New ICMI Study Series

Hilda Borko
Despina Potari *Editors*

Teachers of Mathematics Working and Learning in Collaborative Groups

The 25th ICMI Study

International Commission on Mathematical Instruction

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New ICMI Study Series

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NISS Aims and Scope

The New ICMI Study Series (NISS) presents the results of studies mounted on a regular basis by the International Commission on Mathematical Instruction (ICMI). Among international organizations devoted to mathematics education, ICMI is distinctive because of its close ties to both the mathematics and the mathematics education professional communities, as well as for its breadth — thematic, cultural, and regional.

The ICMI Study Programme, launched in the mid-1980's, supports the general aims of ICMI, such as fostering efforts around the world to improve the quality of mathematics teaching and learning. This work of ICMI stimulates the growth, synthesis, and dissemination of new knowledge (research) and of resources for instruction (curricular materials, pedagogical methods, technology, etc.). ICMI also provides a forum for all stakeholders in mathematics education (teachers, researchers, mathematicians, etc.) promoting reflection, collaboration, exchange and dissemination of ideas and information on all aspects of the theory and practice of mathematics education, as seen from an international perspective.

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Each ICMI Study addresses an issue or topic of particular significance in contemporary mathematics education, and is conducted by an international team of leading scholars and practitioners in that domain. The best contributing professionals from around the world are then invited to a carefully planned and structured international conference/workshop. Beyond the productive interaction and collaborations occasioned by this event, the main product is a Study volume, which aims to offer a coherent, state-of-the art representation of the domain of the Study. It is these Study volumes that constitute the New ICMI Study Series (NISS).

The books published in the NISS series reflect the great variety of issues and concerns in the field of mathematics education and will be of interest to educational researchers, curriculum developers, educational policy makers, teachers of mathematics, and to mathematicians and educators involved in the professional education and development of teachers of mathematics.

This Series is indexed in Scopus.

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Foreword

It almost goes without saying that teaching is a collaborative practice. I write ‘almost’ to indicate that while teacher collaboration might be ubiquitous, this everywhereness masks its diverse forms, functions and effects, raising questions for research and practice as to whether and how this matters, and for whom, in the teaching and learning of mathematics.

In 2019 ICMI launched ICMI Study 25 on Teachers of Mathematics Working and Learning in Collaborative Groups, as this field of inquiry and research had grown extensively. It was time to ask: what is our cumulative knowledge and expertise, and what do we still need to understand and explore? A Discussion Document was produced by the International Program Committee (IPC) appointed to this task, calling for papers towards a conference on this theme to be held at the *Instituto de Educação, Universidade de Lisboa* in Lisbon, Portugal, February 3–7, 2020. The goal, as with all ICMI studies, was to gather together researchers and teachers across the world with interest, experience and expertise in this theme, and through collaboration, engage in the development of a volume that would capture our collective wisdom about this field in mathematics education.

I remember well the opening of the conference. February 2020 was a time when we were newly aware of COVID-19 but had no substantive idea of the pandemic unfolding and how it would impact all of our lives and so many livelihoods. I also remember the enthusiasm through the conference as participants worked together to share and further their knowledge and understanding of the what and how of mathematics teacher collaboration.

The journey from the launch of the study, through the conference and to the study volume is a lengthy one. It required ongoing collaboration and sustained commitment over 4 years amongst the conference co-chairs, the IPC that worked with them, and all participants at the conference. In their introduction, Hilda Borko and Despina Potari (the co-chairs) describe the journey and outline the chapters in the volume and what they offer. I do not rehearse any of that here, but rather use this opportunity to congratulate and thank all authors included in this volume for the insightful and interesting work done.

The volume offers all those working in the field of mathematics teaching and teacher education in general, and with direct interest in the what, where, how and why of teacher collaboration more specifically, an important state of the art in the field, with informed recommendations of what still lies ahead for further research and practice. ICMI, and the field of mathematics education, is indeed fortunate that the considerable work of an ICMI Study and the production of a study volume continues to be undertaken by our leading researchers. I also know, and this is communicated in various ways through the chapters of this volume, that the collaboration is mutually productive, and that all who have participated in this work have themselves benefited a great deal.

On behalf of ICMI

Past ICMI President (2017–2020) Jill Adler

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Part I
Introduction

Chapter 1
Introduction

Hilda Borko and Despina Potari

1.1 The Need for the Study

Across education systems and at all educational levels, mathematics teachers work and learn together through various forms of collaboration. Teachers collaborate both in face-to-face and virtual settings, and in a diverse set of formal and informal groupings, including teams, communities, schools, teacher education programs, professional development courses, and local and national networks. Their collaborations may also include people who support their learning such as coaches, mentors and professional development facilitators.

Teachers' collaborative work has a long tradition in mathematics education as an important way of bringing educational innovation into the everyday practice of teaching. The idea of mathematics teachers working and learning through collaboration has been gaining increasing attention in mathematics education research, particularly since the report on Lesson Study in Japan from the TIMSS classroom video study (Stigler et al., 1999; Hiebert et al., 2003). In 2014, *ZDM: The International Journal on Mathematics Education* published a special issue focused on collaboration entitled, "Interactive practices in promoting professional development of didacticians and teachers of mathematics: An international perspective". As Jaworski and Huang (2014) noted in their introduction to this special issue, "we see collaborative critical inquiry between teachers and didacticians emerging as a significant force for teaching development" (p. 173).

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This attention to teachers learning through collaboration is especially relevant as countries around the world strive to improve educational experiences for all children, and to see these improvements reflected in scores on international assessments such as PISA and TIMSS (Schliecher, 2015). Indeed, Schliecher's OECD report includes a policy recommendation: "Encourage collaboration among teachers, either through professional development activities or classroom practices" (p. 56). It cites research indicating that collaborative professional development is related to a positive impact on: teachers' instructional strategies; their self-esteem and self-efficacy; student learning processes, motivation and outcomes.

Efforts to understand what teachers do as they work in collaborative groups, and how these experiences lead to improvement in their practice and expertise, have led to a growing interest in examining the different activities, processes, contexts and outcomes for teacher collaboration around the world. The work completed by the ICME-13 Survey Team on this theme is further evidence of the considerable international interest in research on teachers working and learning through collaboration (Jaworski et al., 2017; Robutti et al., 2016). Similar to Schliecher's OECD report, the ICME-13 Survey, which was presented at the ICME Congress in 2016, showed that across education systems mathematics teachers work and learn through numerous forms of collaboration, involving different groups of people in a variety of roles. These diverse forms of collaboration and varying combinations of people contribute to learning and development in a variety of ways.

The project also identified several gaps and limitations, not only in the existing research base, but also in the survey's coverage of relevant topics within the theme. For example, Jaworski et al. (2017) reported that their research questions about learning outcomes were the most difficult to address. They did not have consistent clarity on the specific mathematics knowledge and pedagogy that was learned, the ways in which learning occurred or the relationship between collaboration and learning. As they also noted, the survey predominantly reported from the perspective of researchers. In addition, there were issues for which the survey showed that research is not extensive and further studies are needed, such as sustainability and scalability, the role of digital technology in teachers' collaborative learning, working with teachers of different educational levels and making teachers' voices more evident.

ICMI Study 25 aims to build on previous findings about mathematics teacher collaboration and to address the gaps and limitations identified in the ICME-13 Survey. Specifically, as was stated in the Discussion Document (IPC, 2019) that solicited contributions to the Study, it aims: to reflect the diversity of settings and groupings in which mathematics teacher collaboration occurs; to explore the tools and resources that support mathematics teacher collaboration; to address the breadth of outcomes of such collaboration; to represent teachers' experiences and learning through their own voices, as well as the voices of researchers. Because there are different ways of understanding the nature of teacher collaboration and its consequences, the Study includes contributions representing multiple theoretical perspectives and a variety of methodological approaches. (The Discussion Document is included in this Study volume as an appendix.)

1.2 The Scope and Aims of the Study

The scope and the aims of the study were established and developed through the meetings of the International Program Committee (IPC), the Study Conference and preparation of the Study volume. The Study volume was the outcome of the collaboration between the members of the Committee, the participants in the working groups at the conference, and the plenary speakers and panellists. We discuss below the scope and the initial aims of the study as they are stated in the Discussion Document.

The primary aims of the Study as they are phrased in the Discussion Document are:

> to report the state of the art in the area of mathematics teacher collaboration with respect to theory, research, practice, and policy; and to suggest new directions of research that take into account contextual, cultural, national and political dimensions. (p. 3)

To address these aims, we had to define the meaning of teacher collaboration. We discussed possible definitions in the first meeting of the IPC. The distinction between 'co-operation' and 'collaboration' that has been addressed earlier in our field (Peter-Koop et al., 2003) was also considered in our discussions. For Peter-Koop et al., co-operation is usually set up externally and the participants contribute to various aspects of a task. On the other hand, collaboration is initiated by the participants, and it involves the sharing of leadership and control to achieve a goal worthwhile to all participants.

Our conceptualisation of mathematics teacher collaboration adopts a similar perspective. It goes beyond the gathering of teachers in the context of a professional development program or even in everyday school meetings or online networks. Collaboration is characterised by the formation of communities where teachers are involved in joint reflection aiming to develop teaching. We also acknowledge that the form, the goals and the outcomes of collaboration depend on the conditions in which it takes place, as well as on the experiences of the participants and the availability of resources. The ICME-13 survey showed that, although teachers are the central actors in collaborative contexts, their 'voices' typically are not heard outside the context of collaboration. A central goal of the Study is to provide the opportunity for teachers to share their collaborative experiences in the Study Conference and in the Study volume. To this end, we organised a plenary teacher panel at the conference, and one of the chapters of this Study volume is based on the work of this panel.

The research areas and the set of questions that this Study investigates are organised into four themes: A. Theoretical perspectives on studying mathematics teacher collaboration; B. Contexts, forms and outcomes of mathematics teacher collaboration; C. Roles, identities and interactions of various participants in mathematics teacher collaboration; D. Tools and resources used/designed for teacher collaboration and resulting from teacher collaboration.

The IPC formulated several questions for each theme to be addressed in the Study. The questions of Theme A concern the different theoretical perspectives that can enhance our understanding of the processes and the outcomes of teacher collaboration, their strengths and weaknesses, as well as methodological issues related to the study of teacher collaboration. In Theme B, the questions are about the different models of teacher collaboration, their effectiveness in relation to the desired outcomes and the different contexts and conditions for teacher collaboration. Online teacher collaboration, and its benefits and challenges, are also considered in this theme.

The questions of Theme C address the roles and identities of the different stakeholders in the context of teacher collaboration. They also aim to investigate what types of learning environments support professional learning of teachers and of other participants. The Theme D questions refer both to the resources that are available to support teacher collaboration and their impact on the collaboration, and to the design of resources in the context of collaboration. Moreover, issues of scaling-up collaboration and the opportunities for digital environments and resources are also addressed in this theme.

The themes and the questions are reported in the Discussion Document. They were the basis of the submitted papers and of the work of the Working Groups in the Study Conference, and they are reflected in the structure of this volume.

1.3 The Study Conference: Its Program, Structure and Outcomes

The themes and questions identified by the IPC and described in the Discussion Document provided the basis for the Call for Contributions to ICMI Study 25. The call invited submissions of several types including: reports of research studies; syntheses and meta-analyses of empirical studies; discussions of theoretical and methodological issues; examinations of the ways that teacher collaboration has taken place in local or national contexts. To address the complexity of mathematics teacher collaboration, it also encouraged papers reflecting different cultural, political and educational contexts and submissions by researchers, teachers and policy-makers.

We received more than 100 submissions in response to the call. These papers were reviewed by the IPC, which provided feedback and, in many cases, requested revisions. 80 papers were accepted for the Study Conference and were included in the Conference Proceedings. The countries represented include: Algeria, Argentina, Australia, Austria, Brazil, Canada, China, Colombia, Cyprus, Denmark, France, Germany, Greece, India, Iran, Ireland, Israel, Italy, Japan, Malawi, Malta, Mexico, Netherlands, New Zealand, Norway, Portugal, Slovakia, South Africa, South Korea, Spain, Sweden, Switzerland, Taiwan, Turkey, United Kingdom, United States and the US Virgin Islands.

The ICMI Study 25 Conference was hosted by the Instituto de Educação, Universidade de Lisboa in Lisbon, Portugal, on February 3–7, 2020. As is the case for all ICMI Study Conferences, most of the time was spent in Working Groups organised around the themes and led by IPC members. During the Working Group sessions, brief presentations by the participants, based on their papers, served as a springboard for in-depth exploration of the themes and associated questions. These intense discussions were directed toward the preparation of this Study volume. In preparation for the Working Group discussions, a draft of the Conference Proceedings was distributed to participants prior to the conference, so that they would have time to read the papers for their theme's Working Group in advance.

We also invited four renowned scholars to present plenary lectures related to each theme at the Study Conference, and four others to respond to their lectures. The plenary lectures were included in the Conference Proceedings. The lecturers and reactors were:

- Theme A: Susanne Prediger, lecturer; Boris Koichu, reactor;
- Theme B: Masami Isoda, lecturer; Alf Coles, reactor;
- Theme C: Konrad Krainer and Carina Spreitzer, lecturers; Bettina Roesken-Winter, reactor;
- Theme D: Karin Brodie, lecturer; Kara Jackson, reactor.

In addition, to ensure that teachers' voices were well-represented at the Study Conference, we invited four practitioners actively involved in very different collaborative experiences to participate in a plenary panel moderated by a renowned scholar. The panelists' papers and the moderator's introduction and synthesis were also included in the Conference Proceedings. The panelists were: Yiyi Chen, China; Christelle Fitamant, France; Lameck Dition Sandram, Malawi; Shelli Temple, USA. The moderator, whose career includes multiple collaborative projects with teachers, was Hilary Hollingsworth, Australia.

The Study Conference was held after the beginning of the COVID-19 pandemic, before the world was aware of its nature and the rapidity with which it would spread. Several countries, including China, had already begun to implement travel restrictions. To enable participants' attendance, our conference hosts at the Instituto de Educação, Universidade de Lisboa were able to arrange their virtual participation. This experience, although not what we envisioned or would have preferred, presented the opportunity for our group to reflect upon and learn from an additional form of collaboration. We share some of those reflections throughout this Study volume.

1.4 Structure of the Study Volume

The chapters in this book are organised in four parts. The first Part consists of this introductory editorial chapter. Part 2 includes the four chapters that reflect the four organising Theme Working Groups of the ICMI Study Conference:

- Theme A: Theoretical Perspectives on Studying Mathematics Teacher Collaboration (lead authors: João Pedro da Ponte and Takeshi Miyakawa);
- Theme B: Contexts, Forms and Outcomes of Mathematics Teacher Collaboration (lead authors: Cristina Esteley and Rongjin Huang);

- Theme C: Roles, Identities and Interactions of Various Participants in Mathematics Teacher Collaboration (lead authors: Ronnie Karsenty and Shelley Dole);
- Theme D: Tools and Resources Used/Designed for Teacher Collaboration and Resulting from Teacher Collaboration (lead authors: Ornella Robutti and Luc Trouche).

The lead authors for each chapter are the IPC members who led both the working groups during the Study Conference and the writing of the chapters. Working groups members had different levels of collaboration during the writing process; these are reflected in the lists of co-authors and contributors.

Part 3 includes chapters related to the plenary addresses and the plenary panel. There are two chapters related to Theme A, one by the plenary speaker and one by the reactor. Chapters related to Themes C and D were written collaboratively by the plenary speakers and their reactors. The chapter related to the plenary panel was authored by the moderator and the four speakers, under the leadership of the moderator. (Due to unexpected circumstances, there is no chapter for the plenary address for Theme B.) Thus, the five chapters are:

- Theme A (plenary): Using and Developing Content-Related Theory Elements for Explaining and Promoting Teachers' Professional Growth in Collaborative Groups;
- Theme A (reaction): The Art of Being Specific while Theorising for and from Practice of Mathematics Teachers' Collaboration;
- Theme C: Capturing Collaboration in Mathematics Teacher Education, in Terms of Relevant Actors, Targets and Environments;
- Theme D: Resources for and from Collaboration: A Conceptual Framework;
- Plenary Panel: Working and Learning in Collaborative Groups: What's Key to Mathematics Teachers?

The final part (Part IV) of this volume consists of two commentaries, one by Dario Fiorentini and Ana Losano, and one by Rina Zazkis. We invited these distinguished scholars, who have extensive experience with teachers' collaborative work, to comment on the Study themes and chapters in this Volume.

1.5 Editorial Overview of Thematic Working Groups Chapters and Plenary Chapters

1.5.1 Thematic Working Group Chapters

1.5.1.1 Theme A

Chapter 2 reports the outcomes from the working group on theme A (Theoretical perspectives on studying mathematics teacher collaboration) and the presented papers (18) pointing out theories and theoretical frameworks used to study mathematics teacher collaboration. Its aim is to address the four questions stated in the discussion document to indicate the role of the theories in understanding the processes, as well as the outcomes of teacher collaboration, and to illuminate different theoretical perspectives that have been currently developed. The chapter is written by the leaders of the group, da Ponte and Miyakawa, as well as three co-authors and it is structured in five sections.

In Sect. 2.1, the authors discuss the background of Theme A and in particular the ICME-13 research survey and the discussion document. Through this review, they identify theories and theoretical frameworks that originated both outside and inside the mathematics education research that had been used to study mathematics teacher collaboration. Examples of the dominant theories in the first category are Communities of Practice/Inquiry and Activity Theory, while in the second is Meta-Didactical Transposition, Documentational Approach to Didactics.

Epistemological issues about what is theory in general, and in mathematics education in particular, are addressed in Sect. 2.2. The authors point out different interpretations that exist about the nature of theory and argue that, to bring to the fore, these conceptualisations will generate discussions that will advance our field. They support their argument by referring to the dynamic relationship between theory and research, to the implicit and explicit theories, and, in general, to the role of theory in understanding phenomena of practice.

The variety of theories used to study mathematics teacher collaboration, their origins and role, as well as their interrelationships, are discussed in Sect. 2.3. The authors list 22 theories/theoretical perspectives mentioned in the papers presented in this working group, indicating a great diversity in comparison to those reported in the ICME-13 survey. These theories fulfil two main roles: understanding the phenomenon of teacher collaboration and designing the work of a collaborative group according to the context. The authors illustrate these different roles of the theories through specific examples from the presented papers. Moreover, in this section, the origins of the theories are discussed in terms of their specificity to mathematics teaching and learning or to social and cultural aspects that are also central in the context of mathematics teacher collaboration. Examples of the different relationships between the origins of the theories and their contributions to understanding teacher collaboration are also provided.

Finally, the authors report that most studies focus upon the development of theoretical constructs to address specific aspects of mathematics teacher collaboration, while few papers discuss and compare theoretical perspectives. They provide three examples of studies to illustrate ways that the theories have been used: adapting theories mainly outside mathematics education research and used to theorise aspects of mathematics teacher collaboration; studying the networking of theories to understand mathematics teacher collaboration by combining different aspects of it; comparing theoretical models used to design teachers' collaborative work.

In Sect. 2.4, an emphasis is given to the role of theories in understanding the process and the outcomes of mathematics teacher collaboration. Two main forms of collaboration are distinguished in terms of the object of the collaboration. One form concerns the aim to solve a specific problem of practice, where it is initiated mainly from the teachers, and the second form aims to professional development activities that are supported mainly by an 'expert'. Through the analysis of the presented papers these forms are exemplified, and specific issues related to the process of collaboration are identified. Although the theories and perspectives seem to address some of these issues or mathematics teacher collaboration at a generic level, there is the need to enrich them with concepts that allow the study of the complexity of mathematics teacher collaboration as it is addressed also in the other theme chapters.

In Sect. 2.5, the issues addressed in the previous sections are summarised in relation to the questions addressed in the Discussion Document. The study conference presentations of Theme A indicate a diversity of theories and perspectives that generally are not very specific to mathematics teacher collaboration, the different origins and uses of the theories to address aspects of mathematics teacher collaboration and the need to enrich dominant theories (e.g. Activity Theory or Communities of Practice), with theoretical constructs that allow mathematics specific aspects of teacher collaboration, but also affective, cultural and political ones.

1.5.1.2 Theme B

Contexts, forms, and outcomes of mathematics teacher collaboration are the focus of Chap. 3. Esteley, Huang and several co-authors and contributors from the working group examine different forms of mathematics teacher collaboration, the contexts in which collaboration is enacted and outcomes of the collaborative process. Their analysis of the Theme B papers is guided by the five questions posed in the Discussion Document: (a) What models of teacher collaboration have been developed? What are the design features, goals, and outcomes of the different models? (b) How effective are various models for promoting different outcomes? (c) Which forms of collaboration are appropriate in different contexts? (d) What are the affordances and limitations of each form of teacher collaboration? (e) What are the benefits and the challenges that online teacher collaboration poses to the teachers?

Section 3.1 introduces the chapter. It begins with a brief review of the literature on teacher learning through collaboration and the guiding questions that were identified in the Discussion Document. The authors then describe the three levels in their framework for understanding teachers' collaborative work: micro (classroom), meso (institution) and macro (education system), and the approach they used to analyse the Theme B conference papers with respect to the contexts and goals, forms, outcomes and mathematical content of teacher collaboration.

Section 3.2 analyses the goals of the mathematics teacher collaboration projects represented in the Theme B papers using the three-tiered model. Looking across the papers, the authors note that national or multiple-site programs typically address national needs and concerns such as curriculum reform, while programs at the institutional level typically focus on developing teachers' knowledge or teaching practice. In the few programs at the classroom level, teachers' collaboration focuses primarily on improving specific teaching strategies.

The authors organise their discussion of forms of mathematics teacher collaboration in Sect. 3.3 according to four categories or models: adaptations of Lesson Study (LS); researchers–teachers partnerships; networks; forms related to specific purposes. LS adaptations typically focus on the institutional level, and involve a community of researchers and teachers who create original resources and design lessons for teaching specific mathematical content. In researchers–teachers partnerships, the professional learning community, as a collective, decides on goals and processes of inquiry, and engages in activities such as the design of teaching resources. The network model involves collaboration among communities or institutions, for example between communities of researchers from one or more institutions and communities of teachers or preservice teachers. The objectives and processes of the joint work are varied. In models of collaboration connected to specific purposes, specificity of purpose is the focal point of the joint work. In all four models, collaboration may be long-term or short-term and the joint work may focus on the classroom, school or educational system level.

Section 3.4 addresses the outcomes achieved in mathematics teacher collaborations. The authors focus first on products developed through collaborative work, noting that most products are at the classroom level and include resources such as lesson plans and mathematical activities. Next, they consider outcomes related to teacher learning, providing examples of projects that reported changes such as teachers' greater understanding of the curriculum content and better support for all students' learning of algebra. They also present some results that were considered unsuccessful and describe obstacles that hinder collaborative work, such as teachers' lack of time and unforeseen changes in institutional routines and requirements. They then consider sustainability and dissemination of the outcomes of collaborations. With respect to sustainability, they observe that most collaborations that lasted 5 years or more were government initiatives or received sustained funding. The most common forms of dissemination were conferences and digital media. They conclude this section by highlighting the large number of long-term collaborations that exist across the world, and suggesting that nationwide programs seem to be based on the idea that the collaborations themselves are the most relevant outcomes, rather than results that can be disseminated beyond the collaboration.

The authors reflect on the role of mathematics content in teacher collaborations in Sect. 3.5. They observe that mathematical content both catalyses and supports teacher change, and is an outcome of the collaborative process. Mathematics-related content was central to many of the collaborative projects discussed in the Theme B working group. The nature of the mathematics content varied including, for example, mathematical tasks, mathematical practices and learners' mathematical thinking. Topics related to the knowledge required for teaching mathematics included Mathematical Knowledge for Teaching, curricular knowledge and horizon content knowledge (Ball et al., 2008) and pedagogical content knowledge (Shulman, 1986). Mathematics-related activities in teacher collaborations that were addressed included doing mathematics, talking about mathematics and investigating representations of mathematics teaching and learning. In concluding this section, the authors suggest that more research is needed on how mathematics content mediates teacher learning outcomes.

In Sect. 3.6, the authors draw from the analyses presented in Sects. 3.2–3.5 to answer the five questions that guided the work of the Theme B working group. They conclude with suggestions for further research.

1.5.1.3 Theme C

Chapter 4 addresses Theme C: Roles, identities and interactions of various participants in mathematics teacher collaboration. Karsenty, Dole and several co-authors and contributors from the working group explore the roles and identities of the actors in collaborative groups such as teachers, mathematicians and researchers; and the nature of interactions between them. Section 4.1 introduces the chapter, noting that it presents a comprehensive overview of scholarship in this field, while paying particular attention to the emergent literature on the role of facilitator—the professional who manages the activities of the group, and whose responsibilities may include setting norms for interactions, supporting teachers' exchange of experiences and new insights, and monitoring the discussion. Throughout, the authors consider both themes and issues related to various aspects of the roles, identities and interactions of participants in collaborative groups, as well as unanswered questions and directions for future research.

Section 4.2 focuses on methodological issues. It begins with a discussion of the methods and theoretical perspectives represented in the Theme C papers, and an analysis of the relevant actors, targets and environments addressed in a sample of these papers. Then, attending more specifically to the role of facilitator, the authors consider several methodological issues and challenges that have surfaced in research on the profession of facilitators, or mathematics teacher educators more broadly.

Section 4.3 reviews contemporary research that addresses the facilitator's role in designing, maintaining and supporting collaborative activities for mathematics teachers. The authors examine several frameworks and models used in studying the role of the facilitator, the knowledge and skills central to the role, and the practices of successful facilitators. They then consider several situational challenges associated with promoting productive collaborative work: starting and managing a discussion; establishing and maintaining norms; sharing responsibility while keeping the discussion on track.

Section 4.4 focuses on the preparation of facilitators. The topics it addresses include: the trajectory of becoming a facilitator; principles of facilitator preparation; preparation programs for facilitators. The authors examine facilitators' development over time, including changes in their knowledge, beliefs, identity, and practices, based on an analysis of six papers focused on projects to develop and support mathematics professional development facilitators. The section concludes with a discussion of means and models for supporting facilitators.

In Sect. 4.5, the authors examine the environment or setting in which teacher collaboration takes place. The section begins with a discussion of several models of teacher collaboration that address the environment of collaboration. They next focus on the internal environment created by participants within a collaborative community, exploring the roles of various participants and the nature of interactions that support the development of these communities. The concepts of communities of practice (Wenger, 1998) and communities of inquiry (Jaworski, 2006) frame their discussion. They then consider cultural and contextual aspects of the external environments in which collaborative communities exist, and institutionally imposed factors such as the time allocated for teacher collaboration. Here, they consider teacher collaboration both with and without facilitators.

In the final section of the chapter, the authors briefly summarise the foci of the previous sections and offer several suggestions for further research, including research on the impact of different actors in the environment on teacher collaboration and on facilitators, issues related to scaling up programs of teacher collaboration, and institutional factors that impact the sustainability of mathematics teacher collaboration. They conclude by acknowledging, "While there is still much work to be done, we recognise the progress made in recent years in studying different roles in mathematics teacher collaboration, reflected in the considerable body of research that we have drawn upon here to address this important issue" (Chap. 4, p. 192).

1.5.1.4 Theme D

Chapter 5 refers to Theme D: Tools and resources used/designed for teacher collaboration and resulting from teacher collaboration. It aims to illuminate the role of tools and resources designed and used in mathematics teacher collaboration, as well as those developed from mathematics teacher collaboration. It is written by the two group leaders (Robutti and Trouche), four co-authors and 15 contributors, participants in the Theme D Working Group. It is the outcome of the 18 papers presented in this group and discussed in the Working Group sessions. What is particular in the Theme D Working Group in comparison to the other groups was the use of virtual collaboration of the participants and the production of a shared discussion document during and after the conference. The chapter is structured in seven sections.

In Sect. 5.2, the authors present the background on Theme D and the five questions that are stated in the discussion document, the way that the group operated in the conference and their definitions of central concepts and constructs relevant to the theme. Distinctions between tools, resources, instruments and documents are discussed, as well as concepts and theories related to teacher collaboration such as community of practice, community of inquiry, boundary crossing, teacher knowledge and professional learning.

Theme D papers focusing on the design of resources through the collaboration of teachers with researchers and knowledgeable others are discussed in Sect. 5.2. Initially, the authors discuss the different conceptualisations and uses of curriculum resources in designing and enacting mathematics lessons, pointing out the role of language, the balance of prior and new resources, the critical role of the availability of digital resources, as well as the important role of teacher collaborative actions. Next, they address the design of resources for promoting students' understanding of mathematical concepts. Finally, relationships between practices developed in the context of mathematics teacher collaboration and those in the school classroom are elaborated.

In Sect. 5.3, the emphasis is on tools and resources that teacher educators and researchers use to support teachers to enact collaborative inquiry into designing, enacting and redesigning mathematics teaching. The authors note that support was facilitated through the promotion of shared reflections concerning the evolutions of the designed resources and the classroom enactments, the use of theoretical and methodological tools in facilitating teachers' inquiry into teaching, the analysis of teachers' own and other teachers' teaching in real classroom situations and by using representations of practice (e.g. hypothetical lessons).

In Sect. 5.4, the emphasis is given to the role of resources and tools in the process of teacher collaboration. A categorisation of the tools and resources is offered in terms of their purpose to facilitate teacher collaboration (e.g. category 1—those planned for facilitating collaboration, and category 2—those that are not planned but were adapted through the work of the teachers as environments for teacher collaboration). In this categorisation, digital resources seem to be central (e.g. platforms, social-media, video-streaming). The authors elaborate on the use of the tools and resources focusing on their affordances in promoting teacher collaboration.

Section 5.5 addresses the theoretical frameworks and methods currently used to study the impact of teacher collaboration on the participants and the interactions within the teacher collaboration. The impact is studied directly using different sources of data (e.g. recordings of the collaboration, documents, recordings from classroom) or indirectly from teachers' reflections on their experiences through interviews, questionnaires or focus groups. Dominant theoretical frameworks adopted to study the interactions include boundary crossing, existing frameworks of categorising learning opportunities, Documentational Approach to Didactics (DAD), Cultural–Historical Activity Theory (CHAT) and Anthropological Theory of the Didactic (ATD). Finally, the authors stress the need to develop infrastructures that are based on technology that facilitate teacher collaboration, and research agendas that generate rich data, allow its storage and are associated with metadata for future use.

In Sect. 5.6, the threads and perspectives of current research on the role of tools and resources for and from teacher collaboration as well as the future orientations of research are discussed. Some main results indicate that: (a) the resources for and from collaboration are not distinct but influence each other; (b) the resources that are open and dynamic give opportunities for reflection; (c) tools and resources for fostering teacher collaboration are appropriate when they allow teachers to develop teaching materials while they share their ideas and reflections; (d) the necessity of the development and the use of certain tools and resources to research teacher collaboration. Finally, in Sect. 5.7, the authors report findings from a survey during the COVID pandemic to address the importance of contextual and equity issues in the way that the tools and resources mediate the process and the outcomes of teacher collaboration.

1.5.2 Plenary Chapters

1.5.2.1 Theme A

Chapter 6 addresses Theme A, "Theoretical perspectives on studying mathematics teacher collaboration". As Susanne Prediger explains in the first section, the intent of the chapter is "to elaborate a theoretical foundation for explaining and promoting teachers' professional growth in collaborative groups" (p. 2, Chap. 6). The chapter argues that the theoretical foundations should integrate content-specific theory elements at both the classroom and PD levels.

Section 6.2 introduces generic models of professional development in collaborative groups (PDCG) and characterises professional growth as changes in teaching practices and practices of inquiry, underlying orientations, and shared categories for noticing and thinking. Section 6.3 presents a vignette from the first meeting of a researcher facilitator and a community of mathematics teachers and special education teachers who are working on differentiating instruction for at-risk students. The vignette, in which the teachers are discussing one student's approach to multi-digit subtraction, is analysed using a generic framework for PDCG to illustrate limitations of that framework. It is then revisited throughout the chapter and analysed using the content-specific theory elements that Prediger introduces.

In Sect. 6.4, she introduces four content-specific theory elements at the classroom level that are necessary for the design of classroom learning environments, and four parallel theory elements that are lifted from the classroom level to the teacher PD level. At the PD level, for example, the four content-related theory elements are content (elements for specifying and structuring the PD content), growth (elements for explaining mechanisms of teachers' professional growth), facilitating (elements for explaining the nature and background of facilitating PD) and environment (elements for designing and enacting PD environments).

The first part of Sect. 6.5 introduces the four classroom-level theory elements specific to the vignette—Content: multi-digit subtraction; Learning: mathematics content trajectories; Teaching: differentiated instruction, at-risk students; Learning environment: design principles for creating a learning environment for mathematics—and then analyses the vignette with respect to these theory elements. In the second part, Prediger conducts similar analyses at the PD level. She next offers a theoretical framework for explaining teachers' professional growth and providing external resources to support professional growth, first for the specific set of vignettes and then for teachers' communities of inquiry in general. She then offers three "lessons learned" for PDCG in general.

In Sect. 6.6, Prediger provides "meta-theoretical reflections" about the theoretical foundations necessary for explaining and promoting teachers' professional growth in collaborative groups. For example, she emphasises the importance of the four PD-level theory elements and reminds us that PD content includes both classroom mathematics content and teaching practices. Finally, she suggests that whereas general theoretical frameworks such as communities of inquiry and models of professional growth provide a "generic search space", they must be elaborated in content-related ways for specific areas of PD content.

1.5.2.2 Theme A Reaction

Boris Koichu (Chap. 7) begins his reaction to Prediger's Theme A plenary chapter by noting that her chapter makes an important contribution to debates about the role of theorising in mathematics education research and practice. At the same time, he comments that her central suggestion—that more content-specific theorising is needed—is not obvious. He suggests that, in addition to Prediger's vignette, episodes in different situations of mathematics teacher collaboration should be analysed using her model of content-specific theorising. The majority of the Commentary is an analysis of an episode in one of Koichu's PD projects as it addresses three issues: (a) characteristics of Prediger's research strategy that affords and includes content-specific theorising; (b) the application of the content-specific theory elements to a situation of teacher collaboration that differs from Prediger's vignette; (c) how to connect content-specificity and generality of theorising in future research on teacher collaboration.

Reflecting on his analysis, Koichu observes that Prediger's conception of content is multifaceted, including, for example, mathematical, epistemological and PD components. Her theorising about content-specificity aligns with the principles of design research. He analyses an episode in his *Raising the Bar in Mathematics Classrooms* (RBMC) project using her content-specific theory elements, and concludes that he is able to apply classroom-level and PD-level elements of the framework to the episode and, by doing so, deepened his understanding of both the RBMC episode and Prediger's ideas. Koichu identifies two features of her ideas that make them transferable to different contexts—the functionally-oriented scheme of analysis and the bottom-up approach that begins by engaging deeply with the mathematical content and then theorises about learning, teaching and PD-facilitating. He concludes that the connection between the content-specific and the general is achieved by considering the particular content as a case of something more general.

1.5.2.3 Theme C

In this chapter (Chap. 8), the plenary speakers, Konrad Krainer and Carina Spreitzer, in collaboration with the reactor Bettina Roesken-Winter, propose a framework for analysing mathematics teacher collaboration, focusing on the relevant actors, the relevant targets and the relevant environments (RATE). The authors argue that the diversity of participants, goals of collaboration and settings where the collaboration takes place makes the study of collaboration too difficult to go deeply into the process and to compare different initiatives promoted in the collaboration.

The authors discuss the dimensions to the RATE framework and operationalise it in the form of a triangle with vertices, Teachers, Knowledgeable others and

Environment, to analyse seven papers published in mathematics education reporting initiatives from four continents. Seven dimensions/codes were used to describe these papers: relevant actors; relevant targets; relevant environments; authors; types of initiative specificity of collaboration; research results.

By comparing the seven cases, the authors come to similar conclusions with other surveys that: (a) small-scale studies predominate; (b) most teacher education research is conducted and reported by teacher educators/researchers studying the teachers with whom they are working. Other observations and directions for future research are also reported, such as the need to: focus more systematically on teacher educators' learning; make better links between teacher learning and the process of collaboration; emphasise the particularities of the contexts where the collaboration takes place and compare to other similar cases, as well as to stress the importance of sharing reflections.

1.5.2.4 Theme D

Chapter 9, co-authored by Karin Brodie and Kara Jackson, focuses on resources for collaborative professional work and their role in teacher collaboration and learning. They define resources broadly to include representational, knowledge, affective, human and institutional resources. Brodie and Jackson present a framework for conceptualising these resources and their functions in supporting teacher professional collaboration. In describing the framework, they make two key points about the nature of resources: (a) resources travel between teacher collaborative groups and classroom practice and are transformed or redesigned across contexts; (b) missing resources can contribute to unequal opportunities for teacher development, classroom learning and, ultimately, inequities in society.

After describing the framework, Brodie and Jackson apply it to projects they have worked in, the *Data-Informed Practice Improvement* project *(DIPIP)* in South Africa and the *Middle-School Mathematics and the Institutional Setting of Teaching (MIST)* project in the United States. They show how each type of resource supported or constrained teachers' collaborative work in the two projects, and they consider how the resources both shaped and were shaped through teacher collaboration. For example, representations of teaching such as lesson plans, video-recordings of classroom lessons, student work and student assessment data often comprise the shared text of collaborative inquiry; and they were a key resource for teacher collaboration in both projects.

The DIPIP project focused on teachers' use of learner errors as a resource for teaching. Representational resources included tests, learners' responses to tests and video records of the teachers' lessons. In an effort to improve their teaching, teachers' collaborative inquiry communities analysed these representational resources to understand learner errors and the reasons underlying them. In productive teacher collaboration in the MIST project, teachers used representational resources such as lesson plans, curricular materials and student assessment data to engage in conversations about the 'how' and 'why' of instruction.

The design of both projects supported the movement of representational resources between classroom and collaboration, and these resources were often modified as they moved between contexts. Findings from the two projects suggested that the focus and quality of representational resources shape teachers' learning opportunities within collaborative contexts. Brodie and Jackson conclude the chapter by highlighting 'how' and 'why' each plays an important role in productive teacher collaboration and suggesting areas for future research.

1.5.2.5 Plenary Panel

In the panel, the voices of four teachers from different countries and settings were heard about their experiences collaborating with other teachers. Hilary Hollingsworth and the four teachers—Yivi Chen, Christelle Fitamant, Lameck Dition Sandram and Shelli Temple—were the authors of this chapter (Chap. 10). The aim of the chapter is to understand the learning opportunities offered to teachers in collaborative settings, and to identify conditions that support or constrain teachers' professional learning. The collaborative groups had been established in projects taking place in China, England, France, Africa and the US, and included mainly planning, teaching and analysing Lesson Study lessons. All except the US group were face-to-face and small-group collaboration. In the US context, thousands of mathematics teachers from different countries collaborated online. Comparing teachers' experiences, common points are addressed and discussed in the chapter. Improving the quality of mathematics teaching and as a result students' learning was the focus of all the groups. These goals were formed in the context of the collaboration, and the expertise of other teachers and researchers played an important role in fulfilling their goals and developing professional learning.

Then, the reports of the four teachers are presented providing information about: (a) the context, purpose and design of the collaboration; (b) the collaboration outcomes; (c) what is learned—factors supporting or limiting the collaboration, as well as challenges encountered, and professional learning. Synthesising these reports especially related to lessons learned, the chapter reports several cultural, social, environmental and physical factors that support collaboration, such as a culture for life-long learning of teachers, teachers' motivation, available resources and connections with other teachers.

Among the factors limiting the collaboration are unwillingness to participate, resource constraints, difficulties to lead new ideas and approaches, and that communication protocols and tools require time. To sustain collaboration and professional learning involves joint reflection about the lessons, analysis of selected lessons and online opportunities. The chapter ends with the use of the Interconnected Model of Professional Growth to identify growth paths of the four teachers and the formulation of a set of questions related to mathematics teacher collaboration, and the four study themes addressed in this volume.

1.5.3 Commentary Chapters

Dario Fiorentini and Ana Losano organise their Commentary on ICMI Study 25 around four issues: forms and meanings of collaboration; the nature of collaboration and how it is investigated; the relations between collaborative groups and classroom practice; possibilities for scaling up collaborative PD. For each one, they draw upon the earlier chapters in this Study 25 volume and their own experiences of collaboration with teachers in Brazil, in order to reflect on advances and possibilities and to identify challenges related to collaborative PD. With respect to the first issue, they note the diverse meanings of collaboration that were addressed in the Study volume and suggest that effective collaboration requires time, support by participants' institutions and shared negotiation of goals and action.

In their discussion of the second issue, Fiorentini and Losano suggest that narratives written by participants in collaborative groups both enhance the collaborative work and provide rich material for analysing teachers' learning. And they make a case for conducting collaborative research *with* teachers in addition to the more typical research *about* teachers. They stress the importance of studying the complex relations between collaborative groups and classroom practice, and suggest two promising directions for research: (a) how resources are transformed as they travel between the collaborative groups and the classroom setting; (b) teachers' developing professional identity and agency as they regularly cross the boundaries between the collaborative group and classroom practice. They highlight the potential of blended and online approaches for scaling up collaborative PD and emphasise that building trust and ensuring that members feel safe are essential for the success of these approaches. Their commentary concludes with the suggestion that collaboration is "a fertile and still little explored field that demands continuity of studies and socialisation, discussion and systematisation in events" (p. 13, Chap. 11).

Rina Zazkis describes her commentary as a reflection on her noticing and wonderings, and what drew her attention when reading the Study volume chapters. Her reflection focuses on five issues that she labels: teachers' work; broad applicability; mathematics; content-related theorising; effects or products of collaboration. Zazkis points out, for example, that the Study volume chapters consider teachers' work very broadly to include learning and professional development, as well as planning and assessing student work. She suggests that many ideas about collaboration in the chapters are broadly applicable beyond mathematics. Focusing specifically on mathematics, she identifies an extensive list of possible outcomes of teacher collaboration, which she describes as different forms of professional growth. In the final section of her commentary, Zazkis shares a personal reflection on the value of teacher professional collaboration, explaining how and why building professional community became the central goal of her "foundations of Mathematics" course—the first course in a two-year, cohort-based Secondary Mathematics Education master's program for practicing mathematics teachers.

1.6 Reflections on the Study and the Study Volume

Producing this Study volume has been a rather long process. It involved close collaboration among many people, especially the IPC members and leaders of the working groups, and the ICMI president and secretary who initiated the Study. This collaboration took place in multiple physical and virtual spaces in the various phases of the Study—the writing of the Discussion Document, the planning of the conference and the editing of the proceedings and Study volume. Because of the COVID pandemic, a virtual way of collaborating among participants was initiated during the conference, where colleagues from China shared their work online. This virtual form of collaboration was evident in all the working groups and happened before, during and after the conference. Online collaborative meetings are now a part of our everyday realities both as researchers and practitioners. Virtual collaboration was a research topic in the Theme D chapter, where tools and resources are the central focus. It is an area that needs to be researched further in relation to collaboration both during and after the COVID pandemic.

A goal that was promoted in the conference, and to some extent was achieved, was to hear the teachers' voices from their experiences participating in collaborative activities working with other teachers and researchers. The teacher panel in the conference and in Chap. 10 in this Study volume brought these voices to our study and indicate the gains, but also the challenges that the teachers face. The panel brought to the fore the role of contextual and cultural aspects that frame the collaboration and its outcomes.

Emerging issues from each theme chapter and across the chapters indicate the state of the art in our understanding of teachers working and learning together, as well as future directions. Here, we highlight key learnings and areas for future research for each of the four themes that guided ICMI Study 25. With respect to theory, there is a diversity of theories and theoretical frameworks used for studying mathematics teacher collaboration. However, we need to develop theoretical constructs that allow mathematics specific aspects of teacher collaboration to be addressed, as well as affective, cultural and political ones. Focusing on the process of collaboration, there are obstacles that relate to unforeseen changes in institutional routines and requirements, and teachers' lack of time that must be managed in long-term collaborations. Most products of teacher collaboration are at the classroom level, and include resources such as lesson plans and mathematical activities. Long-term collaborations continue to be a challenge for the educational community.

To address this challenge, it is important to identify features of successful, long-term collaborations that are generalisable across contexts. We also need to continue developing and improving models of online and blended collaboration, and to identify the features of these models that make them effective. Facilitators play an important role in the processes and outcomes of teacher collaboration. Professional development programs for facilitators, and their role in developing facilitators' academic knowledge, and social and interpersonal skills, is an important direction for research. The settings in which teacher collaboration takes place, and participants in those settings other than the teachers and facilitators, also impact the outcome of the collaboration. Future research needs to address the impact of facilitators and other actors in collaboration, as well as the role of institutional factors that support the sustainability of mathematics teacher collaboration.

A diverse set of resources for and from teacher collaboration have been identified. The roles of these resources vary, depending on the settings in which the collaboration takes place, as well as the different participants that are involved. Digital tools and resources that offer wide and flexible uses and opportunities for teachers to experience innovative representations of mathematics teaching foster teacher collaboration. Issues that remain open for future research include the quality of resources, the sustainability of tools, the use of resources in scaling up of teacher collaboration and the role of digital tools and resources in mathematics teacher collaboration.

The plenary chapters and the two commentaries on the volume add to the above points, offering directions for future research. Different aspects of mathematics teacher collaboration are discussed in these chapters, such as:

the content-specific character of teacher collaboration;

the need to link teacher collaboration with classroom teaching both as a way of theorising it and for supporting its sustainability;

the emphasis on the quality and the focus of the resources to shape mathematics teacher collaboration;

the need for more research with teachers rather than about teachers, and for encouraging teachers to become co-authors with researchers;

the development of theoretical and methodological perspectives to consider the complexity of mathematics teacher collaboration by focusing on the role of context and on the different agents;

approaches for scaling up mathematics teacher collaboration.

In conclusion, this ICMI-25 Study volume shows that mathematics teacher collaboration is an area that has attracted much research attention during the last several years. Yet, there are still open questions to be addressed at the theoretical, empirical and practical levels. Moreover, there are areas that have not been addressed (or addressed only minimally) in the Study volume, such as the collaboration between teachers of different subjects, the collaboration of teachers at the university level and the collaboration among mathematicians and mathematics educators.

Research in these areas, as well as others not represented in ICMI Study 25, will extend our understanding of teacher collaboration, for example by giving more attention to the characteristics of mathematics (in comparison to other disciplines), collaborative groups that include participants other than K–12 mathematics teachers and additional contextual factors that frame these collaborations. Our understanding and appreciation of the value of teachers working and learning in collaborative groups has grown tremendously in the process of leading ICMI Study 25. We hope that this Study volume encourages additional programs of, and research on, teacher collaboration and looks forward to learning from future projects.

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Part II
Theme Working Group Chapters

Chapter 2
Theoretical Perspectives on Studying Mathematics Teacher Collaboration

João Pedro da Ponte, Takeshi Miyakawa, Nicole Bannister, Boris Koichu, and Birgit Pepin

2.1 Introduction

The aims of this chapter are: (i) to showcase the state-of-the-art for theoretical perspectives for studying mathematics teacher collaboration; (ii) to identify promising theoretical and methodological perspectives for future studies. The ideas that are synthesised in this chapter are based on the presentations, papers, and discussions that occurred as part of the ICMI Study 25 Conference. In this introductory section, we briefly review the results from the previous ICME-13 research survey on mathematics teacher collaboration (Robutti et al., 2016; Jaworski et al., 2017). These results, which functioned as a starting place for the Theme A Working Group, not only informed the organisation of our group but were also reified in the ICMI Study 25 Conference discussion document (IPC, 2019).

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In this section, we connect results from the previous survey with the research reported by ICMI Study 25 Theme A participants. We note points of convergence among the studies and theoretical additions from ICMI Study 25, which helps document the current theoretical and methodological landscape of research on mathematics teacher collaboration. Next, we outline the research questions posed in the discussion document for Theme A and present the methodology of our work. We conclude this section with an overview of the structure of the chapter.

2.1.1 Background

The research on mathematics education often requires a theory or a theoretical framework for different purposes (Assude et al., 2008; Niss, 2007a). Theories, and the inherent concepts that come along with them, provide guideposts for analysing phenomena related to the object of study and for making important distinctions, connections, and relationships. Therefore, the identification of theories that may be used to study mathematics teachers' collaboration is an essential condition for a deeper understanding of the potential and limitations of the collaboration.

2.1.1.1 ICME-13 Survey and Four Theoretical Perspectives

ICME-13 Survey refers to the published results of an international research survey commissioned for ICME-13 held in 2016 focusing on mathematics teachers working and learning through collaboration (Robutti et al., 2016). Importantly, the ICME-13 Survey found that, while many papers did not declare explicitly the theoretical perspectives, four perspectives were evident for studying collaboration involving mathematics teachers: Community (of Practice or of Inquiry), Activity Theory, Valsiner's Zone Theory, and Meta-Didactical Transposition.

