} = 0.$$

By induction, we suppose that for the top $k-1$ rows, $b_{ij} = 0$ if $i < j$ and $i < k < n$. For the $k$th row, we have for $j > k$,

$$0 = (I)_{kj} = \sum_{t=1}^{n} a_{kt}b_{tj} = \sum_{t=1}^{k} a_{kt}b_{tj} = a_{kk}b_{kj} \quad \Longrightarrow \quad b_{kj} = 0.$$

In particular, $b_{n-1,n} = 0$. Therefore, $B$ is also a lower triangular matrix. $\square$

### 1.7.2 Symmetric matrix

A square matrix $A$ is called ***symmetric*** if $A = A^T$. A square matrix $A$ is called ***skew-symmetric*** if $A = -A^T$.

**Example** Consider the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & -2 \\ 1 & -2 & 7 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 3 \\ -4 & -3 & 0 \end{bmatrix}.$$

Then $A$ and $B$ are symmetric and $C$ is skew-symmetric since

$$A^T = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = A, \quad B^T = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & -2 \\ 1 & -2 & 7 \end{bmatrix} = B, \quad C^T = \begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & -3 \\ 4 & 3 & 0 \end{bmatrix} = -C.$$

**Theorem 1.21** *Let $A$ and $B$ be symmetric matrices of the same size. Then*

(a) *$A^T$ is symmetric.*

(b) *$A \pm B$ are symmetric.*

(c) *$kA$ is symmetric, where $k$ is any scalar.*

(d) *$AB$ is symmetric if and only if $AB = BA$.*

(e) *If $A$ is invertible, then $A^{-1}$ is symmetric.*

*Proof* Parts (a), (b), and (c) are obvious. Here we only prove (d) and (e).

For (d), since $A$ and $B$ are symmetric, it follows from Theorem 1.2 (d) that

$$AB = (AB)^T \Longleftrightarrow AB = B^T A^T \Longleftrightarrow AB = BA.$$

For (e), if $A$ is invertible and symmetric, then we have by Theorem 1.10,

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}.$$

Thus, $A^{-1}$ is symmetric. □

**Theorem 1.22** *Let $A$ be an arbitrary matrix. Then $AA^T$ and $A^T A$ are symmetric. Furthermore, if $A$ is square and invertible, then both $AA^T$ and $A^T A$ are symmetric and invertible.*

*Proof* It directly follows from Theorem 1.2 (d) and (a) that

$$(AA^T)^T = (A^T)^T A^T = AA^T \quad \text{and} \quad (A^T A)^T = A^T (A^T)^T = A^T A.$$

If $A$ is square and invertible, it follows from Theorem 1.10 that $A^T$ is also invertible. By Theorem 1.9, we find that $AA^T$ and $A^T A$ are invertible. $\square$

**Remark** Every square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix. This is one of the most famous results in matrix theory. See Exercise 1.35.

## Exercises

### Elementary exercises

**1.1** Determine which equations are linear in variables $x$, $y$, and $z$. If an equation is not linear, explain why not.

(a) $x - \pi y + \sqrt[3]{5} z = 0$. (b) $x^2 + y^2 + z^2 = 1$.
(c) $x^{-1} + 7y + z = \sin(\pi/9)$. (d) $3\cos x - 4y + z = \sqrt{3}$.
(e) $(\cos 3)x - 4y + z = \sqrt{3}$. (f) $x = -7xy + 3z$.
(g) $xy + z + 1 = 0$. (h) $x^{-2} + y + 8z = 5$.

**1.2** Determine whether each system has a unique solution, infinitely many solutions, or no solution. Then try to solve each system to confirm your answer.

$$\text{(a)} \quad \begin{cases} x + y = 0 \\ 2x + y = 3. \end{cases} \qquad \text{(b)} \quad \begin{cases} x + 5y = -1 \\ -x + y = -5 \\ 2x + 4y = 4. \end{cases}$$

**1.3** Find the augmented and coefficient matrices for each of the following linear systems.

$$\text{(a)} \quad \begin{cases} 2x - 3y \phantom{{}+5z} + 5 = 0 \\ 4x + 2y \phantom{{}+5z} - 2 = 0 \\ 3x \phantom{{}+2y} + 5z + 3 = 0. \end{cases} \qquad \text{(b)} \quad \begin{cases} 5x_1 + x_2 - x_4 + 2x_5 = 1 \\ 3x_2 + 2x_3 - x_4 = 3 \\ 5x_1 + 3x_5 = 2. \end{cases}$$

**1.4** Consider

$$\begin{cases} x + y + 3z = a \\ 2x - y + 2z = b \\ x - 2y - z = c. \end{cases}$$

Show that if this system is consistent, then the constants $a$, $b$, and $c$ must satisfy $c = b - a$.

**1.5** For which values of $a$, will the following system have no solution? Exactly one solution? Infinitely many solutions?

$$\begin{cases} x + 2y - 3z = 5 \\ 3x - y + 5z = 1 \\ 4x + y + (a^2 - 14)z = a + 2. \end{cases}$$

**1.6** Determine whether each matrix is in reduced row-echelon form.

(a) $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

**1.7** Solve the following systems by Gauss-Jordan elimination.

(a) $\begin{cases} 2x - y + 2z = 10 \\ x - 2y + z = 8 \\ 3x - y + 2z = 11. \end{cases}$

(b) $\begin{cases} 5x - 2y + 6z = 0 \\ -2x + y + 3z = 1. \end{cases}$

(c) $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = -3 \\ x_1 + 2x_2 - 5x_4 = 1 \\ 2x_1 + 4x_2 - 3x_3 - 19x_4 = 6 \\ 3x_1 + 6x_2 - 3x_3 - 24x_4 = 7. \end{cases}$

(d) $\begin{cases} 2x_1 + 4x_2 + 8x_4 + 6x_5 + 18x_6 - 16 = 0 \\ x_1 + 2x_2 - 2x_3 + 3x_5 = 0 \\ 5x_3 + 10x_4 + 15x_6 - 5 = 0 \\ -2x_1 - 4x_2 + 5x_3 + 2x_4 - 6x_5 + 3x_6 - 1 = 0. \end{cases}$

**1.8** Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(a) Is $M = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$ a linear combination of $A$, $B$, and $C$?

(b) Is $N = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ a linear combination of $A$, $B$, and $C$?

**1.9** Let $A \in \mathbb{R}^{4\times 5}$, $B \in \mathbb{R}^{4\times 5}$, $C \in \mathbb{R}^{5\times 2}$, $D \in \mathbb{R}^{4\times 2}$, and $E \in \mathbb{R}^{5\times 4}$. Determine which of the following matrix operations can be performed. If so, find the size of each resulting matrix.

(a) $E(A+B)$. (b) $AE+BC+D$. (c) $B(EA)$.

**1.10** Find $AB$ and $BA$, where

$$A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

**1.11** Let $A$, $B$, and $C$ be square matrices of the same size. Find an example to show that $AB = AC$ but $B \neq C$.

**1.12** Compute $A = (6E)\left(\frac{1}{3}D\right)$, where

$$D = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 4 \\ -1 & 0 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} -1 & 1 & 2 \\ 6 & 1 & 3 \\ 4 & 1 & 3 \end{bmatrix}.$$

**1.13** Let

$$C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \qquad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

Using as few computations as possible, compute

(a) The second row of $DE$.

(b) The third column of $DE$.

(c) The entry in row 2 and column 3 of $C(DE)$.

**1.14** In each part, compute the product of $A$ and $B$ by the method of product of partitioned matrices in Subsection 1.3.4. Then check your results by multiplying $AB$ directly.

$$\text{(a)}\ A = \left[\begin{array}{cc:cc} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \\ \hdashline 1 & 5 & 6 & 1 \end{array}\right], \quad B = \left[\begin{array}{cc:c} 2 & 1 & 4 \\ -3 & 5 & 2 \\ \hdashline 7 & -1 & 5 \\ 0 & 3 & -3 \end{array}\right].$$

(b) $A = \left[\begin{array}{rrr|r} -1 & 2 & 1 & 5 \\ \hline 0 & -3 & 4 & 2 \\ 1 & 5 & 6 & 1 \end{array}\right], \quad B = \left[\begin{array}{rr|r} 2 & 1 & 4 \\ -3 & 5 & 2 \\ 7 & -1 & 5 \\ \hline 0 & 3 & -3 \end{array}\right].$

(c) $A = \left[\begin{array}{rr} 2 & -5 \\ 1 & 3 \\ 0 & 5 \\ \hline 1 & 4 \end{array}\right], \quad B = \begin{bmatrix} 2 & -1 & 3 & -4 \\ 0 & 1 & 5 & 7 \end{bmatrix}.$

**1.15** Let $A$ and $B$ be partitioned as follows:

$$A = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right], \qquad B = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \vdots \\ \hline \mathbf{b}_n \end{array}\right],$$

where $\mathbf{a}_i$ $(1 \leqslant i \leqslant n)$ are column matrices of $A$ and $\mathbf{b}_i$ $(1 \leqslant i \leqslant n)$ are row matrices of $B$. Then $AB$ can be expressed as

$$AB = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_n\mathbf{b}_n. \tag{1.13}$$

Based on (1.13), compute $AB$ if

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

**1.16** Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. Find two $n \times 1$ matrices $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}^T A \mathbf{y} = a_{ij}$, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$.

**1.17** Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}.$$

Compute

(a) $\operatorname{tr}(3A - 5B^T)$. (b) $\operatorname{tr}(A^2)$. (c) $\operatorname{tr}(AB)$.

**1.18** Show that $\operatorname{tr}(AA^T) = 0$ if and only if $A = \mathbf{0}$.

**1.19** What is $M^T$ for the partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}?$$

**1.20** Prove Theorem 1.4 except (c).

**1.21** Prove Theorem 1.6.

**1.22** Use Theorem 1.8 to compute the inverses of the following matrices.

$$\text{(a) } A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}. \qquad \text{(b) } B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

**1.23** Find $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n$, where $n$ is any positive integer.

**1.24** Let $A$ be a square matrix. Show that if $A^2 = A$ and $A$ is invertible, then $A = I$.

**1.25** Let $A$ be a square matrix. Show that if $A^4 = \mathbf{0}$, then

$$(I - A)^{-1} = I + A + A^2 + A^3.$$

**1.26** Let $A$ be a square matrix. Show that if $A^2 - 3A + 4I = \mathbf{0}$, then $A + I$ is invertible. Find $(A + I)^{-1}$.

**1.27** Let $A, B \in \mathbb{R}^{n\times n}$. Show that if $A$ is invertible and $AB = \mathbf{0}$, then $B = \mathbf{0}$.

**1.28** Show that $P^T = P^{-1}$ for any permutation matrix $P$.

**1.29** Let $A$ be a square matrix and partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11}$ and $A_{22}$ are square matrices. Find a permutation matrix $P$ such that $P^T AP = B$, where

$$B = \begin{bmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{bmatrix}.$$

**1.30** Prove Theorem 1.13.

**1.31** Express each of the following matrices as a product of elementary matrices.

$$\text{(a)} \quad \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

**1.32** Find the inverses of the following matrices.

$$\text{(a)} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}. \qquad \text{(b)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}.$$

**1.33** Find a matrix $A$ such that

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} A = \begin{bmatrix} 4 & -6 \\ 3 & 1 \end{bmatrix}.$$

**1.34** Let $A, B \in \mathbb{R}^{n\times n}$. Show that $\operatorname{tr}(AB) = 0$ if $A$ is symmetric and $B$ is skew-symmetric.

**1.35** Let $A \in \mathbb{R}^{n\times n}$. Show that $A$ can be written as $A = H + K$, where $H$ is a symmetric matrix and $K$ is a skew-symmetric matrix.

### Challenge exercises

**1.36** Determine the value of $\lambda$ such that the following linear system has only the trivial solution.

$$\begin{cases} \lambda x_1 + \phantom{\lambda}x_2 + x_3 = 0 \\ x_1 + \lambda x_2 + x_3 = 0 \\ x_1 + \phantom{\lambda}x_2 + x_3 = 0. \end{cases}$$

**1.37** Let $A, B \in \mathbb{R}^{n\times n}$. If $AB = \mathbf{0}$, show that for any positive integer $k$,

$$\operatorname{tr}[(A+B)^k] = \operatorname{tr}(A^k) + \operatorname{tr}(B^k).$$

**1.38** Show that if $A, B, C, D \in \mathbb{R}^{n\times n}$ such that $ABCD = I$, then

$$ABCD = DABC = CDAB = BCDA = I.$$

**1.39** Find the inverses of the following matrices.

$$\text{(a)} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}. \qquad \text{(b)} \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

**1.40** Find the inverse of the $3 \times 3$ *Vandermonde matrix*

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix},$$

where $a$, $b$, and $c$ are distinct scalars from each other.

**1.41** Let $A, B, C, X, Y, Z \in \mathbb{R}^{n\times n}$. If $A^{-1}$ and $C^{-1}$ exist, find

$$\text{(a)} \quad \begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}^{-1}. \qquad \text{(b)} \quad \begin{bmatrix} I & X & Y \\ \mathbf{0} & I & Z \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}^{-1}.$$

**1.42** Let $A, B \in \mathbb{R}^{n\times n}$. Show that if $A + B$ is invertible, then

$$A(A+B)^{-1}B = B(A+B)^{-1}A.$$

**1.43** Let $A \in \mathbb{R}^{n\times n}$. Show that if $AB = BA$ for all $B \in \mathbb{R}^{n\times n}$, then $A = cI$, where $c$ is a scalar.

**1.44** Let $A$ be a skew-symmetric matrix. Show that

(a) $I - A$ is invertible.

(b) $(I - A)^{-1}(I + A) = (I + A)(I - A)^{-1}$.

(c) $M^T M = I$, where $M = (I - A)^{-1}(I + A)$.

(d) $I + M$ is invertible.

**1.45** Let $A, B \in \mathbb{R}^{n\times n}$. Show that if $A^3 = 2I$ and $B = A^3 - 2A + 3I$, then $B$ is invertible. Find $B^{-1}$.

# Chapter 2

# Determinants

"*The purpose of computation is insight, not numbers.*"

— Richard Hamming

In this chapter, we introduce the determinant of any square matrix, which actually is a function $f$ defined on $\mathbb{R}^{n\times n}$ in the sense that it associates a number $f(A) \in \mathbb{R}$ with any $A \in \mathbb{R}^{n\times n}$. We then study some fundamental properties of determinant functions and discuss their applications to linear systems and matrices.

## 2.1 Determinant Function

We begin with the following definitions before we introduce the determinant function.

### 2.1.1 Permutation, inversion, and elementary product

**Definition** *A* ***permutation*** *of the set* $\{1, 2, \ldots, n\}$*, denoted by* $(j_1, j_2, \ldots, j_n)$*, is an arrangement of* $\{1, 2, \ldots, n\}$ *in some order without omissions or repetitions. An* ***inversion*** *is said to occur in a permutation* $(j_1, j_2, \ldots, j_n)$ *whenever a larger integer precedes a smaller one.*

**Remark** A permutation is called even if the total number of inversions is an even integer and is called odd if the total number of inversions is an odd integer. For instance, the number of inversions in $(2, 4, 3, 1)$ is 4 and therefore it is an even permutation. The number of inversions in $(4, 2, 3, 1)$ is 5 and therefore it is an odd permutation.

**Definition** *An* ***elementary product*** *from an* $n \times n$ *matrix* $A = [a_{ij}]$ *means any product of* $n$ *entries from* $A$*, no two of which come from the same row or column, i.e.,*

$$a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n},$$

*where* $(i_1, i_2, \ldots, i_n)$ *and* $(j_1, j_2, \ldots, j_n)$ *are permutations of the set* $\{1, 2, \ldots, n\}$*. A*

***signed elementary product*** *from $A$ is defined by*

$$(-1)^{\tau(i_1,i_2,\ldots,i_n)+\tau(j_1,j_2,\ldots,j_n)} a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}, \tag{2.1}$$

*where $\tau(i_1,i_2,\ldots,i_n)$ and $\tau(j_1,j_2,\ldots,j_n)$ denote the number of inversions in $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$, respectively.*

In fact, an $n\times n$ matrix $A$ has $n\cdot(n-1)\cdots 2\cdot 1 = n!$ elementary products. Note that if the positions of any two elements in $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ are exchanged, the sign in front of (2.1) keeps unchanged. For instance, consider the following signed elementary product

$$(-1)^{\tau(i_1,\ldots,i_p,i_{p+1},\ldots,i_n)+\tau(j_1,\ldots,j_p,j_{p+1},\ldots,j_n)} a_{i_1j_1}\cdots a_{i_pj_p}a_{i_{p+1}j_{p+1}}\cdots a_{i_nj_n}. \tag{2.2}$$

If the positions of $a_{i_pj_p}$ and $a_{i_{p+1}j_{p+1}}$ are exchanged, then

$$\begin{aligned}
&(-1)^{\tau(i_1,\ldots,i_{p+1},i_p,\ldots,i_n)+\tau(j_1,\ldots,j_{p+1},j_p,\ldots,j_n)} a_{i_1j_1}\cdots a_{i_{p+1}j_{p+1}}a_{i_pj_p}\cdots a_{i_nj_n}\\
&= (-1)^{\tau(i_1,\ldots,i_p,i_{p+1},\ldots,i_n)\pm 1+\tau(j_1,\ldots,j_p,j_{p+1},\ldots,j_n)\pm 1} a_{i_1j_1}\cdots a_{i_{p+1}j_{p+1}}a_{i_pj_p}\cdots a_{i_nj_n}\\
&= (-1)^{\tau(i_1,\ldots,i_p,i_{p+1},\ldots,i_n)+\tau(j_1,\ldots,j_p,j_{p+1},\ldots,j_n)} a_{i_1j_1}\cdots a_{i_{p+1}j_{p+1}}a_{i_pj_p}\cdots a_{i_nj_n}. \qquad (2.3)
\end{aligned}$$

Comparing (2.3) with (2.2) shows that they are equal. Thus, we can rearrange the order of $a_{i_1j_1}a_{i_2j_2}\ldots a_{i_nj_n}$ in (2.1) such that the permutation of row indexes is $(1,2,\ldots,n)$. It follows that (2.1) can be rewritten as

$$(-1)^{\tau(1,2,\ldots,n)+\tau(j_1',j_2',\ldots,j_n')} a_{1j_1'}a_{2j_2'}\cdots a_{nj_n'} = (-1)^{\tau(j_1',j_2',\ldots,j_n')} a_{1j_1'}a_{2j_2'}\cdots a_{nj_n'}.$$

For simplicity, later we usually use

$$(-1)^{\tau(j_1,j_2,\ldots,j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n}.$$

**Example** We list all signed elementary products from the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

| **Elementary Product** | **Associated Permutation** | **Even or Odd** | **Signed Elementary Product** |
|---|---|---|---|
| $a_{11}a_{22}a_{33}$ | $(1,2,3)$ | even | $a_{11}a_{22}a_{33}$ |
| $a_{11}a_{23}a_{32}$ | $(1,3,2)$ | odd | $-a_{11}a_{23}a_{32}$ |
| $a_{12}a_{21}a_{33}$ | $(2,1,3)$ | odd | $-a_{12}a_{21}a_{33}$ |
| $a_{12}a_{23}a_{31}$ | $(2,3,1)$ | even | $a_{12}a_{23}a_{31}$ |
| $a_{13}a_{21}a_{32}$ | $(3,1,2)$ | even | $a_{13}a_{21}a_{32}$ |
| $a_{13}a_{22}a_{31}$ | $(3,2,1)$ | odd | $-a_{13}a_{22}a_{31}$ |

### 2.1.2 Definition of determinant function

We are now in a position to define the determinant function.

**Definition** *Let $A$ be a square matrix. The determinant function is defined to be the sum of all signed elementary products from $A$. This function (number), denoted by* $\det(A)$, *is called the* ***determinant*** *of $A$ usually.*

More precisely, let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we have

$$\det(A) := \sum (-1)^{\tau(j_1,j_2,\ldots,j_n)} a_{1j_1}a_{2j_2}\cdots a_{nj_n} = \sum \pm a_{1j_1}a_{2j_2}\cdots a_{nj_n}. \tag{2.4}$$

Here $\sum$ indicates that the terms are summed over all permutations $(j_1, j_2, \ldots, j_n)$.

**Example** We obtain

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

and

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

## 2.2 Evaluation of Determinants

In this section, we show that determinants can be evaluated by using row (or column) reduction.

### 2.2.1 Elementary theorems

**Theorem 2.1** *Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then*

(a) $\det(A) = 0$ *if $A$ has a zero row (or column).*

(b) $\det(A) = \det(A^T)$.

(c) $\det(A) = a_{11}a_{22}\cdots a_{nn}$ *if $A$ is a triangular matrix.*

*Proof* We only need to prove (b). The proofs of (a) and (c) are left as an exercise. For a general term in $\det(A)$, we have by (2.1),

$$(-1)^{\tau(i_1,i_2,\ldots,i_n)+\tau(j_1,j_2,\ldots,j_n)} a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}.$$

It can be written as

$$(-1)^{\tau(j_1',j_2',\ldots,j_n')} a_{1j_1'}a_{2j_2'}\cdots a_{nj_n'}$$

and also as

$$(-1)^{\tau(i'_1,i'_2,\ldots,i'_n)} a_{i'_1 1} a_{i'_2 2} \cdots a_{i'_n n}.$$

Thus,

$$\begin{aligned}
\det(A) &= \sum (-1)^{\tau(i_1,i_2,\ldots,i_n)+\tau(j_1,j_2,\ldots,j_n)} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n} \\
&= \sum (-1)^{\tau(j'_1,j'_2,\ldots,j'_n)} a_{1 j'_1} a_{2 j'_2} \cdots a_{n j'_n} \\
&= \sum (-1)^{\tau(i'_1,i'_2,\ldots,i'_n)} a_{i'_1 1} a_{i'_2 2} \cdots a_{i'_n n} \\
&= \sum (-1)^{\tau(i'_1,i'_2,\ldots,i'_n)} (A^T)_{1 i'_1} (A^T)_{2 i'_2} \cdots (A^T)_{n i'_n} \\
&= \det(A^T). \qquad \square
\end{aligned}$$

**Remark** Part (c) in Theorem 2.1 shows that it is easy for us to evaluate the determinant of a triangular matrix regardless of its size. A method proposed later for evaluating determinants is to reduce a given matrix to be a triangular matrix.

Since $\det(A) = \det(A^T)$, nearly every statement about determinants that contains rows is also true when rows are replaced by columns.

**Theorem 2.2** *Let $A$ be an $n \times n$ matrix.*

(a) *If $B$ is resulted when a single row (or column) of $A$ is multiplied by any scalar $k$, then*

$$\det(B) = k \cdot \det(A).$$

(b) *If $B$ is resulted when two rows (or columns) of $A$ are interchanged, then*

$$\det(B) = -\det(A).$$

(c) *If $A$ has two same rows (or columns), then* $\det(A) = 0$.

(d) *If $A$ has two proportional rows (or columns), then* $\det(A) = 0$.

(e) *If $B$ is the matrix that results when a multiple of one row (or column) of $A$ is added to another row (or column), then* $\det(B) = \det(A)$.

*Proof* The proofs of (a), (c), (d), and (e) are left as an exercise. Here we only prove (b). For simplicity, we first assume that $B$ is resulted when the first row of $A$ is interchanged with the second row of $A$. Then

$$\det(B) = \sum (-1)^{\tau(i_1,i_2,\ldots,i_n)} b_{1 i_1} b_{2 i_2} b_{3 i_3} \cdots b_{n i_n}$$

$$
\begin{aligned}
&= \sum (-1)^{\tau(i_1,i_2,\ldots,i_n)} a_{2i_1} a_{1i_2} a_{3i_3} \cdots a_{ni_n} \\
&= \sum (-1)^{\tau(i_1,i_2,\ldots,i_n)} a_{1i_2} a_{2i_1} a_{3i_3} \cdots a_{ni_n} \\
&= \sum (-1)^{\tau(i_2,i_1,\ldots,i_n)\pm 1} a_{1i_2} a_{2i_1} a_{3i_3} \cdots a_{ni_n} \\
&= -\sum (-1)^{\tau(j_1,j_2,\ldots,j_n)} a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n} \\
&= -\det(A).
\end{aligned}
$$

Similarly, one can prove that (b) still holds if $B$ is resulted from interchanging any other two rows of $A$. □

### 2.2.2 A method for evaluating determinants

Based on Theorems 2.1 and 2.2, the row (or column) reduction actually gives us a method to evaluate determinants by reducing the given matrix to a triangular matrix which can be computed easily. Here is an example.

**Example** Evaluate the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 7 & 6 \\ 3 & 6 & 0 & 5 \end{bmatrix}.$$

Interchanging row 2 with row 3 and then adding $(-3)\times$ row 1 to row 4, we have

$$\det(A) = -\det \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 3 & 7 & 6 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 0 & 5 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 3 & 7 & 6 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix} = 24.$$

## 2.3 Properties of Determinants

We develop some essential properties of determinants in this section. We will show that if $A$ and $B$ are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B).$$

Specially, a determinant test for the invertibility of a matrix is given.

### 2.3.1 Basic properties

Let $A$ and $B$ be $n \times n$ matrices and $k \in \mathbb{R}$. We consider possible relationships between $\det(A)$, $\det(B)$, and

$$\det(kA), \qquad \det(A+B), \qquad \det(AB).$$

**Theorem 2.3** *Let $A = [a_{ij}]$ be an $n \times n$ matrix and $k$ be any scalar. We have*

$$\det(kA) = k^n \cdot \det(A).$$

*Proof* By noting that $kA = [ka_{ij}]$, it follows from (2.4) that

$$\det(kA) = \sum \pm k a_{1j_1} k a_{2j_2} \cdots k a_{nj_n} = k^n \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n} = k^n \cdot \det(A). \quad \square$$

Usually, $\det(A+B) \neq \det(A) + \det(B)$. For instance, if $A = B = I_2$, then

$$4 = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \neq \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 + 1 = 2.$$

However, we have the following theorem.

**Theorem 2.4** *Let $A$, $B$, and $C$ be $n \times n$ matrices that differ only in a single row, say the $r$th row. Assume that the $r$th row of $C$ can be obtained by adding corresponding entries in the $r$th rows of $A$ and $B$. Then*

$$\det(C) = \det(A) + \det(B).$$

*The same result holds for columns.*

*Proof* Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$. We assume that

$$\begin{cases} c_{ij} = a_{ij} = b_{ij} & \text{if } i \neq r; \\ c_{rj} = a_{rj} + b_{rj} & \text{if } i = r. \end{cases}$$

It follows from (2.4) that

$$\begin{aligned} \det(C) &= \sum \pm c_{1j_1} c_{2j_2} \cdots c_{rj_r} \cdots c_{nj_n} \\ &= \sum \pm c_{1j_1} c_{2j_2} \cdots (a_{rj_r} + b_{rj_r}) \cdots c_{nj_n} \\ &= \sum \pm c_{1j_1} c_{2j_2} \cdots a_{rj_r} \cdots c_{nj_n} + \sum \pm c_{1j_1} c_{2j_2} \cdots b_{rj_r} \cdots c_{nj_n} \\ &= \sum \pm a_{1j_1} a_{2j_2} \cdots a_{rj_r} \cdots a_{nj_n} + \sum \pm b_{1j_1} b_{2j_2} \cdots b_{rj_r} \cdots b_{nj_n} \\ &= \det(A) + \det(B). \end{aligned}$$

$\square$

**Remark** By using Theorem 2.4 and Theorem 2.2 (d), one can prove Theorem 2.2 (e) easily.

### 2.3.2 Determinant of a matrix product

Let $A$ and $B$ be square matrices of the same size. We are now going to show that

$$\det(AB) = \det(A)\det(B).$$

**Lemma 2.1** *Let $k$ be any scalar. For three types of elementary matrices, we have*

(a) $\det\big(E(i(k))\big) = k.$

(b) $\det\big(E(i,j)\big) = -1.$

(c) $\det\big(E(i,j(k))\big) = 1.$

*Proof* It follows from Theorem 2.1 (c) that (a) and (c) are true.

For (b), we first note that $\det(I) = 1$ by Theorem 2.1 (c) again. It follows from Theorem 2.2 (b) that

$$\det\big(E(i,j)\big) = -\det(I) = -1. \qquad \square$$

**Lemma 2.2** *Let $B$ be an $n \times n$ matrix and $E$ be an $n \times n$ elementary matrix. Then*

$$\det(EB) = \det(E)\det(B).$$

*Proof* We consider three types of elementary matrices $E\big(i(k)\big)$, $E(i,j)$, and $E\big(i,j(k)\big)$. If $E = E\big(i(k)\big)$, then by Theorem 1.12, $E\big(i(k)\big)B$ results from multiplying the $i$th row of $B$ by $k$. It follows from Theorem 2.2 (a) that

$$\det\big(E(i(k))B\big) = k \cdot \det(B).$$

But from Lemma 2.1 (a), we have $\det\big(E(i(k))\big) = k$. Thus,

$$\det\big(E(i(k))B\big) = \det\big(E(i(k))\big)\det(B).$$

The proofs of the other two cases are similar to that of $E\big(i(k)\big)$. $\square$

**Remark** It follows by repeated applications of Lemma 2.2 that if $B$ is an $n \times n$ matrix and $E_{(1)}, E_{(2)}, \ldots, E_{(r)}$ are $n \times n$ elementary matrices, then

$$\det\big(E_{(r)} \cdots E_{(2)}E_{(1)}B\big) = \det\big(E_{(r)}\big) \cdots \det\big(E_{(2)}\big)\det\big(E_{(1)}\big)\det(B). \tag{2.5}$$

The next theorem gives a determinant test for the invertibility of a matrix.

**Theorem 2.5** *A square matrix $A$ is invertible if and only if* $\det(A) \neq 0$.

*Proof* Let $E_{(1)}, E_{(2)}, \ldots, E_{(r)}$ be the elementary matrices that correspond to the elementary row operations that produce the reduced row-echelon form $R$ of $A$, i.e.,

$$R = E_{(r)} \cdots E_{(2)} E_{(1)} A.$$

We deduce by (2.5),

$$\det(R) = \det\bigl(E_{(r)}\bigr) \cdots \det\bigl(E_{(2)}\bigr) \det\bigl(E_{(1)}\bigr) \det(A). \tag{2.6}$$

But from Lemma 2.1 the determinants of elementary matrices are all nonzero. It follows from (2.6) that $\det(A) \neq 0$ if and only if $\det(R) \neq 0$.

If $A$ is invertible, then by Theorem 1.19 we have $R = I$, so $\det(R) = 1 \neq 0$ and consequently $\det(A) \neq 0$. Conversely, if $\det(A) \neq 0$, then $\det(R) \neq 0$, so $R$ cannot have a row of zeros. It follows from Theorem 1.6 that $R = I$ and therefore $A$ is invertible by Theorem 1.19 again. □

**Theorem 2.6** *Let $A$ and $B$ be square matrices of the same size. Then*

$$\det(AB) = \det(A)\det(B).$$

*Proof* Let $R$ be the reduced row-echelon form of $A$. Then

$$R = E_{(r)} E_{(r-1)} \cdots E_{(2)} E_{(1)} A, \tag{2.7}$$

where $E_{(1)}, E_{(2)}, \ldots, E_{(r-1)}, E_{(r)}$ are the elementary matrices that correspond to the elementary row operations that produce $R$ from $A$. It follows from Theorem 1.6 that $R$ has a zero row or $R = I$. We have by (2.7),

$$A = E_{(1)}^{-1} E_{(2)}^{-1} \cdots E_{(r-1)}^{-1} E_{(r)}^{-1} R,$$

where $E_{(1)}^{-1}, E_{(2)}^{-1}, \ldots, E_{(r-1)}^{-1}, E_{(r)}^{-1}$ are still elementary matrices. Moreover,

$$AB = E_{(1)}^{-1} E_{(2)}^{-1} \cdots E_{(r-1)}^{-1} E_{(r)}^{-1} RB,$$

where either $RB$ has a zero row if $R$ has a zero row or $RB = B$ if $R = I$.

If $R$ has a zero row, then $\det(A) = 0$ and also $\det(RB) = 0$. Thus, we obtain by (2.5),

$$\begin{aligned}
\det(AB) &= \det\bigl(E_{(1)}^{-1} E_{(2)}^{-1} \cdots E_{(r-1)}^{-1} E_{(r)}^{-1} RB\bigr) \\
&= \det\bigl(E_{(1)}^{-1}\bigr) \det\bigl(E_{(2)}^{-1}\bigr) \cdots \det\bigl(E_{(r-1)}^{-1}\bigr) \det\bigl(E_{(r)}^{-1}\bigr) \det(RB) \\
&= 0 = \det(A)\det(B).
\end{aligned}$$

If $R = I$, then we deduce by (2.5) again,

$$\begin{aligned}\det(AB) &= \det\big(E_{(1)}^{-1}E_{(2)}^{-1}\cdots E_{(r-1)}^{-1}E_{(r)}^{-1}B\big)\\ &= \det\big(E_{(1)}^{-1}\big)\det\big(E_{(2)}^{-1}\big)\cdots\det\big(E_{(r-1)}^{-1}\big)\det\big(E_{(r)}^{-1}\big)\det(B)\\ &= \det\big(E_{(1)}^{-1}E_{(2)}^{-1}\cdots E_{(r-1)}^{-1}E_{(r)}^{-1}\big)\det(B) = \det(A)\det(B).\end{aligned}$$

□

**Remark** The proof of Theorem 1.17 can be given easily by using Theorems 2.5 and 2.6.

**Theorem 2.7** *If $A$ is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

*Proof* Since $A^{-1}A = I$, it follows from Theorem 2.6 that

$$\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1.$$

Thus, the result holds. □

### 2.3.3 Summary

We conclude this section by the following theorem that relates all of the major topics we have studied so far.

**Theorem 2.8** *Let $A$ be an $n \times n$ matrix. Then the following are equivalent.*

(a) *$A$ is invertible.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced row-echelon form of $A$ is $I_n$.*

(d) *$A$ is expressible as a product of elementary matrices.*

(e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*

(f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*

(g) $\det(A) \neq 0$.

**Remark** We now prove Theorem 1.20 (c). Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a triangular matrix with diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$. From Theorem 2.1 (c) and Theorem 2.8, the matrix $A$ is invertible if and only if

$$\det(A) = a_{11}a_{22}\cdots a_{nn} \neq 0,$$

which is true if and only if the diagonal entries are all nonzero.

## 2.4 Cofactor Expansions and Cramer's Rule

We introduce a method for evaluating determinants which is useful from a theoretical viewpoint. As a consequence of the method here, we obtain a formula in terms of determinants for the solution to a certain linear system with a square coefficient matrix.

### 2.4.1 Cofactors

**Definition** *Let $A = [a_{ij}]$ be a square matrix. Then the minor of entry $a_{ij}$, denoted by $M_{ij}$, is defined to be the determinant of the submatrix that remains after the $i$th row and $j$th column are deleted from $A$. The number*

$$C_{ij} := (-1)^{i+j} M_{ij}$$

*is called the **cofactor** of $a_{ij}$.*

**Example** Consider the following matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then the cofactors $C_{11}$ and $C_{12}$ are given as follows:

$$C_{11} = (-1)^{1+1} M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

and

$$C_{12} = (-1)^{1+2} M_{12} = -\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} = a_{23}a_{31} - a_{21}a_{33}.$$

### 2.4.2 Cofactor expansions

Now, we introduce the method of cofactor expansions for evaluating determinants.

**Theorem 2.9** *The determinant of an $n \times n$ matrix $A = [a_{ij}]$ can be evaluated by multiplying the entries in any row (or column) by their cofactors and adding the resulting products. More precisely, for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$,*

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(*cofactor expansion along the ith row*)

*and*

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

(*cofactor expansion along the jth column*)

We omit the proof of Theorem 2.9. For readers who are interested in the proof, we refer to [19, pp. 72–76] and [10, pp. 236–237].

**Example 1** Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -3 & 2 \\ 4 & 3 & -1 \end{bmatrix}.$$

Evaluate $\det(A)$ by cofactor expansion along the first row:

$$\begin{aligned} \det(A) &= 2 \cdot \det \begin{bmatrix} -3 & 2 \\ 3 & -1 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} -1 & 2 \\ 4 & -1 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} -1 & -3 \\ 4 & 3 \end{bmatrix} \\ &= 2 \times (-3) - 1 \times (-7) + 0 = 1. \end{aligned}$$

This agrees with the result obtained directly by using the definition of determinants.

**Example 2** Let

$$A = \begin{bmatrix} 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

where $n \geqslant 3$. One can easily obtain

$$\det(A) = (-1)^{n+1} n!$$

by using Theorem 2.9 (cofactor expansion along the first column).

**Example 3** Let

$$A = \begin{bmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{bmatrix} \in \mathbb{R}^{n \times n},$$

where $n \geqslant 3$. It follows from Theorem 2.9 again (cofactor expansion along the first column) that

$$\det(A) = x^n + (-1)^{n+1} y^n.$$

### 2.4.3 Adjoint of a matrix

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $C_{ij}$ be the cofactor of $a_{ij}$. We first show that for $p \neq q$,

$$a_{p1}C_{q1} + a_{p2}C_{q2} + \cdots + a_{pn}C_{qn} = 0. \tag{2.8}$$

We now construct a new matrix $B = [b_{ij}]$, in which all the rows of $B$ are the same as those in $A$ except the $q$th row which is replaced by the $p$th row in $A$ ($p \neq q$), i.e.,

$$B = \begin{array}{cc} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & b_{p3} & \cdots & b_{pn} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{q1} & b_{q2} & b_{q3} & \cdots & b_{qn} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix} & \begin{array}{c} \\ \\ \text{row } p \\ \\ \text{row } q \\ \\ \\ \end{array} \end{array} = \begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pn} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pn} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{c} \\ \\ \text{row } p \\ \\ \text{row } q \\ \\ \\ \end{array} \end{array}$$

Therefore, the $p$th and $q$th rows of $B$ are the same. Let $\widetilde{C}_{ij}$ be the cofactor of $b_{ij}$. In fact, for this fixed $q$, we have $\widetilde{C}_{qj} = C_{qj}$ for $1 \leqslant j \leqslant n$. It follows from Theorem 2.2 (c) and Theorem 2.9 (cofactor expansion along the $q$th row) that

$$0 = \det(B) = b_{q1}\widetilde{C}_{q1} + b_{q2}\widetilde{C}_{q2} + \cdots + b_{qn}\widetilde{C}_{qn} = a_{p1}C_{q1} + a_{p2}C_{q2} + \cdots + a_{pn}C_{qn}.$$

Thus, (2.8) holds. We have by Theorem 2.9 again,

$$a_{p1}C_{q1} + a_{p2}C_{q2} + \cdots + a_{pn}C_{qn} = \delta_{pq}\det(A), \tag{2.9}$$

where

$$\delta_{pq} = \begin{cases} 1 & \text{if } p = q; \\ 0 & \text{if } p \neq q. \end{cases}$$

Also we have

$$a_{1p}C_{1q} + a_{2p}C_{2q} + \cdots + a_{np}C_{nq} = \delta_{pq}\det(A).$$

We are now in the position to develop a formula for the inverse of an invertible matrix.

**Definition** *Let $A = [a_{ij}]$ be an $n \times n$ matrix and $C_{ij}$ be the cofactor of $a_{ij}$. Then the **adjoint** of $A$ is defined by*

$$\operatorname{adj}(A) := \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

**Theorem 2.10** *Let $A$ be an invertible matrix. Then*

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

*Proof* Let $A = [a_{ij}]$ be an $n \times n$ invertible matrix. First, we show that

$$A \operatorname{adj}(A) = \det(A)I.$$

We have by using (2.9),

$$\begin{aligned} A \operatorname{adj}(A) &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & \cdots & C_{n2} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} \\ &= \det(A)I. \end{aligned} \tag{2.10}$$

Since $A$ is invertible, $\det(A) \neq 0$. Therefore, (2.10) can be rewritten as

$$A\left[\frac{1}{\det(A)}\operatorname{adj}(A)\right] = I.$$

Thus, it follows from Theorem 1.16 (b) that

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A). \qquad \square$$

### 2.4.4 Cramer's rule

Cramer's rule provides a formula for representing the solution of a certain linear system.

**Theorem 2.11** (*Cramer's Rule*) *Let $A\mathbf{x} = \mathbf{b}$ be a system of $n$ linear equations in $n$ unknowns such that $\det(A) \neq 0$. Then the system has a unique solution which is given by*

$$x_1 = \frac{\det\big(A_{(1)}\big)}{\det(A)}, \quad x_2 = \frac{\det\big(A_{(2)}\big)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det\big(A_{(n)}\big)}{\det(A)},$$

*where $A_{(j)}$ is the matrix obtained by replacing the $j$th column of $A$ by the $n \times 1$ matrix*

$$\mathbf{b} = [b_1, b_2, \ldots, b_n]^T.$$

*Proof* Let $A = [a_{ij}]$ be an $n \times n$ matrix and $C_{ij}$ be the cofactor of $a_{ij}$. By using Theorems 2.5, 1.18, and 2.10, we know that the unique solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\mathbf{b},$$

i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} \sum\limits_{i=1}^{n} C_{i1}b_i \\ \sum\limits_{i=1}^{n} C_{i2}b_i \\ \vdots \\ \sum\limits_{i=1}^{n} C_{in}b_i \end{bmatrix}. \tag{2.11}$$

Construct

$$A_{(j)} = \begin{bmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}, \quad 1 \leqslant j \leqslant n.$$

We have by using cofactor expansion of $\det\big(A_{(j)}\big)$ along the $j$th column,

$$\det\big(A_{(j)}\big) = \sum_{i=1}^{n} C_{ij}b_i, \quad 1 \leqslant j \leqslant n. \tag{2.12}$$

Thus, substituting (2.12) into (2.11) yields

$$x_j = \frac{\det\big(A_{(j)}\big)}{\det(A)}, \quad 1 \leqslant j \leqslant n. \qquad \square$$

## Exercises

### Elementary exercises

**2.1** Find the number of inversions in each of the following permutations.

(a) $(3, 2, 4, 1)$. (b) $(2, 4, 1, 3)$. (c) $(4, 1, 3, 5, 2)$.

**2.2** In the determinant of a matrix $A = [a_{ij}] \in \mathbb{R}^{6\times 6}$, what are the signs of the terms $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$ and $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$, respectively?

**2.3** Prove Theorem 2.1 (a) and (c).

**2.4** Prove Theorem 2.2 except (b).

**2.5** Let

$$A = \begin{bmatrix} b & b & b & \cdots & b & b \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where $b \neq 0$. Find $\det(A)$ and $A^{-1}$.

**2.6** Evaluate the determinants of the following matrices.

$$\text{(a)} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 2 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & n-1 & \cdots & 0 & 0 \\ n & 0 & \cdots & 0 & 0 \end{bmatrix}. \qquad \text{(b)} \begin{bmatrix} 1 & 3 & 3 & \cdots & 3 \\ 3 & 2 & 3 & \cdots & 3 \\ 3 & 3 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 3 & 3 & \cdots & n \end{bmatrix}.$$

$$\text{(c)} \begin{bmatrix} x_1+1 & x_1 & \cdots & x_1 \\ x_2 & x_2+1 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n & \cdots & x_n+1 \end{bmatrix}. \qquad \text{(d)} \begin{bmatrix} x & y & \cdots & y \\ y & x & \cdots & y \\ \vdots & \vdots & \ddots & \vdots \\ y & y & \cdots & x \end{bmatrix}.$$

$$\text{(e)} \begin{bmatrix} x_1-y_1 & x_1-y_2 & \cdots & x_1-y_n \\ x_2-y_1 & x_2-y_2 & \cdots & x_2-y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n-y_1 & x_n-y_2 & \cdots & x_n-y_n \end{bmatrix}.$$

**2.7** Show that

$$\det \begin{bmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{bmatrix} = 2 \cdot \det \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}.$$

**2.8** Let $\mathbf{u}$ and $\mathbf{v}$ be $n \times 1$ matrices and $B$ be an $n \times n$ matrix. Show that

$$\det \begin{bmatrix} B & -B\mathbf{u} \\ -\mathbf{v}^T B & \mathbf{v}^T B\mathbf{u} \end{bmatrix} = 0.$$

**2.9** Let $A$ be a matrix defined by $A = \mathbf{a}^T\mathbf{a}$, where $\mathbf{a} = [2, 0, -1]$. If $k$ is a positive integer, find

$$\det\big((2I - A)^k\big),$$

where $I$ is the $3 \times 3$ identity matrix.

**2.10** Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

If $\det(A) = -7$, find

(a) $\det(A^2)$. (b) $\det\big((2A)^{-1}\big)$. (c) $\det \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$.

**2.11** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad C = \begin{bmatrix} a_{11} & b^{-1}a_{12} & b^{-2}a_{13} \\ ba_{21} & a_{22} & b^{-1}a_{23} \\ b^2a_{31} & ba_{32} & a_{33} \end{bmatrix},$$

where $b \neq 0$. Show that $\det(A) = \det(C)$.

**2.12** If $A^2 = A$, find all possible values of $\det(A)$.

**2.13** Let $A$ be an $n \times n$ skew-symmetric matrix. Show that $\det(A^T) = (-1)^n \det(A)$.

**2.14** Let

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Find $\det\big((2A)^{-1}B\big)$ and $\det\big((B^{-1}A^T)^2\big)$.

**2.15** Find all values of $k$ so that each of the following matrices is invertible.

(a) $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$. (b) $B = \begin{bmatrix} k & k & 0 \\ k^2 & 4 & k^2 \\ 0 & k & k \end{bmatrix}$.

**2.16** Let

$$A = \begin{bmatrix} 1 & -3 & 4 \\ -2 & 1 & 3 \\ 7 & 6 & -1 \end{bmatrix}.$$

(a) Evaluate the determinant of $A$ by cofactor expansion along the first row.

(b) Evaluate the determinant of $A$ by cofactor expansion along the second column.

(c) Find $\text{adj}(A)$.

(d) Find $A^{-1}$ by using Theorem 2.10.

(e) Find $\det\big((3A)^{-1} + \text{adj}(2A)\big)$.

**2.17** Let $A \in \mathbb{R}^{4\times 4}$. The elements in the first row of $A$ are $1, 2, -3, 4$. The cofactors of the elements of the third row of $A$ are given by $6, x, 9, 5$. Find the value of $x$.

**2.18** Suppose that $A, B \in \mathbb{R}^{3\times 3}$ such that $\text{adj}(A)BA = 10BA - I_3$. If

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

then find $B$.

**2.19** Let $A \in \mathbb{R}^{n\times n}$. Show that if $A \neq \mathbf{0}$ and $\text{adj}(A) = A^T$, then $A$ is invertible.

**2.20** Let $A, B \in \mathbb{R}^{n\times n}$.

(a) Show that if $A$ is invertible, then $\text{adj}(A)$ is invertible and

$$[\text{adj}(A)]^{-1} = \frac{1}{\det(A)} A = \text{adj}(A^{-1}).$$

(b) Show that

$$\det[\text{adj}(A)] = [\det(A)]^{n-1}.$$

(c) If $\det(A) = 2$ and $\det(B) = -3$, find

$$\det[2 \cdot \text{adj}(A)B^{-1}].$$

### Challenge exercises

**2.21** Using the fact that 2093, 6992, 3496, and 989 are divisible by 23, show that the determinant of the following matrix is also divisible by 23 without computing the determinant directly.

$$A = \begin{bmatrix} 2 & 0 & 9 & 3 \\ 6 & 9 & 9 & 2 \\ 3 & 4 & 9 & 6 \\ 0 & 9 & 8 & 9 \end{bmatrix}.$$

**2.22** Let $A \in \mathbb{R}^{3\times 3}$. Show that

$$\det(\lambda I_3 - A) = \lambda^3 - \lambda^2 \mathrm{tr}(A) + \lambda \mathrm{tr}\big(\mathrm{adj}(A)\big) - \det(A).$$

**2.23** Show that

$$\det \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} = (a^2 + b^2 + c^2 + d^2)^2.$$

**2.24** Show that

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1 \leqslant i < j \leqslant n} (a_j - a_i),$$

where $a_1, a_2, \ldots, a_n$ are distinct scalars from each other.

**2.25** Given four matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$, and $D \in \mathbb{R}^{k\times k}$, define

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Show that

(a) If $B = \mathbf{0}$ and $C = \mathbf{0}$, then $\det(M) = \det(A)\det(D)$.

(b) If $B = \mathbf{0}$ or $C = \mathbf{0}$, then $\det(M) = \det(A)\det(D)$.

(c) If $A$ is invertible, then $\det(M) = \det(A)\det(D - CA^{-1}B)$.

**2.26** Let $A$, $B \in \mathbb{R}^{n\times n}$. Show that if $A^2 = B^2 = I$ and $\det(A) + \det(B) = 0$, then $A + B$ is not invertible.

**2.27** Let $A \in \mathbb{R}^{n\times n}$. Show that if $AA^T = I$ and $\det(A) < 0$, then $\det(A + I) = 0$.

**2.28** Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be an upper triangular matrix. Show that the cofactor $C_{ij}$ of $a_{ij}$ is zero if $i < j$.

# Chapter 3

# Euclidean Vector Spaces

"*An interesting feature of these codes is that they make a very intensive use of subroutines*; *the addition of two vectors, multiplication of a vector by a scalar, inner products, etc, are all coded in this way.*"

— James Hardy Wilkinson

" '*Obvious*' *is the most dangerous word in mathematics.*"

— Eric Temple Bell

In the mid-seventeenth century, people started to use pairs of numbers to denote points in a plane and triples of numbers to denote points in a *3*-dimensional space. Later, mathematicians recognized that they can apply a similar idea to high-dimensional spaces. For instance, an $n$-tuple of numbers can be used to represent a point in an $n$-dimensional space. In this chapter, we begin with the definition of the $n$-vector space and follow by the definition of Euclidean $n$-space. We then introduce linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ and study their properties.

## 3.1 Euclidean *n*-Space

In this section, we first introduce definitions of the $n$-vector space and Euclidean $n$-space. Then we study some geometric properties of Euclidean $n$-space.

### 3.1.1 *n*-vector space

Let $\mathbb{R}^n := \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$, where an ordered $n$-tuple $(a_1, a_2, \ldots, a_n)$ is called a ***vector*** in $\mathbb{R}^n$. Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ are called equal if

$$u_1 = v_1, \quad u_2 = v_2, \quad \ldots, \quad u_n = v_n.$$

**Definition** *Let* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *be two vectors in* $\mathbb{R}^n$*.*

(i) *The sum* $\mathbf{u}+\mathbf{v}$ *is defined by*

$$\mathbf{u}+\mathbf{v} := (u_1+v_1, u_2+v_2, \ldots, u_n+v_n).$$

(ii) *If* $k$ *is a scalar, the scalar multiplication* $k\mathbf{u}$ *is defined by*

$$k\mathbf{u} := (ku_1, ku_2, \ldots, ku_n).$$

*The set* $\mathbb{R}^n$ *with the operations of addition and scalar multiplication is called the* ***n-vector space.***

The most important arithmetic properties of vector operations in $\mathbb{R}^n$ are listed in the following theorem. The proof of the theorem is trivial and is left as an exercise.

**Theorem 3.1** *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$. *Then*

(a) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.

(b) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(c) $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$, *where* $\mathbf{0} = (0, 0, \ldots, 0)$.

(d) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$, *i.e.,* $\mathbf{u}-\mathbf{u} = \mathbf{0}$.

(e) $k(l\mathbf{u}) = (kl)\mathbf{u}$.

(f) $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$.

(g) $(k+l)\mathbf{u} = k\mathbf{u}+l\mathbf{u}$.

(h) $1\mathbf{u} = \mathbf{u}$.

*Here* $k$ *and* $l$ *are scalars in* $\mathbb{R}$.

### 3.1.2 Euclidean *n*-space

To develop geometrical notions of distance, norm, and angle in $\mathbb{R}^n$, we begin with the following definition.

**Definition** *Let* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *be any vectors in* $\mathbb{R}^n$. *Then the* ***Euclidean inner product*** $\mathbf{u}\cdot\mathbf{v}$ *is defined by*

$$\mathbf{u}\cdot\mathbf{v} := u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i. \tag{3.1}$$

*The vector space* $\mathbb{R}^n$ *with the Euclidean inner product is called* ***Euclidean n-space.***

Some arithmetic properties of the Euclidean inner product are listed in the following theorem.

**Theorem 3.2** *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and* $k$ *be any scalar. Then*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

(c) $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$.

(d) $\mathbf{v} \cdot \mathbf{v} \geqslant 0$. *Further,* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.

*Proof* The proofs of (a) and (c) are trivial and we therefore only prove (b) and (d). Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n), \quad \mathbf{v} = (v_1, v_2, \ldots, v_n), \quad \mathbf{w} = (w_1, w_2, \ldots, w_n).$$

For (b), it follows directly from the definition of the Euclidean inner product that

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \cdots + (u_n + v_n)w_n \\ &= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n) \\ &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}.\end{aligned}$$

For (d), we have

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \sum_{i=1}^{n} v_i^2 \geqslant 0.$$

Furthermore,

$$\sum_{i=1}^{n} v_i^2 = 0 \Longleftrightarrow v_i = 0, \quad 1 \leqslant i \leqslant n.$$

Thus, $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$. □

### 3.1.3 Norm, distance, angle, and orthogonality

**Definition** *The **Euclidean norm** (or **Euclidean length**) of a vector* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *in* $\mathbb{R}^n$ *is defined by*

$$\|\mathbf{u}\| := (\mathbf{u} \cdot \mathbf{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

*The distance of* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *is defined by*

$$d(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

The next theorem provides one of the most important inequalities in matrix theory: the ***Cauchy-Schwarz inequality***.

**Theorem 3.3** (*Cauchy-Schwarz Inequality in* $\mathbb{R}^n$) *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$. *Then*

$$|\mathbf{u}\cdot\mathbf{v}| \leqslant \|\mathbf{u}\|\cdot\|\mathbf{v}\|. \tag{3.2}$$

*Proof* If $\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$, then the theorem is obviously true. Now assume $\mathbf{u}\neq\mathbf{0}$ and $\mathbf{v}\neq\mathbf{0}$. Construct a new vector

$$\mathbf{r}=\mathbf{u}+t\mathbf{v}, \quad t\in\mathbb{R}.$$

We have

$$0\leqslant \mathbf{r}\cdot\mathbf{r}=(\mathbf{u}+t\mathbf{v})\cdot(\mathbf{u}+t\mathbf{v})=\mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}t+\mathbf{v}\cdot\mathbf{v}t^2.$$

Considering the discriminant $\Delta$ of the quadratic function of $t$, we have

$$\Delta=(2\mathbf{u}\cdot\mathbf{v})^2-4(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})\leqslant 0,$$

which implies

$$(\mathbf{u}\cdot\mathbf{v})^2\leqslant(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}).$$

Thus,

$$|\mathbf{u}\cdot\mathbf{v}| \leqslant \|\mathbf{u}\|\cdot\|\mathbf{v}\|. \qquad \square$$

**Remark** In $\mathbb{R}^2$, we know that two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ form an angle $\theta$, where $0\leqslant\theta\leqslant\pi$. Then we have by the cosine formula,

$$\cos\theta=\frac{\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-\|\mathbf{v}-\mathbf{u}\|^2}{2\|\mathbf{u}\|\cdot\|\mathbf{v}\|}=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|}.$$

It follows from the Cauchy-Schwarz inequality (3.2) that the cosine of an angle $\theta$ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ can also be defined by

$$\cos\theta:=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|}. \tag{3.3}$$

The next two theorems are concerned with the basic properties of norm and distance in Euclidean $n$-space.

**Theorem 3.4** *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$ *and* $k$ *be any scalar. Then*

(a) $\|\mathbf{u}\|\geqslant 0$.

(b) $\|\mathbf{u}\|=0$ *if and only if* $\mathbf{u}=\mathbf{0}$.

(c) $\|k\mathbf{u}\|=|k|\cdot\|\mathbf{u}\|$.

(d) $\|\mathbf{u}+\mathbf{v}\| \leqslant \|\mathbf{u}\|+\|\mathbf{v}\|$. (*Triangle inequality*)

*Proof* From Theorems 3.2 (c) and (d), one can show that (a), (b), and (c) are true. Here we only prove (d). Based on (3.2), we have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} \leqslant \mathbf{u}\cdot\mathbf{u}+2|\mathbf{u}\cdot\mathbf{v}|+\mathbf{v}\cdot\mathbf{v}\\ &\leqslant \|\mathbf{u}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\|+\|\mathbf{v}\|^2 = \big(\|\mathbf{u}\|+\|\mathbf{v}\|\big)^2.\end{aligned}$$

Thus, (d) holds. □

**Theorem 3.5** *Let* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and* $k$ *be any scalar. Then*

(a) $d(\mathbf{u},\mathbf{v}) \geqslant 0$.

(b) $d(\mathbf{u},\mathbf{v}) = 0$ *if and only if* $\mathbf{u}=\mathbf{v}$.

(c) $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$.

(d) $d(\mathbf{u},\mathbf{v}) \leqslant d(\mathbf{u},\mathbf{w})+d(\mathbf{w},\mathbf{v})$. (*Triangle inequality*)

The proof of Theorem 3.5 is left as an exercise.

We now introduce the concept of orthogonality of vectors.

**Definition** *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$ *are called* ***orthogonal*** *if* $\mathbf{u}\cdot\mathbf{v}=0$.

**Remark** Actually, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ defined by (3.3) is $\pi/2$ .

**Theorem 3.6** (*Pythagorean Theorem in* $\mathbb{R}^n$) *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be orthogonal vectors in* $\mathbb{R}^n$ *with the Euclidean inner product. Then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2.$$

*Proof* Since $\mathbf{u}\cdot\mathbf{v}=0$, we have

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2. \quad \square$$

### 3.1.4 Some remarks

(1) A vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)\in\mathbb{R}^n$ can be written in row matrix notation or column matrix notation if no confusion arises:

$$\mathbf{u}=[u_1,u_2,\ldots,u_n] \qquad \text{or} \qquad \mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}.$$

Notice that there is no essential difference between an $n$-vector and a $1 \times n$ matrix or an $n \times 1$ matrix from an algebraic viewpoint. Thus, in the following, we will use the notations above to denote any vector in $\mathbb{R}^n$ freely.

(2) The Euclidean inner product (3.1) of vectors

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \qquad \text{and} \qquad \mathbf{v} = (v_1, v_2, \ldots, v_n)$$

can also be written in a form of the matrix product:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = [u_1, u_2, \ldots, u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

(3) If the row matrices of an $m \times r$ matrix $A$ are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and the column matrices of an $r \times n$ matrix $B$ are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$, then by using the Euclidean inner product, the matrix product $AB$ can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_n \end{bmatrix}.$$

## 3.2 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

In this section, we study linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

### 3.2.1 Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

A transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a map which maps each point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ to a unique point $T(\mathbf{x}) = \mathbf{w} = (w_1, w_2, \ldots, w_m)$ in $\mathbb{R}^m$. A ***linear transformation*** $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a map which is defined by linear equations of the form

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\;\;\vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

or in matrix notation

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or more briefly by

$$\mathbf{w} = T(\mathbf{x}) = A\mathbf{x}.$$

The $m \times n$ matrix $A = [a_{ij}]$ is called the ***standard matrix*** for the linear transformation $T$ and $T$ is called multiplication by $A$. We sometimes denote this $T$ by $T_A$, i.e.,

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

Hence $T_A$ is also called a ***matrix transformation*** from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**Remark** If $\mathbf{0}$ is the $m \times n$ zero matrix, then for every vector $\mathbf{x}$ in $\mathbb{R}^n$, we have

$$T_{\mathbf{0}}(\mathbf{x}) = \mathbf{0}\mathbf{x} = \mathbf{0} \in \mathbb{R}^m.$$

We call $T_{\mathbf{0}}$ the ***zero transformation*** from $\mathbb{R}^n$ to $\mathbb{R}^m$. Moreover, for any $m \times n$ matrix $A$ and the zero vector $\mathbf{0} \in \mathbb{R}^n$, we have

$$T_A(\mathbf{0}) = A\mathbf{0} = \mathbf{0} \in \mathbb{R}^m.$$

If $I$ is the $n \times n$ identity matrix, then for every vector $\mathbf{x} \in \mathbb{R}^n$,

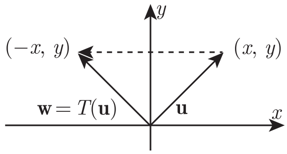
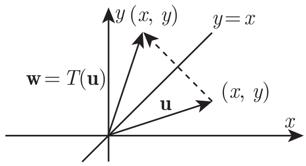
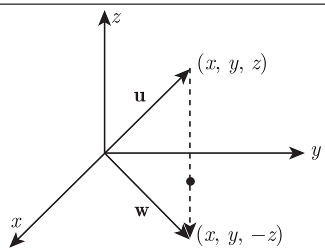
$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x},$$

so multiplication by $I$ maps every vector in $\mathbb{R}^n$ into itself. We call $T_I$ the ***identity transformation*** on $\mathbb{R}^n$.

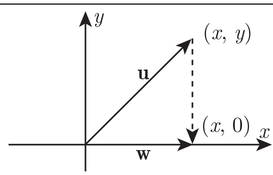
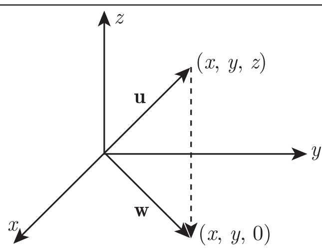
### 3.2.2 Some important linear transformations

Among the most important linear transformations on $\mathbb{R}^2$ and $\mathbb{R}^3$ are reflections, projections, and rotations. We now discuss such linear transformations $T$ one by one [1]. In the following, let $\mathbf{u} = (x, y) \in \mathbb{R}^2$ or $\mathbf{u} = (x, y, z) \in \mathbb{R}^3$ and we denote $\mathbf{w} = T(\mathbf{u})$ by $\mathbf{w} = (w_1, w_2)$ or $\mathbf{w} = (w_1, w_2, w_3)$.

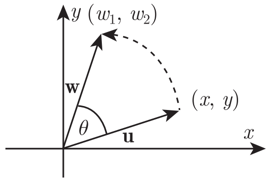
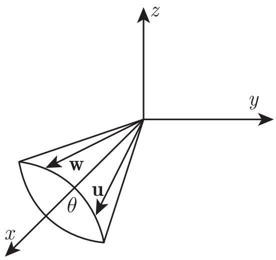
(1) Reflection transformations.

| **Transformation** | **Illustration** | **Equations** | **Standard Matrix** |
|---|---|---|---|
| Reflection about the $y$-axis |  | $w_1 = -x$<br>$w_2 = \phantom{-}y$ | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Reflection about the line $y = x$ |  | $w_1 = y$<br>$w_2 = x$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| Reflection about the $xy$-plane |  | $w_1 = \phantom{-}x$<br>$w_2 = \phantom{-}y$<br>$w_3 = -z$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ |

(2) Projection transformations.

| **Transformation** | **Illustration** | **Equations** | **Standard Matrix** |
|---|---|---|---|
| Orthogonal projection on the $x$-axis |  | $w_1 = x$<br>$w_2 = 0$ | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ |
| Orthogonal projection on the $xy$-plane |  | $w_1 = x$<br>$w_2 = y$<br>$w_3 = 0$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |

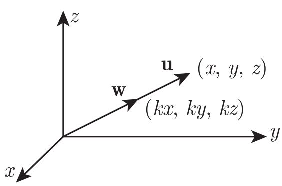
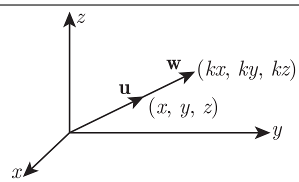
(3) Rotation transformations.

| Transformation | Illustration | Equations | Standard Matrix |
|---|---|---|---|
| Rotation through an angle $\theta$ |   | $w_1 = x\cos\theta - y\sin\theta$<br>$w_2 = x\sin\theta + y\cos\theta$ | $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ |
| Counterclockwise rotation about the $x$-axis with an angle $\theta$ |   | $w_1 = x$<br>$w_2 = y\cos\theta - z\sin\theta$<br>$w_3 = y\sin\theta + z\cos\theta$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ |

**Remark** In $xy$-plane, by using the polar coordinates, we have $x = r\cos\alpha$ and $y = r\sin\alpha$. Thus,

$$\begin{aligned} w_1 &= r\cos(\alpha+\theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta = x\cos\theta - y\sin\theta, \\ w_2 &= r\sin(\alpha+\theta) = r\cos\alpha\sin\theta + r\sin\alpha\cos\theta = x\sin\theta + y\cos\theta. \end{aligned}$$

(4) Contraction and dilation transformations.

| Transformation | Illustration | Equations | Standard Matrix |
|---|---|---|---|
| Contraction with factor $k$ on $\mathbb{R}^3$ $(0 \leqslant k \leqslant 1)$ |   | $w_1 = kx$<br>$w_2 = ky$<br>$w_3 = kz$ | $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ |
| Dilation with factor $k$ on $\mathbb{R}^3$ $(k \geqslant 1)$ |   | $w_1 = kx$<br>$w_2 = ky$<br>$w_3 = kz$ | $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ |

### 3.2.3 Compositions of linear transformations

Let $T_A\colon \mathbb{R}^n \to \mathbb{R}^k$ and $T_B\colon \mathbb{R}^k \to \mathbb{R}^m$ be linear transformations. Then for each $\mathbf{x} \in \mathbb{R}^n$, one can first compute $T_A(\mathbf{x}) \in \mathbb{R}^k$, and then compute $T_B\big(T_A(\mathbf{x})\big) \in \mathbb{R}^m$. Hence the application of $T_A$ followed by $T_B$ produces a transformation from $\mathbb{R}^n$ to

$\mathbb{R}^m$. This transformation, denoted by $T_B \circ T_A$, is called the ***composition*** of $T_B$ with $T_A$. Actually, we have

$$(T_B \circ T_A)(\mathbf{x}) = T_B\big(T_A(\mathbf{x})\big) = B(A\mathbf{x}) = (BA)\mathbf{x}. \tag{3.4}$$

Thus, $T_B \circ T_A$ is multiplication by $BA$, which is also a linear transformation. Formula (3.4) tells us that the standard matrix for $T_B \circ T_A$ is $BA$, i.e.,

$$T_B \circ T_A = T_{BA}. \tag{3.5}$$

**Remark** Formula (3.5) reveals an important idea that actually multiplying matrices is equivalent to composing the corresponding linear transformations in the right-to-left order of the factors.

Compositions can be defined for three or more linear transformations. For instance, we consider the linear transformations

$$T_A\colon \mathbb{R}^n \to \mathbb{R}^k, \qquad T_B\colon \mathbb{R}^k \to \mathbb{R}^l, \qquad T_C\colon \mathbb{R}^l \to \mathbb{R}^m.$$

The composition $T_C \circ T_B \circ T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$(T_C \circ T_B \circ T_A)(\mathbf{x}) = T_C\big(T_B(T_A(\mathbf{x}))\big) = C\big(B(A\mathbf{x})\big) = (CBA)\mathbf{x}.$$

Thus, the standard matrix for $T_C \circ T_B \circ T_A$ is $CBA$, which is a generalization of (3.5). This property can be extended to a finite number of linear transformations without any difficulty.

## 3.3 Properties of Transformations

In this section, we study the linearity conditions and investigate the relationship between the invertibility of a matrix and properties of the corresponding matrix transformation.

### 3.3.1 Linearity conditions

**Theorem 3.7** *A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if the following **linearity conditions** hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and every scalar $c$.*

(a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(b) $T(c\mathbf{u}) = cT(\mathbf{u})$.

*Proof* If $T$ is a linear transformation, then it is easy to see that the linearity conditions hold. Conversely, if the linearity conditions hold, then for any vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, we can express $\mathbf{x}$ by the following linear combination:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the $i$th column matrix of the $n \times n$ identity matrix for $1 \leqslant i \leqslant n$. By using the linearity conditions, we obtain

$$T(\mathbf{x}) = T\Big(\sum_{i=1}^{n} x_i\mathbf{e}_i\Big) = \sum_{i=1}^{n} x_i T(\mathbf{e}_i) = A\mathbf{x},$$

where the successive column matrices of $A$ are $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$, i.e.,

$$A = \Big[\; T(\mathbf{e}_1) \;\vdots\; T(\mathbf{e}_2) \;\vdots\; \cdots \;\vdots\; T(\mathbf{e}_n) \;\Big]. \tag{3.6}$$

Thus, $T$ is a linear transformation and $A$ is the standard matrix for $T$. $\square$

### 3.3.2 Example

Let $l$ be the line in the $xy$-plane that passes through the origin and makes an angle $\theta$ with the positive $x$-axis, where $0 \leqslant \theta \leqslant \pi/2$. As illustrated in Figure 3.1 (a), let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps each vector into its orthogonal projection on $l$.

(a) Find the standard matrix $A$ for $T$.

(b) Find the orthogonal projection of the vector $\mathbf{x} = [2, 3]^T$ onto the line through the origin that makes an angle of $\theta = \pi/6$ with the positive $x$-axis.

*Solution* For (a), it follows from (3.6) that

$$A = \Big[\; T(\mathbf{e}_1) \;\vdots\; T(\mathbf{e}_2) \;\Big],$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

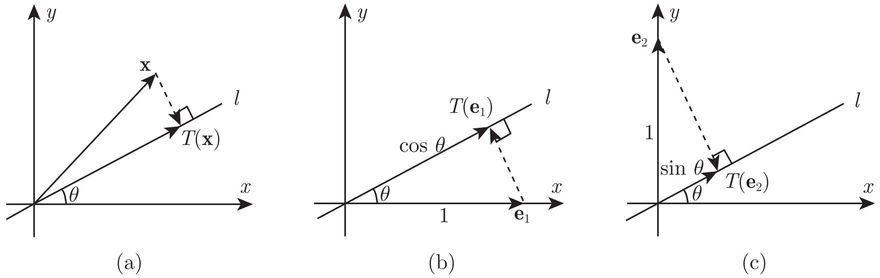



Figure 3.1

Referring to Figure 3.1 (b), the length of the vector $T(\mathbf{e}_1)$ is given by $\|T(\mathbf{e}_1)\| = \cos\theta$, so

$$T(\mathbf{e}_1) = \begin{bmatrix} \|T(\mathbf{e}_1)\|\cos\theta \\ \\ \|T(\mathbf{e}_1)\|\sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \\ \sin\theta\cos\theta \end{bmatrix}.$$

Referring to Figure 3.1 (c), the length of the vector $T(\mathbf{e}_2)$ is given by $\|T(\mathbf{e}_2)\| = \sin\theta$, so

$$T(\mathbf{e}_2) = \begin{bmatrix} \|T(\mathbf{e}_2)\|\cos\theta \\ \\ \|T(\mathbf{e}_2)\|\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\theta \\ \\ \sin^2\theta \end{bmatrix}.$$

Thus, the standard matrix $A$ for $T$ is

$$A = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}.$$

For (b), since $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$, it follows from part (a) that the standard matrix $A$ for this projection transformation is

$$A = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \\ \sqrt{3}/4 & 1/4 \end{bmatrix}.$$

Thus, the orthogonal projection of the vector $\mathbf{x}$ is

$$T(\mathbf{x}) = T\left(\begin{bmatrix} 2 \\ \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \\ \sqrt{3}/4 & 1/4 \end{bmatrix}\begin{bmatrix} 2 \\ \\ 3 \end{bmatrix} = \begin{bmatrix} \dfrac{6+3\sqrt{3}}{4} \\ \\ \dfrac{2\sqrt{3}+3}{4} \end{bmatrix}. \qquad \square$$

### 3.3.3 One-to-one transformations

**Definition** *A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be* ***one-to-one*** *if $T$ maps distinct vectors in $\mathbb{R}^n$ into distinct vectors in $\mathbb{R}^m$.*

For the relationship between the invertibility of a square matrix and properties of corresponding linear transformation, we have the following theorem.

**Theorem 3.8** *Let $A$ be an $n \times n$ matrix and $T_A\colon \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by $A$. Then the following statements are equivalent.*

(a) *$A$ is invertible.*

(b) *The range of $T_A$ is $\mathbb{R}^n$, where the range of $T_A$ is given by $\{T_A(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n\}$.*

(c) *$T_A$ is one-to-one.*

*Proof* We first give three more equivalent statements (a′), (b′), and (c′).

(a) $A$ is invertible. (a′) $A\mathbf{x} = \mathbf{0} \Longrightarrow \mathbf{x} = \mathbf{0}$.

(b) The range of $T_A$ is $\mathbb{R}^n$. (b′) $A\mathbf{x} = \mathbf{w}$ is consistent for all $\mathbf{w} \in \mathbb{R}^n$.

(c) $T_A$ is one-to-one. (c′) $T_A(\mathbf{x}) = \mathbf{0} \Longrightarrow \mathbf{x} = \mathbf{0}$.

Then by using Theorem 2.8 and the definition of linear transformations, one can easily see that

$$\text{(a)} \Leftrightarrow \text{(a}'\text{)} \Leftrightarrow \text{(b}'\text{)} \Leftrightarrow \text{(b)}, \qquad \text{(a}'\text{)} \Leftrightarrow \text{(c}'\text{)}.$$

Therefore, in order to complete the proof of the theorem, we only need to prove (c) $\Leftrightarrow$ (c′).

(c) $\Rightarrow$ (c′): If $T_A$ is one-to-one, then for any nonzero $\mathbf{x} \in \mathbb{R}^n$,

$$T_A(\mathbf{x}) \neq T_A(\mathbf{0}) = \mathbf{0}.$$

Thus, (c′) holds.

(c′) $\Rightarrow$ (c): Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\mathbf{x}_1 \neq \mathbf{x}_2$. We want to show that $T_A(\mathbf{x}_1) \neq T_A(\mathbf{x}_2)$. By contradiction, we assume that $T_A(\mathbf{x}_1) = T_A(\mathbf{x}_2)$. Then

$$T_A(\mathbf{x}_1 - \mathbf{x}_2) = T_A(\mathbf{x}_1) - T_A(\mathbf{x}_2) = \mathbf{0}.$$

By the given condition, we have $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ and then $\mathbf{x}_1 = \mathbf{x}_2$, which contradicts the fact that $\mathbf{x}_1 \neq \mathbf{x}_2$. Thus, (c) holds. □

**Remark** Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one linear transformation. Then it follows from Theorem 3.8 that the matrix $A$ is invertible. Thus, $T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$ is itself a linear transformation and it is called the ***inverse*** of $T_A$. In fact, the linear transformations $T_A$ and $T_{A^{-1}}$ cancel the effect of one another. More precisely, for all $\mathbf{x}$ in $\mathbb{R}^n$,

$$T_A\big(T_{A^{-1}}(\mathbf{x})\big) = AA^{-1}\mathbf{x} = I\mathbf{x} = \mathbf{x}, \qquad T_{A^{-1}}\big(T_A(\mathbf{x})\big) = A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x},$$

or equivalently,

$$T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_I, \qquad T_{A^{-1}} \circ T_A = T_{A^{-1}A} = T_I.$$

### 3.3.4 Summary

**Theorem 3.9** *Let $A$ be an $n \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by $A$. Then the following are equivalent.*

(a) *$A$ is invertible.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced row-echelon form of* $A$ *is* $I_n$.

(d) $A$ *is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) *The range of* $T_A$ *is* $\mathbb{R}^n$.

(i) $T_A$ *is one-to-one.*

## Exercises

**Elementary exercises**

**3.1** Prove Theorem 3.1.

**3.2** Let $\mathbf{v}_1 = [1, 0, 1, 0]$, $\mathbf{v}_2 = [0, 1, 0, 2]$, $\mathbf{v}_3 = [2, 0, 0, 1]$, and $\mathbf{v}_4 = [0, -2, -3, 0]$. Find $k_1$, $k_2$, $k_3$, and $k_4$ such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = [-4, 1, -5, 5].$$

**3.3** Find the inner product $\mathbf{u} \cdot \mathbf{v}$.

(a) $\mathbf{u} = [4, 2, -7], \quad \mathbf{v} = [-1, 2, 5].$ (b) $\mathbf{u} = [-2, 8, 4, -7], \quad \mathbf{v} = [5, -1, -3, 2].$

**3.4** Let $\mathbf{u} = [4, 1, 2, 3]$, $\mathbf{v} = [0, 3, 8, -2]$, and $\mathbf{w} = [3, 1, 2, 2]$. Evaluate each expression.

(a) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|.$ (b) $\dfrac{1}{\|\mathbf{w}\|}\mathbf{w}.$

**3.5** Find $\mathbf{u} \cdot \mathbf{v}$ if $\|\mathbf{u} + \mathbf{v}\| = 1$ and $\|\mathbf{u} - \mathbf{v}\| = 5$.

**3.6** Find $\|\mathbf{u} + \mathbf{v}\|$ if $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\| = 2\sqrt{2}$.

**3.7** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Show that

(a) $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$.

(b) $\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4\mathbf{u} \cdot \mathbf{v}$.

**3.8** If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{n\times 1}$ and $A \in \mathbb{R}^{n\times n}$, show that

$$(\mathbf{u}^T A^T A\mathbf{v})^2 \leqslant (\mathbf{u}^T A^T A\mathbf{u})(\mathbf{v}^T A^T A\mathbf{v}).$$

**3.9** Prove Theorem 3.5.

**3.10** For which value of $k$, are $\mathbf{u} = [2, 1, 3]$ and $\mathbf{v} = [1, 7, k]$ orthogonal?

**3.11** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|.$$

**3.12** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

(a) If $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, is $\mathbf{u}$ orthogonal to $\mathbf{v} + \mathbf{w}$?

(b) If $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$, is $\mathbf{u}$ orthogonal to $\mathbf{v}$ and $\mathbf{w}$?

**3.13** Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathbb{R}^n$. Show that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are pairwise orthogonal, i.e., $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for any $i \neq j$, then

$$\|\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n\|^2 = \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 + \cdots + \|\mathbf{u}_n\|^2.$$

**3.14** Show that

(a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ defines a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(b) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ does not define a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

**3.15** Find the standard matrix for each of the following linear transformations.

(a) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_2 - 3x_3 \end{bmatrix}$. (b) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 + x_4 \\ -x_3 \\ x_2 \end{bmatrix}$.

**3.16** For each part, find the standard matrices for $T_1$ and $T_2$, then determine whether $T_1 \circ T_2 = T_2 \circ T_1$.

(a) $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the $x$-axis, and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the $y$-axis.

(b) $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the $x$-axis, and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection on the $y$-axis.

(c) $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation through an angle $\theta$, and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the $y$-axis.

(d) $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection on the $xy$-plane, and $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection on the $yz$-plane.

(e) $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the counterclockwise rotation about the positive $x$-axis through an angle $\theta_1$, and $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the counterclockwise rotation about the positive $y$-axis through an angle $\theta_2$.

**3.17** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps each vector into its reflection about the line $l$.

(a) Find the standard matrix for $T$ if $l$ is the line in the $xy$-plane that passes through the origin and makes an angle $\theta$ with the positive $x$-axis, where $0 \leqslant \theta \leqslant \pi/2$.

(b) Find the reflection of the vector $\mathbf{x} = [1, 5]^T$ about the line $l$ through the origin that makes an angle of $\theta = \pi/6$ with the positive $x$-axis.

**3.18** Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that counterclockwise rotates each vector about the positive $y$-axis through an angle $\theta$, where $0 \leqslant \theta \leqslant \pi/2$.

(a) Find the standard matrix for $T$.

(b) Find the rotation of the vector $\mathbf{x} = [-5, 1, 2]^T$ through an angle of $\theta = \pi/3$.

**3.19** Let $T_1\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T_2\colon \mathbb{R}^m \to \mathbb{R}^s$ be linear transformations.

(a) If $T_1$ and $T_2$ are one-to-one, is $T_2 \circ T_1$ one-to-one?

(b) If either $T_1$ or $T_2$ is one-to-one, is $T_2 \circ T_1$ one-to-one?

**3.20** Determine if each linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ $(n = 2, 3)$ defined by the given equations is one-to-one; if so, find the standard matrix for the inverse transformation, and find $T^{-1}$.

(a) $\begin{cases} w_1 = \phantom{-}x_1 + 2x_2 \\ w_2 = -x_1 + \phantom{2}x_2. \end{cases}$ (b) $\begin{cases} w_1 = \phantom{-}x_1 + 2x_2 + \phantom{3}x_3 \\ w_2 = -x_1 + \phantom{2}x_2 - \phantom{3}x_3 \\ w_3 = \phantom{-}x_1 + \phantom{2}x_2 + 3x_3. \end{cases}$

### Challenge exercises

**3.21** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Show that if $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ holds for all $\mathbf{w} \in \mathbb{R}^n$, then $\mathbf{u} = \mathbf{v}$.

**3.22** Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that if $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (B\mathbf{y})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, then $B = A^T$.

**3.23** Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Show that

$$\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{z}-\mathbf{x}\|^2 + 2\|\mathbf{z}-\mathbf{y}\|^2 - 4\|\mathbf{z}-(\mathbf{x}+\mathbf{y})/2\|^2.$$

**3.24** Using the Cauchy-Schwarz inequality, show that

(a) If $a_1, a_2, \ldots, a_n > 0$, then

$$(a_1 + a_2 + \cdots + a_n)\Big(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\Big) \geqslant n^2.$$

(b) If $a, b, c > 0$, then

$$\Big(\frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}c\Big)^2 \leqslant \frac{1}{2}a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2.$$

(c) If $a_1, a_2, \ldots, a_n, w_1, w_2, \ldots, w_n > 0$ and $\sum\limits_{k=1}^{n} w_k = 1$, then

$$\Big(\sum_{k=1}^{n} a_k w_k\Big)^2 \leqslant \sum_{k=1}^{n} a_k^2 w_k.$$

**3.25** Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that the following are equivalent.

(a) $\mathbf{x} \cdot \mathbf{y} \leqslant 0$.

(b) $\|\mathbf{x}\| \leqslant \|\mathbf{x} - \alpha\mathbf{y}\|$ for all $\alpha \geqslant 0$.

(c) $\|\mathbf{x}\| \leqslant \|\mathbf{x} - \alpha\mathbf{y}\|$ for all $\alpha \in [0, 1]$.

**3.26** Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that the following are equivalent.

(a) $\mathbf{x} \cdot \mathbf{y} = 0$.

(b) $\|\mathbf{x}\| \leqslant \|\mathbf{x} - \alpha\mathbf{y}\|$ for all $\alpha \in \mathbb{R}$.

(c) $\|\mathbf{x}\| \leqslant \|\mathbf{x} - \alpha\mathbf{y}\|$ for all $\alpha \in [-1, 1]$.

**3.27** Let $T_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix transformation such that for all $\mathbf{x} \in \mathbb{R}^3$,

$$T_A(\mathbf{x}) \cdot \mathbf{x} = A\mathbf{x} \cdot \mathbf{x} = 0.$$

(a) Show that $A$ is not invertible.

(b) Is a similar assertion true for a matrix transformation $T_A \colon \mathbb{R}^2 \to \mathbb{R}^2$?

**3.28** Let $T_A$ be the matrix transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, where $m \leqslant n$. Show that the following are equivalent.

(a) $T_A$ is one-to-one.

(b) There exists a matrix transformation $T_B$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $T_B \circ T_A = T_I$, where $T_I$ is the identity transformation on $\mathbb{R}^m$.

(c) For any matrix transformations $T_C$ and $T_D$ from $\mathbb{R}^r$ to $\mathbb{R}^m$ satisfying $T_A \circ T_C = T_A \circ T_D$, we have $T_C = T_D$.

# Chapter 4

# General Vector Spaces

"*Mathematics is the art of giving the same name to different things.*"

— Henri Poincaré

"*Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field.*"

— Paul Dirac

In this chapter, we generalize the concept of vectors in $\mathbb{R}^n$ further. If a class of objects with two operations satisfies a set of axioms, then we entitle those objects to be called "vectors". Moreover, since the axioms of generalized vectors are based on properties of vectors in $\mathbb{R}^n$, the generalized vectors have many similar properties. Thus, this generalization provides a powerful tool to extend geometric properties of vectors in $\mathbb{R}^n$ to many important mathematical problems where geometric intuition may not be available. Consequently, if we have a problem involving our generalized vectors, say matrices or functions, we may study the problem based on the corresponding one in $\mathbb{R}^n$.

## 4.1 Real Vector Spaces

In this section, we extend the concept of vectors in $\mathbb{R}^n$ by extracting the most fundamental properties from them and turning those properties into axioms for our generalized vectors.

### 4.1.1 Vector space axioms

The following definition is extremely useful for many purposes. It consists of two operations and eight axioms.

**Definition** *Let $V$ be a nonempty set of objects on which two operations are defined, addition and scalar multiplication. It requires that $V$ is closed under the addition* *and scalar multiplication, i.e., for each pair of objects* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $V$, $\mathbf{u}+\mathbf{v}$ *is in* $V$; *for each scalar* $k$ *and each object* $\mathbf{u}$ *in* $V$, $k\mathbf{u}$ *is in* $V$. *Then* $V$ *is called a* ***vector space*** *and the objects in* $V$ *are said to be* ***vectors*** *if the following eight axioms are satisfied for all* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* $V$.

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(ii) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(iii) *There is an object* $\mathbf{0}$ *in* $V$, *called a zero vector for* $V$, *such that for all* $\mathbf{u}$ *in* $V$, $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

(iv) *For each* $\mathbf{u}$ *in* $V$, *there is an object* $-\mathbf{u}$ *in* $V$, *called a negative of* $\mathbf{u}$, *such that* $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(v) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.

(vi) $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$.

(vii) $k(l\mathbf{u})=(kl)\mathbf{u}$.

(viii) $1\mathbf{u}=\mathbf{u}$.

*Here* $k$ *and* $l$ *are scalars. If the scalars are in* $\mathbb{R}$, *then* $V$ *is called a* ***real vector space***.

**Remark** In fact, Axiom (i) is not necessary because it can be deduced by the other axioms. Hence there is no need to list it explicitly. See Appendix A for details.

In the following, all scalars will be real numbers until Section 8.3.

**Examples**

(a) The set $\mathbb{R}^n$ with the operations of addition and scalar multiplication defined in Subsection 3.1.1 is a typical example of a vector space.

(b) The set $\mathbb{R}^{m\times n}$ with the operations of matrix addition and scalar multiplication is a vector space.

(c) For all functions $\mathbf{f}=f(x)$ and $\mathbf{g}=g(x)$ defined on $(-\infty,\infty)$, we define the following operations of function addition and scalar multiplication

$$(\mathbf{f}+\mathbf{g})(x):=f(x)+g(x), \qquad (k\mathbf{f})(x):=kf(x),$$

where $k$ is a scalar. Then the set of functions with these two operations is a vector space, denoted by $F(-\infty,\infty)$. Note that the zero vector $\mathbf{0}\in F(-\infty,\infty)$ is the zero function defined by $\mathbf{0}:=0(x)=0$ for all $x\in(-\infty,\infty)$.

### 4.1.2 Some properties

**Theorem 4.1** *Let $V$ be a vector space, $\mathbf{u}$ be a vector in $V$, and $k$ be any scalar. Then*

(a) $0\mathbf{u} = \mathbf{0}$.

(b) $k\mathbf{0} = \mathbf{0}$.

(c) $(-1)\mathbf{u} = -\mathbf{u}$.

(d) *If* $k\mathbf{u} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{u} = \mathbf{0}$.

*Proof* We only prove (a) and leave the proofs of the remaining parts as an exercise.

For (a), we can write by Axiom (vi),

$$0\mathbf{u} + 0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u}.$$

By Axiom (iv), we know that the vector $0\mathbf{u}$ has a negative $-0\mathbf{u}$. Adding this negative to both sides above obtains

$$[0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u}) = 0\mathbf{u} + (-0\mathbf{u}).$$

Then we have by Axiom (ii),

$$0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})] = 0\mathbf{u} + (-0\mathbf{u}).$$

It follows from Axiom (iv) that

$$0\mathbf{u} + \mathbf{0} = \mathbf{0}.$$

Thus, by Axiom (iii), it holds

$$0\mathbf{u} = \mathbf{0}. \qquad \square$$

## 4.2 Subspaces

We consider a special kind of subset of a vector space $V$ that is itself a vector space under the operations of addition and scalar multiplication defined on $V$.

### 4.2.1 Definition of subspace

**Definition** *A subset $W$ of a vector space $V$ is called a* ***subspace*** *of $V$ if $W$ is itself a vector space with respect to the addition and scalar multiplication defined on $V$.*

The following theorem states that $W \subseteq V$ is a subspace if and only if $W$ is closed under the operations of addition and scalar multiplication.

**Theorem 4.2** *Let $W$ be a nonempty set of vectors in a vector space $V$. Then $W$ is a subspace of $V$ if and only if the following conditions hold.*

(a) *If $\mathbf{u}$ and $\mathbf{v}$ are in $W$, then $\mathbf{u}+\mathbf{v}$ is in $W$.*

(b) *If $k$ is any scalar and $\mathbf{u}$ is in $W$, then $k\mathbf{u}$ is in $W$.*

*Proof* Let $W$ be a subspace of $V$. It follows from the definition of vector space that conditions (a) and (b) hold.

Conversely, assume conditions (a) and (b) hold. Axioms (i), (ii), (v), (vi), (vii), (viii) are automatically satisfied by the vectors in $W$ since they are satisfied by all vectors in $V$. Therefore, to complete the proof, we need only verify that Axioms (iii) and (iv) are satisfied by vectors in $W$. Let $\mathbf{u} \in W$. By condition (b), we know that $k\mathbf{u}$ is in $W$ for every scalar $k$. Setting $k = 0$, it follows from Theorem 4.1 (a) that $0\mathbf{u} = \mathbf{0}$ is in $W$. Setting $k = -1$, it follows from Theorem 4.1 (c) that $(-1)\mathbf{u} = -\mathbf{u}$ is in $W$. Thus, Axioms (iii) and (iv) hold. □

**Examples**

(a) The following subsets are subspaces of $\mathbb{R}^2$ or $\mathbb{R}^3$.

In $\mathbb{R}^2$: $\{\mathbf{0}\}$; lines through the origin; $\mathbb{R}^2$.

In $\mathbb{R}^3$: $\{\mathbf{0}\}$; lines through the origin; planes through the origin; $\mathbb{R}^3$.

For instance, one can easily check that the sum of two vectors on a line $l$ through the origin of $\mathbb{R}^2$ or $\mathbb{R}^3$ also lies on $l$, and a scalar multiple of a vector on this line $l$ is on $l$ as well. Thus, by Theorem 4.2, the line $l$ through the origin is a subspace.

(b) Let $W$ be the set of all $n \times n$ symmetric matrices. For any two matrices $A, B \in W$ and any scalar $k$, it follows from Theorem 1.21 (b) and (c) that $A + B$ and $kA$ are both symmetric matrices. Thus, by Theorem 4.2, $W$ is a subspace of $\mathbb{R}^{n\times n}$.