Theories involving community were used most frequently, and in most cases referred to the theory of *Communities of Practice* (Lave & Wenger, 1991; Wenger, 1998) or the derived theory of *Communities of Inquiry* (Jaworski, 2006). *Communities of Practice* is a social theory of learning, which is understood as a process of social participation. This theory emphasises the negotiation of meaning through participation within the community, the formation of common goals and a repertoire of social practices, and the building of a professional (teaching) identity relative to the shared social norms of the community. Relatedly, the theory of *Communities of Inquiry* puts special emphasis on how *alignment* of the participants takes place in order to achieve the aims of the group, stressing the role of *critical alignment*, in which participants align with the norms of the community while asking questions and critically reflecting through inquiry.

The second most frequently used theoretical perspective in research on collaboration involving mathematics teachers was *Activity Theory* (Engeström, 2001), in which the activity and its object are achieved collaboratively through the mediation of tools and framed by the communities' rules and division of labor. Special attention has been paid to the version presented by Engeström that stresses the role of *expansive learning*, taking place when there are contradictions within an activity system or between different activity systems.

Another theoretical perspective identified in the survey results was Valsiner's (1997) *Theory of Zones*, which expands Vygotsky's *Zone of Proximal Development* (ZPD) learning theory to distinguish among the free movement of learners in their Zone of Free Movement (ZFM) and the restrictions placed on them by an "other" who seeks particular outcomes for them in their Zone of Promoted Action (ZPA). These ideas were applied to investigate the teacher's practice as a learner, in which ZFM suggests the teacher's possible actions and ZPA are the activities offered by the teacher education program.

The fourth identified perspective was that of *Meta-Didactical Transposition* (MDT; Aldon et al., 2013), derived from *Anthropological Theory of the Didactic* (ATD; Chevallard, 1985, 2019). ATD characterises mathematical knowledge and its teaching and learning in terms of 'praxeologies' and 'didactic transposition'. MDT considers the transposition (i.e. teaching and learning) of didactic knowledge and practice on teaching, which is conducted in the professional development.

Interestingly, whereas *Communities of Practice*, *Activity Theory* and *Valsiner's Zones* were developed outside of the field and later applied to mathematics education, MDT is a framework that was developed within the field of mathematics education.

2.1.1.2 Connections with ICMI Study 25: Discussion Document in Review

A summary of the most frequently used theoretical and methodological perspectives to study teacher collaboration (Robutti et al., 2016) is present in the ICMI Study 25 discussion document (IPC, 2019), including aspects of the dynamics of teachers' collaborative work and the communities in which they work. This document also references theoretical perspectives developed outside mathematics education that have focused on the nature of the communities in which teachers collaborate. Given their prevalence, special attention was given to *Communities of Practice* (Wenger, 1998), *Communities of Inquiry* (Jaworski, 2006) and *Activity Theory* (Engeström, 2001). The ICMI Study 25 discussion document also attends to Valsiner's (1997) *Theory of Zones* and its use in mathematics teacher education (Goos, 2005), in which the emphasis is on the professional learning experienced by participants.

In addition to these three broad perspectives, the ICMI Study 25 discussion document also references theoretical perspectives outside mathematics education that have focused on conceptions of teacher learning. For example, *Situative Theories of Learning* assume that knowing and learning are situated in particular physical and social contexts, are social in nature, and distributed across the individual, other persons, and the tools used (e.g. Greeno, 1997). Another perspective presented in this document is the *Practice-Based Theory of Professional Education* (Ball &

Cohen, 1999). This theory considers the mechanisms underlying teacher learning, suggesting that professional development programs should situate teacher learning in the types of practice they wish to encourage. The discussion document notes that the themes and questions addressing theories that focus on teacher learning overlap with those identified by the theoretical perspectives that focus on the nature of communities.

Furthermore, theoretical frameworks developed within mathematics education research that allow us to investigate different aspects of teacher collaboration are also reported in the discussion document. For example, the *Documentational Approach to Didactics* is mentioned, which describes a theoretical approach that focuses on studies of teacher collaboration on the role of participants as designers and users of resources (Pepin et al. 2013). Two frameworks based on ATD (Chevallard, 1985, 2019) are also included: (1) MDT (Aldon et al., 2013; Arzarello et al., 2014; Robutti, 2020), which was referred to in ICME-13 Survey; and (2) *Paradidactic Infrastructure* (Miyakawa & Winsløw, 2013), which characterises the different settings for teacher collaboration inside and outside school.

2.1.1.3 ICME-13 Survey and Methodological Approaches

The ICME-13 Survey (Robutti et al., 2016) specifies two methodological approches: (1) research methodologies used to study the work of collaborative groups; and (2) developmental methodologies, which provide directions for the constitution, and development of collaborative groups. The research methodologies used to study teacher collaboration are mostly qualitative and include participant observation, case studies, action research, and design-based research. Data sources include participant journals, interviews, questionnaires, narratives, and audio and video recordings of the collaborative activities or of the activities that the members of the collaborative groups carry out with other participants.

Concerning developmental methodologies, the ICME-13 Survey highlighted Lesson Study, learning study, action-research, design-based research and developmental research in their findings. The discussion document mentioned all these methodological approaches and data collection techniques and added that, concerning methodology, "the important issue is that data is thorough, systematic, reliable and authentic regarding the perspectives and practices of participants" (IPC, 2019, pp. 4–5).

2.1.2 *Questions of the Discussion Document*

The theoretical and methodological landscapes mentioned above suggest the following questions to be explored within the context of ICMI Study 25. These were framed in the discussion document (IPC, 2019) in the following terms:

- *How do the different theoretical perspectives or networks of theories enhance understanding of the processes of teacher collaboration?*
- *How do they enhance understanding of the outcomes of teacher collaboration?*
- *What is illuminated by the different perspectives and methodologies and what needs further investigation?*
- *What are promising research designs and data collection and analysis methods to study teacher collaboration?*

2.1.3 Work Methodology and Structure of the Chapter

This chapter is based on the work carried out by participants at the ICMI Study 25 Conference whose collective work was organised around Theme A: The Theoretical Perspectives on Studying Mathematics Teacher Collaboration. This body of work included the presentation of 18 papers and associated discussions, as well as overarching discussions on cross-cutting issues and a plenary talk by Susanne Prediger with a reaction from Boris Koichu, which were dedicated to issues of theory and theorising (all included in the references of this chapter).

We began the writing process by creating an overarching structure for the chapter that would address the theories that were discussed at the study conference in the Theme A working group. The strength of our group was both the diversity of theories and the diversity of contexts and aspects of teacher collaboration used by participants in their research. This strength simultaneously presented us with the challenge of writing a clear, coherent narrative capturing the important takeaways from our collective work while avoiding oversimplification or misrepresentation of any of the theories.

We organise this chapter around two complementary perspectives, which allow us to investigate the nature of theories used and developed in the research on mathematics teacher collaboration. The first perspective is the theory itself. We classify the theories we faced in the presentations of the ICMI Study 25 Conference according to their roles, origins, and research issues linked to the theories, and investigate the characteristics and specificities of these theories in our research area. The second perspective is the teacher collaboration. We classify the theories we faced according to the kinds of teacher collaboration, which conceptualise in different settings of teachers' work, and investigate what kinds of theories may help to address different issues of teacher collaboration, and what kinds of theories are still needed to be developed to productively address these issues.

This chapter is arranged under five sections. After this first introductory section, in the second section, we present a general discussion on theories in mathematics education research. In the third section, we discuss the diversity of theories that we identified in the studies related to teacher collaboration, and subsequently, in the fourth section, consider issues specific to teacher collaboration. The fifth section completes the chapter with conclusions and perspectives for future work. All along the chapter, we only make occasional reference to the methodologies used, because these were scarcely discussed in the study conference and in the 18 presented papers.

2.2 Generalities of Theories

In this section, we address the broader questions on the notion of theory and its roles within the context of research on mathematics education. We conclude with the assumption that all research is theoretical in some way, even if researchers are unaware of the underlying epistemology that influences their work. To this end, we argue that making these implicit perspectives explicit through continued dialogue within the broader mathematics education research community is generative for the field.

2.2.1 What Is a Theory?

When answering this question and others closely related to it, researchers in mathematics education often begin with dictionary definitions of the word or its etymology (e.g. Eisenhart, 1991; Jablonka et al. 2013; Mewborn, 2005). For example, the online *Oxford Advanced Learner's Dictionary* lists the following:

theory *noun*

1. a formal set of ideas that is intended to explain why something happens or exists
   According to the theory of relativity, nothing can travel faster than light.
2. the principles on which a particular subject is based
   This is your chance to put theory into practice.
3. an opinion or idea that somebody believes is true but that is not proved
   He has a theory about why dogs walk in circles before going to sleep.

Word Origin: late 16th cent. (denoting a mental scheme of something to be done): via late Latin from Greek *theōria* 'contemplation, speculation', from *theōros* 'spectator'.

Idioms: **in theory**. Used to say that a particular statement is supposed to be true but may in fact be wrong:

> *In theory, all children get an equal chance at school.* (OED, n.d.)

The variability within the everyday usage of *theory* suggests that the "notion of theory is not exactly a monolithic one" (Niss, 2007a, p. 97) within the relatively young field of mathematics education (Schoenfeld, 2000). Thus, one purpose of this linguistic exercise in scholarly publications is to situate contemporary usage among historical origins, and as a byproduct, bring an established problem in mathematics education into focus: "theory is a value-laden term with a long and convoluted history" (Mason & Waywood, 1996, p. 1055).

This history is reflected in some of the critical questions about theory currently under discussion within the broader mathematics education community. As framed by Niss (2007a): Where do the entities referred to as theories invoked in mathematics education come from? How are they developed? What foundations do they have? What roles do they play in the field? Is it problematic that there is "no such thing as a well-established unified or unifying 'theory of mathematics education' that is supported by the mathematics education research community" (Niss, 2007b, p. 1308; see also diSessa, 1991)? Put another way, what this means within our community is that the quintessential scientific and scholarly practice of theorising is often rendered invisible by inconsistent and overlapping usage of *theory*. Such use, in turn, is reified as a taken-as-shared repertoire of practices for the broader mathematics education research community (Mewborn, 2005; Silver & Herbst, 2007; Schoenfeld, 2007).

With that being said, there is also a notion that "implicit theories" (Sternberg, 1985) can guide research practices without being explicitly articulated. In the ICMI Study 25 Theme A group, which was devoted to theory, the authors did explicate their theoretical stances. However, the papers manifest various implicit theories of theory, as the theory notion is sometimes used at its face value. The role of this section is to make *implicit theories of theory* explicit as a background for further analysis and synthesis of the contributed papers.

While a shared understanding of *theory* remains unsettled in the international mathematics education community (e.g. Robutti et al., 2016), some scholars have offered definitions in an effort to address this critical issue in our field. For example, Niss (2007b) proposed the following definition:

A *theory* is a system of concepts and claims with certain properties.

- A theory consists of an *organised* network of *concepts* (including ideas, notions, distinctions, terms, etc.) *and claims* about some extensive domain, or a class of domains, consisting of objects, processes, situations, and phenomena.
- In a theory, the *concepts are linked in a connected hierarchy* (oftentimes, but not necessarily, of a logical or proto-logical nature), in which a certain set of concepts, taken to be basic, are used as building blocks in the formation of the other concepts.
- In a theory, the *claims are either* basic hypotheses, assumptions, or axioms, taken as *fundamental* (i.e. not subject to discussion within the boundaries of the theory itself), or statements obtained from the fundamental claims by means of *formal* (including deductive) or *material* (i.e. experiential or experimental with regard to the domain(s) of the theory) *derivation*. (p. 1308; *italic in original*)

Niss's definition is clearly inspired by how mathematics is (often) conceived, while the second example that follows seeks to encompass the fundamental products of an empirical research process. Radford (2008) specified a theory in mathematics education as a triplet formed by the following elements.

- A system, *P*, of *basic principles*, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
- A *methodology*, *M*, which includes techniques of data collection and data-interpretation as supported by *P*.
- A set, *Q*, of paradigmatic *research questions* (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified). (Radford, 2008, p. 320; *italic in original*)

While a wide range of different theories or theoretical perspectives have been taken up by mathematics education researchers over the last decades, there is also heterogeneity in what is called a theory by different researchers and different scholarly traditions (Niss, 2007b; Prediger et al., 2008b). These conceptualisations may or may not match with the ordinary usage of theory or application of theoretical perspectives in the research community. One consequence of defining the notion of theory in such rigid terms is that it leaves room for the interpretation of elements that are necessarily ambiguous or absent depending on the theories being used. While some researchers understand this as a problem to be corrected, we instead argue that productive discussions around ambiguous or missing elements of purposefully used theories in research can be generative for the field.

2.2.2 The Roles of Theory in Mathematics Education

Another way of conceptualising the notion of theory is through the clarification of its roles in mathematics education. For example, Niss (2007b) suggested six purposes of theory: as an explanation of observed phenomena, as a predictor of future occurrence, as a set of guideposts for research design, as a structured set of lenses through which research is conducted, as a safeguard against unscientific approaches to myriad facets of research, and as a shield against attacks from skeptics and hostile colleagues from other disciplines. As these purposes are inextricably connected to the design decisions of the researcher, we follow the purposes of the participating authors of the Theme A papers, taking them into account at face value.

However, theories do not always specify all of these details, are not necessarily structured in a hierarchical way, and have the potential to emerge from research contexts (Bikner-Ahsbahs & Prediger, 2010; Prediger et al., 2008a; see also Assude et al., 2008). Even well-established or well-known theories such as the *Theory of Didactical Situations* (Brousseau, 1997) or *Anthropological Theory of the Didactic* (ATD; Chevallard, 1992, 2019) are still in a process of evolution and advancement. From this perspective, Bikner-Ahsbahs and Prediger (2010) argue for a more dynamic consideration of theories in mathematics education research in their complex relation to mathematics education reality (see also Prediger et al., 2008b; Prediger, Chap. 6, this volume).

A more dynamic view is also assumed by Silver and Herbst (2007) when they characterise the roles of the theory as a mediator of relationships among practices, problems, and research. In proposing these relationships, which highlight the *diversity in approaches* to understanding theory, the authors discussed different types of theories: “grand theories” of mathematics education, “middle-range theories” that concern subfields of study, and “local theories”, thereby providing one way to conceptualise the *diversity of theories* used in mathematics education. The objective of the papers presented for Theme A is not only to better understand the teacher collaboration and to get insights for the practices, but also to study, examine, compare or develop the theories. This suggests that the theory itself can be an object of study, which is not fully captured in the aforementioned examples.

Accordingly, theory can be understood as both a tool and an object of research (Assude et al., 2008). When considering theory as a tool, its functions are to:

(a) conceive of ways to improve the teaching/learning environment including the curriculum;
(b) develop methodology;
(c) describe, interpret, explain, and justify classroom observations of student and teacher activity;
(d) transform practical problems into research problems;
(e) define different steps in the study of a research problem;
(f) generate knowledge. (English & Sriraman, 2009, p. 1622)

When considering theory as an object, the development of theory is also one of its primary functions (English & Sriraman, 2009). For example, in the way that theory drives methodology, such as “what is taken to be data and what data are selected for interpretation” (Kilbourn, 2006, p. 545), the methodology for data collection and analysis may similarly drive theory development. Taken together, in Theme A we argue that making “implicit theories of theory” explicit is a non-linear essential practice that affords the mathematics education research community with important opportunities to engage with and apply a broader range of scholarly results in their own work.

2.2.3 A Coda on Epistemological Awareness

It has been argued that theories are often “taken to be unproblematically applied to a research study” (Lerman, 2006, p. 12). Simon (2009, p. 486) attributed this phenomenon to the confounding of “what one looks *at*” and “what one looks *with*” when using theory in mathematics education research. This leaves room for the choices in a research study to be based entirely on comfort with what the researcher is looking at, rather than deliberate consideration for theoretical perspectives that may be most useful to look with. As a consequence, the *theories* that are used for and developed from research may be ambiguous or absent to the researcher and/or the broader scholarly community. This can fuel dangerous self-reinforcing research cycles in which assumptions of atheoretical research in mathematics education becomes a self-fulfilling prophecy.

Silver and Herbst (2007) observed this phenomenon as journal editors, noting that manuscripts are often rejected for being atheoretical. Some would argue that it is not possible for research to be atheoretical, rather epistemologically unaware (Koro-Ljungberg et al., 2009). However, the larger point we make here encourages avoidance of the "sort of rigid, blind adherence to a theory that characterises much theory-based research" (Lester, 1991, p. 198; see also Eisenhart, 1991). Instead, we hope to provoke discomforting yet somehow comforting conversations by continually applying pressure to explore theoretical frontiers in and for mathematics education research.

To this end, as we concluded in our ICMI Study 25 Theme A discussions, the practices of interrogating and communicating the theories used for and developed from our research not only support a better understanding of our own studies, but also provide accessible entry points for researchers who use a diversity of theories—and a diversity of approaches to theories—to better understand and learn from the scholarship of others.

2.3 Diversity of Theories Related to Teacher Collaboration

In this section, we introduce the variety of theories used to analyse teacher collaboration, discuss the roles of theories as well as the origins of theories and their foci, and conclude by addressing research issues related to developing, enhancing, networking, and analysing theories.

2.3.1 Diversity as a Result

The 18 papers presented in the study conference referred to a variety of theories and theoretical perspectives, which range from grand theories to local theories in terms of Silver and Herbst (2007). Looking across the 18 papers, we found it difficult to judge what counts as a theory. Some papers referred to well-established theories such as *the Anthropological Theory of the Didactic* or *the Cultural–Historical Activity Theory*. Others referred to models or perspectives that may not be called theory, but they nevertheless play important roles for studying teacher collaboration. In Table 2.1, we list the theoretical references explicitly mentioned in the 18 papers as theoretical underpinnings that were used or proposed to study teacher collaboration. Evidently, the references are of different nature.

This table shows a greater diversity of theoretical perspectives as compared to the results of ICME-13 Survey (Robutti et al., 2016; Jaworski et al., 2017), where apparently only four perspectives were evident (see also earlier):

> Many papers did not declare explicitly the theoretical perspectives behind a project. Of those that did, four perspectives were evident: Community (of Practice or of Inquiry) (69%), Activity Theory (20%), Metadidactical Transposition (6%) and Valsiner's Zone Theory (5%). (Jaworski et al., 2017, p. 267)

The percentages provided in the quote refer to papers that explicitly mentioned theoretical perspectives.

The greater theoretical diversity in our Theme A working group can partly be explained by the fact that the nature of papers examined in that survey was very different from ours—our group explicitly focused on theories, and the papers in the survey did not. Even so, the theoretical diversity in our working group reflects a wider use of theoretical perspectives in the research field of teacher collaboration. While the percentage of the papers referring to a specific perspective may not be significant, as there are only 18 papers in our working group, some theoretical references have been mentioned in more than one paper. Those are ATD, DAD, Community of Practice/Inquiry, Activity Theory and Lesson Study.

Considering the theoretical references in Table 2.1, they can be classified according to different aspects such as purpose, origin, focus, and use. Selected theoretical perspectives are specific to teacher education (e.g. IMPG, Zones of enactment). Some theories were developed in mathematics education research, while others were developed in other disciplines (e.g. sociology, psychology) and later used in mathematics education research studies. Some theories are primarily used to understand teacher collaboration, while others are instrumental in designing the professional development (PD) program or in conceptualising teachers' ordinary work including teacher collaboration. Some studies combine or network the theories

Table 2.1 List of theoretical references explicitly mentioned in the 18 papers

Anthropological theory of the didactic (ATD)
Meta-didactical transposition
Commognition
Documentational approach to the didactics (DAD)
Interconnected model of professional growth (IMPG)
Cultural-historical activity theory (CHAT)
Expansive learning
Boundary crossing and boundary objects
Community of Practice
Community of Inquiry
Zones of enactment
Enactivism
Theory of situated learning
Semiosphere
Semiotic mediation
Culturally figured worlds
Fractals
Lesson study (LS)
Action-education model
Learning study
Community-centered model
Professional learning community (PLC)

to analyse teacher collaborative work. This diversity of theories was one of the results we obtained in the study conference. It reflected the complexity of researching teacher collaboration, its multi-faceted nature, and different forms in different contextual/cultural parts of the world.

Of note is that the theoretical diversity is not specific to research on teacher collaboration. In mathematics education research, a diversity of theories has often been observed and discussed as an issue to be addressed. In fact, several attempts have been carried out to identify the different roles of theories, as well as to explore the relationships between them, in order to develop a coherent view on the theories used and developed in mathematics education research (Niss, 2007a; Assude et al., 2008; Radford, 2008; Bikner-Ahsbahs et al., 2014). In line with the Networking Theories Group (Bikner-Ahsbahs et al., 2014), we consider that a diversity of theories is an indicator for the dynamic character of the field. We also agree with this group in that, "the diversity of theoretical approaches can only become fruitful if connections between them are actively established" (p. 8).

In order to clarify and understand better the roles and functions of different theoretical perspectives and to make connections between them, we investigate in this section the theoretical diversity related to the research on mathematics teacher collaboration, through the 18 papers presented in the study conference. In particular, we identify the special features of theories and perspectives used or developed, for studying and designing mathematics teacher collaboration. For this, we examine the 18 papers, looking for the research issues addressed as well as for roles, origins, and foci of theories and perspectives related to teacher collaboration.

We choose the papers as examples so as to introduce different theoretical perspectives and not to overlap with the ones that will be presented in the next section. In this way, we expect to be able to develop insights that will contribute to the enhancement of theories suitable for studying teacher collaboration. Hence, in addition to providing an overall landscape of different theoretical perspectives used for studies on teacher collaboration, the aim of Sect. 2.3 is to characterise the diversity and suitability of theoretical perspectives as well as to outline some perspectives for future research.

2.3.2 Roles of Theories

As discussed in Sect. 2.2, theories in mathematics education research play different roles. We consider that the two main roles of the theory in the research on teacher collaboration, when it is used as a tool, are: "a way of producing understanding and ways of action" (Radford, 2008, p. 320). Regarding the former role, theories are used to *understand* the educational phenomena related to teacher collaboration, by providing conceptual and/or methodological tools to analyse and understand phenomena from different perspectives. There are several aspects in teacher collaboration that can be the object of study. This is one of the reasons for the diversity of theories in research on teacher collaboration. A variety of theories can be used depending on the aspects to be studied, and different theories enable different understandings of the multifaceted nature of teacher collaboration.

Regarding the second role, the theory is used to *design* the work of a collaborative group, of a PD program or of other settings (including teacher collaboration inside/outside schools). A diversity of contexts and scales of designing a situation or a setting for teacher collaboration also produces a diversity of theories. The object to be designed could be the pre-service or in-service PD program or the community of teachers and/or researchers, at different levels (national, regional, local/district, school, etc.). The different theories would inform how to design a situation according to the contexts. The frameworks used for such studies are often called 'model', which implies how to organise PD programs, for instance. The term 'model' in the research on teacher collaboration may be used for those that "embody a theory of objects and relations among them" (Schoenfeld, 1998, p. 9) like in scientific fields (e.g. a model of the solar system), and also for those that describe or conceptualise the teachers' collaborative practices without explicit theoretical underpinning.

Among the theoretical references mentioned in the papers presented in the study conference, one can easily identify these two main roles. Many papers use or develop theoretical constructs to better understand teacher collaboration. For examples, Capone et al. (2020) utilise multiple theoretical perspectives (*Semiosphere*, *Semiotic mediation*, *Boundary objects*) to advance a multifaceted understanding on Lesson Study (LS) in the Italian context. Pepin and Gueudet (2020) discuss how the *Documentational Approach to Didactics* (DAD), a theoretical perspective which focuses on resources used and/or developed in teachers' work, allows us to understand the professional learning of teachers (in teacher collaboration) in terms of *schemes*.

In contrast, some papers referred to theoretical perspectives in terms of designing a PD program or other forms of teacher collaboration. For example, White (2020) designed a collaborative PD program in Ireland by leaning on the idea of *Professional Learning Community* (PLC) and the insights obtained around this idea in previous studies (see the detail below). Horn and Bannister (2020) developed insights for the intervention design of a form of teacher collaboration that was said to support transformative professional learning, based on the *Interactionist* perspectives.

The distinction between the theory for understanding and the theory for designing is not clear-cut. The former could be also used for designing a PD program or a teacher community. This depended on how the researchers/educators used the theory and in which ways they were engaged in the teachers' practices. In fact, the Interactionist perspective (Horn & Bannister, 2020) mentioned above was used, in addition to designing PD programs, to define teacher learning and allow the researchers to analyse the data and get insights for intervention design.

The *Fractals* perspective (Suurtamm, 2020—see more details below) was used to understand and design the professional learning communities and their relationships, as well as to design research with teachers. The intervention design was also a central concern in the project presented in the plenary lecture by Prediger (2020, and

Chap. 6, this volume), who introduced several content-specific theoretical elements for designing and explaining classroom practices as well as teacher PD practices and teacher educators' practices.

We provide here examples of studies presented in the study conference, which show how a specific theory is used for understanding and for designing PD programs.

2.3.2.1 Example 1: Theoretical Constructs for Understanding Lesson Study Within ATD

The paper presented by Otaki et al. (2020) proposed theoretical constructs within the *Anthropological Theory of the Didactic* (ATD; Chevallard, 2019), in order to better understand two aspects of Lesson Study (LS), which are critical to describe the mechanism of this professional development process.

The first aspect is the *paradidactic* aspect of LS that considers the nature of teachers' activities outside classroom such as designing, discussing, and analysing mathematics lessons. Otaki et al. (2020) proposed a theoretical construct that describes the factors the teachers consider in such activities, in terms of the *dialectics*, which is a notion used in ATD to characterise the two opposed types of constraints that influence different activities (Chevallard with Bosch, 2020). There are *six dialectics* which may happen during the process of LS: *dialectics of stakes and gestures, of period and study program, of milieu and infrastructure, of the predidactic and the postdidactic, of school and noosphere,* and *of the designer and the analyser*. We do not go into the detail of all these dialectics. To explain just one, the dialectic of stakes and gestures identifies teachers' back-and-forth reflection between mathematical knowledge to be taught and way of teaching it, by employing the notions of *didactic stake* and *didactic gesture* used in ATD (Chevallard, 2019). These theoretical constructs allowed the research team to characterise teachers' activities and reflections during the LS.

The second aspect is the sociocultural aspect of LS that questions the viability of LS in a given place. Within the ATD, the so called 'ecological analysis' studies the 'living' of a given phenomenon (e.g. a specific mathematics teaching, teacher collaboration) by identifying the *conditions* that made it viable in a given place (called *institution*) and the *constraints* that might hinder it. It has been one of the main research issues addressed within the ATD (Chevallard, 2019). Otaki et al. (2020) proposed *the scale of levels of paradidactic determinacy* as a theoretical tool to investigate and classify the conditions and constraints that support or hinder the existence of LS in a given institution (e.g. another country). This tool is an evolution of the *scale of levels of didactic co-determinacy*, which has been originally developed in ATD, and that highlight the multilayered nature of conditions on a given phenomenon, going from Humankind to Civilizations, Societies, Schools, Pedagogies and Didactic systems (Otaki et al., 2020).

The theoretical constructs developed in this study provided a terminology and allowed the researchers to arrange "*a set of specific observations and interpretations* of singular but related phenomena into a coherent whole" (Niss, 2007a, p. 105; *italics in original*). It was noticeable that the ATD as a principal theoretical framework provided the basic concepts and ideas, as well as "a coherent whole" to characterise the educational phenomena, in this case the existence of LS.

2.3.2.2 Example 2: PLC as a Perspective for Designing a PD Program

We present here another study wherein a theoretical perspective is used for designing teacher collaboration. White's (2020) paper already mentioned above presents an attempt to design and implement an effective teacher PD for mathematics teachers through an examination of the different models proposed and developed in previous studies in Ireland. In this study, the author examined first the impact and limitation of LS implemented in the Irish context in the last decade, and then proposed a PD based on the notion of *Professional Learning Community* (PLC) as an alternative approach for Irish teachers' collaboration.

PLC is a term widely used in the context of teacher education. Following DuFour (2004), White considered it as "a group of teachers who recognize the need to work collaboratively with a common purpose of improving student learning and achievement" (p. 216). In the literature, several studies have been carried out to characterise PLC, and they identified different elements of effective PLCs. For example, PLCs operate effectively when members have shared values and vision, focus on student learning, take an inquiry stance, make teaching more public, share experience and expertise, and so forth (Scott et al., 2011).

From what we can see in the studies of our ICMI Study group, PLC seems to be a general notion denoting the situation or setting where the professional learning happens, and it is not sufficiently conceptualised. It is more a model that provides ideas to design and structure the PD program, rather than an elaborated theory that is a systemic entity characterising and informing the mechanism of teacher collaboration. This is why several studies have investigated the characteristics of effective PLCs. Similar to LS, PLCs could be also an object of study, and at the same time used for designing PD activities.

2.3.2.3 Example 3: Fractals as a Perspective for Designing Teacher Collaboration and Research

Another example concerning the roles of theories is the case where the theoretical perspective functions as a tool or model for designing the teacher collaboration as well as for engaging in the research with teachers. Suurtamm (2020) proposed the mathematical idea of *Fractals* as a model for networked teacher collaborative communities, and at the same time as a model for the research with such communities. In several of her projects, she identified, as a key mechanism of teacher collaboration, the iterative and self-similar nature of the communities of researchers and teachers, which could be described by fractals. According to her, this nature fits well the ways of thinking about how they work and create the teacher collaborative communities, and design educational research with teachers. For example, she noticed self-similarity between the collaborative community of teachers and the collaborative communities developed in their classrooms, with a reciprocal process of feedback between these two kinds of communities.

In another case, the self-similarity can be found in large-scale projects between the multiple networked PLCs, and the single PLC. Further, in terms of research, the iterative dynamic of research design can be found in the data collection (each collection builds on the previous one), and in the ways some participants were engaged in collaborative research with teachers (e.g. mathematics co-ordinators became research participants as the data collection progressed).

The idea of Fractals here is a model that leads us to focus on specific aspects of teacher professional learning communities and understand their characteristics: iterative, and self-similar. This model is likely to have implications when designing and facilitating the nested communities of practice and when setting up a research project. However, this model alone would not inform the design of other aspects of teacher collaboration and research projects. Hence, she relies on several theoretical perspectives of communities and learning, such as Communities of Practice (Lave & Wenger, 1991; Wenger, 1998), theories of constructive learning (Cobb et al. 1993), sociocultural theory (Vygotsky, 1978, 1986), to name but a few. In other words, she synthesises different perspectives into one overarching one: Fractals.

2.3.3 Origins of Theories and Their Foci

We identified a variety of theories, which could be distinguished in terms of their different origins and different foci. Some have been developed in the area of mathematics education research, and others in for example, general education, sociology, or psychology. Again other theoretical frameworks were constructed specifically for analysing teachers' practice and later adapted to study teachers' collaborative work (e.g. DAD). It can be claimed that most of the theoretical perspectives discussed were not created for studying teacher collaboration; instead, they have been enhanced to deal with the issues of teacher collaboration.

Mathematics teacher collaboration includes a wide range of aspects, it is social by nature and takes place in a specific contextual or cultural setting. It is often discussed in the context of PD or teacher learning, which may be connected to the cognitive perspective. It includes the use and development of curriculum materials and resources, which may be considered in terms of theories of tool use and mediation.

Different theories enable us to understand different aspects of teacher collaboration (e.g. Trouche et al., 2019). The aspects the different theories consider are different according to their origins. The theories that originated from mathematics education research were developed especially to better understand the phenomena specific to the teaching and learning of mathematics, as the object of study. The theories developed in other areas do not focus on the aspects specific to mathematics.

They typically provide insights into other important aspects of teacher collaboration, such as interactions between different participants, teacher learning, and community development and operation.

In general, the origin of theoretical frameworks and perspectives is not always clear. A perspective may be developed based on the different perspectives by combining and adapting them, so that it may fit well to the analysis of the object of study. For example, DAD is a theoretical perspective that originates in French research of mathematics education and is today used in different fields of educational research. It had been developed based on ideas of French didactics of mathematics, the field of technology use (cognitive ergonomics), and socio-cultural theory, and later further developed including the field of curriculum design (e.g. Trouche et al. 2020).

The 18 papers in the study conference included a wide range of theories, and they originated from different fields. It is not easy to determine a single origin to one theory. Selected theories, such as for example, ATD, Commognition and DAD, have been developed in mathematics education research. Some theories, such as Cultural–Historical Activity Theory (CHAT) and Enactivism have their origins in general education or educational psychology. Theoretical perspectives, such as Community of Practice and Culturally Figured Worlds, have been developed within anthropology.

The theories used or developed in the papers presented in the study conference focus on one or several of the following aspects of teacher collaboration:

- social;
- cognition and learning;
- identity;
- resources;
- teacher mathematical knowledge.

We also found that several theoretical perspectives considered the social aspect in different ways. The theoretical construct within ATD presented above, *levels of paradidactic determinacy* (Otaki et al., 2020), account for the cultural factors, beyond the teacher's local setting, that shape the teacher collaborative work. The theoretical framework, *Culturally Figured Worlds* (presented below), focuses on teachers' formation of professional identity, and characterises how teacher behaviour shapes and is shaped by their cultural and social contexts and the mutual relations of power. This implies that the diversity of theories is due to the multifaceted nature of mathematics teacher collaboration, as well as to the approach adopted by a theoretical perspective to study a specific aspect. This would be also the case for the aspect of cognition or learning, which is characterised in different ways according to the theoretical perspectives.

In what follows, we provide three examples of how theories from different origins contribute to a better understanding of mathematics teacher collaboration.

2.3.3.1 Example 1: A Theory Originated in Mathematics Education Research

The first example is a study by Kondratieva (2020). She used the ATD developed in mathematics education research to understand teacher learning in a collaborative setting. The ATD includes several theoretical constructs to investigate the phenomena of mathematics teaching and learning, such as 'didactic transposition', 'praxeology', inquiry with 'media' and 'milieu', for example. In her study, Kondratieva used two kinds of ideas as analytical tools to better understand teachers' collaborative learning in mixed groups of elementary and secondary school teachers, when solving mathematical problems within a teacher education graduate online course.

First, she used the notion of *praxeology*. This is a model of human activity. A praxeology consists of two blocks: the first that describes *praxis* or practices including a *type of tasks* and a *technique* to solve such a type of tasks, and the second, that describes *logos* or discourse including *technology* and *theory* underpinning the praxis. Kondratieva used this notion to describe the development of mathematical knowledge and practice at stake in teacher collaboration, and revealed the overlap of elementary and secondary school praxeologies consisting of generic exemplification, which belongs to logos in the former and to praxis in the latter.

Second, she used the following three *dialectics* (from ATD), to analyse teachers' collaborative learning:

- dialectics of *idionomy and synnomy* (*the individual and the group*);
- dialectics of *conjecture and proof* (also called dialectics of *media and milieu*);
- dialectics of *black boxes and clear boxes*.

These are dialectics which can be found in the mathematical inquiry as well as in the inquiry of other domains. The use of dialectics allows Kondratieva to identify various tensions and discrepancies among teachers' viewpoints, and to consider teacher learning as a result of negotiation and resolution of those viewpoints during the process of mathematical problem solving. Of note is that the notion of *dialectics* is common in ATD and was also employed in the study by Otaki et al. (2020) presented above.

Kondratieva's study focuses on the mathematical inquiry and the collaborative development of mathematical knowledge in teacher collaboration. In such a study, the theory developed in mathematics education research like ATD allows us to capture the specificities of mathematics teacher collaboration.

We have also seen that the same ATD notions, dialectics and praxeology, can be used in different ways to analyse teacher collaboration. We will see later that the notion of praxeology is also used to characterise teachers' and researchers' practices in *Meta-didactical transposition* (Aldon, 2020; Shinno & Yanagimoto, 2020). It implies that a single theoretical construct may be employed and developed to analyse different aspects of the phenomena under investigation. The particular object of study we set up for studying teacher collaboration therefore enhances the theories.

2.3.3.2 Example 2: A Theoretical Perspective Originated in General Education

The next example is taken from the study by Calleja (2020) who investigated teacher learning in the collaborative setting in a continuing PD program. He showed how the theoretical perspective of *Zones of Enactment* could be used to better understand teacher change, that is teacher learning in his case, through their engagement in a collaborative PD program (learning to teach mathematics through inquiry). It showed in particular that the interplay between teachers' personal resources in enacting inquiry (e.g. knowledge, beliefs, and practices) and external factors (e.g. pupils, policy, public, private, and professional sectors) could be investigated through Zones of Enactment.

Zones of Enactment is a theoretical perspective that has been developed in general education and curriculum studies, based on Vygotsky's work regarding *the Zone of Proximal Development* (ZPD). Spillane (1999) proposed it to understand teachers' reconstruction of their practice in the implementation of educational reform from the perspective of educational policy. Zones of Enactment are defined as the space in which teachers "make sense of, and operationalise for their own practice, the ideas advanced by reformers" (p. 159). It characterises the teacher's change or reform of practices by focusing on the role of community and the context of teacher learning within educational settings. Spillane suggested a model to account for the ways teachers responded to and enacted mathematics reform.

Vygotsky's work is very influential in educational research and has been developed in different directions according to the object of study and to the aspects the researchers are interested in. Valsiner's zone theory that we already discussed at the Introduction was also based on Vygotsky's work to characterise child development and then adapted to study teacher learning. In the case of Zones of Enactment, this perspective has initially been a development for the curriculum studies or educational policy. Calleja (2020) further adopted it to understand teacher learning in the collaborative setting, with a special attention not only to the personal or cognitive factors (knowledge, beliefs, and practices), which are often discussed in the context of teacher learning, but also to the external or sociocultural factors. As the theory has been developed in general educational research, the disciplinary specificities of mathematics teacher learning were not taken into consideration in the theory itself. This is a point, which is different from the previous example. However, this is not necessarily a shortcoming. Such specificities would be addressed by combining other theoretical perspectives.

2.3.3.3 Example 3: A Theoretical Approach Originated in a Social Practice Perspective

The third example is taken from Skott's (2020) paper, which adopts a social practice theoretical perspective to better understand how contextual and power related aspects influence teacher collaboration. Specifically, her study concerns the dialectical relationships between teachers' social interactions and the social, cultural and power-related aspects of their local setting and beyond that, when they adapt LS in a Danish educational context. With an example on how an individual teacher participates in teacher collaboration in a LS context, she shows how social practice theory, especially the concepts of *Figured Worlds* and *Cultural models*, enable us to both conceptualise adaptations of LS in countries outside East Asia in a new theoretical way and to study teacher collaboration and individual teacher learning in such adaptations from a contextual perspective. The teacher's learning was characterised by shifts in ways of participating in the collaborative interactions with respect to the Figured Worlds s/he drew on.

The concept of Figured Worlds has been developed in the area of anthropology, in order to investigate how people form their shifting identity in relation to their social worlds and privileges of power. A Figured World is defined as, "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al., 1998, p. 52). This perspective enables researchers to study how people's behaviour shapes and is shaped by their cultural and social contexts and their privileges of power. The original aim was not to study educational phenomena, but the approach is increasingly employed in educational research.

This theoretical perspective provides a terminology and concepts to understand and explain teacher collaboration and individual teacher's learning in terms of cultural, social, and power-related aspects both in and beyond their local setting. As we have seen in other examples, the contextual aspect is critical in teacher collaboration and addressed in different ways according to the adopted theoretical perspective.

2.3.4 Research Issues on Theory: Developing, Enhancing, Networking, Analysing

In addition to using theory as a tool for understanding or designing teacher collaboration, the researchers may examine and develop it as an object of study. The rationale for carrying out explorations on theories should go towards a better theorising of teacher collaboration. As we have shown, most existing theories were not specific to teacher collaboration and addressed only certain aspects of teacher collaboration.

Furthermore, very often they did not attend to the characterisation of teacher collaboration as a whole, including associated teacher activities, student learning, curriculum resources. Prediger's plenary lecture for Theme A presented her attempt to theorise teachers' PD in a collaborative group in more content-specific ways. She based her work on the theory of professional growth (Clarke & Hollingsworth, 2002), and the identification of content-specific theory elements capturing the PD learning content (Prediger, Chap. 6, this volume).

Viewed from this perspective (on theory), one may find the different ways of exploring theories: the theory to be constructed; the theory to be used, adapted, or enhanced; and the theory to be analysed. This corresponds to another diversity of theories, as an object of scientific research from a structural point of view.

In the papers presented in the study conference, we identified two issues on theory:

1. developing theoretical constructs or theorising some aspects of teacher collaboration;
2. analysing theoretical perspectives or models.

Regarding the first, some papers proposed a new theoretical construct or model that allowed the researcher to better understand teacher collaboration. Other papers reported on other proposed theoretical constructs that complement existing theories which are not specific to teacher collaboration. For example, Otaki et al. (2020) developed concepts within ATD, which theorise the sociocultural aspect of teacher collaboration and the constraints the teachers encounter. Suurtamm (2020) proposed the idea of *Fractals* to theorise the self-similar and iterative mechanism within teachers' and researchers' work and nested communities. In addition, some papers provided a new methodological perspective on how the researcher can analyse or design the teacher collaboration, by adapting, enhancing, or combining the existing theoretical frameworks.

Regarding the second issue, some papers investigated the theories or models themselves, in order to better understand their affordances and limitations. It is especially in these studies that two or more perspectives were compared or contrasted.

In the study conference many studies considered more than one theoretical perspective. However, the ways to relate two or more perspectives were different. In some cases, there was a dominant theoretical perspective, with the integration of some small ideas or concepts into the dominant one. In other cases, theoretical perspectives were equally combined. In a study by the Networking Theories Group (Prediger et al., 2008b), different networking strategies have been identified (Fig. 2.1). We can also find some of these strategies in the papers of the study conference. For example:

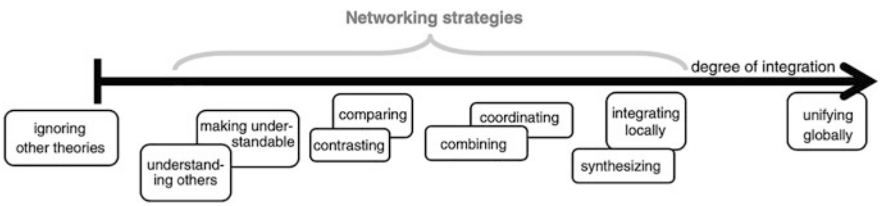

Fig. 2.1 A landscape of networking strategies. (Prediger et al., 2008b)

- synthesising, in which a new theory emerges: Fractals model (Suurtamm, 2020) is a result of synthesising the ideas of different theoretical perspectives, mathematical idea of fractals, communities of practice (Lave & Wenger, 1991; Wenger, 1998), theories of constructive learning (Cobb et al. 1993), sociocultural theory (Vygotsky, 1978, 1986), and so forth;
- combining, in which the used theories keep their identity: Hoyos and Garza's (2020) study combines DAD with Ernest's idea of personal philosophy or image of mathematics, and also with Clarke and Hollingsworth's (2002) *IMPG*. Capone et al.'s (2020) paper also combines three theoretical frameworks: *Semiosphere*, *Semiotic Mediation* and *Boundary Objects*;
- comparing, in which the theories keep themselves apart: Ding and Jones (2020) carried out a comparative analysis of different models of mathematics teacher collaboration: Action-Education model; Learning Study model; Community-Centered model.

In addition to carrying out the theorising work by combining existing theories, we note that it is also possible to undertake such theorising by a methodology involving the construction of hypotheses and theories through the collection and analysis of empirical data, similar to the grounded theory approach. However, we have no examples of such studies in Theme A papers at ICMI Study conference.

We now present some specific studies that exemplify different ways of exploring theories.

2.3.4.1 Example 1: Adapting and Theorising

The first example is Brown and Coles's (2020) study which has attempted to theorise a way of working with teachers of mathematics in collaborative work. They investigated and theorised how teacher learning might take place, based on: (1) their experiences of working in a PD course; (2) what they perceived as *enactivist* ideas. They proposed a model that conceptualises the process of mathematics teacher learning in collaborative groups. Through the analyses of the specific cases, they showed how this model allowed them to identify mathematics teacher learning in the process of collaborative work.

Brown and Coles' basic idea of theorising is that 'learning to teach mathematics' can be viewed in terms of the development of awareness. They characterised this development of awareness, relying on enactivist ideas, especially the three levels of categorisation of how humans perceive the world, namely at a detailed or subordinate level, a basic level, and an abstract or superordinate level (Varela et al. 1991, p. 177). According to their view, teacher learning takes place through a descent into the detail of experience, from the basic level to the subordinate level. A cyclical process of mathematics teacher learning in five phases was proposed: describing the detail of events; identifying new distinctions; developing new labels; trying out new actions; developing new basic-level categories.

Enactivism is a perspective of cognitive science that claims to model mechanisms of human cognition. It is a general perspective, which is not specific to education nor teacher collaboration but offers ideas which enables the researcher to theorise the process of mathematics teacher learning in collaborative work. Brown and Coles' study is a case of adapting the enactivist perspective to theorise mathematics teacher learning.

2.3.4.2 Example 2: Combining Theories as a Networking Practice

In terms of combining multiple theories, Capone et al. (2020) explored the possibilities of combining theoretical frameworks in order to understand mathematics teachers' PD, in particular LS in the Italian context. This paper is reminiscent of Skott's (2020) paper, which provides a theoretical perspective to analyse the implementation or adaptation of LS in a specific cultural context outside East Asia. While Skott's paper adapts a single social practice perspective (i.e. the framework of Figured Worlds), Capone et al. propose a study showing how dissimilar theoretical frameworks can highlight different aspects related to the adaptation of LS, and how LS improves teachers' practices in terms of PD. The following three perspectives were employed: *Semiosphere and Semiotic of Cultures*; *Semiotic Mediation*; and *Boundary Objects*.

They characterised the abovementioned three theoretical frameworks in terms of the three elements proposed by Radford (2008), which describe the theory: principles, methodology, and research questions. They then showed how each theoretical framework enabled the researcher to understand the different aspects of implementation of LS. Semiosphere, which is a concept developed in semiotics, permitted to see the deconstruction of practices and beliefs, and the production of a new awareness on the part of teachers. Semiotic mediation, which is originally Vygotsky's idea and has been further developed by an Italian group of mathematics education research (Bartolini Bussi & Mariotti, 2008), facilitated a better understanding (and importance) of the teacher's role for the appropriate choice of artefact.

The concept of Boundary Object, which has been developed within CHAT (Akkerman & Bakker, 2011) gains a significant role here. A boundary object is an object that lies at the intersection of several social worlds facilitating communication between them (Star, 2010). Capone et al. (2020) show how LS helped the boundary crossing of the prospective teachers towards the practicing teaching community. A specificity of their study is that it was carried out explicitly from the perspective of networking theories. The object of study was the LS, and at the same time the theories themselves. Capone et al. combined the theories, in order to answer the common research questions on LS. This was carried out through contrasting the specificities of each theory, in particular the basic principles that constitutes the system of theory and the methodology supported by this system.

2.3.4.3 Example 3: Comparing and Analysing Different Models for Teacher Collaboration

The last example presents a case of analysing multiple models of teachers' collaborative work. Ding and Jones (2020) comparatively examined three models of mathematics teacher collaboration and learning, and identified the affordances and limitations of each model. The models analysed were *Action–Education model* (Gu & Gu, 2016), which is practiced in China; *Learning Study* (Lo & Marton, 2012), which is a combination of LS and design study; and *the Community-Centered model* (Borko et al., 2005), which is developed for university-based PD program in US. The authors analysed these models by using Boylan et al.'s (2018) framework that theorises the nature and processes of teachers' professional learning, in terms of components and relationships, scope, theory of learning and location of agency.

The results of analysis showed the differences in particular aspects of PD: goals, learning processes, learning outcomes and contexts. For example, the learning outcome of the Action–Education model was, "Core elements of practical knowledge & its relationship such as task design and lesson implementation", while that of Learning Study was, "Lesson design and implementation of necessary conditions for learning according to VT [Variation Theory]" and that of the Community-Centered model was, "Content knowledge, mathematics-specific pedagogical knowledge, and recognition of the importance of learning community" (p. 116).

This paper is similar to Capone et al.'s (2020) study, in the sense that both studies aimed to compare/contrast the multiple theoretical perspectives or models related to teacher collaboration in order to better understand them. In contrast, the natures of perspectives or models seemed very different in the two studies. The three theoretical perspectives investigated in their study were constructed mainly as tools to better understand the educational phenomena, which was not necessarily specific to teacher collaboration, while the three models in Ding and Jones' study were mainly to design teacher activities or PD, which presupposed teacher collaboration. The roles of perspectives and models and the position of teacher collaboration were also different.