(c) Let $F(-\infty, \infty)$ be the vector space of functions discussed in Subsection 4.1.1, $n$ be a positive integer, and $P_n$ be the subset of $F(-\infty, \infty)$ consisting of all polynomials in the following form

$$\mathbf{p} := p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n$ are real numbers. Then $P_n$ consists of all real polynomials of degree $n$ or less. We now show that $P_n$ is a subspace of $F(-\infty, \infty)$. For any two polynomials $\mathbf{p}, \mathbf{q} \in P_n$ and any scalar $k$, one can directly check that $\mathbf{p}+\mathbf{q}$ and $k\mathbf{p}$ are both in $P_n$. Thus, by Theorem 4.2, $P_n$ is a subspace of $F(-\infty, \infty)$. Moreover, we have the following chain of subspaces:

$$F(-\infty, \infty) \supset \cdots \supset P_n \supset P_{n-1} \supset \cdots \supset P_1 \supset P_0.$$

(d) We consider the solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where $A$ is an $m \times n$ matrix. Let $\mathbf{u}$, $\mathbf{v}$ be any two solutions and $k$ be any scalar. Then, $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. We therefore have

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}, \qquad A(k\mathbf{u}) = kA\mathbf{u} = k\mathbf{0} = \mathbf{0}.$$

So $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ lie in the solution set. It follows from Theorem 4.2 that the solution set is a subspace of $\mathbb{R}^n$. Thus, the solution set will be called the ***solution space*** of $A\mathbf{x} = \mathbf{0}$.

(e) Let $W$ and $U$ be two subspaces of a vector space $V$. Then $W \cap U$ is a subspace of $V$ and $W + U$ is also a subspace of $V$, where

$$W \cap U := \{\mathbf{v} \mid \mathbf{v} \in W \text{ and } \mathbf{v} \in U\}, \qquad W + U := \{\mathbf{w} + \mathbf{u} \mid \mathbf{w} \in W,\ \mathbf{u} \in U\}. \tag{4.1}$$

See Exercise 4.2.

### 4.2.2 Linear combinations

Addition and scalar multiplication are frequently used in combination to construct new vectors.

**Definition** *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be vectors in a vector space $V$. A vector $\mathbf{w}$ in $V$ is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if it can be written in the form*

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r,$$

*where $k_1, k_2, \ldots, k_r$ are scalars.*

**Theorem 4.3** *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be vectors in a vector space $V$. Then*

(a) *The set $W$ of all linear combinations of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *is a subspace of* $V$.

(b) $W$ *is the smallest subspace of* $V$ *that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *in the sense that every other subspace of* $V$ *that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *must contain* $W$.

*Proof* For (a), let $\mathbf{u}$ and $\mathbf{w}$ be vectors in $W$. Since $W$ is the set of all linear combinations of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the vectors $\mathbf{u}$ and $\mathbf{w}$ can be written as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r, \qquad \mathbf{w} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_r\mathbf{v}_r.$$

For any scalar $k$, we have

$$\begin{aligned}\mathbf{u} + k\mathbf{w} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r + k(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_r\mathbf{v}_r)\\ &= (c_1 + kd_1)\mathbf{v}_1 + (c_2 + kd_2)\mathbf{v}_2 + \cdots + (c_r + kd_r)\mathbf{v}_r,\end{aligned}$$

i.e., $\mathbf{u} + k\mathbf{w}$ is in $W$. It follows from Theorem 4.2 that $W$ is a subspace of $V$.

For (b), let $U$ be another subspace of $V$ that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. It follows from Theorem 4.2 again that $U$ must contain all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. For any $\mathbf{w} \in W$, it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r.$$

Then $\mathbf{w}$ should be in $U$. Thus, $W \subseteq U$. □

**Definition** *Let* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *be a set of vectors in a vector space* $V$ *and* $W$ *be the subspace of* $V$ *consisting of all linear combinations of the vectors in* $S$. *Then* $W$ *is called the subspace spanned by* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *and denoted by*

$$W := \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} \quad \textit{or} \quad W = \mathrm{span}(S).$$

**Example 1** The polynomials $1, x, x^2, \ldots, x^n$ span the vector space $P_n$ since each polynomial $\mathbf{p} \in P_n$ can be written as

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n,$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. Thus,

$$P_n = \mathrm{span}\{1, x, x^2, \ldots, x^n\}.$$

**Example 2** Determine whether $\mathbf{v}_1 = [1, 3, 2]$, $\mathbf{v}_2 = [1, 0, 2]$, and $\mathbf{v}_3 = [2, 3, 4]$ span the vector space $\mathbb{R}^3$.

*Solution* We must determine whether an arbitrary vector $\mathbf{b} = [b_1, b_2, b_3] \in \mathbb{R}^3$ can be written in the form of

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3.$$

Expressing this equation in terms of components yields

$$[b_1, b_2, b_3] = k_1[1, 3, 2] + k_2[1, 0, 2] + k_3[2, 3, 4],$$

i.e.,

$$[b_1, b_2, b_3] = [k_1 + k_2 + 2k_3, 3k_1 + 3k_3, 2k_1 + 2k_2 + 4k_3]$$

or

$$\begin{cases} k_1 + \phantom{2}k_2 + 2k_3 = b_1 \\ 3k_1 \phantom{+ 2k_2} + 3k_3 = b_2 \\ 2k_1 + 2k_2 + 4k_3 = b_3. \end{cases}$$

Thus, the problem reduces to determining whether the system is consistent for all values of $b_1$, $b_2$, and $b_3$. By Theorem 3.9 (e) and (g), a system with a square coefficient matrix is consistent for every vector on the right-hand side if and only if the determinant of the coefficient matrix of the system is not equal to zero. However,

$$\det \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ 2 & 2 & 4 \end{bmatrix} = 0.$$

Thus, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ can not span $\mathbb{R}^3$. □

**Theorem 4.4** *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be two sets of vectors in a vector space $V$. Then*

$$\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \mathrm{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

*if and only if each vector in $S$ is a linear combination of those in $S'$, and conversely each vector in $S'$ is a linear combination of those in $S$.*

The proof of Theorem 4.4 is left as an exercise.

## 4.3 Linear Independence

We knew that a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ spans a given vector space $V$ if every vector $\mathbf{u} \in V$ can be written as a linear combination of the vectors in $S$, i.e.,

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r,$$

where $k_j$ $(1 \leqslant j \leqslant r)$ are scalars. In general, there may be many different ways to express a vector in $V$ as a linear combination of the vectors in $S$. In this section, we study conditions of $S$ under which each vector in $V$ can be expressed as a linear combination of the vectors in $S$ in a unique way.

### 4.3.1 Linear independence and linear dependence

**Definition** *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a given nonempty set of vectors. Then the vector equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

*has at least one solution obviously*

$$k_1 = k_2 = \cdots = k_r = 0.$$

*If this is the only solution, then $S$ is called a* ***linearly independent*** *set (or $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are said to be linearly independent). If there exist nonzero solutions, then $S$ is called a* ***linearly dependent*** *set (or $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are said to be linearly dependent).*

**Examples**

(a) Determine whether $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent or not, where

$$\mathbf{v}_1 = [2, -2, 1], \qquad \mathbf{v}_2 = [5, 3, -2], \qquad \mathbf{v}_3 = [7, 1, -1].$$

*Solution* Since

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0},$$

$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. □

(b) Show that the polynomials $1, x, x^2, \ldots, x^n$ are linearly independent in $P_n$.

*Proof* We consider the following equation

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0(x), \quad x \in (-\infty, \infty), \tag{4.2}$$

where $0(x)$ is the zero function. Recall from the Fundamental Theorem of Algebra [11] that any nonzero polynomial of degree $n$ in one variable has at most $n$ distinct complex roots. However, the polynomial $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ in (4.2) has infinitely many roots. Therefore, its coefficients should be all zero, i.e.,

$$c_0 = c_1 = c_2 = \cdots = c_n = 0.$$

Thus, $1, x, x^2, \ldots, x^n$ are linearly independent in $P_n$. □

### 4.3.2 Some theorems

The following theorems are concerned with some basic properties of linear independence.

**Theorem 4.5** *Let $S$ be a set with two or more vectors. Then*

(a) *$S$ is linearly dependent if and only if at least one of the vectors in $S$ is expressible as a linear combination of the other vectors in $S$.*

(b) *$S$ is linearly independent if and only if no vector in $S$ is expressible as a linear combination of the other vectors in $S$.*

*Proof* Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ with $r \geqslant 2$.

For (a), based on the definition of a linearly dependent set, $S$ is linearly dependent if and only if

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

has nontrivial solutions, i.e., there exists at least a nonzero $k_t$ for some $t$ such that

$$\mathbf{v}_t = -\frac{k_1}{k_t}\mathbf{v}_1 - \cdots - \frac{k_{t-1}}{k_t}\mathbf{v}_{t-1} - \frac{k_{t+1}}{k_t}\mathbf{v}_{t+1} - \cdots - \frac{k_r}{k_t}\mathbf{v}_r.$$

Thus, (a) holds. Part (b) follows immediately from (a). □

**Theorem 4.6** *A set of a finite number of vectors that contains the zero vector is linearly dependent.*

*Proof* Let $S = \{\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and consider the following equation

$$k_0\mathbf{0} + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}. \tag{4.3}$$

Let $k_0 = 2$ and $k_1 = k_2 = \cdots = k_r = 0$. Then

$$2 \cdot \mathbf{0} + 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + \cdots + 0 \cdot \mathbf{v}_r = \mathbf{0},$$

i.e., equation (4.3) has a nonzero solution. Thus, $S$ is linearly dependent. □

**Theorem 4.7** *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then $S$ is linearly dependent.*

*Proof* First, we assume that the vectors in $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ are column vectors. Consider the following equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}.$$

We can rewrite it in the following matrix form

$$\left[\begin{array}{c:c:c:c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{array}\right]\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

which is a homogeneous system of $n$ linear equations in $r$ unknowns. Since $r > n$, it follows from Theorem 1.1 that the system above has infinitely many solutions (nonzero solutions). Thus, $S$ is linearly dependent. $\square$

## 4.4 Basis and Dimension

How can the vectors in a vector space be generated? There exist some linearly independent subsets which can span the entire vector space. For instance, $\mathbb{R}^2 = \text{span}\{[1,0],[0,1]\}$ and $P_2 = \text{span}\{1, x, x^2\}$. Concepts of basis and dimension are proposed from such kinds of subsets.

### 4.4.1 Basis for vector space

**Definition** *Let $V$ be any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in $V$. Then $S$ is called a **basis** for $V$ if the following two conditions hold.*

(i) *$S$ is linearly independent.*

(ii) *$V = \text{span}(S)$.*

**Theorem 4.8** *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $V$. Then every vector $\mathbf{u}$ in $V$ can be expressed in the form*

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

*in exactly one way.*

*Proof* We only need to show that there is only one way to express a vector $\mathbf{u} \in V$ as a linear combination of the vectors in $S$. Suppose that $\mathbf{u}$ can be written as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

and also as

$$\mathbf{u} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n.$$

By subtracting the second equation from the first one, we obtain

$$\begin{aligned}\mathbf{0} = \mathbf{u} - \mathbf{u} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) - (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n) \\ &= (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_n - d_n)\mathbf{v}_n. \qquad (4.4)\end{aligned}$$

Since $S$ is linearly independent, (4.4) implies

$$c_i - d_i = 0, \quad \text{i.e.,} \quad c_i = d_i, \quad 1 \leqslant i \leqslant n.$$

Thus, the two expressions for $\mathbf{u}$ are the same. □

### 4.4.2 Coordinates

**Definition** *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $V$ and*

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

*be the expression for a vector $\mathbf{v}$ in terms of the basis $S$. Then the scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates** of $\mathbf{v}$ relative to the basis $S$. The vector $[c_1, c_2, \ldots, c_n]$ in $\mathbb{R}^n$ is called the **coordinate vector** of $\mathbf{v}$ relative to $S$ and is denoted by*

$$[\mathbf{v}]_S = [c_1, c_2, \ldots, c_n].$$

**Remark** Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a set of vectors in $\mathbb{R}^n$, where

$$\mathbf{e}_1 = [1, 0, 0, \ldots, 0], \quad \mathbf{e}_2 = [0, 1, 0, \ldots, 0], \quad \ldots, \quad \mathbf{e}_n = [0, 0, 0, \ldots, 1].$$

One can show that $S$ is a basis which is called the standard basis for $\mathbb{R}^n$. For every vector $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$, it can be expressible as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Then the coordinate vector of $\mathbf{x}$ relative to the standard basis $S$ is

$$[\mathbf{x}]_S = [x_1, x_2, \ldots, x_n].$$

Thus, $\mathbf{x} = [\mathbf{x}]_S$, i.e., a vector $\mathbf{x}$ and its coordinate vector relative to the standard basis are the same.

**Example** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = [1, 1, 2]$, $\mathbf{v}_2 = [0, 2, 1]$, and $\mathbf{v}_3 = [2, 1, 3]$.

(a) Show that $S$ is a basis for $\mathbb{R}^3$.

(b) Find the coordinate vector of $\mathbf{v} = [-3, 5, -1]$ relative to $S$.

(c) Find the vector $\mathbf{v}$ in $\mathbb{R}^3$ whose coordinate vector relative to $S$ is $[\mathbf{v}]_S = [-1, 2, 1]$.

*Solution* For (a), to show that $S$ spans $\mathbb{R}^3$, we must show that any vector $\mathbf{b} = [b_1, b_2, b_3] \in \mathbb{R}^3$ can be expressed as a linear combination of the vectors in $S$, i.e.,

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

Expressing this equation in terms of components gives

$$[b_1, b_2, b_3] = c_1[1, 1, 2] + c_2[0, 2, 1] + c_3[2, 1, 3],$$

or in matrix form

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{4.5}$$

Since

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = -1 \neq 0,$$

it follows from Theorem 3.9 that (4.5) has a unique solution for every $\mathbf{b}$. Thus, $S$ spans $\mathbb{R}^3$.

To prove that $S$ is linearly independent, we must show that the following equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{4.6}$$

has only the zero solution $c_1 = c_2 = c_3 = 0$. In matrix form, (4.6) can be written as a homogeneous system

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.7}$$

which is a special case of (4.5) when $\mathbf{b} = \mathbf{0}$. Hence (4.7) has only the trivial (zero) solution by Theorem 3.9 again. Thus, $S$ is a basis for $\mathbb{R}^3$.

For (b), we consider the following equation

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3,$$

i.e.,

$$[-3, 5, -1] = c_1[1, 1, 2] + c_2[0, 2, 1] + c_3[2, 1, 3].$$

Equating corresponding components gives

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}.$$

Solving this system, we obtain $[c_1, c_2, c_3] = [1, 3, -2]$. Therefore,

$$[\mathbf{v}]_S = [1, 3, -2].$$

For (c), we obtain by using the definition of the coordinate vector $[\mathbf{v}]_S$,

$$\mathbf{v} = (-1)\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = [1, 4, 3]. \qquad \square$$

### 4.4.3 Dimension

**Definition** *A nonzero vector space $V$ is called* ***finite-dimensional*** *if it contains a set of a finite number of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ that forms a basis. If no such set exists, $V$ is called* ***infinite-dimensional****. In addition, the zero vector space is said to be finite-dimensional.*

**Theorem 4.9** *Let $V$ be a finite-dimensional vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis. Then*

(a) *Every set with more than $n$ vectors is linearly dependent.*

(b) *No set with fewer than $n$ vectors spans $V$.*

*Proof* For (a), let $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be any set of $m$ vectors in $V$, where $m > n$. Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis, each $\mathbf{w}_j$ can be expressed as a linear combination of the vectors in $S$:

$$\begin{cases} \mathbf{w}_1 = a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{n1}\mathbf{v}_n \\ \mathbf{w}_2 = a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{n2}\mathbf{v}_n \\ \quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\ \mathbf{w}_m = a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \cdots + a_{nm}\mathbf{v}_n. \end{cases} \tag{4.8}$$

To show that $S'$ is linearly dependent, we must find scalars $k_1, k_2, \ldots, k_m$, not all zero, such that

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_m\mathbf{w}_m = \mathbf{0}. \tag{4.9}$$

Using (4.8), equation (4.9) can be rewritten as

$$(k_1a_{11} + k_2a_{12} + \cdots + k_ma_{1m})\mathbf{v}_1 + (k_1a_{21} + k_2a_{22} + \cdots + k_ma_{2m})\mathbf{v}_2$$
$$+ \cdots + (k_1a_{n1} + k_2a_{n2} + \cdots + k_ma_{nm})\mathbf{v}_n = \mathbf{0}.$$

Since $S$ is linearly independent, we have

$$\begin{cases} a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_m = 0 \\ a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_m = 0 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nm}k_m = 0. \end{cases}$$

Since there are more unknowns than equations ($m > n$), Theorem 1.1 guarantees the existence of nontrivial (nonzero) solutions, i.e., equation (4.9) has nonzero solutions. Thus, $S'$ is linearly dependent.

The proof of (b) is left as an exercise. □

We immediately have the following corollary.

**Corollary 1** *All bases for a finite-dimensional vector space have the same number of vectors.*

**Definition** *The **dimension** of a finite-dimensional vector space $V$, denoted by* $\dim(V)$*, is defined to be the number of vectors in a basis for $V$. In addition, the dimension of zero vector space is defined to be zero.*

**Corollary 2** *If* $\dim(V) = n$*, then*

(a) *Every set with more than $n$ vectors is linearly dependent.*

(b) *No set with fewer than $n$ vectors spans $V$.*

**Example 1** Dimensions of some vector spaces:

$$\dim(\mathbb{R}^n) = n, \qquad \dim(P_n) = n + 1, \qquad \dim(\mathbb{R}^{m\times n}) = mn.$$

**Example 2** Determine a basis for and the dimension of the solution space of the homogeneous system

$$\begin{cases} x_1 + 3x_2 + x_3 - x_4 = 0 \\ x_1 - x_2 + 2x_3 - 3x_4 - x_5 = 0 \\ x_3 - 2x_4 - x_5 = 0 \\ -2x_1 + x_2 - x_3 - x_5 = 0. \end{cases}$$

*Solution* The solution of the given homogeneous system is

$$\begin{cases} x_1 = -s - t \\ x_2 = 0 \\ x_3 = s + 2t \\ x_4 = t \\ x_5 = s \end{cases} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

which shows that $\{[-1,0,1,0,1]^T,\ [-1,0,2,1,0]^T\}$ is a basis for the solution space. Thus, the dimension of the solution space is 2. □

### 4.4.4 Some fundamental theorems

The following theorems reveal the subtle relationships among the concepts of spanning sets, linear independence, basis, and dimension.

**Theorem 4.10** (*Plus/Minus Theorem*) *Let $S$ be a nonempty set of a finite number of vectors in a vector space $V$.*

(a) *If $S$ is linearly independent, and if $\mathbf{v}$ is in $V$ but is outside of* $\operatorname{span}(S)$*, then the set $S \cup \{\mathbf{v}\}$ is still linearly independent.*

(b) *Let $\mathbf{v}$ be in $S$ and it can be expressed as a linear combination of other vectors in $S$. If $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from $S$, then*

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\}).$$

*Proof* For (a), let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$. Then

$$S \cup \{\mathbf{v}\} = \{\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}.$$

Consider the following equation

$$k_0\mathbf{v} + k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r = \mathbf{0}. \tag{4.10}$$

Then we must have $k_0 = 0$. Otherwise $\mathbf{v}$ can be expressed as a linear combination of the vectors in $S$, i.e., $\mathbf{v} \in \operatorname{span}(S)$, which contradicts the fact that $\mathbf{v} \notin \operatorname{span}(S)$. Hence (4.10) simplifies to

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r = \mathbf{0}.$$

Since $S$ is linearly independent, we deduce

$$k_1 = k_2 = \cdots = k_r = 0.$$

Thus, (4.10) only has the zero solution, i.e., $S \cup \{\mathbf{v}\}$ is still linearly independent.

For (b), let $S = \{\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$. It is obvious that

$$\operatorname{span}(S - \{\mathbf{v}\}) = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \subseteq \operatorname{span}\{\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} = \operatorname{span}(S).$$

For any vector $\mathbf{u} \in \operatorname{span}(S)$, it can be expressed as

$$\mathbf{u} = c_0\mathbf{v} + c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r. \tag{4.11}$$

Since $\mathbf{v} \in S$ and $\mathbf{v}$ can be expressed as a linear combination of other vectors in $S$, we have

$$\mathbf{v} = \sum_{j=1}^{r} d_j\mathbf{w}_j. \tag{4.12}$$

We can replace $\mathbf{v}$ in (4.11) with (4.12) and then

$$\mathbf{u} = c_0\Big(\sum_{j=1}^{r} d_j\mathbf{w}_j\Big) + c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r = \sum_{j=1}^{r}(c_0d_j + c_j)\mathbf{w}_j.$$

Therefore,

$$\mathbf{u} \in \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} = \operatorname{span}(S - \{\mathbf{v}\}).$$

Thus,

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\}).$$

□

**Theorem 4.11** *Let $V$ be a vector space with* $\dim(V) = n$ *and $S$ be a set in $V$ with exactly $n$ vectors. Then $S$ is a basis for $V$ if either $V =$* $\operatorname{span}(S)$ *or $S$ is linearly independent.*

*Proof* We first assume that $V = \operatorname{span}(S)$ and $S$ has exactly $n$ vectors. To show that $S$ is a basis, we must prove that $S$ is linearly independent. By contradiction, we assume that $S$ is linearly dependent. It follows from Theorem 4.5 (a) that at least one of the vectors in $S$, say $\mathbf{v}$, can be expressed as a linear combination of the other vectors in $S$. Then we have by Theorem 4.10 (b),

$$\operatorname{span}(S - \{\mathbf{v}\}) = \operatorname{span}(S) = V.$$

Since $S - \{\mathbf{v}\}$ contains $n-1$ vectors only, it follows from Theorem 4.9 (b) and the given condition $\dim(V) = n$ that $V$ can not be spanned by $S - \{\mathbf{v}\}$. A contradiction! Thus, $S$ should be linearly independent.

We next assume that $S$ is linearly independent. To show that $S$ is a basis, we must prove that $V = \operatorname{span}(S)$. By contradiction, we assume that there is a vector $\mathbf{w} \in V$ but $\mathbf{w} \notin \operatorname{span}(S)$. By Theorem 4.10 (a), $S \cup \{\mathbf{w}\}$ is still linearly independent. However, $S \cup \{\mathbf{w}\}$ has $n+1$ vectors. It follows from Theorem 4.9 (a) and the given condition $\dim(V) = n$ that $S \cup \{\mathbf{w}\}$ should be linearly dependent. A contradiction! Thus, $V = \operatorname{span}(S)$. □

**Theorem 4.12** *Let $S$ be a set of a finite number of vectors in a finite-dimensional vector space $V$.*

(a) *If $S$ spans $V$ but is not a basis for $V$, then $S$ can be reduced to a basis for $V$ by removing appropriate vectors from $S$.*

(b) *If $S$ is a linearly independent set that is not already a basis for $V$, then $S$ can be enlarged to a basis for $V$ by inserting appropriate vectors into $S$.*

*Proof* Note that $V$ is a finite-dimensional vector space. Therefore, the following removing process and inserting process can be completed in finite steps.

For (a), since $V = \mathrm{span}(S)$, by removing appropriate vectors from $S$, it follows from Theorem 4.10 (b) that $S$ can be reduced to a subset of $S$ which forms a basis for $V$.

For (b), since $S$ is linearly independent, by inserting appropriate vectors into $S$, it follows from Theorem 4.10 (a) that $S$ can be enlarged to a basis for $V$. □

**Theorem 4.13** *Let $W$ be a subspace of a finite-dimensional vector space $V$. Then we have $\dim(W) \leqslant \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $W = V$.*

The proof of Theorem 4.13 is left as an exercise.

### 4.4.5 Dimension theorem for subspaces

**Theorem 4.14** (*Dimension Theorem for Subspaces*) *Let $V_1$ and $V_2$ be two subspaces of a vector space $V$. Then*

$$\dim(V_1) + \dim(V_2) = \dim(V_1 + V_2) + \dim(V_1 \cap V_2).$$

*Proof* We assume that

$$\dim(V_1) = n_1, \qquad \dim(V_2) = n_2, \qquad \dim(V_1 \cap V_2) = m.$$

We can choose a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ for $V_1 \cap V_2$. By Theorem 4.12 (b), there exist $n_1 - m$ vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_1-m}$ such that the set

$$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_1-m}\}$$

is a basis for $V_1$. Similarly, by Theorem 4.12 (b) again, there exist $n_2 - m$ vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n_2-m}$ such that the set

$$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n_2-m}\}$$

is a basis for $V_2$. Since

$$V_1 = \mathrm{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_1-m}\}$$

and

$$V_2 = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n_2-m}\},$$

it follows from (4.1) that

$$V_1 + V_2 = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_1-m}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n_2-m}\}.$$

Next, we want to show that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_1-m}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n_2-m}\}$ is linearly independent. Consider the following equation

$$k_1\mathbf{x}_1 + \cdots + k_m\mathbf{x}_m + p_1\mathbf{y}_1 + \cdots + p_{n_1-m}\mathbf{y}_{n_1-m} + q_1\mathbf{z}_1 + \cdots + q_{n_2-m}\mathbf{z}_{n_2-m} = \mathbf{0}. \tag{4.13}$$

Let

$$\mathbf{v} = k_1\mathbf{x}_1 + \cdots + k_m\mathbf{x}_m + p_1\mathbf{y}_1 + \cdots + p_{n_1-m}\mathbf{y}_{n_1-m} \in V_1. \tag{4.14}$$

It follows from (4.13) that

$$\mathbf{v} = -q_1\mathbf{z}_1 - \cdots - q_{n_2-m}\mathbf{z}_{n_2-m} \in V_2. \tag{4.15}$$

Therefore, $\mathbf{v} \in V_1 \cap V_2$ and $\mathbf{v}$ can also be expressed as

$$\mathbf{v} = l_1\mathbf{x}_1 + l_2\mathbf{x}_2 + \cdots + l_m\mathbf{x}_m. \tag{4.16}$$

Combining (4.15) and (4.16), we obtain

$$l_1\mathbf{x}_1 + l_2\mathbf{x}_2 + \cdots + l_m\mathbf{x}_m + q_1\mathbf{z}_1 + \cdots + q_{n_2-m}\mathbf{z}_{n_2-m} = \mathbf{0}.$$

By the fact that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n_2-m}\}$ is a linearly independent set, we have

$$l_1 = \cdots = l_m = q_1 = \cdots = q_{n_2-m} = 0.$$

Then $\mathbf{v} = \mathbf{0}$. Furthermore, (4.14) becomes

$$\mathbf{0} = k_1\mathbf{x}_1 + \cdots + k_m\mathbf{x}_m + p_1\mathbf{y}_1 + \cdots + p_{n_1-m}\mathbf{y}_{n_1-m}. \tag{4.17}$$

Since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_1-m}\}$ is a linearly independent set, it follows from equation (4.17) that

$$k_1 = \cdots = k_m = p_1 = \cdots = p_{n_1-m} = 0.$$

Therefore, from (4.13) again, we know that

$$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_1-m}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n_2-m}\}$$

is linearly independent and it forms a basis for $V_1 + V_2$. Hence

$$\dim(V_1 + V_2) = n_1 + n_2 - m = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

The result holds. $\square$

## 4.5 Row Space, Column Space, and Nullspace

In this section, we study three important vector spaces that are associated with matrices.

### 4.5.1 Definition of row space, column space, and nullspace

We first introduce the following definition of row vectors and column vectors.

**Definition** *For an $m \times n$ matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

*the vectors*

$$\mathbf{r}_1 = [a_{11}, a_{12}, \ldots, a_{1n}], \quad \mathbf{r}_2 = [a_{21}, a_{22}, \ldots, a_{2n}], \quad \ldots, \quad \mathbf{r}_m = [a_{m1}, a_{m2}, \ldots, a_{mn}] \tag{4.18}$$

*in $\mathbb{R}^n$ are called the **row vectors** of $A$, and the vectors*

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{4.19}$$

*in $\mathbb{R}^m$ are called the **column vectors** of $A$.*

**Definition** *Let $A$ be an $m \times n$ matrix. Then*

(i) ***row space** of $A := \operatorname{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$,*

(ii) ***column space** of $A := \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$,*

(iii) ***nullspace** of $A :=$ solution space of $A\mathbf{x} = \mathbf{0}$,*

*where $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ are the row vectors given in (4.18), and $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the column vectors given in (4.19).*

**Theorem 4.15** *A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of $A$.*

*Proof* Let

$$A = \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right], \qquad \mathbf{x} = [x_1, x_2, \ldots, x_n]^T,$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the column vectors of $A$. Then by using (1.5), the linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$$

has solutions, which means that $\mathbf{b}$ can be written as a linear combination of $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$, i.e.,

$$\mathbf{b} \in \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}. \qquad \square$$

### 4.5.2 Relation between solutions of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

**Theorem 4.16** *Let $\mathbf{x}_0$ be any single solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is a nonzero vector. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for the nullspace of $A$, then every solution of $A\mathbf{x} = \mathbf{b}$ can be written as the following form*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k. \tag{4.20}$$

*Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in (4.20) is a solution of $A\mathbf{x} = \mathbf{b}$.*

*Proof* Let $\mathbf{y}$ be any other solution of $A\mathbf{x} = \mathbf{b}$, i.e.,

$$A\mathbf{y} = \mathbf{b}.$$

We already knew that $A\mathbf{x}_0 = \mathbf{b}$. Therefore,

$$A(\mathbf{y} - \mathbf{x}_0) = A\mathbf{y} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

which implies that $\mathbf{y} - \mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{y} - \mathbf{x}_0$ is in the nullspace of $A$. Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for the nullspace of $A$, we have

$$\mathbf{y} - \mathbf{x}_0 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

It follows that

$$\mathbf{y} - \mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars. Thus,

$$\mathbf{y} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Conversely, for any choices of scalars $c_1, c_2, \ldots, c_k$, we can construct a vector as

$$\mathbf{z} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Multiplying both sides by $A$ yields

$$A\mathbf{z} = A\mathbf{x}_0 + c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_k A\mathbf{v}_k = \mathbf{b} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{b}.$$

Therefore, $\mathbf{z}$ is a solution of $A\mathbf{x} = \mathbf{b}$. $\square$

**Remark** There is some terminology associated with (4.20). The vector $\mathbf{x}_0$ is called a ***particular solution*** of $A\mathbf{x} = \mathbf{b}$. The expression

$$\mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

is called the ***general solution*** of $A\mathbf{x} = \mathbf{b}$, and the expression

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

is called the ***general solution*** of $A\mathbf{x} = \mathbf{0}$.

**Example** Solve the linear system

$$\left\{\begin{array}{rcrcrcrcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & & & - & 3x_5 & & & = & -4 \\ -x_1 & + & 2x_2 & + & 4x_3 & + & x_4 & + & 4x_5 & - & 2x_6 & = & 2 \\ & & & & 3x_3 & + & 3x_4 & + & 3x_5 & + & x_6 & = & 1 \\ 2x_1 & - & 4x_2 & & & + & 6x_4 & & & + & 3x_6 & = & -5 \end{array}\right. \tag{4.21}$$

and obtain

$$x_1 = 2r - 3s - 4, \quad x_2 = r, \quad x_3 = -s - t, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 1.$$

This result can be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2r - 3s - 4 \\ r \\ -s - t \\ s \\ t \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{x}_0} + \underbrace{r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{y}}, \tag{4.22}$$

which is the general solution of (4.21). The vector $\mathbf{x}_0$ in (4.22) is a particular solution of (4.21) and the linear combination $\mathbf{y}$ in (4.22) is the general solution of

$$\left\{\begin{array}{rcrcrcrcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & & & - & 3x_5 & & & = & 0 \\ -x_1 & + & 2x_2 & + & 4x_3 & + & x_4 & + & 4x_5 & - & 2x_6 & = & 0 \\ & & & & 3x_3 & + & 3x_4 & + & 3x_5 & + & x_6 & = & 0 \\ 2x_1 & - & 4x_2 & & & + & 6x_4 & & & + & 3x_6 & = & 0. \end{array}\right.$$

**Remark** In fact, for a consistent linear system $A\mathbf{x} = \mathbf{b}$, the number of free variables is equal to the number of parameters in the general solution of the system. Thus, the number of those parameters is equal to the number of vectors in a basis for the nullspace of $A$.

### 4.5.3 Bases for three spaces

It is well-known that any elementary row operation does not change the solution set of linear system $A\mathbf{x} = \mathbf{0}$. Thus, we have the following theorem.

**Theorem 4.17** *Elementary row operations do not change the nullspace of a matrix $A$.*

Moreover, the following theorem is concerned with the row space of a matrix $A$.

**Theorem 4.18** *Elementary row operations do not change the row space of a matrix $A$.*

*Proof* Assume that $B$ is a matrix obtained from $A$ by implementing an elementary row operation on $A$. Let

$$\text{row space of } A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}, \qquad \text{row space of } B = \text{span}\{\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n\},$$

where $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}$ is the set of row vectors of $A$ and $\{\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n\}$ is the set of row vectors of $B$. We want to show that

$$\text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\} = \text{span}\{\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n\}.$$

For any $\mathbf{r}'_i \in B$, corresponding to three kinds of elementary row operations performed on $A$, we consider the following three cases:

$$\begin{cases} \mathbf{r}'_i = \mathbf{r}_j, & \text{where } \mathbf{r}_j \text{ is the } j\text{th row vector of } A; \\[2em] \mathbf{r}'_i = c\mathbf{r}_i, & \text{where } c \text{ is a nonzero scalar and } \mathbf{r}_i \text{ is the } i\text{th row vector of } A; \\[2em] \mathbf{r}'_i = \mathbf{r}_i + k\mathbf{r}_j, & \text{where } k \text{ is a scalar and } \mathbf{r}_p \text{ is the } p\text{th row vector of } A \text{ for } p = i, j. \end{cases}$$

We then have for all $i$,

$$\mathbf{r}'_i \in \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}.$$

Therefore,

$$\text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\} \supseteq \text{span}\{\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n\}. \tag{4.23}$$

Since $A$ can be obtained from $B$ by performing inverse elementary row operations on $B$, one can show that similarly

$$\text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\} \subseteq \text{span}\{\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n\}. \tag{4.24}$$

It follows from (4.23) and (4.24) that

$$\operatorname{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\} = \operatorname{span}\{\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n\}. \qquad \square$$

**Remark** However, elementary row operations can change the column space of a matrix $A$. For instance, consider

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

If we add $-2$ times the first row of $A$ to the second row, we obtain

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}.$$

Note that

$$\text{column space of } A = \operatorname{span}\{[1,2]^T\} \neq \operatorname{span}\{[1,0]^T\} = \text{column space of } B.$$

Although elementary row operations can change the column space of a matrix, whatever relationships of linear independence or linear dependence that exist among the column vectors of a matrix prior to a row operation will keep holding for the corresponding columns of the matrix that results from that row operation. More precisely, we have the following result.

**Theorem 4.19** *Let $E$ be any elementary matrix. Then a given set of column vectors of $A$ is linearly independent if and only if the corresponding column vectors of $EA$ are linearly independent.*

*Proof* Let $E$ be any elementary matrix and

$$A = \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right],$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the column vectors of $A$. Then

$$EA = \left[\begin{array}{c:c:c:c} E\mathbf{c}_1 & E\mathbf{c}_2 & \cdots & E\mathbf{c}_n \end{array}\right].$$

Without loss of generality, we consider the following equations:

$$\sum_{i=1}^{r} k_i \mathbf{c}_i = \mathbf{0} \qquad \text{and} \qquad \sum_{i=1}^{r} k_i E\mathbf{c}_i = \mathbf{0},$$

where $r \leqslant n$. In fact,

$$\sum_{i=1}^{r} k_i E\mathbf{c}_i = \mathbf{0} \Longleftrightarrow E\Big(\sum_{i=1}^{r} k_i \mathbf{c}_i\Big) = \mathbf{0} \Longleftrightarrow \sum_{i=1}^{r} k_i \mathbf{c}_i = \mathbf{0}. \tag{4.25}$$

Thus, the given set of column vectors of $A$ is linearly independent if and only if the set of corresponding column vectors of $EA$ is linearly independent. $\square$

**Remark** In fact, (4.25) implies an even deeper result that whatever linear combinations that exist among the column vector of $A$ keep holding for the corresponding column vectors of $EA$.

**Theorem 4.20** *Let $R$ be a matrix in row-echelon form. Then the row vectors with the leading 1's form a basis for the row space of $R$, and the column vectors with the leading 1's of the row vectors form a basis for the column space of $R$.*

The result of Theorem 4.20 is virtually self-evident and the proof of the theorem is left as an exercise.

**Remark** Theorem 4.20 makes it possible to find bases for the row and column spaces of a matrix in row-echelon form by inspection.

### 4.5.4 A procedure for finding a basis for span$(S)$

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset \mathbb{R}^n$. Then by the following procedure, one can find a basis for span$(S)$ and simultaneously express the vectors in $S$ as a linear combination of the basis vectors.

(1) Form the matrix $A$ having $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ as its column vectors.

(2) Reduce $A$ to its reduced row-echelon form $R$, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ be the column vectors of $R$.

(3) Identify the columns that contain the leading 1's in $R$. The corresponding column vectors of $A$ are the basis vectors for span$(S)$.

(4) Express each column vector $\mathbf{w}_j$ of $R$ that does not contain a leading 1 as a linear combination of preceding column vectors that do contain leading 1's.

(5) In each linear combination obtained in (4), replace $\mathbf{w}_j$ with $\mathbf{v}_j$ for $j = 1, 2, \ldots, k$.

**Example** Let $\mathbf{v}_1 = [2, -1, 1, 0]$, $\mathbf{v}_2 = [-4, 2, -2, 0]$, $\mathbf{v}_3 = [1, 0, -2, 1]$, $\mathbf{v}_4 = [0, 7, -2, 3]$, and $\mathbf{v}_5 = [3, 5, 2, 2]$.

(a) Find a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ that forms a basis for span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

(b) Express each vector not in the basis as a linear combination of the basis vectors.

*Solution* For (a), we begin by constructing a matrix $A$ that has $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$ as its column vectors:

$$A = \begin{array}{c} \begin{bmatrix} 2 & -4 & 1 & 0 & 3 \\ -1 & 2 & 0 & 7 & 5 \\ 1 & -2 & -2 & -2 & 2 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{array} \end{array}.$$

We reduce the matrix $A$ to its reduced row-echelon form $R$ and denote the column vectors of the resulting matrix by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, and $\mathbf{w}_5$. We yield

$$R = \begin{array}{c} \begin{bmatrix} 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 & \mathbf{w}_5 \end{array} \end{array}.$$

The leading 1's occur in columns 1, 3, and 4. It follows from Theorem 4.20 that $\{\mathbf{w}_1, \mathbf{w}_3, \mathbf{w}_4\}$ forms a basis for the column space of $R$. Consequently, $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for the column space of $A$ by Theorem 4.19.

For (b), we have the following linear combinations by inspection of $R$,

$$\mathbf{w}_2 = -2\mathbf{w}_1, \qquad \mathbf{w}_5 = 2\mathbf{w}_1 - \mathbf{w}_3 + \mathbf{w}_4.$$

The corresponding relationships in $A$ are

$$\mathbf{v}_2 = -2\mathbf{v}_1, \qquad \mathbf{v}_5 = 2\mathbf{v}_1 - \mathbf{v}_3 + \mathbf{v}_4. \qquad \square$$

## 4.6 Rank and Nullity

For a given matrix $A$, we have the following four fundamental matrix spaces:

(1) row space of $A$;

(2) column space of $A$;

(3) nullspace of $A$;

(4) nullspace of $A^T$.

In this section, we are concerned with relationships between the dimensions of these four vector spaces. The results obtained here are fundamental and will provide a deeper insight into the relationship between a linear system and its coefficient matrix.

### 4.6.1 Rank and nullity

**Theorem 4.21** *Let $A$ be any matrix. Then the row space and column space of $A$ have the same dimension.*

*Proof* Let $R$ be the reduced row-echelon form of $A$. It follows from Theorems 4.18 and 4.20 that

$$\dim(\text{row space of } A) = \dim(\text{row space of } R) = \text{number of leading 1's} \tag{4.26}$$

and it follows from Theorems 4.19 and 4.20 that

$$\dim(\text{column space of } A) = \dim(\text{column space of } R) = \text{number of leading 1's.} \tag{4.27}$$

Thus, we have by (4.26) and (4.27),

$$\dim(\text{row space of } A) = \dim(\text{column space of } A). \qquad \square$$

**Definition** *The common dimension of the row space and column space of a matrix $A$ is called the **rank** of $A$ and is denoted by* $\text{rank}(A)$. *The dimension of the nullspace of $A$ is called the **nullity** of $A$ and is denoted by* $\text{nullity}(A)$.

**Theorem 4.22** *Let $A$ be any matrix. Then* $\text{rank}(A) = \text{rank}(A^T)$.

*Proof* We have

$$\text{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A^T) = \text{rank}(A^T). \qquad \square$$

**Theorem 4.23** (*Dimension Theorem for Matrices*) *Let $A$ be a matrix with $n$ columns. Then*

$$\text{rank}(A) + \text{nullity}(A) = n.$$

*Proof* Since $A$ has $n$ columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has $n$ variables. These variables fall into two categories: the leading variables and the free variables. Then

$$\begin{bmatrix} \text{number of leading} \\ \text{variables} \end{bmatrix} + \begin{bmatrix} \text{number of free} \\ \text{variables} \end{bmatrix} = n,$$

i.e.,

$$[\text{number of leading 1's}] + [\text{number of free variables}] = n.$$

Thus,

$$\text{rank}(A) + [\text{number of free variables}] = n.$$

We recall that the number of free variables is equal to the nullity of $A$. This is so because the nullity of $A$ is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, which is the same as the number of parameters in the general solution, which is the same as the number of free variables. Thus,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n. \qquad \square$$

**Example** Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 2 & -8 & 1 & 3 & -4 \\ -1 & 4 & 0 & -5 & 3 \\ -2 & 8 & -2 & 4 & 2 \\ 0 & 0 & 1 & -7 & 2 \end{bmatrix}.$$

*Solution* Consider solving the linear system $A\mathbf{x} = \mathbf{0}$. The reduced row-echelon form of $A$ is

$$\begin{bmatrix} 1 & -4 & 0 & 5 & -3 \\ 0 & 0 & 1 & -7 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are two nonzero rows (or equivalently, two leading 1's), the row space and column space are both two-dimensional, i.e., $\operatorname{rank}(A) = 2$. The corresponding system is

$$\begin{cases} x_1 - 4x_2 \qquad + 5x_4 - 3x_5 = 0 \\ \qquad\qquad x_3 - 7x_4 + 2x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 4x_2 - 5x_4 + 3x_5 \\ x_3 = \qquad\quad 7x_4 - 2x_5. \end{cases}$$

It follows that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = 4r - 5s + 3t \\ x_2 = r \\ x_3 = 7s - 2t \\ x_4 = s \\ x_5 = t \end{cases}$$

or equivalently,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}. \tag{4.28}$$

The three vectors on the right-hand side of (4.28) form a basis for the solution space of $A\mathbf{x} = \mathbf{0}$. Therefore, $\operatorname{nullity}(A) = 3$. $\square$

**Remark** Let $A$ be an $m \times n$ matrix and $\operatorname{rank}(A) = r$. Then $\operatorname{rank}(A) \leqslant \min\{m, n\}$ and we have the following table relating the dimensions of the four fundamental spaces of $A$.

| **Fundamental Space** | **Dimension** |
|---|---|
| Row space of $A$ | $r$ |
| Column space of $A$ | $r$ |
| Nullspace of $A$ | $n - r$ |
| Nullspace of $A^T$ | $m - r$ |

### 4.6.2 Rank for matrix operations

**Theorem 4.24** *For any $n \times n$ matrices $A$ and $B$, we have*

(a) $\operatorname{rank}(A + B) \leqslant \operatorname{rank}(A) + \operatorname{rank}(B)$.

(b) $\operatorname{rank}(AB) \leqslant \min\{\operatorname{rank}(A),\ \operatorname{rank}(B)\}$.

(c) $\operatorname{rank}(PAQ) = \operatorname{rank}(A)$, *where $P$ and $Q$ are invertible matrices.*

*Proof* We only prove (b). The proofs of (a) and (c) are left as an exercise. Let

$$A = [a_{ij}] = \left[\begin{array}{c:c:c:c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right], \qquad B = [b_{ij}] = \left[\begin{array}{c:c:c:c} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{array}\right],$$

where $\mathbf{a}_k$ and $\mathbf{b}_k$ $(1 \leqslant k \leqslant n)$ are the column vectors of $A$ and $B$, respectively. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ denote the column vectors of $AB$. Then

$$\begin{aligned} AB &= \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right] = A\left[\begin{array}{c:c:c:c} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{array}\right] \\ &= \left[\begin{array}{c:c:c:c} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array}\right] \\ &= \left[\begin{array}{c:c:c:c} \displaystyle\sum_{j=1}^{n} b_{j1}\mathbf{a}_j & \displaystyle\sum_{j=1}^{n} b_{j2}\mathbf{a}_j & \cdots & \displaystyle\sum_{j=1}^{n} b_{jn}\mathbf{a}_j \end{array}\right]. \end{aligned}$$

Thus,

$$\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} \subseteq \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

We therefore have

$$\operatorname{rank}(AB) = \dim\bigl(\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}\bigr) \leqslant \dim\bigl(\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}\bigr) = \operatorname{rank}(A). \tag{4.29}$$

Moreover, it follows by Theorem 4.22 and (4.29) that

$$\operatorname{rank}(AB) = \operatorname{rank}\big((AB)^T\big) = \operatorname{rank}(B^T A^T) \leqslant \operatorname{rank}(B^T) = \operatorname{rank}(B). \tag{4.30}$$

Combining (4.29) and (4.30), we obtain

$$\operatorname{rank}(AB) \leqslant \min\{\operatorname{rank}(A),\ \operatorname{rank}(B)\}. \qquad \square$$

**Example** For any square matrices $A$ and $B$ of the same size, show that

$$\operatorname{rank}(I - AB) \leqslant \operatorname{rank}(I - A) + \operatorname{rank}(I - B),$$

where $I$ is the identity matrix.