Interestingly, in Ding and Jones' study, another theoretical tool was employed to analyse the three models of teacher collaboration: the analytical framework developed by Boylan et al. (2018) to study teachers' professional learning. This framework directed the focus of study to specific elements that constituted professional learning: components, scope, theory of learning, location of agency, and philosophical paradigms. Compared with Capone et al.'s study which adopted Radford's (2008) framework of theory, Boylan et al.'s framework is specific to teacher learning and provided insights into the models of teacher collaboration for designing and practicing PD. In contrast, Radford's framework provided insights into the perspectives for designing research on mathematics education.

2.3.5 To Conclude

In this section, we reported on our observations from investigating the diversity and specificities of theories related to the research on mathematics teacher collaboration, in terms of the nature of theory. Our investigation strategy was to enter and clarify this diversity regarding the nature of theory in terms of three aspects: roles of theories, origins and foci of theories, and research issues related to theories.

We illustrated the diversity in each of these aspects with the papers presented in the study conference. This investigation allowed us to better understand the characteristics of theories used or developed in research of mathematics teacher collaboration, as well as the research issues related to the individual theories and to the set of theories.

Regarding the specificities of theories in research on teacher collaboration, we have observed that there were two main lines (while some frameworks deal with both): frameworks that conceptualise teacher learning in collaborative groups (e.g. teacher knowledge development) and teacher collaborative activities (e.g. PD program, teacher communities); and those that theorise the socio-cultural aspects of teacher collaboration (e.g. under what conditions this is likely to happen).

2.4 Conceptualising Teacher Collaboration

2.4.1 Introduction

In Sect. 2.3, we discussed the nature of theories how theories were used, developed, and adapted to study or design teacher collaboration. In this section, we discuss theories taking into account the nature of teacher collaboration. According to the kinds of teacher collaboration, we look for, through the papers presented at ICMI Study 25 Conference Theme A, how different theoretical perspectives enhance our understandings of the processes and outcomes of teacher collaboration situations, and what needs further investigation and theorising. To this end, in this section we first conceptualise how we understand teacher collaboration.

While teacher collaboration may take different forms in different contexts, some aspects are essential. We assume that mathematics teacher collaboration involves a group of participants, who work together pursuing a common aim, by establishing some joint working processes in which active involvement, balanced roles and caring relationships are central features. A collaborative group always develops its activity in a given context that provides the elements that justify the need for collaboration as well as the resources to make it happen. At the same time, the context provides constraints that affect the working processes. In this section, we begin by making distinctions between different situations or settings where teacher collaboration can occur.

Looking at different situations, we may understand what they have in common and how they differ, and which specific features may be considered in each kind of teacher collaboration. Each kind of collaboration is illustrated by examples from the study conference papers presented, and for each kind of collaboration, we investigate how the different theoretical perspectives allow us to address different questions related to teacher collaboration. For each situation we begin by making a general description, especially in terms of the features of collaboration; subsequently, we present the research questions that are addressed for this situation; and finally, we describe the theories used to address such questions.

From the papers at the study conference Theme A, a first distinction to make is (a) the collaboration in order to *solve a problem or to deal with an issue,* that has emerged in a given professional context and is perceived by a group of professionals as necessitating attention and (b) collaboration in the frame of a *professional development activity*. In this kind of activity, a group of professionals is formed (around a particular issue/problem/question) and typically the group has a 'minder' who guides and takes initiative for setting the aims and carrying out the work, as it is the case of LS.

All 18 papers of the study conference Theme A fall under (a) or (b) (with selected ones at the border between the two). Both situations are interesting to discuss as they involve collaborative processes and have a role in enhancing mathematics teachers' pedagogic practice. However, each paper shows specific features of situation (a) or (b), and each refers to a specific theory or framework, sometimes to several ones, and we pay special attention to these.

2.4.2 Collaboration to Solve a Problem or Deal with an Issue

In the collaboration to solve a problem or deal with an issue, one may find different kinds of situations in terms of institutional frames that shape mathematics teachers' collaboration. An important kind is the *collaborative project* (Sect. 2.4.2.1). A project may be regarded as an activity which aims at a particular outcome and is typically limited in time. When a 'project' continues over a prolonged period of time, it tends to become an organisational or institutional activity, which can be classified as another kind of situation, *collaborative activity* (Sect. 2.4.2.2). In addition, it is possible to speak of *collaboration in the frame of an existing organisation* (Sect. 2.4.2.3). We can exemplify these kinds of collaborations by selected papers presented at the study conference; when we cannot, we provide examples from the mathematics education literature.

2.4.2.1 Collaborative Projects

Collaboration-in-time-bound projects have particular features. A collaborative project departs from a 'problem situation', that is a situation which demands a solution;

this becomes the aim of the project. The tension between the 'problem situation' and the desired aim requires a structured organisation of work, including the mobilisation of internal and external resources required to achieve the set aim, and leadership of the project. The issue of leadership, or minder of the project, and relationship between participants is critical in collaborative projects.

An important issue in a collaborative project is the diversity of participants. In some cases, the collaborative group can encompass participants with similar status and experience (e.g. all are teachers with a similar role in their schools, as in Stephens, 2020). In other cases, a collaborative group consists of people with different academic and institutional status (e.g. some are schoolteachers and others are academic researchers, as in Pericleous, 2020). How the participants are organised, how they monitor their work, how they deal with internal tensions and conflicts are important issues to consider.

Another issue is the diversity of aims and outcomes of projects. In a collaborative project, the aim is explicitly set up, and the outcome may be evaluated in the project. These aims and outcomes may vary according to the project (e.g. resources, lessons, teacher learning, research results, diffusion of something, scaling-up). Further, the aim and outcome within a project may be different for different participants (e.g. teachers, researchers, etc.). Theoretical perspectives that enable investigating differing aims and outcomes are valuable for research on mathematics teacher collaboration. Each collaborative project is suited for a specific purpose, and it faces different challenges.

At the study conference, there were examples of papers in this sub-category. For example, in the case of Wake et al. (2020) study, the aim of the collaborative project was to advance understanding on how to organise teaching that holds curriculum coherence. The authors identified as research outcome the roles of the didactical devices/tools that provide connections across topics of conceptual understanding of mathematics and development over time. They studied this using the perspective of *Cultural–Historical Activity Theory* (CHAT) and the associated notion of *boundary object*. In this case, the theoretical perspective allowed them to better understand the nature of outcomes, and the different levels of activities the different participants engaged in.

In another project, involving the collaboration of a researcher and a mathematics teacher, Pericleous (2020), also drawing on CHAT, studied the activity of proving in the classroom. The author addressed the collaboration between the two participants, focusing on the design of tasks and lessons and on the classroom implementation in order to gain access to the aims and motivations of the teacher and to understand what drives the teacher decisions during teaching. The paper discusses the conflicts between the two participants that occurred during this collaboration and the role of classroom resources to shape the process of proving.

These two papers use CHAT. In CHAT the unit of analysis is an activity, which is an endeavor directed to an identifiable goal or object (Engeström, 2001). Such activity works at both an individual level (subject, tool, object) and a social level (rules, community, division of labour). The object of a collective activity is constantly evolving, both in its material features and also as a social entity. A main notion in this theory is that of contradictions. These may arise in and between components of the activity system, between different phases of development, and between different activity systems. Contradictions may lead to transformations and expansions of the system, therefore supporting participants' motivation and learning.

The participation in multiple activity systems leads to the theoretical concept of *boundary crossing*, as a socio-cultural gap creating discontinuities in the actors' actions or interactions. Boundary crossing is facilitated by *boundary objects*, a theoretical notion used in several papers at the ICMI Study conference. Bowker and Star (2000) talk about boundary infrastructures consisting of boundary objects that allow different communities to work together without fully resolving their conflicts or reaching a consensus (Star, 1989, 2010). Historically, Activity Theory has undergone several developments, with the third generation CHAT including the theory of *expansive learning* (Engeström, 2001), which stresses the role of communities and of transformation of culture, through the construction of new objects and concepts, and the development of new practices.

The notions of division of labor, rules and community seem quite apt to address the composition and activity of collaborative projects. These notions, however, are very general and may be merged to further notions that qualify or modify them, to consider the specific nature of collaborative projects. A similar observation also applies to *Meta-Didactical Transposition Framework* (MDTF—see later in this section), as both frameworks are well suited to study time-bound projects. Therefore, an open issue is to know how to enrich these frameworks with further notions specifically apt to investigate collaborative projects.

2.4.2.2 Collaborative Activities

Regarding collaborative activities, the main feature is their flexible nature. In contrast to the collaborative projects, the aims may be more diffuse. Instead of setting a timeline for achieving them, the group might keep progressing as long as benefits are perceived. The aim is not something that, once achieved, empties the need for collaboration, but, on the contrary, is something that may only be achieved by the continuation of the collaboration. In this case, there will be very likely some people that are more central to the collaborative activity than others, but the differentiation of roles is not as stringent as in the collaborative project. An important theoretical issue is to know what kind of bond may keep the activity together, progressing and developing. At the study conference, there were three examples of papers in this subcategory, one using the theoretical frame of *Fractals*, already presented in Sect. 2.3 (Suurtamm, 2020), and two others (Hoyos & Garza, 2020; Stephens, 2020) using the *Interconnected Model of Professional Growth* (IMPG).

In the study of Stephens (2020), the aim was improving the understanding of the processes through which teachers, working in collaboration, integrate new knowledge and improve their practice. The participants were all the teachers, except a novice teacher, of the mathematics department of a U.S. high school. The collaborative activities were rather unstructured and included very varied activities of the mathematics department. The results suggest that collaboration, despite being unstructured, played a very important role in the professional growth of the participating teachers. In the study of Hoyos and Garza (2020), the aim of the study was to build a model for the professional development of mathematics teachers. The participants were 14 middle-school teachers who carried out collaborative activities in relation to a mathematics curriculum reform based on the notion of competence. The results of the study show a change in the participant teachers regarding their conceptions of mathematics teaching.

The IMPG (Clarke & Hollingsworth, 2002) models professional learning through the consideration of interactions of four main domains: (i) the personal domain (e.g. teacher knowledge, beliefs, and attitudes); (ii) the domain of practice (e.g. professional experimentation); (iii) the domain of consequence (e.g. the salient outcomes that are perceived by the teacher); (iv) the external domain (which refers to external sources of information). Between these domains, there are mediating processes of (1) *enactment*, as participants incorporate new ideas within existing ideas, or carry out a new practice within existing practices and (2) *reflection*, as participants consider new knowledge, ideas, practices, and outcomes.

The IMPG is intentionally framed in terms of individual teachers. It models the change sequences and growth pathways for individual teachers in relation to the learning experiences they engage in, their knowledge, beliefs and attitudes, their practices, and their perceptions of outcomes. A suggestion made at the study conference was to consider if it would be possible to transpose the model from the level of the individual to the level of the group, to address the idea that collaboration plays an essential role in the professional growth of participants in collaborative groups of mathematics teachers. Whereas in the original model, collaboration would be located in the external domain, in a proposed modified version collaboration might fit in the domain of practice—the practice of the collaborative group.

This model seems especially suited to study professional development because of the 'external domain' (the context and activities arranged by the facilitators), which has an important influence in the unfolding of learning processes. In the study conference, it was used in one case with that purpose, and in another case to study a collaborative activity (with the external domain being called upon from the interactions with sources, such as web-based teacher sites). This modification and adaptation of existing models to study new situations may be a fruitful line of theoretical development. At the same time, in this case the model would require further specifications, both to attend to the features of collaborative activities and to effectively model collective rather than individual change and growth.

2.4.2.3 Collaboration in the Frame of Existing Organisations

Whereas in *collaborative projects* and in *collaborative activities* collaboration takes place in settings of informal organisations, collaboration may also take place in settings of a formal organisation. The organisation may have a name, a legal status, working rules, responsibilities for participants, formal procedures for admission, for example. The collaborative group may be a subgroup of a larger organisation, such as a working group of a teacher association. How different levels of the organisation relate to each other, how the work is monitored, how conflicts are handled, and how new members are introduced, are important issues in this kind of collaboration. *The Theory of Communities of Practice* (Wenger, 1998), which has been framed in the setting of formal organisations, may be useful to study this kind of collaborative activity.

At the study conference, an example of a paper fitting into this subcategory is Pepin and Gueudet (2020). One of the examples presented in this paper concerns the Sésamath Association in France (https://www.sesamath.net/index.php). In this association, volunteer participants, all practicing teachers, have collaborated to write e-textbooks and to create software and other digital curriculum resources (e.g. Gueudet & Trouche, 2012). Another example is the Grupo de Trabalho de Investigação (GTI) of the Associação de Professores de Matemática in Portugal (http://www.apm.pt/gt/gti/), a collaborative group of teachers and researchers that organises multiple activities including the writing of edited books with theoretical essays and practical examples of themes of interest to mathematics teachers (da Ponte, 2008). The first book of this series includes an essay (Boavida & da Ponte, 2002) about mathematics teacher collaboration that provided the blueprint for subsequent work.

2.4.3 Professional Development Activities

In terms of collaborative professional development activities, there is a variety of situations concerning the role of the different participants. We have 15 examples of this kind of collaboration in the papers presented at the study conference. One of them concerns pre-service mathematics teacher education (Shinno & Yanagimoto, 2020), 14 concern in-service mathematics teacher education.

2.4.3.1 Pre-service Teacher Education

At the study conference, there was one paper on pre-service mathematics teacher education (Shinno & Yanagimoto, 2020). In pre-service teacher education, collaboration may exist among different participants. However, it is influenced and shaped by the institutional setting, with roles of participants and power relations established by national or institutional norms. In this case, theories and issues about pre-service teacher education should be considered. The aims and processes of the activity should be known, including the negotiation of roles and activities to be carried out, the evaluation of prospective teachers, and the teaching style of the instructors, as well as the contextual affordances and constraints (e.g. institutional rules, previous preparation of participants, time and resources available).

Some pre-service teacher education activities take place in schools, involving the 'supervising' mentor (e.g. co-operating subject teacher) and in some cases further actors (e.g. other teachers, parents, etc.). The collaboration may take place in different ways (e.g. different teacher education models), and with different participants. This makes it a very complex but interesting object of study. The main issue here is how to develop and sustain collaborative relationships among participants given the institutional constraints and each participant's role in this situation.

The study by Shinno and Yanagimoto aimed to understand the planning skills of primary school pre-service teachers and the changes through the experience in LS, which involved researchers and practicing teachers. The LS was carried out outside university at an annual half-day conference with several open lessons centered on mathematics. This conference included primary and secondary school teachers and mathematics education university professors. Interestingly, the pre-service teachers were not really engaged in this LS activity, but worked separately in the university course to write a lesson plan on the same topic used in the open lesson of the conference before participating it. Then after attending the open lesson and the discussion of LS, they had another discussion in the university to re-design a new lesson plan. The results of this study indicate how pre-service teachers adapted their lesson plans in relation to the work carried out by in-service teachers in the LS.

As in the study by Shinno and Yanagimoto, meta-didactic transposition has mostly been used to analyse pre-service teachers' learning. However, we claim that this framework could also be used to describe collaboration involving in-service teachers and researchers. Indeed, it could be used in a more general way to study collaborations involving two or more different groups of participants. The meta-didactic transposition framework has been described in Arzarello et al. (2014), Robutti (2020) and Aldon (2020). It considers four main concepts: (1) Meta-didactic praxeologies; (2) double dialectics; (3) brokering processes; (4) internal and external components. As mentioned earlier, praxeology is a concept developed within ATD (Chevallard, 2019).

In Shinno and Yanagimoto's (2020) words, "Didactic praxeologies describe teachers' didactic activities. Meta-didactic praxeologies describe researchers' (or teacher educators') activities related to those of teachers" (p. 175). Concerning double dialectics, the first dialectic occurs in the classroom, with poles on the students' personal meanings and the scientific meanings. The second dialectic occurs in the interaction between teachers and researchers, and concerns the interpretations of the first dialectic by these two groups of actors.

The brokering processes concern the dialogues that take place between the two groups of actors. The internal and external components refer to relative position of the elements of each block of praxeologies. These brokering processes are at the heart of the interaction between participants in the process (in our case, the members of the collaborative activity) and this framework also uses the notion of boundary object (Star & Griesemer, 1989) to describe those interactions.

Meta-didactic transposition provides a language of description for collaborative activities, which may be useful to identify important elements of the activity that must be considered, in order to understand how the activity started, how it developed, and how it enabled the eventual creation of shared praxeologies between different groups of participants.

2.4.3.2 In-Service Teacher Education

At the study conference, a variety of in-service collaborative activities were conducted, in which collaboration was assumed to play a prominent role. In this respect it would be important to know about the aims and processes of the activity, as well as the features of the context such as teachers' motivation to participate and disposition for an active participation, teachers' previous preparation, time and resources available, style of facilitation, to name but a few. Collaboration may take place among participating teachers, and between teachers and facilitators. How may these collaborative relationships develop? What factors may sustain or inhibit them? What is the potential and what are the limits of such collaborative relationships? We present three examples of theories used to address these processes: (1) the theory of *Commognition*; (2) *the Documentational Approach to Didactics* (DAD); (3) the theory of *Situated Learning*.

(1) The Commognitive theory identifies thinking as communication (Sfard, 2008). This theory is used in the study by Elbaum-Cohen and Tabach (2020). The aim of the study was to know if participating in a professional development program that encouraged the integration of technology in the classroom and reflection led to changes in teaching practice. The authors indicate that the commognitive theory allowed to identify changes in the professional identity of a participant teacher. They suggest that providing such opportunity for reflection may lead to the development of the teachers' professional identity.

The Commognitive theory claims that thinking is a human capacity that develops through the social activity of communication. It considers learning as becoming a proficient participant in a discourse, which is identified by the changes in the participation in such discourse. It also emphasises the concept of personal identity, defined as the set of all stories that are reified (saying something about what the person is), meaningful (indicating key characteristics), and endorsable (supported with empirical evidence) (Sfard & Prusak, 2005). The identity of a person may be actual, as the stories are told about this person at the present or 'designated', if the stories are told about the expected future of the person. This theory distinguishes discourse that concerns mathematical objects ('mathematising') and discourse about people that participate in the discourse ('subjectifying'). The stories related to the identity of a person come from 'subjectifying' discourse.

This theory originates in the field of mathematics education, and was initially not intended to study collaborative processes. However, it has been shown that this theory can be beneficial for the study of human interactions and its consequences in participants' learning. Further, it may be used to study specific phenomena occurring during collaborations, in particular those that take place in professional development processes.

(2) Another theory used to study collaboration in professional development processes is *the Documentational Approach to Didactics* (DAD) (Hoyos & Garza, 2020; Pepin & Gueudet, 2020). Pepin and Gueudet present three studies that address the use of resources in mathematics teacher collaboration. One of the studies focused on the documentation work of two secondary school mathematics teachers working in collaboration, another studied the production of documents in a group of teachers of the Sésamath association, and the third concerned the documentation work of a Norwegian teacher working in a large European project. The results of the three projects suggest that collective work can promote the evolution of individual schemes.

In DAD, the focus is on teachers' interaction with resources and on its consequences (Gueudet et al. 2013). The DAD draws on the instrumental approach (Rabardel, 2002) which makes a distinction between artefact and instrument: artefact + utilisation schemes = instrument. In a similar way, the DAD distinguishes between resource and document: resource + utilisation schemes (for a particular goal) = document. The notion of 'scheme' (Vergnaud, 1998) is central to the DAD. According to Vergnaud, a 'scheme' has four components: (i) the goal of the activity; (ii) rules of action; (iii) operational invariants (concepts-in-action, and theorems-in-action); (iv) possibilities of inferences.

Utilisation schemes can be (i) procedural schemes, regarding the use of a given resource or (ii) mental/cognitive schemes, regarding the knowledge about the resource and strategies how to use it. The DAD has been used to study both individual teachers' work, as well as teacher collaboration, in terms of their use of resources. In collaborative processes, the group of participants may develop 'agreed schemes'; in this respect an interesting question is how these schemes develop and how they may influence individual schemes. It is interesting to note that the DAD developed within mathematics education in close relation to theories outside mathematics education, such as instrumental genesis (Rabardel, 2002).

(3) Finally, another theory used to study teacher collaboration is the *Situative Theory of Learning* (Horn & Bannister, 2020), which is also called *Situated Learning* (Cobb & Bowers, 1999). Horn and Bannister present two examples of studies drawing on this perspective. One of these projects used principles about teacher collaborative inquiry to create an activity based on video formative feedback aiming to encourage the development of innovative mathematics teaching. The other project used principles about video-based collaborative teacher learning to design what they call a "responsive professional development model" aiming to respond to participant teachers learning needs. The authors suggest that features such as considering the novelty of deep collaboration, working with a shared vision of teaching and provide adequate visions of practice increase the possibilities of mathematics teachers' learning through collaboration.

Situated Learning (Greeno, 1998; Putnam & Borko, 2000) stands in contrast with cognitive theories that look at learning of individuals; it investigates learning of individuals-in-context. Hence, *Situated Learning* considers not only individual teachers but also groups of teachers working together, and their social environment. Under this theory, interactionist analysis may address teachers' professional conversations, looking how these conversations may provide teachers with conceptual resources for their activities. In these conversations, an important distinction is made between interpretative viewpoints that allow teachers to make sense of events, and epistemic stances that refer to claims or statements that teachers consider to be true. In contrast with the other two former theories, *Situated Learning* had its origin outside mathematics education and is currently applied to the study of general teacher education issues, for example.

Papers dealing with in-service teacher education were, by far, the largest set at the study conference. The sharp distinction between participants in a professional development activity and facilitators would require a careful analysis in terms of its consequences regarding the collaborative activities. What important elements bear in the initial negotiations regarding teachers' participation in the professional development activity? How do the relationships between participants and facilitators develop during the activity and what factors influence such development? However, these are issues that have not been addressed in the papers and do not stand in a clear way in the theories/frameworks presented in those papers.

2.4.3.3 Lesson Study

Lesson Study (LS) has been mentioned several times in this chapter: it is one of the professional development activities which is from time to time adopted for the pre-service and/or in-service PD. Several frameworks have been used to study different aspects of LS. At the study conference, six papers discussed LS processes. One paper (Skott, 2020) was already presented in Sect. 2.3, Social Practice Theory/Figured Worlds. Two further papers, already referred to above, concerned the Meta-Didactical Transposition Framework in pre-service teacher education (Shinno & Yanagimoto, 2020) and Cultural–Historical Activity Theory regarding a collaboration between UK and Japanese researchers (Wake et al., 2020).

Another paper using ATD is also referred to in Sect. 2.3 (Otaki et al., 2020). Still, two further papers that present LS as teacher development contexts are White (2020), based on the notion of Professional Learning Community, and Capone et al. (2020), who present several theoretical frameworks. In these papers, LS was regarded as a specific form or model of teacher professional development and not as a particular theory to study mathematics teacher collaboration.

2.4.4 To Conclude

Each kind of collaboration faces different challenges regarding issues such as: (i) establishing and developing the aims of the activity; (ii) establishing and developing the working processes; (iii) establishing and developing relationships among the participants; (iv) establishing and developing forms of leadership that support the development and the regulation of the activity; (v) framing the relationship between the collaborative activity and important issues in mathematics, in mathematics education or in mathematics teacher education (such as mathematical focus, curriculum approach, focus on specific didactical issues or materials, focus on the student, and so on).

The examples of theories show powerful ways of looking at specific aspects of teacher collaboration that may be used in collaboration to solve a problem or deal with an issue as well as in professional development. All these theories provide elements for describing teacher collaborations that are useful to understand these processes. However, none of these theories is specifically geared to study teacher collaborations and there are central features of teacher collaboration that are not addressed by any of these theories (e.g. affect in teacher collaboration).

In fact, this analysis shows that there are different kinds of situations. Some theories and frameworks are very general and may be used to study different kinds of social processes that extend beyond collaboration. These theories and frameworks could be enriched with further specific concepts targeted at the study of mathematics teachers' collaboration. Other theories and frameworks only address selected aspects of the collaborative activities or change in the participants, and these could be networked with other theories and frameworks to study the most essential aspects of mathematics teachers' collaboration (e.g. those related to the different forms of collaboration, or the participants in the collaboration and the tools used, as addressed in other chapters of this book).

2.5 Conclusion

In this final section, we summarise the state of the art of our Theme A and the perspectives for future studies. We also discuss what this ICMI Study advances in our understanding of the theoretical and methodological perspectives on the studies in mathematics teacher collaboration.

2.5.1 Summary of Our Reflections

The analysis of the different theories used in the study of teacher collaboration showed a great diversity of theories, frameworks and models. We identified theories used for understanding teacher collaboration, for designing teacher collaboration, and for both understanding and designing teacher collaboration. We also noted that some theories have their origin in mathematics education, whereas others come from general education, sociology, social psychology, cognitive sciences, to name just a few. In addition, the position of the theories in the research work varies—some present theories to be constructed, others theories to be used or adapted, and still others theories as an object of analysis.

The analysis of the different contexts for teacher collaboration in Theme A papers showed that in many cases collaboration is seen as an element of professional development processes, and more attention is necessary to the work of collaborative groups. We also observed that collaborations are created to solve specific problems, as collaborative projects or activities, or that collaborators develop their activity within existing organisations. Wide-ranging theories, such as Communities of Practice or Activity Theory, have been used to study collaboration, and they could be enriched with further concepts to better address the specific features of the collaborations. In addition, several theories have been used to study some aspects of the work of collaborative groups but leave out of the picture some other aspects of the work or the context of the collaboration. These theories may be further developed to address the work of the collaboration in their complexity.

2.5.2 Responses to the Initial Questions

In what follows, we respond the four questions initially posed in the discussion document of ICMI Study 25, and addressed in the 18 papers presented in the study conference for Theme A. We provide our answers obtained through the whole discussion on our theme, the theoretical perspectives on studying and mathematics teacher collaboration, in the study conference as well as in this chapter, and show in which ways the studies presented at the conference advanced our knowledge of this theme.

How do the different theoretical perspectives or networks of theories enhance understanding of the processes of teacher collaboration? How do they enhance understanding of the outcomes of teacher collaboration?

There have been three broad groups of theories addressing teacher collaboration. The first group includes theories developed outside mathematics education that focus on the nature of communities (e.g. CHAT, Communities of practice). These theories facilitate a close look on the activity and working processes of collaborative groups. The second group includes theories, also developed outside mathematics education, that address professional learning occurring inside collaborative groups (e.g. Interactionist perspectives, Enactivism, Zones of enactment). These theories support an understanding of the outcomes of teacher collaboration, especially concerning the learning of participants. The third group includes theories originally developed inside mathematics education (e.g. Meta-didactic transposition, ATD, DAD, and Commognition theory). These theories facilitate a close attention to particular processes and outcomes regarding the activity of collaborative groups that are specific for the work of mathematics teachers.

The outcomes of teacher collaboration can be viewed in terms of the professional learning of the participants, but also in terms of developing their identity as a group and in terms of their impact within and on the context in which they operate. Whereas those theories that address the professional learning that occurs inside collaborative groups allow an understanding of the participants' learning, it seems that other kinds of theories, addressing organisational and political issues, will be necessary to study the learning of groups and their impact in the underlying contexts.

What is illuminated by the different perspectives and methodologies, and what needs further investigation?

The different theoretical perspectives allow the study of processes and outcomes of teacher collaboration as we indicated above. However, some aspects of these outcomes and processes remain largely invisible. These include, for example, the aspects related to the dispositions, motivations and other volitional elements concerning the participants in teacher collaborative groups. Another issue that has not been fully addressed in previous studies is the development of collaboration relative to the context in which it develops. Several papers in the study conference address this issue, in particular in the case of LS, with some theoretical perspectives (e.g. Social Practice theory, ATD). Still another area in need of further study is the content of teacher collaboration—the 'what question'—especially when the collaboration is carried out to solve problems or to deal with an issue, outside or inside existing organisations. Further investigation is also necessary for other kinds of teacher collaboration by asking questions about, for example, the features of this context that support or hinder the creation and development of collaborative groups; the impact of the work of collaborative groups within or on the context in which they operate.

What are promising research designs, data collection and analysis methods to study teacher collaboration?

In the study conference, the discussion focussed mainly on the theoretical perspective, whilst methodological aspects have only been scarcely addressed. While some papers referred to the design method of teacher collaboration, research designs did not have significant differences in the methodologies regarding what was found in the ICMI Study 15 Survey. Research designs depend on the object of study (or aspects of teacher collaboration) and the theories within which the study is carried out. Given the high complexity of the phenomena involved in teacher collaboration, it is not surprising that the most common research designs are qualitative.

These designs can be observational, such as participant observation, and case studies of existing collaborative groups; or interventional, such as action research, design-based research, and other developmental designs. Grounded theory studies may also provide important new insights regarding mathematics teacher collaboration, complementing what we may learn based on existing theories. Lesson Study, with its inherent collaborative nature, is a context highly favorable to study teacher collaboration. Regarding data collection and analysis methods, and as well as for many other topics of mathematics education research, an intensive use of technology, such as video recording and data analysis software, may yield interesting and new results that have not been possible with more conventional methods.

2.5.3 Perspectives for Future Studies

In this chapter, we strived to take the most out of the papers that were presented at Theme A of the ICMI Study conference. The issues that we could address follow from what was presented in the group's papers; they represent a follow-up from what has already been apparent from the work of the ICMI survey team (Robutti et al., 2016; Jaworski et al., 2017). Regarding wide ranging theories, further theorisation will be necessary to add further concepts to address the issues specific to mathematics teacher collaboration and in turn to support practice. Regarding theories that only address some elements or outcomes of teacher collaboration, further theorisation will be necessary to address the complexity and outcomes of collaborative phenomena. In addition, further theorisation regarding affective, organisational and political issues will be necessary to study issues that so far have been largely invisible in the study of teacher collaboration.

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Chapter 3
Contexts, Forms and Outcomes of Mathematics Teacher Collaboration

Cristina Esteley, Rongjin Huang, Maria Mellone, Gabriel Soto, Raewyn Eden, and Alf Coles

3.1 Introduction

The chapter aims to synthesise a comprehensive analysis of papers included in Theme B of ICMI Study 25 that focused on forms and outcomes of mathematics teacher collaboration enacted in different contexts and provided insight into studies on teachers' collaborative learning. By doing so, this chapter helps to elucidate the

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relationships between different forms chosen to design collaborative settings and their effects on the outcomes of collaboration regarding teachers' learning, as well as exploring affordances and limitations of some forms of collaboration.

The main sources used for writing this chapter include the papers, presentations, reactions, and discussions that took place at the ICMI Study 25 conference. To carry out our work, we first identify research gaps related to Theme B by reviewing relevant seminal research findings and presenting research questions for Theme B (Sect. 3.1.1). Then, we provide overarching conceptual frameworks which help us to examine the papers in depth (Sect. 3.1.2). After that, we describe methods for analysing the papers and organising the results (Sect. 3.1.3). Finally, the structure of the rest of this chapter is presented (Sect. 3.1.4).

3.1.1 Backgrounds and Purpose

Teacher learning through collaboration has been a research field over the past two decades. Some related and relevant works are summarised here in order to understand the importance and purposes of Theme B of ICMI Study 25.

3.1.1.1 Early Contributions

It is worth acknowledging the early contribution of Peter-Koop et al. (2003) and Even and Ball (2009) who, considering collaborations mainly within professional development programmes, emphasise the contextualised aspect of the collaborative work and its corresponding outcomes. Peter-Koop et al. (2003) highlight that the *form* and *path* that a collaborative project can take depend on the *context of work*. They also point out that interactions within its context may require some changes in the collaborative work and emphasise that *products or outcomes* of collaboration emerge as the process unfolds, including *undesired outcomes*. Even and Ball (2009) emphasise the transformation of teacher professional learning from acquisition-centred to participation-oriented models. They discuss various forms of collaborative work that involve teachers and their professional practices. In this regard, they mention lesson studies, communities of practice, communities of inquiry and collaborative groups.

3.1.1.2 ICME 13 Survey: Expanding Ideas and Contributions

The survey on "Teachers working and learning through collaboration", presented at ICME 13, expands on previous contributions focusing on collaboration as a process connected to the work of teachers (Jaworski et al., 2017; Robutti et al., 2016). "Teachers' work includes all dimensions of teaching in and beyond face-to-face activity with students in the classroom" (Jaworski et al., 2017, p. 263). They argue that participants within a collaborative context, will address issues that challenge teachers professionally and promote reflections on the role of teachers in school and society. However, the ways in which collaborative work with teachers can be organised depend on who the partners are, the type of initiative, the foci of the work, and its aims.

Regarding the type of initiative, Robutti et al. (2016) identify five main types: (1) initiatives mandated by ministries and national/regional institutions; (2) collaborations supported by ministries and national/regional institutions; (3) research collaborations initiated by researchers; (4) professional development initiated by researchers/didacticians; (5) school-based collaborations that were both initiated and sustained by the teachers without the direct involvement of others. Yet, they recognised that these categories of initiatives could often overlap.

Concerning *foci*, Robutti et al. recognise two broad categories. The *first category* is related to some aspects of *innovation* about mathematical content, the development of new curricula, different pedagogical approaches and the integration of new tools and resources (mainly digital tools). The *second category* is focused on practices that foster teachers who are able to implement innovative ideas, curriculum and tools in their classes. In terms of the *aims* of the collaboration, the multiplicity of objectives is highlighted. The multiplicity depends on the type of initiative on what the participants want to focus on.

As noted above, the ways in which collaborative work can be organised or shaped may vary depending on the project under consideration. Despite this variability, Robutti et al. identify common characteristics such as: (a) the implementation of tasks or approaches that encourage teachers' willingness to participate; (b) the role played by some experts; (c) the fostering of teacher engagement within the communities; (d) the use of theoretical lenses to introduce a specific subject or a specific topic or to support the analysis and the sharing of reflections of the participants. However, Robutti et al. note that, in a significant number of the papers reviewed, the aims of the collaborations were not specified in detail, nor was the impact of local contexts on the collaboration process or its outcomes.

Building on the existing studies and the ICME 13 survey, Theme B of ICMI Study 25 specifies its guiding questions as below.

3.1.1.3 Guiding Questions of Theme B

As described in the Discussion Document for ICMI Study 25, the studies presented in the Theme B group aimed to address the following questions:

1. *What models of teacher collaboration have been developed? What are the design features, goals, and outcomes of the different models?*
2. *How effective are various models for promoting different outcomes?*
3. *Which forms of collaboration are appropriate in different contexts?*
4. *What are the affordances and limitations of each form of teacher collaboration*?
5. *What are the benefits and the challenges that online teacher collaboration poses to the teachers?*

In this chapter, we provide answers to these questions and rise new issues through synthesising the conference papers and presentations to expand and advance the research field on teachers' collaborative learning.

3.1.2 Conceptual Framework for Synthesising Papers

The complexity of teacher collaboration is highlighted in the Discussion Document for ICMI Study 25 (Borko & Potari, 2020) as follows:

> The goals of teacher collaboration are multi-faceted and might be related to the mathematics content, to the learning experience of students, to the development of mathematics teaching that promotes students' learning [. . .], to the design of resources, [. . .], to the creation of a community in which ongoing professional learning is supported, or even to day-to-day teaching [. . .]. (p. 5)

In line with the findings of the ICME 13 Survey, Borko and Potari also point out the diversity of forms that collaborative work can take, of foci that are chosen, or of the results achieved. It is argued that decisions on the different purposes, forms and focus of the collaborative work not only depend on the collaborative project but also on the context which frames the collaborative work and the level of complexity of such work.

The contextualised aspect of teachers' collaborative work, as described in the Discussion Document for ICMI Study 25, can be clearly demonstrated by the case of adaptation of Japanese Lesson Study (JLS) around the world. Even though key elements of JLS are similar, adapted forms and intended goals of learning vary tremendously (Fujii, 2014). Miyakawa and Winsløw (2019) further argued that, even within Japan, JLS can function in different ways depending on institutional conditions and the motives teachers have in the context of their practice.

To have a common language to synthesise the major ideas across different papers in Theme B, we now delimitate the key terms and concepts, and present our overarching framework.

3.1.2.1 Forms of Collaboration

Although there is a variety of meanings of the noun *form*, we highlight three meanings of form related to teacher collaboration as defined by Oxford Advanced Learner's Dictionary (https://www.oxfordlearnersdictionaries.com/): (1) a type or variety of something; (2) the particular way something is, seems, looks, or is presented; (3) the usual way of doing something. Specifically, we consider that the forms of collaborative work represent the types of professional learning environments for teachers' collaborative work to take place. With each form, the corresponding activities are created and organised according to the goals set by those who initiate and/or participate in the collaboration.

3.1.2.2 Context

Horn et al. (2017) acknowledge that specifying a notion of context is problematic due to the polysemy of the concept; context is a notion that is continuously open to be revised within the social and human sciences. According to Lave (1988), context consists of two components, namely a fixed arena and a setting (or scenario) that is defined as "a *relation* between acting persons and the arenas in relation with which they act" (p. 150). The setting refers to what is created by subjects who develop their activities in interaction with the arena and others. The activities and the experiences are "dialectically constituted in relation to the setting" (p. 151). The setting generates the activities and these, in turn, generate the setting. In this way, descriptions of the contexts of collaboration may reveal arenas, settings, actions, activities, and participants. A detailed presentation of the context would provide elements not only to understand the work carried out by the participants, but also the significance of the outcomes achieved.

3.1.2.3 The Nested Nature of Teachers' Professional Work

Esteley (2014) considers that a teacher's professional work can occur at three levels: the micro didactic level of the classroom (e.g. interactions between content, students, and teachers); the institutional level of the school where he/she performs his/her duties as a teacher (e.g. interactions with colleagues or principals to agree and organise assignments for students, assessments, meetings with parents, etc.), which is called the meso or institutional level; and the macro level, referring to the educational system (ministries or other broad institutions) to which the teacher can make various contributions (see Fig. 3.1). Although the three levels of a teacher's work are interrelated, a teacher's collaborative work with others may focus on one or more of these levels. However, the collaborative work at each level informs one another (as highlighted by the arrows), forming a nested hierarchical system.

3.1.2.4 Teachers' Professional Development as Multi-level System

In accordance with Davis and Sengupta (2020), we understand collaborative work for teachers as a complex phenomenon due to the multiple levels involved in the work of teachers (Jaworski et al., 2017) and to the work being a context-sensitive phenomenon (Dowling, 2020; Mellone et al., 2020).

The complexity of professional development of teachers has been well-recognised and described by several studies (e.g. Prediger, 2020; Prediger et al., 2019; Krainer, 2014; Opfer & Pedder, 2011; Loucks-Horsley et al., 2010). For example, Loucks-Horsley et al. (2010) emphasise that the process of professional learning includes five interconnected phases: goal setting, planning, execution, review of results, and reflection on the entire process. Specifically, from a

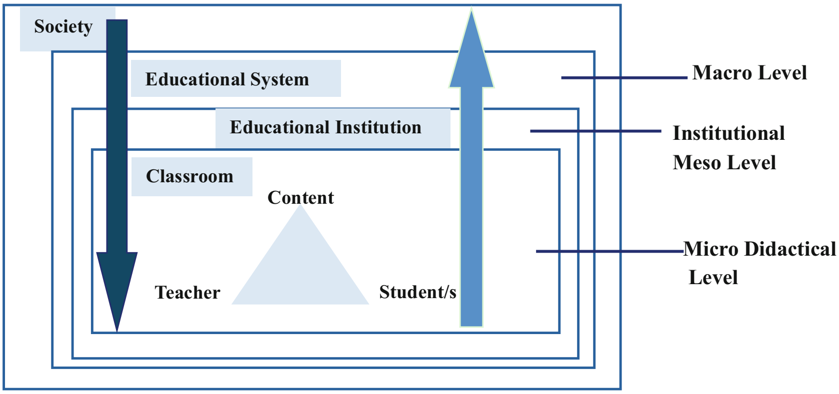

Fig. 3.1 Nested diagram of teachers' professional work. (The diagram in this figure is an English translation of the original Spanish version. The diagram presented here was elaborated with contributions from Esteley's fellow teachers and researchers.) (Esteley, 2014)

complexity theory perspective, Opfer and Pedder (2011) conceptualise professional learning as the dynamics of interactions across three levels of context:[1] micro (individual teachers), meso (school level) and macro (school system).

Krainer (2014) conceptualised the complexity of professional learning in relation to the scale of Professional Development Programs (PDPs), in terms of the number of participants, institutions or communities involved in the PDPs. In her plenary talk, Prediger (2020) presented a nested three-level model to examine the area of teacher development as well as the facilitators' development in collaborative contexts. Each level was represented by a tetrahedron related to classroom teaching, teachers' professional development (TPD) and facilitators' development.

At the *first level* of the model is a tetrahedron representing the teacher's work in the classroom. It is composed of the three elements of the didactic triangle (mathematics, students and teacher) and classroom resources as a fourth element. At the *second level or TPD level*, the tetrahedron has as one of the vertices the teachers, the facilitator is at another vertex, the content (aspects of teaching and learning that are addressed in the program) is placed at a third vertex while the fourth vertex represents the TPD resources (i.e. the materials, the activities of that program). Prediger (2020) uses a similar structure for a *third level* or facilitator professional development (FPD) level. Considering the nature of collaborative work, she also incorporates the concepts of community of practice (Wenger, 1998) and community of inquiry (Jaworski, 2006).

From this brief review, we can see there are different terms applied to levels of professional learning from different perspectives. In this chapter, we use *micro*

[1] It is important to note that, as the teacher's work is the core of the collaboration, the levels considered by these, or other authors are connected to the levels of the teacher's professional work (Esteley, 2014), and in that regard, they can be named in a similar way.

(classroom), *meso* (institution) and *macro* (education system) to represent three levels.

3.1.3 Analytical Approach for Analysing the Papers

The main resources informing this chapter are the 28 conference papers and 25 presentations and reactions, along with the discussion notes generated at the study conference. The 28 papers accepted for Theme B represent a variety of collaborations carried out in different geographic and cultural contexts: 11 from Europe, eight from Latin America,[2] three from North America and six from the Asia–Pacific region. In addition to these primary data resources, additional relevant literature was referenced to help us to frame and sharpen major arguments about the guiding questions for Theme B. To develop this chapter, we have gone through three main phases, which are iterative rather than linear due to the complexity of the five guiding questions.

During the first phase, the participants at the study conference generated four sub-themes (big ideas and major issues) based on the presentations and discussion of all papers. These sub-themes were: (1) the context, needs and goals of collaboration; (2) the forms of collaboration; (3) the outcomes of collaboration; (4) the mathematics content of and/or for collaboration. Participants were then grouped in terms of the sub-themes and developed initial ideas about the major constructs of the sub-theme and a potential structure for organising the sub-theme.

To capture the key features of each sub-theme, each group identified several categorising parameters (key words). For example, for the sub-theme (1) the context, needs and goals for collaborations, the group proposed: needs expressed by teachers; needs expressed by teacher educators; needs expressed by researchers. These parameters were combined into one spreadsheet and shared with all participants who were asked to enter relevant information of their paper into the sheet. Thus, by the end of the Study conference, a spreadsheet was created which contained a short description of each paper along with a brief narrative about each sub-theme.

The second phase entailed the leading authors updating the spreadsheet. Some items were reorganised or combined, and some missing information was added. This sheet then provided a comprehensive and overarching picture of the major features of all papers in Theme B, which created a foundation for each sub-theme leader to develop further sections.

In *the third phase*, the leading authors in each sub-theme identified additional details through reading relevant papers and grouping papers into different categories

[2]Latin America refers to all Central and South American countries as well as Mexico. Although Mexico is geographically and politically part of North America, it is considered part of what it is named as Latin America, because it shares a similar cultural context to the other countries of Central and South America.

based on the framework established in each sub-theme. The reaction presentations and discussion notes were also consulted to ensure the accuracy of our understanding. The papers were compared to synthesise the major ideas of the sub-theme and arguments, which were supported using the original papers.

Within each phase, several dimensions of analysis were identified and chosen based on the relevance to each of the sub-themes. We used the sub-themes and these analyses to structure this chapter.

3.1.4 Layout of the Rest of the Chapter

The rest of this chapter is organised into five main sections. Following this introduction (Sect. 3.1), other sections focus on different sub-themes as follows.

Section 3.2 highlights the institutional or cultural contexts, the origins and the goals of collaborative work. Section 3.3 focuses on the different forms of collaboration. We identify forms as well as specific goals for the joint work, the stakeholders involved and the scale (e.g. number of stakeholders, time) of the collaboration. Section 3.4 presents different outcomes achieved according to the structure and context of the collaborative work. The outcomes of the collaboration are related to products developed, professional learning, knowledges, beliefs and practices. Unexpected results are also identified and dissemination of the results of the collaborative work is discussed. Section 3.5 discusses the role of mathematical content in the forms or results of collaborations that such content can promote, as well as the limitations or difficulties involved.

Although each section has its own major focus, they are interconnected due to the nature of collaborative work. Thus, the same paper may be analysed and cited in more than one section to illustrate purposely the relevant features. At the end of this chapter (Sect. 3.6), we provide answers to the guiding questions, identify further research questions and indicate directions that could be grounds for further studies.

3.2 The Multifaceted Nature of Collaborative Work: Contexts, Origins, Needs and Goals

3.2.1 Introduction

A globalised society offers a complex web of factors that bring people, cultures, beliefs and educational practices into greater proximity with one another. Institutions around the world, pressed by international comparative assessment leagues, are called upon to improve school education through various initiatives such as massive teacher professional development programs. On the other hand, the sense of isolation of teachers who find themselves pressed by constant shifts from ever-changing educational needs and requests, creates a push from the bottom for teacher professional development.

To understand teachers' collaborative learning in various teacher professional development programs, it is critical to understand the contexts, origins, and goals of the collaborations among teachers and between teachers and researchers. We recognise that all these collaborations are rooted and take place within different contexts (Lave, 1988). Cross-culturally, contexts are culture-bound although the changing and dynamic nature of cultures over time sometimes makes it difficult to recognise the boundaries between different cultures. According to the three-layered model of teacher professional development programs (Opfer & Pedder, 2011; Prediger et al., 2019), as described in Sect. 3.1, we analyse and organise the major points based on the three context levels: micro (classroom level), meso (school or institution level) and macro (school system or multiple-institution level).

3.2.2 Contexts and Origins of the Collaboration

Identifying the contexts and the origins of teacher collaboration in the Theme B papers is not easy. First, the different scales and levels of collaboration, such as *multiple-institution level* (schools, universities, communities) and *single institution level* are related to very different origins, and most of these origins are not explicitly reported in the papers. Second, the origins of teacher collaboration programs reported in research papers, sometimes may be different from those endorsed by the collaborating teachers. As noted in Sect. 3.2.1, we would like to focus on the different visions of collaboration and their relationship with the different cultural (philosophical, political, social, economic) contexts in which these visions developed. Indeed, different cultural contexts and visions generate different forms of collaboration which can be classified at three levels.

Some research studies report the origins of teacher collaboration at multiple-institution or national levels. In these cases, the teachers' collaborations are aimed to improve mathematical teaching practice. An example is the case of Lesson Study involving cycles of study, design, enact and reflect, that was introduced in Iranian schools by the Ministry of Education (Rafiepour, 2020). Another example is the research work of Heck et al. (2020), which represents teacher collaboration organised at a multiple-institution level.

This research was designed to address the lack of professional opportunities for secondary mathematics teachers to learn challenging mathematics in collaboration with colleagues in the United States. In this project, groups of teachers from different cities of the United States have been engaged in PD organised in distance learning mode, using both synchronous and asynchronous approaches. The goals of the PD were to provide participating teachers with experiences of being immersed in mathematical activities which are connected to their teaching, and to establish a blended teacher professional learning community (Heck et al., 2020).

In other research studies, the origins of the collaboration between teachers took place at the *institution level*. These programs focused on professional development courses organised by educators in and for universities or other institutions/providers of PD. Many of these studies were carried out in European countries like France and Spain. For example, Coppé and Roubin (2020) report a French experience of more than 15 years whose origin is linked to the working philosophy of the Institute of Research on Mathematics Education (IREM). In this long tradition, researchers work with teachers in a collaborative research group to develop and disseminate resources on specific mathematical content, like algebra or probability (see, for example, Coppé & Roubin, 2020; Masselin et al., 2020). Similarly, a Spanish experience is reported by Climent et al. (2020), in which reflection on practice (one's own and others) and reciprocal learning represent key features of the collaboration.