*Proof* We have by Theorem 4.24 (a),

$$\operatorname{rank}(I - AB) = \operatorname{rank}(I - A + A - AB) \leqslant \operatorname{rank}(I - A) + \operatorname{rank}(A - AB).$$

Moreover, it follows from Theorem 4.24 (b) that

$$\operatorname{rank}(A - AB) = \operatorname{rank}\big(A(I - B)\big) \leqslant \min\{\operatorname{rank}(A),\ \operatorname{rank}(I - B)\} \leqslant \operatorname{rank}(I - B).$$

Thus, the proof is completed. $\square$

### 4.6.3 Consistency theorems

The following theorem guarantees a linear system to be consistent.

**Theorem 4.25** *Let $A\mathbf{x} = \mathbf{b}$ be a linear system of $m$ equations in $n$ unknowns. Then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{b}$ *is consistent.*

(b) $\mathbf{b}$ *is in the column space of $A$.*

(c) $\operatorname{rank}(A) = \operatorname{rank}([\ A \mid \mathbf{b}\ ])$, *where $[\ A \mid \mathbf{b}\ ]$ is the augmented matrix.*

*Proof* (a) $\Leftrightarrow$ (b): See Theorem 4.15.

(b) $\Leftrightarrow$ (c): Let

$$A = \left[\ \mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_n\ \right],$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the column vectors of $A$. We have by Theorem 4.10 (b),

$$\begin{aligned}
&\mathbf{b} \in \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} \Longleftrightarrow \mathbf{b} = \sum_{i=1}^{n} k_i \mathbf{c}_i \\
&\Longleftrightarrow \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n, \mathbf{b}\} \\
&\Longleftrightarrow \dim(\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}) = \dim(\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n, \mathbf{b}\}) \\
&\Longleftrightarrow \operatorname{rank}(A) = \operatorname{rank}([\ A \mid \mathbf{b}\ ]),
\end{aligned}$$

where $[\ A \mid \mathbf{b}\ ] = \left[\ \mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_n \mid \mathbf{b}\ \right]$. $\square$

**Corollary** *Let $A\mathbf{x} = \mathbf{b}$ be a linear system of $m$ equations in $n$ unknowns. Then $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if* $\operatorname{rank}(A) = \operatorname{rank}([\ A \mid \mathbf{b}\ ]) = n$.

The proof is left as an exercise (see Exercise 4.26).

The following theorem guarantees a linear system to be consistent for any possible choices of $\mathbf{b}$.

**Theorem 4.26** *Let $A\mathbf{x} = \mathbf{b}$ be a linear system of $m$ equations in $n$ unknowns. Then the following are equivalent.*

(a) *$A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix $\mathbf{b}$.*

(b) *The column vectors of $A$ span $\mathbb{R}^m$.*

(c) $\operatorname{rank}(A) = m$.

*Proof* Let

$$A = \left[\ \mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_n\ \right],$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the column vectors of $A$.

(a) $\Rightarrow$ (b): We want to show that

$$\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \mathbb{R}^m.$$

Since every $\mathbf{c}_j \in \mathbb{R}^m$ for $j = 1, 2, \ldots, n$, we have

$$\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} \subseteq \mathbb{R}^m.$$

On the other hand, it follows from (a) and Theorem 4.15 that for every $\mathbf{b} \in \mathbb{R}^m$,

$$\mathbf{b} \in \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}.$$

Thus,

$$\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \mathbb{R}^m.$$

(b) $\Rightarrow$ (c): We have

$$\operatorname{rank}(A) = \dim(\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}) = \dim(\mathbb{R}^m) = m.$$

(c) $\Rightarrow$ (a): Since

$$\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} \subseteq \mathbb{R}^m,$$

and also

$$\dim(\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}) = \operatorname{rank}(A) = m = \dim(\mathbb{R}^m),$$

we obtain by Theorem 4.13,

$$\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \mathbb{R}^m.$$

It follows from Theorem 4.15 again that $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b} \in \mathbb{R}^m$. $\square$

**Theorem 4.27** *Let $A$ be an $m \times n$ matrix. Then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) *The column vectors of $A$ are linearly independent.*

(c) $A\mathbf{x} = \mathbf{b}$ *has at most one solution (none or one) for every $m \times 1$ matrix* $\mathbf{b}$.

The proof of the theorem is left as an exercise.

### 4.6.4 Summary

**Theorem 4.28** *Let $A$ be an $n \times n$ matrix and $T_A\colon \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by $A$. Then the following are equivalent.*

(a) *$A$ is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row-echelon form of $A$ is $I_n$.*

(d) *$A$ is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) *The range of $T_A$ is $\mathbb{R}^n$.*

(i) *$T_A$ is one-to-one.*

(j) *The column vectors of $A$ are linearly independent.*

(k) *The row vectors of $A$ are linearly independent.*

(l) *The column vectors of $A$ span $\mathbb{R}^n$.*

(m) *The row vectors of $A$ span $\mathbb{R}^n$.*

(n) *The column vectors of $A$ form a basis for $\mathbb{R}^n$.*

(o) *The row vectors of $A$ form a basis for $\mathbb{R}^n$.*

(p) $\operatorname{rank}(A) = n$.

(q) *$A$ has nullity* $0$.

## Exercises

### Elementary exercises

**4.1** Prove Theorem 4.1 (b), (c), and (d).

**4.2** Let $W$ and $U$ be two subspaces of a vector space $V$. Show that $W \cap U$ and $W + U$ are subspaces of $V$.

**4.3** Use Theorem 4.2 to determine which of the following are subspaces.

(a) The set of all polynomials $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.

(b) The set of all polynomials $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ for which $a_0$, $a_1$, $a_2$, and $a_3$ are integers.

(c) The set of all polynomials $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ for which $a_1 \times a_3 = 0$.

(d) The set of all vectors in $\mathbb{R}^3$ with the first coordinate component nonzero.

(e) The set of all diagonal matrices in $\mathbb{R}^{n \times n}$.

(f) The set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \neq \mathbf{0}$.

(g) The set of all differentiable functions $\mathbf{f} = f(x)$ in $F(-\infty, +\infty)$ that satisfy

$$\frac{df(x)}{dx} = 0.$$

**4.4** Express the following vectors as linear combinations of $\mathbf{u} = [2, 1, 4]$, $\mathbf{v} = [1, -1, 3]$, and $\mathbf{w} = [3, 2, 5]$.

(a) $[-9, -7, -15]$. (b) $[1, 0, 3]$.

**4.5** Determine whether the vector $\mathbf{v} = [0, 5, 6, -3]$ is contained in the subspace spanned by $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$, where

$$\mathbf{u}_1 = [-1, 3, 2, 0], \quad \mathbf{u}_2 = [2, 0, 4, -1], \quad \mathbf{u}_3 = [7, 1, 1, 4], \quad \mathbf{u}_4 = [6, 3, 1, 2].$$

**4.6** Let $A_{(1)}$, $A_{(2)}$, $A_{(3)}$, and $B$ be matrices in $\mathbb{R}^{2 \times 2}$, where

$$A_{(1)} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \quad A_{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}, \quad A_{(3)} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 \\ 5 & 3 \end{bmatrix}.$$

Determine whether $B$ is contained in $\operatorname{span}\{A_{(1)}, A_{(2)}, A_{(3)}\}$.

**4.7** Let $W$ be the set of all vectors of each given form, where $a$, $b$, and $c$ represent arbitrary real numbers. Determine whether $W$ is a subspace of $\mathbb{R}^4$. If so, find a set $S$ of vectors that spans $W$.

(a) $[2a+3b, -1, 2a-5b, 5a]$. (b) $[2a-b, 3b-c, 3c-a, 3b]$.

**4.8** In each part, determine whether the given vectors span $\mathbb{R}^3$.

(a) $\mathbf{v}_1 = [2, -1, 3]$, $\mathbf{v}_2 = [4, 1, 2]$, $\mathbf{v}_3 = [8, -1, 8]$.

(b) $\mathbf{v}_1 = [1, 2, 6]$, $\mathbf{v}_2 = [3, 4, 1]$, $\mathbf{v}_3 = [4, 3, 1]$, $\mathbf{v}_4 = [3, 3, 1]$.

**4.9** Prove Theorem 4.4.

**4.10** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be linearly independent vectors in $\mathbb{R}^n$. Show that

(a) $\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is linearly independent.

(b) $\{\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}$ is linearly independent.

(c) $\{\mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{w}, \mathbf{v}-\mathbf{w}\}$ is linearly dependent.

**4.11** Determine whether each set of vectors is a basis for the given vector space.

(a) $\mathbf{u}_1 = [2, 1, 3]$, $\mathbf{u}_2 = [1, 1, 0]$, $\mathbf{u}_3 = [2, 0, 0]$ for $\mathbb{R}^3$.

(b) $\mathbf{u}_1 = [2, -3, 1]$, $\mathbf{u}_2 = [4, 1, 1]$, $\mathbf{u}_3 = [0, -7, 1]$ for $\mathbb{R}^3$.

(c) $\mathbf{p}_1 = 2 + x^2$, $\mathbf{p}_2 = 1 + x$, $\mathbf{p}_3 = 3 + 2x + x^2$ for $P_2$.

(d) $A_{(1)} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, A_{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}, A_{(3)} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ for $\mathbb{R}^{2\times 2}$.

**4.12** Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for $\mathbb{R}^2$.

(a) $\mathbf{u}_1 = [1, 0]$, $\mathbf{u}_2 = [0, 1]$, $\mathbf{w} = [3, -7]$.

(b) $\mathbf{u}_1 = [1, 1]$, $\mathbf{u}_2 = [0, 2]$, $\mathbf{w} = [a, b]$.

**4.13** Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for a vector space $V$, where $n \geqslant 2$.

(a) Show that the set $\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \ldots, \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n\}$ is also a basis for $V$.

(b) Is the set $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2 + \mathbf{u}_3, \ldots, \mathbf{u}_{n-1} + \mathbf{u}_n, \mathbf{u}_n + \mathbf{u}_1\}$ a basis for $V$?

**4.14** Prove Theorem 4.9 (b).

**4.15** Prove Theorem 4.13.

**4.16** Prove Theorem 4.20.

**4.17** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$, where

$$\mathbf{v}_1 = [2, 1, 0, -2], \qquad \mathbf{v}_2 = [4, 2, 0, -4], \qquad \mathbf{v}_3 = [0, -2, 5, 5],$$

$$\mathbf{v}_4 = [8, 0, 10, 2], \qquad \mathbf{v}_5 = [6, 3, 0, -6], \qquad \mathbf{v}_6 = [18, 0, 15, 3].$$

(a) Find a subset of $S$ that forms a basis for the space spanned by these vectors.

(b) Express each vector not in the basis as a linear combination of these basis vectors.

**4.18** Let $A, B, C \in \mathbb{R}^{n\times n}$. Show that

(a) $\mathrm{rank}(AB) = \mathrm{rank}(B)$ if and only if the systems $(AB)\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions.

(b) $\mathrm{rank}(ABC) = \mathrm{rank}(BC)$ if $\mathrm{rank}(AB) = \mathrm{rank}(B)$.

**4.19** Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$. If $AB = \mathbf{0}$, show that

$$\mathrm{rank}(A) + \mathrm{rank}(B) \leqslant n.$$

**4.20** Prove Theorem 4.24 (a) and (c).

**4.21** Let $A \in \mathbb{R}^{n\times n}$. Show that $A^2 = A$ if and only if

$$\mathrm{rank}(A) + \mathrm{rank}(A - I) = n.$$

**4.22** Let $A, B \in \mathbb{R}^{n\times n}$. Show that

$$\max\{\mathrm{rank}(A),\ \mathrm{rank}(B)\} \leqslant \mathrm{rank}\left(\left[\begin{array}{c:c} A & B \end{array}\right]\right) \leqslant \mathrm{rank}(A) + \mathrm{rank}(B).$$

**4.23** Let $A$ and $B$ be any matrices. Show that

$$\mathrm{rank}(A) + \mathrm{rank}(B) = \mathrm{rank}\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}\right).$$

**4.24** Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(A) = 1$.

(a) Show that $A$ can be expressed as the following form

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [b_1, b_2, \ldots, b_n].$$

(b) Show that $A^2 = kA$, where $k$ is a scalar.

**4.25** How does the rank of $A$ vary with $t$?

$$A = \begin{bmatrix} t & 1 & 1 & 1 \\ 1 & t & 1 & 1 \\ 1 & 1 & t & 1 \\ 1 & 1 & 1 & t \end{bmatrix}.$$

**4.26** Let $A\mathbf{x} = \mathbf{b}$ be a linear system of $m$ equations in $n$ unknowns. Show that $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\operatorname{rank}(A) = \operatorname{rank}([\ A \mid \mathbf{b}\ ]) = n$.

**4.27** Prove Theorem 4.27.

## Challenge exercises

**4.28** Let $W_1$ and $W_2$ be subspaces of a vector space $V$.

(a) Show that $W_1 \cap W_2 \subseteq W_1 \cup W_2 \subseteq W_1 + W_2$.

(b) When is $W_1 \cup W_2$ a subspace of $V$?

(c) Show that if $U$ is a subspace of $V$ containing $W_1 \cup W_2$, then $W_1 + W_2 \subseteq U$.

**4.29** Let $W$ be the subspace of all $n \times n$ symmetric matrices in $\mathbb{R}^{n\times n}$. Find a basis for and the dimension of $W$.

**4.30** Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. Determine whether the following statements are true or not. If true, prove it. Otherwise, give a counterexample.

(a) If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent, then $\mathbf{u}_i$ and $\mathbf{u}_j$ are linearly independent for each pair of $i, j$, where $1 \leqslant i, j \leqslant k$ and $i \neq j$.

(b) If $\mathbf{u}_i$ and $\mathbf{u}_j$ are linearly independent for each pair of $i, j$, where $1 \leqslant i, j \leqslant k$ and $i \neq j$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent.

**4.31** Let $P_n$ be the set that consists of all real polynomials of degree $n$ or less.

(a) Show that $\{1, 1+x, 1+x+x^2\}$ is a basis for $P_2$.

(b) Let $W = \{\ p(x) \mid p(-x) = p(x),\ p(x) \in P_n\}$. Show that $W$ is a subspace of $P_n$ and find a basis for $W$.

**4.32** Let $A \in \mathbb{R}^{2\times 2}$. Show that if $A^2 = I$ but $A \neq \pm I$, then

$$\operatorname{rank}(A + I) = \operatorname{rank}(A - I) = 1.$$

**4.33** Let $A$ and $B$ be any given matrices. Show that

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leqslant \operatorname{rank}\left(\begin{bmatrix} A & C \\ \mathbf{0} & B \end{bmatrix}\right),$$

where $C$ is an arbitrary matrix.

**4.34** Let $A, B \in \mathbb{R}^{n\times n}$. Show that if $ABAB = I$, then

$$\operatorname{rank}(I + AB) + \operatorname{rank}(I - AB) = n.$$

**4.35** Let $A \in \mathbb{R}^{n\times n}$. Show that

(a) $\operatorname{rank}\big(\operatorname{adj}(A)\big) = n$ if $\operatorname{rank}(A) = n$.

(b) $\operatorname{rank}\big(\operatorname{adj}(A)\big) = 1$ if $\operatorname{rank}(A) = n - 1$.

(c) $\operatorname{rank}\big(\operatorname{adj}(A)\big) = 0$ if $\operatorname{rank}(A) < n - 1$.

**4.36** Let $A, B \in \mathbb{R}^{n\times n}$. Show that

$$\operatorname{rank}(AB) \geqslant \operatorname{rank}(A) + \operatorname{rank}(B) - n.$$

**4.37** Let $A_{(1)}, A_{(2)}, \ldots, A_{(k)} \in \mathbb{R}^{n\times n}$. Show that if $A_{(1)}A_{(2)}\cdots A_{(k)} = \mathbf{0}$, then

$$\operatorname{rank}\big(A_{(1)}\big) + \operatorname{rank}\big(A_{(2)}\big) + \cdots + \operatorname{rank}\big(A_{(k)}\big) \leqslant (k-1)n.$$

# Chapter 5

# Inner Product Spaces

"*Inner product gives a structure to vector space which allows mathematician to build geometry out of bare manifold.*"

— Shing-Tung Yau

We introduced the Euclidean inner product on $\mathbb{R}^n$ in Chapter 3. In this chapter, we extend the concept of the Euclidean inner product to general vector spaces. We extract the most important properties of the Euclidean inner product on $\mathbb{R}^n$ and turn them into axioms that are applicable in general vector spaces. Then, it is reasonable to use these generalized inner products to define notions of length, distance, and angle in general vector spaces.

## 5.1 Inner Products

In this section we use the most important properties of the Euclidean inner product as axioms to define the general concept of an inner product. We then explain how an inner product defines notions of length and distance in general vector spaces other than $\mathbb{R}^n$.

### 5.1.1 General inner products

The fundamental properties of the Euclidean inner product on $\mathbb{R}^n$ that were listed in Theorem 3.2 are precisely the axioms in the following definition.

**Definition** *An **inner product** on a real vector space $V$ is a function that associates a real number with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$, denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$, in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and all scalars $k$.*

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. [*Symmetry axiom*]

(ii) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$. [*Additivity axiom*]

(iii) $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$. [*Homogeneity axiom*]

(iv) $\langle \mathbf{v}, \mathbf{v} \rangle \geqslant 0$; $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$. [*Positivity axiom*]

*A real vector space with an inner product is called a* ***real inner product space****.*

**Definition** *Let* $V$ *be a real inner product space. Then the* ***norm*** (*or* ***length***) *of a vector* $\mathbf{u}$ *in* $V$ *is defined by*

$$\|\mathbf{u}\| := \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}.$$

*The* ***distance*** *between two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $V$ *is defined by*

$$d(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\|.$$

*The* ***unit vector*** *is defined to be a vector* $\mathbf{u}$ *with* $\|\mathbf{u}\| = 1$.

The following theorem lists some properties of inner products.

**Theorem 5.1** *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in a real inner product space* $V$, *and* $k$ *be any scalar. Then*

(a) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$.

(b) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

(c) $\langle \mathbf{u}, k\mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$.

(d) $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$.

(e) $\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$.

*Proof* We only prove (a). The proofs of remaining parts are trivial and we therefore omit them. We have by Theorem 4.1 (a) and Axiom (iii) [Homogeneity axiom],

$$\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0\mathbf{u}, \mathbf{v} \rangle = 0 \cdot \langle \mathbf{u}, \mathbf{v} \rangle = 0. \qquad \square$$

### 5.1.2 Examples

(1) Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]^T$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]^T$ be in $\mathbb{R}^n$. Then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = \mathbf{u}^T \mathbf{v}$$

defines $\langle \mathbf{u}, \mathbf{v} \rangle$ to be the Euclidean inner product on $\mathbb{R}^n$.

(2) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $A$ be an invertible $n \times n$ matrix. It can be shown that the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle_A := \langle A\mathbf{u}, A\mathbf{v} \rangle = (A\mathbf{u})^T A\mathbf{v} = \mathbf{u}^T A^T A\mathbf{v}$$

defines a new inner product on $\mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. When $A = I$, $\langle \mathbf{u}, \mathbf{v} \rangle_A$ is turned back to the Euclidean inner product. In the following, we only show that it satisfies Axiom (ii) [Additivity axiom] and Axiom (iv) [Positivity axiom]. One can verify that it also satisfies Axiom (i) [Symmetry axiom] and Axiom (iii) [Homogeneity axiom].

For Axiom (ii), we have

$$\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle_A = (\mathbf{u}+\mathbf{v})^T A^T A \mathbf{w} = \mathbf{u}^T A^T A \mathbf{w} + \mathbf{v}^T A^T A \mathbf{w} = \langle \mathbf{u}, \mathbf{w} \rangle_A + \langle \mathbf{v}, \mathbf{w} \rangle_A.$$

For Axiom (iv), we have

$$\langle \mathbf{u}, \mathbf{u} \rangle_A = \langle A\mathbf{u}, A\mathbf{u} \rangle = \mathbf{u}^T A^T A \mathbf{u} = \mathbf{y}^T \mathbf{y} \geqslant 0,$$

where $\mathbf{y} = A\mathbf{u}$. When $\langle \mathbf{u}, \mathbf{u} \rangle_A = 0$, it follows that

$$\mathbf{0} = \mathbf{y} = A\mathbf{u}.$$

Since $A$ is invertible, we obtain $\mathbf{u} = \mathbf{0}$.

(3) Let $C[a, b]$ denote the vector space of all continuous functions on $[a, b]$ with the following operations of function addition and scalar multiplication

$$(\mathbf{f}+\mathbf{g})(x) = f(x) + g(x), \qquad (k\mathbf{f})(x) = kf(x),$$

where $\mathbf{f} = f(x)$, $\mathbf{g} = g(x) \in C[a, b]$ and $k$ is a scalar. Define

$$\langle \mathbf{f}, \mathbf{g} \rangle := \int_a^b f(x)g(x)dx.$$

We show that this formula defines an inner product on $C[a, b]$ by verifying four axioms one by one for functions $\mathbf{f} = f(x)$, $\mathbf{g} = g(x)$, and $\mathbf{s} = s(x)$ in $C[a, b]$.

For Axiom (i), we have

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = \langle \mathbf{g}, \mathbf{f} \rangle.$$

For Axiom (ii), we have

$$\begin{aligned}\langle \mathbf{f}+\mathbf{g}, \mathbf{s} \rangle &= \int_a^b [f(x)+g(x)]s(x)dx = \int_a^b f(x)s(x)dx + \int_a^b g(x)s(x)dx \\ &= \langle \mathbf{f}, \mathbf{s} \rangle + \langle \mathbf{g}, \mathbf{s} \rangle.\end{aligned}$$

For Axiom (iii), we have for all scalar $k$,

$$\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x)dx = k\int_a^b f(x)g(x)dx = k\langle \mathbf{f}, \mathbf{g} \rangle.$$

Finally, for Axiom (iv), if $\mathbf{f} = f(x)$ is any function in $C[a, b]$, then $f^2(x) \geqslant 0$ for all $x$ in $[a, b]$. Therefore,

$$\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b f^2(x)dx \geqslant 0.$$

Further, because $f^2(x) \geqslant 0$ and $\mathbf{f} = f(x)$ is continuous on $[a, b]$, it follows that

$$\int_a^b f^2(x)dx = 0 \quad \Longleftrightarrow \quad f(x) = 0, \quad x \in [a, b].$$

Therefore,

$$\langle \mathbf{f}, \mathbf{f} \rangle = 0 \quad \Longleftrightarrow \quad \mathbf{f} = \mathbf{0}.$$

(4) Let $\mathbb{R}^{n\times n}$ denote the vector space of all $n \times n$ real matrices. An inner product on $\mathbb{R}^{n\times n}$ is defined by

$$\langle X, Y \rangle := \operatorname{tr}(XY^T),$$

where $X, Y \in \mathbb{R}^{n\times n}$. We recall that the trace of a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is given by

$$\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}.$$

In the following, we only show that it satisfies Axioms (ii) and (iv). One can verify that it also satisfies Axioms (i) and (iii).

For Axiom (ii), we have by Theorem 1.3 (c),

$$\begin{aligned}\langle X + Y, Z \rangle &= \operatorname{tr}\big((X + Y)Z^T\big) = \operatorname{tr}(XZ^T + YZ^T) = \operatorname{tr}(XZ^T) + \operatorname{tr}(YZ^T)\\ &= \langle X, Z \rangle + \langle Y, Z \rangle,\end{aligned}$$

where $X, Y, Z \in \mathbb{R}^{n\times n}$.

For Axiom (iv), let $X = [x_{ij}] \in \mathbb{R}^{n\times n}$. Then

$$\langle X, X \rangle = \operatorname{tr}(XX^T) = \sum_{i=1}^n \sum_{j=1}^n x_{ij}^2 \geqslant 0.$$

Thus,

$$\langle X, X \rangle = 0 \quad \Longleftrightarrow \quad x_{ij} = 0, \;\; 1 \leqslant i, j \leqslant n,$$

i.e., $X$ is the zero matrix.

**Remark** The ***Frobenius norm*** of an $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\|A\|_F := \langle A, A \rangle^{1/2} = [\operatorname{tr}(AA^T)]^{1/2} = \left( \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \right)^{1/2}.$$

## 5.2 Angle and Orthogonality

In this section we define the notion of an angle between two nonzero vectors in an inner product space. With this concept, we study some basic relations between vectors in an inner product space.

### 5.2.1 Angle between two vectors and orthogonality

We first introduce the ***Cauchy-Schwarz inequality*** before we define an angle between two vectors in general inner product spaces. The proof of the theorem is left as an exercise.

**Theorem 5.2** (*Cauchy-Schwarz Inequality*) *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors in a real inner product space* $V$*. Then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leqslant \|\mathbf{u}\| \cdot \|\mathbf{v}\|.$$

In $\mathbb{R}^n$, by using the notation of the Euclidean inner product, the cosine of an angle $\theta$ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by (3.3). We are now going to define the notion of an angle between two nonzero vectors in a general inner product space $V$. For any nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$, by using the Cauchy-Schwarz inequality, we deduce

$$-1 \leqslant \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} \leqslant 1. \tag{5.1}$$

Thus, we can define the cosine of the unique ***angle*** $\theta$ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$ by (5.1) as follows:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}, \quad 0 \leqslant \theta \leqslant \pi. \tag{5.2}$$

Observe that in $\mathbb{R}^n$ with the Euclidean inner product, (5.2) agrees with (3.3).

In $\mathbb{R}^n$, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$, i.e., the angle $\theta$ between them is $\pi/2$. It follows from (5.2) that $\cos\theta = 0$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. This suggests the following definition in a general inner product space.

**Definition** *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in an inner product space* $V$ *are called* ***orthogonal*** *if*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

**Example 1** Let $P_2$ have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle := \int_{-1}^{1} p(x)q(x)dx$$

for $\mathbf{p}, \mathbf{q} \in P_2$. Then the polynomials $\mathbf{p} = 2x$ and $\mathbf{q} = 3x^2$ are orthogonal, since

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx = \int_{-1}^{1} 6x^3 dx = 0.$$

**Example 2** Let $\mathbb{R}^{2\times 2}$ have the inner product $\langle U, V \rangle = \mathrm{tr}(UV^T)$ for $U, V \in \mathbb{R}^{2\times 2}$. Then the matrices

$$U = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$$

are orthogonal, since

$$\langle U, V \rangle = 1 \times (-2) + 3 \times 1 + 2 \times 0 + (-1) \times 1 = 0.$$

### 5.2.2 Properties of length, distance, and orthogonality

The following two theorems list some basic properties of length and distance in general inner product spaces.

**Theorem 5.3** *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in an inner product space* $V$, *and* $k$ *be any scalar. Then*

(a) $\|\mathbf{u}\| \geqslant 0$.

(b) $\|\mathbf{u}\| = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.

(c) $\|k\mathbf{u}\| = |k| \cdot \|\mathbf{u}\|$.

(d) $\|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$. (*Triangle inequality*)

*Proof* We only prove (d) and the proofs of remaining parts are trivial. We have by the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &\leqslant \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \end{aligned}$$

Thus, (d) holds. $\square$

**Theorem 5.4** *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in an inner product space* $V$, *and* $k$ *be any scalar. Then*

(a) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \geqslant 0$.

(b) $d(\mathbf{u}, \mathbf{v}) = 0$ *if and only if* $\mathbf{u} = \mathbf{v}$.

(c) $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$.

(d) $d(\mathbf{u},\mathbf{v}) \leqslant d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$. (*Triangle inequality*)

The proof of the theorem is left as an exercise.

The following theorem extends the result in Theorem 3.6 from $\mathbb{R}^n$ to general inner product spaces. The proof of the theorem is exactly the same as that of Theorem 3.6 and we therefore omit it.

**Theorem 5.5** (*Generalized Theorem of Pythagoras*) *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be orthogonal vectors in an inner product space* $V$*. Then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

### 5.2.3 Complement

We extend the orthogonality of two vectors to that of sets of vectors in inner product spaces.

**Definition** *Let* $W$ *be a subspace of an inner product space* $V$*.*

(i) *A vector* $\mathbf{u}$ *in* $V$ *is said to be* ***orthogonal to W*** *if it is orthogonal to every vector in* $W$*.*

(ii) *The set of all vectors in* $V$ *that are orthogonal to* $W$ *is called the* ***orthogonal complement*** *of* $W$*, and denoted by* $W^\perp$*.*

The following theorem shows three basic properties of orthogonal complements.

**Theorem 5.6** *If* $W$ *is a subspace of a finite-dimensional inner product space* $V$*, then*

(a) $W^\perp$ *is a subspace of* $V$*.*

(b) $W \cap W^\perp = \{\mathbf{0}\}$.

(c) $W \subseteq (W^\perp)^\perp$.

*Proof* For (a), let $\mathbf{u},\mathbf{v} \in W^\perp$ and $k \in \mathbb{R}$. Then for all $\mathbf{w} \in W$, we have

$$\langle \mathbf{u},\mathbf{w}\rangle = 0, \qquad \langle \mathbf{v},\mathbf{w}\rangle = 0.$$

Therefore,

$$\langle \mathbf{u}+k\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{w}\rangle + k\langle \mathbf{v},\mathbf{w}\rangle = 0.$$

Thus, $\mathbf{u}+k\mathbf{v} \in W^\perp$ and it follows from Theorem 4.2 that $W^\perp$ is a subspace.

For (b), for any vector $\mathbf{u} \in W \cap W^{\perp}$, let $\mathbf{u} = \mathbf{u}_1 = \mathbf{u}_2$. Then

$$\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0,$$

because

$$\mathbf{u}_1 \in W \cap W^{\perp} \subseteq W, \qquad \mathbf{u}_2 \in W \cap W^{\perp} \subseteq W^{\perp}.$$

By Axiom (iv) [Positivity axiom], we have $\mathbf{u} = \mathbf{0}$, i.e., $W \cap W^{\perp} = \{\mathbf{0}\}$.

For (c), for any vector $\mathbf{u} \in W$, $\mathbf{u}$ is orthogonal to $W^{\perp}$. Besides, $(W^{\perp})^{\perp}$ is the set of all vectors in $V$ that are orthogonal to $W^{\perp}$. Then $\mathbf{u} \in (W^{\perp})^{\perp}$, i.e., $W \subseteq (W^{\perp})^{\perp}$. □

## 5.3 Orthogonal Bases and Gram-Schmidt Process

In many problems involving inner product spaces, we choose an appropriate basis for the vector space to simplify the solution of a problem. Frequently we consider a basis in which each pair of vectors is orthogonal. In this section, we reveal how to find such a basis.

### 5.3.1 Orthogonal and orthonormal bases

**Definition** *A set of vectors in an inner product space $V$ is called **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is called **orthonormal**.*

If $\mathbf{v}$ is a nonzero vector in an inner product space, then by Theorem 5.3 (c), the vector

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has norm 1, since

$$\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

The process of multiplying a nonzero vector $\mathbf{v}$ by $1/\|\mathbf{v}\|$ to obtain a unit vector is called normalizing $\mathbf{v}$. An orthogonal set of nonzero vectors can always be converted to an orthonormal set by normalizing each of its vectors.

**Remark** A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is orthonormal if and only if

$$\begin{cases} \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, & i \neq j; \\ \langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1, & 1 \leqslant i \leqslant n. \end{cases}$$

Two nonzero orthogonal vectors are linearly independent. The following theorem generalizes the property to an orthogonal set of nonzero vectors.

**Theorem 5.7** *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal set of nonzero vectors in an inner product space $V$. Then $S$ is linearly independent.*

*Proof* Consider the following equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0}. \tag{5.3}$$

We want to show that

$$k_1 = k_2 = \cdots = k_n = 0.$$

Beginning with $k_1$, we have by taking the inner product on both sides of (5.3) with $\mathbf{v}_1$,

$$\langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n, \mathbf{v}_1\rangle = \langle \mathbf{0}, \mathbf{v}_1\rangle.$$

Since $S$ is orthogonal, we obtain

$$k_1\langle \mathbf{v}_1, \mathbf{v}_1\rangle + k_2 \cdot 0 + \cdots + k_n \cdot 0 = 0,$$

i.e.,

$$k_1\langle \mathbf{v}_1, \mathbf{v}_1\rangle = 0.$$

Since $\mathbf{v}_1 \neq \mathbf{0}$, it follows that $\langle \mathbf{v}_1, \mathbf{v}_1\rangle \neq 0$ by Axiom (iv) [Positivity axiom]. Then $k_1 = 0$. Similarly,

$$k_j = 0, \quad 2 \leqslant j \leqslant n.$$

Thus, $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent. □

**Definition** *In an inner product space, a basis consisting of orthogonal vectors is called an **orthogonal basis**, and a basis consisting of orthonormal vectors is called an **orthonormal basis**.*

Orthonormal bases for inner product spaces are always convenient to solve problems because they simplify the expression of a vector and some related formulas as the following two theorems show.

**Theorem 5.8** *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space $V$. Then for any $\mathbf{u}$ in $V$,*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n\rangle\mathbf{v}_n.$$

*Proof* Since $\mathbf{u} \in V$ and $S$ is a basis for $V$, we have

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n. \tag{5.4}$$

Because $S$ is orthonormal, taking the inner product of $\mathbf{u}$ and $\mathbf{v}_1$, it follows from (5.4) that

$$\langle \mathbf{u}, \mathbf{v}_1\rangle = \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n, \mathbf{v}_1\rangle$$

$$= k_1\langle \mathbf{v}_1, \mathbf{v}_1\rangle + k_2\langle \mathbf{v}_2, \mathbf{v}_1\rangle + \cdots + k_n\langle \mathbf{v}_n, \mathbf{v}_1\rangle$$
$$= k_1 \times 1 + k_2 \times 0 + \cdots + k_n \times 0 = k_1.$$

Similarly,

$$\langle \mathbf{u}, \mathbf{v}_j\rangle = k_j, \quad 2 \leqslant j \leqslant n. \qquad \square$$

**Theorem 5.9** *Let $S$ be an orthonormal basis for an $n$-dimensional inner product space $V$. If the coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$ relative to the basis $S$ are given by*

$$[\mathbf{u}]_S = [u_1, u_2, \ldots, u_n] \qquad \textit{and} \qquad [\mathbf{v}]_S = [v_1, v_2, \ldots, v_n],$$

*then*

(a) $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$.

(b) $d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$.

(c) $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

*Proof* We only prove (c) and leave the proofs of the remaining parts as an exercise. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$. Then

$$\mathbf{u} = \sum_{i=1}^{n} u_i\mathbf{w}_i, \qquad \mathbf{v} = \sum_{j=1}^{n} v_j\mathbf{w}_j.$$

Since $S$ is orthonormal, we have

$$\langle \mathbf{u}, \mathbf{v}\rangle = \left\langle \sum_{i=1}^{n} u_i\mathbf{w}_i, \sum_{j=1}^{n} v_j\mathbf{w}_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} u_iv_j\langle \mathbf{w}_i, \mathbf{w}_j\rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n. \quad \square$$

**Remark** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for a vector space $V$. Then normalizing each of these vectors yields the orthonormal basis

$$S' = \left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \right\}.$$

For any vector $\mathbf{u} \in V$, it follows from Theorem 5.8 that

$$\mathbf{u} = \left\langle \mathbf{u}, \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \right\rangle \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \left\langle \mathbf{u}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\rangle \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} + \cdots + \left\langle \mathbf{u}, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \right\rangle \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|},$$

which can be rewritten as

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

### 5.3.2 Projection theorem

**Theorem 5.10** (*Projection Theorem*) *Let $W$ be a finite-dimensional subspace of an inner product space $V$. Then every vector $\mathbf{u}$ in $V$ can be expressed in exactly one way as*

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2,$$

*where $\mathbf{w}_1$ is in $W$ and $\mathbf{w}_2$ is in $W^\perp$.*

*Proof* First, we prove the existence of $\mathbf{w}_1$ and $\mathbf{w}_2$. Let $\dim(W) = n$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for $W$. For any $\mathbf{u} \in V$, we construct the following two vectors

$$\mathbf{w}_1 = \sum_{i=1}^{n} \langle \mathbf{u}, \mathbf{v}_i \rangle \mathbf{v}_i, \qquad \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1.$$

Then $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$. Obviously, $\mathbf{w}_1 \in W$. We want to show that $\mathbf{w}_2 \in W^\perp$, i.e., $\langle \mathbf{w}_2, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$. Let $\mathbf{w} \in W$ and then $\mathbf{w} = \sum_{j=1}^{n} k_j \mathbf{v}_j$. Thus,

$$\langle \mathbf{w}_2, \mathbf{w} \rangle = \langle \mathbf{u} - \mathbf{w}_1, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{w}_1, \mathbf{w} \rangle. \tag{5.5}$$

Since

$$\langle \mathbf{u}, \mathbf{w} \rangle = \left\langle \mathbf{u}, \sum_{j=1}^{n} k_j \mathbf{v}_j \right\rangle = \sum_{j=1}^{n} k_j \langle \mathbf{u}, \mathbf{v}_j \rangle$$

and

$$\langle \mathbf{w}_1, \mathbf{w} \rangle = \left\langle \sum_{i=1}^{n} \langle \mathbf{u}, \mathbf{v}_i \rangle \mathbf{v}_i, \sum_{j=1}^{n} k_j \mathbf{v}_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \mathbf{u}, \mathbf{v}_i \rangle k_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$

$$= \sum_{i=1}^{n} k_i \langle \mathbf{u}, \mathbf{v}_i \rangle, \qquad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

substituting them into (5.5), we have $\langle \mathbf{w}_2, \mathbf{w} \rangle = 0$ for any $\mathbf{w} \in W$. Hence $\mathbf{w}_2 \in W^\perp$.

Second, we prove the uniqueness of the expression. Let $\mathbf{u} = \mathbf{w}_1' + \mathbf{w}_2'$ with $\mathbf{w}_1' \in W$ and $\mathbf{w}_2' \in W^\perp$. Also $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$. We then have

$$\mathbf{u} - \mathbf{u} = \mathbf{0} = (\mathbf{w}_1' - \mathbf{w}_1) + (\mathbf{w}_2' - \mathbf{w}_2).$$

Hence

$$\underbrace{\mathbf{w}_1 - \mathbf{w}_1'}_{W} = \underbrace{\mathbf{w}_2' - \mathbf{w}_2}_{W^\perp}.$$

Let $\mathbf{q} = \mathbf{w}_1 - \mathbf{w}_1' = \mathbf{w}_2' - \mathbf{w}_2$. By Theorem 5.6 (b), we know that $\mathbf{q} \in W \cap W^\perp = \{\mathbf{0}\}$. Then

$$\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{0} = \mathbf{w}_2' - \mathbf{w}_2.$$

Thus,

$$\mathbf{w}_1 = \mathbf{w}_1', \qquad \mathbf{w}_2 = \mathbf{w}_2'. \qquad \square$$

**Corollary** *If $W$ is a subspace of an inner product space $V$ with $\dim(V) = m$, then*

$$\dim(W) + \dim(W^\perp) = m. \tag{5.6}$$

*Proof* By using Theorem 4.14, Theorem 5.10, and Theorem 5.6 (b), we have

$$\dim(W) + \dim(W^\perp) = \dim(W + W^\perp) + \dim(W \cap W^\perp) = \dim(V) + 0 = m. \quad \square$$

**Remark** Let $V$ be an inner product space with $\dim(V) = m$. Since $W^\perp$ is a subspace of $V$, we have by the corollary above,

$$\dim(W^\perp) + \dim\big((W^\perp)^\perp\big) = m. \tag{5.7}$$

By (5.6) and (5.7), we obtain

$$\dim(W) = \dim\big((W^\perp)^\perp\big).$$

It follows from Theorem 5.6 (c) and Theorem 4.13 that

$$W = (W^\perp)^\perp.$$

Because $W$ and $W^\perp$ are orthogonal complements of one another, we say that $W$ and $W^\perp$ are orthogonal complements.

In Theorem 5.10, the vector $\mathbf{w}_1$ is called the ***orthogonal projection*** of $\mathbf{u}$ on $W$ and is denoted by $\mathrm{proj}_W\mathbf{u}$. The vector $\mathbf{w}_2$ is called the component of $\mathbf{u}$ orthogonal to $W$ and is denoted by $\mathrm{proj}_{W^\perp}\mathbf{u}$. Thus,

$$\mathbf{u} = \mathrm{proj}_W\mathbf{u} + \mathrm{proj}_{W^\perp}\mathbf{u} \quad \text{or} \quad \mathrm{proj}_{W^\perp}\mathbf{u} = \mathbf{u} - \mathrm{proj}_W\mathbf{u}.$$

The following theorem gives formulas to compute orthogonal projections onto a finite-dimensional subspace.

**Theorem 5.11** *Let $W$ be a finite-dimensional subspace of an inner product space $V$.*

(a) *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for $W$ and $\mathbf{u}$ is any vector in $V$, then*

$$\mathrm{proj}_W\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r\rangle \mathbf{v}_r.$$

(b) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *is an orthogonal basis for* $W$ *and* $\mathbf{u}$ *is any vector in* $V$*, then*

$$\mathrm{proj}_W\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r\rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r. \tag{5.8}$$

*Proof* We only prove (a) and the proof of (b) is trivial by using (a). We have by Theorem 5.10,

$$\mathbf{u} = \mathrm{proj}_W\mathbf{u} + \mathrm{proj}_{W^\perp}\mathbf{u}.$$

Since $\mathrm{proj}_W\mathbf{u} \in W$, we have

$$\mathrm{proj}_W\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r.$$

Then

$$\mathbf{u} = \sum_{j=1}^{r} k_j\mathbf{v}_j + \mathrm{proj}_{W^\perp}\mathbf{u}.$$

Hence

$$\langle \mathbf{u}, \mathbf{v}_i\rangle = \left\langle \sum_{j=1}^{r} k_j\mathbf{v}_j + \mathrm{proj}_{W^\perp}\mathbf{u}, \mathbf{v}_i\right\rangle = \sum_{j=1}^{r} k_j\langle \mathbf{v}_j, \mathbf{v}_i\rangle + \langle \mathrm{proj}_{W^\perp}\mathbf{u}, \mathbf{v}_i\rangle = k_i, \quad 1 \leqslant i \leqslant r.$$

Thus, (a) holds. □

We next provide a geometric perspective to depict the relations between the nullspace and the row space of a matrix.

**Theorem 5.12** *Let* $A$ *be an* $m \times n$ *matrix. Then*

(a) *The nullspace of* $A$ *and the row space of* $A$ *are orthogonal complements in* $\mathbb{R}^n$ *with respect to the Euclidean inner product.*

(b) *The nullspace of* $A^T$ *and the column space of* $A$ *are orthogonal complements in* $\mathbb{R}^m$ *with respect to the Euclidean inner product.*

*Proof* For (a), let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ be the row vectors of $A$. First, we want to show that the orthogonal complement of $\mathrm{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$ is the nullspace of $A$. To do this we must show that if a vector $\mathbf{v}$ is orthogonal to $\mathrm{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$, then $\mathbf{v}$ is in the nullspace of $A$. Conversely, if a vector $\mathbf{u}$ is in the nullspace of $A$, then $\mathbf{u}$ is orthogonal to $\mathrm{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$.

Assume first that $\mathbf{v}$ is orthogonal to $\mathrm{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\}$. Then $\mathbf{v}$ is orthogonal to all row vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$, i.e., $\mathbf{r}_i\mathbf{v} = 0$ for $1 \leqslant i \leqslant m$. We have

$$A\mathbf{v} = \begin{bmatrix} \mathbf{r}_1 \\ -- \\ \mathbf{r}_2 \\ -- \\ \vdots \\ -- \\ \mathbf{r}_m \end{bmatrix} \mathbf{v} = \begin{bmatrix} \mathbf{r}_1\mathbf{v} \\ -- \\ \mathbf{r}_2\mathbf{v} \\ -- \\ \vdots \\ -- \\ \mathbf{r}_m\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ -- \\ 0 \\ -- \\ \vdots \\ -- \\ 0 \end{bmatrix} = \mathbf{0}. \tag{5.9}$$

Thus, $\mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{v}$ is in the nullspace of $A$.

Conversely, assume that $\mathbf{u}$ is in the nullspace of $A$. Then $A\mathbf{u} = \mathbf{0}$. It follows from (5.9) that $\mathbf{r}_i\mathbf{u} = \langle \mathbf{u}, \mathbf{r}_i \rangle = 0$ for $1 \leqslant i \leqslant m$. Let $\mathbf{w} \in \mathrm{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\}$ and then $\mathbf{w} = \sum_{i=1}^{m} c_i\mathbf{r}_i$. Taking the inner product of $\mathbf{u}$ and $\mathbf{w}$ yields

$$\langle \mathbf{u}, \mathbf{w} \rangle = \left\langle \mathbf{u}, \sum_{i=1}^{m} c_i\mathbf{r}_i \right\rangle = \sum_{i=1}^{m} c_i \langle \mathbf{u}, \mathbf{r}_i \rangle = 0.$$

Then $\mathbf{u}$ is orthogonal to the row space of $A$.

Thus, the orthogonal complement of the row space of $A$ is the nullspace of $A$. Since $W = (W^{\perp})^{\perp}$ for a subspace $W$ in a finite-dimensional space, the orthogonal complement of the nullspace of $A$ is the row space of $A$.

For (b), by applying the results in (a) to $A^T$, it follows that the nullspace of $A^T$ and the row space of $A^T$ are orthogonal complements in $\mathbb{R}^m$. Thus, the nullspace of $A^T$ and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$. $\square$

### 5.3.3 Gram-Schmidt process

In order to produce orthogonal (or orthonormal) bases, we introduce the Gram-Schmidt process. Let

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

be a linearly independent set. The following process converts $S$ to be an orthogonal set.