These experiences are connected to a community of inquiry (Jaworski & Huang, 2014) where teachers and teacher educators jointly work in the processes of developing mathematics teaching. Thus, these collaborative experiences between teachers and researchers can happen in cultural contexts that allow participants to overcome some stereotypes and beliefs regarding hierarchy in academia. For example, the assumption that researchers who work in the academy have superiority in comparison with teachers who work in school. On the contrary, these research works present an authentic and collaborative dynamic between researchers and teachers (Radford, 2019), that creates important connections between research and practice.

Another interesting study is about a particular experience of collaboration among teachers that took place in Nordic and Baltic countries (Hreinsdóttir, 2020). This project was born and developed through a series of conferences and several smaller meetings regarding the use of *GeoGebra* at school. A group of teachers regularly participated in these conferences and meetings over the past 10 years.

A few studies refer to the origins of teacher collaboration *at the classroom level*. Among these studies, the experience of co-teaching in New Zealand was reported by Eden (2020). The research work describes the joint activity of a group of teachers in a New Zealand primary (elementary) school as they collaboratively inquired into their practice. The aim of their collaboration focused on experiences of *co-teaching* was to improve mathematics learning for low-achieving students.

3.2.3 Needs and Concerns at the Origin of Collaborative Work

There are different needs and requirements that give life to the experiences of collaboration among teachers in different cultural contexts and countries. At the institution level, we can recognise different needs of researchers and teachers with regard to reflecting on and improving teaching practices, but also needs expressed by teachers or teacher educators in responding to changes and demands from the institution. For example, the researcher and teacher partnerships reported by

Coppé and Roubin (2020) and Climent et al. (2020) demonstrate how collaborative work between teachers and researchers can change teachers' beliefs and develop their mathematics knowledge for teaching (e.g. MKT, as defined by Ball et al., 2008, or Mathematics Teacher Specialized Knowledge—MTSK—by Carrillo-Yañez et al., 2018) and get insights into students' mathematical thinking as well. The need to develop teachers' MKT is presented in several studies (see, for example, Pacelli et al., 2020; Collura & Di Paola, 2020).

These studies emphasise the need to support mathematics teachers' development of MKT effectively to orchestrate students' discussions and eventually to promote their mathematics learning. Furthermore, these studies focusing on collaboration between researchers and teachers highlight how this type of experience is useful for researchers to test their research hypotheses, to develop teaching resources for teachers and to explore effective ways of collaborating with teachers.

In contrast, there are cultural contexts in which, for historical and socio-political reasons, collaboration among teachers is made mandatory through teacher professional development programs. This is the case for the practice of teacher professional development through Lesson Study (LS) in Asia, which in recent years has become a model adapted in many countries (Huang et al., 2019). In Japan and China, LS is a job-embedded and system-wide practice with a long history. Particularly in China, the development of LS is the key component of the Teaching Research Group system, which has been promoted throughout China as a powerful form of school-based collaborative professional development for implementing curriculum reform (Li & Huang, 2013).

The growing popularity of LS around the world engenders research needs such as those to establish the potential and limits for adapting LS in other cultural contexts (see, for instance, Rafiepour's (2020) study in the Iranian context); to understand the characteristics of LS, for example by experimenting with LS within different implementation phases (Richit & Tomkelski, 2020); to investigate the role of LS in establishing collaborative and reflective professional learning.

At the classroom level, the need to improve mathematics learning for students identified as at risk of underachievement is the starting point for the experience of co-teaching in New Zealand reported by Eden (2020). Even if there are many self-managed experiences of teachers' collaboration and training around the world (see also Hargreaves & O'Connor, 2018, where the case of a self-managed collaborative group of teachers from one institution is analysed), there are few studies at this level. This is in part because there is restricted access to some institutions, and it can be difficult to access cases.

3.2.4 Main Goals of the Teacher Collaboration

The goals of teacher collaboration are related to the context of the collaboration and the needs and concerns of the people who design and initiate the collaboration. Beyond the two broad categories (innovation and practices) by Robutti et al. (2016), we identified the following goals of collaborative work: (1) development of mathematics teachers' knowledge needed for teaching (e.g. MKT as defined by Ball et al. (2008), designing lessons, and noticing student thinking) and disposition toward mathematics teaching and learning; (2) development of interdisciplinary knowledge needed for STEM education; (3) improvement of teaching practice that promotes student learning; (4) development of teachers' resources (curricula, tasks, lesson plans, and so on); (5) development of teacher professional learning community. Although a collaborative program was often designed to achieve multiple goals, the intended goals of projects reported are organised in terms of the three context levels.

A few of projects reported are multiple-institution level (or national level) which targeted changes of teachers' knowledge and their teaching practice (e.g. Asgari et al., 2020; Canavarro & Serrazina, 2020; Ekici et al., 2020; Heck et al., 2020). For example, Canavarro and Serrazina (2020) reported a national teacher professional development program, which was grounded in teachers' classrooms with a focus on reflecting on students' mathematical production. In their project, Heck et al. (2020) examined how to develop practicing teachers' MKT and their teaching practice through collaboratively working on mathematics problems and sharing their teaching of the problems through a virtual environment. Ekici et al. (2020) presented a STEM education project which was aimed at developing culturally responsive teaching using mathematical and scientific modelling practices and developing teachers' interdisciplinary knowledge as well.

Most of the projects reported are at institution level with various intended goals (e.g. Acevedo-Rincón, 2020; Coppé & Roubin, 2020; Jahn et al., 2020; Masselin et al., 2020; Soto et al., 2020). Some studies focused on specific goals: for example, Pacelli et al. (2020) examined how to develop teacher knowledge about interpreting student work through a course for a master's degree in mathematics education. Kooloos et al. (2020) examined how to develop teachers' noticing of students' thinking and increase linking of teacher actions to students' thinking through teachers' collaborative reflection on their own videos of classroom practice which was designed to "foster students to think, to articulate their thinking, and to discuss each other's ideas" (p. 372).

Yet other studies focused on broad or interdisciplinary goals. For example, Acevedo-Rincón (2020) investigated how a prospective teacher course could develop participants' interdisciplinary knowledge through bringing together trainees from different degree programs to understand and confront the reality of the classroom in a school setting. In Pinzón and Gómez's (2020) study, they reported a master's degree program through which the practicing teachers collaboratively worked together in studying mathematics topics, designing and implementing lessons for the topics, and reflecting on their implementation, with the support of researchers and mentors in a blended manner (online and on-site). The goals of the program were to develop practicing teachers' PCK and ability to design lessons by using curricular materials and predicting students' solutions and errors.

At the institution level, there are two big sub-groups of studies. One is about the partnership between researchers and teachers and the other is about LS. There is a long tradition of establishing researcher–teacher partnership in Europe (e.g. IREMs in France). Several studies examine how such traditional partnerships could: (1) develop mathematical tasks and teaching resources in algebra, and train teachers how to use these materials in their classrooms (Coppé & Roubin, 2020; Horoks et al., 2020); (2) develop good tasks in probability using simulation and train teachers to use these in their classes, and subsequently change teaching practice and teachers' specific knowledge for teaching (Masselin et al., 2020); (3) change teachers' teaching practice and develop their profession in general (Modeste & Yvain-Prébiski, 2020).

Climent et al. (2020) reported a longitudinal collaboration between a group of practicing teachers and university faculty in Spain, which aimed to develop teachers' collaborative and reciprocal learning community and reflection on their practice through adopting a problem-solving based approach. In contrast, some studies examined initiatives of researcher–teacher partnerships with different goals of collaboration. For example, Soto et al. (2020) reported a study on a community of inquiry, where researchers created problems for teachers to solve collaboratively, implement in their own classrooms and then come back to collectively reflect on their implementation. The intended goals were to "create a database of good transition problems that can be used by peers" (p. 424) and develop teachers' MKT and positive dispositions needed for helping their students to transition from "primary to secondary school and to help teachers and educators to reflect on professional tensions and practices" (p. 420).

Koichu et al. (2020) revealed the nature of teachers' voice in a teachers' and researchers' community of inquiry while Castro Superfine and Pitvorec (2020) examined how a teacher and researcher collaborative inquiry approach supported teachers' understanding and use of learning trajectory-based formative assessment. Some international researcher-teacher collaborative projects are reported. Hernández et al. (2020) focused on The Seminar on Re-thinking Mathematics (SRM) as an environment of collaboration, which offers resources to enhance different mathematics teaching approaches for teachers and researchers through the process of the dialogue between researchers and teachers. Hreinsdóttir (2020) reported an international collaboration using *GeoGebra* Network to facilitate collaboration and sharing, among teachers and researchers, on the use of technology in the teaching and learning of mathematics.

Several studies examined how an Asian PD tradition of LS could be adopted in other cultures. For example, Richit and Tomkelski (2020) focused on developing professional collaboration through different phases of LS, facilitated by a teacher educator. Rafiepour (2020) examined the affordances and constraints of adapting lesson LS in Iran. Trevisan and Elias (2020) explored how LS could contribute to establishing a collaborative and reflective professional learning community and identified many challenges.

Very few papers focused on the classroom level (e.g. Collura & Di Paola, 2020; Ell, 2020). Collura and Di Paola reported how high-school teachers from different subjects in an Italian school co-design and co-teach mathematics lessons to develop their interdisciplinary knowledge linking scientific and humanistic knowledge. Research by Ell explored how schools could make substantial changes to organisational, leadership and teacher practices, and to student learning in mathematics, through utilising a Spiral of Inquiry approach to teaching and learning in mathematics.

3.2.5 Summary and Comments

This section of the chapter presents an analysis of contexts, origins, needs and goals of teacher collaboration projects. A three-tiered model comprising a multiple-institution level (e.g. national and/or multiple sites, facilitated by multiple facilitators), an institution level (e.g. organised by a single PD provider) and a classroom level (e.g. within a school) is used to frame the discussion. The national or multiple-site teacher collaboration programs are typically developed to address national needs and concerns, such as curriculum reforms and innovative initiatives.

At the institution level, the major concerns are about developing teachers' MKT or interdisciplinary knowledge, changing their teaching practice, or building partnerships between teachers and researchers to promote teachers' growth, and linking research to classroom practice. At the classroom level, the teachers' collaboration mainly focuses on improving specific teaching strategies which promote innovative ideas and address practical issues. In general, teacher collaborations have multi-dimensional goals. These goals focus on some of the following aspects: developing interdisciplinary knowledge or MKT and dispositions; developing curriculum and teaching materials; improving teaching practice; strengthening professional learning communities.

However, it is hard to identify the origins of teacher collaboration projects due to a lack of explicit explanation in research reports and/or the disparity in explanations from teacher professional development providers and participating teachers. Moreover, there are few projects focusing on teacher collaboration at the classroom or school level. These findings suggest that more attention needs to be paid to teachers' collaboration within their own schools and classrooms, which directly impact on teachers' teaching practice and eventually impact on student learning outcomes. In addition, when projects are reported, the context and origin should be presented explicitly.

3.3 Exploring Forms, Settings and Conditions Related to Teachers' Collaboration

3.3.1 Introduction

In this section, we analyse and report on settings that are implemented to support participants' collaborative work, learning and professional practices. For analytical purposes, we define a setting for collaboration as consisting of four main components: a form of collaboration; the participants; the topic they will focus on; the scale of the collaboration. It is emphasised that, as underlined in Sect. 3.1, the setting and its components are contextualised in the cultural environments in which they are embedded (Lave, 1988). The implementations of a collaborative setting are the result of a process that involves three interconnected phases: setting goals, planning and execution (in the sense of Loucks-Horsley et al., 2010). Once the goals are agreed upon, as reported in Sect. 3.2, a set of decisions are made during the planning phase for achieving those goals within the context that frames them. For instance, the forms of the implemented settings reported in this section are the result of this process which is itself complex.

Studies on the collaborative work of teachers have highlighted the diversity of forms of collaboration or professional work, and settings implemented. The forms identified are grouped into broad categories such as professional learning communities, communities of inquiry, LS, open classes, and collaborative planning networks (Borko & Potari, 2020; Hargreaves & O'Connor, 2018; Robutti et al., 2016). Rodrigues et al. (2016) analysed various implementations of a form of collaborative work known as "collaborative groups". Considering the case of the Brazilian groups, they point out that, although they share certain characteristics, it is also possible to identify variations among them. Similarities and differences among the forms of collaboration, as pointed out by Rodrigues et al., are facets of the forms of joint work that could make it difficult to categorise them.

Regardless of the forms of collaboration, the main participants and stakeholders of the joint work are teachers and researchers (Jaworski & Huang, 2014; Krainer, 2014); however, other participants may be involved as the collaboration unfolds. Participants and their relationships characterise their communities (Fiorentini, 2013). Some studies discuss the scale of collaboration (size and time). Regarding size, several studies (e.g. Krainer, 2014; Krainer & Spreitzer, 2020; Robutti et al., 2016) emphasise the value of small-scale collaboration; however, the need to promote larger size interventions and studies is also noted. The duration of the collaboration is an aspect that has been discussed mainly from the perspective of the sustainability of collaboration and its effects on long-term outcome analysis (e.g. Hargreaves & Fink, 2003). Regarding the topic on which the collaborations are focused, as discussed in Sect. 3.1, Robutti et al. (2016) provide an extensive discussion. Prediger (2020) offers a valuable insight by focusing on the classroom-level tetrahedron as the content of collaboration, revealing the complexity of such content.

In accordance with what has been previously presented, we report next our analysis of the settings related to the papers presented for Theme B and discussed by the participants in the Lisbon Study conference. The themes[3] for this section, selected by drawing insights from ICMI Study 25 Conference, are: (1) the settings, forms and contexts of collaboration (Sect. 3.3.2); (2) the foci, scale, and participants

[3]During the work in Lisbon, Ana Canavarro, Núria Climent, Alf Coles, Sylvie Coppé, Matthew McLeod, Adriana Richit and Gabriel Soto contributed to selecting the themes considered in this section.

of collaborative settings (Sect. 3.3.3); (3) conditions for promoting or inhibiting collaboration (Sect. 3.3.4). In Sect. 3.3.5, we discuss the limitations of the proposed categories of collaborative forms and synthesise them in terms of models, revealing their essential aspects. Finally, we draw conclusions related to the themes of this section.

3.3.2 Settings, Forms and Contexts of Collaboration

In the Theme B studies, it is possible to recognise diverse forms framing collaborative works. Despite the diversity, some similar aspects can still be recognised. In all cases, communities are established to sustain the joint work. In most of the works, the communities were developed with a declared objective of collaboration, while in others, communities emerge within the framework of a professional development program. In all cases, the participants are mainly teachers and researchers.

In several of the studies, researchers play the role of main facilitator for collaboration. Differences emerge when considering details about the contexts that frame the joint work and consequently differences in the ways in which collaboration is designed and organised are revealed. One aspect that differentiates the papers is the way in which interactions between participants are mediated. In that sense, two main categories are distinguished: face-to-face and blended settings. 25 out of 28 studies described a face-to-face setting while three involved a blended setting.

3.3.2.1 Face-to-Face Settings and Forms

Among the studies mediated by a face-to-face modality, we were able to distinguish three categories of collaborative form identified by the authors themselves: *JLS adapted* and implemented outside Japan; *researchers–teachers partnership*; *workshops.* Another group of studies did not identify their joint work with a particular form, but did so from a broader perspective, referring to the type of community or communities involved. Among the studies that were identified only in terms of the type of community, aspects that especially distinguished them were considered. For example, the type of community and/or number of communities involved the impact of goals on the form and environment of collaboration, and the impact of the context in which the joint work is framed.

Based on this analysis, two categories are proposed: *networks* and *forms related to specific purposes*. Finally, we observed that although several papers used workshops, in only one study (large-scale), the collaboration was entirely mediated by a workshop form. Given that this study involved a network of institutions, it was included in the network category. Thus, four categories of forms are considered: adaptation of *JLS*, *researchers–teachers partnership*, *networks* and *forms related to specific purposes*. We acknowledge that the proposed categories may not be totally discrete which could be recognised as a limitation of our work.

Adaptation of LS As highlighted in Sect. 3.2, four studies report work on LS adaptations. For instance, Richit and Tomkelski (2020) and Rafiepour (2020) focused on JLS adaptations. Richit and Tomkelski report on a study about the development of a professional collaboration which involved six Brazilian teachers from a public primary school and concerned the measurement of length at a fourth-grade level (students 8 or 9 years old). Participating teachers focused on the planning of a research lesson, its enactment and the reflection of the observed classes. Specifically, the paper focuses essentially on the process of the planning of the research lesson.

In that process, all teachers first analysed the contexts of the students and their families, in order to plan the initial activity of the research lesson. They focused on a study of the measurement strategies and instruments used by their families. Then, they conducted a detailed study of the school's interior and exterior space, in order to plan an activity for which the students would use units of length measurement. Throughout the research lesson work process, the group discussed and reflected on the progress made. All the information collected plus the interactions and reflections made within the group were fundamental inputs for the design of the lesson.

Richit and Tomkelski reported that the gradual growth in the level of confidence within the group provided a framework of security for the teacher who finally implemented the collectively designed lesson. In a similar way, Rafiepour (2020) reports work carried out developing a newly modified research lesson on mathematical trigonometric ratios. In this case, before or during engaging in the classical LS cycle, the researcher-facilitator invited the teachers to start reflecting on their trigonometry teaching experiences. After this first step, the refined lesson was taught and observed, followed by a reflection on the whole process.

In both studies, collective work is developed to collect and systematise information about local contexts and teachers' knowledge and experiences. This occurs both at the beginning of the LS process and in moments of lesson design, lesson redesign or reflection (Lewis et al., 2009). By comparing these studies with relevant aspects of the JLS, we observe that both cases are practice-embedded collaborations in which a group of teachers come together to study ways of teaching particular mathematical content (Shimizu, 2014). However, the teachers who are involved in these cases do not have previous experience in working with LS, as is the case with Japanese teachers (Lewis et al., 2009).

Both include cycles of collaborative activities such as lesson planning and delivering planned lessons along with team observation, post-lesson debriefing, and reflection. In both cases, however, collective activity is added to contextualise the content to be taught and the modes of support offered by the mathematics educators. The role of knowledgeable others (Huang & Shimizu, 2016; Shimizu, 2014) who adapt LS outside Japan becomes especially relevant in the whole process of the joint work. For both studies, the adaptations showed affordance in their local contexts as literature indicated (Willems & Van den Bossche, 2019; Cheung & Wong, 2014; Lewis et al., 2009).

Researchers–Teachers Partnership Several of the studies presented in Theme B reported collaborations designated as researchers-teachers partnership. In some of them the group was referred to as a community made up of researchers and teachers. In other cases, the authors referred to the group in terms of some type of community (e.g. community of practice, community of inquiry). Some examples of this form of work are the studies by Castro Superfine and Pitvorec (2020), Climent et al. (2020), Soto et al. (2020), Coppé and Roubin (2020), and Horoks et al. (2020).

For instance, Soto et al. worked with Argentinean teachers with the support of the provincial Ministry of Education and the university where the researchers work. In this case, the continuity of the collaboration depends on constant changes in the policies of the Ministry of Education or the public universities from Argentina. We refer to these examples of collaboration, as well as to the cases of Castro Superfine and Pitvorec and of Climents et al., as *project-based forms,* since the continuity or progress of the collaboration depends on financial or academic support for the project or group fostering the collaboration.

As noted in Sect. 3.2, the study of Coppé and Roubin (2020), and the one by Horoks et al. (2020) are both developed in France. While Coppé and Roubin developed their study in the framework of the IREM tradition, Horoks et al. report on a study developed at a LéA (LéA stands for *lieux d'éducation associés*—associated places of education), which were created in 2010 by the initiative of the French Institute for Education as places for promoting the development of collaborative projects among teachers and researchers interested in relevant issues.

Horoks and colleagues have worked in the LéA RMG,[4] a Parisian school with students aged 12–15, focused on the teaching of algebra, since 2014. During the collaborative work, questions that concern the teachers are collectively identified and transformed into research issues for the researcher and for the whole group. By considering the daily school practices of teachers, researchers do not offer methods to be applied, but rather offer teaching alternatives that could be used to enrich practices, to support collective work or to share and analyse the classes. For the design of resources, an iterative design process is adopted that includes the testing of resources that are appropriate to the work context.

All participants are involved and take responsibility for various tasks. For example, management tasks are divided between the two LéA leaders (a teacher and a researcher). Everyone collects and analyses data that is then shared, for example for use in the resource development cycle. This form of collaborative work is sustained by the co-construction of a common issue, the consideration of teachers' practices and the context that frames them, the design and implementation of resources, and the sharing of tasks and responsibilities.

A characteristic of the work under the IREM tradition or the collaborative culture (Hargreaves & O'Connor, 2018) of the LéA is that they are settings that, with some

[4]RMG is the acronym of the name of the secondary school where the project is being developed. The authors inform that that school is considered as rather disadvantaged and the students are usually assessed as low achieved.

variations, can go beyond the implementation of the named studies. We could point out that they are institutionalised collaborative works, as they are accessible to teachers and researchers and have resources of time, human resources and some money available (Hargreaves & Fink, 2003). The institutionalised character of the latter two studies offers means for the affordances of collaboration and evidences a difference with project-related forms. However, it should be noted that the latter form also offers means of rapid adaptability to diverse social or educational contexts requiring fewer human or material resources. Institutionalised *researchers–teachers partnership* can even have an impact outside the collaborative group as is the case with the work of the IREMs which, for example, has an impact on research communities inside and outside France.

In relation to the former form of collaboration, we are interested in highlighting the work of Coburn et al. (2013), Coburn and Penuel (2016), and Penuel and Hill (2019). We note that the forms of collaboration characterised as researchers-teachers partnership in the studies in Theme B and discussed in the three papers above, reveal some salient commonalities about collaboration. What is presented in these three papers focuses on what the authors refer to as research-practice partnerships. For example, Penuel and Hill note that, "Research-practice partnerships are collaborative research arrangements that seek to transform relationships between researchers, educators and communities" (p. 1).

Despite the differences in the names of the two forms of collaboration, they are consistent in the sense that in both, teachers and researchers work together towards the realisation of a common goal linked to the three levels of teaching work (Esteley, 2014). In the group of works cited at the beginning of this paragraph, it is emphasised that common goals are agreed and worked towards, the work goes beyond a single project and open engagement of the partners with each other is identified. Also, Penuel and Hill discuss issues related to the time and resources required to sustain partnerships. In the researchers–teachers partnership form, the name is chosen because the participants at the ICMI 25 conference themselves identified themselves by that way. Moreover, in some of the cases analysed, this name would emphasise that the research can become a product of the partnership itself. This last observation could be connected to the modifications or changes of projects in the case of the form research-practice partnerships.

Network The network form involves collaboration among groups, communities or institutions and is compatible with a kind of collaboration identified as a *network model* by Hargreaves and O'Connor (2018).[5] These networks may include joint work between communities of researchers from one or several institutions with teachers from several schools and different levels of education. In this case, collaborations can be established not only between communities but also within each community. This aspect of collaboration may imply different content and modes of communication within each community and between the communities involved. In

[5]It should be noted that the authors use the term 'model' in the sense of model or theoretical structure.

some of the reported studies that take this form, networked groups are referred to as communities of practice (Wenger, 1998) or communities of inquiry (Jaworski, 2006, 2008). Other studies, on the other hand, only highlight institutional networks.

In the studies reported, communities are made up of in-service teachers, pre-service teachers, and researchers, and it is possible to identify collaborative interactions inside each community or between communities. The interactions occur in different instances of work, reflection, study, or joint production. Acevedo-Rincón (2020), Koichu et al. (2020), Dörr and Neves (2020) and Canavarro and Serrazina (2020) are examples of collaborative networks. While Acevedo-Rincón shows in detail an example organised in Brazil, Koichu et al. (2020) report a collaboration in Israel.

Both papers report on a complex network of communities. For instance, Koichu et al. report on a collaboration in Israel involving two main communities of inquiry (Jaworski, 2006). One community consists of secondary school teachers and the other of researchers. The communities are organised in the TRIAL (Teacher–Researcher Alliance to Investigate Learning) theoretical–organisational framework. The teachers' community is divided and reorganised into communities or TRIAL teams according to different Israeli regions. The work of each of these teaching communities is done collectively with two researchers. Within each team, "teachers and researchers study pedagogical questions of importance and mutual interest by going through the stages of a research cycle as partners" (p. 369). There were regular work meetings of the teams and a conference that brought together the two main communities.

The work of Dörr and Neves (2020) and Canavarro and Serrazina (2020) are examples of institutional networks. The first is carried out in Brazil through a series of workshops, while the second is developed in Portugal (the latter case will be presented in detail both in Sects. 3.3.3 and in Sect. 3.4).The work reported by Dörr and Neves involves researchers (from a Brazilian university), in-service and pre-service teachers, and students from several public schools that gather and work to "promote school students' development of their mathematical learning processes and contribute to the initial and continuing training of mathematics teachers in the Brazilian Federal District" (p. 294).

A significant number of the interactions and collaborative work activities between researchers, teachers, pre-service teachers and/or students are carried out in a workshop format. The volume of material created in or for these workshops has grown significantly since the start of the collaborations and as the collaborative network has expanded. The activities developed in the framework of the workshops not only offer mathematical experiences, but also bring the possibility of co-planning and co-acting on a shared goal. The authors point out the affordances of this form of collaborative work, and two main limitations. One limitation is linked to the possibility of scientifically validating some of the activities due to the difficulty of formalising the project as an inter-institutional research project. Other limitations concern the dissemination of the activities outside the schools where they took place due to the amount of material produced.

In line with the Dörr and Neves' study, Goodchild et al. (2013) report on collaborative work, carried out in Norway, designing and implementing a series of workshops. This involved extensive fieldwork which was recorded in an extensive database. Lachance and Confrey (2013) report work with teachers from a high school in the United States by implementing workshops. One of the major results reported by these authors is that the workshop provided the catalyst for developing a professional community among the teachers in the school and for their joint work. It seems that workshop-mediated collaborations offer a flexible means of collaborating and have the potential to promote professional community.

The ICMI studies mentioned here show the inter-institutional, organisational and human resource supports that are required for collaboration in the form of a network. These aspects are also highlighted by Hargreaves and O'Connor (2018). Such resources are the supports that give affordances to this form. Perhaps a lack of support for a network could be a limitation to its implementation.

Forms Connected to Specific Purposes We use the term *forms* connected to specific purposes to mean that the collaborative work is organised for achieving a specific goal. Usually, the collaboration is for a specific period and, in some cases, the work may not continue, at least for a similar purpose.

In several of the cases, participants are organised into communities or groups according to the activities that are designed. We highlight groups of an interdisciplinary nature (e.g. Ekici et al., 2020), groups that work in a co-teaching or co-planning design (Eden, 2020; Collura & Di Paola, 2020), groups involving teachers, researchers and curriculum development specialists (Asgari et al., 2020), and groups developed within courses for master's students or with pre-service teachers which focused on issues important to student teachers' future teaching practice (e.g. Pacelli et al., 2020).

This form of collaboration was especially evident in studies that reported on collaborative groups between researchers and teachers in the framework of Ph.D. research (e.g. Kooloos et al., 2020; Masselin et al., 2020) or collaborative works developed by governmental agencies (e.g. Asgari et al., 2020).

For instance, Asgari et al. report that the Iranian Office of Planning and Compilation of School Textbooks (OPCST) produces textbooks and sends them to schools for teachers to use. However, teachers often do not adopt the textbooks produced by the OPCST. In order to find a way to address this issue, Asgari et al. conducted joint work with a heterogeneous group of 21 mathematics teachers (from different regions of Iran) together with experts in mathematics curriculum development. Based on an algebra book produced by the OPCS, the entire collaborative group initiated a cyclical process of analysis–critique production that culminated in the development of an algebra course package for seventh grade. The full cycle and the joint work ended when the collaboratively produced material was sent to the OPCST for further distribution of that material to the Iranian schools.

Focusing on purpose and motivations can, in some cases, promote small and homogeneous collaborative communities. In other cases, there is an emergence of heterogeneous communities that may require time to agree on ways to communicate ideas or work practices. These and other aspects bring both challenges and richness to collaboration. Details of these forms of collaborative work are provided in Sects. 3.3.3 and 3.3.4.

3.3.2.2 Blended Forms

Only three papers report on blended collaborations. The paper by Hernández et al. (2020) analyses the Seminar on Re-thinking Mathematics (SRM), which brings together teachers and researchers mainly from Latin America (including Spanish or Portuguese speakers) to discuss ways to relate theory and practice. The SRM provides interaction between teachers and researchers via video conferences, online forum discussions and e-mails. Prior to the video conference, the participants read one or more assigned research articles. Once the video conference is over, asynchronous and diachronic interactions are initiated via forum discussions, face-to-face[6] and/or virtual.

Hreinsdóttir (2020) reports on a collaboration developed by the group "The Nordic and Baltic *GeoGebra* Network". This group was founded in 2010 by seven teachers from seven different Nordic or Baltic countries. They privileged online interactions and met face-to-face regularly through conferences. Heck et al. (2020) present a study on the implementation of a PD program, based on the blended 'Mathematics Immersion for Secondary School Teachers' (MIST) model. The program aims for teachers, constituted in communities, to collaborate in activities of mathematical production and pedagogical reflection. They work in sites with four to seven teachers who work face-to-face gathered at the same physical location to promote collaborative learning.

Three or four of these sites work with a facilitator connected through videoconferencing to enhance the learning. One participant at each site is a table leader whose role is to be the eyes and ears for the facilitator who was not in the same room as they were. Table leaders post pictures and periodic verbal updates of participants' work to a shared space for the facilitator. The facilitator's role is to share examples of the participants' work and lead the discussions between the sites. Groups use a collaborative application for asynchronous discussions and to facilitate interactions between table leaders and facilitators.

The three studies exemplified the collaborative work in blended settings. The great advantage of blended collaborations is the scaling up of collaboration that transcends even geographic spaces. A difficulty in the first two cases may stem from the different languages of the participants. In all three cases, time differences between regions can be a problem, as well as the teachers' access to the technology necessary to participate.

[6]Face-to-face meetings take place in each university engaged in the collaboration and involve researchers and/or teachers. For example, different face-to-face meetings take place in Mexico, Colombia, Argentina, etc.

It should be noted that, as discussed in Borba and Llinares (2012) and Borba et al. (2016), the development and study of collaborative activities in a blended format is an issue that is in its infancy. That is reflected in the scarcity of papers presented in Theme B reporting on blended collaborations. Borba and Llinares point out that while the Internet can facilitate the creation of communities, it can also invite members of a community to join and change communities at a rate that could hinder professional development. Recently studies on teacher collaborative work in blended form discussed the strengths, challenges, and further directions (e.g. Chan et al., 2021; Huang et al., 2021).

3.3.3 Collaborative Settings: Focus, Scale and Participants

In the next section, we describe and analyse the main focus and the scale (time and size) of teachers' collaborative work. We also explore possible relationships among them and their influences in the collaborative settings or the related communities.

3.3.3.1 Foci and Goals Beyond Innovation

In some of the studies, as discussed in Robutti et al. (2016), collaborative work focused on the type of innovation including those related to mathematical content, student thinking, task design, curriculum, formative assessment or innovations linked to practices designed to foster pre- or in-service teachers' professional learning. However, other foci also became evident. These were linked to the teaching or learning of certain content and practices to sustain the development of professional learning, but as an integral part of the collaboration.

In such cases, the work focused on developing and sustaining a collaborative community (e.g. Soto et al., 2020) or network (e.g. Hreinsdóttir, 2020), on mathematical discourse (Kooloos et al., 2020) and noticing process (Eden, 2020). The above foci are connected to the goals delimited according to the contexts, origins and needs for collaboration, or the different levels considered, as outlined and discussed in Sect. 3.2 of this chapter.

3.3.3.2 Scale (Time and Size)

Timescale is an important factor in collaborative efforts, and it could be related to different forms and origins of collaboration. Two main categories are identified here, long-term collaborations which last five or more years and those that last less than 5 years. Long-term collaboration can be associated with the institutionalisation and sustainability (Hargreaves & Fink, 2003) of the collaborative work. Collaborations that last less than 5 years may also be institutionalised and provide relevant information for researchers and agencies interested in the collaborative work. In Sect. 3.3.2, some long-term works are identified in both face-to-face and blended formats.

Researchers–teachers partnerships such as IREM/LéA are examples of long-term collaborations that have an impact from the micro level of the classroom to the macro level of the educational system. (e.g. Modeste & Yvain-Prébiski, 2020). Other examples of long-term collaboration are the case of Asian LS or the GDS (Saturday Group), developed at the State University of Campinas (Brazil). The GDS, is constituted by a network of researchers and teachers (Gonçalves et al., 2014; Rodrigues et al., 2016).

One issue related to long-time collaboration that needs to be addressed is the turnover rate of participants, which could be high among teachers and potentially has a strong impact on the collaboration. One potential impact is connected to changes in the focus and outcomes of such collaborative work because new participants can bring new interests without necessarily changing the main goal of the collaboration set by the community.

Collaborations of less than 5 years can include those developed in the form of project-based researchers–teachers partnership, and those associated with specific purposes connected to a master's or Ph.D. thesis (e.g. Jacques & Clark-Wilson, 2020). We also found collaborations connected to Master's or Ph.D. theses connected to collaborative networks (e.g. Acevedo-Rincón, 2020). Although not exclusive to research associated with doctorates, in such cases teachers are particularly highlighted as stakeholders for research and in this context researchers as stakeholders for teachers' professional learning (Krainer, 2014).

It is worth noting that, in either timescale, the collaborative work was sustainable within the framework of the objectives and contexts in which it was developed (this point is discussed in more detail in Sects. 3.4 and 3.5). For both long- and short-term timescales, there are instances of dissemination of the progress of the work and theoretical contributions either through the groups' own publications (e.g. GDS publications) or through thesis dissertations.

Regarding the *size* of collaborations, according to the three levels proposed by Opfer and Pedder (2011) which interact with teachers' learning and work, we identify three categories: *classroom*, *school* and *educational system.* The *classroom size* corresponds to collaborations involving in-service teachers working together with facilitators in Master's or Ph.D. theses (e.g. Jacques & Clark-Wilson, 2020). The *school size* corresponds to collaborations involving more than one teacher from the same school or teachers from a particular school in the same educational level (e.g. Collura & Di Paola, 2020; Ell, 2020; Kooloos et al., 2020; Richit & Tomkelski, 2020).

At the *educational system size* collaborations involve more than one school, teachers from different educational levels and government or other agencies. At this level, we identify collaborations occurring within one or a few different cities or regions (Climent et al., 2020), at a national level (Heck et al., 2020) and at an international level (Hernández et al., 2020).

When we consider time and size together, we highlight the study of Canavarro and Serrazina (2020). This is a large-scale (time and size) collaborative work with a nationwide initiative developed in Portugal, in all 18 inland districts of the country. The program was developed from 2005–2006 to 2010–2011. It involved more than 14,000 primary school teachers (grades 1–4) and 18 public higher education institutions, each one responsible for primary teacher education in one of the Portuguese districts. The collaboration was developed with the purpose of supporting teachers' work with a new mathematics curriculum (more details on this case are given in Sect. 3.4).

Outside of the studies presented for ICMI 25, the case of the collaborative network known as Escuelas Nuevas[7] (New Schools) stands out as an example of large-scale (time and size) collaboration. This network started in the seventies in Colombia and takes as a reference a transformative pedagogy for small, multi-grade rural schools. Currently it has been extended to multiple Colombian rural and urban schools. Adaptations have been made for the urban contexts and for other countries (Hargreaves & O'Connor, 2018).

3.3.3.3 Participants and Communities

The size of the collaboration has an important link to the participants in the collaboration, both in number and in the communities to which they belong. In this regard, examples of participants for the first four forms are given in Sect. 3.3.2. Other examples can be considered with reference to collaborations that are *connected to specific purposes*. In Ekici et al. (2020), the collaboration was based on implementing a culturally responsive pedagogy (Gay, 2010). They worked with an interdisciplinary community which focused on STEM and mathematical modelling approaches for teaching.

Another example of interdisciplinary collaboration is the study reported by Collura and Di Paola (2020) developed in an Italian scientific upper secondary school. In both cases, the communities that support the collaboration are considered as interdisciplinary communities. They are so-called, not only because the teachers and researcher-facilitators involved come from different areas of knowledge (not only mathematics), but also because they collaborate to plan a lesson or provide answers to local problems of an interdisciplinary nature.

Fiorentini (2013), in considering collaborations between researchers and teachers in the framework of GDS, proposes the idea of borderline communities. For the author, such communities are those that are on "the border between school and university and usually have more freedom of action and ability to define their own agenda of work and study, since they are not institutionally supervised by the school or university" (p. 157). Even group meetings can occur in spaces and times outside the schools or the university. Of course, for the author, the GDS community is a

[7] https://escuelanueva.org/

borderline community. Similarly, when considering two cases of a blended form of collaboration (Hreinsdóttir, 2020; Hernández et al., 2020), the communities related with those studies can also be considered borderline communities.

3.3.4 Collaborative Settings and Arenas: A Brief Reflection on Conditions That May Inhibit or Foster Collaboration

As noted in Sect. 3.1, settings refer to what is created by the subjects who develop their activities in interaction with the arena that frames the settings, and the proposed activities. In this sense, activities and experiences are dialectically constituted with the settings (Lave, 1988).

Attendance in collaborative groups implies voluntary participation of the group members. However, in some collaborative settings, and their corresponding social or educational arenas, participation could be required. Although this may be considered a contradiction in terms of a joint work, it is feasible to distinguish settings that can enable the start of a genuine collaborative work while others may inhibit it. However, regarding both required and voluntary participation, the role played by the type and role of the participants in the setting is highlighted as a relevant aspect that can inhibit or favour collaboration.

The required participation of in-service teachers in collaborative initiatives usually originates from government agencies, teachers' unions, non-governmental organisations or a school's requirements. In the studies reported by Trevisan and Elias (2020); Soto et al. (2020) and Canavarro and Serrazina (2020), we identified two possible conditions arising from required participation in collaborative initiatives that could inhibit or promote authentic collaboration. One of these conditions has to do with the way in which the form and focus of the collaboration is agreed upon. Where the form and focus of the collaboration is not sufficiently communicated by the researcher-facilitator, discussed or reworked with the teachers, the collaboration may be compromised (e.g. Trevisan & Elias, 2020—this case will be presented in detail in Sect. 3.4).

However, when the form and focus of the collaboration are thoroughly discussed within the group, collaboration is promoted, as reported by Soto et al. (2020). Another condition that might inhibit or foster collaboration arises from the participants involved and the professional relationship they establish with each other. For instance, Canavarro and Serrazina (2020), due to recognising the importance of this condition as a possible obstacle, highlight certain actions to foster good relations. Among these is the value of promoting trust and complicity between facilitators and teachers, mainly when facilitators enter teachers' classrooms.

Required collaboration in undergraduate or graduate courses designed for teacher training has potential for fostering initiatives for future collaborations when facing the transition from student teacher to teacher (Gueudet et al., 2016). These opportunities for early collaboration offer prospective teachers a learning context for promoting what Geijsel and Meijers (2005) call professional identity learning, which is considered as a process for "meaning-giving and sense-making essential to bringing fundamental educational change" (p. 420). The process of identity learning could occur only when social construction and individual sense-making become closely related to each other (Anderson et al., 2018).

Pacelli et al. (2020) offer an example of promoting identity learning through an individual−collective−individual cyclical activity. In this case, prospective teachers were asked to solve a mathematical problem individually, and to compare their solution with secondary school students' solutions for a similar problem. Then, the prospective teachers engaged in a collective discussion focused on their individual solutions and their interpretations on secondary school students' solutions. These activities allowed them to work on a typical professional practice, in which collaboration becomes a powerful tool.

In cases in which participation is voluntary and the objectives are shared from the very beginning, it is possible to recognise aspects common to the different collaborative forms that promote collaboration. For example, the value teachers place on participants contributing to interpreting their practices in new ways is highlighted (e.g. Canavarro & Serrazina, 2020; Horoks et al., 2020; Koichu et al., 2020; Richit & Tomkelski; 2020). In relation to the above, the analysis of video recordings of classes is also valued (e.g. Kooloos et al., 2020). Another aspect that can promote collaboration is interaction between secondary and primary school teachers, with their different knowledge, even though, at the beginning, it may seem an obstacle (e.g. Soto et al., 2020). These aspects of teachers' work that promote collaboration align with the work of Robutti et al. (2016).

The above examples highlight the role of participants in promoting or inhibiting collaboration. Although issues related to the participants' role will be expanded in Chap. 4 (Theme C, this volume), what has been discussed seeks to highlight the relationship between the activities developed related to different forms and the arenas that support the application of such forms, and how this relationship can give rise to a truly collaborative work context (we refer to 'context' in the sense of Lave, 1988).

3.3.5 Summary and Comments

Attending to the notion of form and the characterisation of context (Lave, 1988), we have described and analysed illustrative examples offering fine-grained details of the professional learning settings and the social–cultural contexts framing the collaboration. The descriptions and analysis became a way for conveying information on the concepts or actions related to the context of joint work, in connection with the ideas of the people who interact with the activities they engage in (Chaiklin & Lave, 2003).

Considering Theme B studies, as well as other studies, we identified five categories of forms of joint work. We recognise as a limitation of our categorisation the fact that the categories are not mutually exclusive. Perhaps this fact could be connected to the double facets (differences and similarities) that seem to interweave the diversity of forms. We note that, in a broad way, Denzin and Lincoln (2018) point out that there are certain social phenomena, that at first may come to resist distinct categorisations. Despite the limitation mentioned above, being able to offer categories of form for designing joint work could be a starting point or step towards designing or studying forms of collegiality that comprise joint work and joint responsibility in the different stages of collaboration (Little, 1990). The essential characteristics of each empirical form are summarised below. In such a synthesis, we try to go beyond particular cases to consider the forms in a decontextualised way or as general models.

LS adaptations are a collaborative model characterised by being essentially focused on the micro and/or institutional level; the collaboration involves a community of researchers and teachers who create and apply original resources for the teaching of mathematical content. In all the joint work, the typical LS cycle is followed (Lewis et al., 2009; Fujii, 2014) and research processes are privileged to make evident relevant information from the local context that contributes to the design of the lessons.

The researchers–teachers partnership model is characterised by the development of joint work involving researchers and teachers in professional learning communities. Within each community, objectives are set, processes of inquiry and joint reflection are carried out, and guidelines are established to develop joint work. The community, as a collective agent, carries out activities such as the creation of resources for the teaching of mathematics and joint work to solve issues recognised as problematic in one or several institutions, bringing together teachers from the same or different levels.

The *network* model involves joint work between communities. For example, there may be interactions between a community of researchers and several communities of teachers or pre-service teachers. The communities can be defined according to a school district in the same country or several communities of teachers or researchers in a wide geographical region. The objectives and processes of joint work are varied, as are the communities of practice that compose them. The co-existence of theoretical research activities and scholarly inquiry stands out. Within this model, workshop-mediated collaborations, as well as blended forms, are included, and borderline communities are identified. In the case of blended forms, we highlight the important mediation of technologies for the joint work.

The *connected to specific purposes* model is characterised by the fact that the specificity of the purpose is taken as the focal point of the joint work. In this case, the collaborations may vary in the spectrum of scale (time and size) with a prevalence of short-term and small-size works. The levels of interest range from micro to macro. In these cases, researchers, teachers of the same or different levels or pre-service teachers are involved. Different motives for participation were associated with this model (see Sect. 3.3.4) and learning communities, inquiry communities and interdisciplinary communities were identified as examples.

In these decontextualised characterisations, similarities and differences can be observed that stem from the collaborative nature of the work. The latter comes from the decisions and agreements that are reached among those who promote or participate in the joint work. It is highlighted that the way in which the models are presented is in line with ideas expressed by Matos et al. (2009), who propose that a model of joint work encompasses: "A community of individuals, sharing cultural specificities [. . .], who have particular forms of engagement in the professional development sessions and whose topic of focus or domain pertains to specific aspects of their practice [. . .]" (p. 176).

We note that the models presented above can be refined and modified *a posteriori* in an iterative process as joint work studies progress. However, these models can be useful tools in the framework of research or collaborative work design. In any case, collaborative setting designs are contextualised to the working conditions of each group. In all cases, the settings experienced depend on who is involved, the purposes pursued, the scale of work and the levels (micro, meso, macro) on which the group is focused.

In all models, the settings implemented in the Theme B studies (or other collaborative experiences reported in this section) can account for shared work and responsibilities. Some collaborations included in the researchers–teachers partnership and network models are sustained over time, perhaps due to their institutionalised nature provided by the educational system itself. This aspect is shared by the JLS model. These models have the potential to be scaled up in relation to the number of participants. Models that focus mainly on the micro level, and include a small number of participants, could be used as a means of scaling-up collaborative work. However, in this case, it is important to note that, just as the context adaptations of forms/models are important, adaptations in their scale also deserve special attention.

In this section, the context as a relevant aspect for collaboration was evident in all cases. The collaborative context was manifested as the result of the interrelation between the settings and the arenas (or terrains) upon which the joint work is based (Lave, 1988). Finally, we are interested in highlighting that the experience of each participant in any joint work can signify a valuable point in their professional trajectories (Vezub, 2013). Perhaps this aspect can be made more evident in Sects. 3.4 and 3.5.

3.4 A Comprehensive Examination of Outcomes Related to Teachers' Collaboration

3.4.1 Introduction

In this section, we report and analyse outcomes achieved in collaborative processes in which teachers, mathematics educators and others participated as stakeholders in the whole process. We consider outcomes as the result of an activity or process, where an activity is defined as a situation in which things are happening, or people are engaged in actions in order to achieve particular, more or less well-defined, aims (Oxford Dictionaries online: www.oxfordlearnersdictionaries.com/definition).

Taking what was presented and discussed in the previous sections as a reference and, in accordance with the introduction to this chapter, it is pointed out that we do not suggest a cause-and-effect relationship, but rather a necessary interplay between the collaborative activities and the results achieved. Considering that most of the studies presented in Theme B have involved research approaches of a qualitative nature, we point out that, in general, qualitative studies highlight processes which lead to outcomes (Denzin & Lincoln, 2018).

It is in that sense that we will examine results, or outcomes, in this section. When we talk about results or outcomes, we are not only thinking about those that were successful in terms of the planned achievements, but we have also tried to recognise those results that were considered not positive, in order to think about what happened in those cases, What can we learn from them? It is important, before getting to the studies themselves, to offer some framing remarks about teacher learning, development, growth and change; we set out, below, how these terms relate to outcomes of collaboration and, then, how they can be studied.

One of the most influential models of professional growth comes from Clarke and Hollingsworth (2002). Following these authors, we take professional growth (which for us is also synonymous with development) as "an inevitable and continuing process of learning" (p. 947). Clarke and Hollingsworth's model offers an image of the change environment as four interconnected, non-linear domains: "the personal domain (teacher knowledge, beliefs and attitudes), the domain of practice (professional experimentation), the domain of consequence (salient outcomes), and the external domain (sources of information, stimulus or support)" (p. 950).

The interactions between these domains occur through the mediating processes of reflection and of enactment. The evident power of this model is indicated by its extensive use. In terms of studying the outcomes of collaboration, we have considered each domain. We follow this model in viewing change as what occurs in one domain, which then triggers change in other domains. Activity is taken to be the catalyst of change.

We now consider how the domains of the change environment can be studied. Changes in the personal domain have been researched via interviews, questionnaires and observations, for instance using attitude surveys or analyses of different forms of teacher knowledge. Coles (2018) notes the perhaps obvious, but still sometimes overlooked, point that questionnaires or surveys can only tell us what teachers *say* about their beliefs and attitudes, which may be different from the beliefs and attitudes enacted in the classroom. Changes in the domain of practice are frequently researched through the use of video and sometimes through self-reports. Changes in outcomes relating to instructional practice, in a review of work on teacher learning by Goldsmith et al. (2014), can be classified into three areas: "mathematical content of lessons, classroom discourse, and students' intellectual autonomy" (p. 13), which can be researched through recordings of lessons and subsequent analysis by researchers.

The Goldsmith et al.'s review also considered the external domain, for instance, curriculum change, and concluded that, as a field, we know little about the impact of curriculum reform on teacher learning. Indeed, that review confirmed a finding from Goldsmith et al. (2009) that we know little about, namely "*how* teachers develop knowledge, beliefs, or instructional practices" (2014, p. 21; *italic in original*). This conclusion relates strongly to our focus here on outcomes of collaboration. In other words, as we review what is said about outcomes, we have been concerned to also note where studies offer us some insight into *how* changes have occurred.