**Step 1.** Let $\mathbf{v}_1 = \mathbf{u}_1$.

**Step 2.** As illustrated in Figure 5.1, we can obtain a vector $\mathbf{v}_2$ that is orthogonal to $\mathbf{v}_1$ by computing the component of $\mathbf{u}_2$ orthogonal to the space $W_1 = \mathrm{span}\{\mathbf{v}_1\}$. By using (5.8), we have

$$\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1.$$

Of course, if $\mathbf{v}_2 = \mathbf{0}$, then $\mathbf{v}_2$ is not a basis vector. However, this cannot happen, since it would then follow from the preceding formula for $\mathbf{v}_2$ that

$$\mathbf{u}_2 = \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{u}_1$$

which implies that $\mathbf{u}_2$ is a multiple of $\mathbf{u}_1$, contradicting the linear independence of $S$.

**Step 3.** To construct a vector $\mathbf{v}_3$ that is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, we compute the component of $\mathbf{u}_3$ orthogonal to the space $W_2 = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. See Figure 5.2. From (5.8) again,

$$\mathbf{v}_3 = \mathbf{u}_3 - \mathrm{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2.$$

As in Step 2, the linear independence of $S$ ensures that $\mathbf{v}_3 \neq \mathbf{0}$.

**Step 4.** To determine a vector $\mathbf{v}_4$ that is orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we compute the component of $\mathbf{u}_4$ orthogonal to the space $W_3 = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It follows from (5.8) that

$$\mathbf{v}_4 = \mathbf{u}_4 - \mathrm{proj}_{W_3}\mathbf{u}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$$

and $\mathbf{v}_4 \neq \mathbf{0}$.

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**Step n.** $\mathbf{v}_n = \mathbf{u}_n - \sum_{i=1}^{n-1} \frac{\langle \mathbf{u}_n, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}\mathbf{v}_i$ and $\mathbf{v}_n \neq \mathbf{0}$.

The preceding step-by-step construction for converting an arbitrary linearly independent set into an orthogonal set is called the ***Gram-Schmidt process***.

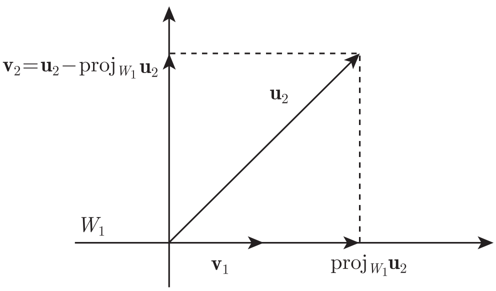


Figure 5.1

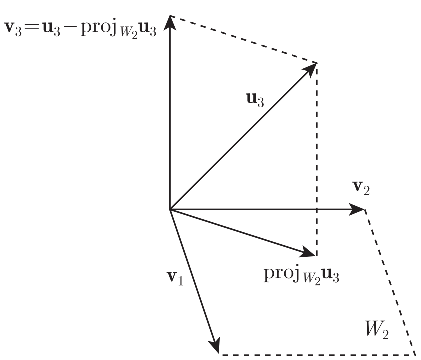


Figure 5.2

**Remark** We have

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\}, \quad 1 \leqslant j \leqslant n.$$

Moreover, $\mathbf{v}_{k+1}$ is orthogonal to $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for any $k$. Thus, every nonzero finite-dimensional inner product space has an orthogonal (or orthonormal) basis.

**Example** Consider $\mathbb{R}^3$ with the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = [1, -1, 0], \qquad \mathbf{u}_2 = [-1, 1, 1], \qquad \mathbf{u}_3 = [1, 1, 1]$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

*Solution* By using the Gram-Schmidt process, we have

**Step 1.** $\mathbf{v}_1 = \mathbf{u}_1 = [1, -1, 0]$.

**Step 2.** $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = [-1, 1, 1] + [1, -1, 0] = [0, 0, 1]$.

**Step 3.** $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

$$= [1, 1, 1] - \frac{0}{2}[1, -1, 0] - [0, 0, 1] = [1, 1, 0].$$

Thus,

$$\mathbf{v}_1 = [1, -1, 0], \qquad \mathbf{v}_2 = [0, 0, 1], \qquad \mathbf{v}_3 = [1, 1, 0]$$

form an orthogonal basis for $\mathbb{R}^3$. The norms of these vectors are

$$\|\mathbf{v}_1\| = \sqrt{2}, \qquad \|\mathbf{v}_2\| = 1, \qquad \|\mathbf{v}_3\| = \sqrt{2}.$$

Therefore, an orthonormal basis for $\mathbb{R}^3$ is $S = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$, where

$$\mathbf{q}_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right], \qquad \mathbf{q}_2 = [0, 0, 1], \qquad \mathbf{q}_3 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]. \qquad \square$$

### 5.3.4 *QR*-decomposition

Let $A$ be an $m \times n$ ($m \geqslant n$) matrix with linearly independent column vectors. If $Q$ is the matrix with orthonormal column vectors that results from applying the Gram-Schmidt process to the column vectors of $A$, what is the relationship between $A$ and $Q$?

To solve this problem, let

$$A = \left[\begin{array}{c:c:c:c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{array}\right], \qquad Q = \left[\begin{array}{c:c:c:c} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array}\right],$$

where $\mathbf{u}_j$ and $\mathbf{q}_j$ $(1 \leqslant j \leqslant n)$ are column vectors of $A$ and $Q$, respectively. It follows from Theorem 5.8 that

$$\begin{cases} \mathbf{u}_1 = \langle \mathbf{u}_1, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_1, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_1, \mathbf{q}_n \rangle \mathbf{q}_n \\ \mathbf{u}_2 = \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_2, \mathbf{q}_n \rangle \mathbf{q}_n \\ \quad \vdots \\ \mathbf{u}_n = \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_n, \mathbf{q}_n \rangle \mathbf{q}_n. \end{cases}$$

It can be written in matrix form

$$\begin{aligned} &\left[\begin{array}{c:c:c:c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{array}\right] \\ &= \left[\begin{array}{c:c:c:c} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array}\right] \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{q}_2 \rangle & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{q}_n \rangle & \langle \mathbf{u}_2, \mathbf{q}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix}, \end{aligned}$$

or more briefly as

$$A = QR, \tag{5.10}$$

where

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{q}_2 \rangle & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{q}_n \rangle & \langle \mathbf{u}_2, \mathbf{q}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix}.$$

It is a property of the Gram-Schmidt process that for $j \geqslant 2$, the vector $\mathbf{q}_j$ is orthogonal to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{j-1}$. Therefore, all entries below the main diagonal of $R$ are zero,

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix}.$$

All the diagonal entries $\langle \mathbf{u}_j, \mathbf{q}_j \rangle$ $(1 \leqslant j \leqslant n)$ of $R$ are nonzero. We prove this fact as follows:

For $\langle \mathbf{u}_1, \mathbf{q}_1 \rangle$, since $\mathbf{q}_1 = \dfrac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \neq \mathbf{0}$, we know that $\langle \mathbf{u}_1, \mathbf{q}_1 \rangle \neq 0$.

For $\langle \mathbf{u}_2, \mathbf{q}_2 \rangle$, if $\langle \mathbf{u}_2, \mathbf{q}_2 \rangle = 0$, then

$$\mathbf{u}_2 = \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \mathbf{q}_1 = \frac{\langle \mathbf{u}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1,$$

i.e., $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent, which contradicts the fact that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

Moreover, for $\langle \mathbf{u}_j, \mathbf{q}_j \rangle$, if $\langle \mathbf{u}_j, \mathbf{q}_j \rangle = 0$, then

$$\mathbf{u}_j \in \operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{j-1}\}.$$

But

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{j-1} \in \operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{j-1}\},$$

which implies

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{j-1}, \mathbf{u}_j \in \operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{j-1}\}.$$

Thus, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{j-1}, \mathbf{u}_j$ are linearly dependent by Theorem 4.9 (a). A contradiction again! Therefore, $\langle \mathbf{u}_j, \mathbf{q}_j \rangle \neq 0$ for $1 \leqslant j \leqslant n$. Then $R$ is invertible.

Formula (5.10) is a decomposition of $A$ in the form of the product of a matrix $Q$ with orthonormal column vectors and an invertible upper triangular matrix $R$. We call (5.10) the ***QR-decomposition*** of $A$. In summary, we have the following theorem.

**Theorem 5.13** (*QR-Decomposition*) *Let $A$ be an $m \times n$ matrix with linearly independent column vectors. Then $A$ can be decomposed as*

$$A = QR,$$

*where $Q$ is an $m \times n$ matrix with orthonormal column vectors and $R$ is an $n \times n$ invertible upper triangular matrix.*

**Remark** Recall from Theorem 4.28 that if $A$ is an $n \times n$ matrix, then the invertibility of $A$ is equivalent to linear independence of the column vectors. Thus, every invertible matrix has a $QR$-decomposition.

We conclude this section by adding three more results to the following theorem, which involves all major topics we have studied so far.

**Theorem 5.14** *Let $A$ be an $n \times n$ matrix and $T_A \colon \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by $A$. Then the following are equivalent.*

(a) *$A$ is invertible.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced row-echelon form of* $A$ *is* $I_n$.

(d) $A$ *is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) *The range of* $T_A$ *is* $\mathbb{R}^n$.

(i) $T_A$ *is one-to-one.*

(j) *The column vectors of* $A$ *are linearly independent.*

(k) *The row vectors of* $A$ *are linearly independent.*

(l) *The column vectors of* $A$ *span* $\mathbb{R}^n$.

(m) *The row vectors of* $A$ *span* $\mathbb{R}^n$.

(n) *The column vectors of* $A$ *form a basis for* $\mathbb{R}^n$.

(o) *The row vectors of* $A$ *form a basis for* $\mathbb{R}^n$.

(p) $A$ *has rank* $n$.

(q) $A$ *has nullity* $0$.

(r) *The orthogonal complement of the nullspace of* $A$ *is* $\mathbb{R}^n$.

(s) *The orthogonal complement of the row space of* $A$ *is* $\{\mathbf{0}\}$.

(t) $A$ *has a QR-decomposition.*

## 5.4 Best Approximation and Least Squares

We show how orthogonal projections can be used to solve certain approximation problems. The results obtained here have a wide variety of applications in both mathematics and science.

### 5.4.1 Orthogonal projections viewed as approximations

If a point $P \in \mathbb{R}^3$ and $W$ is a plane through the origin $O$, then the point $Q$ in $W$ closest to $P$ is obtained by dropping a perpendicular from $P$ to $W$. See Figure 5.3. Therefore, let $\mathbf{u} = \overrightarrow{OP}$ and then the distance between $P$ and $W$ is given by

$$\min_{\mathbf{w}\in W} \|\mathbf{u} - \mathbf{w}\| = \|\mathbf{u} - \mathrm{proj}_W \mathbf{u}\|.$$

See Figure 5.4. Thus, $\mathrm{proj}_W \mathbf{u}$ is the "best approximation" to $\mathbf{u}$ by vectors in $W$.

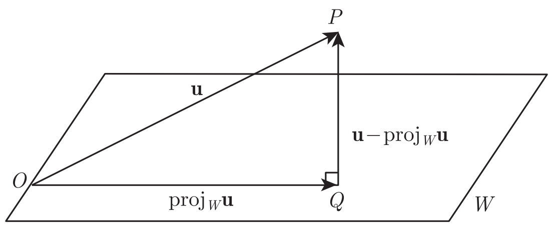


Figure 5.3 $Q$ is the point in $W$ closest to $P$

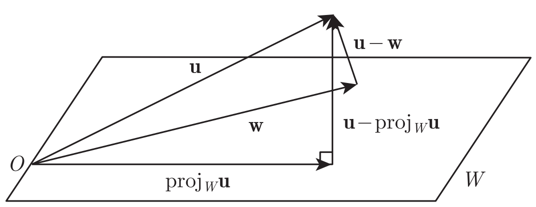


Figure 5.4 $\|\mathbf{u} - \mathbf{w}\|$ is minimized by $\mathbf{w} = \mathrm{proj}_W \mathbf{u}$

**Theorem 5.15** (*Best Approximation Theorem*) *Let $W$ be a finite-dimensional subspace of an inner product space $V$ and $\mathbf{u}$ be in $V$. Then $\mathrm{proj}_W \mathbf{u}$ is the best approximation to $\mathbf{u}$ from $W$ in the sense that*

$$\|\mathbf{u} - \mathrm{proj}_W \mathbf{u}\| < \|\mathbf{u} - \mathbf{w}\|$$

*for every vector $\mathbf{w}$ in $W$ with $\mathbf{w} \neq \mathrm{proj}_W \mathbf{u}$.*

*Proof* For every vector $\mathbf{w}$ in $W$, we can write by Theorem 5.10 (Projection Theorem),

$$\mathbf{u} - \mathbf{w} = \underbrace{(\mathbf{u} - \mathrm{proj}_W \mathbf{u})}_{W^\perp} + \underbrace{(\mathrm{proj}_W \mathbf{u} - \mathbf{w})}_{W}.$$

Thus, by Theorem 5.5,

$$\|\mathbf{u} - \mathbf{w}\|^2 = \|\mathbf{u} - \mathrm{proj}_W \mathbf{u}\|^2 + \|\mathrm{proj}_W \mathbf{u} - \mathbf{w}\|^2.$$

If $\mathbf{w} \neq \mathrm{proj}_W \mathbf{u}$, then the second term in this sum is positive, so that

$$\|\mathbf{u}-\mathbf{w}\|^2 > \|\mathbf{u}-\text{proj}_W\mathbf{u}\|^2,$$

which implies that

$$\|\mathbf{u}-\mathbf{w}\| > \|\mathbf{u}-\text{proj}_W\mathbf{u}\|. \qquad \square$$

### 5.4.2 Least squares solutions of linear systems

Up to now, we have been mainly concerned with consistent systems of linear equations. However, inconsistent linear systems also appear in science and engineering. If $A\mathbf{x}=\mathbf{b}$ has no solution, then for any $\mathbf{x}$, $\|A\mathbf{x}-\mathbf{b}\| \neq 0$ with the Euclidean norm. We therefore study the following least squares problem.

(1) Least squares problem. Given a linear system $A\mathbf{x}=\mathbf{b}$ of $m$ equations in $n$ unknowns, find a vector $\mathbf{x} \in \mathbb{R}^n$ that minimizes $\|A\mathbf{x}-\mathbf{b}\|$ with respect to the Euclidean norm on $\mathbb{R}^m$. Such $\mathbf{x}$ is called a ***least squares solution*** of $A\mathbf{x}=\mathbf{b}$.

Let $A = [\ \mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_n\ ]$, where $\mathbf{c}_i$ $(1 \leqslant i \leqslant n)$ are the column vectors of $A$. Then

$$W = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \{\mathbf{r} = A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}, \tag{5.11}$$

i.e., $A\mathbf{x} \in W$.

(2) Review of Theorem 5.12 (b). If $A$ is an $m \times n$ matrix, then the nullspace of $A^T$ and the column space $W$ of $A$ are orthogonal complements in $\mathbb{R}^m$ with respect to the Euclidean inner product, i.e.,

$$W^\perp = \text{nullspace of } A^T.$$

(3) Find a solution of the least squares problem. It follows from Theorem 5.15 (Best Approximation Theorem) that the closest vector in $W$ to $\mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ on $W$. Thus, for a vector $\mathbf{x} \in \mathbb{R}^n$ to be a least squares solution of $A\mathbf{x}=\mathbf{b}$, $\mathbf{x}$ must satisfy

$$A\mathbf{x} = \text{proj}_W\mathbf{b},$$

and then

$$\mathbf{b} - A\mathbf{x} = \mathbf{b} - \text{proj}_W\mathbf{b}.$$

Since $\mathbf{b} - \text{proj}_W\mathbf{b} \in W^\perp$, it follows from (2) that

$$\mathbf{b} - A\mathbf{x} \in W^\perp = \text{nullspace of } A^T.$$

Therefore, a least squares solution of $A\mathbf{x}=\mathbf{b}$ must satisfy

$$A^T(\mathbf{b} - A\mathbf{x}) = \mathbf{0},$$

i.e.,

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

This is called the ***normal system*** associated with $A\mathbf{x} = \mathbf{b}$. Thus, the problem of finding a least squares solution of $A\mathbf{x} = \mathbf{b}$ has been reduced to the problem of finding an exact solution of the associated normal system. The following observations are about the normal system:

(a) The normal system has $n$ equations in $n$ unknowns.

(b) The normal system is consistent, since it is satisfied by a least squares solution of $A\mathbf{x} = \mathbf{b}$.

(c) The normal system may have infinitely many solutions and all of its solutions are least squares solutions of $A\mathbf{x} = \mathbf{b}$.

### 5.4.3 Uniqueness of least squares solutions

We establish conditions under which a linear system is guaranteed to have a unique least squares solution. We need the following theorem.

**Theorem 5.16** *Let $A$ be an $m \times n$ matrix. Then the following are equivalent.*

(a) *$A$ has linearly independent column vectors.*

(b) *$A^T A$ is invertible.*

*Proof* (a) $\Rightarrow$ (b): Assume that $A$ has linearly independent column vectors. The size of matrix $A^T A$ is $n \times n$, so we can prove that this matrix is invertible by showing that the linear system $A^T A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Assuming $\mathbf{x}$ is any solution of this system, then

$$\langle A\mathbf{x}, A\mathbf{x}\rangle = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0,$$

which implies $A\mathbf{x} = \mathbf{0}$ by Axiom (iv) [Positivity axiom]. Assuming $A$ has linearly independent column vectors, so $\mathbf{x} = \mathbf{0}$ by Theorem 4.27.

(b) $\Rightarrow$ (a): Assume that $A^T A$ is invertible. To prove that $A$ has linearly independent column vectors, it suffices to prove that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution by Theorem 4.27. But if $\mathbf{x}$ is any solution of $A\mathbf{x} = \mathbf{0}$, then $A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ from the invertibility of $A^T A$. $\square$

The following theorem is a direct consequence of Theorem 5.16 and one can prove it easily.

**Theorem 5.17** (*Uniqueness of Least Squares Solutions*) *Let $A$ be an $m \times n$ matrix with linearly independent column vectors. Then for every $\mathbf{b}$ in $\mathbb{R}^m$, the linear system $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution given by*

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}.$$

**Example** Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{cases} x_1 - \phantom{2}x_2 = 2 \\ x_1 \phantom{+2x_2} = -8 \\ x_1 + \phantom{2}x_2 = 12 \\ x_1 + 2x_2 = 2 \end{cases}$$

and find the orthogonal projection of $\mathbf{b}$ on the column space of $A$.

*Solution* Note that

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -8 \\ 12 \\ 2 \end{bmatrix}.$$

Since $A$ has linearly independent column vectors, we know in advance that there is a unique least squares solution. We have

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -8 \\ 12 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}.$$

So the normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}.$$

Solving this system yields the least squares solution

$$x_1 = 1, \qquad x_2 = 2.$$

Thus, the orthogonal projection of $\mathbf{b}$ on the column space of $A$ is

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 5 \end{bmatrix}. \qquad \square$$

## 5.5 Orthogonal Matrices and Change of Basis

A basis that is suitable for one problem may not be suitable for another. We will study a process that changes one basis to another basis in a vector space and also discuss various problems related to the changes of basis.

### 5.5.1 Orthogonal matrices

We first introduce the following important matrices and then study their fundamental properties.

**Definition** *A square matrix $A$ is said to be an **orthogonal matrix** if*

$$A^{-1} = A^T$$

*or equivalently, $A^T A = AA^T = I$.*

**Theorem 5.18** *The following are equivalent for an $n \times n$ matrix $A$.*

(a) *$A$ is orthogonal.*

(b) *The row (or column) vectors of $A$ form an orthonormal set in $\mathbb{R}^n$ with respect to the Euclidean inner product.*

*Proof* We only consider the case of row vectors and the proof of the case of column vectors is similar. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ be the row vectors of $A$. We have by using $AA^T = I$,

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ -- \\ \mathbf{r}_2 \\ -- \\ \vdots \\ -- \\ \mathbf{r}_n \end{bmatrix} \left[\, \mathbf{r}_1^T \mid \mathbf{r}_2^T \mid \cdots \mid \mathbf{r}_n^T \,\right] = \begin{bmatrix} \mathbf{r}_1\mathbf{r}_1^T & \mathbf{r}_1\mathbf{r}_2^T & \cdots & \mathbf{r}_1\mathbf{r}_n^T \\ \mathbf{r}_2\mathbf{r}_1^T & \mathbf{r}_2\mathbf{r}_2^T & \cdots & \mathbf{r}_2\mathbf{r}_n^T \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n\mathbf{r}_1^T & \mathbf{r}_n\mathbf{r}_2^T & \cdots & \mathbf{r}_n\mathbf{r}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

if and only if

$$\begin{cases} \langle \mathbf{r}_i, \mathbf{r}_i \rangle = \mathbf{r}_i \mathbf{r}_i^T = 1, & 1 \leqslant i \leqslant n; \\ \langle \mathbf{r}_i, \mathbf{r}_j \rangle = \mathbf{r}_i \mathbf{r}_j^T = 0, & i \neq j. \end{cases}$$

Thus, the result holds. □

**Remark** In fact, if an $n \times n$ matrix $A$ is orthogonal, then the set of row vectors of $A$ forms an orthonormal basis for $\mathbb{R}^n$ and the set of column vectors of $A$ forms an orthonormal basis for $\mathbb{R}^n$ as well.

The following two theorems are concerned with some properties of orthogonal matrices.

**Theorem 5.19** *We have*

(a) *The inverse of an orthogonal matrix is orthogonal.*

(b) *A product of orthogonal matrices is orthogonal.*

(c) *If $A$ is orthogonal, then* $\det(A) = \pm 1$.

*Proof* For (a), let $A$ be an orthogonal matrix. Then

$$A^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1} = (A^T A)^{-1} = I^{-1} = I$$

and

$$(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = I^{-1} = I.$$

Therefore, $A^{-1}$ is also orthogonal.

For (b), let $A$ and $B$ be orthogonal matrices. Then

$$(AB)^T AB = B^T A^T AB = B^T B = I$$

and

$$AB(AB)^T = ABB^T A^T = AA^T = I.$$

Thus, $AB$ is also orthogonal.

For (c), we have

$$1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = [\det(A)]^2.$$

Thus, $\det(A) = \pm 1$. □

**Theorem 5.20** *The following are equivalent for a square matrix $A$.*

(a) *$A$ is orthogonal.*

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$.

*Proof* We only prove (a) $\Leftrightarrow$ (c). The proof of (a) $\Leftrightarrow$ (b) is left as an exercise.

(a) $\Rightarrow$ (c): We have for any $\mathbf{x}$ and $\mathbf{y}$,

$$A\mathbf{x} \cdot A\mathbf{y} = (A\mathbf{x})^T A\mathbf{y} = \mathbf{x}^T A^T A\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

(c) $\Rightarrow$ (a): Since for any $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{x}^T A^T A\mathbf{y} = A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T I\mathbf{y},$$

we obtain

$$\mathbf{x}^T (A^T A - I)\mathbf{y} = 0. \tag{5.12}$$

Because (5.12) holds for all vectors $\mathbf{x}$ and $\mathbf{y}$, it follows that

$$A^T A - I = \mathbf{0}, \quad \text{i.e.,} \quad A^T A = I.$$

Therefore, $A$ is orthogonal. □

### 5.5.2 Change of basis

If we change the basis for a vector space $V$ from an old basis $B$ to a new basis $B'$, how is the old coordinate vector $[\mathbf{v}]_B$ of a vector $\mathbf{v} \in V$ related to the new coordinate vector $[\mathbf{v}]_{B'}$? More precisely, let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for $V$. For any vector $\mathbf{v} \in V$, we have

$$\mathbf{v} = \sum_{j=1}^{n} k_j \mathbf{u}_j.$$

Denote the coordinate vector $[\mathbf{v}]_B$ of $\mathbf{v}$ by

$$[\mathbf{v}]_B = [k_1, k_2, \ldots, k_n] \qquad \text{or} \qquad [\mathbf{v}]_B = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

If we change the basis for $V$ from the old basis $B$ to a new basis

$$B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\},$$

then the old coordinate vector $[\mathbf{v}]_B$ of $\mathbf{v}$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ of the same vector $\mathbf{v}$ by the equation

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \tag{5.13}$$

where the columns of the matrix $P$ are the coordinate vectors of the new basis vectors relative to the old basis, i.e.,

$$P = \left[\ [\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \cdots \mid [\mathbf{u}'_n]_B\ \right].$$

The matrix $P$ usually is called the ***transition matrix*** from $B$ to $B'$.

Now we want to show that (5.13) is true. Actually,

$$\mathbf{u}'_i = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n][\mathbf{u}'_i]_B$$

for $1 \leqslant i \leqslant n$. Thus,

$$[\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n] = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n]P.$$

For any $\mathbf{v} \in V$, we have

$$\mathbf{v} = \sum_{j=1}^{n} k'_j \mathbf{u}'_j = [\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n]\begin{bmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{bmatrix} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n]P\begin{bmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{bmatrix} \tag{5.14}$$

and

$$\mathbf{v} = \sum_{j=1}^{n} k_j \mathbf{u}_j = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n]\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}. \tag{5.15}$$

Combining (5.14) and (5.15) yields

$$[\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n]\left[P\begin{bmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}\right] = \mathbf{0}.$$

Since $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly independent, we have

$$P\begin{bmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \mathbf{0}, \qquad \text{i.e.,} \quad P[\mathbf{v}]_{B'} - [\mathbf{v}]_B = \mathbf{0}.$$

Hence

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'}.$$

**Example** Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = [1, 0], \quad \mathbf{u}_2 = [0, 1]; \quad \mathbf{u}'_1 = [1, 2], \quad \mathbf{u}'_2 = [3, 1].$$

(a) Find the transition matrix from $B$ to $B'$.

(b) Use (5.13) to find $[\mathbf{v}]_B$ if $[\mathbf{v}]_{B'} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

*Solution* For (a), we must find the coordinate vectors of the new basis vectors $\mathbf{u}'_1$ and $\mathbf{u}'_2$ relative to the old basis $B$. We have by inspection

$$\begin{cases} \mathbf{u}'_1 = \mathbf{u}_1 + 2\mathbf{u}_2 \\ \mathbf{u}'_2 = 3\mathbf{u}_1 + \mathbf{u}_2 \end{cases}$$

so that

$$[\mathbf{u}'_1]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad [\mathbf{u}'_2]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then the transition matrix $P$ from $B$ to $B'$ is given by

$$P = \left[\, [\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \right] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}.$$

For (b), using (5.13) and the transition matrix in (a), we obtain

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}. \qquad \square$$

In the above example, the transition matrix $Q$ from $B'$ to $B$ is

$$Q = -\frac{1}{5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}.$$

If we compute the product of the two transition matrices above, we find that $PQ = I$, which implies that $Q = P^{-1}$. The following theorem shows that this holds for every case.

**Theorem 5.21** *Let $P$ be the transition matrix from a basis $B$ to a basis $B'$. Then $P$ is invertible and $P^{-1}$ is the transition matrix from $B'$ to $B$.*

*Proof* Let $Q$ be the transition matrix from $B'$ to $B$. Then

$$[\mathbf{x}]_B = P[\mathbf{x}]_{B'}, \qquad [\mathbf{x}]_{B'} = Q[\mathbf{x}]_B.$$

Thus, for any $\mathbf{x}$, we obtain

$$[\mathbf{x}]_B = PQ[\mathbf{x}]_B.$$

Therefore,

$$PQ = I, \qquad \text{i.e.,} \quad Q = P^{-1}.$$

□

Theorem 5.21 illustrates that a transition matrix is always invertible. The following theorem shows that the transition matrix from one orthonormal basis to another orthonormal basis is orthogonal.

**Theorem 5.22** *Let $P$ be the transition matrix from one orthonormal basis to another orthonormal basis for an inner product space $V$. Then $P$ is an orthogonal matrix.*

*Proof* Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be the two orthonormal bases for $V$, and $P = [p_{ij}]$ be the $n \times n$ transition matrix from $B$ to $B'$. Then

$$[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}.$$

Because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis, we have

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{5.16}$$

On the other hand, by using the orthonormal property of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, we have for any $i$ and $j$,

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \left\langle \sum_{s=1}^n \mathbf{u}_s p_{si}, \sum_{r=1}^n \mathbf{u}_r p_{rj} \right\rangle = \sum_{s=1}^n \sum_{r=1}^n p_{si} p_{rj} \langle \mathbf{u}_s, \mathbf{u}_r \rangle = \sum_{s=1}^n p_{si} p_{sj}. \tag{5.17}$$

Combining (5.16) and (5.17), we deduce

$$p_{1i}p_{1j} + p_{2i}p_{2j} + \cdots + p_{ni}p_{nj} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Thus, $P^TP = I$, i.e., $P$ is an orthogonal matrix.

□

# Exercises

### Elementary exercises

**5.1** Let $\mathbf{u} = [u_1, u_2]$, $\mathbf{v} = [v_1, v_2] \in \mathbb{R}^2$. Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 3u_2v_2$$

defines an inner product.

**5.2** Let $\mathbf{p} = a_0 + a_1x + a_2x^2$ and $\mathbf{q} = b_0 + b_1x + b_2x^2$ be any two polynomials in $P_2$.

(a) Show that $\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + a_2b_2$ defines an inner product on $P_2$.

(b) Use the inner product in (a) to find $d(\mathbf{p}, \mathbf{q})$ if

$$\mathbf{p} = -3 + x + x^2, \qquad \mathbf{q} = 1 + 2x - 4x^2.$$

**5.3** Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^n$. Show that for any $A \in \mathbb{R}^{n\times n}$,

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \langle A^T\mathbf{u}, \mathbf{v} \rangle.$$

**5.4** Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a real inner product space. Show that

(a) Theorem 5.2 holds, i.e., $|\langle \mathbf{u}, \mathbf{v} \rangle| \leqslant \|\mathbf{u}\| \cdot \|\mathbf{v}\|$.

(b) The equality holds in the Cauchy-Schwarz inequality if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

**5.5** Prove Theorem 5.4.

**5.6** Consider $\mathbb{R}^3$ with the Euclidean inner product. For which values of $k$, are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

(a) $\mathbf{u} = [1, 3, -4]$, $\mathbf{v} = [-2, k, 6]$. (b) $\mathbf{u} = [k, k, 1]$, $\mathbf{v} = [k, 4, 4]$.

**5.7** Let $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Check whether the vector $\mathbf{u} \in S^\perp$.

(a) $\mathbf{v}_1 = [1, 0, 0, 0]$, $\mathbf{v}_2 = [0, 3, 0, 0]$, $\mathbf{v}_3 = [5, 2, 1, 0]$, and $\mathbf{u} = [0, 0, 0, 1]$.

(b) $\mathbf{v}_1 = [1, -2, 3, 1]$, $\mathbf{v}_2 = [2, 0, 3, 5]$, $\mathbf{v}_3 = [0, 1, 2, 5]$, and $\mathbf{u} = [0, 1, 3, 0]$.

(c) $\mathbf{v}_1 = [3, 4, 1, 7]$, $\mathbf{v}_2 = [1, 0, 3, 1]$, $\mathbf{v}_3 = [-1, 2, -1, 1]$, and $\mathbf{u} = [-1, -1, 0, 1]$.

**5.8** Show that if $\mathbf{u}$ is orthogonal to each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then it is also orthogonal to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

**5.9** Prove Theorem 5.9 (a) and (b).

**5.10** Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthogonal set in an inner product space $V$. Show that

$$\|\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n\|^2 = \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 + \cdots + \|\mathbf{u}_n\|^2.$$

**5.11** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space $V$. Show that

$$\|a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2,$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

**5.12** Find the orthogonal projection of $\mathbf{v}$ onto the subspace $W$ spanned by vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{v} = [1, 2, 3]$, $\mathbf{u}_1 = [2, -2, 1]$, and $\mathbf{u}_2 = [-1, 1, 4]$.

**5.13** Let $\mathbf{u} = [1, 1, 1, 1]$ and $S = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = [0, -3, 2, 2], \quad \mathbf{v}_2 = [1, -1, 0, 1], \quad \mathbf{v}_3 = [3, 0, -2, 1], \quad \mathbf{v}_4 = [1, 2, -2, -1].$$

(a) Find a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ that forms a basis for the space $S$.

(b) Express each vector which is not in the basis as a linear combination of the basis vectors.

(c) Find the orthogonal projection of $\mathbf{u}$ onto $S$ and the component of $\mathbf{u}$ orthogonal to $S$.

**5.14** Consider $\mathbb{R}^3$ with the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = [1, 0, 0], \qquad \mathbf{u}_2 = [0, 4, 1], \qquad \mathbf{u}_3 = [3, 7, -2]$$

into an orthonormal basis.

**5.15** Consider $\mathbb{R}^4$ with the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = [1, 2, 0, -1], \quad \mathbf{u}_2 = [1, 0, 0, 1], \quad \mathbf{u}_3 = [0, 2, 1, 0], \quad \mathbf{u}_4 = [1, -1, 0, 0]$$

into an orthonormal basis.

**5.16** Find an orthogonal basis for $\mathbb{R}^3$ that contains a vector $\mathbf{v}_1 = [0, 2, 3]$.

**5.17** Find an orthonormal basis for $\mathbb{R}^4$ that contains vectors

$$\mathbf{v}_1 = [1, 0, 0, 0], \quad \mathbf{v}_2 = \left[0, \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right].$$

**5.18** Find the $QR$-decomposition of the matrix.

(a) $\begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$.

**5.19** Let $\mathbf{u}_1 = [1, 2, -1]$, $\mathbf{u}_2 = [5, -2, 1]$, and $\mathbf{v} = [3, 2, 5]$.

(a) Find the best approximation to $\mathbf{v}$ from $W = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(b) Find $\min_{\mathbf{w} \in W} \|\mathbf{v} - \mathbf{w}\|$.

**5.20** Find the least squares solution of each linear system $A\mathbf{x} = \mathbf{b}$, and find the orthogonal projection of $\mathbf{b}$ on the column space of $A$.

(a) $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 1 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. (b) $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

**5.21** Determine which of the following matrices are orthogonal.

(a) $\dfrac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$. (b) $\dfrac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$.

**5.22** Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $C = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = [1, 0], \quad \mathbf{u}_2 = [1, 1]; \quad \mathbf{v}_1 = [1, -1], \quad \mathbf{v}_2 = [0, 1].$$

(a) Find the transition matrix $P$ from $B$ to $C$, and the transition matrix $Q$ from $C$ to $B$.

(b) Find the coordinate vector $[\mathbf{x}]_C$ if $\mathbf{x} = [3, -4]$.

(c) Find $[\mathbf{x}]_B$ according to the coordinate vector $[\mathbf{x}]_C$ in (b).

**5.23** Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $C = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{u}_1 = [1, 0, -1], \quad \mathbf{u}_2 = [2, 1, 1], \quad \mathbf{u}_3 = [1, 1, 1];$$

$$\mathbf{v}_1 = [-1, 1, 0], \quad \mathbf{v}_2 = [0, 1, 1], \quad \mathbf{v}_3 = [1, 2, 1].$$

(a) Find the transition matrix $P$ from $B$ to $C$, and the transition matrix $Q$ from $C$ to $B$.

(b) Find $[\mathbf{w}]_B$ if $[\mathbf{w}]_C = [1, -3, -5]$.

## Challenge exercises

**5.24** Show that

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

holds for any vectors $\mathbf{u}$ and $\mathbf{v}$ in a real inner product space.

**5.25** Let $f(x)$ and $g(x)$ be two functions in $C[0,1]$. Show that

$$\left(\int_0^1 f(x)g(x)dx\right)^2 \leqslant \left(\int_0^1 f^2(x)dx\right)\left(\int_0^1 g^2(x)dx\right)$$

and

$$\left(\int_0^1 [f(x)+g(x)]^2 dx\right)^{\frac{1}{2}} \leqslant \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}} + \left(\int_0^1 g^2(x)dx\right)^{\frac{1}{2}}.$$

**5.26** Find the nullspace $S$ of the augmented matrix of the following system and then find $S^{\perp}$.

$$\begin{cases} x_1 - 2x_2 + 3x_3 - 4x_4 = 0 \\ x_1 + 5x_2 + 3x_3 + 3x_4 = 0. \end{cases}$$

**5.27** Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace $W$ of $\mathbb{R}^n$. Show that for any $\mathbf{y} \in \mathbb{R}^n$,

$$\text{proj}_W \mathbf{y} = UU^T\mathbf{y},$$

where $U$ is an $n \times p$ matrix given by $U = \left[\begin{array}{c:c:c:c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{array}\right]$.

**5.28** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthonormal set in $\mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Show that

$$\|\mathbf{x}\|^2 \geqslant \langle\mathbf{x}, \mathbf{v}_1\rangle^2 + \langle\mathbf{x}, \mathbf{v}_2\rangle^2 + \cdots + \langle\mathbf{x}, \mathbf{v}_k\rangle^2.$$

**5.29** Let $A$ be a symmetric matrix. Show that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to the nullspace of $A$.

**5.30** Let $W$ be the plane $x - y + 2z = 0$ in $\mathbb{R}^3$, and $\mathbf{v} = [3, -1, 2]$.

(a) Find the orthogonal projection of $\mathbf{v}$ onto $W$.

(b) Find the component of $\mathbf{v}$ orthogonal to $W$.

**5.31** Let $A\mathbf{x} = \mathbf{b}$ be consistent. Show that if $A$ has linearly independent column vectors, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ is the same as the exact solution of $A\mathbf{x} = \mathbf{b}$.

**5.32** Let $A$ be a matrix with linearly independent column vectors and $\mathbf{b}$ be a vector orthogonal to the column space of $A$. Show that the least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{0}$.

**5.33** Show that $A$ is orthogonal if and only if $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$.

**5.34** Let $B$ and $C$ be bases for $\mathbb{R}^2$ and $C = \{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = [1, 2]$, $\mathbf{u}_2 = [2, 3]$. Find the basis $B$ if the transition matrix from $B$ to $C$ is

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

**5.35** Let $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| = 1$. Suppose that $\mathbf{x}$ has a partition as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ -- \\ \mathbf{y} \end{bmatrix},$$

where $\mathbf{y} \in \mathbb{R}^{n-1}$, and

$$Q = \begin{bmatrix} x_1 & \mathbf{y}^T \\ \mathbf{y} & I - \left(\dfrac{1}{1 - x_1}\right) \mathbf{y}\mathbf{y}^T \end{bmatrix},$$

where $I$ is the $(n-1) \times (n-1)$ identity matrix. Show that $Q$ is orthogonal.

# Chapter 6

# Eigenvalues and Eigenvectors

"*Eigenvalues are in everything. There is an eigenvalue in the burrito you are going to eat for lunch today.*"

— A linear algebra professor

"*Nature hides her secrets because of her essential loftiness, but not by means of ruse.*"

— Albert Einstein

In this chapter, we study eigenvalues and eigenvectors of a square matrix $A$. An eigenvector $\mathbf{x} \neq \mathbf{0}$ of $A$ is a special vector which does not change its direction when it is multiplied by $A$, i.e., $A\mathbf{x} = \lambda\mathbf{x}$ for some value $\lambda$. Such a value is then called an eigenvalue of $A$. An essential highlight of this chapter is the diagonalization problem.

## 6.1 Eigenvalues and Eigenvectors

We introduce the concepts of eigenvalue, eigenvector, and eigenspace and then present methods to compute them through examples.

### 6.1.1 Introduction to eigenvalues and eigenvectors

**Definition** *Let $A$ be a square matrix. If a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, where $\lambda$ is a scalar, then $\lambda$ is called an **eigenvalue** of $A$ and $\mathbf{x}$ is called an **eigenvector** of $A$ corresponding to $\lambda$.*

We observe that if $A$ is an $n \times n$ matrix and $\lambda$ is a scalar, then the following are equivalent.

(1) There exists $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

(2) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

(3) $\det(\lambda I - A) = 0$.

The equation $\det(\lambda I - A) = 0$ is called the ***characteristic equation*** of $A$. Any scalar satisfying the equation $\det(\lambda I - A) = 0$ is an eigenvalue of $A$. Obviously, if $A$ is a triangular matrix, then the eigenvalues of $A$ are the entries on the main diagonal of $A$. In fact, for an $n \times n$ matrix $A$, $\det(\lambda I - A)$ is a polynomial in $\lambda$ of degree $n$, which is called the ***characteristic polynomial*** of $A$.

**Example** Find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

*Solution* The characteristic polynomial of $A$ is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda - 3 \end{bmatrix} = \lambda^3 - 6\lambda^2 - 15\lambda - 8.$$

The eigenvalues of $A$ therefore satisfy

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0 \quad \Longrightarrow \quad (\lambda - 8)(\lambda + 1)^2 = 0.$$

Thus, the distinct eigenvalues of $A$ are

$$\lambda = 8, \qquad \lambda = -1.$$

□

**Remark** A real matrix may have complex eigenvalues. For instance, let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of $A$ is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1.$$

The eigenvalues of $A$ are the roots of $\lambda^2 + 1 = 0$ and therefore $\lambda = \mathbf{i}$ and $\lambda = -\mathbf{i}$, where $\mathbf{i}^2 = -1$. For a review of complex numbers, we refer to Subsection 8.3.1 for details.

### 6.1.2 Two theorems concerned with eigenvalues

We list two theorems concerned with some properties of eigenvalues and eigenvectors. The first theorem indicates a simple way to find the eigenvalues and eigenvectors of any positive integer powers of a matrix $A$ once the eigenvalues and eigenvectors of $A$ are found. The second one demonstrates a relationship between eigenvalues and the invertibility of a matrix.

**Theorem 6.1** *Let $k$ be a positive integer, $\lambda$ be an eigenvalue of a matrix $A$, and $\mathbf{x}$ be a corresponding eigenvector. Then $\lambda^k$ is an eigenvalue of $A^k$ and $\mathbf{x}$ is a corresponding eigenvector.*

*Proof* Since $A\mathbf{x} = \lambda\mathbf{x}$, we have

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}.$$

By induction, one can prove easily that $A^k\mathbf{x} = \lambda^k\mathbf{x}$ for any positive integer $k$. □

**Theorem 6.2** *A matrix $A$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $A$.*

*Proof* In fact, the statement in the theorem is equivalent to that $A$ is not invertible if and only if $\lambda = 0$ is an eigenvalue of $A$. So we consider its equivalent statement. If $\lambda = 0$ is an eigenvalue of $A$, then

$$A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$$

has nonzero solution $\mathbf{x}$, i.e., $A$ is not invertible. Conversely, it is obviously true. □

### 6.1.3 Bases for eigenspaces

We know that the eigenvectors corresponding to $\lambda$ are the nonzero vectors in the solution space of $(\lambda I - A)\mathbf{x} = \mathbf{0}$. This solution space is called the ***eigenspace*** of $A$ corresponding to $\lambda$. The following example is of finding bases for the eigenspaces of a matrix $A$.

**Example** Find bases for the eigenspaces of

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$

*Solution* The characteristic equation of $A$ is

$$\lambda^3 - 4\lambda^2 - 3\lambda + 18 = (\lambda + 2)(\lambda - 3)^2 = 0.$$

Thus, the distinct eigenvalues of $A$ are $\lambda = -2$ and $\lambda = 3$. There are two eigenspaces of $A$. By definition, we know that $\mathbf{x} = [x_1, x_2, x_3]^T$ is an eigenvector of $A$ corresponding to $\lambda$ if and only if $\mathbf{x}$ is a nontrivial solution of

$$(\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda - 1 & -1 & -3 \\ 0 & \lambda - 3 & 0 \\ -2 & -2 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.1}$$

If $\lambda = -2$, then (6.1) becomes

$$\begin{bmatrix} -3 & -1 & -3 \\ 0 & -5 & 0 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this system yields

$$x_1 = -s, \quad x_2 = 0, \quad x_3 = s.$$

Thus, the eigenvectors corresponding to $\lambda = -2$ are the nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $[-1, 0, 1]^T$ is a basis for the eigenspace corresponding to $\lambda = -2$.

If $\lambda = 3$, then (6.1) becomes

$$\begin{bmatrix} 2 & -1 & -3 \\ 0 & 0 & 0 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this system yields

$$x_1 = 3s, \quad x_2 = 0, \quad x_3 = 2s.$$

Thus, the eigenvectors of $A$ corresponding to $\lambda = 3$ are the nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} 3s \\ 0 \\ 2s \end{bmatrix} = s \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

Hence $[3, 0, 2]^T$ is a basis for the eigenspace corresponding to $\lambda = 3$. $\square$

## 6.2 Diagonalization

We are concerned with the problem of finding a basis for $\mathbb{R}^n$ that includes all eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ because such a basis helps us to simplify numerical computations involving $A$. In this section, our goal is to show that this problem is actually equivalent to a diagonalization problem.

### 6.2.1 Diagonalization problem

Given a square matrix $A$, does there exist an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix? Such kind of problem is called the diagonalization problem. A square matrix $A$ is called ***diagonalizable*** if there is an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix.