In an overview article on a special double issue on teacher change, Reid and Zack (2010a) identified two themes salient to how teachers learn. Firstly, they noted, "the importance of emotional engagement to the process of change" and secondly, "the vulnerability that occurs as part of change" (p. 372). These themes touch on the personal and external domains mentioned above. Reid and Zack (2010b) also raise the question of whether there is anything specific to *mathematics* teacher change. One potential parallel is between images of teaching mathematics as a process of inquiry, and teachers' investigations into their own practice as a process of inquiry. Reid and Zack (2010b) highlight how changing views of mathematics (as a discipline) can lead to changing approaches to teaching mathematics. In addition, they note the significance of time in the change process, e.g. how destructive it can be in complex change processes to have rigid restrictions on time.

With these preparatory comments in mind, we now turn to giving an account of the results that emerged from studies of collaborative work across 28 papers submitted for Theme B which were analysed by participants of the Study Conference in Lisbon. The themes identified,[8] which grew out of the ICMI conference discussion document, were: (1) *products developed* during the collaborative work as a result of such work, assuming that the subjects involved have opportunities to set their ideas, contributions and knowledge into play or together create new knowledge (Sect. 3.4.2); (2) *outcomes related to learning, professional development, knowledge, beliefs and practices* (Sect. 3.4.3); (3) *difficulties or problematic issues encountered at work* (Sect. 3.4.4); (4) *the sustainability, dissemination and sharing of outcomes* (Sect. 3.4.5). It is evident that these themes cut across Clarke and Hollingsworth's change environment domains.

3.4.2 Products Developed as Result of Collaboration and Their Levels of Application

Since the different outcomes or learning that result from the collaborative work can be related to the activities developed, the objectives set and the participants involved

[8]During the work in Lisbon, Jenny Acevedo-Rincón, Chris Kooloos, Raquel Carneiro Dörr, André Pinzón and Henrique Rizek Elias contributed for selecting some of the themes considered in this section.

(see Sects. 3.2 and 3.3), it is expected that the nature and levels of the outcomes will be related to these factors. Since the collaborative work we have in focus is centred on the practice of mathematics teaching, it is worth remembering that such practices are framed by the interplay between three contexts related to the work of teaching: the classroom, the school and the educational system itself (Opfer & Pedder, 2011).

To start a description and analysis of the levels and nature of the results coming from the processes of collaborative work, we will first consider those results related to developed products that materialise in the form of resources to sustain the collaborative work. In considering what was reported in Sect. 3.3, it is noted that, for several of the collaborative scenarios chosen, specific objectives were set in relation to the development of certain products. In some cases, the expected products are linked to the micro didactical context of the classroom, such as tasks, lessons, textbooks, websites, etc. Despite the didactical nature and the micro level of several of these products, some of them are valuable resources for school institutions and other products could be useful means or inputs for both the educational system and mathematics educators.

Other products are related to curricular aspects, designed mainly to have an impact at the meso (institutional) and macro (educational system) levels. Some examples of these products are curricular designs or materials created specifically to support curricular reforms. Finally, resources designed and produced to support full collaborative work or to carry out research linked to professional development processes are also highlighted. These products could be identified as being transversal since they have implications for all levels of collaborative work.

Due to the complex and intertwined nature of collaborative work, the results in terms of products developed in and for collaboration can be linked in one way or another to different context levels as well as subject levels. However, for analytical purposes, Table 3.1 that follows provides examples of products distributed by the levels to which they can be mainly related.

This table includes two categories of products: material and intellectual. The material products seem to be evident on the micro and meso levels, while the intellectual ones are on the macro level. What is presented here is illustrative and it is assumed that at all levels one can also consider material and intellectual products. Chapter 5 (Theme D, this volume) advances details about some products as resources for, or from, collaboration. Below are examples of collaborative work in which the emergence of some such products is evident.

Among others, the study reported in Canavarro and Serrazina (2020) makes evident the development of products that are related to the micro, meso and macro levels, and with products illustrated in Table 3.1. At the micro didactical level, the teachers chose mathematical content and worked on the didactical knowledge needed for the planning of lessons tackling that content. They also participated in the collective planning of challenging teaching experiments requiring new mathematical knowledge, new kinds of tasks, different teaching strategies and new resources in the classroom, which edges into the meso level. The programme was grounded in teachers' classrooms and placed great emphasis on teacher collaboration in schools, adopting the analysis of students' mathematical productions as a focus to

Table 3.1 Products distributed by levels (Own source)

Main context/levels related to the collaborative work	Main products or resources developed in or for the collaboration
Micro-didactical level Teachers' level or students' level	Lessons plans Mathematical activities Lessons and portfolios Task sequences Tasks of different natures Guidelines for preparing and orchestrating classroom discourse and lesson plans Different resources for teaching Websites STEM and modelling tasks: modelling problems, experimentations, etc. Assessment activities Students' productions
Meso-institutional level and macro level	Curriculum design Curricular materials Teaching material for a new national curriculum
Collaborative work and/or research level	Resources for professional development processes: artefacts (e.g. video records, reactivated forum discussions) A designed collaborative inquiry model involving co-teaching. Interdisciplinary enterprises PD Theoretical and methodological results Portfolios A spiral of inquiry model approach to teaching and learning in mathematics in collaborative settings Models for teacher collaboration in curriculum development

promote reflection on how to improve teaching practice. In that sense, students' productions could be considered as a developed product (by the students) that materialises in the form of resources to sustain collaborative work.

At the end of the year, each teacher was asked to complete a reflective portfolio. Such activity has the aim of "fostering teachers' meta-reflection and learning" (p. 250). The portfolio created by each teacher included a personal reflection on the challenges of teaching within the scope of the new curricular guidelines and on how the Programme of Professional Development in Mathematics (PFCM) had contributed to meet the teacher's needs. "This option acknowledges that teachers learn when they prepare new topics for teaching, but also when they reflect on what happens in the classroom" (p. 249). Some selected portfolios became resources for the professional development (TPD) level of teachers from other schools in the country in the context of PFCM.

Regarding collaborative products connected to the work of teachers or their practice, we take as a reference the study of Dörr and Neves (2020). This paper presents an extensive collaborative work called "Circuito de Vivências em Matemática" (Circuit of Experiences in Mathematics). One result, among others reported, is a web-site[9] where recreational activities created for the teaching of mathematics were uploaded. Such activities were developed between 2004 and 2018 with the intervention of higher education teachers, basic schoolteachers,[10] researchers and future teachers, and they involved a large number of public or private schools from the same school district.

In the papers mentioned above, as in many other works, several of the products created can be related to results at the micro level and the meso level. In that respect, we mention Ekici et al. (2020). The authors discuss an experience of a collaborative community constituted by 40–50 mathematics and science teachers, education faculty and community partners to work in interdisciplinary collaborative groups which focus their work on the development and implementation of locally relevant integrated STEM projects using mathematical and scientific modelling practices. The group approach begins by integrating STEM problems into schools to foster teacher and student interest, to serve as a catalyst for locally relevant curricula development, and to support "researcher identity in both students and teachers" (p. 309). As discussed in the paper, the products developed during 2 years of collaborative work had impact on the students:

> We observed that students that participated in STEM projects developed [. . .] improved attitudes towards STEM learning, a heightened interest in attending college, and higher degrees of engagement, as they studied problems and concerns situated within their community. (p. 312)

Combining outcomes for teachers and students, as well as the application of a spiral model for inquiry (Timperley et al., 2014), we mention the study by Ell (2020). This study reports on a two-year experience of collaborative work between a researcher and teachers from two Australian primary schools. Those schools were part of a professional learning initiative focused on collaborative inquiry as a "way of tailoring professional learning to local contexts and students' needs" (p. 316). In that sense and considering teaching as an inquiry process, Ell adopted a spiral of inquiry, which began from a students' learning challenge. As reported in the paper, the collection of information about students, both from conversations with students about their learning and from specific diagnostic tools, led to curiosity and enthusiasm on the teachers' behalf. "Results suggest that changes in teacher practice and improvements in student learning can be made by applying a Spiral of Inquiry approach to mathematics teaching and learning" (p. 316).

Just as Ell applies an available model to her work as a valuable tool for her study, among the works presented for Theme B, we highlight papers that propose different and new models as results of their own study. For example, Masselin et al. (2020) present and discuss potentialities and limitations of a collaborative work model, which relates to a doctoral research project from France. The collaboration model displayed on p. 241 is structured in three loops involving a network of three different collaborative groups. Probably, the main limitation of this model is the strong

[9] Website: http://circuitodevivencias.mat.unb.br

[10] In Brazil, the basic school provides services to students between 6 and 14 years of age.

connotation it has in relation to the specificity of the research context, but it also brings contributions:

> it is possible to assert that the model developed provided results for research on the simulation of random experiments. While it has limitations related to the constraints of doctoral research and training over a limited period of time, this collaborative work has helped to explore task simulation use and conceptualize training engineering based on class study (Masselin, 2019). (p. 245)

In Asgari et al. (2020), a cyclical model is presented (p. 236). The model is the result of collaborative face-to-face work focused on the development of curricular material; while Heck et al. (2020, p. 327), set out their “Mathematics Immersion for Secondary Teachers at Scale” model, for a blended form of collaboration.

All the examples presented in this section highlight the products developed as interconnected objects and as contextualised tools that support the professional practices of both teachers and mathematics educators. Many of these objects are aligned to the group’s work objectives, while in other cases they become means for achieving them. All of them involve the bringing into play of new or known knowledge and expertise by the group members. When considering the last examples about models, it should be noted that the models created are also related to the context, the problem and the objectives of the work carried out.

This fact highlights one of the main characteristics of a model since, according to Fourez (1995), models are project related. Regarding the applied or created models, they stand out for being considered as valuable tools that can guide the design and analysis of contextualised collaborative settings. We note that the creation of a model may be a valuable form of collaborative activity for those involved.

3.4.3 Outcomes Related to Learning, Professional Development, Knowledge, Beliefs and Practices

In this Sect., to analyse results related to teachers’ learning, we take as our starting model the one offered by Clarke and Hollingsworth (2002) and discussed in the introduction. However, some other contributions will also be considered. On the one hand, it is noted that, among the papers presented in Theme B, few offer discussions that conceptualise the process of change itself. For the most part, the papers recognise changes or growth in teachers during the collaborative immersion process and, at the same time, connect those changes or growth to learning.

In several papers, the learning is mainly related to the domains of personal and professional practice. In some cases, such learning is intertwined with the consequence or external domains, perhaps something to be expected since most studies take as their central focus the professional practices of teachers in their school environments. The following is a group of examples in which we seek not only to make evident their outcomes, but also to give some details about the contexts and settings in which they were developed, in order to make evident the connections between results, contexts and collaborative scenarios.

One example, from the ICMI 25 Study Conference, of a project where there is attention to conceptualising the change process, is the work of Kooloos et al. (2020). This study took place in The Netherlands, involved five mathematics teachers and one researcher, and included guidelines for "discourse-based lessons" which were collaboratively developed. The suggestion from the authors is that, through participating in the study, discourse-based lessons became a "boundary space" (Akkerman & Bakker, 2011) between the teachers' regular teaching practice and academic discourse on mathematics teaching. The study did not aim to collect data on the cross-over or connection between discourse-based lessons and teachers' more regular teaching practice, but Kooloos et al. offered some suggestive comments about likely or possible influences. Change, in this model, is about the development of boundary objects and spaces, initially only minimally interacting with a sphere of practice, and the gradual transformation of the sphere of practice through the influence of the boundary (which itself is also transformed).

The study reported by Asgari et al. (2020) represents an example where the external domain (Clarke & Hollingsworth, 2002) is evident as a stimulus for collaboration. In this case, the authors point out as problematic, in the Iranian context, the gap between the curriculum proposed by experts and the curriculum implemented by teachers. Among the results reported, the authors highlight an increase in teachers' knowledge of the content of the curriculum, a change in their beliefs about their role and opportunities to participate in curriculum development activities, and increased confidence in collaborative curriculum development. In addition, "teachers were found to believe in the effectiveness of teamwork in improving knowledge and sharing their experiences" (p. 230). The above results show that changes in the personal and professional domain of teachers can be related to the external domain.

Horoks et al. (2020) report a long-term collaboration among researchers and teachers. The group worked together to achieve three main objectives. The complete group had the objective of designing resources for the teaching and learning of algebra. Teachers set themselves the goal of enriching their professional practices to offer better support for the learning of all their students while working in algebra. Researchers–facilitators were interested not only in studying the effects of collaborative work, both on the teachers and their students, but also in providing knowledge to the field of mathematical education. Among the several results reported in this work, we highlight those that indicate changes in the teachers and the students. It is worth noticing that the reported changes were not immediately visible but, on the contrary, they became noticeable after the first year of immersion in the collaborative work.

These considerations are in line with those expressed by Reid and Zack (2010b) and with Potari et al. (2010). The first changes of the teachers were linked to the kind of tasks selected by them for their classes, by selecting tasks that favoured the construction of meaning in the algebraic work. Later, changes were noted in their interactions with the students, with the increasing prevalence of interactions that promoted a certain autonomy among the students while working on sophisticated algebraic tasks. Another important aspect observed concerns professional development at the time of the induction of new members. With respect to the students, changes stand out as they move from an essentially arithmetical way of working to the use of algebraic procedures when solving problems. The results reported make evident an interconnection between three domains: the personal domain, the domain of practice and the domain of consequences.

Richit and Tomkelski (2020) highlight some teachers' growth from participation in jointly carried-out activities. For example, it is noted that, as a result of their involvement in the collaborative development of lesson plans, the teachers identified changes in the group itself that had enabled them to grow closer to each other. They also emphasise the increasing confidence within the group, considering that such achievement "was developed through the respect and care with the needs, interests and anguishes of all participants in the Lesson Study" (p. 419). The authors acknowledge that the participation of the teachers in the LS activities promoted the growth of the teachers in their personal and professional domains.

These results are supported by the teachers' voices:

> As we finish our meeting, I became very happy with what I learned through planning of classes [. . .] I feel the lack of collective planning in the everyday routine. [teacher Erika, p. 415]

> We think alone, we analyse alone, and we find some way out. But what we made here in the group [. . .] is think together and reflect together. There are many people analysing the same question, thinking in the same class. And this made a difference [. . .]. [teacher Ivy, p. 415]

Due to time constraints, at the suggestion of the mathematics educators, part of the teachers' production work was done via *Google Drive*. These activities implied, for the teachers, changes in routines and learning about the use of these resources. As mathematics educators, the authors recognise that, to promote collaborative work in the field of LS, it is necessary to put into practice: "the care, attention and appropriate intervention of those who lead it, so that the group feels welcomed, respected and valued in the collective" (p. 419).

In line with the previously mentioned, we find it interesting to highlight routines and time, as two dimensions of school practices. School routines institute the daily work of teachers and changes in these routines give possibilities for reconfiguration of certain social practices. This is so, because "teachers are not simply learners of techniques; they are also social learners" (Hargreaves, 1996, p. 39). In relation to time, it is a key consideration to understand teachers' problems of change and the structuring of their work. In this regard, as Hargreaves points out, for teachers, time is not only an objective restriction that can be oppressive, but also a horizon of subjectively defined possibilities or limitations.

What is presented in Sects. 3.4.2 and 3.4.3 are some of the main successful results achieved during collaborations. Several of them are considered successful by linking them to expected achievements. Other successful results, as well as issues considered problematic or difficult, emerged during the process itself. In the next section, some of these difficulties are highlighted.

3.4.4 Unexpected Outcomes: Difficulties or Problematic Issues Encountered at Work

Approximately half of the papers report some difficulties or issues considered problematic in the development of collaborative work or research itself as emerging results. Difficulties or problematic issues are those elements or situations that, in one way or another, influenced the collaborative work, or the objectives, being only partially achieved. They are hard to categorise, because, as we have mentioned before, they arise from the interaction between the complex process of collaboration and the research that accompanies or dialogues with that process.

The difficulties and problems, as well as the results, are embedded in the contexts and practices developed, such as the collaborative, teaching, learning or research practices. Aspects related to the teachers' lack of time, the need for a long period to observe work progress, the dynamics of school routines and conditions imposed by the institutional or cultural context are recognised, in some way, as limitations to collaborative work or research carried out in these scenarios.

Rafiepour (2020) indicates that the teachers involved in their project admit that the demands on their time and administrative structures limit their possibilities for working with JLS in their schools. Trevisan and Elias (2020) present work developed between 2018 and 2019 in Brazil. This work aims to establish a collaborative group appealing to an adaptation of the LS model and involving primary school teachers and mathematics educators to support the development of their professional learning. The project had support by the Municipal Education Secretariat. In 2018, the work of three primary school teachers with the facilitators–researchers would focus on the development of lesson plans for teaching division and fractions.

However, a series of difficulties made it impossible to achieve the aims. The main difficulties reported are restrictions of context, previous teachers' knowledge or beliefs about the chosen content and some inexperience of the facilitators–researchers in dealing with such issues. The facilitators–researchers, and authors of the report, highlight their recognition and understanding of these issues as a learning experience for them. This allowed them to develop a new collaborative setting in 2019 to work in another school and under different conditions. It could be noted that, in part, the difficulties reported in this case highlight the complexity in achieving productive adaptations of the JLS model.

During discussions at the study conference, the reactor to this paper suggested several aspects that could perhaps have been considered in this study in more detail. Among them were the non-voluntary aspect of starting the work together as an obstacle in this case for collaboration, and the absence of references about the extended use of LS in some countries within a Latin American context.

Heck et al. (2020) focus their analysis on two groups they worked with. From their analysis, they recognise problems related to the partial achievement of certain objectives of their work. Such problems refer to the links that teachers create between producing mathematics and reflecting on their teaching. The authors point out that, even though the teachers showed evidence of their involvement in the mathematical community, group discussions about the connections between what they experienced in mathematical immersion and teaching were infrequent or lacked depth. The authors also report certain perceived differences in the group's performances according to the facilitators (non-researchers) who intervened in each group. The researchers aim to increase the scale of the study to analyse these problems further.

Horoks et al. (2020) report difficulties associated with school routines and some issues related to collecting data on students' learning. The researchers had planned to evaluate a large number of students. However, due to school organisation issues and the lack of access to digital technologies, this number decreased significantly. This limited their conclusions on the impact of collaborative work on students. Regarding students' learning, it is recognised that some of their developed algebraic skills encounter limitations when operating with complex expressions.

We stress the importance of reporting difficulties, challenges or issues recognised as problematic during collaboration. These difficulties may have been obstacles for those involved, but they are also important inputs for thinking about collaborative practices or for all those who engage in such work. In the next section, we move forward with the analysis of *sustainability*, *dissemination* and *sharing of outcomes.*

3.4.5 The Sustainability, Dissemination and Sharing of Outcomes

In this section, we firstly address the question of the sustainability of outcomes, and then the dissemination and sharing of outcomes. There are two aspects to the sustainability of outcomes of collaboration with teachers. The first is the sustainability of the collaboration itself and the second is the sustainability of any change or learning in relation to teachers' practices, i.e. were changes made in the context of collaboration sustained into long-term changes in the classroom? We will take these aspects in turn.

A significant number of the studies reporting at the ICMI Conference were of collaboration structures that lasted 5 years or more. Most common, of these longer-term projects were structures where the teachers engaged for a year and then a new group of teachers joined (e.g. Koichu et al., 2020). However, some of the research involved even longer-term collaborations. In some cases, this involved a fluid design with some stability of participants and some new members joining (e.g. Hreinsdottir, 2020). In other cases, networks met with the same participants, including one group in Spain who have met for 20 years (Climent et al., 2020). Inevitably, these longer-term collaborations of the same group tend to involve smaller numbers of teachers than those with more fluid memberships or with deliberately changing memberships.

Canavarro and Serrazina (2020) report on a national professional development programme for primary teachers from Portugal, relating to mathematics teaching and discussing and analysing students' mathematical productions. The programme ran from 2005 to 2011 and, although voluntary, involved over half of the primary teachers in the country, over its lifetime. The programme was dispersed across 18 higher education institutions and involved meetings of teachers in groups of 8–10.

Although not represented in studies for this conference, a programme of professional development on a similar scale has been underway in England for the last 5 years, organised around 42 "Maths Hubs", which are based in schools and funded by the government, and are responsible for offering professional development to their local communities (courses tending to last 1 year). The aim is to create a school-led system for the leadership of professional development (see: https://www.ncetm.org.uk/maths-hubs). A similar aspiration is present in a project in the Virgin Islands (Ekici et al., 2020) that, since 2014, has established yearlong professional learning communities with the aim of developing culturally responsive teaching practices. The work of Dörr and Neves (2020) has sustained itself since 2004, originating from a collaboration between the Brazilian Society of Mathematics Education and associated members in the city of Brasilia. The collaboration appears to be growing in scale.

Three studies in the ICMI 25 programme (Coppé & Roubin, 2020; Masselin et al., 2020; Modeste & Yvain-Prébiski, 2020) related to the French IREMs. IREMs were set up in the 1960s by the French government, to facilitate reform of mathematics teaching via interaction between teachers and academics; involvement of teachers is voluntary and in their free time. IREMs get established in different areas, linked to a university. The work reported at our ICMI 25 Study involved IREMs that had been running for 15 years or more. These groups have an explicit aim of linking theory and practice, and their longevity supports moves towards co-design and co-production of knowledge. They also model the processes of their own collaboration.

The capacity for collaborative groups to emerge into new ways of working appears as significant in several studies, in terms of the sustainability of the co-operation. Coppé and Roubin (2020) discuss three different phases in the evolution of their group, since 2002. An American study (Castro Superfine & Pitvorec, 2020), funded for 5 years, focused on developing practices within teaching algebra and of formative assessment, with 10 teachers. The emergent design of the project allowed a shift in years 4 and 5, towards co-designed collaborations in researcher–teacher dyads.

An unsurprising commonality across many of the collaborations, or collaboration structures, that have lasted 5 years or more, is that they were government initiatives and/or received sustained funding. Another commonality is that several research reports on these long-term collaborations mention the significance of the different levels of interaction facilitated by the work, for instance, between academics and teacher educators (where these are different roles), between academics and teachers, and between teachers and teachers.

Some activities, such as conferences (Hreinsdóttir, 2020) help get new and different actors involved in collaborative processes. Some national contexts, such as government support for professional development, encourage the involvement of volunteers (Koichu et al., 2020). Sustainability is also developed by taking advantage of working conditions or available resources, but at the same time sustaining a rhythm that adapts conditions and resources to the pace of changes and demands of the context (IREM). Another aspect is that it brings into play flexible and adaptable processes to create, through teamwork, rapid solutions to emerging problems (Horoks et al., 2020).

In terms of our second meaning of sustainability, the extent to which collaborations led to teacher development that sustains in the long-term, there are a diverse set of findings, arising from a range of methodological approaches and a range of conceptualisations of learning and knowledge. For instance, Chen et al. (2020) report on differences in teacher descriptions and interpretations of video-recordings, from having taken part in a Lesson Study group, based on a coding scheme developed by Vrikki et al. (2017). Coppé and Roubin (2020) report on changes in relation to knowledge of specific topics, assessed via pre and post questionnaires. Collura and Di Paola (2020) use interviews to conduct a qualitative analysis of change.

Pacelli et al. (2020) observed changes in teacher beliefs/attitudes, assessed against their novel construct of interpretive knowledge. Eden (2020) researched developments in teacher noticing, both in their observations of video recordings of lessons and also from co-teaching. Jacques and Clark-Wilson (2020) analysed changes in questions or prompts used by teachers in their lessons. Horoks et al. (2020) observed changes in teachers' planning and in their interactions during collaborative meetings. This diversity inevitably makes comparisons across collaborations and collaborative structures next to impossible. Furthermore, the studies reported here analysed change during the process of collaboration, and so we do not have evidence, either way, about the sustainability of changes beyond the collaboration. This is a question relevant for future research.

Two studies (Ekici et al., 2020; Horoks et al., 2020) attempted to track student learning in the classrooms of those teachers involved in collaboration. Horoks et al. aimed to collect data on student knowledge and skills in algebra. Although they reported difficulties with data collection, leading to a small sample, their findings were encouraging. Ekici et al. surveyed student (pre- and post-) attitudes for those students who had taken part in projects linked to their teachers' involvement in professional learning communities and found significant improvements. We suggest further work tracing the impacts of collaboration into the classroom would be highly beneficial to the field.

We now move to consider, more briefly, dissemination and sharing. By far the most common forms of dissemination of outcomes of collaboration were via conferences (for a range of relevant actors) and digital media (websites and resource depositories). However, it seems clear that the main benefit of collaboration is for the collaborators themselves (for instance, in LS collaborative groups taking place in Iran and Brazil). In other words, we surmise that it is the process of reflection, via engaging in collaboration, which is transformative. We suggest, therefore, that a key element of collaboration is the sustainability of the structure within which it takes place (our first element of sustainability, discussed above).

Thus, if the lasting benefits of collaboration are mainly on the actors taking part, then long-term funding and organisation are required such that ultimately all, or a majority of, teachers are able to benefit. The examples from Portugal and England show that, with ambition, it is possible to achieve involvement at a country-wide scale within a relatively short timescale (5 years), through funding of a nationwide network of centres or hubs. There is a need for more research on the long-term impacts of involvement in one-year professional development collaborations.

Returning to our distinction at the beginning of this section, there is clearly a compromise to be made between long-term collaborations involving a few teachers and an iterative design that might involve teachers for only 1 year or less, but then potentially reaches more people. It appears there is little evidence which might, at present, give us confidence as to what length of collaboration, in different contexts, might be needed in order to provoke teacher learning that sustains beyond the time of the collaboration.

3.4.6 Summary and Comments

In general, and independently of the model of collaborative work that is adopted, most studies on papers reported for Theme B take, as a central core, different practices that involve future teachers or in-service teachers. In this sense, some of these results report on classroom practices, but, at the same time, given the complex and collective aspect of collaboration, the results expand to other practices such as research or curriculum development. Of course, the type of results reported acquire particularities according to the purposes of the collaborative project, the scale of these and the contexts that host them.

In Sect. 3.4.2, we considered collectively developed products, as situated results of the collaboration. In Table 3.1, presented in Sect. 3.4.2, the main material or intellectual products are visualised as results of different collaborative work distributed by levels of incidence. It is at the micro level (e.g. lesson plans, task sequences) where there is the largest quantity of products linked to teachers' school practice. The expansion of results outside a micro didactical level shows not only the influence of the aims and focus of the collaborative work, but also the expansion of teachers' work beyond face-to-face activity with their students in the classroom, which implies interactions not only with other teachers, but also with curriculum developers, facilitators or researchers. For example, the second row shows results of the work that impacts on curriculum, while the third row shows results of special interest for researchers.

It is worth noting that, in models of a systemic or structured nature, such as the Japanese LS or the work at French IREMs, the phenomenon of expansion of results to all three levels (micro, meso, macro), in one way or another, forms part of the purposes of such collaborative projects. Similarly, those models of blended collaborative work focus on products linked to websites or other types of material, for example, but it is important to indicate that this is not unique to those models.

What is reported in Sect. 3.4.3 is a range of outcomes related to learning, professional development, knowledge, beliefs and practices. We note that what is reported as an outcome is inevitably coloured by the particular model of collaboration and theoretical perspective of the author. For example, Kooloos et al. (2020) use the notion of boundary objects and, hence, report on the spheres of practice of participants. Horoks et al. (2020) are influenced by the Theory of Didactical Situations, which inevitably influences what is offered to teachers in collaboration and, hence, the outcomes of their work. As well as seeing strong connections between the type of products developed, as a result of the form of collaboration, the feelings of togetherness generated by involvement in a group over time, which has a common focus, appears in several reports. Sections 3.4.2 and 3.4.3 report successful results according to the objectives set for the project and certain variability in results depending on the collaborative work models chosen.

On the other hand, Sect. 3.4.4 presents certain results perceived as unsuccessful by the authors or as emerging results of the work that, in one way or another, hindered the full achievement of the proposed objectives. One such obstacle, which often hinders collaborative work, has to do with certain changes in institutional routines that could not be fully foreseen. Perhaps the most frequent of these are changes in teachers' obligations, activities or times according to emerging institutional requirements during the collaboration. This fact became more evident in those models that involve significant immersion in school institutions, such as the LéA's collaborative working model or those application of the LS model in new contexts.

However, in relation to the application of LS, it should be noted that, when comparing two cases of its application, carried out in the south of Brazil in similar socio-cultural contexts, the results were quite different. In one case, particularly good results were obtained and in another not. When comparing both cases and without wishing to detract from any work, we found two different aspects of the two works that seem important to highlight. One of them is linked to the ways of accessing or approaching school spaces (i.e. whether teachers were volunteers or not) and the other to the expertise of those who intervened as facilitators.

For more details on the first point, see what was discussed in Sect. 3.3.4. Regarding the effect of the role of the facilitator in collaborative work, in another investigation, not within a LS model but in a blended model, certain differences in results depending on the facilitator are also pointed out. This issue of the role of the facilitator in different models deserves more careful and extensive studies, an issue addressed in Chap. 4 (Theme C, this volume).

By way of closing, we wish to highlight and celebrate the large number of long-term collaborations taking place across the world. The model of Japanese LS is one that has been disseminated widely. Nationwide programmes (such as those described above, taking place in Portugal and England) appear to be based on the idea that collaborations are not so much for the purpose of achieving results and outcomes that can then be disseminated outside the collaboration, but rather that the collaborations themselves are the most relevant outcomes. In other words, it is the act of engaging in a collaboration that, time and again, appears to be transformative.

3.5 Mathematics Content and Context for the Collaborative Activity or the Case of a Small-Scale Frame for Collaboration

3.5.1 Introduction

Hiebert and Wearne (1993) proposed that, "what students learn is largely defined by the tasks they are given" (p. 395), and researchers have taken this to apply to pre-service and in-service teachers as well, including in relation to how mathematical tasks are presented, developed, engaged with and resolved (Watson & Mason, 2007). Due to the critical importance of mathematical tasks on student learning and teacher learning, great efforts have been made to explore the nature and roles of mathematical tasks (*Journal of Mathematics Teacher Education*, *10*(4–6) Special Issue edited by Jaworski, 2007), principles and strategies of task design (ICMI Study 22 volume, edited by Watson & Ohtani, 2015), and teachers as partners in task design (*Journal of Mathematics Teacher Education*, *19*(2–3), Special Issue, edited by Jones & Pepin, 2016).

Within the context of teacher PD, Prediger et al. (2019) and Prediger (2020) described a content-specific theory for explaining and enhancing teacher learning in collaborative settings. According to Prediger, two levels of classroom tetrahedron (students, mathematics content, classroom resources and teacher) and PD tetrahedron (teacher, PD content, PD resources and PD facilitator) are interconnected through lifting or nesting. The classroom tetrahedron forms the PD content which must be nested in PD tetrahedron. Mathematics-related content in PD (such as mathematical tasks, student learning artifacts, and teaching artifacts (lesson plan, videotaped lesson) are critically important for what teachers experience, do and learn. By focusing on mathematics content and teacher collaborative learning, we analyse and organise this section into following three aspects: nature and roles of mathematics-related content; design of mathematics-related content; interacting with mathematics-related content and teacher learning.[11]

3.5.2 Nature and Role of Mathematics-Related Content

Building on, and taking a broad view of, the notion of tasks, mathematics-related content was central to the PD activity of many, but not all, of the collaborative groups described in the studies. The mathematics-related content of teachers' collaborative interactions varied in relation to the specificity of the mathematics focus and the nature of teachers' mathematical knowledge to be developed. In relation to

[11] During the work in Lisbon, Raewyn Eden, Freyja Hreinsdóttir, Chris Plyley, Liliana Suárez Téllez and Rachel Zaks contributed for selecting some of the themes considered in this section.

the mathematical content collaborative groups engaged with, a range of learning and teaching objects and processes were of interest including mathematical tasks and problems; mathematical practices such as discourse, modelling and argumentation; the mathematical thinking of learners such as errors, misconceptions and non-standard solutions.

In several studies, the work of the group was not specific to mathematics, and in others the collaborative activity was centered on mathematics generally. For instance, Dörr and Neves (2020) describe a project aimed at promoting interest and engagement in mathematics for public school students in Brazil, through the design and execution of workshops that present mathematics in playful, practical and creative ways. In contrast, other studies focused on specific mathematical domains including algebra and algebraic thinking, probability, measurement, trigonometry, and number and operations. Associated mathematical concepts of interest in the studies included place value, fractions and units of rate and measurement.

The knowledge required for teaching mathematics well is substantial, complex, and evolving (Ball et al., 2008). Building on Ball and colleagues' model of Mathematical Knowledge for Teaching (MKT), Carrillo-Yañez et al. (2018) propose a model of Mathematics Teachers' *Specialised* Knowledge (MTSK), in which teachers' beliefs are central and there is an emphasis on the specialised nature of the knowledge used in teaching mathematics. Drawing on the breadth of mathematical knowledge for teaching, the mathematics focus of some studies included teachers' knowledge of mathematics curricula such as knowledge of standards, content such as knowledge of domains of mathematics including algebra and geometry, and mathematical practices and processes, such as modelling.

Several studies focused on teachers' understanding of the mathematical content including both specialised content knowledge, "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400); and horizon content knowledge, "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Other studies centred on how knowledge is used in mathematics teaching including teachers' interpretive knowledge involved in making sense of student productions and responding in ways that promote mathematical understanding (Ribeiro et al., 2013).

Making connections between specialised and horizon content knowledge, one study centred on teachers' understanding of learning trajectories for identified aspects of mathematical content; that is "how student understanding develops along the conceptual strands of a discipline" (Castro Superfine & Pitvorec, 2020, p. 255). Arguing that teachers require mathematics knowledge that is both broad and deep, in order to make sense of students' reasoning and anticipate and respond to common obstacles to understanding, Castro Superfine and Pitvorec's study involved teachers and researchers co-constructing a micro learning trajectory for unit rate which was identified as a key aspect of mathematics curriculum content. They found that as they co-constructed, used and reflected on micro-trajectories, teachers were able to deepen their knowledge of mathematics and student learning in the context of their classroom practice, thus expanding the possibilities for responding productively to students' mathematical thinking.

Similarly, Pacelli et al.'s (2020) study is premised on the assumption that teachers require deep and broad knowledge of mathematics, in order to respond to students' mathematical reasoning and productions and support their developing mathematical understandings. Conceptualising such knowledge as interpretive knowledge (IK), the authors argue that collaborative discussion supported prospective teachers to develop novel insights into, and new attitudes towards, students' productions. The study involved the use of interpretive tasks whereby prospective teachers solved mathematics problems and then interpreted a range of students' productions pertaining to the same task. The authors were particularly interested in how prospective teachers could support students to build understanding from their own mathematical productions, including in relation to those students' non-standard, ambiguous and erroneous responses. The authors found that collaborative discussion supported shifts for prospective teachers from evaluative towards more interpretative reasoning about student productions.

Expanding the focus from mathematics alone, interdisciplinary knowledge was highlighted in several studies. For instance, Ekici et al. (2020) reported on a project involving science and mathematics teachers and practitioners collaborating to model locally relevant STEM problems in classrooms as a catalyst to promote interest in these disciplines amongst students and teachers. A continuous cycle of collaborative inquiry involving careful support and planning supported teachers to make culturally responsive adjustments to their practice and fostered students' understanding of the interconnectedness of science and mathematics. Mathematical and scientific modelling of local community issues promoted increased student agency as they started to develop enhanced STEM identities, posed more STEM questions and were more interested in developing mathematical and scientific arguments.

Such interdisciplinary context allowed teachers to expand their disciplinary perspectives and provide more equitable access to mathematics and science experiences for previously underserved students. In her study, Acevedo-Rincón (2020) found that collaborating with peers across disciplinary boundaries expanded the perspectives from which prospective mathematics teachers could understand their practice, their role and the mathematics they were teaching. In both studies, interdisciplinary collaboration afforded teachers of mathematics opportunities to engage with different perspectives and, as such, served as both the context and the catalyst for change. For instance, Ekici et al. (2020) found that the interdisciplinary context provided a pathway for teachers to develop different disciplinary perspectives along the course of their professional lives; a necessity when working in isolated communities. Similarly, Acevedo-Rincón found that engaging within and across interdisciplinary contexts of teaching and learning helped prospective teachers to problematise their professional practices and develop new meanings for the contexts, knowledges and processes of school mathematics, whereby engaging across difference brought new meanings to light.

Developing teachers' pedagogical content knowledge of mathematics and their associated practices was the focus of many of the studies. In his seminal article, Shulman (1986) argued that pedagogical content knowledge (PCK) lies at the intersection of pedagogical and content knowledge—where teachers put their content knowledge to use in their instructional practice—is key to understanding teachers' knowledge. Studies focused on teachers' pedagogical knowledge of mathematics included for instance teachers' use of "talk moves", formative assessment, mathematics as a focus of teacher inquiry to address student learning needs, and the selection and design of mathematics tasks and task sequences.

A number of studies were centred on teachers' knowledge of students including noticing, interpreting and understanding students' mathematical thinking. For instance, Pacelli et al. (2020) found that engaging prospective teachers in the collaborative analysis of students' productions stimulated deeper reflection and supported norms of mathematical/pedagogical reasoning that were increasingly interpretive.

Teachers' knowledge is relational and mediated by affective factors (Bobis et al., 2012) and teacher collaboration is both supported and constrained by emotions (Brodie, 2020). Brodie suggests that "all learning involves emotion" (p. 40) and emotion can be seen as both a tool for, and an object of, teachers' collaborative activity. The role of emotion in teachers' collaboration is highlighted in several studies, including those focused on teachers' self-knowledge and beliefs. For instance, Trevisan and Elias (2020) in the context of their study aimed at disrupting conservative classroom practices reported one of the teachers in their study as saying, "I felt a little embarrassed. [...] Because I do not have mathematical training" and another as reflecting "the exchange of experiences with the other teachers [...] helps us to see a way, to feel a little more confident of our practice or even to speak" (p. 439).

In professional development contexts where the focus is on individual teachers, a tension can emerge whereby exposing classroom challenges can direct attention to the quality of the individual teacher's practice and, thus, make the teacher vulnerable to the risk of negative critique. In contrast, in the context of primary and secondary mathematics teachers' collaborating to solve, discuss, implement and reflect on classroom problems, teachers described a shift from negative emotions, such as fear of exposure, to feeling valued and confident, as they jointly reflected on and learned from practice dilemmas (Soto et al., 2020). One of the study's participants, for instance, was described as feeling "valued by the community [...] encouraging her more and more to present proposals to their peers to work collaboratively. Her enthusiasm had a multiplier effect" (p. 423). Such shifts illustrate the dual role of trust as both a requirement for, and a product of, teacher collaboration (Eden, 2018), whereby mutual trust is an important element in any such activity.

The notion of teachers' knowledge of themselves, of the epistemologies they bring to their mathematics teaching, might be thought of as meta-knowledge. That is teachers' knowledge of their thinking, beliefs, assumptions and emotions about mathematics teaching and learning can be viewed as an overarching conception of teachers' knowledge.

Together, the studies illustrate the complexity and scope of mathematics as an object of mathematics teachers' professional development through collaborative activity, and suggest that different components of knowledge for mathematics teaching may inform one another in complex ways. In particular, teachers' collaborative professional development activity focused on specific mathematics-related content might both provide pedagogical tools to expand the possibilities for classroom practice and "influence the collective domain, namely the shared pedagogical content focus on further basic conceptual needs in different mathematical content" as suggested by Prediger (2020, p. 11). The section that follows elaborates on the purpose, selection and design of the specific mathematics-related content that was the focus of the collaborative activity reported.

3.5.3 Selection, Design and Use of Mathematical Content/Tasks

In accordance with the theory elements identified by Prediger (2020), mathematics can be viewed as PD content and/or PD resources at the level of teacher PD, as well as mathematical content at the level of the classroom. Tools that mediate the work and learning of collaborative groups of teachers of mathematics are discussed in Chap. 5 (Theme D, this volume); of interest here are the mathematical content and processes that characterise the collaborative activity of different groups and the ways in which these were selected and designed.

We return first to the importance of mathematical tasks for the professional development of teachers of mathematics, as established at the beginning of this section. A number of studies focused on mathematical tasks and problems, particularly those for use in the classroom, and several studies centred on the selection, design and/or development of tasks for the teaching of mathematics, including learning tasks and diagnostic tools. Of particular interest in several studies focusing on task design and development was the organisation of mathematical content so that it is feasible to be taught. One such study (Asgari et al., 2020), exploring how collaborative curriculum design impacted upon teachers' knowledge of, and attitudes towards, teaching mathematics, found that teachers' knowledge of how algebra is sequenced and organised for teaching, and their trust in the efficacy of the curriculum, were both enhanced.

Soto et al. (2020) describe how the design and use of mathematical tasks for classroom instruction were the focus for the collaborative activity of a community of in-service mathematics teachers from primary and secondary schools, and university-based mathematics educators. The purpose of the collaborative activity was to bridge students' transition from primary- to secondary-level mathematics while deepening teachers' mathematical knowledge for teaching. The design of mathematical tasks was central to an iterative cycle of solving problems together, planning for implementation, using the tasks in classrooms and collectively reflecting on the enactment. The mathematics tasks were thus central to understanding the mathematical transition from primary to secondary school through an iterative process of collective design, enactment and reflection.

Other studies focused on mathematical processes such as modelling, argumentation and simulation (e.g. Ekici et al., 2020; Masselin et al., 2020; Modeste & Yvain-Prébiski, 2020) and practices such as classroom discourse (e.g. Kooloos et al., 2020). The study by Koichu et al. (2020) reports on teachers' co-learning inquiry focused on questions posed by each of four teacher-research communities. One group's focus was on thinking flexibility, and another was on the design and enactment of talking tasks, described as involving opportunities for problem solving, discussion and reflection. Professional development was often framed as addressing weaknesses or gaps in teachers' knowledge, and this was the aim of a number of studies.

For instance, one study examined teachers' professional development related to the role and use of simulation in the teaching of probability and statistics, following its introduction into the French curriculum in 2010 (Masselin et al., 2020). The authors found that a teacher's choice of mathematical objects influenced the mathematical activity and subsequent learning opportunities at the professional development and classroom levels.

Decisions about the focus of the collaborative activity were variously made by teachers or researchers, or by the mutual agreement of both whereby members of the group decided together what to discuss and reflect upon. In a number of studies, the mathematical focus was chosen for its authenticity, and for drawing connections between mathematics and learners' lives. Ekici et al. (2020) report on one such study, in which multidisciplinary groups worked with the community to identify and investigate a locally relevant STEM-focused problem—in this case, water quality—and worked to develop culturally responsive practices aimed at providing equitable access to quality STEM instruction for all learners. In a study by Trevisan and Elias (2020) involving LS, choosing mathematical content in the form of classroom tasks that were close to teachers' current practice appeared to be one of the factors that mitigated resistance and promoted openness to changes in practice.

3.5.4 Interacting with Mathematics-Related Tasks and Teacher Learning

The nature of teachers' interactions with mathematics in the context of their collaborative activity tended to centre on three interrelated dimensions: doing mathematics; talking about mathematics; investigating representations of mathematics teaching and learning. Engaging in *doing* mathematics highlights the active and social nature of mathematics, whereby groups with a shared interest generate problem solutions using mathematics, including experiencing tasks and associated practices as a rehearsal for subsequent classroom use.

For example, in Modeste and Yvain-Prébiski's (2020) study, the researchers proposed to the group an authentic problem involving mathematical modelling such as predicting tree growth or optimising the positioning of a warehouse. The elaboration of the problem was based on characteristics that came from research results and provided indicators to develop a situation favouring modelling activity in the classrooms. From there, all members of the group collaborated to develop a problem suitable for a modelling activity in the classroom from sixth to twelfth grade. The group designed a first statement and *a priori* analysis; that is, they anticipated student responses to the task prior to using it with their students.

Prior to the group's next meeting, teachers experimented with this problem in their classrooms to contribute to the next iteration of the group's reflections in the development of the problem; a process involving several meetings to reach the final version of the problem. The mathematical content appeared to mobilise the whole process of collaboration that later materialised in the final version of the problem. The problem, then, can be considered a collective product of the group's collaborative activity whereby mathematical objects can be seen as both a vehicle for and the product of teachers' collaboration.

Many studies described collaborative contexts that involved group members' active *discussion about and listening* to mathematics, including raising and answering questions related to mathematical concepts, subjects, processes and procedures in connection with classroom work, sometimes in collaboration with expert partners. In many studies, conversations about mathematics involved groups' collective reflection on classroom events including stories, observations and experiences of individual teachers' classroom practice (e.g. Rafiepour, 2020), the shared practice of co-teaching pairs (e.g. Eden, 2020), and on representations of practice including the analysis of videos of mathematics lessons (e.g. Castro Superfine & Pitvorec, 2020).

The focus of such reflections was the collaborative review of teaching practices related to the specified mathematics content to promote practice improvements: for instance, in Trevisan and Elias' (2020) study focused on the challenges inherent in the reflective and collaborative process of implementing lesson study. In such studies, mathematical discourse can be seen as linking the collaborative activity and the development of mathematical knowledge for teaching. For instance, Eden (2020) found shared planning appeared to set teachers up to be responsive during a co-taught lesson, whereby anticipating how students might respond to a mathematics task supported enhanced noticing. As co-teaching pairs acted to make sense of and reconcile contradictions between what was anticipated and what actually occurred in the lesson, this in turn served to expand the breadth and depth of their reflective thinking after the lesson.

In Chen et al.'s (2020) study, they explored how discourse threads during teachers' collaborative discussion about a video of teaching a geometrical topic may be related to their MKT growth. As such, talking about mathematical content at the classroom level, including related teacher actions and student impacts, serves as both the means of accessing the classroom level as PD content (PD1) and the mechanism for teachers' professional growth (PD2) as described by Prediger (2020). That is, as teachers engage in collaborative discourse about the mathematics of the classroom, they are engaging in sense-making about the mathematics itself, alongside the co-generation of theory about what it means to know, do, teach and/or learn mathematics. Talking about mathematics thus positions mathematical content simultaneously as a resource or a mechanism for teacher development and the object of that development.

A number of the studies centred on collaborative groups *investigating* authentic problems of practice through collaborative inquiry. An approach common to a number of studies involved collaborative groups developing *research lessons* to address particular aspects of mathematics content including the design, teaching, reflection, revision and re-teaching of mathematics lessons. For instance, Climent et al.'s (2020) study involves two-year projects, each focused on the teaching and learning of an aspect of mathematical process or content, such as problem solving, problem posing or fractions. A group of primary teachers and mathematics education researchers read about and discuss content, and then plan lessons and implement them. The collaboration is centred on authentic problem solving involving reflection on practice as a mechanism for promoting professional development and knowledge generation as connected and reciprocal outcomes.

Again, discourse was a critical element in the professional development process, whereby engagement in reflective discussion was a central mechanism through which teachers could interrogate teaching decisions and their impacts. Through a process of shared critical reflection, participants were able to identify and interrogate the importance of the specialised knowledge of mathematical content they brought to bear in their classroom practice, where previously the teachers had been reluctant to engage with conversations about their own mathematical knowledge. It seems that the collaborative work space created an atmosphere of trust, so that the teachers were willing to communicate ideas about their own practices when teaching fractions.

The collaborative activity of many groups was directed at outcomes related to students' mathematical learning whereby activity at the classroom level is nested within the PD level. For instance, some studies focused on groups collectively interpreting students' productions when solving mathematics problems (e.g. Pacelli et al., 2020). In many cases, mathematics content was central to all aspects of the collaborative process, including teachers developing understanding of mathematical content, anticipating student thinking in relation to that content and designing classroom tasks to engage students with the content.

For instance, Castro Superfine and Pitvorec (2020) report on a study involving teachers and researchers working to co-construct professional learning experiences aimed at developing understandings of learning trajectories in algebra. Reflection on video of classroom discussions that elicited student understandings of algebra engaged teachers in discussion of, and deepened their understandings of, algebra learning trajectories, and expanded the possibilities for their responses to students' thinking in algebra.

Although mathematics learning goals for students were implied rather than specifically addressed in some of the studies, some had an explicit focus on mathematics goals at the classroom level. In a study involving groups of teachers working across disciplines, Collura and Di Paola (2020) found that co-planning and co-teaching mathematics lessons contributed to an expanded view of the nature of mathematics, whereby students saw it "no longer as a discipline in itself, detached from reality and written in a language incomprehensible to many, but as a discipline that has evolved with others, in history and time, which finds applications in various branches of knowledge" (p. 283).

The connection between mathematics learning at the classroom level and content at the PD level is made explicit in studies involving teachers in inquiry processes, whereby classroom level student learning data informs activity aimed at improving mathematics teaching and learning. In one such study, Ell (2020) found that participating in PD activity aimed at understanding and implementing spirals of inquiry promoted "teacher learning about mathematics concepts, student learning in mathematics, assessment and mathematics teaching" (p. 322), where the chosen inquiry focus was mathematics teaching and learning. In particular, engagement with explicit evidence of students' mathematical thinking was central to the collaborative process, supporting the generation of trust and "sparking curiosity and raising questions" (p. 323) within the inquiry groups.

3.5.5 Summary and Comments: Mathematics Content Is a Mediating Factor Influencing the Nature and Effects of Collaboration

Across the papers reviewed in the preceding sections, mathematical objects (or content) appear both to catalyse and to support professional change by acting as a source of difference. As groups engage in collaborative discussion about mathematical objects, such as classroom tasks, student productions or mathematical processes, those objects simultaneously serve as resources for and objects of the teachers' learning and development. Through teachers' collaborative activity, professional learning and development are catalysed, and new understandings and practices are forged by negotiating across differences. That is, as mathematical content is introduced into, or shaped by, the collaborative activity, new mathematical concepts, tasks and processes serve as contradictions to the status quo and, as Katz and Dack (2014) argue, paying attention to that which interrupts the status quo "holds the potential to yield new professional learning" (p. 36).

Participation in dialogue about mathematical objects simultaneously promotes expanded opportunities for the teachers to act towards the dual objects of their own and their students' learning goals. As teachers consider the mathematics teaching and learning of their students, they also expand their own understandings and practices. Consequently, in the context of collective (or collaborative) interactions, each participant reviews his or her own knowledge and perspectives, and contributes to the co-generation of new shared understandings.

Thus, talking about mathematics positions mathematical content as both a resource or mechanism for teacher development and the object of that development whereby differences act as reflexive objects and catalysts for the on-going transformation of teachers' understandings and work. Making differences in practice visible in the context of teachers collaborative activity appears to open opportunities to learn about and from those differences (Tobin, 2014). However, more empirical studies on the mechanisms of how mathematics content mediates the teacher learning outcomes are needed.

3.6 Conclusions

In this final section, the progress of the work on Theme B (Sect. 3.6.1) is summarised. Then the answers to the questions that guided the work are presented (Sect. 3.6.2). Finally, further research directions are discussed (Sect. 3.6.3).

3.6.1 A Summary

Throughout this chapter, the main goals, forms and achieved outcomes related to collaborative settings between teachers and others were analysed, synthesised and presented. In all cases, the socio-cultural contexts in which collaboration resided were highlighted. In the analysed studies, there was diversity in goals, design and implementation of settings and outcomes. In this diversity, mathematical knowledge, as part of the collaborative content, emerged as a substantial element for promoting collaborative work.

The studies focused their collaboration on one or more of the following levels: the micro didactic level of the classroom, the meso institutional level or the macro level of the educational system. Accordingly, the outcomes achieved contributed especially to one of these levels. Certain unexpected outcomes emerged while other results may provide input for future studies. Although most of the studies included in-service teachers and researchers, there were also cases in which prospective teachers participated.

3.6.2 Answers to Five Interconnected Questions

From Sects. 3.2, 3.3, 3.4, and 3.5, we present results of a systematic analysis of the studies that were presented and discussed in Theme B to respond to the following five questions:

1. What models of teacher collaboration have been developed? What are the design features, goals and outcomes of the different models?
2. How effective are various models for promoting different outcomes?
3. Which forms of collaboration are appropriate in different contexts?
4. What are the affordances and limitations of each form of teacher collaboration?

5. What are the benefits and the challenges that online teacher collaboration poses to the teachers?

From the analysed sources, we can offer proper answers to the first four questions. Regarding question 5, since only three of the 28 papers, linked to Theme B, presented case studies of collaborations developed in blended format, it is hard to offer an accurate answer to it. But we will provide a brief discussion of relevant issues related to this kind of form. Given the interconnected nature of the questions, we clustered questions 1 and 4, questions 2 and 3, and question 5, and answered each group of questions separately. However, we recognised these questions are strongly interconnected.

3.6.2.1 Forms of Teachers' Collaborative Work: Affordances and Limitations

Drawing on Sects. 3.2 and 3.3, we present our answers to the following questions:

What models of teacher collaboration have been developed?
What are the design features, goals and outcomes of the different models?
What are the affordances and limitations of each form of teacher collaboration?

Although the collaboratives work analysed differed in their forms, the majority of them (25 out of 28) chose to organise their work in a face-to-face format. However, in a few cases, shared work via Google Drive or e-mail were reported (e.g. Richit & Tomkelski, 2020). Of the three cases that adopted blended formats, in two of them, perhaps due to the wide geographical dispersion of the participants, there was a prevalence of general meetings in virtual mode. However, local face-to-face meetings were also held.

Among the *face-to-face* forms, we identify the following forms: *Adaptations of lesson studies*, *researchers–teachers partnerships*, *networks* and *forms connected to specific purposes.* Depending on the participants involved in the joint work, four types of communities have been identified: *Inquiry communities*, *collaborative groups*, *borderline communities* and *interdisciplinary communities*. The first two communities are mainly linked to the first two forms. Borderline communities can be related to a network form. Both the interdisciplinary and inquiry communities are linked to forms connected to specific purposes. We note that the collaborations related to this last form, are mainly those that emerge from an initiative associated with graduate research projects (mainly Ph.D. theses), courses for future teachers or the development of specific materials.

The goals for collaboration are related to the three levels of joint work. At the educational system level, national or multiple-site teacher collaboration programs are typically developed to address national needs and concerns (curriculum reforms and innovative initiatives). At the institutional level, the major concerns for the collaborative work are about developing teachers' MKT or interdisciplinary knowledge, changing teaching practice, building partnerships between teachers and researchers to promote teachers' growth, and linking research to classroom practice. At the classroom level, the teachers' collaboration mainly focuses on implementing specific teaching strategies and/or research-based, reform-oriented teaching practices.

Regarding scale in relation to duration and size, five studies were large-scale in terms of numbers of participants (teachers, researchers, facilitators, schools or other institutions) and lasted over a long period. Among the five projects, three were of a blended form, while the other two were a face-to-face form. If we consider duration only, many of the works provide long-term collaborative experiences (five or more years). However, most of the long-term collaborations involved issues related to changes in the number (and type) of participants, focus and objectives. Some of the issues are caused by changes in economic, institutional or environmental conditions that afford collaborative work over time, while others are related to unpredictable emerging factors.

All reported forms of collaboration involve in-service teachers or prospective teachers (from one or several schools and/or from one or several teaching levels), researchers/facilitators, facilitators (non-researchers), various institutions and/or agencies. In the case of forms related to interdisciplinary communities, teachers or researchers from areas of knowledge other than mathematics or mathematics education also participate. In one way or another, they become the stakeholders for the collaboration and the corresponding settings. Each form, according to its characteristics, objectives and the activity that is developed will shape its settings. The following provides, for each form a brief summary of its settings, affordances and limitations.

The *LS-adaptative form* could be school-based or district-based, depending on the context and needs. These adaptations include the typical LS cyclical process by considering the local social and school contexts, in order to design and implement the corresponding research lessons. The affordance of this adapted LS depends on certain constraints of the local context (for instance, schools' characteristics, teachers' working environments and educational systems) and/or the expertise of the researchers–facilitators involved.

The *researchers–teachers partnership form* usually includes heterogeneous groups of teachers who may work at different educational levels and at different schools. The joint work is mainly related to processes of collective study of various topics (such as the teaching and learning of certain areas or topics of mathematics, the production and dissemination of resources for the classroom among others). The corresponding settings, although variable, generally involve periodic meetings, including instances of production of educational materials and reflection on what has been worked upon.

In some cases, enactment and class observation are also included. We note that IREM and LéA partnerships have shown a great affordance over time. A great number of the papers presented for Theme B relate to this form. Some of them show limitations with regards to sustainability of the continued participation of teachers. In the case of IREMs, teachers' participation is not sufficiently recognised as part of their regular work. Therefore, for teachers, participation in IREM may require extra time outside of school. This fact could become an obstacle to sustaining their participation in the joint work.

Considering the *network* form, it is highlighted that, despite differences in the work and settings of networks, they also have certain similarities, in terms of the complex network of participants or communities that support the collaboration. The communities that interact to collaborate involve diverse types of participants (e.g. teachers from more than one school, researchers, facilitators, agencies) and even, in some cases, the number of them. Different groups of stakeholders choose diverse settings (e.g. collaboration in schools or out of schools, learning communities, communities of practice, inquiry communities, joint work within a community and between communities) and foci for their work (applying innovative teaching materials, developing, and enacting large-scale and long-term collaborations).

In all those cases, there was evidence of the affordances of the settings chosen to carry out small- or large-scale, as well as short- or long-term, collaborations. For those large-scale and long-term collaborations, major limitations are related to processes of compilation and analysis of the information (or resources) generated. Working in a long-term network model could require careful scheduling for organising interactions among the members, as well as for enacting the joint work. However, collaborative work through those forms could have favourable implications for the micro, meso and macro levels.

In the case of *forms connected to specific purposes*, the communities are small-size, and the collaboration usually lasts for a short time. The foci are diverse and graduate-project-related. Usually, the form of the setting is connected to a community of inquiry or an interdisciplinary community. In some cases, these forms include co-planning of teaching activities, enacting lessons, co-teaching, collective reflection and interpretative activities. These forms are valuable in terms of the affordances of collaboration, knowledge and/or resource production, and teacher and student learning. When the project is related to future teachers, this model shows the capacity to promote early collaborative work.

When considering *adaptation of LS* models, scalability and sustainability are crucial, while cultural, institutional and leadership factors have an influence on them. For the partnership model, sustainability is crucial, and institutional and leadership factors are important. Some unique effects of using this model could be its capacity for connecting research and practice, as well as developing knowledge which has theoretical and practical implications. Regarding network models, they could be valuable to document what participants learned, as well as the research methodology applied, and to share what was produced.

Each of the forms presented has shown the ability to promote certain affordances or to have some constraints. In any case, each form is promoted and sustained within a particular social, educational, or cultural context (Lave, 1988). In that sense, one similarity across many of the collaborations, or collaboration structures that have lasted 5 years or more, is that they were government initiatives and/or received sustained funding: that is, they are institutionalised models (Hargreaves & Fink, 2003). Examples of this case are the work developed in the IREM or LéA or even the

Asian LS. All of them show important aspects of sustainability and sharing of outcomes to many teachers, researchers or government agencies.

At the same time, we find that several of the long-term collaborations inform processes of changes over time for adapting their work to new emerging conditions. According to Davis and Sengupta (2020), adaptability and the occurrence of emerging events are linked to the complexity of a phenomenon. In this case, we could say that forms for the collaborative work of teachers are connected to a complex phenomenon characterised as being multi-dimensional (multi-level), adaptable, context-sensitive and to have capacity for coping with new, emerging conditions.

Given the particularity and the few cases presented on blended models, next we will focus on their affordances, limitations, benefits and challenges.

3.6.2.2 Blended Form: Affordances, Limitation, Benefits and Challenges

Below, we present partial response to question 5:

What are the benefits and the challenges that online teacher collaboration poses to the teachers?

The three studies that report blended learning are characterised by working in the form of a complex network. They illustrate potential benefits for the collaborative work among teachers such as removing the obstacles of geographical distances and sharing perspectives, resources or documents asynchronously. The potential of this form to sustain and expand collaborative work is also promising. However, the challenges of accessing appropriate technologies, sharing documents synchronously, facilitating productive online discussions and, in some cases, overcoming language barriers (as is the case for the SRPM or the NGGN networks) need to be addressed. These three studies provide valuable insight into development of blended PD, which emphasise connections between online PD and teachers' classroom practices.

However, it seems to be difficult for documenting how the online immersion experience impacts teachers' practices. Previous research suggests that the following factors are crucial for developing a productive online community of practice: trustful relationships among the members; highly qualified mediators for cultivating knowledge generation; ensuring the contents are closely related to teachers' practices (Lantz-Andersson et al., 2018). At the same time, we emphasise that teacher collaboration in online communities of practice has become a popular and necessary approach due to the COVID-19 pandemic (Ferdig et al., 2020). Thus, being able to advance with studies on collaboration in such formats seems to be a need to be covered. For instance, more studies are needed to explore strategies for developing effective, online teacher communities of practice.

3.6.2.3 Models, Outcomes, Content, and Contexts

Next, we provide responses to the following questions:

How effective are various models for promoting different outcomes?
Which forms of collaboration are appropriate in different contexts?

As reported in the previous sections and synthesised in Sects. 3.4 and 3.5, several papers presented for Theme B framed their outcomes on the teaching dimension related to student activities in the classroom, but also on other dimensions, such as those highlighted by Jaworski et al. (2017). Therefore, the outcomes of collaboration are not only dependent on the chosen forms, but also on other factors, such as the purposes and contextual conditions of the work. At the same time, given the complex and collective aspect of collaborative work, as well as the types of communities involved in the collaboration, the outcomes are extended to other practices. These include research practices linked to the production of diverse resources, and educational practices related to teaching disciplines outside of mathematics, as shown in Table 3.1.

For instance, when considering the application of the adaptive LS form at the micro level of the classroom, we argue that they have a great capacity to promote deep teacher learning and produce useful instructional products. However, these results are not independent of contextual aspects linked to the educational institutions or the experience of the facilitators. While considering the possibility of working simultaneously with heterogeneous groups of teachers, the category researchers–teachers partnership has the capability to produce outcomes related to partners' reflections or learning, to the resources for different school levels or for different mathematical content to be taught. In this case, the outcomes can inform aspects connected to the micro level or to the interactions between different levels of the school educational system. Institutionalised researchers–teachers partnership shows affordances for expanding the outcomes to all three levels (micro, meso, macro).

This form also has affordances for supporting the sustainability of collaborations. Meanwhile, the network form has some possibilities for promoting outcomes related to the micro, meso, macro or all three levels. It depends on the project and the type of the community established. In that sense, it has possibilities to yield similar outcomes to the previous model. However, the network model has other capabilities. Among them, we can mention the affordances for yielding outcomes related to the impact of long-term and large-scale collaborative projects on teacher training or on the educational system. It also has the potential to produce outcomes related to the particularities that teacher training requires when working with interdisciplinary projects. Under this last condition, the model also has the potential to promote outcomes related to the impact that the work of teachers and students may have in the social or environmental contexts close to their schools.

Across different models of collaboration, mathematical content catalyses participants' professional learning. Through working on challenging mathematical objects (both content- and pedagogy-related) and reflecting on teachers' existing knowledge and teaching practice, new understanding of MKT and teaching practice may occur. In the context of collective (or collaborative) interactions, each participant reviews his or her own knowledge and perspectives, and contributes to the co-generation of new shared understandings. Meanwhile, through teachers' collaboration, mathematical objects (e.g. tasks, lesson design) which are more useful or applicable in classroom teaching could be produced. Thus, mathematical objects could be both a vehicle for teacher collaborative learning and the product of teacher collaboration.

For all models, we have highlighted the educational context level with which their outcomes could be connected. In that sense, we offer an answer to the second question selected for this part (which forms of collaboration are appropriate in different contexts?). Regarding this question in a broader notion of context (cultural, social, political), it is important to recognise that collaborative work is a context-sensitive phenomenon. Thus, it is difficult to offer a broader answer, if not impossible. We have presented cases related to the adaptation of LS to some contexts. However, in a very similar social, cultural, and educational context level, some cases were successful while others were not. In any case, as it was discussed, the collaborative work of teachers has a strong relationship with the projects they engage with, and this gives them ample versatility to choose a form, to create a new form or to adapt a known form to a particular context in which the collaboration will develop.

Finally, as pointed out in the bibliography (Sect. 3.1) and evidenced in the developments presented in Sects. 3.2, 3.3, 3.4 and 3.5, we recognise the importance of diversity in collaborative work in terms of objectives, forms and collaboration results. Within the framework of such diversity, we highlight the main contributions of this chapter. Firstly, with the empirical and analytical work carried out, it is possible to establish a categorisation of the forms of collaboration enacted and the corresponding decontextualised models associated with them.

With these categories, it is feasible to unveil the relationships between environments, arenas, subjects, activities and scenarios in collaborative work contexts (Lave, 1988). Secondly, after careful collaborative deconstruction work (Esteley, 2014), detailed and complex pictures of the contexts, forms, outcomes and logics underlying such work are described. With that contribution, on the one hand we consider enriching and extending relevant existing studies (Jaworski et al., 2017; Robutti et al., 2016).

On the other hand, in the process of such deconstruction, it was possible to present voices of participating teachers and researchers that capture the senses attributed to collaboration. After all, we emphasise that, considering the complexity of collaborative work (Davis & Sengupta, 2020), this chapter resorts to its own units of analysis or those of other authors (e.g. Prediger, 2020), as useful tools, to examine such complexity trying not to decomplexify. For instance, as an emergent of the work done in Lisbon, we seek to recover mathematical content in association with didactic systems and its potential to catalyse collaboration among teachers, as described in Sect. 3.5. This chapter also suggests the importance of facilitators and resources of or for collaboration. Both aspects are discussed in depth in the following: Chap. 4, Theme C, and Chap. 5, Theme D.

3.6.3 Further Research Directions

Based on the synthesis of the major findings presented from Sects. 3.2, 3.3, 3.4 and 3.5, this final section provides answers to the questions proposed in Theme B. However, when attempting to answer these questions, some more new issues emerge which need further exploration. This analysis highlights the complexity of the collaborative work of teachers, and the diversity of forms and outcomes achieved (such as multi-dimensionality and multiple levels, adaptability to emergent conditions and context-sensitivity). The realisation of the diversity of the forms of collaborative work represents an interesting contribution to the field and offers insights into future work. However, this diversity also makes comparisons across different collaborative forms and their outcomes a very complicated task, and yields difficulties in creating generalisable knowledge about and patterns in teacher collaboration, if indeed these are possible.

Particularly, if we consider many papers on long-term collaborations, we also note the difficulty of generalising across these contexts and the complexity of comparing approaches and their efficacy. It remains a challenge for the research community to be able to provide generalisable knowledge about and research-based effective models for teacher collaboration across different contexts. It is not yet clear how the results of teacher collaboration are related to school improvements, changes in classroom practices and student learning outcomes. Specifically, when the collaboration work is not job-embedded, there were insufficient outcomes to report impacts on schools where teachers who participated in the collaborative groups work. There is a need to uncover the mechanisms through which teacher collaborations result in their outcomes, and how to measure those outcomes.

Moreover, the unprecedented COVID-19 pandemic has suddenly changed the manner of teacher collaboration tremendously. During 2020, 2021, teacher collaborations through online and/or blended forms of teacher professional development projects became popular, and both advantages and challenges are evidenced (Chan et al., 2021; Huang et al., 2021). Although the pandemic is now over, teacher collaboration will not be returned to previous traditions (Quezada et al., 2020). Developing effective online or blended teacher collaboration models is deemed to be a crucial and challenging task (Desimone, 2020).

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Chapter 4
Roles, Identities and Interactions of Various Participants in Mathematics Teacher Collaboration

Ronnie Karsenty, Shelley Dole, Stéphane Clivaz, Birgit Griese, and Birte Pöhler

4.1 Introduction

Mathematics teacher collaboration often involves additional participants and stakeholders besides the collaborating teachers themselves. These may include facilitators, mathematicians, researchers, administrators, policy makers or other professionals who assume a variety of roles in regard to the endeavoured collaboration. Such actors may have a significant impact on the collaboration's productiveness, either from within the collaboration, for example when diverse perspectives are shared amongst group members with different expertise (Robutti et al., 2016), or

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from the outside, for example when external actors encourage (or hinder) the creation of suitable environments for collaboration (Vangrieken et al., 2015).

The nature of roles taken up by participants in teacher collaboration can vary in different countries and contexts. For instance, in the Lesson Study model, the role of the 'knowledgeable other' alters across cultural environments in which the model is implemented (e.g. Adler & Alshwaikh, 2019; Gu & Gu, 2016; Takahashi, 2014). Moreover, within teacher collaborative groups, the roles of participants may shift over time (Jaworski, 2006; Krainer, 2008).

In this chapter, we aim to explore the roles and identities of various actors involved in mathematics teacher collaboration, as well as the nature of interactions between them. This topic was at the centre of discussions within the Theme C group in the ICMI Study 25 conference. In particular, the group was interested in the role of facilitators of collaboration, as we shall detail below. The papers submitted to the Theme C group aimed to address the following questions, appearing in the Discussion Document published as part of the call for papers towards the conference:

- What is the role of lead teachers, facilitators, mentors and teacher educators in supporting teacher collaboration?
- How are different roles and identities shaped and developed among various actors (teachers, leaders, mathematicians, researchers, etc.) within a collaborative group? How do lead teachers negotiate their dual roles and identities as both teachers and facilitators of peer-collaboration?
- What are characteristics of a good facilitator of teacher collaboration? How can these facilitators be prepared and supported?
- How can different stakeholders impact teacher collaboration?
- What types of learning environments enhance or hinder mutual learning of teachers and other participants in collaborative interactions?

A total of 16 papers were accepted to this theme (henceforth, we refer to these as Theme C papers) and the group included 19 participants, representing 4 continents and 13 countries:[1] thus, we enjoyed a diversity of cultural perspectives that enriched our conversations. Unfortunately, we did not receive any papers from Africa, and we note this as a shortcoming that requires the community's attention.

This chapter draws on the Theme C papers as one source of reviewing the state of the art in researching various roles within mathematics teacher collaboration. Other sources are the plenary talk given by Konrad Krainer at the conference, and current research as reflected in Special Issues, conferences, books and articles dedicated to the topic of professionals involved in collaborative learning of mathematics teachers.

We begin this chapter by reviewing various methodologies for researching roles in collaboration, as revealed in the Theme C papers, as well as from other sources. This review provides general insights on the types of research that are prevalent in the field, pointing also to certain gaps that need to be addressed in the future. Then,

[1] In alphabetical order: Australia, Austria, Brazil, China, France, Germany, Greece, Israel, Norway, Portugal, Slovakia, Switzerland and USA.

we devote the lion's share of the chapter to a central actor in mathematics teacher collaboration: the professional who leads, maintains and supports teachers' collaborative activities. There was a consensus among the Theme C group participants that of all actors who may be involved in a collaboration – be it a professional development (PD) course, a community of practice, a school-based initiative or any other form of collaboration – the role of the person who facilitates the work of the collaborative group is most critical.

This stance is in line with the growing interest of the mathematics education community in the work of facilitators, which began in the late 1990s and early 2000s with sporadic pioneering studies (e.g. Even, 2005; Schifter & Lester, 2002; Zaslavsky & Leikin, 1999, 2004) and has, since then, been accumulating, and more so in recent years (e.g. Borko et al., 2014, 2017; Coles, 2014, 2019; Jaworski & Huang, 2014; Karsenty, 2016; Karsenty et al., 2023; Kuzle & Biehler, 2015; Lesseig et al., 2017; Prediger & Pöhler, 2019; Prediger et al., 2019; van Es et al., 2014). The interest in facilitators is also linked to the need to understand their crucial role in processes of scaling-up PD models (Cobb & Jackson, 2011; Jackson et al., 2015; Roesken-Winter et al., 2015; Zehetmeier, 2015).

At this point, a note on terminology and definitions is required. Various names are used in the literature in reference to professionals who support the learning of mathematics teachers. As Even and Krainer (2014) have noted, the term 'Mathematics Teacher Educator' (MTE) commonly refers both to those who educate prospective teachers (e.g. Appova & Taylor, 2019) and to those who educate practicing teachers (e.g. Zaslavsky & Leikin, 2004). However, in the literature specifically referring to professionals who work with practicing teachers, we find the terms *facilitators* (e.g. Karsenty et al., 2023; Prediger & Pöhler, 2019; Tekkumru-Kisa & Stein, 2017a); *leaders* (e.g. Borko et al., 2014; Elliott et al., 2009; Lesseig et al., 2017); *mentor-teachers* (e.g. Kuzle & Biehler, 2015); *didacticians* (e.g. Coles, 2014; Jaworski & Huang, 2014), *PD providers* (e.g. Even, 2005); *multipliers* of a PD program (e.g. Maaß & Doorman, 2013; Roesken-Winter et al., 2015); *pedagogical instructors* (e.g. Yow & Lotter, 2016) and *brokers* (Eriksen & Solomon, 2022). Sometimes, various terms are used within the same study, for example Lewis (2016), who explored how novice practitioners learn to conduct Lesson Study in the United States, used the terms *teacher leaders*, *facilitators* and *teacher developers* interchangeably.

As we shall show later in this chapter, this multiplicity of terms may result in methodological challenges, particularly when researchers from diverse contexts attempt to communicate about their methods of study. Thus, for the sake of clarity, from now on we use the term 'facilitator', which we define as follows:

> A facilitator of a group of mathematics teachers involved in collaborative work is the person who manages the activities of the group, by supporting teachers' exchange of experiences and new insights, monitoring the discussion, ensuring adherence to the norms set for the collaboration and assisting teachers in forming a trajectory toward the goals set for the group's work.

A facilitator may or may not be the initiator of the collaboration, and s/he also may or may not be the provider of resources (e.g. contents, methods, artifacts) for the teachers' collaborative work. This is appreciably dependent on the specific context of the collaboration at hand. Also, facilitators may come from diverse backgrounds; they could be experienced mathematics teachers, university mathematics educators, mathematicians, researchers, etc., who take on the role described above within a certain group. Hence, being a facilitator is always attached to a particular context: the same person can be a facilitator in one group, a participant in another group and an external observer in yet a third group. Moreover, this role can change within the same group over time, as shown, for example, by Ribeiro (2020).

We explore the role of a facilitator in two consecutive parts of this chapter. Firstly, we focus on ways to conceptualise this role. Within this focus, we provide a brief account of constructs, frameworks and models developed to describe what being a facilitator may entail, and elaborate upon research findings concerning the knowledge, practices and skills needed for productive facilitation of teacher collaboration. We also refer to challenges faced by facilitators while attempting to lead groups of teachers, and touch upon the model of co-facilitation. Secondly, we review the research concerning professional trajectories of facilitators, including their preparation to take on this role, development and changes that may occur over time in different aspects of their work, and existing means and models for supporting them along their work period.

Facilitators of mathematics teacher collaboration do not act in a void. Even the most capable facilitator needs to negotiate and manage the environment in which the collaboration takes place. The complexity of factors associated with the environment of collaboration is the focus of the penultimate section of this chapter, where we consider the interrelations between the environment and the various actors within it. Included in this account is a review of models that specifically relate to the environment of collaboration; how environments are created and shaped by different participants; and the cultural and contextual aspects of collaborative environments in which various participants act. We also refer to institutionally-imposed environmental factors that impact teacher collaboration. Finally, we challenge the facilitator–environment relationships by looking at collaborative environments that operate without a facilitator. The chapter concludes with a short summary, looking ahead at future challenges in this field.

4.2 Methodologies for Researching Roles in Collaboration

In general, research on teacher collaboration reflects diversity of theoretical framing and choice of methodology (Robutti et al., 2016). Methodological issues are particularly important when analysing the knowledge, practices, identities and interactions of people in their different roles within a collaborative group, an analysis that, as we show in this chapter, is complicated and multi-variabled. The literature on mathematics teacher collaboration includes a spectrum of foci, ranging from investigating knowledge facets and affective motivational variables at the teacher level (e.g. Stahnke et al., 2016) to the impact of the collaboration on achievement at the student level (e.g. Lomos et al., 2011).

Within this extensive and diversified body of research, investigations concerning the roles of different actors in teacher collaboration also reveal a span of methods and research perspectives. Specific consideration is warranted to extend and consolidate the growing body of knowledge around methodologies for researching such roles, if we are to understand better this new field of research. This is the focus with which we begin this chapter.

In Sect. 4.2.1, we provide a summary of our analysis of methods, approaches and theoretical perspectives described in the set of the Theme C papers. We further explore the variety of methods employed in these studies by applying the RATE tool (Relevant Actors, Targets and Assignments) presented at the ICMI Study 25 conference (Krainer & Spreitzer, 2020) and elaborated in Krainer, Roesken-Winter & Spreitzer (Chap. 8, this volume), as a means for depicting and identifying essential methodological features. We then compare the results of our analysis with previous surveys of research into teacher collaboration. In Sect. 4.2.2, we address methodological challenges, including lack of commonly shared constructs resulting from the infancy of this field, and highlight the need for studies that go beyond self-studies, in order to develop theory and more operationalisable constructs. In Sect. 4.2.3, we underline tensions observed within several aspects of possible research methods. We conclude in Sect. 4.2.4 by reflecting on the methodological issues identified.

4.2.1 Various Methods Used to Research Different Roles Within Mathematics Teacher Collaboration

Research into the roles of various participants within mathematics teacher collaboration employs a variety of methodologies, grounded in diverse theoretical perspectives. While a full survey of these methodologies and perspectives is beyond the scope of this chapter, we provide here an analysis of methods and perspectives represented in the Theme C papers, which can be seen as an international sample of state-of-the-art research in this domain.

The 16 papers provided descriptions of methods used to explore the different roles that participants undertook within mathematics teacher collaboration. Of this set of papers, 87% used qualitative approaches. The majority of the Theme C papers were written by researchers working in the field of teacher education and, in more than half of the cases, the researcher was part of the studied process. In 67% of the papers, the study included groups of fewer than ten members, or even fewer than three in 40% of the papers. In 73% of the papers, the context was professional development for practicing teachers, whereas initial teacher education or day-to-day work were present in only one or two papers each. Regarding the types of collected data, video or audio recordings of meetings and interviews were the most prevalent means (53% and 40%, respectively) and they were generally transcribed, at least partially. Other types of data included field notes, questionnaires, lesson observations, online forums and protocols. The theoretical underpinning reported by the authors of the Theme C papers varied; however, 93% indicated that their study was situated within a sociocultural perspective, one author took a cognitive approach and one indicated application of a metacognitive approach. In 40% of the papers, the identified themes and categories inductively emerged from the data. In the other 60% of papers, themes and categories were derived deductively from the literature. Two papers employed both inductive and deductive approaches (Nieman et al., 2020; Quaresma, 2020).

In previous surveys that align the theme of mathematics teacher collaboration (Adler et al., 2005; Robutti et al., 2016), the majority of studies were authored by researchers working in the field of mathematics teacher education, most studies were self-studies, qualitative methodologies were applied and the majority of studies were small-scale studies. We see similar trends in the Theme C papers. In order more readily to depict the range of methods used in the Theme C papers we now present selected research papers mapped via the RATE model, which was introduced in the ICMI Study 25 conference (Krainer & Spreitzer, 2020); see also Krainer, Rösken-Winter & Spreitzer (Chap. 8, this volume). RATE assists in identifying the essential features of an interaction, by focusing on the Relevant Actors, Targets and Environments (RATE) of an initiative. The use of arrows serves to indicate the relationship between actors. As applied here, RATE enabled us to make observations related to methodological issues and to summarise key features regarding the different roles involved in the collaboration.

Four studies from Theme C papers were selected for RATE mapping, based on selection criteria aiming to cover a variety of methods, actors, foci and scale: since the majority of papers reported on small-scale qualitative studies, 3 of the 4 selected papers (Clivaz & Daina, 2020; Nieman et al., 2020; Schwarts, 2020) represent this choice: however, these three exemplify different methods of data collection and analysis (e.g. stimulated-recall interviews; field notes; fine-grained analysis of PD interactions), as well as different involved actors (e.g. facilitators; a facilitator's mentor; school principal). In addition, they represent different kinds of relationships between the researcher(s) and the research subjects (the researcher as an external investigator versus a double-role of being a researcher *and* a facilitator of the collaboration). The fourth selected paper (Cao et al., 2020) represents a larger scale study, employing a combination of quantitative and qualitative approaches. Finally, the four papers represent very different cultures, as they come from USA, Israel, Switzerland and China. To broaden the picture further, we also used RATE to map a study outside the set of the Theme C papers (Roesken-Winter et al., 2015). Below we detail the RATE analysis for each of these five studies.

The small-scale study by Schwarts (2020) investigated how one novice facilitator perceived her role within a collaborative, video-based, PD project. The focus of the paper was on one person: however, the project as a whole was broader. Mapped using RATE (see Fig. 4.1), we readily see the relationship between the facilitators, the team that supports them, the researchers and the participating teachers. The arrows indicate that the novice facilitator and the mentor (a senior facilitator) who supported her worked together in a coaching model. The target of the larger collaboration was improvement of mathematics teaching via peer-analysis of video

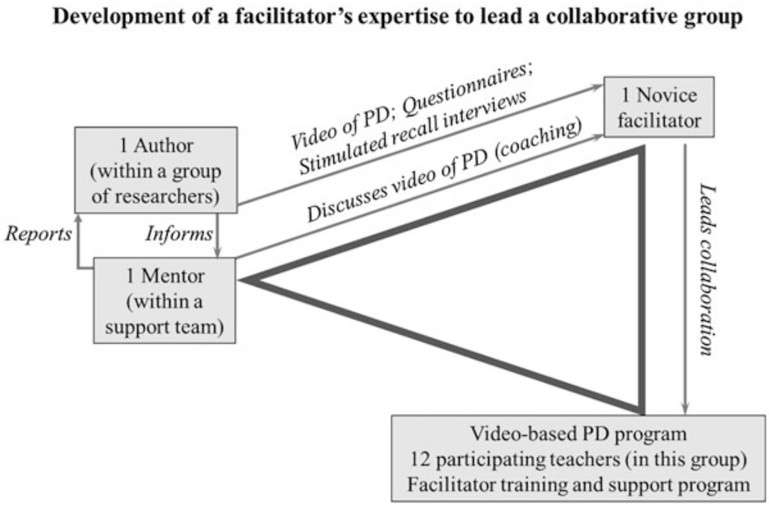

Fig. 4.1 RATE representation for Schwarts (2020)

recordings of lessons taught by unfamiliar teachers. The target of the specific study was to explore the facilitator's development of expertise, thus enabling "insight into how novice facilitators' practices and identities evolve, as well as what is required to support this process" (Schwarts, 2020, p. 540). The data collection means, including questionnaires, stimulated-recall interviews and reports by the facilitator's mentor are also depicted.

The study by Nieman et al. (2020) investigated the roles that one school administrator and two facilitators played in establishing an inquiry-oriented mathematics teacher community. Interviews with the three leaders, as well as transcripts of the conversations during planning meetings, were used and analysed, resulting in categories that emerged from the data. The results revealed the influence of the relevant environment and its important role. The influence of the broader community on how leaders see their roles was analysed at many levels: the teachers, the school and the normative expectations set for teachers and leaders in the USA. Mapped onto RATE (Fig. 4.2), we see the three authors following a school leadership team, comprising the principal and two facilitators of professional learning, collaborating with the target of developing a strong professional community of middle school teachers.

Clivaz and Daina (2020) reported their micro-analysis of teachers' interactions while participating in a mathematics lesson study group. Through the RATE mapping (Fig. 4.3), we see eight teachers in the study collaborating with the four researchers and two facilitators; one of the researchers is a facilitator and one of the teachers is also a facilitator. The role of the school was to enable the collaboration through allocating time to the teachers; similarly, the role of the university was to allocate time for work to facilitators and researchers. The focus of the collaboration

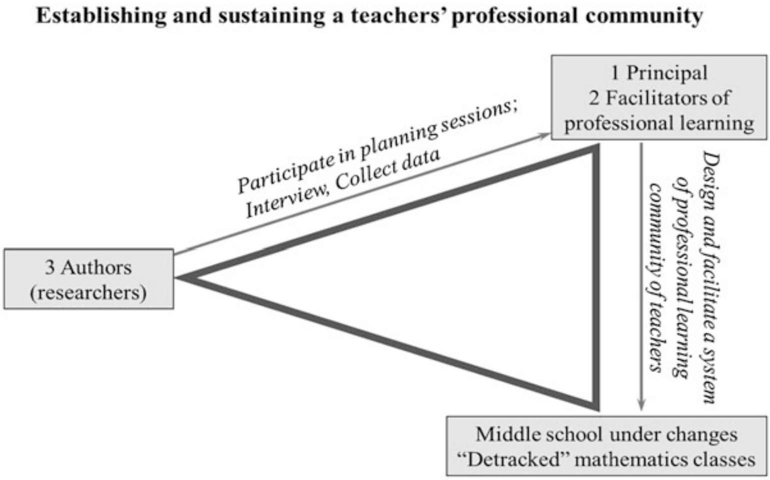

Fig. 4.2 RATE representation for Nieman et al. (2020)

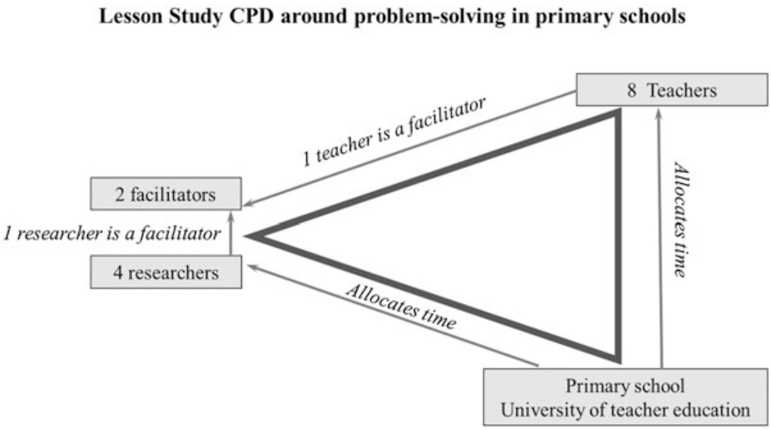

Fig. 4.3 RATE representation for Clivaz and Daina (2020)

was teaching oriented towards problem solving. In this study, video-recorded interactions between teachers and facilitators were analysed to interrogate the influence of those interactions on the development of teachers' mathematical knowledge for teaching. An existing dialogic framework was adapted to support the development of categories, in order to identify the "specific dialogic role of each facilitator and of each teacher during the phases of the lesson study process" (Clivaz & Daina, 2020, p. 465), which was inferred from the analysis of the interactions. The small scale of the study enabled the development of a very fine-grained analysis of the roles of the participants in this Lesson Study research.

The three studies presented above are all small-scale studies, with a focus on 12 or fewer participants. In contrast, the study reported by Cao et al. (2020), with a relatively high number of teachers ($n = 72$) from three cities in different regions of China, would be regarded as a large-scale study. To distinguish it further from any other Theme C paper, its focus resided in mathematics teachers' daily interactions and collaborations in middle schools, in the absence of any designated facilitators. Data collection included questionnaires which were statistically analysed, but also semi-structured interviews (not presented in the paper). The survey methodology employed in this study is arguably of a typical 'classic' nature, but at the same time uncommon in the context of researching roles and identities within collaboration. The RATE mapping of this study (Fig. 4.4) shows authors surveying 72 teachers from different middle schools in China regarding informal teacher collaboration, and captures the context in which Chinese teachers interact and collaborate, and the influence of groups such as the Teaching Research Groups and the Lesson Planning Groups, as well as the influence of the specific Chinese school collective culture. Despite the rich results reported, the authors concluded that, "results from the semi-structured interviews are needed to understand how teachers seek learning and growth through interactions with colleagues" (p. 451).

Moving beyond the Theme C papers, we now consider another unique study to exemplify further variety in methodologies used to research various roles in teacher collaboration. The study by Roesken-Winter et al. (2015), in which 12 facilitators participated, can be distinguished from more prevalent, small-scale studies, due to its

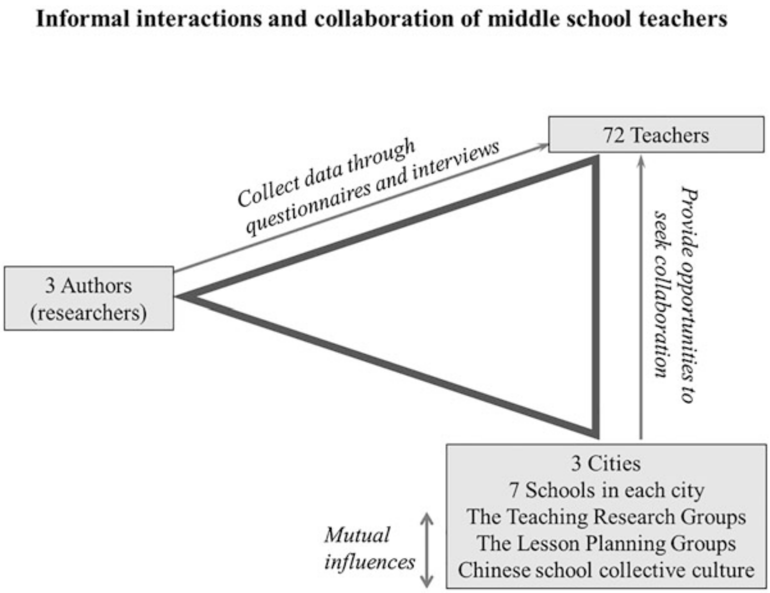

Fig. 4.4 RATE representation for Cao et al. (2020)

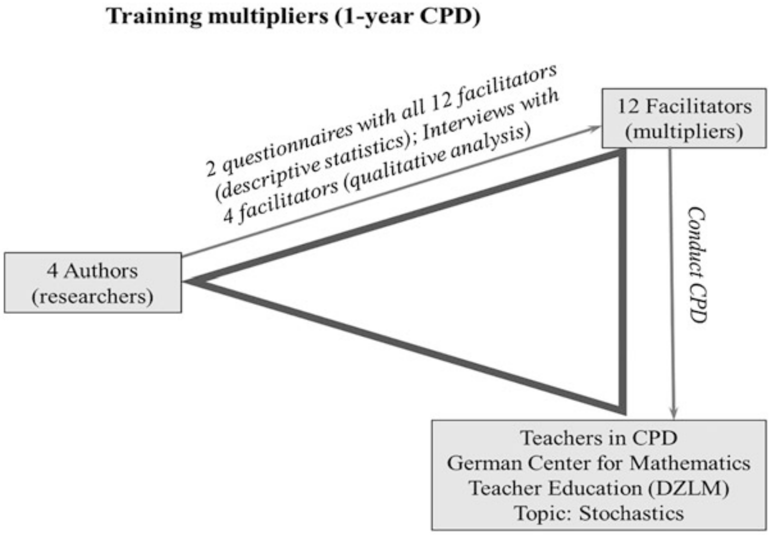

Fig. 4.5 RATE representation for Roesken-Winter et al. (2015)

comprehensive methodology. The time scale and the complementarity of the descriptive statistical analysis of the two questionnaires (at intervals of 4 months) on the one hand, and the qualitative treatment of the semi-structured interviews on the other, allow examination of many elements about the Continuous Professional Development (CPD) program in which the participants collaborated. However, we acknowledge that details of roles, identities and interactions between the various participants in this study remain challenging to extract. Mapping this study onto RATE (Fig. 4.5) shows both the purpose of the study, i.e. to conduct an in-depth investigation on a one-year CPD course for facilitators, and the purpose of the whole project, i.e. to support facilitators in conducting their own CPD courses for mathematics teachers via the German Centre for Mathematics Teacher Education (DZLM). The topic at the focus of the CPD was stochastics, a relatively new topic within the middle-school mathematics curriculum.

The five studies pictorially represented above by RATE reveal a range of participants taking on various roles within mathematics teacher collaboration and the application of a range of methodologies used to investigate these roles. To round out our discussion of the various methods used to research different roles and their underlying theoretical perspectives, we consider the methods of meta-studies.

First, we refer to Krainer et al. (Chap. 8, this volume) who presented and used the RATE model as a tool to assist in identifying the relevant actors, targets and environments in teacher collaboration across seven selected studies. To apply their model, they selected research studies for analysis according to strict criteria: a clear focus on the topic of mathematics teacher collaboration (via keywords); recency of publication (2018–2019); journal quality; and geographical spread (one or two papers from each continent). As a result of mapping these studies onto RATE, the authors formulated six observations, four of them related to methodological issues: that small-scale qualitative research predominates; that most research is conducted by teacher educators studying the teachers with whom they work; that only a few initiatives describe the context and relevant environments having a potential impact on the initiative; that most initiatives describe extensively their particular approach. The analysis of the five studies we have conducted, presented above, concurs to a large extent with the observations resulting from Krainer et al.'s meta-study.

Second, we refer to the survey by Robutti et al. (2016). The choice of papers for inclusion in that meta-study was made according to publication date (2005–2015), sources (mathematics education journals, conference proceedings, books and handbooks) and keywords. This led to a set of 316 sources that then were analysed, resulting in the identification of three main themes. We note that one of the themes was entitled, "Theories and methodologies framing the studies", indicating the importance of considering this aspect of this field and pointing to possible gaps.

In this sub-section, we have highlighted the variability in choice of methodology employed in research into mathematics teacher collaboration through a brief analysis of selected studies. We found the RATE mapping to be useful for exemplifying key methods and as a means for visually depicting the essential elements of studies on teacher collaboration while drawing attention to the various participants. Our analysis concurs with methodological issues identified in previous studies (e.g. Adler et al., 2005; Robutti et al., 2016; Krainer & Spreitzer, 2020), mainly that small-scale studies dominate and that research designs are predominantly qualitative. This analysis has also highlighted the centrality of the facilitator in teacher collaboration, as shown in the RATE diagrams, in most of the analysed papers facilitators played a critical role. We now turn our attention more specifically to methodological considerations regarding the role of the facilitator, broadening our view to include mathematics teacher educators (MTEs) in general.

4.2.2 Methodological Challenges, Issues and Considerations Framed Within Research on Mathematics Teacher Educators

As with any other research domain, research in mathematics education can be viewed as accumulating through trends, resulting from the recognition of important issues for which the community has yet to develop deep understanding (Karsenty, 2020). Nowadays, we witness the rise of a research trend that can be referred to as *studying the profession of the mathematics teacher educator (MTE)*. Our analysis in the previous sub-section indicated the centrality of facilitators in initiating, promoting and supporting teacher collaboration. Combining this with the fact that facilitators are a specific and important sub-group of MTEs,[2] our interest in facilitators is well aligned with the current trend of researching MTEs, which is the focus of our discussion here.

As mentioned above, although early work in researching MTEs began around the turn of the millennium (e.g. Even, 2005, 2008; Goos, 2009; Jaworski & Wood, 2008; Zaslavsky & Leikin, 2004), it is only in recent years that the literature is considerably growing (e.g. Borko et al., 2014; Beswick & Chapman, 2020; Lesseig et al., 2017; Roesken-Winter et al., 2015). The recent initiatives of conference groups dedicated to this topic – for instance, ETE (Educating the Educators) conferences beginning in 2014, the new Thematic Working Group in CERME named "the professional practices, preparation and support of mathematics teacher educators" formed in 2022 and several Special Issues in leading mathematics education journals (e.g. Jaworski & Huang, 2014; Krainer et al., 2021; Tekkumru-Kisa & Stein, 2017b) – indicate that this trend's apogee is still ahead of us. In this sub-section, we focus on MTEs, and explore methodological challenges, issues and considerations that arise in the accumulated research around this role.

One of the characteristics of a research domain in its early period is that theory-building is in the making, and this is often reflected in the following phenomena: (a) a lack of commonly shared constructs; (b) a search for conceptual frameworks, research methods and strategies that would be appropriate for the developing field; and (c) the predominance of self-studies. In the following, we provide a brief account of how these phenomena are manifested in the research domain, focusing on MTEs as supporters of teacher learning and collaboration.

4.2.2.1 A Lack of Commonly Shared Constructs

To begin with, research on MTEs is far from having a shared language with which to speak. As shown in the introduction section above, apart from the term mathematics teacher educator, many other different terms are used to describe the role of a person who promotes and manages teacher learning in a collaborative group, such as: facilitator; mentor; teacher leader; PD provider; didactician; pedagogical instructor. This abundance of terms not only reflects cultural differences between research projects across the world, but may also sometimes imply, for individual researchers, a choice of a particular theoretical lens. This leads to a challenge when researchers attempt to communicate their work to peers. To illustrate this, we refer to two anecdotes that we have recently witnessed.

1. At the ICMI Study 25 conference in Lisbon, it was apparent that some terms had very different meanings to participants from various countries. For instance, for some researchers, the label 'facilitator' related to a role which is hierarchically

[2] As already noted, the term MTEs can refer to educators of prospective, as well as practicing teachers. Here, we focus on facilitators, i.e. MTEs who lead groups of practicing teachers (see Sect. 4.1).

lower, in terms of prestige and status, than the role of a teacher educator, whereas, for other researchers, a 'facilitator' could be a senior university researcher who is head of the project. These different interpretations produced some awkward moments in the peer conversation, resulting in the consensus that a glossary is needed for the international community of researchers, in order to reach a shared understanding of objects under investigation within this domain. To the best of our knowledge, such glossary does not yet exist, and creating one can be a challenge for future researchers aiming to review the field.