**Theorem 6.3** *Let $A$ be an $n \times n$ matrix. Then the following are equivalent.*

(a) *$A$ is diagonalizable.*

(b) *$A$ has $n$ linearly independent eigenvectors.*

*Proof* (a) $\Rightarrow$ (b): Since $A$ is diagonalizable, there exists an invertible matrix $P$ such that

$$P^{-1}AP = D,$$

where $D$ is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, i.e.,

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Let

$$P = \left[\begin{array}{c:c:c:c} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{array}\right],$$

where $\mathbf{p}_i$ $(1 \leqslant i \leqslant n)$ are the column vectors of $P$. Then from $P^{-1}AP = D$, we have

$$AP = PD,$$

which implies

$$\left[\begin{array}{c:c:c:c} A\mathbf{p}_1 & A\mathbf{p}_2 & \cdots & A\mathbf{p}_n \end{array}\right] = \left[\begin{array}{c:c:c:c} \lambda_1\mathbf{p}_1 & \lambda_2\mathbf{p}_2 & \cdots & \lambda_n\mathbf{p}_n \end{array}\right].$$

Therefore, for any $i$,

$$A\mathbf{p}_i = \lambda_i\mathbf{p}_i,$$

i.e., $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are eigenvectors of $A$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Since $P$ is invertible, the column vectors $\mathbf{p}_i$ $(1 \leqslant i \leqslant n)$ are linearly independent. Thus, $A$ has $n$ linearly independent eigenvectors. Conversely, one can show that it is also true. $\square$

### 6.2.2 Procedure for diagonalization

Recall that an $n \times n$ matrix $A$ with $n$ linearly independent eigenvectors is diagonalizable. The following procedure provides a method for diagonalizing $A$.

(1) Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A$.

(2) If there are $n$ linearly independent eigenvectors of $A$, say, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we can construct a matrix

$$P = \left[\begin{array}{c:c:c:c} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{array}\right].$$

(3) The matrix $P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

**Example 1** Find a matrix $P$ that diagonalizes

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}.$$

*Solution* The characteristic equation of $A$ is

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0.$$

Thus, the eigenvalues of $A$ are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$. The corresponding eigenvectors are given as follows:

$$\lambda = 1, \quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}; \qquad \lambda = 2, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \qquad \lambda = 3, \quad \mathbf{p}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

In fact, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent. We can construct an invertible matrix

$$P = \left[ \begin{array}{c:c:c} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{array} \right] = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

such that

$$P^{-1}AP = \operatorname{diag}(1, 2, 3).$$

□

**Example 2** Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Is $A$ diagonalizable?

*Solution* The characteristic equation of $A$ is

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 2)^2 = 0.$$

Thus, the distinct eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$. The corresponding eigenvectors are given as follows:

$$\lambda = 1, \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \qquad \lambda = 2, \quad \mathbf{p}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Since $A$ is a $3 \times 3$ matrix and there are only two linearly independent eigenvectors in total, $A$ is not diagonalizable. □

**Remark** If an $n \times n$ matrix $A$ is diagonalizable, then it is much easier for us to compute the power of $A$. More precisely, if $P^{-1}AP = D$, where

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

we then have

$$A = PDP^{-1},$$

which implies

$$A^k = \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{k} = P\underbrace{DD\cdots D}_{k}P^{-1} = PD^kP^{-1},$$

where $D^k = \operatorname{diag}(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)$.

### 6.2.3 Two theorems concerned with diagonalization

From the examples in previous subsection, one may guess that basis vectors from various eigenspaces of $A$ are linearly independent. The following theorem gives the proof.

**Theorem 6.4** *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be eigenvectors of $A$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.*

*Proof* By contradiction, we assume that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent. Since an eigenvector is nonzero by definition, $\{\mathbf{v}_1\}$ is linearly independent. Without loss of generality, let $r$ be the largest integer such that

$$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$$

is linearly independent. Then we have $1 \leqslant r < k$. Moreover, $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{r+1}\}$ is linearly dependent. Thus, there are scalars $c_1, c_2, \ldots, c_{r+1}$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}. \tag{6.2}$$

Multiplying both sides of (6.2) by $A$ and using

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \quad \ldots, \quad A\mathbf{v}_{r+1} = \lambda_{r+1}\mathbf{v}_{r+1},$$

we deduce

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{v}_{r+1} = \mathbf{0}. \tag{6.3}$$

Multiplying both sides of (6.2) by $\lambda_{r+1}$ and subtracting the resulting equation from (6.3) yields

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{r+1})\mathbf{v}_2 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v}_r = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent, this equation implies that

$$c_1(\lambda_1 - \lambda_{r+1}) = c_2(\lambda_2 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0.$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$ are distinct, we have

$$c_1 = c_2 = \cdots = c_r = 0. \tag{6.4}$$

Substituting these values into (6.2) yields

$$c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}.$$

Note that the eigenvector $\mathbf{v}_{r+1}$ is nonzero and it follows that

$$c_{r+1} = 0. \tag{6.5}$$

Equations (6.4) and (6.5) contradict the fact that $c_1, c_2, \ldots, c_{r+1}$ are not all zero. The proof is completed. □

As a result of the theorem above, we have the following important theorem.

**Theorem 6.5** *Let an $n \times n$ matrix $A$ have $n$ distinct eigenvalues. Then $A$ is diagonalizable.*

*Proof* Since $A$ has $n$ distinct eigenvalues, by Theorem 6.4, we know that there are $n$ linearly independent eigenvectors of $A$. It follows from Theorem 6.3 that $A$ is diagonalizable. □

## 6.3 Orthogonal Diagonalization

In this section, we focus on another problem of finding an orthonormal basis that consists of eigenvectors of a square matrix. Equivalently, we study a diagonalization employing an orthogonal matrix. Given $A \in \mathbb{R}^{n\times n}$, does there exist an orthogonal matrix $P$ such that

$$P^{-1}AP = P^TAP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)?$$

Such kind of problem is called the orthogonal diagonalization problem. A square matrix $A$ is called ***orthogonally diagonalizable*** if there is an orthogonal matrix $P$ such that $P^TAP$ is a diagonal matrix.

**Theorem 6.6** *For an $n \times n$ matrix $A$, the following are equivalent.*

(a) *$A$ is orthogonally diagonalizable.*

(b) *$A$ has an orthonormal set of $n$ eigenvectors.*

(c) *$A$ is symmetric.*

*Proof* (a) $\Rightarrow$ (b): Since $A$ is orthogonally diagonalizable, there is an orthogonal matrix $P$ such that $P^{-1}AP$ is diagonal. As shown in the proof of Theorem 6.3, the $n$ column vectors of $P$ are eigenvectors of $A$. Since $P$ is orthogonal, by Theorem 5.18 these column vectors are orthonormal. Hence $A$ has $n$ orthonormal eigenvectors.

(b) $\Rightarrow$ (a): Assume that $A$ has $n$ orthonormal eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$, i.e.,

$$A\mathbf{p}_j = \lambda_j \mathbf{p}_j, \qquad 1 \leqslant j \leqslant n.$$

Construct a matrix with $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ as column vectors:

$$P = \left[\begin{array}{c:c:c:c} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{array}\right].$$

We then have

$$AP = PD,$$

where $P$ is orthogonal and $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Thus, $P^{-1}AP$ is a diagonal matrix, i.e., $A$ is orthogonally diagonalizable.

(a) $\Rightarrow$ (c): Since $A$ is orthogonally diagonalizable, there exists an orthogonal matrix $P$ such that

$$P^T AP = D,$$

where $D$ is a diagonal matrix. It implies

$$A = PDP^T.$$

It follows that

$$A^T = \left(PDP^T\right)^T = PD^T P^T = PDP^T = A.$$

Thus, $A$ is symmetric.

(c) $\Rightarrow$ (a): We prove this by using induction. If $A$ is a $1 \times 1$ matrix, then (c) obviously implies (a). Suppose that all $(n-1) \times (n-1)$ symmetric matrices can be orthogonally diagonalizable. Now we consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Let $\lambda$ be an eigenvalue of $A$ and $\mathbf{v}$ be the corresponding eigenvector. Since $\mathbf{v}$ is a nonzero vector, we construct a unit vector

$$\mathbf{v}' = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Then $\mathbf{v}'$ is an eigenvector of $A$ with $\|\mathbf{v}'\| = 1$. We can always find vectors $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n-1}$ together with $\mathbf{v}'$ to form an orthonormal basis for $\mathbb{R}^n$. Construct a matrix with $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n-1}$ as column vectors:

$$Y = \left[\begin{array}{c:c:c:c} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_{n-1} \end{array}\right] \in \mathbb{R}^{n \times (n-1)}.$$

We then have

$$Y^TY = \begin{bmatrix} \mathbf{y}_1^T \\ -- \\ \mathbf{y}_2^T \\ -- \\ \vdots \\ -- \\ \mathbf{y}_{n-1}^T \end{bmatrix} [\ \mathbf{y}_1 \mid \mathbf{y}_2 \mid \cdots \mid \mathbf{y}_{n-1}\ ] = \begin{bmatrix} \mathbf{y}_1^T\mathbf{y}_1 & \mathbf{y}_1^T\mathbf{y}_2 & \cdots & \mathbf{y}_1^T\mathbf{y}_{n-1} \\ \mathbf{y}_2^T\mathbf{y}_1 & \mathbf{y}_2^T\mathbf{y}_2 & \cdots & \mathbf{y}_2^T\mathbf{y}_{n-1} \\ \vdots & \vdots & & \vdots \\ \mathbf{y}_{n-1}^T\mathbf{y}_1 & \mathbf{y}_{n-1}^T\mathbf{y}_2 & \cdots & \mathbf{y}_{n-1}^T\mathbf{y}_{n-1} \end{bmatrix}$$

$$= I_{n-1}$$

and

$$\mathbf{v}'^TY = [\ \mathbf{v}'^T\mathbf{y}_1 \mid \mathbf{v}'^T\mathbf{y}_2 \mid \cdots \mid \mathbf{v}'^T\mathbf{y}_{n-1}\ ] = \mathbf{0}^T \in \mathbb{R}^{1\times(n-1)}. \tag{6.6}$$

Note that the matrix $Y^TAY \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric. Then by the inductive hypothesis there exists an orthogonal matrix $P \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$P^TY^TAYP = D \tag{6.7}$$

is diagonal. Constructing

$$B = [\ \mathbf{v}' \mid YP\ ] \in \mathbb{R}^{n\times n},$$

we therefore have by using (6.6),

$$B^TB = \begin{bmatrix} \mathbf{v}'^T \\ --- \\ (YP)^T \end{bmatrix} [\ \mathbf{v}' \mid YP\ ] = \begin{bmatrix} \mathbf{v}'^T\mathbf{v}' & \mathbf{v}'^TYP \\ (\mathbf{v}'^TYP)^T & (YP)^TYP \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_{n-1} \end{bmatrix} = I_n.$$

Thus, $B$ is orthogonal. By (6.6) and (6.7), we obtain

$$\mathbf{v}'^TAYP = (A^T\mathbf{v}')^TYP = (A\mathbf{v}')^TYP = \lambda\mathbf{v}'^TYP = \mathbf{0}^T \tag{6.8}$$

and

$$(YP)^TAYP = P^TY^TAYP = D. \tag{6.9}$$

Finally, it follows from (6.8) and (6.9) that

$$B^TAB = \begin{bmatrix} \mathbf{v}'^TA\mathbf{v}' & \mathbf{v}'^TAYP \\ (\mathbf{v}'^TAYP)^T & (YP)^TAYP \end{bmatrix} = \begin{bmatrix} \mathbf{v}'^TA\mathbf{v}' & \mathbf{0}^T \\ \mathbf{0} & D \end{bmatrix},$$

i.e., $B^TAB$ is diagonal. The proof is completed. □

Before we develop a procedure of orthogonal diagonalization, we need the following important theorem about eigenvalues and eigenvectors of symmetric matrices. We defer the proof of the theorem until Chapter 8 (see Theorem 8.12).

**Theorem 6.7** *Let $A$ be a symmetric matrix. Then*

(a) *The eigenvalues of $A$ are all real.*

(b) *Eigenvectors from different eigenspaces are orthogonal.*

Therefore, combining the above theorems, we provide the following method for diagonalizing a symmetric matrix $A$.

(1) Find a basis for each eigenspace of $A$.

(2) Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

(3) Find the matrix $P$ whose columns are the basis vectors constructed in (2). Then $P^T AP$ is diagonal.

**Example** Find a matrix $P$ that orthogonally diagonalizes

$$A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

*Solution* The characteristic equation of $A$ is

$$\det(\lambda I - A) = (\lambda + 1)^2(\lambda - 2) = 0,$$

and the distinct eigenvalues are $\lambda = -1$ and $\lambda = 2$. The following are bases for the eigenspaces:

$$\lambda = -1, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \qquad \lambda = 2, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

There are three basis vectors in total. Applying the Gram-Schmidt process to each of these bases, we have the following bases for each eigenspaces:

$$\lambda = -1, \quad \mathbf{p}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}; \qquad \lambda = 2, \quad \mathbf{p}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Construct an orthogonal matrix

$$P = \left[\begin{array}{c:c:c} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{array}\right] = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{3}} \\ 0 & \dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \end{bmatrix}.$$

It is easy to verify

$$P^T AP = \operatorname{diag}(-1, -1, 2). \qquad \square$$

## 6.4 Jordan Decomposition Theorem

The following theorem is fundamental and useful in linear algebra.

**Theorem 6.8** (*Jordan Decomposition Theorem*) *Let $A$ be any $n \times n$ matrix. Then there exists an invertible matrix $X$ such that*

$$X^{-1}AX = J := \begin{bmatrix} J_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_2 & \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & J_3 & \ddots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & J_p \end{bmatrix},$$

*where $J_i$ is an $n_i \times n_i$ matrix for $1 \leqslant i \leqslant p$ given by*

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ 0 & 0 & \lambda_i & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

*with $\lambda_i$ $(1 \leqslant i \leqslant p)$ being the eigenvalues of $A$ and $n_1 + n_2 + \cdots + n_p = n$. The matrix $J$ is called the **Jordan canonical form** of $A$ and $J_i$ $(1 \leqslant i \leqslant p)$ are called **Jordan blocks**. The Jordan canonical form of $A$ is unique up to the permutation of diagonal Jordan blocks.*

The proof of the Jordan decomposition theorem is beyond the scope of this text. We refer the interested readers to [14, pp. 164–171].

**Remark** It is well-known that there exist some square matrices which are not diagonalizable. However, the Jordan decomposition theorem tells us that for every square matrix $A$, there exists an invertible matrix $X$ such that $X^{-1}AX$ is a bi-diagonal matrix.

**Example 1** Let

$$J = \left[\begin{array}{cc:c:ccc} 2 & 1 & & & & \\ & 2 & & & & \\ \hdashline & & 4 & & & \\ \hdashline & & & 3 & 1 & \\ & & & & 3 & 1 \\ & & & & & 3 \end{array}\right]$$

be the Jordan canonical form of a $6 \times 6$ matrix $A$. Then the Jordan blocks are

$$J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \qquad J_2 = [4], \qquad J_3 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Moreover, from the Jordan canonical form of $A$, we find that $A$ has three distinct eigenvalues $\lambda = 2, \lambda = 4$, and $\lambda = 3$.

**Example 2** Let

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 1 & -1 \\ 2 & 4 & 5 \end{bmatrix}.$$

Is it diagonalizable? Find its Jordan canonical form.

*Solution* The characteristic equation of $A$ is

$$\det(\lambda I - A) = (\lambda - 2)^2(\lambda - 3) = 0.$$

Therefore, the distinct eigenvalues and the corresponding eigenvectors of $A$ are

$$\lambda = 2, \quad \mathbf{p}_1 = [-1, -1, 2]^T; \qquad \lambda = 3, \quad \mathbf{p}_2 = [-1, 0, 1]^T.$$

Since $A$ is a $3 \times 3$ matrix and has only two linearly independent eigenvectors, $A$ is not diagonalizable. However, we can find an invertible matrix

$$X = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

such that

$$X^{-1}AX = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 3 \end{array}\right]$$

is a bi-diagonal matrix (the Jordan canonical form of $A$). □

## Exercises

### Elementary exercises

**6.1** Find the eigenvalues and bases for the eigenspaces of the following matrices.

$$\text{(a)} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}. \quad \text{(b)} \begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}. \quad \text{(c)} \begin{bmatrix} 1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

**6.2** For a positive integer $k \geqslant 2$, compute

$$\text{(a)} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^k. \quad \text{(b)} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^k. \quad \text{(c)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^k. \quad \text{(d)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^k.$$

**6.3** Find the eigenvalues and bases for the eigenspaces of $A^9$, where

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

**6.4** Suppose that

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

(a) Find $A^{10}\mathbf{b}$.

(b) Find $A^{10}$ and check whether your solution of (a) is true or not.

**6.5** Find a matrix $A \in \mathbb{R}^{3\times 3}$ such that

$$A\mathbf{u}_1 = \mathbf{u}_1, \qquad A\mathbf{u}_2 = 2\mathbf{u}_2, \qquad A\mathbf{u}_3 = 3\mathbf{u}_3,$$

where $\mathbf{u}_1 = [1, 2, 2]^T$, $\mathbf{u}_2 = [2, -2, 1]^T$, and $\mathbf{u}_3 = [-2, -1, 2]^T$.

**6.6** Show that if

$$A = \begin{bmatrix} 0 & 0 & 1 \\ a & 1 & b \\ 1 & 0 & 0 \end{bmatrix}$$

has three linearly independent eigenvectors, then $a + b = 0$.

**6.7** Show that if $\lambda$ is an eigenvalue of an invertible matrix $A$ and $\mathbf{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of $A^{-1}$ and $\mathbf{x}$ is a corresponding eigenvector.

**6.8** Show that if $\lambda$ is an eigenvalue of a matrix $A$, $\mathbf{x}$ is a corresponding eigenvector, and $\alpha$ is a scalar, then $\lambda - \alpha$ is an eigenvalue of $A - \alpha I$ and $\mathbf{x}$ is a corresponding eigenvector.

**6.9** Show that if $\lambda$ is an eigenvalue of an invertible matrix $A$ and $\mathbf{x}$ is a corresponding eigenvector, then $\det(A)/\lambda$ is an eigenvalue of $\mathrm{adj}(A)$ and $\mathbf{x}$ is a corresponding eigenvector.

**6.10** Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

Find the eigenvalues and bases for the eigenspaces of $A$, $A^{-1}$, $A - 2I$, and $A + 3I$, where $I$ is the $3 \times 3$ identity matrix.

**6.11** Let $A \in \mathbb{R}^{n \times n}$. Show that if $\lambda$ is an eigenvalue of $A$, then $\lambda^3 + 3\lambda^2 - 2\lambda + 5$ is an eigenvalue of the matrix $A^3 + 3A^2 - 2A + 5I$, where $I$ is the $n \times n$ identity matrix.

**6.12** Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & a \\ 4 & 0 & 5 \end{bmatrix}.$$

Find the value of $a$ such that $A$ is diagonalizable.

**6.13** Let $A \in \mathbb{R}^{n \times n}$ with $A^m = \mathbf{0}$ for some $m > 1$. Show that if $A$ is diagonalizable, then $A$ must be the zero matrix.

**6.14** Determine whether $A$ is diagonalizable. If so, find an invertible matrix $P$ that diagonalizes $A$, and determine $P^{-1}AP$.

(a) $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. (c) $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

**6.15** Let $A$ be a diagonalizable matrix.

(a) Show that $A^T$ is also a diagonalizable matrix.

(b) Show that if $A$ is invertible, then $A^{-1}$ is diagonalizable.

**6.16** Find $A^2$ and $A^6$, where

$$A = \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

**6.17** Show that if $A$ is a symmetric matrix, then all eigenvalues of $A$ are nonnegative if and only if there exists a symmetric matrix $B$ such that $A = B^2$.

**6.18** Find a matrix $P$ that orthogonally diagonalizes each of the following matrices.

$$\text{(a)} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}. \qquad \text{(b)} \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}.$$

**6.19** If $b \neq 0$, find a matrix $P$ that orthogonally diagonalizes

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

**6.20** Let $A, B \in \mathbb{R}^{n \times n}$ be two orthogonally diagonalizable matrices.

(a) Show that $A + B$ is orthogonally diagonalizable.

(b) Show that if $AB = BA$, then $AB$ is orthogonally diagonalizable.

**6.21** Show that if $\mathbf{v}$ is any $n \times 1$ matrix and $I$ is the $n \times n$ identity matrix, then $I - \mathbf{v}\mathbf{v}^T$ is orthogonally diagonalizable.

### Challenge exercises

**6.22** Find $\det(A)$ given that $A$ has $p(\lambda)$ as its characteristic polynomial.

$$\text{(a) } p(\lambda) = \lambda^3 + 2\lambda^2 - \lambda + 4. \qquad \text{(b) } p(\lambda) = \lambda^4 + 3\lambda^3 + 6.$$

**6.23** Show that the characteristic equation of $A \in \mathbb{R}^{2 \times 2}$ can be expressed as

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0.$$

**6.24** Let $A \in \mathbb{R}^{n\times n}$. Show that $A$ and $A^T$ have the same eigenvalues and may not have the same eigenspaces.

**6.25** Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Show that for any positive integer $k \geqslant 2$,

$$\operatorname{tr}(A^k) = \operatorname{tr}(A^{k-1}) + \operatorname{tr}(A^{k-2}).$$

**6.26** Let $\lambda_1$ and $\lambda_2$ be two distinct eigenvalues of a matrix $A$, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of $A$ corresponding to $\lambda_1$ and $\lambda_2$, respectively. Show that $\mathbf{v}_1 + \mathbf{v}_2$ is not an eigenvector of $A$.

**6.27** Let $\lambda_1$ and $\lambda_2$ be two distinct eigenvalues of a matrix $A$, and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$ be linearly independent eigenvectors of $A$ corresponding to $\lambda_1$ and $\lambda_2$, respectively. Show that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ is linearly independent.

**6.28** Let $A = \mathbf{u}\mathbf{v}^T$, where $\mathbf{u} = [1, 2, 3]^T$ and $\mathbf{v} = [4, 5, 6]^T$. Find $A^n$, where $n$ is an integer and $n > 1$.

**6.29** Show that if $A$ is diagonalizable, then $\operatorname{rank}(A)$ is equal to the number of nonzero eigenvalues of $A$.

**6.30** Let $E_n = [e_{ij}] \in \mathbb{R}^{n\times n}$, where $e_{ij} = 1$ for all $i, j$. Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} \mathbf{0} & E_n \\ E_n & \mathbf{0} \end{bmatrix}.$$

**6.31** Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$, where $m \geqslant n$. Show that

$$\det(\lambda I_m - AB) = \lambda^{m-n} \cdot \det(\lambda I_n - BA).$$

**6.32** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be nonzero column vectors orthogonal to each other. Find all eigenvalues of $A = \mathbf{u}\mathbf{v}^T$ and corresponding eigenvectors.

**6.33** Prove the Cayley-Hamilton theorem [14, pp. 109–111]: If $A \in \mathbb{R}^{n\times n}$ with characteristic equation

$$\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0,$$

where $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$, then

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = \mathbf{0},$$

where $I$ is the $n \times n$ identity matrix.

# Chapter 7

# Linear Transformations

"*We do not need magic to transform our world.*"

— Joanne Rowling

"*Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.*"

— Joseph Fourier

In Chapter 3, we introduced linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this chapter, we will study linear transformations between general vector spaces. The results obtained here many important applications in science and engineering.

## 7.1 General Linear Transformations

In Section 3.2, we defined and studied linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this section, we will define and study the more general concept of a linear transformation from a general vector space to another.

### 7.1.1 Introduction to linear transformations

By inspection of Theorem 3.7 about the linearity conditions of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, we will use these conditions as the starting point to define general linear transformations.

**Definition** *Let $T\colon V \to W$ be a function from a vector space $V$ to a vector space $W$. Then $T$ is called a* ***linear transformation*** *from $V$ to $W$ if for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$ and all scalars $k$:*

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$.

**Examples**

(a) Let $\mathbf{p} = p(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial in $P_n$ and define the function $T\colon P_n \to P_{n+1}$ by

$$T(\mathbf{p}) = T\big(p(x)\big) = xp(x) = c_0 x + c_1 x^2 + \cdots + c_n x^{n+1}.$$

For any polynomials $\mathbf{p}_1, \mathbf{p}_2 \in P_n$ and any scalar $k$, we have

$$\begin{aligned} T(\mathbf{p}_1 + k\mathbf{p}_2) &= T\big(p_1(x) + kp_2(x)\big) = x\big(p_1(x) + kp_2(x)\big) \\ &= xp_1(x) + kxp_2(x) = T(\mathbf{p}_1) + kT(\mathbf{p}_2). \end{aligned}$$

Thus, $T$ is a linear transformation.

(b) Let $V$ be an inner product space and $\mathbf{v}_0 \in V$ be any fixed vector. Let $T\colon V \to \mathbb{R}$ be the transformation that maps a vector $\mathbf{v}$ into its inner product with $\mathbf{v}_0$, i.e.,

$$T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_0 \rangle.$$

From the axioms of an inner product, we have for any $\mathbf{u}, \mathbf{v} \in V$ and any scalar $k$,

$$T(\mathbf{u} + k\mathbf{v}) = \langle \mathbf{u} + k\mathbf{v}, \mathbf{v}_0 \rangle = \langle \mathbf{u}, \mathbf{v}_0 \rangle + k\langle \mathbf{v}, \mathbf{v}_0 \rangle = T(\mathbf{u}) + kT(\mathbf{v}).$$

Hence $T$ is a linear transformation.

(c) Consider the trace defined on $\mathbb{R}^{n\times n}$. For $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{n\times n}$ and any scalar $k$, we have

$$\operatorname{tr}(A + kB) = \sum_{i=1}^{n} (a_{ii} + kb_{ii}) = \sum_{i=1}^{n} a_{ii} + k\sum_{i=1}^{n} b_{ii} = \operatorname{tr}(A) + k\operatorname{tr}(B).$$

Thus, $\operatorname{tr}(\cdot)$ is a linear transformation.

(d) Let $V = C^1(-\infty, \infty)$ be the vector space of all functions with continuous first derivatives on $(-\infty, \infty)$ and $W = C(-\infty, \infty)$ be the vector space of continuous functions defined on $(-\infty, \infty)$. Let $D\colon V \to W$ be the transformation that maps a function $\mathbf{f} = f(x)$ into its derivative, i.e.,

$$D(\mathbf{f}) = f'(x) = \frac{df(x)}{dx}.$$

It follows from the properties of differentiation that for any $\mathbf{f} = f(x), \mathbf{g} = g(x) \in V$ and any scalar $k$,

$$D(\mathbf{f} + k\mathbf{g}) = \big(f(x) + kg(x)\big)' = f'(x) + kg'(x) = D(\mathbf{f}) + kD(\mathbf{g}).$$

Thus, $D$ is a linear transformation.

(e) Let $V = C(-\infty,\infty)$ be the vector space of all continuous functions on $(-\infty,\infty)$, $W = C^1(-\infty,\infty)$ be the vector space of functions with continuous first derivatives on $(-\infty,\infty)$, and $J\colon V \to W$ be the transformation that maps $\mathbf{f} = f(x)$ into its integral, i.e.,

$$J(\mathbf{f}) = \int_0^x f(t)dt.$$

It follows from the properties of integration that for any $\mathbf{f} = f(x), \mathbf{g} = g(x) \in V$ and any scalar $k$,

$$J(\mathbf{f} + k\mathbf{g}) = \int_0^x [f(t) + kg(t)]dt = \int_0^x f(t)dt + k\int_0^x g(t)dt = J(\mathbf{f}) + kJ(\mathbf{g}).$$

Hence $J$ is a linear transformation.

The following theorem lists some basic properties that hold for all linear transformations.

**Theorem 7.1** *Let $T\colon V \to W$ be a linear transformation. Then*

(a) $T(\mathbf{0}) = \mathbf{0}$.

(b) $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v}$ *in* $V$.

(c) $T(\mathbf{v} - \mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w})$ *for all* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $V$.

(d) $T\Big(\sum_{i=1}^{n} k_i\mathbf{v}_i\Big) = \sum_{i=1}^{n} k_iT(\mathbf{v}_i)$ *for all* $\mathbf{v}_i$ *in* $V$ *and all* $k_i$ *in* $\mathbb{R}$ $(1 \leqslant i \leqslant n)$.

*Proof* For (a), we have

$$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}.$$

One can prove the remaining parts easily. □

Let $T\colon V \to W$ be a linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for $V$. Then for any vector $\mathbf{v} \in V$, $T(\mathbf{v})$, the image of $\mathbf{v}$ under $T$, can be calculated from $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$, which are the images of the basis vectors. In fact, let

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

It follows from Theorem 7.1 (d) that

$$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n).$$

### 7.1.2 Compositions

In Subsection 3.2.3, we defined the composition of linear transformations on the Euclidean vector spaces. We extend that concept to general linear transformations.

**Definition** *Let* $T_1: U \to V$ *and* $T_2: V \to W$ *be linear transformations. The* ***composition*** *of* $T_2$ *with* $T_1$*, denoted by* $T_2 \circ T_1$*, is the function defined for any vector* $\mathbf{u}$ *in* $U$ *by the formula*

$$(T_2 \circ T_1)(\mathbf{u}) = T_2\big(T_1(\mathbf{u})\big).$$

The following theorem shows that the composition of two linear transformations is still a linear transformation.

**Theorem 7.2** *Let* $T_1 : U \to V$ *and* $T_2 : V \to W$ *be linear transformations. Then*

$$T_2 \circ T_1 : U \to W$$

*is also a linear transformation.*

*Proof* Since $T_1$ and $T_2$ are linear transformations, we have for any $\mathbf{u}, \mathbf{v} \in U$ and $k \in \mathbb{R}$,

$$\begin{aligned}(T_2 \circ T_1)(\mathbf{u} + k\mathbf{v}) &= T_2\big(T_1(\mathbf{u} + k\mathbf{v})\big) = T_2\big(T_1(\mathbf{u}) + kT_1(\mathbf{v})\big)\\ &= T_2\big(T_1(\mathbf{u})\big) + kT_2\big(T_1(\mathbf{v})\big)\\ &= (T_2 \circ T_1)(\mathbf{u}) + k(T_2 \circ T_1)(\mathbf{v}).\end{aligned}$$

Thus, $T_2 \circ T_1$ is a linear transformation from $U$ to $W$. □

**Example** Let $T_1 : P_1 \to P_2$ and $T_2 : P_2 \to P_2$ be the linear transformations given by the formulas

$$T_1\big(p(x)\big) = xp(x) \quad \text{and} \quad T_2\big(p(x)\big) = p(3x + 2).$$

Then the composition $T_2 \circ T_1 : P_1 \to P_2$ is given by the formula

$$(T_2 \circ T_1)\big(p(x)\big) = T_2\big(T_1(p(x))\big) = T_2\big(xp(x)\big) = (3x + 2)p(3x + 2).$$

In particular, if $p(x) = c_0 + c_1x$, then

$$\begin{aligned}(T_2 \circ T_1)\big(p(x)\big) &= (T_2 \circ T_1)(c_0 + c_1x) = (3x + 2)\big(c_0 + c_1(3x + 2)\big)\\ &= c_0(3x + 2) + c_1(3x + 2)^2.\end{aligned}$$

If $T : V \to V$ is any linear transformation and if $I : V \to V$ is the identity transformation, then for all vectors $\mathbf{v} \in V$,

$$(T \circ I)(\mathbf{v}) = T\big(I(\mathbf{v})\big) = T(\mathbf{v}), \qquad (I \circ T)(\mathbf{v}) = I\big(T(\mathbf{v})\big) = T(\mathbf{v}).$$

It follows that

$$T \circ I = T, \qquad I \circ T = T.$$

We conclude this section by noting that compositions can be defined for more than two linear transformations. For example, let $V_0$, $V_1$, $V_2$, and $V_3$ be vector spaces. If $T_1 \colon V_0 \to V_1$, $T_2 \colon V_1 \to V_2$, and $T_3 \colon V_2 \to V_3$ are linear transformations, then the composition $T_3 \circ T_2 \circ T_1$, defined by

$$(T_3 \circ T_2 \circ T_1)(\mathbf{v}) = T_3\big(T_2(T_1(\mathbf{v}))\big)$$

for $\mathbf{v} \in V_0$, is a linear transformation from $V_0$ to $V_3$. See Figure 7.1.

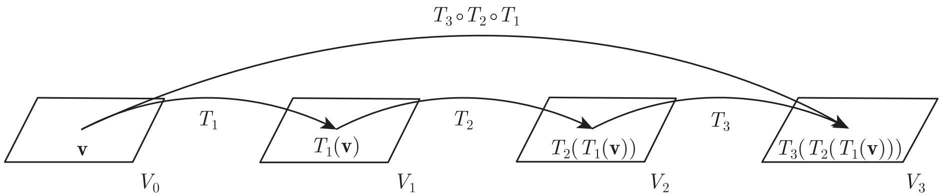


Figure 7.1

In general, if $T_j$ is a linear transformation from the vector space $V_{j-1}$ to another vector space $V_j$ for $1 \leqslant j \leqslant n$, then the composition $T_n \circ T_{n-1} \circ \cdots \circ T_2 \circ T_1$, defined by

$$(T_n \circ T_{n-1} \circ \cdots \circ T_2 \circ T_1)(\mathbf{v}) = T_n\big(T_{n-1} \cdots (T_2(T_1(\mathbf{v})))\big)$$

for $\mathbf{v} \in V_0$, is a linear transformation from $V_0$ to $V_n$.

## 7.2 Kernel and Range

In this section, we develop some fundamental properties of linear transformations.

### 7.2.1 Kernel and range

**Definition** *Let $T \colon V \to W$ be a linear transformation. Then the set of vectors in $V$ that $T$ maps into $\mathbf{0}$ is called the* ***kernel*** *of $T$, denoted by* $\ker(T)$*. The set of all vectors in $W$ that are images under $T$ of at least one vector in $V$ is called the* ***range*** *of $T$, denoted by $R(T)$.*

Note that $\ker(T) \subseteq V$ and $R(T) \subseteq W$.

**Examples**

(a) If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T_A(\mathbf{x}) = A\mathbf{x}$, where $A$ is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$, then $\ker(T_A)$ is the nullspace of $A$, and $R(T_A)$ is the column space of $A$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection on the $xy$-plane. The kernel of $T$ is the set of points that $T$ maps into $\mathbf{0} = [0, 0, 0]$. These are the points on the $z$-axis (Figure 7.2). Since $T$ maps every point in $\mathbb{R}^3$ into $xy$-plane, the range of $T$ must be in this plane. However, every point $[x_0, y_0, 0]$ in the $xy$-plane is the image under $T$ of some points. Actually, it is the image of all points on the vertical line that passes through $[x_0, y_0, 0]$ (Figure 7.3). Thus, $R(T)$ is the entire $xy$-plane.

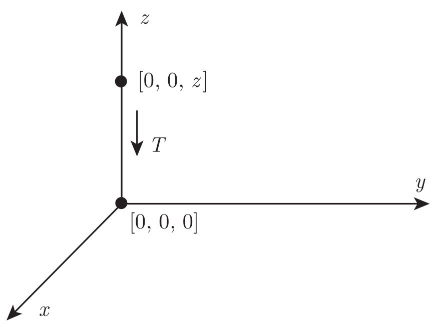


Figure 7.2 $\ker(T)$ is the $z$-axis

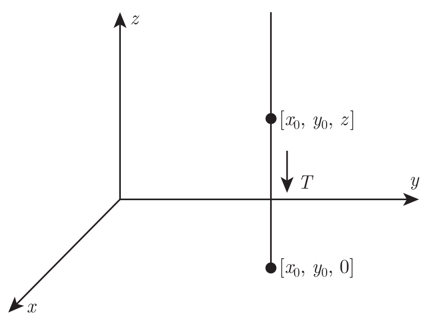


Figure 7.3 $R(T)$ is the entire $xy$-plane

(c) Let $V = C^1[a, b]$ be the vector space of functions with continuous first derivatives on $[a, b]$ and $W = C[a, b]$ be the vector space of continuous functions on $[a, b]$. Let $D : V \to W$ be the differentiation transformation

$$D(\mathbf{f}) = f'(x) = \frac{df(x)}{dx},$$

where $\mathbf{f} = f(x) \in V$. The kernel of $D$ is the set of functions in $V$ with derivative zero. From calculus, this is the set of all constant functions on $[a, b]$. The range of $D$ is given by $R(D) = W = C[a, b]$.

In all the examples above, $\ker(T)$ and $R(T)$ turned out to be subspaces. This is actually a consequence of the following result.

**Theorem 7.3** *Let $T : V \to W$ be a linear transformation. Then*

(a) *The kernel of $T$ is a subspace of $V$.*

(b) *The range of $T$ is a subspace of $W$.*

*Proof* For (a), let $\mathbf{u}, \mathbf{v} \in \ker(T)$ and $k \in \mathbb{R}$. We have

$$T(\mathbf{u} + k\mathbf{v}) = T(\mathbf{u}) + kT(\mathbf{v}) = \mathbf{0} + k\mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{u} + k\mathbf{v} \in \ker(T)$, i.e., $\ker(T)$ is a subspace by Theorem 4.2.

For (b), let $\mathbf{u}, \mathbf{v} \in R(T)$ and $k \in \mathbb{R}$. We know that there are $\mathbf{p}, \mathbf{q} \in V$ such that

$$T(\mathbf{p}) = \mathbf{u}, \qquad T(\mathbf{q}) = \mathbf{v}.$$

It follows that

$$T(\mathbf{p} + k\mathbf{q}) = T(\mathbf{p}) + kT(\mathbf{q}) = \mathbf{u} + k\mathbf{v},$$

where $\mathbf{p} + k\mathbf{q} \in V$. Thus, $\mathbf{u} + k\mathbf{v} \in R(T)$, i.e., $R(T)$ is a subspace by Theorem 4.2 again. □

### 7.2.2 Rank and nullity

**Definition** *Let $T: V \to W$ be a linear transformation. Then the dimension of the range of $T$ is called the* ***rank*** *of $T$ and is denoted by* $\operatorname{rank}(T)$; *the dimension of the kernel is called the* ***nullity*** *of $T$ and is denoted by* $\operatorname{nullity}(T)$.

Let $T_A: \mathbb{R}^n \to \mathbb{R}^m$ be multiplication by $A \in \mathbb{R}^{m\times n}$. Then we have the following relationship between the rank and nullity of the matrix $A$ and the rank and nullity of the corresponding linear transformation $T_A$.

**Theorem 7.4** *Let $A$ be an $m \times n$ matrix and $T_A$ be the matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then*

(a) $\operatorname{nullity}(T_A) = \operatorname{nullity}(A)$.

(b) $\operatorname{rank}(T_A) = \operatorname{rank}(A)$.

*Proof* For (a), we have

$$\operatorname{nullity}(T_A) = \dim\big(\ker(T_A)\big) = \dim(\text{nullspace of } A) = \operatorname{nullity}(A).$$

For (b), we have

$$\operatorname{rank}(T_A) = \dim\big(R(T_A)\big) = \dim(\text{column space of } A) = \operatorname{rank}(A).$$ □

### 7.2.3 Dimension theorem for linear transformations

**Theorem 7.5** (*Dimension Theorem for Linear Transformations*) *Let $T: V \to W$ be a linear transformation from an $n$-dimensional vector space $V$ to a vector space $W$. Then*

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = n.$$

*Proof* We only prove the case of $1 \leqslant \dim\big(\ker(T)\big) < n$. The proofs of cases of $\dim\big(\ker(T)\big) = 0$ and $\dim\big(\ker(T)\big) = n$ are left as an exercise. We must show that

$$\dim\big(R(T)\big) + \dim\big(\ker(T)\big) = n.$$

Assume $\dim\big(\ker(T)\big) = r$, and let

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

be a basis for the kernel. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent, Theorem 4.12 (b) states that there are $n-r$ vectors $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ such that

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$$

is a basis for $V$. We want to show that the $n-r$ vectors in $S = \{T(\mathbf{v}_{r+1}), \ldots, T(\mathbf{v}_n)\}$ form a basis for $R(T)$.

First, we show that $S$ spans $R(T)$. If $\mathbf{w}$ is any vector in $R(T)$, then $\mathbf{w} = T(\mathbf{v})$ for some vector $\mathbf{v}$ in $V$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is a basis for $V$, we have

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} + \cdots + c_n\mathbf{v}_n.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ lie in $\ker(T)$, we obtain $T(\mathbf{v}_1) = T(\mathbf{v}_2) = \cdots = T(\mathbf{v}_r) = \mathbf{0}$, so that

$$\mathbf{w} = T(\mathbf{v}) = c_{r+1}T(\mathbf{v}_{r+1}) + \cdots + c_nT(\mathbf{v}_n).$$

Thus, $R(T) = \mathrm{span}(S)$.

We next show that $S$ is a linearly independent set. Suppose that

$$k_{r+1}T(\mathbf{v}_{r+1}) + \cdots + k_nT(\mathbf{v}_n) = \mathbf{0}. \tag{7.1}$$

Since $T$ is linear, (7.1) can be rewritten as

$$T(k_{r+1}\mathbf{v}_{r+1} + \cdots + k_n\mathbf{v}_n) = \mathbf{0},$$

which says that $k_{r+1}\mathbf{v}_{r+1} + \cdots + k_n\mathbf{v}_n \in \ker(T)$. This vector can therefore be written as a linear combination of the basis vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, say

$$k_{r+1}\mathbf{v}_{r+1} + \cdots + k_n\mathbf{v}_n = k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r.$$

Thus,

$$k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r - k_{r+1}\mathbf{v}_{r+1} - \cdots - k_n\mathbf{v}_n = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is linearly independent, all of the $k$'s are zero. In particular,

$$k_{r+1} = \cdots = k_n = 0.$$

Hence $S = \{T(\mathbf{v}_{r+1}), \ldots, T(\mathbf{v}_n)\}$ is linearly independent. Consequently, $S$ forms a basis for $R(T)$. Therefore,

$$\dim\big(R(T)\big) + \dim\big(\ker(T)\big) = (n-r) + r = n. \qquad \square$$

**Remark** In Theorem 7.5, if $T = T_A$ is a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, where $A$ is an $m \times n$ matrix, then it follows from Theorem 7.4 that

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

Thus, Theorem 4.23 actually is a special case of Theorem 7.5.

## 7.3 Inverse Linear Transformations

In Subsection 3.3.3, we discussed some properties of one-to-one linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this section, we extend those ideas to general linear transformations.

### 7.3.1 One-to-one and onto linear transformations

**Definition** *A linear transformation $T\colon V \to W$ is said to be **one-to-one** if $T$ maps distinct vectors in $V$ into distinct vectors in $W$, i.e., for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$, if $\mathbf{u} \neq \mathbf{v}$, then $T(\mathbf{u}) \neq T(\mathbf{v})$.*

**Definition** *A linear transformation $T\colon V \to W$ is said to be **onto** if every vector in $W$ is the image of at least one vector in $V$, i.e., for every vector $\mathbf{w}$ in $W$, there is a vector $\mathbf{v}$ in $V$ such that $T(\mathbf{v}) = \mathbf{w}$.*

The following theorem establishes a relationship between a one-to-one linear transformation and its kernel.

**Theorem 7.6** *Let $T\colon V \to W$ be a linear transformation. Then the following are equivalent.*

(a) *$T$ is one-to-one.*

(b) $\ker(T) = \{\mathbf{0}\}$.

*Proof* (a) $\Rightarrow$ (b): Let $\mathbf{u} \in V$. By (a), if $\mathbf{u} \neq \mathbf{0}$, then

$$T(\mathbf{u}) \neq T(\mathbf{0}) = \mathbf{0},$$

i.e., $\mathbf{u} \notin \ker(T)$. Therefore, $\ker(T) = \{\mathbf{0}\}$.

(b) $\Rightarrow$ (a): If $\mathbf{u} \neq \mathbf{v}$, then $\mathbf{u}-\mathbf{v} \neq \mathbf{0}$. Hence $\mathbf{u}-\mathbf{v}$ is not in $\ker(T)$ by (b). We obtain

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u}-\mathbf{v}) \neq \mathbf{0},$$

i.e.,

$$T(\mathbf{u}) \neq T(\mathbf{v}).$$

Thus, $T$ is one-to-one. $\square$

Furthermore, if the vector spaces $V$ and $W$ have the same dimension, then the following theorem shows one more equivalent property. The proof of the theorem is left as an exercise.

**Theorem 7.7** *Let $V$ and $W$ be finite-dimensional vector spaces with the same dimension, and $T\colon V \to W$ be a linear transformation. Then the following are equivalent.*

(a) *$T$ is one-to-one.*

(b) $\ker(T) = \{\mathbf{0}\}$.

(c) *$R(T) = W$, i.e., $T$ is onto.*

**Example** In each part, determine whether the linear transformation is one-to-one, onto, both, or neither.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ rotates each vector through the angle $\theta$.

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection on the $xy$-plane.

(c) $T_A\colon \mathbb{R}^4 \to \mathbb{R}^3$ is multiplication by the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & 5 & -1 & 2 \\ 5 & 3 & 1 & 6 \end{bmatrix}.$$

*Solution* For (a), note that $\ker(T) = \{\mathbf{0}\}$, and then $T$ is both one-to-one and onto.

For (b), since $\ker(T)$ is the $z$-axis which contains nonzero vectors, $T$ is neither one-to-one nor onto.

For (c), note that $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 1$. Since $\dim\bigl(\ker(T_A)\bigr) = \operatorname{nullity}(A) = 1$, i.e., $\ker(T_A) \neq \{\mathbf{0}\}$, it follows from Theorem 7.6 that $T_A$ is not one-to-one. However, since $\operatorname{rank}(A) = 3$, it follows from Theorem 4.26 that the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b} \in \mathbb{R}^3$. Thus, $R(T_A) = \mathbb{R}^3$, i.e., $T_A$ is onto. $\square$

### 7.3.2 Inverse linear transformations

The inverse transformation of a one-to-one transformation $T\colon V \to W$, denoted by $T^{-1}$, is defined as a new function that maps $\mathbf{w} = T(\mathbf{v}) \in R(T) \subseteq W$ back into $\mathbf{v}$ for any $\mathbf{v} \in V$. See Figure 7.4.