2. Two of the co-authors of this chapter, who submitted a manuscript about the professionalisation process of PD facilitators, received a comment from a reviewer asserting that the term 'facilitator' "belongs to the constructivist paradigm", which was not the explicit theoretical stance taken in the research. This seemingly minor detail suggests that the choice of terms within this emerging field carries a potential for miscommunication about theoretical lenses underlying studies, which is a point we find worth thinking about.

Yet, a lack of shared terms is not the only methodological challenge we witness in the field of studying the role of MTEs. It seems that the constructs used in various studies in regard to this role are ambiguous and may have different interpretations in diverse contexts. Thus, we hear about MTEs' knowledge, skills, practices, strategies and moves without a clear shared definition of these constructs and what differentiates them from one another. For instance, Jaworski (2008) described the connection between MTEs knowledge and mathematics teachers' knowledge as a *partial overlap* (Fig. 4.6), which implies that MTEs' are not necessarily former (or present) teachers, whereas others (e.g. Perks & Prestage, 2008; Appova & Taylor, 2019) portray MTEs' knowledge as *including* teachers' knowledge (for further examples of models pertaining to MTEs' knowledge, see Karsenty, 2020).

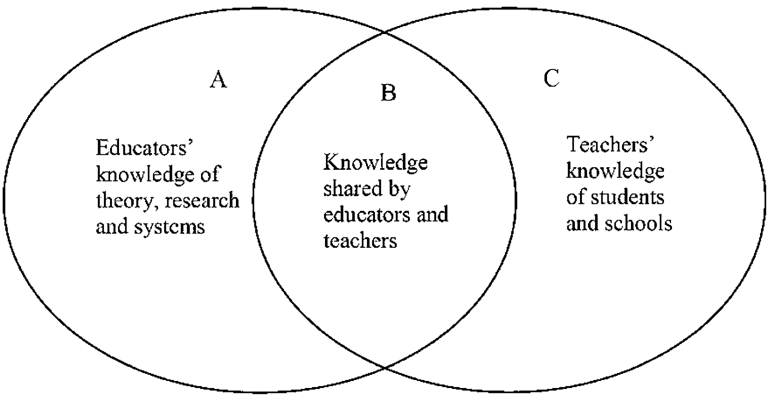

Fig. 4.6 Interconnections between teacher educators' knowledge and teachers' knowledge (Jaworski, 2008, p. 336)

Another example is the use of the construct of *practice*: in van Es et al. (2014), the construct of facilitators' practice is characterised as a *set of moves* (e.g. the practice of 'sustaining an inquiry stance' comprises the moves of highlighting, lifting up, pressing, offering an explanation, countering and clarifying; see also Sect. 4.3.2). Similarly, Lesseig et al. (2017) define facilitation practices as useful *skills* for leaders to enact in PD sessions (e.g. monitoring, selecting and sequencing teachers' sharing of their solutions for mathematical tasks). In contrast, Appova and Taylor (2019) related to MTEs' practices not as moves or skills, but rather in terms of *knowledge resources* that expert MTEs draw upon (e.g. the MTE uses student work; refers to professional literature; connects to the curriculum). Thus, it seems that the research of the MTE profession is still in need of clarifying and developing further understanding of core constructs.

4.2.2.2 The Pursuit of Conceptual Frameworks, Research Methods and Strategies

A methodological challenge typical of an emerging research domain is *how* to develop useful frameworks (for examples of such frameworks, developed for researching the profession of MTEs, see Sect. 4.3.1 below). One of the very few works that address this challenge of creating frameworks is that of Konuk (2018). Konuk described four main approaches by which frameworks for conceptualising the knowledge and/or practices of MTEs may be generated: the standards-based approach; the practice-oriented approach; the inquiry-based approach; and finally, the method of extending or revising existing frameworks. Due to space limitations, we do not detail here the characteristics of each of these approaches (which can be found in Konuk, 2018; see also Karsenty, 2020, for a short summary), but focus briefly on the last approach of extending or revising existing frameworks, which is becoming more and more prevalent.

Prediger et al. (2019) presented three strategies for setting research agendas (i.e. developing design principles, research practices and methods) when moving from the classroom level to the PD level. These strategies are *lifting*, *nesting* and *unpacking*. Karsenty (2020) suggests that the lifting and nesting strategies, and sometimes their combination, are also useful in the process of forming new frameworks. According to Prediger et al. (2019), lifting a research practice means that certain types of research questions and/or methods from the classroom level are implicitly or explicitly transferred, and adapted to the PD level and applied in an analogous way. Similarly, Karsenty (2020) refers to lifting a framework as the idea of creating an analogy between an existing framework at the teacher level and a new framework at the MTE level. An example of this idea is how the Mathematical Knowledge for Professional Development (MKPD) framework (Borko et al., 2014; see Sect. 4.3.1) lifts the Mathematical Knowledge for Teaching (MKT) framework (Ball et al., 2008). The nesting strategy is defined by Prediger et al. (2019) as using a similar structure in different levels, so that the content, design principles or practices of one level are included as a component in the next level. Borrowing this strategy to the forming of frameworks, Karsenty (2020) asserts that nesting a framework (or a model) that was originally created for the teacher level within the MTE level results in a complex structure, where the elements that comprise the knowledge and practices of MTEs usually include, as a subset, the elements that comprise the knowledge and practices of teachers. An example of this strategy can be found in the work of Zaslavsky and Leikin (2004), who nested Jaworski's (1992) Teaching Triad of mathematics teachers within the Teaching Triad of MTEs. It is reasonable to anticipate that in the next few years, we will witness plentiful research practices (including frameworks, methods, strategies, etc.) to study the profession of the MTE, that extend, adapt or revise existing research practices used in the already well-developed study domain focusing on the profession of the mathematics teacher.

4.2.2.3 The Predominance of Self-Studies

Even and Krainer (2014, p. 203) have noted that, "most research on the professional education and development of mathematics teacher educators includes reflections of teacher educators on their own personal development". In this type of research, known as self-studies (i.e. "the study of oneself, in particular, one's thinking and actions" – Chapman, 2008, p. 1), MTEs are researching themselves or are part of the research team. Self-studies are not only an effective way to develop expertise among MTEs (e.g. Baker et al., 2019), but are also the field's way to accumulate first-hand knowledge about what is involved in the study of MTEs' knowledge and practices (e.g. Bragg & Lang, 2018; Coles, 2014; van Es et al., 2014). Self-studies enable researchers to pursue the unpacking and the operationalisation of core constructs in the field, which is, as described above, necessary for its advancement. The tendency to engage in self-study is therefore typical of the early stage in which the MTE literature currently stands, and was also reflected in the ICMI Study 25 conference, where many of the papers presented about the role of MTEs (see Sect. 4.2.1) were self-studies.

However, in the ICMI Study 25 conference there were also several researchers who presented work outside the realm of self-study, that is who explored the role of MTEs from 'an external' perspective (see, for example, Pöhler, 2020; Schwarts, 2020). Such research is gradually accumulating in the past decade, as exemplified in the work of Borko et al. (2014), Jackson et al. (2015), Lesseig et al. (2017), Karsenty et al. (2023), Prediger and Pöhler (2019), and others. This might be an important step towards the maturity of this research field and, moreover, necessary for the design of large-scale implementations of programs that centralise teacher collaboration, facilitated and supported by professional and skillful MTEs. In the words of Tekkumru-Kisa and Stein (2017b, p. 2), we need, "to learn how designers' foci, tools, and resources evolve as they transition from a single program to one that is scalable beyond the initial developers' vision".

To sum up, we illustrated how three phenomena typical of a developing research field (i.e. the lack of commonly shared constructs; the search for appropriate frameworks and methods; the high proportion of self-studies) are manifested in the study of MTEs' role. As mentioned earlier, the roles of other participants in teacher collaboration are still under-researched (we briefly touch upon this in Sect. 4.5.3 below). However, it can be expected that similar methodological challenges would be encountered and reported as this research domain progresses.

4.2.3 Tensions Within Research Methods

We now turn to the tensions that researchers negotiate as they employ research methods to investigate various roles within teacher collaboration. We draw in part on aspects of the keynote presentation by Prediger (2020) at the ICMI Study 25 conference (see also Prediger, Chap. 6, this volume) for this purpose. First, we consider tensions in investigating generic versus content-specific aspects of roles within teacher collaboration. We then reflect on specification of roles versus the complexity of roles. Lastly, we mention tensions that may arise between taking a situated perspective, focusing on analysing the practices of an individual and attempting to draw more general conclusions pertaining to groups of teachers.

4.2.3.1 Tension in Investigating Content-Specific Versus Generic Aspects of Practice

Prediger (2020) described a study of teacher collaboration, where a community of inquiry was established for mathematics teachers and special needs teachers, in order to develop inclusive teaching practices of mathematics in heterogeneous classrooms. The episode upon which we draw is associated with helping students with learning difficulties in the area of basic arithmetic operations. The following citation, by one of the teachers in this community, represents one outcome of the collaboration, after 9 months of intense work:

> Suleika can calculate the subtraction well, only the carries pose problems for her. But we can handle this successfully by differentiated tasks: I only give her subtractions without carries. (Prediger, 2020, p. 4)

Superficially, we see the work within this collaborative group resulting in a teacher believing that the approach for differentiation and teaching inclusively in a heterogeneous classroom is to provide students with individualised exercises that guarantee their success. From a methodological perspective, this incident may be evaluated in different ways, depending on the evaluator's viewpoint. If analysed from a *generic viewpoint*, for example by using the TALIS (Teaching and Learning International Survey) distinction of different types of teacher collaboration (OECD, 2014) or the Gräsel et al.'s (2006) framework, one could have assigned a highly-valued category to this teacher collaboration, since the teachers have been working collectively in an intensive manner over a long period of time and took an active part in collaborative professional learning.

However, from a *content-specific viewpoint*, the conclusion drawn with respect to Suleika's learning is that the collaboration could benefit from an expert opinion, helping teachers to find ways to foster the student's learning, going beyond just task completion. This small episode reveals a possible tension, depending on whether the theoretical and methodological approaches are grounded in a content-specific perspective or not. One might go as far as questioning the merit of reporting such data from a mathematics teacher collaboration, if it is being investigated solely from a generic viewpoint. This perspective prompted thought as we discussed methodological considerations in our Theme C meetings. However, whilst other tensions were enunciated in some of the Theme C papers (e.g. Zhao et al., 2020; see Sect. 4.5), the tension of content-specific versus generic aspects of practice was not explicitly addressed in the Theme C collection of papers.

4.2.3.2 Tensions Between Specifying Roles and Attending to the Complexity of Roles

Methodological issues also arise when considering a potential tension between specifying roles in the collaboration and attending to the complexity of roles. Returning to the collaborating teachers in Prediger's (2020) study, we see the benefits of the extended collaboration involving an experienced facilitator, who brought in her perspective and well-designed classroom material to help progress Suleika's learning and the teachers' mathematical knowledge for teaching. This hints to the tension stated above. In relation to the specification of roles, the teachers are involved as learners – the facilitator is learning as well, and both bring in their expertise. The complexity of these roles calls for attentive scrutiny as to what may be the focus of the investigation in such situations. One could imagine investigating the outcome of the teachers' collaboration with the facilitator; the learning pathways of the teachers; the learning pathway of the facilitator; the degree to which the facilitator's expert viewpoint is explicitly brought in; the positioning of the teachers as experts of their classrooms; and more. The question then is: how can roles be specified and, at the same time, how can the complexity of roles be captured? We further ask: what are adequate research methods for resolving this tension? While the answer is likely to depend on how the different aspects of the study are operationalised, what seems apparent is that resolving the tension of specification versus complexity of roles may require qualitative methods such as observations, interviews and the use of narratives that allow for a fine-grained and sensitive analysis, rather than employing quantitative methods.

4.2.3.3 Tension in Pursuing a Situated Research Approach While Investigating Groups of Teachers

A further methodological tension arises in the pursuit of investigating groups of teachers through a situated research approach. The more a situated approach is pursued, the more the practices of individual teachers, facilitators and other actors are at the focus of investigation, usually involving qualitative research methods (Depaepe et al., 2013). However, within a cognitive methodological perspective, constructs such as knowledge and affective-motivational variables would likely be investigated on a larger scale, involving groups of teachers, often by applying quantitative methods with the use of standardised tests or scales. It is acknowledged, however, that a dichotomy of qualitative versus quantitative methods is not a clear demarcation. In their systematic review on empirical mathematics education research pursued from a situated perspective, Stahnke et al. (2016) distinguished between studies elaborating on perception, interpretation and decision-making, and pointed out how a variety of qualitative and quantitative approaches can be realised.

In the preceding paragraphs, we identified three sources of methodological tensions that have arisen in researching roles in teacher collaboration. It appears that a systematic review, with a focus on research methodologies, differentiating between confirmatory studies, intervention studies and case studies would help progress research into the roles of various participants within teacher collaboration. In the next sub-section, we present concluding comments to summarise our discussion on the methodological aspects of such research, while pointing at needed future directions.

4.2.4 Methodological Issues in Researching Roles Within Teacher Collaboration: Looking Ahead

From our analysis of methodologies for researching roles in collaboration, we can point to a need for more research from 'an external' perspective (i.e. going beyond self-studies) and for more large-scale studies. There is also, however, room for further research using observations or interviews, as these allow fine-grained analysis of the roles of various participants in teacher collaboration. As such, there appears to be a need for the development of explicit research agendas that incorporate the individual and move 'upstream' to consider the wider range of participants in the collaboration, but also for a systematic review of research methodologies directed 'downstream', in order to favour the development of commonly-shared constructs.

Methodological issues arise as a result of the choice and the development of theoretical frameworks. Various theoretical perspectives on roles within teacher collaboration impact the methodological choices and approaches. There is a variety of theoretical frameworks that can be applied with respect to investigating roles in teacher collaboration, and one might ask whether we even need more theoretical frameworks, or perhaps more specified methodological approaches that allow for fine-tuning of existing theories. Maybe this is not an 'either/or' question, but more about where we stand and where we want to go. We hope that our discussion above provides a contemporary basis for thinking about these questions. We now conclude our inspection of methodological issues in researching roles of various participants in mathematics teacher collaboration, and proceed with looking specifically at the important role of one significant participant, namely, the facilitator of collaboration.

4.3 Conceptualising the Role of Facilitators in Promoting Mathematics Teacher Collaboration

As already noted, it is universally agreed that facilitators take a central role in teachers' professional development and collaboration (e.g. Borko et al., 2015; Even, 2008; Jackson et al., 2015; Jaworski, 2008), both from a systemic perspective, as they are key to scaling up innovations to a larger number of classrooms (as addressed for example by Banilower et al., 2006, Borko et al., 2011; Zehetmeier, 2015), and from a local perspective, as they catalyse the individual group's expertise and actions into sustainable progress (Jackson et al., 2015; Tekkumru-Kisa & Stein, 2017a).

In this part of the chapter, we review contemporary research, as well as suggested future directions, regarding the conceptualisation of the facilitator's role in designing, maintaining, and supporting collaborative activities for mathematics teachers. We begin in Sect. 4.3.1 with a brief account of constructs, frameworks, and models developed since the turn of the millennium to describe what this role might entail. Then, in Sect. 4.3.2, we elaborate on what research so far tells us about the knowledge needed for productive facilitation of teacher collaboration, and which practices and skills for facilitation were examined in various contexts. Challenges faced by facilitators, as found in current studies, are described in Sect. 4.3.3. Finally, a less-studied type of facilitation, where co-facilitators work together to support teacher collaboration, is presented in Sect. 4.3.4.

4.3.1 A Brief Account of Constructs, Frameworks and Models to Conceptualise the Facilitator's Role

In the past two decades, several researchers have suggested various conceptualisations of the role of facilitators (specifically, their knowledge and practices), introducing new constructs, frameworks and models. These conceptualisations vary in grainsize and detail. Even (2008) introduced the term *knowtice*, a combination of 'knowledge' and 'practice', as a construct that describes the knowledge that PD facilitators need to learn and develop. She suggested that this construct is comprised of four elements: knowledge of mathematics; knowledge of current views of mathematics teaching; knowledge of current views of teaching teachers and of teacher learning; ways of educating practicing mathematics teachers. Borko et al. (2014) have introduced the framework of *Mathematical Knowledge for*

Professional Development (MKPD), which builds on the well-known framework of MKT (Mathematical Knowledge for Teaching, developed by Ball et al., 2008, for characterising the knowledge required by mathematics teachers). MKPD includes specialised content knowledge, pedagogical content knowledge and learning community knowledge. Another framework that builds on MKT is *Mathematical Knowledge for Teaching Teachers* (MKTT), proposed by Zopf (2010), which, although introduced for the pre-service level, may also be useful for conceptualising the work of PD facilitators. MKTT includes: knowledge on how to unpack MKT for teachers; knowledge on how to develop a precise mathematical language; a connected mathematical knowledge; knowledge of the epistemology of mathematics. A more recent framework for facilitators' knowledge, published by Lesseig et al. (2017), lays emphasis on three elements: identifying the learning terrain for teachers in the PD and articulating consequent learning goals; orchestrating discussions; and cultivating norms that support the attainment of goals. Karsenty (2020) suggests the *meta-lenses framework* (MLF) for unpacking knowledge and practices needed for facilitators in a video-based PD. MLF includes six components: the PD's agenda, ideas and norms; explicit and implicit facilitator goals; PD tasks and activities; facilitator–teacher interactions; facilitator dilemmas and decision-making; facilitator beliefs about mathematics teaching, about how teachers learn and about the facilitator's role. Another framework for exploring facilitators' decision-making, and their professionalisation over time, is the ROGI framework (Karsenty et al., 2023), which adapts Schoenfeld's (2010) ROG framework, i.e. the triad of teachers' *resources*, *orientations* and *goals,* into a quartet of facilitators' *resources*, *orientations, goals* and *identity*.

Other researchers have proposed visual models to describe the interrelation between various aspects concerning the role of facilitators (or MTEs in general). For example, the tetrahedron structure suggested by Perks and Prestage (2008) relates to vertexes, such as practical wisdom, professional traditions, and teacher knowledge. The Teaching Triad of MTEs (Zaslavsky & Leikin, 2004, based on Jaworski, 1992) includes challenging content for mathematics teaching, management of mathematics teachers' learning and sensitivity to mathematics teachers.

As can be seen, these frameworks and models differ from one another regarding their foci, yet, in general most if not all of them refer both to academic knowledge and to social and interpersonal skills needed by facilitators. In the next sub-section, we review empirical research about these aspects in various contexts.

4.3.2 *The Facilitator of Collaborative Teacher Learning as a Professional Expert*

Following our definition of a facilitator, we take the position that facilitators are not necessarily the ones who determine the goals for teachers' collaborative work, but rather the role of facilitation is focused around organising and supporting the activities of the group towards agreed goals. Such goals may be set by the teachers themselves or by some external source, which can be either the PD project represented by the facilitator or educational authorities at the local, regional or national level. For example, facilitators may be expected to promote innovations, such as pedagogy oriented towards problem-solving or language sensitivity (e.g. Prediger & Pöhler, 2019), introduce new teaching content such as statistics (e.g. Kuzle & Biehler, 2015) or present and support the use of technology for distant learning, a challenge that became especially pressing since the global spread of COVID-19 (Bakker & Wagner, 2020).

The role of facilitation emphasises skills such as guiding the exchange of ideas, monitoring the discussions and elevating reflections within the group of teachers. In filling that role, facilitators need to maintain a careful balance between communicating at eye-level and leading the group towards the pre-determined goals. In sum, although the collaborative work under facilitation is likely to have clear goals, the process for reaching these goals is often under-defined, and it is the facilitator's responsibility to orchestrate it.

These expectations from facilitators mean high demands on their professional knowledge and expertise, which contrasts with the fact that, as we discuss in Sect. 4.4 of this chapter, there are hardly any institutionalised paths to become a facilitator. The frameworks mentioned in Sect. 4.3.1 imply that facilitators, like teachers, must master academic knowledge aspects as well as pedagogical and social skills. For facilitators, these knowledge and skills need to be framed in terms of educating adults (Knowles, 1990), specifically, mathematics teachers in various contexts. This undertaking involves, for example, unpacking the learning goals and their rationale, since teachers are more likely to participate in PD activities if they see their relevance to their teaching (Pinto & Cooper, 2017). Successful facilitation includes promoting knowledge domains for teachers (Park Rogers et al., 2007), so in PD contexts, these domains are to be viewed from the perspective of leading groups of teachers rather than that of teaching students. However, facilitators' knowledge of the PD content comprises within it also knowledge domains at the teacher level (e.g. as described in the MKT framework), which can be seen as *nested* in the knowledge at the PD level (Zaslavsky & Leikin, 2004; Luft & Hewson, 2014; Perks & Prestage, 2008; Prediger et al., 2019). In addition, Borko et al. (2014) stress that facilitators, "must hold a deeper and more sophisticated knowledge of mathematics than their colleagues, just as teachers must hold a deeper and more sophisticated knowledge than their students" (p. 165). For instance, facilitators ideally have an extensive *specialised content knowledge*, in reference to different solution strategies for tasks in a certain content area under analysis, as well as knowledge about the tasks' conditions and interconnections, and common difficulties which solvers may encounter. Or, if the PD content involves various mathematical representations, the facilitator is expected to have robust knowledge about their advantages, constraints and generalisations. Indeed, in a perfect world, teachers should possess such knowledge as well, but the requirements are stricter for facilitators who are often regarded as experts. According to the MKTT framework (Zopf, 2010), the mathematical knowledge of those who work with teachers needs to be "unpacked, connected, language focused, and discipline oriented" (p. 185). This means that facilitators should have a solid knowledge of mathematical structures and procedures, and be experts in choosing appropriate interpretations, representations and examples, and in highlighting interconnections between them, while at the same time supporting and developing a language that does not compromise mathematical integrity, but is still practical in the classroom. Moreover, facilitators should know how teachers learn to teach the content, which requires them to be familiar with contemporary models of teacher learning with its conventions and restrictions. At best, facilitators have passed through a special training, focusing on how teachers' professional development can be promoted in general and in regard to the PD content (as described for example in Borko et al., 2015; Maaß & Doorman, 2013; Kuzle & Biehler, 2015; see Sect. 4.4 below).

It is worthwhile to explore not only the knowledge of successful facilitators, but also their practices and orientations, i.e. what they *do* in collaborative settings in order to promote teacher professional growth, and which attitudes they adopt when they "support learning toward carefully defined goals without undermining learners' sense of agency" (Tekkumru-Kisa & Stein, 2017a, p. 3). A central issue is promoting fruitful discussions on mathematics and its teaching (Karsenty et al., 2023). For example, this may be done by lifting Smith and Stein's (2011) five practices from the classroom level to the PD level, as was done by van Es et al. (2014). By utilising facilitation moves (such as highlighting, lifting up, pressing, etc.), van Es and her colleagues suggest that facilitators can direct participants' attention to interesting and important mathematical ideas, invite them to elaborate their own ideas and encourage them to explore various explanations and interpretations.

Similar facilitation moves were considered in some of the studies presented in Theme C (e.g. Griese et al., 2020; Schwarts, 2020). In addition, van Es et al. (2014) refer to facilitator practices that explicitly address orienting the group towards the task, maintaining the focus on the mathematics and supporting group collaboration. They conclude that *coherence* of facilitator practices is essential for productive professional development. This means that it is not enough for facilitators to be able verbally to express an attitude of debate and exchange in general, for instance by utilising conversational prompts that spark the discussion, but they need also to keep the intended goal in mind and be aware of the different paths that lead to it. Re-focusing the group may also mean to call a halt to certain threads of the discussion, and to follow up others. This needed coherence is mirrored also in the Theme C papers. For example, Pöhler (2020) found that, "identifying facilitation moves can be insufficient" (p. 522) and that identifying content-related principles that were or were not addressed by the facilitators proved more fruitful in understanding facilitation.

An interesting and somewhat different case of looking into facilitation practices is the case when the facilitators are mathematics researchers (e.g. Goos, 2014; Sztajn et al., 2014), and the PD is framed in terms of "boundary crossing" (Cooper, 2019). The notion of 'boundary' (Robutti et al., 2020; Wenger, 1998) implies that mathematics researchers and mathematics teachers belong to different, yet connected, communities of practice. While the two communities have different values and different knowledge, a PD environment provides a space for researchers and teachers “to exchange knowledge from their communities impacting both researchers’ and teachers’ practices without reducing the importance of either” (Wenger, 1998, p. 201). Various artefacts are used as boundary objects to negotiate and establish shared meanings among researchers and teachers. For example, Goor (2022) used videotaped lessons filmed in high-school mathematics classrooms as boundary objects in a collaborative group of secondary mathematics teachers and mathematicians. In this case, facilitation was conducted by mathematics education researchers, who served as brokers between the two communities. Sztajn et al. (2014) reported a shift in their program from using artifacts from the researchers’ community to those from the teachers’ community (e.g. curriculum materials and lesson plans) to support and sustain meaningful collaboration. In this PD, they drew attention to the researchers’ practices, as they took the role of facilitators in their work with elementary school teachers, designed around research-based knowledge on students’ mathematics. These practices included: (1) drawing on teachers’ expertise and understanding of school mathematics; (2) using research-informed evidence (e.g. clinical interviews with students, written diagnostic assessment) to develop teachers’ learning trajectories; (3) bringing into play teachers’ contextualised knowledge of students’ learning to foster exchange of knowledge.

Generally speaking, the exploration of facilitation practices and moves within mathematics teachers’ collaborative settings in different countries has provided important insights into the expertise of facilitators and the outcomes of such expertise. These insights relate both to the PD content (e.g. learning goals and learning pathways for teachers) and to the PD arrangements (e.g. design principles, process quality, facilitation moves), thus defining the specificities of *what* is to be learned in the PD and of *how* this is to be orchestrated (Prediger, 2019). For example, in the Unites States, Carlson et al. (2007) found that facilitator behaviour that, “attempts to understand the mathematical thinking and/or perspective of someone else” (p. 841) had a positive influence on the quality of the mathematical discourse of the learning community. The Theme C papers add to the variety of perspectives. In Germany, Griese et al.’s (2020) explorations of a collaborative PD on teaching stochastics resulted in proposing general heuristics for successful facilitation, such as moving from assessing a concrete phenomenon to understanding a structure, or mediating between aspiration and reality by elaborating on students’ learning trajectories. Research from Portugal (Quaresma, 2020) has shown that specific critical incidents may be crucial for the development of collaborative experiences, thus facilitators need to have specific moves for such incidents available.

Another aspect that can impact the success of a PD concerns the facilitator’s orientations, which can exert their influence via the prioritising of goals, for example giving precedence to ‘atmosphere’ goals (i.e. maintaining a stress-free environment) over content goals (Karsenty et al., 2023) exemplify such a case. One risk in such prioritising is that, “politeness is valued over professional debate and controversy” (Jackson et al., 2015, p. 95). Some researchers have found that such phenomena can be overcome if addressed accordingly in facilitator training (e.g. Pöhler, 2020, in Theme C). Jackson et al. suggested that it is important for PD leaders to regard teacher learning as a progression, rather than rectifying deficits in understanding which could be dealt with simply by informing the group how to understand the logic of a teaching approach, as many leaders in this research initially tended to believe. In order to shift facilitators' orientations, guided reflections on the underlying motives for choosing a certain activity or a specific move can be useful (Masingila et al., 2018). Just as at classroom level, "it is principles (or beliefs) rather than methods or material that underlie practice at a level that makes a difference for students" (Beswick, 2007, p. 116), at the PD level this rationale retains its significance. Since we already know that changing teacher orientations is a multi-step process that requires time, as "it is not the professional development *per se*, but the experience of successful implementation that changes teachers' attitudes and beliefs" (Guskey, 2002, p. 383), it follows that the same holds when lifted and applied to facilitator orientations. This stresses the necessity of research-inspired facilitator qualification programs which include facilitation *and* reflection phases, so that facilitator expertise is supported and monitored over a period of time.

4.3.3 Situational Challenges of Promoting Collaboration

The challenges for facilitators are located on different levels. We are interested in the specific situational challenges that appear in regard to the interactions of the collaborative group. Researchers agree that promoting a fruitful communication and discussion is a crucial part of successful collaborative facilitation (e.g. Borko et al., 2014; Coles, 2013; Krainer, 2015; Zehetmeier, 2015). Thus, we present below three situational challenges, associated with promoting successful communication: starting and managing a discussion; establishing and maintaining norms; and observing, redirecting and sharing responsibility.

4.3.3.1 Starting and Managing the Discussion

To spark-off an exchange, often stimuli like video vignettes or problem-solving tasks are utilised (Borko et al., 2014; Coles, 2019). Depending on the framing of the collaborative work, the facilitators might be able to choose from a range of stimuli specifically designed for certain goals, or they might face the challenge of having to find an appropriate stimulus themselves. When there is no material available, the challenge is increased by the necessity to find a discussion starter that is in line with the group's agenda, and to create activities for the teachers that involve and activate the whole group. One option raised in the Theme C papers was to create relevant professional learning tasks (PLTs), as demonstrated by Ribeiro (2020). In order to be able to decide which tasks are deemed, suitable, facilitators, particularly those who have not been involved in the design phase of the PD program, are in need of clarification and explication of how to select or create tasks, so that they can use them with integrity to the goals of the PD. Ideally, this should be achieved within their qualification as facilitators of the specific program. Even if there is a range of stimuli available from the PD developers, the question remains as to which aspects of which specific stimulus is to be addressed by the facilitator. Karsenty and Arcavi (2017) suggest that records of practice that serve as a base for teachers' discussion can be inspected by using six different lenses, representing six aspects of lesson analysis (mathematical and meta-mathematical ideas; the teacher's goals; tasks that students engage in; interactions in the lesson; teacher dilemmas; teacher beliefs). The facilitator can choose one aspect or several different aspects, in order to frame the discussion and enhance various viewpoints.

4.3.3.2 Establishing and Maintaining Norms

Although it can be assumed that all collaborations build on norms of trust and respect among the group members, still different collaborative settings may embrace further different norms that commonly stem from the approach underlying the collaborative work. For example, one possible approach to a fruitful discussion around video clips, developed in the Open University in the UK (e.g. Coles, 2013, 2014; Jaworski, 1990), advocates that teachers' contributions should focus at first on describing in detail what they have observed, and only at a second stage they are encouraged to elicit different interpretations and address a wide range of possible alternative reactions. Other approaches encourage teachers to watch videotaped lessons (e.g. Santagata, 2009) or live lessons (as in the Japanese Lesson Study approach and its adaptations, e.g. Fernandez & Yoshida, 2004; Lewis & Lee, 2017; Lim et al., 2005) and explicitly discuss strengths and weaknesses of the lesson (see also the plan–teach–evaluate cycles in Wright, 2020). There are also approaches that deliberately wish to avoid evaluation, and set a norm of restricting the discussion to non-judgmental exchanges (e.g. Karsenty & Arcavi, 2017; Schwarts et al., 2022). This norm has been identified as a considerable challenge for facilitators in certain collaborative settings, as shown in the Theme C paper by Schwarts (2020), since teachers, like other human beings, tend to criticise actions that differ from their own, and so it takes time and effort to uphold a conversation that is not evaluative in nature (Karsenty et al., 2019). The other side of this coin is that teachers may fear being exposed to criticism by peers, and thus they refrain from taking part, for example, in video documentation of practices. In such cases the facilitator's challenge resides in overcoming such fears, as described in another Theme C paper (Quaresma, 2020). In situations where norms are violated, or when criticism becomes harsh, facilitators are expected to remind the group of the rules or assumptions agreed before, for example the "basic assumption that the [filmed] teacher is acting in the best interest of his/her students" (Karsenty & Arcavi, 2017, p. 438), or the agreement of "starting discussion with *accounts of* and only later moving to *accounts for*" (Coles, 2014, p. 269; *italics added*). This also means that the facilitator must consciously fill the role of a *leader*, whereas at other times facilitators might stress their alternative role as *members* of the group (Knapp, 2017). With the objective of supporting collaboration among the group of teachers, facilitators are advised that it is worthwhile sometimes to stand back and simply appreciate the ideas coming up in the exchange (van Es et al., 2014), but this must be utilised in balance with intervening moves of re-establishing norms.

4.3.3.3 Observing, Redirecting and Sharing Responsibility

Once the discussion has gained momentum, it may divert to themes unconnected to the professional goals. The facilitator should carefully observe the conversation and, if necessary, intervene to redirect the focus of the discussion, while being aware that the responsibility for reaching the PD goals is to be shared between the facilitator and the participating teachers. For the purpose of keeping the conversation on the right track, facilitators can utilise practices designated by van Es et al. (2014), such as "sustaining an inquiry stance" or "maintaining a focus on [the video and] the mathematics" (p. 347), which include several possible moves that guide and encourage reflection and promote discussion around the desired goals. The purposes of these moves are many and varied, and the facilitator is challenged to notice nuances in the discourse, to analyse the situation *ad hoc* and to decide on a move, following certain principles such as communal investigation, co-operative teaching and open-mindedness towards the teachers (see the Theme C paper by Nieman et al., 2020). Since these principles may be deployed either by the facilitator or by the participating teachers, the facilitator may choose to use them only when there is no teacher doing so. For example, re-introducing an important idea mentioned before for further discussion, or connecting different ideas, is not exclusive to the facilitator role. Some researchers (e.g. Felton & Koestler, 2015) suggest allowing the group a maximum of agency, which they perceive as crucial when aiming to support changes in teaching. By actively sharing the responsibility for the professional development, the facilitator can promote the group's agency.

In sum, it appears that the work of a facilitator involves specific challenges that portray it as a dual-objective endeavour: creating an environment of support and trust on the one hand, and assuring that the discussions will meet demanded standards and depth on the other, to allow participating teachers to gain new knowledge (Sherin & Han, 2004). In light of this complexity, one can understand the rationale for appointing more than one facilitator per group. The next sub-section deals with the advantages and pitfalls of co-facilitation.

4.3.4 Co-facilitation

Here, we focus on a special form of facilitation, namely co-facilitation, which brings into consideration not only how this unique kind of facilitation supports collaboration within a group, but also how the collaboration of the co-facilitators themselves is formed and characterised. Co-facilitation is a case when a shared responsibility exists between two (or rarely, more than two) facilitators in regard to leading a certain collaborative group of teachers. Even though several authors (e.g. Gitterman & Shulman, 1994; Rothman, 1981) consider solo-facilitation as the most effective way of leading a group, more recent sources (e.g. Cohen & DeLois, 2002; Nachlieli, 2011; Reid & Demissie-Sanders, 2014) describe cases when co-facilitation has brought a new dimension to this role.

Edwards (2009) understands the knowledge of 'who can do what in the best way', as the core of co-working, since collaborative tasks require resourceful use of the expertise of others and mutual alignment of the professional practices of collaborating persons. The ability to offer support and to ask for support from others is closely connected to the sharing of agency, as in general a collaboration includes "negotiating roles and relationships" (Quaresma, 2020, p. 530).

In several reported cases of successful co-facilitation (e.g. Cohen & DeLois, 2002; Nachlieli, 2011; Novotná et al., 2013), at least one of the collaborating facilitators was based in university, and at least one of the two had experience as a teacher. Considering the high expectations from facilitators in collaborative groups, their varied backgrounds may be a positive factor in the successful sharing of responsibilities while leading groups. For example, in the case reported in Theme C by Pöhler (2020), one facilitator was a teacher educator and the other was an experienced teacher. The former provided the theoretical introductive part of the session, and the latter orchestrated the whole-group discussions that followed the small-group work. Nachlieli (2011) presented another example of a researcher and an experienced teacher co-facilitating. Their distinct backgrounds and diverse areas of knowledge enabled them to accommodate different roles during the PD: the experienced teacher encouraged the discussion about specific classes, while the researcher attended to generalising, theorising or hypothesising by moving the conversation to the general classroom. Interestingly, both used the same type of communication move (in this case, confrontation), but with different objectives and effects.

In another Theme C paper, Medová et al. (2020) reported on a process of developing a relationship of trust and of sharing responsibilities between two facilitators from their very first co-facilitated session. In this case, when the less experienced facilitator encountered challenges while leading the PD, the more experienced facilitator first stood back and provided her with an opportunity to gain more experience. He switched to the role of observer of the session, which enabled him to reflect-*in*-action, which in turn led several minutes later to his decision to intervene. This scenario provided information used in consecutive, post-session reflections shared by the collaborating facilitators. Medová et al. concluded that the time invested in co-facilitation contributed not only to the facilitation itself, but also to the professional growth of the facilitators. This may be connected to the construct of *relational knowing* (Hollingsworth et al., 1993), constructed in open conversations during which participants spend their time learning, enriching and trying to understand each other's ideas, rather than just completing the forming of an idea (Hollingsworth & Dybdahl, 2007). In school contexts, such sharing of knowledge can be particularly valuable, especially in view of the long-term relationships between colleagues, whose identities and roles may develop.

Cohen and DeLois (2002) stressed the importance of collaborators feeling pleased with the collaboration from the very first session, despite possible objective discords. Successful co-facilitation depends on finding a balance between co-facilitators filling their individual roles and performing as collaborating practitioners. In order to allow for such successful co-facilitation, it is imperative that the collaborating facilitators spend time together between sessions. A considerable amount of interaction and exchange is necessary for developing productive collaboration, including open communication, mutual regard and trust (Reid & Demissie-Sanders, 2014). In some cases (e.g. Cohen & DeLois, 2002; Medová et al., 2020), a previous friendship or acquaintance may be an asset, but it might also cause an impediment, as sometimes uneasy conversations between co-facilitators may be necessary, particularly when long-held routines are challenged while working with practicing teachers. Meetings between sessions should not be limited to planning future activities and dividing primary responsibility for each part of the content. The practice of shared reflection of co-facilitators on previous sessions appears to be fruitful, not only for the evaluation of the session (Eriksen & Solomon, 2022; Medová et al., 2020; Zaslavsky & Leikin, 2004). The presence of two persons may offer an adequate reflective distance and allow deeper understanding of the situation. The shared reflection can also be the means of enhancing the relational knowing and professional knowledge of the co-facilitators involved, but it requires an openness to honest feedback.

The question of how a collaboration between facilitators may influence the building of community and collaborative relationships within the group of teachers is under-researched. There is a need to explore further co-facilitation settings and identify characteristics that have impact on teacher collaborative work.

To sum up this section of the chapter, the facilitator role was examined from various perspectives, including: a theoretical perspective providing different categories for facilitators' knowledge; a profession-oriented perspective on facilitators' practices and moves, elaborating on facilitators; implementation of their knowledge and skills; a pragmatic perspective concentrating on the challenges involved in facilitation, illustrating the complexity of this endeavor; lastly, the unique perspective of co-facilitation. In the next section of this chapter, we explore the question of how one becomes a facilitator of teacher collaboration, and what may support the development of professionals assuming this role.

4.4 Exploring Professional Trajectories of Facilitators of Mathematics Teacher Collaboration

This section focuses on facilitators' own professional development. First, we unpack what *becoming a facilitator* might mean, taking into account that often facilitators hold other positions as well, and may have different starting points such as teachers or researchers (Sect. 4.4.1). Then, we review and compare different types of existing preparation programs for facilitators and the design principles upon which they are based (Sect. 4.4.2). Next, we refer to the development and changes in different aspects of facilitators' work that may occur over time (Sect. 4.4.3), and review some findings regarding identity and agency shifts (Sect. 4.4.4). Finally, means and models for supporting facilitators along their work period are discussed (Sect. 4.4.5).

4.4.1 Becoming a Facilitator

The process of becoming a facilitator (or an MTE in general) has only started to gain research attention at the beginning of the 2000s (e.g. Even, 2008; Llinares & Krainer, 2006; Zaslavsky & Leikin, 2004). There is no designated route for becoming a facilitator of mathematics teacher collaboration, especially since the backgrounds of facilitators may vary considerably. For instance, facilitators may hold the following positions: mathematics teachers; researchers; teacher coaches; mathematicians; leaders at different school levels; independent consultants and more. These backgrounds may also overlap and, moreover, they may take different forms within different cultural contexts or traditions. For example, in the Chinese Lesson Design Study (Li et al., 2011; Yang, 2014), facilitators are expert teachers, who also conduct research and publish papers. These expert teachers have gone through a process of shifting from being 'ordinary' teachers into exemplary role models that serve as leaders for other teachers.

Even (2014), in her Commentary on the ZDM Special Issue on the practices and professional development of didacticians (Jaworski & Huang, 2014), noted that professional development of those who lead PDs for mathematics teachers is commonly not an organised activity, but rather occurs as a by-product of the project. She pointed to the need to understand better what facilitators should learn, as well as how and when they can learn it. The spectrum of backgrounds that facilitators may have makes this a complex endeavour. Yet, the case of facilitators who are former (or still practicing) mathematics teachers is relatively common (as in the Chinese case above) and merits special attention. Dinkelman et al. (2006) claimed that most practicing facilitators were teachers at some point. As several of the Theme C papers reveal, being a classroom teacher before or simultaneously to being a PD facilitator is a prevalent situation, at least within the cultures represented in these papers (this might not be the case in some Latin American countries, for example).

Perry and Boylan (2018) drew on the concepts of first- and second-order roles of facilitators (Murray & Male, 2005). A first-order role is that of teacher in the classroom, whereas a second-order role is a step removed from the classroom (e.g. researchers or independent consultants). A teacher who is also a facilitator constantly moves between the first-order and the second-order roles. Considering such transition can draw upon other relevant transitions. For example, Labaree (2003) pointed out that, "In many important ways, the transition from teacher to educational researcher is a natural and easy one. As prospective researchers, teachers bring many traits that are ideal for this new role, including maturity, professional experience, and dedication" (p. 15). Similar arguments can be used regarding those who make the teacher–facilitator transition. In other words, an experienced teacher could be considered well equipped to become a facilitator, based on maturity, professional experience and motivation (assuming that facilitation is commonly chosen rather than being imposed).

However, just as Labaree (2005) also noted aspects that may make the teacher–researcher transition complicated, the same could be said about the shift from teacher to facilitator. Indeed, Even (2005) asserts that, "being a good teacher does not necessarily imply the ability to help others develop their teaching" (p. 344), and papers from different countries presented within Theme C demonstrate that switching from a teacher position to a facilitator position is far from being trivial (e.g. Schwarts, 2020; Widjaja & Vale, 2020). The question of whether and how the ability to be a good facilitator can be learned is still an open question in contemporary research, which is being explored in recent years in various contexts.

Maaß and Doorman (2013), based on Müller (2003), have suggested that a longitudinal model of the learning and development of PD facilitators (they use the term 'multipliers') should include three phases: *Learning-off-job*, which consists of gaining fundamental knowledge, preferably within organised seminars or preparation courses; *Learning-by-job*, which consists of using the knowledge acquired in the first phase for planning and implementing a PD, with a close support and counselling by experts; finally, *Learning-on-job*, which consists of further growth enhanced by experience, reflection and peer support (alternatively, Zaslavsky & Leikin, 2004, integrated the second and third phases into one complex model of *growth-through-practice*). This three-phase model was utilised in the EU PRIMAS (Promoting Inquiry-based learning in Mathematics and Science), in which 12 European countries participated (Maaß & Doorman, 2013; reported also by Sikko & Ding, 2020 within Theme C). In some of these countries, the prospective facilitators were schoolteachers who were qualified as facilitators by going through a training period at the university. Schwarts (2020) described a trajectory along which teachers became facilitators in a video-based PD program in Israel: first, they experienced the PD as teachers, then they developed into prospective facilitators by taking a facilitator training course, after which they became novice facilitators supported by more experienced peers. Finally, over time, they became experienced facilitators themselves and could mentor new peers.

In the following, we focus on research exploring the process of becoming a facilitator and developing as a professional within that role. We begin with studies examining the preparation stage, i.e. facilitators' pre-service period, and continue with studies looking into changes occurring within facilitators' in-service periods.

4.4.2 Preparation of Facilitators of Mathematics Teacher Collaboration

Literature on the preparation of facilitators towards leading mathematics teacher collaboration is still relatively limited (Jackson et al., 2015; Roesken-Winter et al., 2015). In particular, little is known about how facilitators may be prepared to become skilled in putting different knowledge aspects into practice. However, the body of research that unpacks what the role of a facilitator may entail, reviewed in Sect. 4.3, can be drawn upon when considering the preparation of facilitators.

Some research suggests, in accordance with the needed requirements of academic knowledge as well as social and interpersonal skills (see Sect. 4.3.1), that key to successful preparation of facilitators is the enhancement of solid acquaintance with mathematical content, skills in establishing norms and using prompts that support productive discussions, and an inquiry stance towards practice (Borko et al., 2008; Elliott et al., 2009). Attention to creating a safe and supportive learning environment and establishing trust, and explicit articulation of key facilitation practices, were also found crucial to the process of preparing facilitators (Borko, 2004; Borko et al., 2014). The importance of having facilitators first experience the PD as learners, before shifting to their role as facilitators, was highlighted as well (Koellner et al., 2008; Kuzle & Biehler, 2015).

Regarding learning artefacts that can serve in the process of preparing facilitators, analogies can be created between preparing teachers and preparing facilitators. At the teacher level, video vignettes and students' work samples have been found to be useful to capture the complexity and dynamic interaction of classroom practice, and to support teachers in developing their ability to notice, critically reflect and direct productive discussions among students (Borko et al., 2008; Brophy, 2004; Rosaen et al., 2008; van Es & Sherin, 2008). By analogy, repeated viewing of video vignettes with different foci can potentially support the development of facilitation skills and foster productive professional discussion among teachers (Borko et al., 2017). Several researchers have argued that it is critical to select video vignettes (either from mathematics classrooms, as proposed for example by Borko et al., 2017, or from PD sessions, as reported by Lesseig et al., 2017) with clear goals, and embed these in activities that are planned thoroughly to assist prospective and beginning facilitators to learn how to support teachers' pedagogical reasoning.

4.4.2.1 Design Principles for the Preparation of Facilitators

Although still relatively scarce, some research has been published on preparation programs for facilitators within various programs across the world, and these allow discussing curricula and design principles developed for this purpose.

In Germany, Roesken-Winter et al. (2015) reported on a 1-year PD course they conducted for 12 PD facilitators. Participants were mathematics teachers (at the secondary school level) who were to become responsible for PD in their federal state. The course was designed by utilising design principles derived from literature on effective PD for teachers, and one of the research aims was to inspect whether such principles could be effectively adapted to the level of facilitators' PD. The study focused on two design principles, named *participant-orientation* and *competence-orientation*, that were rated by the facilitators as the most relevant principles, in surveys that were conducted 6 and 10 months after the course. The principle of participant-orientation means that several modes of participants' active involvement in their own learning were employed (e.g. self-study, e-learning, practical try-outs, portfolio writing). The principle of competence-orientation involves the operationalisation of several types of facilitators' expected competencies (e.g. in the cognitive, affective-motivational and technical domains) into a detailed framework used explicitly in the course for discussing how these competencies may be strengthened.

In the Unites States, Elliott et al. (2009) reported on a training model consisting of a series of seminars aiming at facilitators' learning of how to cultivate mathematically rich PD environments, within a project named Researching Mathematics Leader Learning (RMLL). The design principles of the seminars were based on two frameworks adapted from classroom-based research to the PD level: socio-mathematical norms (Yackel & Cobb, 1996) and the five practices for orchestrating productive mathematical discussions (Stein et al., 2008). These principles include: (1) mathematical coherence of the tasks; (2) socio-mathematical norms as a perspective to explore what counts as adequate mathematical explanations and justifications within teachers' discussions; (3) purpose-based learning of facilitators; (4) opportunities for facilitators to connect the work in the seminars with their own work of facilitating teacher learning; and (5) cultivating a stance of inquiry, through which facilitators consider the affordances and limitations of specific pedagogical moves and recognise that there is no single best 'protocol' for facilitating mathematics PD.

In a second phase of the RMLL project (Lesseig et al., 2017), the researchers have added the MKT framework (Ball et al., 2008) to the other two previously considered, and introduced a revised set of design principles for PD of facilitators, that emphasise mathematical goals. These include the following: (1) facilitators' mathematical work should be based on clear learning goals for teachers, in order to differentiate distinctly between teachers' and students' learning needs; (2) tasks designed for facilitators are more effective if they explicitly relate to Specialised Content Knowledge (SCK); and (3) using video-cases exemplifying instances when mathematical goals for teachers are pursued is helpful in supporting facilitators' awareness to pedagogical and mathematical aspects of facilitating teacher learning.