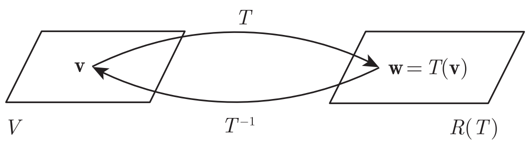


Figure 7.4

We now show that $T^{-1}\colon R(T) \to V$ is a linear transformation. Note that from the definition of $T^{-1}$,

$$T^{-1}\big(T(\mathbf{v})\big) = T^{-1}(\mathbf{w}) = \mathbf{v}, \qquad T\big(T^{-1}(\mathbf{w})\big) = T(\mathbf{v}) = \mathbf{w},$$

i.e.,

$$T^{-1} \circ T = I_V, \qquad T \circ T^{-1} = I_{R(T)}, \tag{7.2}$$

where $I_V$ is the identity transformation on $V$ and $I_{R(T)}$ is the identity transformation on $R(T)$. Thus, for any $\mathbf{u}$, $\mathbf{w} \in R(T)$, we deduce by using (7.2),

$$\begin{aligned} T^{-1}(\mathbf{u} + \mathbf{w}) &= T^{-1}[(T \circ T^{-1})(\mathbf{u}) + (T \circ T^{-1})(\mathbf{w})] = T^{-1}[T(T^{-1}(\mathbf{u})) + T(T^{-1}(\mathbf{w}))] \\ &= T^{-1}[T(T^{-1}(\mathbf{u}) + T^{-1}(\mathbf{w}))] = (T^{-1} \circ T)[T^{-1}(\mathbf{u}) + T^{-1}(\mathbf{w})] \\ &= T^{-1}(\mathbf{u}) + T^{-1}(\mathbf{w}), \end{aligned}$$

and for any scalar $k$,

$$\begin{aligned} T^{-1}(k\mathbf{w}) &= T^{-1}[k(T \circ T^{-1})(\mathbf{w})] = T^{-1}[k(T(T^{-1}(\mathbf{w})))] \\ &= T^{-1}[T(kT^{-1}(\mathbf{w}))] = (T^{-1} \circ T)[kT^{-1}(\mathbf{w})] \\ &= kT^{-1}(\mathbf{w}). \end{aligned}$$

Hence $T^{-1}$ is a linear transformation.

The following theorem lists an important property of one-to-one linear transformations.

**Theorem 7.8** *Let $T_1\colon U \to V$ and $T_2\colon V \to W$ be one-to-one linear transformations. Then $T_2 \circ T_1$ is one-to-one.*

*Proof* Since both $T_1$ and $T_2$ are one-to-one linear transformations, for vectors $\mathbf{u}$ and $\mathbf{v}$ in $U$ and $\mathbf{u} \neq \mathbf{v}$, we have

$$T_1(\mathbf{u}) \neq T_1(\mathbf{v}),$$

where $T_1(\mathbf{u})$ and $T_1(\mathbf{v})$ are vectors in $V$. Moreover,

$$(T_2 \circ T_1)(\mathbf{u}) = T_2\big(T_1(\mathbf{u})\big) \neq T_2\big(T_1(\mathbf{v})\big) = (T_2 \circ T_1)(\mathbf{v}),$$

where $(T_2 \circ T_1)(\mathbf{u})$ and $(T_2 \circ T_1)(\mathbf{v})$ are vectors in $W$. Thus, $T_2 \circ T_1$ is one-to-one. □

**Remark** In general, if $T_j$ is a one-to-one linear transformation from the vector space $V_{j-1}$ to another vector space $V_j$ for $1 \leqslant j \leqslant n$, then $T_n \circ T_{n-1} \circ \cdots \circ T_2 \circ T_1$ is one-to-one.

## 7.4 Matrices of General Linear Transformations

In this section, we show that if $V$ and $W$ are finite-dimensional vector spaces, then by using bases for $V$ and $W$, any linear transformation $T\colon V \to W$ can be regarded as a matrix transformation.

### 7.4.1 Matrices of linear transformations

Let $T$ be a linear transformation between two finite-dimensional vector spaces $V$ and $W$ with $\dim(V) = n$ and $\dim(W) = m$, respectively. We have the following relationship. See Figure 7.5.

$$\begin{tikzcd}
\text{A vector in } V & \mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] & \text{A vector in } W \\
\text{A vector in } \mathbb{R}^n & {[\mathbf{x}]_B} & {[T(\mathbf{x})]_{B'}} & \text{A vector in } \mathbb{R}^m
\end{tikzcd}$$

Figure 7.5

Here $[\mathbf{x}]_B$ is the coordinate vector of $\mathbf{x}$ relative to a basis $B$ for $V$ and $[T(\mathbf{x})]_{B'}$ is the coordinate vector of $T(\mathbf{x})$ relative to a basis $B'$ for $W$. In the following, we show that there exists a matrix $A$ such that

$$A[\mathbf{x}]_B = [T(\mathbf{x})]_{B'}. \tag{7.3}$$

See Figure 7.6.

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
[\mathbf{x}]_B \arrow[r, "A"] & [T(\mathbf{x})]_{B'}
\end{tikzcd}$$

Figure 7.6

We are now going to construct $A$. Let

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \subset V, \qquad B' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subset W.$$

Note that

$$V = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}, \qquad W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}.$$

Since $B' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a basis for $W$ and $T(\mathbf{u}_j) \in W$ for $1 \leqslant j \leqslant n$, we have

$$T(\mathbf{u}_j) = \sum_{i=1}^{m} k_{ij}\mathbf{v}_i = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m] \begin{bmatrix} k_{1j} \\ k_{2j} \\ \vdots \\ k_{mj} \end{bmatrix}.$$

Therefore,

$$[T(\mathbf{u}_j)]_{B'} = [k_{1j}, k_{2j}, \ldots, k_{mj}]^T \in \mathbb{R}^m. \tag{7.4}$$

It implies

$$[T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)] = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m] \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mn} \end{bmatrix}. \tag{7.5}$$

Let

$$\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{u}_i = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in V.$$

Then

$$[\mathbf{x}]_B = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n.$$

It follows from (7.5) that

$$T(\mathbf{x}) = T\Big(\sum_{i=1}^{n} x_i\mathbf{u}_i\Big) = \sum_{i=1}^{n} x_i T(\mathbf{u}_i) = [T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m] \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in W. \tag{7.6}$$

Thus, we have by (7.6) and (7.4),

$$[T(\mathbf{x})]_{B'} = [k_{ij}][\mathbf{x}]_B = \left[\begin{array}{c:c:c:c} [T(\mathbf{u}_1)]_{B'} & [T(\mathbf{u}_2)]_{B'} & \cdots & [T(\mathbf{u}_n)]_{B'} \end{array}\right] [\mathbf{x}]_B. \tag{7.7}$$

Comparing (7.7) with (7.3), we therefore obtain

$$A = \left[\begin{array}{c:c:c:c} [T(\mathbf{u}_1)]_{B'} & [T(\mathbf{u}_2)]_{B'} & \cdots & [T(\mathbf{u}_n)]_{B'} \end{array}\right] \in \mathbb{R}^{m \times n},$$

where $A$ is called the matrix for $T$ relative to the bases $B$ and $B'$ and is denoted by $[T]_{B',B}$ usually. Furthermore, (7.5) can be written as

$$[T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)] = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m][T]_{B',B}. \tag{7.8}$$

**Remark** When $V = W$, it is usual to take $B' = B$ when constructing a matrix for $T$. In this case the resulting matrix is called the matrix for $T$ relative to the basis $B$ and is usually denoted by $[T]_B$ rather than $[T]_{B,B}$. If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, then in this case we obtain

$$[T]_B = \left[\begin{array}{c:c:c:c} [T(\mathbf{u}_1)]_B & [T(\mathbf{u}_2)]_B & \cdots & [T(\mathbf{u}_n)]_B \end{array}\right] \tag{7.9}$$

and

$$[T]_B[\mathbf{x}]_B = [T(\mathbf{x})]_B. \tag{7.10}$$

Phrased informally, (7.9) and (7.10) state that the matrix for $T$ times the coordinate vector for $\mathbf{x}$ is the coordinate vector for $T(\mathbf{x})$.

**Example 1** Let $T\colon P_1 \to P_2$ be the linear transformation defined by

$$T\big(p(x)\big) = xp(x).$$

Find the matrix for $T$ relative to the bases

$$B = \{\mathbf{u}_1, \mathbf{u}_2\}, \qquad B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\},$$

where

$$\mathbf{u}_1 = 1, \quad \mathbf{u}_2 = x; \quad \mathbf{v}_1 = 1, \quad \mathbf{v}_2 = x, \quad \mathbf{v}_3 = x^2.$$

*Solution* We have by (7.8),

$$[T(\mathbf{u}_1), T(\mathbf{u}_2)] = [T(1), T(x)] = [x, x^2] = [1, x, x^2] \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{[T]_{B',B}}. \qquad \square$$

**Example 2** Let $T\colon P_2 \to P_2$ be the linear transformation defined by

$$T\big(p(x)\big) = p(2x+1),$$

i.e., $T(c_0 + c_1 x + c_2 x^2) = c_0 + c_1(2x+1) + c_2(2x+1)^2$.

(a) Find $[T]_B$ relative to the basis $B = \{1, x, x^2\}$.

(b) Compute $T(2 + 3x + 7x^2)$ by using (7.10).

(c) Check the result in (b) by computing $T(2 + 3x + 7x^2)$ directly.

*Solution* For (a), we have from the definition of $T$,

$$T(1) = 1, \quad T(x) = 2x + 1, \quad T(x^2) = (2x+1)^2 = 4x^2 + 4x + 1.$$

It follows from (7.8) that

$$[T(1), T(x), T(x^2)] = [1, 2x+1, 4x^2+4x+1] = [1, x, x^2] \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}}_{[T]_B}.$$

For (b), the coordinate vector relative to $B$ of the vector $\mathbf{p} = 2 + 3x + 7x^2$ is

$$[\mathbf{p}]_B = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}.$$

Thus, we have by using (7.10),

$$[T(\mathbf{p})]_B = [T]_B [\mathbf{p}]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 12 \\ 34 \\ 28 \end{bmatrix}.$$

It follows that

$$T(2 + 3x + 7x^2) = [1, x, x^2] \begin{bmatrix} 12 \\ 34 \\ 28 \end{bmatrix} = 12 + 34x + 28x^2.$$

For (c), we have by direct computation,

$$T(2+3x+7x^2) = 2+3(2x+1)+7(2x+1)^2 = 12+34x+28x^2,$$

which agrees with the result in (b). □

### 7.4.2 Matrices of compositions and inverse transformations

The following theorem is a generalization of (3.5) in Subsection 3.2.3. The proof of the theorem is left as an exercise.

**Theorem 7.9** *Let $T_1 : V_0 \to V_1$ and $T_2 : V_1 \to V_2$ be linear transformations, and let $B_0$, $B_1$, and $B_2$ be bases for $V_0$, $V_1$, and $V_2$, respectively. Then*

$$[T_2 \circ T_1]_{B_2,B_0} = [T_2]_{B_2,B_1}[T_1]_{B_1,B_0}. \tag{7.11}$$

**Remark** In (7.11), observe how the interior subscript $B_1$ (the basis for the intermediate space $V_1$) seems to "cancel out", leaving only the bases for the domain and image space of the composition as subscripts

$$[T_2 \circ T_1]_{B_2,B_0} = [T_2]_{B_2,\underset{\uparrow}{B_1}}[T_1]_{\underset{\uparrow}{B_1},B_0}.$$

Cancelation

This cancelation of interior subscripts suggests the following extension of (7.11) to composition of three linear transformations. Let $B_0$, $B_1$, $B_2$, and $B_3$ be bases for vector spaces $V_0$, $V_1$, $V_2$, and $V_3$, respectively, and $T_j$ be a linear transformation from $V_{j-1}$ to $V_j$ for $j = 1, 2, 3$. See Figure 7.7.

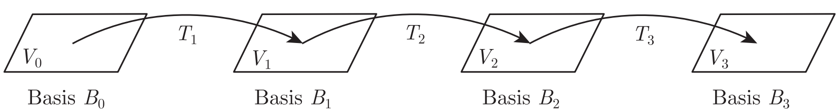


Figure 7.7

Therefore,

$$[T_3 \circ T_2 \circ T_1]_{B_3,B_0} = [T_3]_{B_3,B_2}[T_2]_{B_2,B_1}[T_1]_{B_1,B_0}.$$

In general, we have

$$[T_n \circ T_{n-1} \circ \cdots \circ T_2 \circ T_1]_{B_n,B_0} = [T_n]_{B_n,B_{n-1}}[T_{n-1}]_{B_{n-1},B_{n-2}} \cdots [T_2]_{B_2,B_1}[T_1]_{B_1,B_0},$$

where $B_k$ is a basis for a vector space $V_k$ for $0 \leqslant k \leqslant n$, and $T_j$ is a linear transformation from $V_{j-1}$ to $V_j$ for $1 \leqslant j \leqslant n$.

**Theorem 7.10** *Let $T\colon V \to V$ be a linear transformation and $B$ be a basis for $V$. Then the following are equivalent.*

(a) *$T$ is one-to-one.*

(b) *$[T]_B$ is invertible. Moreover, $[T]_B^{-1} = [T^{-1}]_B$.*

*Proof* Note that $T$ is one-to-one if and only if $T^{-1}$ exists and $T^{-1} \circ T = I_V$, where $I_V$ is the identity transformation on $V$. Then

$$T^{-1} \circ T = I_V \quad \Longleftrightarrow \quad [T^{-1}]_B[T]_B = [T^{-1} \circ T]_B = [I_V] = I,$$

i.e., $[T]_B$ is an invertible matrix and $[T]_B^{-1} = [T^{-1}]_B$. □

## 7.5 Similarity

The corresponding matrix for a linear transformation $T\colon V \to V$ relies on a basis we choose for $V$. Selecting an appropriate basis for $V$ can simplify the matrix for $T$ to be a diagonal or triangular matrix. We first introduce the following definition.

**Definition** *If $A$ and $B$ are square matrices, we say that $B$ is **similar** to $A$ if there is an invertible matrix $P$ such that $B = P^{-1}AP$.*

**Theorem 7.11** *Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be two bases for a vector space $V$, and let $T$ be a linear transformation on $V$. Then $[T]_{B'}$ is similar to $[T]_B$. More precisely,*

$$[T]_{B'} = P^{-1}[T]_B P,$$

*where $P$ is the transition matrix from $B$ to $B'$.*

*Proof* Let $P = [p_{ij}] \in \mathbb{R}^{n\times n}$. Since

$$[\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n] = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]P,$$

we have for $1 \leqslant j \leqslant n$,

$$T(\mathbf{w}_j) = T\Big(\sum_{i=1}^{n} p_{ij}\mathbf{v}_i\Big) = \sum_{i=1}^{n} p_{ij}T(\mathbf{v}_i) = [T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)] \begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{bmatrix},$$

which implies

$$[T(\mathbf{w}_1), T(\mathbf{w}_2), \ldots, T(\mathbf{w}_n)] = [T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)]P. \tag{7.12}$$

Since $P$ is invertible by Theorem 5.21, we obtain

$$[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]P^{-1}. \tag{7.13}$$

Furthermore, we have by (7.8),

$$[T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)] = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n][T]_B \tag{7.14}$$

and

$$[T(\mathbf{w}_1), T(\mathbf{w}_2), \ldots, T(\mathbf{w}_n)] = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n][T]_{B'}. \tag{7.15}$$

It follows from (7.12), (7.14), and (7.13) that

$$[T(\mathbf{w}_1), T(\mathbf{w}_2), \ldots, T(\mathbf{w}_n)] = [T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)]P$$

$$= [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n][T]_B P = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]P^{-1}[T]_B P.$$

Comparing with (7.15), we deduce

$$[T]_{B'} = P^{-1}[T]_B P. \qquad \square$$

**Remark** It follows from the Jordan decomposition theorem that every square matrix $A$ is similar to a bi-diagonal matrix. Moreover, if $A$ is symmetric, then $A$ is similar to a diagonal matrix.

Similar matrices always share some important properties and we list a few of them in Table 7.1. The proofs of the results in table are left as an exercise. See Exercise 7.23.

**Table 7.1**

| **Property** | **Description** |
|---|---|
| Determinant | $A$ and $P^{-1}AP$ have the same determinant. |
| Invertibility | $A$ is invertible if and only if $P^{-1}AP$ is invertible. |
| Rank | $A$ and $P^{-1}AP$ have the same rank. |
| Nullity | $A$ and $P^{-1}AP$ have the same nullity. |
| Trace | $A$ and $P^{-1}AP$ have the same trace. |
| Characteristic polynomial | $A$ and $P^{-1}AP$ have the same characteristic polynomial. |
| Eigenvalues | $A$ and $P^{-1}AP$ have the same eigenvalues. |
| Eigenspace dimension | If $\lambda$ is an eigenvalue of $A$ and $P^{-1}AP$, then the eigenspace of $A$ corresponding to $\lambda$ and the eigenspace of $P^{-1}AP$ corresponding to $\lambda$ have the same dimension. |

**Remark** Let $M$ be an $n \times n$ invertible matrix. Define a transformation $T\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $T(A) = M^{-1}AM$ for all $A$ in $\mathbb{R}^{n\times n}$. It is easy to show that $T(A)$, called the ***similarity transformation***, is linear.

## Exercises

### Elementary exercises

**7.1** Show that none of the following transformations is linear.

(a) $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $T(A) = \det(A)$.

(b) $T\colon \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 2^x$.

(c) $T\colon \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$.

**7.2** Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = [1,1,1]^T, \qquad \mathbf{v}_2 = [1,1,0]^T, \qquad \mathbf{v}_3 = [1,0,0]^T.$$

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that

$$T(\mathbf{v}_1) = [1,0]^T, \qquad T(\mathbf{v}_2) = [2,-1]^T, \qquad T(\mathbf{v}_3) = [4,3]^T.$$

(a) Find a formula for $T(\mathbf{x})$ for all $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$.

(b) Use the formula in (a) to compute $T(\mathbf{x})$ if $\mathbf{x} = [2,-3,5]^T$.

**7.3** Suppose that $T\colon \mathbb{R}^2 \to P_2$ is the linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2 - 3x + x^2, \qquad T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 1 - x^2.$$

Find $T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$.

**7.4** Let $V$ be an $n$-dimensional vector space and $T\colon V \to V$ be defined by

$$T(\mathbf{v}) = 2\mathbf{v}.$$

Find the kernel, range, rank, and nullity of $T$.

**7.5** Show that Theorem 7.5 holds in the cases of $\dim\big(\ker(T)\big) = 0$ and $\dim\big(\ker(T)\big) = n$.

**7.6** In each part, determine whether the linear transformation is one-to-one by finding the kernel or the nullity.

$$\text{(a) } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}. \qquad \text{(b) } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4x-3y \end{bmatrix}.$$

$$\text{(c) } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \\ y \end{bmatrix}. \qquad \text{(d) } T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-z \\ y \end{bmatrix}.$$

**7.7** Let $T_1 : U \to V$ and $T_2 : V \to W$ be linear transformations. Show that if $T_2 \circ T_1$ is one-to-one, so is $T_1$.

**7.8** Suppose that the linear transformations $T_1 : P_2 \to P_2$ and $T_2 : P_2 \to P_3$ are given as follows:

$$T_1\bigl(p(x)\bigr) = p(x+1), \qquad T_2\bigl(p(x)\bigr) = xp(x).$$

Find $(T_2 \circ T_1)(a_0 + a_1 x + a_2 x^2)$.

**7.9** Let $T : P_2 \to P_2$ be the linear transformation given by the formula $T\bigl(p(x)\bigr) = p(2x+1)$.

(a) Find a matrix for $T$ relative to the basis $B = \{1, x, x^2\}$.

(b) Find the rank and nullity of $T$.

(c) Use the result in (b) to determine whether $T$ is one-to-one.

**7.10** Prove Theorem 7.7.

**7.11** Show that the linear transformation $T : \mathbb{R}^2 \to P_1$ defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x$$

is both one-to-one and onto.

**7.12** If $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T_A(\mathbf{x}) = A\mathbf{x}$, then determine whether $T_A$ has an inverse.

$$\text{(a) } A = \begin{bmatrix} 3 & 6 \\ 5 & -1 \end{bmatrix}. \qquad \text{(b) } A = \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}.$$

**7.13** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ 3x_1 - 4x_2 \end{bmatrix},$$

and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$ be bases for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = [0, 2]^T, \quad \mathbf{u}_2 = [2, -1]^T; \quad \mathbf{v}_1 = [1, 2]^T, \quad \mathbf{v}_2 = [-1, 0]^T.$$

Find the matrix for $T$ relative to the basis $B$, and find the matrix for $T$ relative to the bases $B$ and $B'$.

**7.14** Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix},$$

and let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, where

$$\mathbf{u}_1 = [1, 0, 0]^T, \quad \mathbf{u}_2 = [0, 1, 0]^T, \quad \mathbf{u}_3 = [0, 0, 1]^T; \quad \mathbf{v}_1 = [0, 1]^T, \quad \mathbf{v}_2 = [1, 0]^T.$$

(a) Find the matrix for $T$ relative to the bases $B$ and $B'$.

(b) Find $[T(\mathbf{v})]_{B'}$ if $[\mathbf{v}]_B = [1, 3, -2]^T$.

**7.15** Let $V$ be an $n$-dimensional vector space and $I$ be the identity transformation on $V$. What is the matrix for $I$ relative to two distinct bases $B$ and $B'$ for $V$?

**7.16** Let $T\colon P_1 \to P_1$ be the linear transformation defined by

$$T\big(p(x)\big) = p(x+1),$$

and let $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ be bases for $P_1$, where

$$\mathbf{p}_1 = 6 + 3x, \quad \mathbf{p}_2 = 10 + 2x; \quad \mathbf{q}_1 = 2, \quad \mathbf{q}_2 = 3 + 2x.$$

(a) Find the matrix $[T]_{B',B}$ relative to the bases $B$ and $B'$.

(b) If $\mathbf{p} = 1 + 3x$, then find $[T(\mathbf{p})]_{B'}$ by using the matrix $[T]_{B',B}$.

**7.17** Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$$

be the matrix for $T\colon P_2 \to P_2$ relative to the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, where

$$\mathbf{p}_1 = 3x + 3x^2, \quad \mathbf{p}_2 = -1 + 3x + 2x^2, \quad \mathbf{p}_3 = 3 + 7x + 2x^2.$$

(a) Find $T(\mathbf{p}_1)$, $T(\mathbf{p}_2)$, and $T(\mathbf{p}_3)$.

(b) Find $[T(\mathbf{p}_1)]_B$, $[T(\mathbf{p}_2)]_B$, and $[T(\mathbf{p}_3)]_B$.

**7.18** Let

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$

be the matrix for $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = [1, 3]^T, \qquad \mathbf{v}_2 = [-1, 4]^T.$$

(a) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.

(b) Find $[T(\mathbf{v}_1)]_B$, $[T(\mathbf{v}_2)]_B$, and $T(\mathbf{u})$, where $\mathbf{u} = [1, 1]^T$.

**7.19** Let $T\colon P_2 \to P_2$ be the linear transformation defined by $T\big(p(x)\big) = p(x+1)$.

(a) Find the matrix $[T]_{B',B}$ relative to the bases $B = \{1, x, x^2\}$ and $B' = \{x, x^2, 1\}$.

(b) If $\mathbf{p} = 1 + 2x + 3x^2$, find $[T(\mathbf{p})]_{B'}$ by using the matrix $[T]_{B',B}$.

**7.20** Verify that the linear transformations $T_1\colon \mathbb{R}^2 \to P_1$ and $T_2\colon P_1 \to \mathbb{R}^2$ defined by

$$T_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x, \qquad T_2(c + dx) = \begin{bmatrix} c \\ d - c \end{bmatrix}$$

are inverses of each other.

**7.21** Let $T_1\colon U \to V$ and $T_2\colon V \to W$ be one-to-one linear transformations, where $U$, $V$, and $W$ are vector spaces with the same dimension. Show that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

**7.22** Prove Theorem 7.9.

**7.23** Prove all the properties in Table 7.1.

**7.24** Suppose that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & a & 2 \\ 5 & 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & b & 0 \\ 6 & 3 & 1 \end{bmatrix}.$$

Find the values of $a$ and $b$ if $A$ is similar to $B$.

**7.25** Show that if $A$ and $B$ are similar, then $A^T$ and $B^T$ are similar.

**7.26** Show that if two invertible matrices $A$ and $B$ are similar, then $A^{-1}$ and $B^{-1}$ are similar.

## Challenge exercises

**7.27** For any linear transformations $T_1$ and $T_2$ from a vector space $V$ into a vector space $W$, the operations of addition and scalar multiplication are defined by

$$(T_1 + T_2)(\mathbf{x}) := T_1(\mathbf{x}) + T_2(\mathbf{x}), \qquad (kT_1)(\mathbf{x}) := kT_1(\mathbf{x}),$$

where $\mathbf{x} \in V$ and $k$ is a scalar. Show that the set of all linear transformations from $V$ to $W$ with these two operations is a vector space.

**7.28** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in a vector space $V$ and $T\colon V \to W$ be a linear transformation.

(a) Show that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in $W$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent in $V$.

(b) Show that the converse of (a) is false, i.e., it is not necessarily true that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent in $V$, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in $W$.

(c) Show that if $T$ is one-to-one, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent in $V$ if and only if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in $W$.

**7.29** Determine whether each function $T\colon P_2 \to P_2$ is a linear transformation.

(a) $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x+1) + a_2(x+1)^2$.

(b) $T(a_0 + a_1x + a_2x^2) = (a_0 + 1) + (a_1 + 1)(x+1) + (a_2 + 1)(x+1)^2$.

**7.30** Let $T\colon P_2 \to P_3$ be the linear transformation defined by

$$T\big(p(x)\big) = xp(x).$$

Find the bases for the kernel and range of $T$.

**7.31** Find the kernel, range, rank, and nullity of the linear transformation $T\colon P_3 \to P_2$ defined by

$$T\big(p(x)\big) = \frac{dp(x)}{dx}.$$

**7.32** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the formula

$$T(x, y) = (3x + y, -4x + 3y).$$

Find the bases for the kernel and range of $T$.

**7.33** Find the kernel and the nullity of the linear transformation $T\colon P_1 \to \mathbb{R}$ defined by

$$T\big(p(x)\big) = \int_{-1}^{1} p(x)dx.$$

**7.34** Let $V$ be a finite-dimensional vector space and $T\colon V \to V$ be a linear transformation. Show that

(a) $\ker(T) \cap R(T) = \{\mathbf{0}\}$ if and only if $\operatorname{rank}(T) = \operatorname{rank}(T \circ T)$.

(b) $\ker(T) = R(I - T)$ if $T = T \circ T$.

**7.35** Let $V$ be the vector space of all symmetric $2 \times 2$ matrices. Define a linear transformation $T\colon V \to P_2$ by

$$T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (a-b) + (b-c)x + (c-a)x^2.$$

Find the rank and nullity of $T$.

**7.36** Let $T_1\colon P_2 \to P_3$ and $T_2\colon P_3 \to P_3$ be the linear transformations given by the formulas

$$T_1\big(p(x)\big) = xp(x), \qquad T_2\big(p(x)\big) = p(x+1).$$

Find the formulas for $T_1^{-1}\big(p(x)\big)$, $T_2^{-1}\big(p(x)\big)$, and $(T_2 \circ T_1)^{-1}\big(p(x)\big)$.

**7.37** Let $V$ and $W$ be finite-dimensional vector spaces and $\dim(W) < \dim(V)$. Show that there is no one-to-one linear transformation $T\colon V \to W$.

**7.38** Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$ and $P = [p_{ij}] \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that if

$$\mathbf{u}_i = p_{1i}\mathbf{v}_1 + p_{2i}\mathbf{v}_2 + \cdots + p_{ni}\mathbf{v}_n, \quad 1 \leqslant i \leqslant n,$$

then $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$ and $P$ is the transition matrix from $B$ to $B'$.

**7.39** Let $T\colon \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2c & a+c \\ b-2c & d \end{bmatrix}.$$

Find the matrix $[T]_B$ relative to the basis $B = \{A_{(1)}, A_{(2)}, A_{(3)}, A_{(4)}\}$, where

$$A_{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{(2)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{(3)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{(4)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

# Chapter 8

# Additional Topics

"*God used beautiful mathematics in creating the world.*"

— Paul Dirac

"*The art of doing mathematics consists in finding that special case which contains all the germs of generality.*"

— David Hilbert

In this chapter, we study several important topics in linear algebra. We introduce quadratic forms, complex inner product spaces, and some special structured matrices. Finally, we discuss the Böttcher-Wenzel conjecture.

## 8.1 Quadratic Forms

In this section we study functions in which the terms are squares of variables or products of two variables. Such functions arise in a variety of applications, including geometry, vibrations of mechanical systems, statistics, and electrical engineering.

### 8.1.1 Introduction to quadratic forms

Up to now, we have been interested primarily in linear equations of the following form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The expression on the left-hand side of this equation is a linear form, in which all variables occur to the first power. Now, we are concerned with ***quadratic forms***, which are functions of the form

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 + (\text{all possible terms of form } 2a_kx_ix_j \text{ for } i < j).$$

For instance, a quadratic form in the variables $x_1$ and $x_2$ is

$$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 \tag{8.1}$$

and a quadratic form in the variables $x_1, x_2$, and $x_3$ is

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3. \tag{8.2}$$

The terms in a quadratic form that involve products of different variables are called the cross-product terms.

Note that (8.1) can be written in matrix form as

$$[x_1,\ x_2]\begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8.3}$$

and (8.2) can be written as

$$[x_1,\ x_2,\ x_3]\begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{8.4}$$

The products in (8.3) and (8.4) are both of the form $\mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}$ is the column vector of variables, and $A$ is a symmetric matrix whose diagonal entries are the coefficients of the squared terms and whose entries off the main diagonal are half the coefficients of the cross-product terms. By using the Euclidean inner product, we can write the quadratic form as

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(A\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x}\rangle = \langle \mathbf{x}, A\mathbf{x}\rangle. \tag{8.5}$$

There are two important mathematical problems related to quadratic forms.

(1) Find the maximum and minimum values of $\mathbf{x}^T A\mathbf{x}$ if $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ is constrained so that

$$\|\mathbf{x}\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} = 1.$$

(2) What conditions must $A$ satisfy for a quadratic form to satisfy $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$?

We study the problems above in the next two subsections.

### 8.1.2 Constrained extremum problem

The goal in the subsection is to consider the problem of finding the maximum and minimum values of $\mathbf{x}^T A\mathbf{x}$ subject to $\|\mathbf{x}\| = 1$. By Theorem 6.7 (a), we know that all the eigenvalues of a symmetric matrix $A$ are real. Therefore, we can arrange the eigenvalues of $A$ in a decreasing size order.

**Theorem 8.1** *Let $A$ be a symmetric $n \times n$ matrix whose eigenvalues in decreasing size order are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. If $\mathbf{x}$ is constrained so that $\|\mathbf{x}\| = 1$ with respect to the Euclidean inner product on $\mathbb{R}^n$, then*

(a) $\lambda_1 \geqslant \mathbf{x}^T A\mathbf{x} \geqslant \lambda_n$.

(b) $\mathbf{x}^T A\mathbf{x} = \lambda_1$ *if* $\mathbf{x}$ *is an eigenvector of* $A$ *corresponding to* $\lambda_1$ *and* $\mathbf{x}^T A\mathbf{x} = \lambda_n$ *if* $\mathbf{x}$ *is an eigenvector of* $A$ *corresponding to* $\lambda_n$ *.*

*Proof* We only prove (a) and the proof of (b) is left as an exercise. Since $A$ is symmetric, it follows from Theorem 6.6 that there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is such a basis, where $\mathbf{v}_i$ is the eigenvector corresponding to the eigenvalue $\lambda_i$. Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product. It follows from Theorem 5.8 that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Thus,

$$\begin{aligned} A\mathbf{x} &= \langle \mathbf{x}, \mathbf{v}_1 \rangle A\mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle A\mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle A\mathbf{v}_n \\ &= \langle \mathbf{x}, \mathbf{v}_1 \rangle \lambda_1 \mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle \lambda_2 \mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle \lambda_n \mathbf{v}_n \\ &= \lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \lambda_2 \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \lambda_n \langle \mathbf{x}, \mathbf{v}_n \rangle \mathbf{v}_n. \end{aligned}$$

The coordinate vectors of $\mathbf{x}$ and $A\mathbf{x}$ relative to the basis $S$ are

$$[\mathbf{x}]_S = [\langle \mathbf{x}, \mathbf{v}_1 \rangle, \langle \mathbf{x}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{x}, \mathbf{v}_n \rangle]^T$$

and

$$[A\mathbf{x}]_S = [\lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle, \lambda_2 \langle \mathbf{x}, \mathbf{v}_2 \rangle, \ldots, \lambda_n \langle \mathbf{x}, \mathbf{v}_n \rangle]^T.$$

Thus, from Theorem 5.9 (c) and the fact that $\|\mathbf{x}\| = 1$, we obtain

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{v}_1 \rangle^2 + \langle \mathbf{x}, \mathbf{v}_2 \rangle^2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle^2 = 1$$

and

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle^2 + \lambda_2 \langle \mathbf{x}, \mathbf{v}_2 \rangle^2 + \cdots + \lambda_n \langle \mathbf{x}, \mathbf{v}_n \rangle^2.$$

Using (8.5) and these two equations, we can prove that $\mathbf{x}^T A\mathbf{x} \leqslant \lambda_1$ as follows:

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} = \langle \mathbf{x}, A\mathbf{x} \rangle &= \lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle^2 + \lambda_2 \langle \mathbf{x}, \mathbf{v}_2 \rangle^2 + \cdots + \lambda_n \langle \mathbf{x}, \mathbf{v}_n \rangle^2 \\ &\leqslant \lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle^2 + \lambda_1 \langle \mathbf{x}, \mathbf{v}_2 \rangle^2 + \cdots + \lambda_1 \langle \mathbf{x}, \mathbf{v}_n \rangle^2 \\ &= \lambda_1 \Big( \langle \mathbf{x}, \mathbf{v}_1 \rangle^2 + \langle \mathbf{x}, \mathbf{v}_2 \rangle^2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle^2 \Big) = \lambda_1. \end{aligned}$$

Similarly, one can show that $\mathbf{x}^T A\mathbf{x} \geqslant \lambda_n$. $\square$

### 8.1.3 Positive definite matrix

**Definition** *A symmetric matrix $A$ and the quadratic form $\mathbf{x}^T A\mathbf{x}$ are called*

(i) ***positive definite*** *if* $\mathbf{x}^T A\mathbf{x} > 0$ *for all* $\mathbf{x} \neq \mathbf{0}$.

(ii) ***positive semidefinite*** *if* $\mathbf{x}^T A\mathbf{x} \geqslant 0$ *for all* $\mathbf{x}$.

(iii) ***negative definite*** *if* $\mathbf{x}^T A\mathbf{x} < 0$ *for all* $\mathbf{x} \neq \mathbf{0}$.

(iv) ***negative semidefinite*** *if* $\mathbf{x}^T A\mathbf{x} \leqslant 0$ *for all* $\mathbf{x}$.

**Theorem 8.2** *A symmetric matrix $A$ is positive definite if and only if all the eigenvalues of $A$ are positive.*

*Proof* Assume that $A$ is positive definite and $\lambda$ is an eigenvalue of $A$. Let $\mathbf{x}$ be an eigenvector of $A$ corresponding to $\lambda$, i.e., $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. Then

$$0 < \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T \lambda\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} = \lambda\|\mathbf{x}\|^2,$$

where $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x}$. Since $\|\mathbf{x}\|^2 > 0$, we have $\lambda > 0$.

Conversely, assume that all eigenvalues of $A$ are positive. We must show that $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. However, if $\mathbf{x} \neq \mathbf{0}$, we can normalize $\mathbf{x}$ to obtain the vector $\mathbf{y} = \mathbf{x}/\|\mathbf{x}\|$ with the property $\|\mathbf{y}\| = 1$. It now follows from Theorem 8.1 that

$$\mathbf{y}^T A\mathbf{y} \geqslant \lambda_n > 0,$$

where $\lambda_n$ is the smallest eigenvalue of $A$. Thus,

$$0 < \mathbf{y}^T A\mathbf{y} = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)^T A\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = \frac{1}{\|\mathbf{x}\|^2}\mathbf{x}^T A\mathbf{x},$$

which implies

$$\mathbf{x}^T A\mathbf{x} > 0,$$

i.e., $A$ is positive definite. □

Similarly we have the following corollary for positive semidefinite matrices.

**Corollary** *A symmetric matrix $A$ is positive semidefinite if and only if all the eigenvalues of $A$ are nonnegative.*

Our next objective is to give a criterion that can be used to determine whether a symmetric matrix is positive definite without finding its eigenvalues. To do this it is helpful to introduce some terminology. If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is a square matrix, then the ***leading principal submatrices*** of $A$ are the submatrices formed from the first $r$ rows and $r$ columns of $A$ for $1 \leqslant r \leqslant n$. These submatrices are

$$A_{(1)} = a_{11}, \quad A_{(2)} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_{(3)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \ldots,$$

and

$$A_{(n)} = A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

**Theorem 8.3** *A symmetric matrix $A$ is positive definite if and only if every leading principal submatrix of $A$ is positive definite.*

The proof of Theorem 8.3 is left as an exercise.

A ***principal submatrix*** of an $n \times n$ matrix $A = [a_{ij}]$ is a square submatrix obtained by removing certain rows and columns from $A$. In fact, for any $1 \leqslant k \leqslant n$, a $k \times k$ principal submatrix of $A$ is given by

$$\begin{bmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{bmatrix},$$

where $i_1, i_2, \ldots, i_k$ are integers with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. For instance,

$$\begin{bmatrix} a_{22} & a_{24} & a_{28} \\ a_{42} & a_{44} & a_{48} \\ a_{82} & a_{84} & a_{88} \end{bmatrix}$$

is a $3 \times 3$ principal submatrix of an $n \times n$ matrix with $n \geqslant 8$.

**Theorem 8.4** *A symmetric matrix $A$ is positive definite if and only if every principal submatrix of $A$ is positive definite.*

The proof of Theorem 8.4 is left as an exercise.

## 8.2 Three Theorems for Symmetric Matrices

We list three important theorems which are concerned with eigenvalues of symmetric matrices.

**Theorem 8.5** (*Courant-Fischer's Minimax Theorem*) *If $A$ is an $n \times n$ symmetric matrix whose eigenvalues in decreasing size order are $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$, then for $1 \leqslant k \leqslant n$,*

$$\begin{aligned}\lambda_k(A) &= \max_{\substack{\mathcal{X} \subseteq \mathbb{R}^n \\ \dim(\mathcal{X})=k}} \min_{\mathbf{0} \neq \mathbf{x} \in \mathcal{X}} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \max_{\substack{\mathcal{X} \subseteq \mathbb{R}^n \\ \dim(\mathcal{X})=k}} \min_{\substack{\mathbf{x} \in \mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A \mathbf{x} \\ &= \min_{\substack{\mathcal{X} \subseteq \mathbb{R}^n \\ \dim(\mathcal{X})=n-k+1}} \max_{\mathbf{0} \neq \mathbf{x} \in \mathcal{X}} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \min_{\substack{\mathcal{X} \subseteq \mathbb{R}^n \\ \dim(\mathcal{X})=n-k+1}} \max_{\substack{\mathbf{x} \in \mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A \mathbf{x},\end{aligned}$$

*where $\mathcal{X}$ denotes a subspace of $\mathbb{R}^n$ and $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x}$ in $\mathbb{R}^n$. In particular,*

$$\lambda_1(A) = \max_{\|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x}, \qquad \lambda_n(A) = \min_{\|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x}.$$

*Proof* Let $\dim(\mathcal{X}) = k$. Suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are the orthonormal eigenvectors of $A$ corresponding to $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$, respectively. Let $\mathcal{Y} = \operatorname{span}\{\mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$. We have

$$\dim(\mathcal{X}) + \dim(\mathcal{Y}) = n + 1.$$

Note that by Theorem 4.14,

$$\dim(\mathcal{X} \cap \mathcal{Y}) = \dim(\mathcal{X}) + \dim(\mathcal{Y}) - \dim(\mathcal{X} + \mathcal{Y}) \geqslant n + 1 - n = 1.$$

We have for any $\mathbf{x} \in \mathcal{X} \cap \mathcal{Y}$ with $\|\mathbf{x}\| = 1$,

$$\mathbf{x} = \sum_{j=k}^{n} \xi_j \mathbf{u}_j, \qquad \sum_{j=k}^{n} |\xi_j|^2 = 1.$$

Then

$$\mathbf{x}^T A \mathbf{x} = \sum_{j=k}^{n} |\xi_j|^2 \lambda_j(A) \leqslant \sum_{j=k}^{n} |\xi_j|^2 \lambda_k(A) = \lambda_k(A).$$

Hence

$$\min_{\substack{\mathbf{x} \in \mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A \mathbf{x} \leqslant \min_{\substack{\mathbf{x} \in \mathcal{X} \cap \mathcal{Y} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A \mathbf{x} \leqslant \lambda_k(A).$$

On the other hand, if we take

$$\mathcal{X}_0 = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\},$$

then $\dim(\mathcal{X}_0) = k$ and we obtain

$$\min_{\substack{\mathbf{x} \in \mathcal{X}_0 \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A \mathbf{x} = \mathbf{u}_k^T A \mathbf{u}_k = \mathbf{u}_k^T \lambda_k(A) \mathbf{u}_k = \lambda_k(A).$$

Thus,

$$\lambda_k(A) = \max_{\substack{\mathcal{X}\subseteq\mathbb{R}^n \\ \dim(\mathcal{X})=k}} \min_{\substack{\mathbf{x}\in\mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A\mathbf{x}.$$

Applying the above equality on $-A$, and noting that

$$-\lambda_k(A) = \lambda_{n-k+1}(-A), \quad 1 \leqslant k \leqslant n,$$

one can deduce

$$\begin{aligned}\lambda_k(A) = -\lambda_{n-k+1}(-A) &= -\max_{\substack{\mathcal{X}\subseteq\mathbb{R}^n \\ \dim(\mathcal{X})=n-k+1}} \min_{\substack{\mathbf{x}\in\mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T(-A)\mathbf{x} \\ &= -\max_{\substack{\mathcal{X}\subseteq\mathbb{R}^n \\ \dim(\mathcal{X})=n-k+1}} (-\max_{\substack{\mathbf{x}\in\mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A\mathbf{x}) = \min_{\substack{\mathcal{X}\subseteq\mathbb{R}^n \\ \dim(\mathcal{X})=n-k+1}} \max_{\substack{\mathbf{x}\in\mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A\mathbf{x}.\end{aligned}$$

In particular, we have

$$\lambda_1(A) = \max_{\|\mathbf{x}\|=1} \mathbf{x}^T A\mathbf{x}, \qquad \lambda_n(A) = \min_{\|\mathbf{x}\|=1} \mathbf{x}^T A\mathbf{x},$$

which coincide with Theorem 8.1 (b). □

**Theorem 8.6** (*Cauchy's Interlace Theorem*) *Let $A$ be an $n \times n$ symmetric matrix whose eigenvalues in decreasing size order are*

$$\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A).$$

*Let $B$ be any $m \times m$ principal submatrix of $A$ whose eigenvalues in decreasing size order are*

$$\mu_1(B) \geqslant \mu_2(B) \geqslant \cdots \geqslant \mu_m(B).$$

*Then for $1 \leqslant j \leqslant m$,*

$$\lambda_j(A) \geqslant \mu_j(B) \geqslant \lambda_{j+n-m}(A).$$

*Proof* We can assume that $A$ is given as the following form

$$A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}.$$

In fact, we can always take a similarity transformation on $A$ by permutation matrices if necessary. By using Theorem 8.5, there exists a subspace $\mathcal{X} \subseteq \mathbb{R}^m$ with $\dim(\mathcal{X}) = j$ which satisfies

$$\mu_j(B) = \min_{\substack{\mathbf{x}\in\mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T B\mathbf{x}.$$

For any $\mathbf{x} \in \mathbb{R}^m$, we construct $\widetilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^n$. Let $\widetilde{\mathcal{X}} = \{\widetilde{\mathbf{x}} \mid \mathbf{x} \in \mathcal{X}\} \subseteq \mathbb{R}^n$. Then $\dim(\widetilde{\mathcal{X}}) = j$. Moreover,

$$\mathbf{x}^T B\mathbf{x} = \widetilde{\mathbf{x}}^T A\widetilde{\mathbf{x}}.$$

We have by Theorem 8.5 again,

$$\mu_j(B) = \min_{\substack{\mathbf{x}\in\mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T B\mathbf{x} = \min_{\substack{\tilde{\mathbf{x}}\in\tilde{\mathcal{X}} \\ \|\tilde{\mathbf{x}}\|=1}} \widetilde{\mathbf{x}}^T A\widetilde{\mathbf{x}} \leqslant \max_{\substack{\mathcal{Y}\subseteq\mathbb{R}^n \\ \dim(\mathcal{Y})=j}} \min_{\substack{\mathbf{y}\in\mathcal{Y} \\ \|\mathbf{y}\|=1}} \mathbf{y}^T A\mathbf{y} = \lambda_j(A).$$

Applying the above result on $-A$ and $-B$, and noting that

$$-\lambda_i(A) = \lambda_{n-i+1}(-A), \quad 1 \leqslant i \leqslant n \tag{8.6}$$

and

$$-\mu_j(B) = \mu_{m-j+1}(-B), \quad 1 \leqslant j \leqslant m, \tag{8.7}$$

we have by taking $i = j + n - m$ in (8.6) and then followed by using (8.7),

$$-\lambda_{j+n-m}(A) = \lambda_{n-(j+n-m)+1}(-A) = \lambda_{m-j+1}(-A) \geqslant \mu_{m-j+1}(-B) = -\mu_j(B),$$

i.e.,

$$\lambda_{j+n-m}(A) \leqslant \mu_j(B). \qquad \square$$

**Theorem 8.7** (*Weyl's Theorem*) *Let $A$ and $B$ be $n \times n$ symmetric matrices whose eigenvalues in decreasing size order are*

$$\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A), \qquad \lambda_1(B) \geqslant \lambda_2(B) \geqslant \cdots \geqslant \lambda_n(B),$$

*respectively. Let $\lambda_1(A+B), \lambda_2(A+B), \ldots, \lambda_n(A+B)$ denote the eigenvalues of $A+B$ in decreasing size order as*

$$\lambda_1(A+B) \geqslant \lambda_2(A+B) \geqslant \cdots \geqslant \lambda_n(A+B).$$

*Then for all $1 \leqslant j \leqslant n$,*

$$\max_{r+s=j+n} \{\lambda_r(A) + \lambda_s(B)\} \leqslant \lambda_j(A+B) \leqslant \min_{r+s=j+1} \{\lambda_r(A) + \lambda_s(B)\}.$$

*Proof* We prove the left inequality first. Let $r + s = j + n$. By Theorem 8.5, there exist two subspaces $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$ with $\dim(\mathcal{X}) = r$ and $\dim(\mathcal{Y}) = s$ such that

$$\lambda_r(A) = \min_{\substack{\mathbf{x}\in\mathcal{X} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T A\mathbf{x}, \qquad \lambda_s(B) = \min_{\substack{\mathbf{x}\in\mathcal{Y} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T B\mathbf{x}.$$

Since

$$\dim(\mathcal{X} \cap \mathcal{Y}) = \dim(\mathcal{X}) + \dim(\mathcal{Y}) - \dim(\mathcal{X} + \mathcal{Y}) \geqslant r + s - n = j,$$

there exists a subspace $\mathcal{T}_0 \subseteq \mathcal{X} \cap \mathcal{Y}$ which satisfies $\dim(\mathcal{T}_0) = j$. Thus,

$$\lambda_j(A+B) = \max_{\substack{\mathcal{T}\subseteq\mathbb{R}^n \\ \dim(\mathcal{T})=j}} \min_{\substack{\mathbf{x}\in\mathcal{T} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T(A+B)\mathbf{x} \geqslant \min_{\substack{\mathbf{x}\in\mathcal{T}_0 \\ \|\mathbf{x}\|=1}} (\mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x})$$

$$\begin{aligned}&\geqslant \min_{\substack{\mathbf{x}\in\mathcal{T}_0\\ \|\mathbf{x}\|=1}} \mathbf{x}^T A\mathbf{x} + \min_{\substack{\mathbf{x}\in\mathcal{T}_0\\ \|\mathbf{x}\|=1}} \mathbf{x}^T B\mathbf{x} \geqslant \min_{\substack{\mathbf{x}\in\mathcal{X}\\ \|\mathbf{x}\|=1}} \mathbf{x}^T A\mathbf{x} + \min_{\substack{\mathbf{x}\in\mathcal{Y}\\ \|\mathbf{x}\|=1}} \mathbf{x}^T B\mathbf{x}\\ &= \lambda_r(A) + \lambda_s(B).\end{aligned}$$

Applying the above inequality on $-A-B$, and noting that

$$\begin{cases} \lambda_j(-A-B) = -\lambda_{n-j+1}(A+B), & 1 \leqslant j \leqslant n;\\ \lambda_r(-A) = -\lambda_{n-r+1}(A), & 1 \leqslant r \leqslant n;\\ \lambda_s(-B) = -\lambda_{n-s+1}(B), & 1 \leqslant s \leqslant n, \end{cases}$$

we deduce

$$-\lambda_{n-j+1}(A+B) = \lambda_j(-A-B) \geqslant \lambda_r(-A) + \lambda_s(-B) = -\lambda_{n-r+1}(A) - \lambda_{n-s+1}(B). \tag{8.8}$$

Let $j' = n-j+1$, $r' = n-r+1$, and $s' = n-s+1$. Then (8.8) can be simplified to

$$\lambda_{j'}(A+B) \leqslant \lambda_{r'}(A) + \lambda_{s'}(B),$$

where

$$r' + s' = (n-r+1) + (n-s+1) = (n-j+1) + 1 = j' + 1.$$

Thus, the right inequality holds. □

## 8.3 Complex Inner Product Spaces

A complete presentation of linear algebra must include complex numbers. We therefore review some basic knowledge of complex numbers before we study complex inner product spaces.

### 8.3.1 Complex numbers

**Definition** *A complex number $z$ is defined by*

$$z := a + b\mathbf{i},$$

*where $a$ and $b$ are real numbers, and $\mathbf{i}^2 = -1$. The real numbers $a$ and $b$ are called the real and imaginary parts of $z$, respectively.*

Let $z_1 = a + b\mathbf{i}$ and $z_2 = c + d\mathbf{i}$ be two complex numbers. Then $z_1$ and $z_2$ are said to be equal if and only if their real parts are equal and their imaginary parts are equal, i.e.,

$$z_1 = z_2 \iff a = c \text{ and } b = d.$$

Also, $z_1$ and $z_2$ can be added, subtracted, and multiplied in accordance with the standard rules of algebra but with $\mathbf{i}^2 = -1$. For instance,

$$z_1 \pm z_2 = (a + b\mathbf{i}) \pm (c + d\mathbf{i}) = (a \pm c) + (b \pm d)\mathbf{i}$$

and

$$z_1 \cdot z_2 = (a + b\mathbf{i}) \cdot (c + d\mathbf{i}) = (ac - bd) + (ad + bc)\mathbf{i}.$$

**Definition** *For a complex number $z = a + b\mathbf{i}$, the complex conjugate of $z$, denoted by the symbol $\bar{z}$, is defined by*

$$\bar{z} := a - b\mathbf{i}.$$

*The modulus of a complex number $z = a + b\mathbf{i}$, denoted by $|z|$, is defined by*

$$|z| := \sqrt{a^2 + b^2}.$$

The following theorem establishes some essential properties of complex numbers.

**Theorem 8.8** *Let $z$, $z_1$, and $z_2$ be any complex numbers. Then*

(a) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$.

(b) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.

(c) $\bar{\bar{z}} = z$.

(d) $z \cdot \bar{z} = |z|^2$.

*Proof* We only prove (a) and (d). The proofs of (b) and (c) are left as an exercise. For (a), let $z_1 = a_1 + b_1\mathbf{i}$ and $z_2 = a_2 + b_2\mathbf{i}$. Then

$$\begin{aligned}
\overline{z_1 \pm z_2} &= \overline{(a_1 + b_1\mathbf{i}) \pm (a_2 + b_2\mathbf{i})} = \overline{(a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i}} \\
&= (a_1 \pm a_2) - (b_1 \pm b_2)\mathbf{i} = (a_1 - b_1\mathbf{i}) \pm (a_2 - b_2\mathbf{i}) \\
&= \bar{z}_1 \pm \bar{z}_2.
\end{aligned}$$

For (d), let $z = a + b\mathbf{i}$. Then

$$\begin{aligned}
z \cdot \bar{z} &= (a + b\mathbf{i}) \cdot \overline{(a + b\mathbf{i})} = (a + b\mathbf{i}) \cdot (a - b\mathbf{i}) \\
&= a^2 - ab\mathbf{i} + ab\mathbf{i} - b^2\mathbf{i}^2 = a^2 + b^2 \\
&= |z|^2.
\end{aligned}$$

□

### 8.3.2 Complex inner product spaces

In the definition of a general vector space $V$ in Subsection 4.1.1, if the scalars are in

$$\mathbb{C} := \left\{ a + b\mathbf{i} \mid a, b \in \mathbb{R},\ \mathbf{i} := \sqrt{-1}\ \right\},$$

then $V$ is called a ***complex vector space***. The notions of linear combination, linear independence, spanning sets, basis, dimension, and subspace carry over without change to complex vector spaces. Moreover, the theorems developed in previous chapters for real vector spaces continue to hold with real vector spaces changed to complex vector spaces.

**Definition** *An **inner product** on a complex vector space $V$ is a function that associates a complex number with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$, denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$, in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and all scalars $k$ in $\mathbb{C}$.*

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

(ii) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

(iii) $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$.

(iv) $\langle \mathbf{v}, \mathbf{v} \rangle \geqslant 0; \quad \langle \mathbf{v}, \mathbf{v} \rangle = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.

*A complex vector space with an inner product is called a **complex inner product space**.*

**Remark** In a complex inner product space $V$, the norm of a vector $\mathbf{u} \in V$ is defined by

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}.$$

The Cauchy-Schwarz inequality is also available for complex inner product spaces. Moreover, the definitions of orthogonal set, orthonormal set, orthogonal basis, and orthonormal basis carry over to complex inner product spaces without change. The Gram-Schmidt process can be used to convert an arbitrary basis into an orthogonal (or orthonormal) basis for a complex inner product space.

**Example** Let $\mathbb{C}^n := \{(c_1, c_2, \ldots, c_n) \mid c_i \in \mathbb{C}\}$ with the operations of vector addition and scalar multiplication. For vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{C}^n$, the complex Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^{n} u_i \overline{v}_i,$$

which satisfies the four axioms of the inner product. A complex vector space $\mathbb{C}^n$ with this inner product is call the complex Euclidean space. We can define the norm and distance as follows:

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{1/2} = \sqrt{\sum_{i=1}^{n} |u_i|^2}, \qquad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\sum_{i=1}^{n} |u_i - v_i|^2}.$$

Let $\mathbb{C}^{m\times n}$ denote the vector space of all $m \times n$ complex matrices with the operations of matrix addition and scalar multiplication. In fact, almost all the concepts concerned with the matrix operations can be generalized from real matrices to complex matrices straightforwardly. Let $A = [a_{ij}] \in \mathbb{C}^{m\times n}$, i.e., $a_{ij} \in \mathbb{C}$ for any $i$ and $j$. Then the matrix defined by

$$A^* := \bar{A}^T = [\bar{a}_{ji}]$$

is call the ***conjugate transpose*** of $A$. We have the following theorem concerned with some basic properties of $A^*$. The proof of the theorem is trivial and we therefore omit it.

**Theorem 8.9** *Let $A$ and $B$ be complex matrices and $k$ be any complex number. Then*

(a) $(A^*)^* = A$. (b) $(A+B)^* = A^*+B^*$. (c) $(kA)^* = \bar{k}A^*$. (d) $(AB)^* = B^*A^*$.

**Example** An inner product on $\mathbb{C}^{n\times n}$ is defined by

$$\langle X, Y\rangle := \mathrm{tr}(XY^*),$$

where $X, Y \in \mathbb{C}^{n\times n}$. One can check easily that $\langle X, Y\rangle$ satisfies the four axioms of the inner product. The ***Frobenius norm*** for any $X = [x_{ij}] \in \mathbb{C}^{n\times n}$ is defined as

$$\|X\|_F := \langle X, X\rangle^{1/2} = [\mathrm{tr}(XX^*)]^{1/2} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |x_{ij}|^2\right)^{1/2}.$$

## 8.4 Hermitian Matrices and Unitary Matrices

We study Hermitian matrices and unitary matrices in this section.

**Definition** *If a square matrix $A$ with complex entries satisfies $A = A^*$, then $A$ is called a* ***Hermitian matrix****. If a square matrix $A$ with complex entries satisfies $A^{-1} = A^*$, i.e.,*

$$A^*A = AA^* = I,$$

*then $A$ is called a* ***unitary matrix****.*

**Theorem 8.10** *Let $A$ be an $n \times n$ complex matrix. Then the following are equivalent.*

(a) *$A$ is unitary.*

(b) *The row (or column) vectors of $A$ form an orthonormal set in $\mathbb{C}^n$ with respect to the Euclidean inner product.*

The proof of Theorem 8.10 is similar to that of Theorem 5.18 and is left as an exercise.

We note that our earlier definitions of eigenvalue, eigenvector, eigenspace, characteristic equation, and characteristic polynomial carry over without change to complex matrices.

For a square matrix $A$ with complex entries, if there exists a unitary matrix $P$ such that

$$P^*AP = D,$$

where $D$ is a diagonal matrix, then $A$ is called ***unitarily diagonalizable***.

**Theorem 8.11** *If $A$ is a Hermitian matrix, then $A$ is unitarily diagonalizable.*

The proof of Theorem 8.11 is left as an exercise.

**Theorem 8.12** *Let $A$ be Hermitian. Then*

(a) *The eigenvalues of $A$ are all real.*

(b) *Eigenvectors from different eigenspaces are orthogonal.*

*Proof* For (a), let $\lambda$ be an eigenvalue of a Hermitian matrix $A$ and $\mathbf{v}$ be the corresponding eigenvector. Then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Multiplying both sides by $\mathbf{v}^*$ yields

$$\mathbf{v}^*A\mathbf{v} = \lambda\mathbf{v}^*\mathbf{v},$$

and then

$$\lambda = \frac{\mathbf{v}^*A\mathbf{v}}{\|\mathbf{v}\|^2}.$$

Therefore,

$$\bar{\lambda} = \lambda^* = \left(\frac{\mathbf{v}^*A\mathbf{v}}{\|\mathbf{v}\|^2}\right)^* = \frac{\mathbf{v}^*A^*\mathbf{v}}{\|\mathbf{v}\|^2} = \frac{\mathbf{v}^*A\mathbf{v}}{\|\mathbf{v}\|^2} = \lambda.$$

For (b), let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1$ and $\lambda_2$ of $A$. Then we have by (a),

$$\lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle A\mathbf{v}_1, \mathbf{v}_2\rangle = \mathbf{v}_2^* A\mathbf{v}_1 = (A^*\mathbf{v}_2)^*\mathbf{v}_1 = (A\mathbf{v}_2)^*\mathbf{v}_1 = \bar{\lambda}_2\mathbf{v}_2^*\mathbf{v}_1 = \lambda_2\langle \mathbf{v}_1, \mathbf{v}_2\rangle,$$

which implies

$$(\lambda_1 - \lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0.$$

Since $\lambda_1 - \lambda_2 \neq 0$, we have

$$\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0. \qquad \square$$

**Remark** If $A$ is a real symmetric matrix, then $A$ is also Hermitian. Therefore, the results in Theorem 8.12 hold for all real symmetric matrices. See Theorem 6.7.

**Example** The matrix

$$A = \begin{bmatrix} 4 & 1-\mathbf{i} \\ 1+\mathbf{i} & 5 \end{bmatrix}$$

is unitarily diagonalizable because it is Hermitian. Find a matrix $P$ that unitarily diagonalizes $A$.

*Solution* The characteristic equation of $A$ is

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - 4 & -1+\mathbf{i} \\ -1-\mathbf{i} & \lambda - 5 \end{bmatrix} = \lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6) = 0$$

and the eigenvalues are $\lambda = 3$ and $\lambda = 6$. The corresponding eigenvectors are given as follows:

$$\lambda = 3, \quad \mathbf{v}_1 = \begin{bmatrix} -1+\mathbf{i} \\ 1 \end{bmatrix}; \qquad \lambda = 6, \quad \mathbf{v}_2 = \begin{bmatrix} \dfrac{1-\mathbf{i}}{2} \\ 1 \end{bmatrix}.$$

Since each eigenspace has only one basis vector, we have $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$ by Theorem 8.12. Normalizing these basis vectors yields

$$\mathbf{p}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \dfrac{-1+\mathbf{i}}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \end{bmatrix}, \qquad \mathbf{p}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} \dfrac{1-\mathbf{i}}{\sqrt{6}} \\ \dfrac{2}{\sqrt{6}} \end{bmatrix}.$$

Thus, $A$ is unitarily diagonalized by the matrix

$$P = \left[\ \mathbf{p}_1 \mid \mathbf{p}_2\ \right] = \begin{bmatrix} \dfrac{-1+\mathbf{i}}{\sqrt{3}} & \dfrac{1-\mathbf{i}}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{6}} \end{bmatrix}.$$

It is easy to verify

$$P^*AP = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}. \qquad \square$$

**Theorem 8.13** *Let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be eigenvalues of an* $n \times n$ *Hermitian matrix* $A$. *Then*

$$\|A\|_F^2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2.$$

*Proof* It follows from Theorem 8.11 and Theorem 8.12 (a) that there exists a unitary matrix $P$ such that

$$P^*AP = D,$$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_k \in \mathbb{R}$ $(1 \leqslant k \leqslant n)$. Hence

$$\begin{aligned}
\|A\|_F^2 &= \|PDP^*\|_F^2 = \operatorname{tr}[(PDP^*)(PDP^*)^*] = \operatorname{tr}(PDD^*P^*) \\
&= \operatorname{tr}(DD^*P^*P) = \operatorname{tr}(DD^*) = \|D\|_F^2 \\
&= \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2.
\end{aligned}$$

Here we used the property of $\operatorname{tr}(VW) = \operatorname{tr}(WV)$ for all $V, W \in \mathbb{C}^{n\times n}$. $\square$

Finally, for certain Hermitian matrices, we introduce the following definition.

**Definition** *A Hermitian matrix* $A$ *is called*

(i) ***positive definite*** *if* $\mathbf{x}^*A\mathbf{x} > 0$ *for all* $\mathbf{x} \neq \mathbf{0}$.

(ii) ***positive semidefinite*** *if* $\mathbf{x}^*A\mathbf{x} \geqslant 0$ *for all* $\mathbf{x}$.

(iii) ***negative definite*** *if* $\mathbf{x}^*A\mathbf{x} < 0$ *for all* $\mathbf{x} \neq \mathbf{0}$.

(iv) ***negative semidefinite*** *if* $\mathbf{x}^*A\mathbf{x} \leqslant 0$ *for all* $\mathbf{x}$.

**Theorem 8.14** *A Hermitian matrix* $A$ *is positive definite (or semidefinite) if and only if all the eigenvalues of* $A$ *are positive (or nonnegative).*

The proof of the theorem is similar to that of Theorem 8.2 and is left as an exercise.

**Remark** The results of Courant-Fischer's Minimax Theorem, Cauchy's Interlace Theorem, and Weyl's Theorem in Section 8.2 also hold for Hermitian matrices.

## 8.5 Böttcher-Wenzel Conjecture

In the final section of the book, we study the ***Böttcher-Wenzel conjecture***.

### 8.5.1 Introduction

A fundamental fact in matrix theory is that the matrix product is not commutative, i.e., there are $n \times n$ matrices $X$ and $Y$ such that

$$XY \neq YX.$$

See Example 2 in Subsection 1.3.1. The difference $XY - YX$ is called the ***commutator*** or ***Lie product*** of $X$ and $Y$. The commutator plays an important role in diverse areas in mathematics, for instance, Lie algebra and Lie group theory [3] and matrix computation [12]. Böttcher and Wenzel [5] proposed the following conjecture in 2005: the upper bound of the Frobenius norm of the commutator of all $n \times n$ matrices $X$ and $Y$ is given by

$$\|XY - YX\|_F \leqslant \sqrt{2}\, \|X\|_F \|Y\|_F.$$

Note that the constant $\sqrt{2}$ is best possible as shown by a simple example

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The conjecture was first proved for all $n \times n$ real matrices in 2008 by Vong and Jin [24]. Later, the result had been generalized to complex matrices [2, 6, 9]. The result is important and fundamental. This can be reflected by the fact that the result is immediately included in the encyclopedic book [4].

### 8.5.2 Proof of the Böttcher-Wenzel conjecture

As defined in Subsection 8.3.2, the Frobenius norm is given by

$$\|A\|_F^2 = \langle A, A \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 = \mathrm{tr}(AA^*).$$

In order to prove the Böttcher-Wenzel conjecture, we need the following lemmas.

**Lemma 8.1** *Let $p_j \geqslant 0$ for $1 \leqslant j \leqslant n$ with $\sum\limits_{j=1}^{n} p_j = 1$ and $q_j$ be real numbers for $1 \leqslant j \leqslant n$. Then*

$$\sum_{j=1}^{n} p_j q_j^2 - \Big(\sum_{j=1}^{n} p_j q_j\Big)^2 \leqslant \sum_{j=1}^{n} \frac{q_j^2}{2}.$$

*Proof* From direct calculations, we have

$$\sum_{j=1}^{n} p_j q_j^2 - \Big(\sum_{j=1}^{n} p_j q_j\Big)^2 = \sum_{j=1}^{n} p_j \Big[q_j - \big(\sum_{k=1}^{n} p_k q_k\big)\Big]^2.$$

Assuming that $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_n$ and denoting $d = \frac{1}{2}(q_1 + q_n) - \sum_{j=1}^{n} p_j q_j$, we deduce

$$\begin{aligned}
&\sum_{j=1}^{n} p_j q_j^2 - \left(\sum_{j=1}^{n} p_j q_j\right)^2 \leqslant \sum_{j=1}^{n} p_j \left[q_j - \left(\sum_{k=1}^{n} p_k q_k\right)\right]^2 + d^2 \\
&= \sum_{j=1}^{n} p_j \left[q_j - \left(\sum_{k=1}^{n} p_k q_k\right)\right]^2 - 2d \sum_{j=1}^{n} p_j \left(q_j - \sum_{k=1}^{n} p_k q_k\right) + d^2 \\
&= \sum_{j=1}^{n} p_j \left[q_j - \left(\sum_{k=1}^{n} p_k q_k\right) - d\right]^2 \\
&= \sum_{j=1}^{n} p_j \left(\frac{q_j - q_n}{2} - \frac{q_1 - q_j}{2}\right)^2 \\
&\leqslant \sum_{j=1}^{n} p_j \left(\frac{q_1 - q_n}{2}\right)^2 \leqslant \frac{1}{4}(2q_1^2 + 2q_n^2) \leqslant \sum_{j=1}^{n} \frac{q_j^2}{2}. \qquad \square
\end{aligned}$$

**Lemma 8.2** *Let $A$ and $B$ be Hermitian matrices. Then the trace of $AB$ is real.*

*Proof* We have

$$\begin{aligned}
\overline{\operatorname{tr}(AB)} &= \operatorname{tr}(\overline{AB}) = \operatorname{tr}(\bar{A}\bar{B}) = \operatorname{tr}\big((\bar{A}\bar{B})^T\big) \\
&= \operatorname{tr}(\bar{B}^T \bar{A}^T) = \operatorname{tr}(B^* A^*) = \operatorname{tr}(BA) \\
&= \operatorname{tr}(AB).
\end{aligned}$$

Thus, the trace of $AB$ is real. □

**Lemma 8.3** (*Cartesian Decomposition* [8]) *Let $M$ be any square matrix with complex entries. Then $M$ can be decomposed as*

$$M = A + \mathbf{i}B,$$

*where $A$ and $B$ are Hermitian matrices and $\mathbf{i} = \sqrt{-1}$.*

*Proof* Let

$$A = \frac{M + M^*}{2}, \qquad B = \mathbf{i} \cdot \frac{M^* - M}{2}.$$

Then $A$ and $B$ are Hermitian and $M = A + \mathbf{i}B$. □

We now state the Böttcher-Wenzel conjecture as the following theorem. The idea of the following proof is elementary.

**Theorem 8.15** *For any $n \times n$ complex matrices $X$ and $Y$, we have*

$$\|XY - YX\|_F^2 \leqslant 2\|X\|_F^2\|Y\|_F^2. \tag{8.9}$$

*Proof* If $X = \mathbf{0}$, then (8.9) holds obviously. Now suppose $X \neq \mathbf{0}$ and then $\|X\|_F > 0$. In the following, we repeatedly use the property of

$$\mathrm{tr}(VW) = \mathrm{tr}(WV)$$

for all $V, W \in \mathbb{C}^{n\times n}$. We deduce

$$\begin{aligned}\|XY - YX\|_F^2 &= \mathrm{tr}[(XY - YX)(XY - YX)^*]\\ &= \mathrm{tr}(XYY^*X^* - XYX^*Y^* - YXY^*X^* + YXX^*Y^*)\\ &= \mathrm{tr}(X^*XYY^* - XYX^*Y^* - YXY^*X^* + XX^*Y^*Y)\end{aligned}$$

and

$$\begin{aligned}\|X^*Y + YX^*\|_F^2 &= \mathrm{tr}[(X^*Y + YX^*)(X^*Y + YX^*)^*]\\ &= \mathrm{tr}(X^*YY^*X + X^*YXY^* + YX^*Y^*X + YX^*XY^*)\\ &= \mathrm{tr}(XX^*YY^* + YXY^*X^* + XYX^*Y^* + X^*XY^*Y).\end{aligned}$$

Thus,

$$\begin{aligned}&\|XY - YX\|_F^2 + \|X^*Y + YX^*\|_F^2\\ &= \mathrm{tr}(X^*XYY^* + XX^*Y^*Y + XX^*YY^* + X^*XY^*Y)\\ &= \mathrm{tr}[(X^*X + XX^*)(Y^*Y + YY^*)].\end{aligned} \tag{8.10}$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|\mathrm{tr}[Y(X^*X + XX^*)]| &= |\mathrm{tr}[(X^*Y + YX^*)X]| = |\langle X^*Y + YX^*, X^*\rangle|\\ &\leqslant \|X^*\|_F\|X^*Y + YX^*\|_F = \|X\|_F\|X^*Y + YX^*\|_F.\end{aligned}$$

Consequently,

$$\|X^*Y + YX^*\|_F^2 \geqslant |\mathrm{tr}[Y(X^*X + XX^*)]|^2/\|X\|_F^2. \tag{8.11}$$

Combining (8.10) and (8.11) then gives

$$\|XY - YX\|_F^2 \leqslant \mathrm{tr}[(X^*X + XX^*)(Y^*Y + YY^*)] - |\mathrm{tr}[Y(X^*X + XX^*)]|^2/\|X\|_F^2. \tag{8.12}$$

Let

$$D = (X^*X + XX^*)/(2\|X\|_F^2).$$

We can simplify (8.12) by using $D$ as follows:

$$\|XY - YX\|_F^2 \leqslant 4\|X\|_F^2 \left[\mathrm{tr}[D(Y^*Y + YY^*)/2] - |\mathrm{tr}(DY)|^2\right]. \tag{8.13}$$

Note that $D$ is positive semidefinite with

$$\mathrm{tr}(D) = \mathrm{tr}[(X^*X + XX^*)/(2\|X\|_F^2)] = \mathrm{tr}(XX^*)/\|X\|_F^2 = \|X\|_F^2/\|X\|_F^2 = 1.$$

Now, it remains to show that the right-hand side of (8.13) satisfies the following inequality:

$$\mathrm{tr}[D(Y^*Y + YY^*)/2] - |\mathrm{tr}(DY)|^2 \leqslant \frac{\|Y\|_F^2}{2}.$$

Following Lemma 8.3, we suppose that

$$Y = A + \mathbf{i}B,$$

where $A$, $B$ are Hermitian and $\mathbf{i} = \sqrt{-1}$. Obviously,

$$\frac{1}{2}(Y^*Y + YY^*) = A^2 + B^2,$$

and then

$$\|Y\|_F^2 = \mathrm{tr}(YY^*) = \mathrm{tr}(A^2 + B^2) = \mathrm{tr}(AA^*) + \mathrm{tr}(BB^*) = \|A\|_F^2 + \|B\|_F^2. \tag{8.14}$$

Using Lemma 8.2 that the trace of the product of two Hermitian matrices is a real number, we therefore have

$$|\mathrm{tr}(DY)|^2 = |\mathrm{tr}(DA) + \mathbf{i}\mathrm{tr}(DB)|^2 = [\mathrm{tr}(DA)]^2 + [\mathrm{tr}(DB)]^2.$$

Hence

$$\begin{aligned} &\mathrm{tr}[D(Y^*Y + YY^*)/2] - |\mathrm{tr}(DY)|^2 \\ &= \mathrm{tr}[D(A^2 + B^2)] - [\mathrm{tr}(DA)]^2 - [\mathrm{tr}(DB)]^2 \\ &= \left(\mathrm{tr}(DA^2) - [\mathrm{tr}(DA)]^2\right) + \left(\mathrm{tr}(DB^2) - [\mathrm{tr}(DB)]^2\right). \end{aligned} \tag{8.15}$$

It follows from (8.13) and (8.15) that

$$\|XY - YX\|_F^2 \leqslant 4\|X\|_F^2 \left[\left(\mathrm{tr}(DA^2) - [\mathrm{tr}(DA)]^2\right) + \left(\mathrm{tr}(DB^2) - [\mathrm{tr}(DB)]^2\right)\right]. \tag{8.16}$$

Next, we want to show that for any Hermitian matrix $H \in \mathbb{C}^{n\times n}$,

$$\mathrm{tr}(DH^2) - [\mathrm{tr}(DH)]^2 \leqslant \frac{\|H\|_F^2}{2}. \tag{8.17}$$

By Theorems 8.11 and 8.12, we have

$$H = U\Lambda U^*,$$

where $U$ is unitary and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let

$$P = U^* D U = [p_{ij}].$$

Then $P$ is also a positive semidefinite matrix with

$$\operatorname{tr}(P) = \operatorname{tr}(U^* D U) = \operatorname{tr}(D) = 1.$$

Thus, for every column vector $\mathbf{e}_j$ of the $n \times n$ identity matrix, it follows from the definition of positive semidefinite matrices that for $1 \leqslant j \leqslant n$,

$$p_{jj} = \mathbf{e}_j^* P \mathbf{e}_j \geqslant 0.$$

Since $p_{jj} \geqslant 0$ and $\sum\limits_{j=1}^{n} p_{jj} = \operatorname{tr}(P) = 1$, we have by Lemma 8.1 and Theorem 8.13,

$$\begin{aligned}
&\operatorname{tr}(DH^2) - [\operatorname{tr}(DH)]^2 = \operatorname{tr}(P\Lambda^2) - [\operatorname{tr}(P\Lambda)]^2 \\
&= \sum_{j=1}^{n} p_{jj}\lambda_j^2 - \Big(\sum_{j=1}^{n} p_{jj}\lambda_j\Big)^2 \leqslant \sum_{j=1}^{n} \frac{\lambda_j^2}{2} = \frac{\|H\|_F^2}{2}.
\end{aligned}$$

Then (8.17) holds. Applying (8.17) and (8.14) to (8.16), we finally obtain

$$\|XY - YX\|_F^2 \leqslant 4\|X\|_F^2 \frac{\|A\|_F^2 + \|B\|_F^2}{2} = 2\|X\|_F^2 \|Y\|_F^2. \qquad \square$$

## Exercises

### Elementary exercises

**8.1** Express the following quadratic forms in the matrix notation $\mathbf{x}^T A \mathbf{x}$, where $A$ is a symmetric matrix.

(a) $x_1x_2 + x_1x_3 + x_2x_3$. (b) $5x_1^2 + 5x_1x_2$. (c) $4x_1^2 - 9x_2^2 - 6x_1x_2$.

**8.2** Determine which of the following matrices are positive definite.

(a) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

**8.3** Find the maximum and minimum values of each given quadratic form subject to the constraint $x_1^2 + x_2^2 = 1$. Then determine values of $x_1$ and $x_2$ at which the maximum and minimum occur.

(a) $f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2$. (b) $f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$.

**8.4** Prove Theorem 8.1 (b).

**8.5** Determine which of the following quadratic forms are positive definite.

(a) $90x_1^2 + 130x_2^2 + 71x_3^2 - 12x_1x_2 + 48x_1x_3 - 60x_2x_3$.

(b) $-5x_1^2 - 6x_2^2 - 4x_3^2 + 4x_1x_2 + 4x_1x_3$.

**8.6** In each part, find all values of $k$ for which the quadratic form is positive definite.

(a) $x_1^2 + kx_2^2 - 4x_1x_2$.

(b) $2x_1^2 + (2+k)x_2^2 + kx_3^2 + 2x_1x_2 - 2x_1x_3 + x_2x_3$.

**8.7** Show that if $A, B \in \mathbb{R}^{n\times n}$ are positive semidefinite and $\alpha, \beta \in \mathbb{R}$ are nonnegative, then $\alpha A + \beta B$ is positive semidefinite.

**8.8** Let $\mathbf{x}^T A\mathbf{x}$ be a quadratic form and $T\colon \mathbb{R}^n \to \mathbb{R}$ defined by $T(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

(a) Show that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + 2\mathbf{x}^T A\mathbf{y} + T(\mathbf{y})$.

(b) Show that $T(k\mathbf{x}) = k^2 T(\mathbf{x})$, where $k$ is a scalar.

**8.9** Prove Theorem 8.3.

**8.10** Prove Theorem 8.4.

**8.11** Prove Theorem 8.8 (b) and (c).

**8.12** In each part, find real numbers $\alpha$ and $\beta$ that satisfy the following equation.

(a) $\alpha\mathbf{i} + \beta(1+\mathbf{i}) = 3 + 6\mathbf{i}$. (b) $\alpha(2+3\mathbf{i}) + \beta(1-4\mathbf{i}) = -1 + 4\mathbf{i}$.

**8.13** Let $\mathbf{u} = [1, 0, -\mathbf{i}]$, $\mathbf{v} = [1+\mathbf{i}, 1, 1-2\mathbf{i}]$, and $\mathbf{w} = [0, \mathbf{i}, 2]$. Express the following vectors as linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

(a) $[1, 1, 1]$. (b) $[\mathbf{i}, 0, -\mathbf{i}]$. (c) $[2-\mathbf{i}, 1, 1+\mathbf{i}]$.

**8.14** Which of the following sets of vectors in $\mathbb{C}^3$ are linearly independent?

(a) $\mathbf{u}_1 = [1-\mathbf{i}, 1, 0]$, $\mathbf{u}_2 = [2, 1+\mathbf{i}, 0]$, $\mathbf{u}_3 = [1+\mathbf{i}, \mathbf{i}, 0]$.

(b) $\mathbf{u}_1 = [1, 0, -\mathbf{i}]$, $\mathbf{u}_2 = [1+\mathbf{i}, 1, 1-2\mathbf{i}]$, $\mathbf{u}_3 = [0, \mathbf{i}, 2]$.

(c) $\mathbf{u}_1 = [\mathbf{i}, 0, 2-\mathbf{i}]$, $\mathbf{u}_2 = [0, 1, \mathbf{i}]$, $\mathbf{u}_3 = [-\mathbf{i}, -1-4\mathbf{i}, 3]$.

**8.15** If $\mathbf{u} = [u_1, u_2], \mathbf{v} = [v_1, v_2] \in \mathbb{C}^2$, determine which of the following functions $f$ are inner products.

(a) $f(\mathbf{u}, \mathbf{v}) = 3u_1\bar{v}_1 + 2u_2\bar{v}_2$.

(b) $f(\mathbf{u}, \mathbf{v}) = u_1\bar{v}_1 + (1+\mathbf{i})u_1\bar{v}_2 + (1-\mathbf{i})u_2\bar{v}_1 + 3u_2\bar{v}_2$.

**8.16** Find $\|\mathbf{x}\|$ using the Euclidean inner product on $\mathbb{C}^2$.

(a) $\mathbf{x} = [1, \mathbf{i}]$. (b) $\mathbf{x} = [1-\mathbf{i}, 1+\mathbf{i}]$. (c) $\mathbf{x} = [-\mathbf{i}, 3\mathbf{i}]$.

**8.17** Show that the vectors $\mathbf{u}_1 = [\mathbf{i}, \mathbf{i}, \mathbf{i}]$, $\mathbf{u}_2 = [-2\mathbf{i}, \mathbf{i}, \mathbf{i}]$, and $\mathbf{u}_3 = [0, -\mathbf{i}, \mathbf{i}]$ form an orthogonal basis for $\mathbb{C}^3$ with the Euclidean inner product. By normalizing each of these vectors, find an orthonormal set.

**8.18** Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in a complex inner product space, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 + \frac{\mathbf{i}}{4}\|\mathbf{u}+\mathbf{i}\mathbf{v}\|^2 - \frac{\mathbf{i}}{4}\|\mathbf{u}-\mathbf{i}\mathbf{v}\|^2.$$

**8.19** Show that if $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for a complex inner product space $V$, then for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w}_1 \rangle \overline{\langle \mathbf{v}, \mathbf{w}_1 \rangle} + \langle \mathbf{u}, \mathbf{w}_2 \rangle \overline{\langle \mathbf{v}, \mathbf{w}_2 \rangle} + \cdots + \langle \mathbf{u}, \mathbf{w}_n \rangle \overline{\langle \mathbf{v}, \mathbf{w}_n \rangle}.$$

**8.20** Let $A \in \mathbb{C}^{n\times n}$. Show that $A = \mathbf{0}$ if and only if $\mathbf{x}^*A\mathbf{x} = 0$ for any $\mathbf{x} \in \mathbb{C}^n$.

**8.21** Prove Theorem 8.10.

**8.22** In each part, find a unitary matrix $P$ that diagonalizes $A$, and find $P^*AP$.

(a) $A = \begin{bmatrix} 2 & -\mathbf{i} \\ \mathbf{i} & 2 \end{bmatrix}$. (b) $A = \begin{bmatrix} 6 & 2+2\mathbf{i} \\ 2-2\mathbf{i} & 4 \end{bmatrix}$.

**8.23** Let $A$ and $B$ be $n \times n$ Hermitian matrices.

(a) Show that $A + B$ is a Hermitian matrix.

(b) Show that $AB$ is a Hermitian matrix if and only if $AB = BA$.

**8.24** In each part, verify that the matrix is unitary and find its inverse.

(a) $\frac{1}{5}\begin{bmatrix} 3 & 4\mathbf{i} \\ -4 & 3\mathbf{i} \end{bmatrix}$. (b) $\frac{1}{2}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -(1+\mathbf{i}) & 1+\mathbf{i} \end{bmatrix}$.

**8.25** Prove Theorem 8.14.

**Challenge exercises**

**8.26** Show that if $A \in \mathbb{R}^{n\times n}$ is symmetric and $A^2 = \mathbf{0}$, then $A = \mathbf{0}$.

**8.27** Let

$$f(x,y,z) = \frac{2x^2 + y^2 - 4xy - 4yz}{x^2 + y^2 + z^2}, \qquad x^2 + y^2 + z^2 \neq 0.$$

Find the maximum and minimum values of the function $f(x,y,z)$, and determine values of $x$, $y$, and $z$ at which the maximum and minimum occur.

**8.28** Let $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{y} = [y_1, y_2]^T$. Find an orthogonal matrix $Q$ such that the change of variable $\mathbf{x} = Q\mathbf{y}$ transforms the quadratic form

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 5x_2^2$$

into a new quadratic form in the variables $y_1$ and $y_2$ with no cross-product terms.

**8.29** Let $A$ be a symmetric matrix such that

$$A^3 - 4A^2 + 5A = 2I,$$

where $I$ is the identity matrix. Show that $A$ is symmetric positive definite.

**8.30** Let $A$ be a symmetric positive definite matrix. Show that there exists a symmetric positive definite matrix $B$ such that $A = B^2$.

**8.31** Let $A \in \mathbb{R}^{n\times n}$ and

$$B = \lambda I + A^T A,$$

where $\lambda > 0$ and $I$ is the identity matrix. Show that $B$ is symmetric positive definite.

**8.32** Let $A$ and $B$ be symmetric positive semidefinite matrices of the same size. Show that $\mathrm{tr}(AB) \geqslant 0$.

**8.33** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be symmetric positive definite matrices of the same size. Show that $C = [a_{ij}b_{ij}]$ is a symmetric positive definite matrix.

**8.34** Let $A = [a_{ij}]$ be an $n \times n$ symmetric positive semidefinite matrix. Show that

(a) $a_{ii} \geqslant 0$ for $1 \leqslant i \leqslant n$.

(b) If $a_{ii} = 0$, then the $i$th row and $i$th column of $A$ consist entirely of 0.

**8.35** Prove Theorem 8.11.

**8.36** Let $B \in \mathbb{C}^{n\times n}$ be invertible. Show that $A = B^*B$ is Hermitian positive definite.

**8.37** Let $A$ be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that

$$(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I) = \mathbf{0},$$

where $I$ is the identity matrix.

**8.38** Let $A$ and $B$ be Hermitian matrices of the same size. Show that if $AB$ is Hermitian, then every eigenvalue $\lambda$ of $AB$ can be written as $\lambda = \mu\nu$, where $\mu$ is an eigenvalue of $A$ and $\nu$ is an eigenvalue of $B$.

**8.39** Let $A \in \mathbb{C}^{m \times n}$ and $\mathbf{x} \in \mathbb{C}^n$. Show that $(A^*A)\mathbf{x} = \mathbf{0}$ if and only if $A\mathbf{x} = \mathbf{0}$.

**8.40** Let $A \in \mathbb{C}^{m \times n}$. Show that $\operatorname{tr}(A^*A) = 0$ if and only if $A = \mathbf{0}$.

# Appendix A

# Independence of Axioms

An axiom is independent if it can not be proved by using other axioms. To reach the conclusion of a reduced set of axioms, independence is desired. In this appendix, we study the independence of the axioms of vector spaces. For convenience, we copy the definition in Subsection 4.1.1 to here.

**Definition** *Let $V$ be a nonempty set of objects on which two operations are defined, addition and scalar multiplication. It requires that $V$ is closed under the addition and scalar multiplication, i.e., for each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in $V$, $\mathbf{u}+\mathbf{v}$ is in $V$; for each scalar $k$ and each object $\mathbf{u}$ in $V$, $k\mathbf{u}$ is in $V$. Then $V$ is called a **vector space** and the objects in $V$ are said to be **vectors** if the following eight axioms are satisfied for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$.*

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(ii) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(iii) *There is an object $\mathbf{0}$ in $V$, called a zero vector for $V$, such that for all $\mathbf{u}$ in $V$, $\mathbf{u}+\mathbf{0}=\mathbf{u}$.*

(iv) *For each $\mathbf{u}$ in $V$, there is an object $-\mathbf{u}$ in $V$, called a negative of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.*

(v) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.

(vi) $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$.

(vii) $k(l\mathbf{u})=(kl)\mathbf{u}$.

(viii) $1\mathbf{u}=\mathbf{u}$.

*Here $k$ and $l$ are scalars.*

Actually, Axiom (i) is not independent because it can be deduced by the other axioms [13, 20]. We next use other axioms to prove Axiom (i).

**Lemma A.1** *We have* $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$ *for each vector* $\mathbf{u}$ *in* $V$.

*Proof* For any $\mathbf{u} \in V$, we have $-\mathbf{u} \in V$ by Axiom (iv). Then it follows from Axiom (iv) again that $-(-\mathbf{u}) \in V$. Thus,

$$\begin{aligned}
(-\mathbf{u}) + \mathbf{u} &= (-\mathbf{u}) + \mathbf{u} + \mathbf{0} && \text{[Axiom (iii)]}\\
&= (-\mathbf{u}) + \mathbf{u} + \big[(-\mathbf{u}) + [-(-\mathbf{u})]\big] && \text{[Axiom (iv)]}\\
&= (-\mathbf{u}) + [\mathbf{u} + (-\mathbf{u})] + [-(-\mathbf{u})] && \text{[Axiom (ii)]}\\
&= (-\mathbf{u}) + \mathbf{0} + [-(-\mathbf{u})] && \text{[Axiom (iv)]}\\
&= (-\mathbf{u}) + [-(-\mathbf{u})] && \text{[Axiom (iii)]}\\
&= \mathbf{0}. && \text{[Axiom (iv)]}
\end{aligned}$$

□

**Lemma A.2** *We have* $\mathbf{0} + \mathbf{u} = \mathbf{u}$ *for each vector* $\mathbf{u}$ *in* $V$.

*Proof* We have

$$\begin{aligned}
\mathbf{0} + \mathbf{u} &= \mathbf{u} + (-\mathbf{u}) + \mathbf{u} && \text{[Axiom (iv)]}\\
&= \mathbf{u} + [(-\mathbf{u}) + \mathbf{u}] && \text{[Axiom (ii)]}\\
&= \mathbf{u} + \mathbf{0} && \text{[Lemma A.1]}\\
&= \mathbf{u}. && \text{[Axiom (iii)]}
\end{aligned}$$

□

**Theorem A.1** *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $V$, *we have*

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

*Proof* We deduce

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= \mathbf{0} + \mathbf{u} + \mathbf{v} + \mathbf{0} && \text{[Lemma A.2, Axiom (iii)]}\\
&= [(-\mathbf{u}) + \mathbf{u}] + \mathbf{u} + \mathbf{v} + [\mathbf{v} + (-\mathbf{v})] && \text{[Lemma A.1, Axiom (iv)]}\\
&= (-\mathbf{u}) + (\mathbf{u} + \mathbf{u} + \mathbf{v} + \mathbf{v}) + (-\mathbf{v}) && \text{[Axiom (ii)]}\\
&= (-\mathbf{u}) + (1\mathbf{u} + 1\mathbf{u} + 1\mathbf{v} + 1\mathbf{v}) + (-\mathbf{v}) && \text{[Axiom (viii)]}\\
&= (-\mathbf{u}) + (2\mathbf{u} + 2\mathbf{v}) + (-\mathbf{v}) && \text{[Axioms (ii), (vi)]}\\
&= (-\mathbf{u}) + 2(\mathbf{u} + \mathbf{v}) + (-\mathbf{v}) && \text{[Axiom (v)]}\\
&= (-\mathbf{u}) + [(\mathbf{u} + \mathbf{v}) + (\mathbf{u} + \mathbf{v})] + (-\mathbf{v}) && \text{[Axiom (vi)]}\\
&= [(-\mathbf{u}) + \mathbf{u}] + \mathbf{v} + \mathbf{u} + [\mathbf{v} + (-\mathbf{v})] && \text{[Axiom (ii)]}
\end{aligned}$$

$$= \mathbf{0} + \mathbf{v} + \mathbf{u} + \mathbf{0} \qquad \text{[Lemma A.1, Axiom (iv)]}$$

$$= \mathbf{v} + \mathbf{u}. \qquad \text{[Lemma A.2, Axiom (iii)]}$$

$\square$

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