In Israel, within a video-based teacher PD project named VIDEO-LM (Viewing, Investigating and Discussing Environments of Learning Mathematics; Karsenty & Arcavi, 2017), a preparation course for prospective facilitators was implemented. Reported design principles for the preparation model (Karsenty, 2016) included the following: (1) *Relevance* – course activities are directly linked to realistic issues that the facilitators are expected to deal with, according to the experience accumulated in the PDs already conducted; (2) *Feasibility of learning goals* – prospective facilitators need to learn from the experiences of other facilitators, thus vivid cases of facilitation should be presented in the course, along with a set of tools to analyse these cases (this set of tools was named 'meta lenses', see Karsenty, 2020); (3) *Commitment to the project's agenda and norms* – in analysing cases, explicit and recurring references need to be made to ideas at the core of the project (e.g. the six-lens framework used with teachers desired norms such as maintaining nonjudgmental discussions); and (4) *Modeling* – leaders of the facilitator course should maintain facilitating that is aligned with what participants are expected to do as course leaders in the future (e.g. maintain a supportive atmosphere, use diversified and engaging activities).

Looking at these cases from different countries, some general features of what preparation courses for facilitators focus upon can be distilled. The most salient features are the attempted efforts to link the course experiences to actual expected facilitators' practice; the centrality of carefully-designed course tasks (e.g. analysing video-cases) as springboards to discuss what teachers' learning within PDs may entail; the use of specific analytic tools provided to participants for making sense of the cases they inspect (e.g. socio-mathematical norms and the five practices in the case of RMLL, meta-lenses in the case of VIDEO-LM); finally, the emphasis put on active involvement of facilitators in their learning. Broadening this last feature, several researchers have advocated that opportunities given to facilitators, actively to try out facilitation skills in their own schools, are critical (Borko et al., 2021; Elliott et al., 2009; Roesken-Winter et al., 2015).

4.4.3 Development and Change in Different Aspects of Facilitators' Work

Findings regarding changes and development in facilitators' proficiency (e.g. their knowledge, practices, competencies and identities) are beginning slowly, yet steadily, to accrue. In this sub-section, we analyse findings from six projects, reported in the following papers (ordered chronologically): Zaslavsky and Leikin (2004); Shagrir (2010); Jackson et al. (2015); Roesken-Winter et al. (2015); Perry and Boylan (2018); Schwarts (2020). The six studies are briefly overviewed in Table 4.1. We draw on collective salient findings from these projects to discuss development and change in facilitators' identity, knowledge, beliefs and attitudes; development and change in facilitators' practices; the influence of collaboration and interaction on facilitators' changes and development.

4.4.3.1 Development and Change in Facilitators' Identity, Knowledge, Beliefs and Attitudes

Different authors have reported an increased involvement of facilitators in the program they represent, over time (Schwarts, 2020; Zaslavsky & Leikin, 2004). With regard to identity aspects, several studies found that facilitators gradually consolidated their identity pertaining to this role, and saw themselves as more proficient and confident as time passed (Schwarts, 2020; Shagrir, 2010; Zaslavsky & Leikin, 2004). Schwarts (2020) characterised the professionalisation process of a novice facilitator, who facilitated a collaborative PD within a video-based project for the first time, over 1 year. Findings showed that, while in her first session the

Table 4.1 Overview of the six studies

Paper	Project description	Facilitators	PD aim	Research focus	Data sources	Methodology/approaches
Zaslavsky and Leikin (2004)	Five-year in-service PD program for secondary school mathematics teachers	Twenty, with little or no experience as facilitators	Helping teachers to see new possibilities for their own practice	Growth of facilitators through their practice	Videos of PD sessions Written self-reports Protocols of individual interviews and of staff meetings	Qualitative research; three-layer model of growth through practice
Shagrir (2010)	One-year program at an intercollegiate professional centre	Eleven novice facilitators	Fostering preservice teachers' learning	Facilitators' PD; ways of becoming a facilitator	Questionnaire with close and open sections	Analysis according to three dimensions
Jackson et al. (2015)	Researcher-practitioner partnership in a large US school district	Three facilitators, based at a central office	Supporting the learning of middle-grades mathematics teachers	Facilitators' practices in designing and leading high-quality PD	Videos of facilitators' pilot teacher PD sessions Audio-recorded interviews	Retrospective analysis of data collected across four PD cycles
Roesken-Winter et al. (2015)	One-year PD course of DZLM (The German Centre for Mathematics Teacher Education)	Twelve, working as mathematics teachers in secondary education	Providing continuous PD for mathematics teachers	Effects of the continuous PD design on facilitators' development	Questionnaires Interviews	Quantitative and qualitative data analyses
Perry and Boylan (2018)	Developing the developers' program	Seven, with different backgrounds and much experience as facilitators	Providing continuous PD for science teachers	Facilitators' professional learning, practices and change	Written evaluations Semi-structured interviews Follow-up questionnaires	Analysis based on the interconnected model of teacher professional growth
Schwarts (2020)	The VIDEO-LM project: video-based PD for secondary school mathematics teachers	A novice facilitator accompanied over 1 year	Fostering teachers' reflective skills and MKT	Evolving of facilitator's practices and identities	Questionnaires Journals Videos from PD sessions and stimulated-recall interviews	Qualitative case-study approach

facilitator did not take most of the relevant components of the project's goals and agenda into account, at the end of her first year she structured the PD sessions according to these components. By that time, the facilitator finally saw herself as a leader whose role in the VIDEO-LM project was to involve teachers in meaningful peer-discussions and to foster collective reflection. Similarly, Zaslavsky and Leikin (2004) pointed out that novice facilitators are often not very confident in their qualification to undertake this role when they begin working, and expressed a need for guidance from more experienced peers or from the project's team. They noted that, in many cases, the development of facilitators during the project's duration included growth and transition from the role of mathematics teacher to that of a facilitator. According to Zaslavsky and Leikin, this development is accompanied by facilitators' enhancement of knowledge.

Roesken-Winter et al. (2015) maintained that the improvement and deepening of facilitators' competencies over time, for example in terms of pedagogical content or content knowledge, is crucial to the success of the project. Some of the studies have shown facilitators' changes and development in this area (Perry & Boylan, 2018; Roesken-Winter et al., 2015; Shagrir, 2010; Zaslavsky & Leikin, 2004), with different foci on aspects and dimensions of knowledge and skills. Shagrir (2010), for example, stated that most of the participants in her study (94%) testified that they improved their performance as facilitators during their participation in the program, referring to their professional knowledge and skills, as well as the professional language they use. The facilitators emphasised that the project kept them well-informed of global research, theories and existing approaches in the context of teacher education.

The findings of Roesken-Winter et al. (2015) showed a significant positive development of self-estimated competences of facilitators, in each of the seven considered knowledge and skills dimensions (e.g. mathematical content knowledge; pedagogical content knowledge; PD management), over the course of the continuous PD program. Interestingly, the facilitators acknowledged those dimensions in which they themselves had strong self-efficacy, as important for their participating teachers.

Perry and Boylan (2018) investigated facilitators' professional learning, practice and change, and found that the most salient outcomes of their 'Developing the Developers' program were related to facilitation knowledge and skills, and to knowledge about professional development. Facilitators who participated in this study felt less need for learning in the area of content knowledge (e.g. subject-specific content knowledge or pedagogical content knowledge) or in regard to skills for teaching, which reflects the 'second order' nature of their role (see Sect. 4.4.1). Another type of change documented in this study, exemplified by one of the facilitators, was a newly constituted "belief that sharing theories of professional learning with the participants leads to greater impact and engagement" (p. 263). This stemmed from the facilitator's positive experience with an activity in his own PD session with teachers, in which he explicitly used Clarke and Hollingsworth's (2002) interconnected model to stimulate a discussion on how teacher learning can occur through professional development.

4.4.3.2 Development and Change in Facilitators' Practices

The authors of the six papers referred to facilitation strategies (Perry & Boylan, 2018), facilitators' managing of teachers' contributions or ideas (Jackson et al., 2015; Schwarts, 2020), facilitators' sensitivity to, and view of, teachers' learning (Jackson et al., 2015; Zaslavsky & Leikin, 2004), facilitators' design of activities for teachers (Zaslavsky & Leikin, 2004) and facilitators' pursuit of goals for teachers' learning (Jackson et al., 2015).

With regard to facilitation strategies, Perry and Boylan (2018) noted that changes in the domain of practice could occur by trialing new strategies by the facilitators. As an example, the authors presented a case of a facilitator, whose PD session openings were too long at the beginning. In later PD sessions, she therefore tried a new opening technique and perceived it as an improvement, since teachers were evidently more quickly engaged with their professional learning.

Schwarts (2020) reported on a development concerning one facilitator's dealing with teachers' contributions: whereas the facilitator initially demonstrated a practice of mainly listening, later she perceived her role as facilitator in a more active manner and was able to be more adaptive towards teachers' contributions. She also drew on her teaching resources and no longer saw her teaching and facilitating practices as mutually exclusive. This case study shows that keeping the identity of an experienced teacher alongside a facilitator identity "can enrich practice and keep the facilitator practice-oriented while working with her colleagues" (p. 547). In contrast, Jackson et al. (2015) could not identify a linear progression in terms of facilitators' capacity to capitalise on teachers' ideas in PD sessions. Indeed, they reported that their facilitators initially elicited teachers' ideas, but tended to give teachers little direction and did not press for elaboration or build on teachers' contributions in meaningful ways. The quality of the facilitators' ability to elicit and build on ideas did not increase successively over time, but varied across activities, apparently depending on their nature and on the associated goals for teachers' learning.

Other research findings concern an evolvement of facilitators' sensitivity to teachers' ways of thinking, facilitators' awareness to what may be expected of teachers within a PD (Zaslavsky & Leikin, 2004) and facilitators' views of teachers' learning (Jackson et al., 2015). As mentioned in Sect. 4.3.2, the facilitators in the study of Jackson et al. (2015) initially tended to approach teacher learning as rectifying deficits in their understanding and practice, which could be realised by using an isolated activity or by showing good practice. However, with time, they began to approach teacher learning as a developmental progression and to conceptualise the PD sessions in terms of a sequence of linked activities.

In addition to the reported improvements in facilitators' ability to design PD activities as a sequence, the focused studies showed an expansion of facilitators' repertoire of PD activities (Zaslavsky & Leikin, 2004) or a shift from designing PD activities that focused on peripheral aspects to more central aspects of instruction (Jackson et al., 2015). However, despite the finding that the designed PD activities were increasingly focused on the core issues, the facilitators "frequently evidenced a 'show-and-tell' approach when they facilitated those activities" (Jackson et al., 2015, p. 102). The authors therefore concluded that, sometimes, the PD activities did not seem to support teachers' learning in the goal-oriented way intended.

In conclusion, it must be noted that changes and development in facilitators' practices would be necessary, but not sufficient, for instructional improvement at scale (Jackson et al., 2015.). This is related to the reality that the impact of the PD is also "mediated by other aspects of the contexts of teachers' work" (p. 94), for example the instructional expectations that school leaders communicate to teachers.

4.4.3.3 Influence of Collaboration and Interaction on Facilitators' Changes and Development

Across the six studies, a great importance was ascribed to facilitators' collaboration with each other as a resource for their professional learning (Perry & Boylan, 2018; Shagrir, 2010; Zaslavsky & Leikin, 2004). Facilitators themselves also perceived such collaboration as positive, seeing their colleagues as a support group that helped them cope with the difficulties and challenges of their new job (Shagrir, 2010).

Correspondingly, facilitators' changes are often related to input they have received from peers, in terms of peer observations, feedback or suggestions such as new facilitation techniques (Perry & Boylan, 2018). In this context, Zaslavsky and Leikin (2004) argued that facilitators' learning by collaboration occurs via awareness of differences between their own and their colleagues' positions, which then leads to discussions of their different stances in an attempt to find a shared position. Furthermore, the exchange of ideas with more experienced persons (for example, project leaders), or the observation of their facilitation moves or strategies, could have a positive effect on facilitators' change and development as well (Zaslavsky & Leikin, 2004). In addition, they report facilitators could also learn from teachers' contributions and from a careful reflection on how teachers react to the offered PD activities. As a result of such reflection, facilitators often vary, modify, reduce or expand the PD activities over time.

To conclude, the different studies reviewed here show that facilitators "can undergo a professionalisation process and become proficient facilitators" (Schwarts, 2020, p. 546) and that participation in a longer-term program could motivate a novice facilitator "to act like a professional one" (Shagrir, 2010, p. 52). In this context, Roesken-Winter et al. (2015) stated that, from their data, "a clear picture emerges: convincing multipliers that they can trust what they were taught in their CPD [continuous PD] courses is a direct link to implementation into their own CPD practice" (p. 22).

The six papers discussed here are promising steps towards understanding facilitators' change and development over time; however, this aspect is still under-researched and more studies are needed to consolidate the results reported and to explore further directions. In particular, the place of mathematics within facilitators' development of knowledge and skills is still strikingly absent from most existing research on facilitators' professionalisation processes, and remains a challenge for future research, as was noted in the Theme C group discussion.

4.4.4 Shifts in Agency of Facilitators of Mathematics Teacher Collaboration

Much has been written about mathematics teachers' agency and the importance of maintaining this agency in PD initiatives, but 'lifting' this construct to the facilitator level has not yet been established. One possible conceptualisation of facilitators' agency could be made by considering the widely-used definition by Holland et al. (1998), which refers to teachers' agency as their "realized capacity [. . .] to act upon their world [. . .] purposively and reflectively" (p. 42), and which may be appropriate for facilitators as well. It appears that the notion of agency is highly connected to the notion of identity. As Goos and Bennison (2019) stress in regard to mathematics teachers, "there are many factors influencing identity development, and [. . .] individuals can exercise agency by authoring their own identities" (p. 416). Thus, we may conclude that, both for teachers and for facilitators, such increasing capacity for agency is connected to the possibility of informed identity building. Some questions that are of interest in this emerging field are:

- How is facilitators' agency manifested?
- To what extent can studies on facilitators' agency draw on the research regarding teachers' agency?
- What design principles may inform programs that wish to ensure facilitators' agency?
- What are the relationships between fidelity, integrity and agency?

These questions are complex to investigate and lie heavily on how one defines each term. For example, Brodie (2019) argued that, "teachers always enact agency, even when they choose not to act, or might seem to 'passively' accept policies or practices from others" (p. 562). This implies that understanding the different ways facilitators enact agency requires investigations of their underlying motives. Since there are almost no studies published in this field, we can only refer to few preliminary results available so far regarding facilitators' agency.

From the analysis and comparison made in the Theme C paper by Sikko and Ding (2020), it can be concluded that agency may take different forms in different cultural contexts, that is different places in the world with different educational systems and cultural assumptions. In addition, Sikko and Ding highlighted that agency may help bridge the gap between research and practice, and serve as the missing link for the successful execution of reforms. Still, developers of PD programs do not always consider teachers' and facilitators' agency within their design.

In Müller's (2003) model, used in the PRIMAS program (see Sect. 4.4.1), the third developmental stage (*Learning-on-job*) suggests that:

> it is the multiplier's self-education that leads them to develop their competences. [. . .] Once a need is discovered, the teacher educator must search for related information to improve their knowledge and carry out 'experiments' by asking questions about their way of teaching or running professional development courses. (Maaß & Doorman, 2013, p. 892)

From these comments, it appears reasonable to suggest that facilitators' capacity to detect their needs and actively to find ways to respond to them (that is, their agency) is required for facilitators' learning and development.

The transfer from teacher to facilitator also involves a shift in the focus of agency. A teacher's prime focus is the development of students at school, while a facilitator's prime focus is the development of teachers, with student development taken as a secondary goal or as a sign of teacher development. As Perry and Boylan (2018) argued, the impact of PD facilitators on student outcomes is mediated by the 'second-order' nature of their role (Murray & Male, 2005; see Sect. 4.4.1) which separates them from direct classroom impact. Thus, for facilitators "to act upon their world" (Holland et al., 1998, p. 42) means different things from what it means for teachers.

Dinkelman et al. (2006) investigated two teachers' transitions from classroom teachers to teacher educators. They found that the shift in agency was not an easy one, and that both subjects retained elements of their identity as classroom teachers, while at the same time acquiring the new identity as facilitators. Kohen et al.'s (2020) Theme C paper provided insights regarding a PD in the form of a problem-solving forum, that enabled teachers to experience both the role of problem solvers and the role of mentors. A case study of one experienced teacher showed the development from a pure problem solver, with similar behaviour to those of students wanting to demonstrate how clever and quick they are in their solutions, into a mentor who supported the thinking processes of others. This case demonstrates the importance of first-hand experience as a student for development as a good facilitator. It also demonstrated the shifts in agency involved in this development.

Schwarts (2020) provided insight into how facilitators' practices and identities evolve, via an in-depth case study of a novice facilitator working within a video-based PD program (see detailed account in Sect. 4.4.3). The facilitator's conflicting identities, that of a teacher and colleague, and that of a facilitator, yielded challenges that were prominent both in the planning of the PD sessions and in managing teachers' discussions and reflections. Schwarts suggested several means by which the facilitator's identity as a leader may be strengthened, which, as Goos and Bennison (2019) have argued, may have positive consequences for the shifting of agency. Such means are elaborated in the next sub-section.

4.4.5 Supporting Facilitators' Professional Development

The support and guidance provided for facilitators during their professionalisation processes (also framed as part of continuous professional development – CPD – for facilitators; see Roesken-Winter et al., 2015) is another important element that influences the success of facilitators in supporting mathematics teachers' collaboration. This is especially crucial for novices in the field of facilitation.

As already discussed in previous sub-sections, reviews such as that conducted by Even (2014) have shown that facilitators' qualification is often informal and spontaneous, with few projects providing formal preparation (see Sect. 4.4.2). Accordingly, institutionalised on-job support for facilitators is also not common and rarely investigated and reported, even within the accumulated literature on facilitators in recent years. The professional support that facilitators may receive is almost always reviewed as background or context for studies on facilitators, and not as the subject under investigation. Thus, within PD research in mathematics education, it appears that minimal attention has been given so far to the question of how to support facilitators in designing and leading high-quality PD (Elliott et al., 2009; Weißenrieder et al., 2015). As Elliott et al. have stated more than a decade ago:

> The research community has lagged behind in providing insights on how to best support these new leaders as they facilitate teacher learning. Filling the knowledge gap in the research on leading PD is an urgent issue if teacher learning is to be improved and adequately addressed. (p. 365)

Although some progress has been achieved since then, the gap still exists. In this sub-section, we relate to several core questions regarding this issue, and provide an overview of some suggested answers, gathered from the limited amount of works dedicated to facilitators' professional support.

Herein, we define professional support for facilitators as the in-service education offered to those taking up the role of facilitating mathematics teachers' collaboration. Support may include a CPD course, mentoring or guidance sessions, or any other space that allows for learning and reflection about the practice of facilitation. The core questions we address are the following:

- What are the goals set for supporting facilitators and what challenges may be involved in this endeavour?
- Which models and means exist for this purpose?
- What is an effective support and how can it be studied?

4.4.5.1 The Need for Support and Challenges Associated with It

In making the case for the need to support facilitators in developing their skill and expertise, we reflect upon the literature on novice *teachers* and their transition into the profession. It has long been established that various challenges associated with entrance to the profession of teaching may be addressed using professional support, such as sessions with a personal, content-specific mentor at the school, practicum sessions for first-year teachers, induction programs that include PD designed at the school or even the district level, etc. The model of apprenticeship, a key term in situated learning (Collins et al., 1988), is common in many professions such as law and medicine, and is based on the assumption that novices may learn the fundamentals of the profession by observing, imitating and holding continuous conversations with experts in the field. Similar to other novices, beginning facilitators experience various challenges, some of which are discussed in Sect. 4.3.3 above. For example, they might find it challenging to manage productive discussions (Borko et al., 2014), to implement suitable activities (Jackson et al., 2015) and to navigate between their professional identities (Knapp, 2017; Schwarts, 2020). Hence, there is an essential need for professional support to assist them with these encounters. However, such support requires human and financial resources that are not always available in PD programs. It appears that provision of such support is dependent on the educational system in which the program is embedded. In many countries, PDs are not institutionalised, and may, for example, be initiated by universities as part of grant-funded projects. As such, the design of scaling up, and henceforth the education of facilitators, will vary according to circumstances and could impact the sustainability of the project. It follows that the fundamental challenge in providing support to build facilitators' capacity is finding a feasible model that may be implemented within the system's resources and organisational configuration. The subsequent challenge for PD designers is to incorporate appropriate and relevant content to offer facilitators within the chosen model.

4.4.5.2 Models of Support

Although scant, the literature provides some examples of models of support for facilitators, and these can be categorised as falling into two groups: the first group comprises models where support is blended with facilitators' initial training, such as in the iPSC (Implementing the Problem-Solving Cycle) program reported in Borko et al. (2014) and in the CMP2 (Connected Mathematics Project 2) described in Jackson et al. (2015); in the second group, training and support occur in different distinct times. More common to the first group are embedded approaches where the support team and the facilitators meet before and after each PD session, to reflect on the previous session and build the next session together. This approach is typical for programs with a unified curriculum, where all the novice facilitators are about to provide the same content. An example of a model that would fall into our second category is the model offered by Müller (2003), employed in the EU PRIMAS program (Maaß & Doorman, 2013; see Sect. 4.4.1). The support provided in the Language-Responsive Mathematics Teaching Project in Germany (e.g. Prediger & Pöhler, 2019; Pöhler, 2020) and in the VIDEO-LM program in Israel (Karsenty, 2018) are examples of this category as well.

Another differentiation between models is related to their target audience and to the supporters' other professional roles. Roesken-Winter et al. (2015) described a CPD for facilitators where the participants were facilitators of different PD programs, thus the content provided evolved around general design principles of mathematics teachers' PD. However, in the majority of the reported models, the target audience comprised the facilitators of a specific program, and the supporters were part of the program's team, usually researchers, or, much less frequently, district directors or instructors outside the team. This state of affairs does not necessarily reflect the situation in the field; perhaps only support designed and provided within an academic context is published in research journals. Therefore, there is need for more reports of support with an emphasis on sustainable models where other actors in the educational field provide their guidance.

The support itself may be offered in the more formal setting of a PD course, by personal mentoring or coaching, or in a more ‘natural environment’, such as co-facilitation or the program’s team meetings. In the latter setting, novice facilitators who are also members of the team can learn, in a situated manner, from expert facilitators. Zaslavsky and Leikin (2004) frame this kind of support as the learning of newcomers into the facilitators’ community of practice, and emphasise that these interactions are beneficial both for novices and for experts.

4.4.5.3 Means of Support

The literature describes several means of support that can be divided into reflection-related and content-related means. Reflection-related means are different methods or scenarios that provide facilitators with opportunities to discuss their practice. For example, in the support system developed within the VIDEO-LM program (Karsenty, 2018; Schwarts, 2020), facilitators have opportunities to reflect upon their practice in three different types of communication: with themselves, via writing journals and filling reflective questionnaires; with their peers, in facilitators’ bi-monthly group meetings; with experts, in personal mentoring sessions with senior team members who are experienced facilitators. Another arena for reflection is provided by participation in research. In fact, in many cases means of support overlap with research data collection, when sometimes the first leads to the second or vice versa (Zaslavsky & Leikin, 2004). For instance, Karsenty et al. (2023) described a research design where stimulated-recall interviews served as a research tool; however, a preliminary finding in this research suggested that facilitators considered these interviews as milestones in their professional development. Content-related means for support are the materials provided for facilitators such as suggested activities, manuals, video-clips to use, and so on. Facilitators’ meetings often integrate both kinds of means: they allow time for peer-reflection on practice (for example, while watching and analysing videos of facilitation) and time for engaging with content, for example working collaboratively on planning sessions, including rehearsals or simulations of ‘approximations of practice’ (Borko et al., 2017; Jackson et al., 2015) or discussing theories of professional learning (Perry & Boylan, 2018).

4.4.5.4 Features of Effective Support for Facilitators’ Professionalisation

Effective support for facilitators may be defined by its results, that is the extent to which facilitators are competent in executing the goals set for the relevant teacher collaboration. The following features appear to contribute to an effective support for novice facilitators’ professionalisation, as found in studies looking at such results:

- observation of other facilitators in conjunction with post-session reflection (Psycharis & Kalogeria, 2018);

- help of a personal supporter who is an expert facilitator for the specific PD program (Schwarts, 2020);
- the opportunity to investigate prior PD sessions (via analysis of videos) and to plan jointly for upcoming PD sessions with accomplished others (Jackson et al., 2015).

However, systematic understanding of how support contributes to facilitators' development is still lacking. More research is needed in order to unpack how exactly support affects novice facilitators, and what kinds of support can be sustainable in projects that aim for large-scale dissemination.

In sum, it is clear that the professional development of facilitators is complex. Research shows that facilitators may have different backgrounds and starting points, and that they move through diversified trajectories of becoming facilitators and professionalising in their role. Several models for facilitators' preparation and for their in-service support have been outlined here, and changes in different features of facilitators' work such as knowledge, practices, beliefs, identities, attitudes and agency have been addressed. Knowledge about the professional trajectories of facilitators of mathematics teacher collaboration is accumulating, but more research is needed to extend our knowledge on becoming and developing as a facilitator. A further important consideration in the development of facilitators is how a facilitator negotiates and manages the environment in which the facilitation occurs. Even if a facilitator might be identified as reaching a high level of expertise, environmental factors over which the facilitator will have little or no control may impact the success of the collaboration. The complexity of environmental factors at play that may impact teacher collaboration is the focus of the next section.

4.5 The Role of the Environment in Mathematics Teacher Collaboration

At this point in the chapter, we move away from exploring the role of facilitators in mathematics teacher collaboration to look expansively at the environment of collaboration. Apart from facilitators, various participants make up such environments, and their actions and interactions will impact the outcomes of a collaboration. For the purposes of this section, we define environment as the setting in which teacher collaboration takes place.

The complexity of considering the impact of the environment on the roles, identities and interactions of various participants will become evident throughout this section. First, we overview models for analysing teacher collaboration, selected due to their specific reference to the environment of collaboration (Sect. 4.5.1). We then look internally to the environment created by the actors within a collaboration (Sect. 4.5.2). Next, we turn our attention outward to consider cultural and contextual aspects of environments in which various participants collaborate (Sect. 4.5.3). This is followed by an analysis of an often overlooked but frequently mentioned institutionally-imposed environmental factor that impacts teacher collaboration, that is time allocated for teacher collaboration (Sect. 4.5.4). The facilitator–environment relationship is then challenged through reviewing teacher collaboration *without* facilitators (Sect. 4.5.5).

At several points within this section, we refer specifically to two particular research reports presented at the ICMI Study 25 conference, because they describe teacher collaboration without a facilitator (Cao et al., 2020; Jütte & Lüken, 2020). As such, they provide valuable anchor points for analysis both of the environment and of the role of facilitators within it. As we return repeatedly to these studies, the complexity of analysing the environment in relation to teacher collaboration is highlighted.

4.5.1 Environments of Collaboration in Research: Review of Existing Models

In their plenary paper to the ICMI Study 25, Krainer and Spreitzer (2020) considered collaborative groups in mathematics teacher education to interrogate the diversity of roles, identities and interactions. They framed their analysis in accordance with three elements in teacher collaboration, one of which was the environment, thus emphasising the importance of this component. The other two elements were identified as actors and targets. To highlight the extensive and expansive nature of the environments in which teacher collaboration occurs, they listed "departments, schools, school boards, districts, committees, ministries or enterprises" (p. 23) as examples of environments that impact teacher collaboration. To analyse teacher collaboration on an international scale, Krainer and Spreitzer identified reports of mathematics teacher collaboration projects recently published (2018–2019) undertaken in Africa, Asia, Australia, North America, South America and Europe. Environments were only minimally addressed in these reports, with Krainer and Spreitzer concluding that there was little discussion about the environments in which the collaborations occurred, or even environments being mentioned as a factor impacting the success or otherwise of the collaborations. As they stated, "it would be interesting to read more about context" (p. 34), as "only a few initiatives describe the context and relevant environments having a potential impact on the initiatives" (p. 35).

Krainer and Spreitzer's analysis revealed that it is not common for the role of the environment to be the focus of interrogation when large studies of teacher collaboration are reported. Yet, it is an important aspect of consideration in relation to sustainability and scalability if mathematics teacher collaboration is regarded as teacher learning. Underscoring the complexity of learning through collaboration, Krainer (2008) stated that, "it is important to take into account that teachers' learning is a complex process and is to a large extent influenced by person, social, organizational, cultural and political factors" (p. 2). This statement reminds us of the need to pause and think about the impact of the environment on teacher collaboration, and to give due consideration to the fact that successful collaboration requires more than artful and expert facilitators.

In exploring teacher learning, Llinares and Krainer (2006) discussed the work of Peter (1995), who proposed a model that incorporates four analytic domains (personal, practice, inference, external) and four associated mediating processes (teacher knowledge and beliefs, classroom experimentation, valued outcomes, stimulus or support). This model emphasises the expansive nature of factors that impact teacher learning and, by extension for our chapter, teacher collaboration. It is the external domain and the mediating processes of support that are of interest here. As stated by Llinares and Krainer, "The nature of changes undergone by teachers suggests a link between individual change processes and external conditions determined by school culture" (p. 446). They also referred to Krainer's (1994) model that includes four dimensions of action, reflection, autonomy and networking, and that can be used both to "design in-service courses and explain how teachers' learning is generated" (p. 447). In Krainer and Zehetmeier (2013), this model is framed as a learning system, "where action, reflection, autonomy and networking are regarded as important educational dimensions to consider when analysing educational practice" (p. 875). The following statement elaborates the operation of these four dimensions in a teacher professional development project: "the focus is on discussing in groups, negotiating meanings and norms, sharing knowledge, collaborative learning, designing didactical contracts, institutional constraints and organizational and systemic aspects that foster or hinder teachers' learning" (Llinares & Krainer, 2006, p. 450). In this summary statement, we can see the interconnected nature of the individual and the environment, with individual teachers engaging in group discussions which have their own meanings and norms, of applying their learning and sharing with others the outcomes of their experimentation, together with operating within the constraints and affordances of their working environment. While this model does not explicitly capture the role of the facilitator, it does consider the role of the environment on teacher collaboration. However, the role of the facilitator can be extrapolated here as taking responsibility for managing discussions among teachers, or between teachers and teacher educators, establishing the meanings and norms, facilitating the sharing of knowledge and negotiating the environment of collaboration.

Another model that serves to encapsulate the potentially problematic or supportive role of the environment in teacher collaboration was proposed by Goos (2013). In this model, Goos adapted Valsiner's (1997) zone theory of child development to propose a zone theory model of teacher development and, hence, teacher learning. Valsiner's zone theory incorporates Vygotsky's well-known zone of proximal development (ZPD) and two other zones: the zone of free movement (ZFM) and the zone of promoted action (ZPA). According to Goos, the ZPA can be "interpreted as activities offered via teacher education programs, formal PD, or informal interaction with colleagues that promote *certain* teaching actions" (p. 523; *italics in original*). Further, "The ZFM structures the teacher's environment, or professional context" (p. 523), and elements of the ZFM could include teachers' knowledge and beliefs, curriculum and assessment requirements, but also *organisational structures* (school timetable, class allocations, student groupings) and *organisational cultures*. A succinct summary of these two zones is offered by Bennison and Goos (2013), who stated that, "The ZFM is related to the individual's environment and includes the set of actions that the individual is *allowed to perform*, while the ZPA is the set of actions that are *promoted* by the individuals" (p. 4; *italics in original*).

From this perspective, the ZPA can be initiated by external influences and suggestions of facilitators of professional learning. The ZPA can also be initiated as a result of informal collaborations with others, and therefore may take place in the absence of a designated facilitator. ZFM is about teaching actions that are permitted; that is, the freedom with which teachers may be allowed to move in relation to developing and applying their learning. This zone captures both restrictions and affordances internal to the individual and external in the environment. As stated by Goos (2013), "Zone theory offers a useful framework for research that aims to understand the complexities of teachers' learning [. . .] But another important line of inquiry [. . .] is concerned with promoting *change* in classroom practice, and [. . .] zone theory can also provide guidelines for designing such projects and interpreting the outcomes" (p. 528; *italics in original*). It is an interesting exercise to consider zone theory in relation to the role of the facilitator and the role of the environment in teacher collaboration. The role of the facilitator is to promote teacher action, and therefore the facilitator operates and is located in the ZPA. The ZFM relates to the environment in which teachers are permitted to apply their new learning, but the environment can also be seen as a factor in promoting teacher action. In fact, as these two zones work together, Valsiner suggested that they be considered as a 'ZFM/ZPA complex' (Bennison & Goos, 2013, p. 4).

A further approach for interrogating teacher learning is through the concept of teacher agency. In Sect. 4.4.4, shifts in agency of facilitators were discussed. Here, we outline Imants and Van der Wal's (2020) model of professional development, teacher agency and school reform, noting that it does not specifically address the role of the facilitator, but is included here as another lens for analysing the role of the environment in teacher collaboration. The basis of this model lies in theorising the extent to which teachers enact new learning, which, according to them, must be considered in relation to the environment in which teachers work. Both agency theory and organisational theory underlie this model. If we consider teacher collaboration as a process for seeking improvement of the current state, then the outcomes of such collaboration are both related to the actions of the teacher and in accordance with the environment in which s/he works. The complexity is that the role of the environment and the role of the facilitator in teacher collaboration are mediated by the activity of the teacher. As stated by Imants and Van der Wal:

> Teachers' enactments of their work environment in the context of professional development and school reform primarily are located in the teachers' own sphere of influence, this is in the interactions where they practice their agency. Besides, general school conditions will affect teachers' action in their own spheres mediated by teachers' interpretations. (p. 5)

Agency theory, then, is potentially useful, by extension, for considering the role of the facilitator in teacher collaboration and how the environment may enhance or hinder mutual learning of teachers and other participants in collaborative teacher interaction.

The models presented in this sub-section are important as they bring our consideration of the role of the environment within teacher collaboration to the fore. They provoke us to look more deeply into various issues associated with the environment in relation to teacher collaboration. In the next sub-section, we explore different actors and the interactions between them, as we look internally to the environment created by various stakeholders in the collaboration.

4.5.2 The Environment Created by Different Actors Within a Collaboration

Different actors within collaborative communities, and the interactions between them, create the human environment in which teachers' communities can develop (Cestari et al., 2006; Jaworski et al., 2017). Collaboration between teachers and other members of the school community not only contributes to the professional development of teachers, but also leads to better student achievements (Darling-Hammond et al., 2017; Robutti et al., 2016). In order to realise the foundation for collaboration that supports interaction between various actors, we turn to a fundamental theoretical perspective for collaboration within communities; that is, communities of practice (Wenger, 1998).

'Community of practice' is a well-known term used in reference to collectives of people committed to learning and developing through sharing their knowledge and practice (Wenger, 1998, 2011; Wenger & Snyder, 2000). Although regarded as a contemporary term, the premise of groups of people forming together to learn from and with each other is an enduring social practice that is age-old (Wenger, 2011). Of particular note is Wenger's discussion of spontaneously formed groups in educational settings, where teachers meet together during break times, sit together in the staff room, sharing practice, commenting on student progress and discussing various approaches they have tried and found successful.

A community of practice is a joint enterprise in which members mutually engage to produce a shared practice that is reflective of their collective processes of learning. In communities of practice, different members can interact in different ways to support the community, to maximise its benefits. In some cases, communities of practice develop to solve a particular problem or to achieve a certain outcome, but, fundamentally, they are a group of people who mutually engage "in a process of collective learning in a shared domain of human endeavor" (Wenger, 2011, p. 1). Extending from this perspective, Jaworski (2006) proposed a shift from learning within a community of practice to forming a community of inquiry. This involves the process of critical alignment in which members of the community critically question and reflect upon their roles and on the purposes of the collaboration as they progress in their collaborative learning toward a shared product.

It is important to note that Wenger (2011) defined a community of practice as a group where members join through self-selection. He distinguished this from other organisational working groups in which members may have been forced to join. This is a critical distinction when considering the environment created by the actors in the collaboration. One of the Theme C studies describes how teachers work collaboratively in a self-selected manner (Cao et al., 2020). From this study, we see Chinese teachers' informal, daily interactions with different people, including friends, leaders of local school-based groups, colleagues in the same subject group, the 'neighbour who is sitting next to you in the office' and mentors in apprenticeship programs. Cao et al.'s study suggests that, when seeking help with regard to teaching, Chinese teachers tend to pursue interactions with colleagues in teaching research groups or in lesson planning groups. Whilst not presented as a collaboration that reflects a community of practice or a community of inquiry, this study is interesting to consider in relation to the environment of collaboration and the voluntary roles of the actors.

Research also indicates that, through the experiences and interactions between different actors in a collaboration, actors might shift or modify their roles over time (Krainer, 2008; Kramarski & Kohen, 2017; see also Ribeiro's, 2020 Theme C paper). According to Hunter (2010), the possible change of roles might be affected by several key aspects, including the nature of the communication and the collaborative partnership between the various actors, the setting for collaboration and the social and cultural backgrounds of the members in the community.

The interest in how teachers learn through interaction with other stakeholders is widely reflected in research on the Japanese Lesson Study model. This form of collaboration focuses on teachers' interactions and involvement with colleagues and 'knowledgeable others' (Takahashi, 2014), through a practice-based approach within a professional community. Japanese Lesson Study was elaborated by Isoda (2020) at the ICMI Study 25 Plenary Lecture for Theme B. A Lesson Study cycle includes the development of a lesson plan, the implementation of the lesson plan in a classroom by one of the group members, while other teachers observe, and a post-lesson discussion to analyse its impact on students' learning, in which the group may decide to plan an improved version of the lesson, and the cycle begins again (Chen & Yang, 2013; Fernandez & Yoshida, 2004; Lewis, 2002; Lewis et al., 2009). In some Lesson Study scenarios, and particularly so in Japan, actors operate in an environment where roles are clearly defined (as discussed in the Theme C paper by Clivaz & Daina, 2020; see also Clivaz & Takahashi, 2018, and Sect. 4.5.3 below). However, this might not be the case for other scenarios where Lesson Study is implemented. For example, as we mentioned earlier, the role of the 'knowledgeable other' varies across and within cultural contexts (e.g. Adler & Alshwaikh, 2019; Gu & Gu, 2016; Lewis, 2016; Takahashi, 2014).

Several Theme C papers described the roles of various participants in the collaboration. While not explicitly stated, the environment created by different actors within a collaboration can be extrapolated as a community is formed and we see actors shifting roles. It is reasonable to assume that the environment of the collaboration enabled forward progress of the group's work. Pereira et al. (2020) describe a collaboration between three groups of actors: supervising teachers (counselors) who were university lecturers in mathematics, and who formed a link between the university and school administrators; the teachers in the project schools who acted as facilitators and helped pre-service teachers to adapt to the life of the school community; pre-service teachers who were studying for a mathematics degree course. Ribeiro (2020) describes how teachers and teacher educators, working collaboratively, created opportunities to reflect on their knowledge and to share their practical classroom experiences. A dynamic process of changing of roles was reported, with participants sometimes acting as learners and, at other times, acting as educators. Kohen et al. (2020) report on a collaboration amongst mathematics teachers who solved challenging problems in an online environment termed Problem-Solving Forums (PSF). The study's focus was on the importance of collaboration for the development of teachers' dual roles – from students in the PSF to mentors of others in this environment. These studies, and others as outlined previously (see Sect. 4.4.4), show a range of actors taking on various roles, as well as shifting roles in teacher collaboration.

At this point in this sub-section, and as we grapple with considering the environment created by the various actors in a collaboration, it is worth considering the seminal study of Horn and Kane (2015), in relation to opportunities for professional learning in mathematics teacher workgroup conversations. From their large corpus of data collected over a six-year professional development initiative, and working with groups of teachers located at their various schools in one district, Horn and Kane noted the progress of some groups over others. In their detailed analysis of collected survey, interview and observation data (during workgroups and in classroom practice), they concluded that, "a teacher community cannot be the only lever for change" (p. 415). They determined that teachers with greater knowledge for mathematics teaching brought much richer conversations to the workgroup, which in turn provided greater opportunities to learn for the whole group. It was noted that the presence of the facilitator promoted greater opportunities in all workgroups, but, in their absence, these opportunities varied considerably. The district contributed to the environment through provision of time for workgroups to meet. Internally, actors impacted that environment.

In this sub-section, we briefly overviewed Wenger's (2011) concept of communities of practice and Jaworski's (2006) concept of a community of inquiry. The former concept is useful for reflecting upon collaboration that occurs without a facilitator. We then provided examples of studies that included a range of participants who, in some cases, took on various roles. All participants contribute to the environment of the collaboration and serve to impact it. In the next sub-section, we turn our attention to considering cultural aspects of environments in which participants collaborate, and consider more deeply the facilitator–environment relationships.

4.5.3 Cultural Aspects of Environments in which Various Actors Collaborate

Facilitators operate to promote collaboration, but they do not operate in a vacuum. Facilitators move and act in varying environments. Negotiating the environment is an important part of the role of the facilitator in teacher collaboration. It is also interesting to consider the role of the environment when teacher collaboration occurs without a facilitator. The impact of the environment on teacher collaboration is considered in this sub-section by interrogating environments and how they served to assist, or worked to impede, the collaboration. From this analysis, we see how facilitators manage environments, but also view how environments without facilitators may support or otherwise hinder teacher collaboration.

Few Theme C papers specifically addressed the role that the environment plays in teacher collaboration, yet some papers referred to environments as contexts in which the studies took place, while showing the influence of context and/or culture. Several studies referred to the solitary, or private, nature of teachers' work in schools. For example, Nieman et al. (2020) mentioned the "deeply established norms of privatization in US schools" (p. 500), with reference to Little (1990) and Lortie (1975). They further stated that only recently time has been built into the teachers' workday to enable collaboration. This was echoed in another Theme C paper by Griese et al. (2020), who stated that the "last 40 years have been characterized by a shift from teachers working in isolation to collegial collaboration" (p. 468), and in line with the work of Krainer (2003) pointing to the gradual shift towards teachers' work in more collaborative ways. Pereira et al. (2020) also mentioned the tradition in their country (Brazil) of teachers "working in a solitary way" (p. 510). Widjaja and Vale (2020) contextualised their study through reference to the work of Robinson and Timperley (2007), who discussed the difficult role of facilitators when teacher autonomy and privacy of classroom practice is the norm. They further referenced the work of Vangrieken et al. (2015), who described the impact of school culture on teacher collaboration where there is a "longstanding culture of teacher isolation and individualism, together with teachers' preference to preserve their individual autonomy" (p. 35, as cited in Widjaja & Vale, 2020, p. 556). Through analysis of just this small collection of studies, it appears that, although gradually changing, there is a long-standing tradition of teachers working in isolation, where their practice is very private and individualised, in many countries across the world.

In contrast, in another Theme C paper, Cao et al. (2020) stated that their study was located in schools that have an established environment of teachers working collaboratively. They emphasised that, "Chinese teachers have traditionally worked in groups" (p. 444). This feature of Chinese teachers' practice has been previously noted by Hargreaves and O'Connor (2018). Zhao et al. (2020) elaborate the specific roles of participants in Lesson Study in China, showing that clearly defined roles contribute to the environment for Lesson Study to take place. As stated by Zhao et al., the university educators/researchers bring theory on teaching and learning for teachers to draw upon as they develop lessons that are then viewed and critiqued by all. Zhao et al. reported on the initially passive role taken by the teachers in their Lesson Study research, due to cultural norms of deference to university educators/ researchers as knowledgeable others. However, through the course of the project, group members began to engage in collaborative dialogue, facilitated by a teacher educator. Zhao et al. applied Activity Theory to analyse their data, concluding that effective Lesson Study requires “active boundary brokers”, “shared tools” and “more equal rules and division of labor” to “promote interaction between teachers and researchers” (p. 570).

Three specific Theme C studies are outlined here because of their contribution to considering the impact of the environment on teacher collaboration. Jütte and Lüken (2020) investigated teacher collaboration without a facilitator (elaborated in Sect. 4.5.4 below). They described two teachers who were required to teach together due to an organisational shift in policy that resulted in a more inclusive approach for teaching children with special needs in regular classrooms. The researchers reported that this collaboration, which, as was already said, occurred without a facilitator, was not overly successful for a number of reasons. When interrogating the environment, it can be seen that the collaboration was forced upon the participants, that there was little time provided by the organisation for the participants to plan together and get to know each other’s teaching style and strengths, and that, as identified by one of the participants, there was no “structural framework and guidance by the school administration which accompanied the collaboration” (p. 480). Yet, both teachers joined the teaching team voluntarily and shared similar views about the importance of inclusive education. In this case study, we see that the participants pointed to environmental factors as impacting their collaboration. The researchers described how the team was “not yet situated in a positive school culture” (p. 482), and posited the potential value of a facilitator to progress this collaboration in a productive way. In contrast, Cao et al. (2020), who investigated Chinese mathematics teachers’ informal interactions and collaborations without a facilitator (also elaborated in Sect. 4.5.4 below), reported positive outcomes. Their research discussed the environment in which teachers work in Chinese schools, and specifically emphasised that the “Chinese school collective culture provides teachers with opportunities to seek help ‘within their reach’” (p. 450). In this study, we see that the environment played a significant role in supporting teacher collaboration: as collaboration is a typical and unexceptional process in schools, teachers readily collaborated with colleagues, even without the presence of formal facilitation.

From another viewpoint, Choutou (2020) described her research into assisting teachers collaboratively to plan and teach interdisciplinary units in mathematics and arts classes. The researcher was the facilitator and Choutou’s study provides a personal account of the difficulties she encountered in her role. Of the environmental factors that impacted the collaboration, she mentions the difficulty in finding a suitable and quiet space to meet with participants, and the importance of scheduling meetings at a time when all participants could meet. She makes reference to wishing to avoid a ‘top-down’ model (as per Cestari et al., 2006, p. 1357) as the facilitator, but that potentially “a stakeholder from ‘above’ would make things easier” (p. 458), indicating how the environment could have provided greater support in progressing the collaboration in this study.

Of these three case studies presented in Theme C, the study by Cao et al. (2020) clearly emphasises the potential of the environment to impact teacher collaboration positively. In the other two cases presented, we see one report on teacher collaboration without a facilitator and one with a facilitator, and in both studies the environment seemed to have a non-positive impact on the collaboration. Other Theme C papers also touched upon the role of the environment in different collaborations. Clivaz and Daina (2020) discussed Lesson Study where all participants have well-defined roles, and the participation is led by a 'knowledgeable other'. In this case, it can be seen that the environment provided clear demarcation of roles. Further, Nieman et al. (2020) discussed a whole-school vision of high-quality mathematics instruction emanating from the school principal. As emphasised by Nieman et al., it was the work of the school leader, who valued teachers' collaboration and who created conditions necessary for it to occur, that ensured that the collaboration was effective. Interestingly, Nieman et al. mentioned the fragile nature of the environment at this school, because the work of the principal and senior leadership team did not readily align with the established tradition of practice in schools in the district, where privatisation of practice was the norm and whole-school visions for teaching and learning mathematics were typically unapparent.

Whilst scarcely emphasised or addressed in studies that discuss the role of facilitators within teacher collaboration, it is clear that the environment does play a role in collaboration. However, the management of the environment to foster collaboration is not clear cut. As stated by Nieman et al., "creating a structure in which teachers are expected to collaborate, on its own, does not necessarily lead to strong community nor does it necessarily benefit teachers or students" (p. 501). To explore this issue more deeply, we consider institutional systems that impact the environment of collaboration.

4.5.4 Institutional Systems and the Provision of Time for Participants to Collaborate

One of the repeated themes emanating from studies that address the role of facilitators in teacher collaboration is time. While not explicitly addressed as an issue in many reports, it is a theme that is repeated in many studies on teacher collaboration. In the Theme C meetings, it was mentioned in various studies; hence, it warrants elaboration. In this sub-section, a brief summary of how time, as an institutional factor that impacts teacher collaboration, is overviewed. In her study of facilitating collaboration between mathematics and arts teachers, Choutou (2020) frequently makes reference to time, although it was not overly emphasised as a key finding associated with the collaboration. Early in her report, however, she refers to the issue of strict timelines that limit teachers' options to meet, resulting in situations where teachers may not have sufficient time for collaboration. Throughout her study, we see teachers arriving "tired and under time pressure" (p. 456) to the collaborative meetings. We read of teachers making demands on the facilitator as a result: "Give me already designed tasks or even ideas to use them" or "I would like that what is brought to the group meetings would be something that concerns everyone and that everyone can learn from it [. . .] Why should I waste my time on something that I already know?" (p. 456). These words highlight that teachers need to feel that the time they invest in a collaboration is worth spending, especially if they are under considerable time constraints. This concurs with Cramer and Stivers' (2007) claim that, ideally, "Collaborative relationships can be a rich source of professional and personal growth, well worth the investment of time and effort that may be necessary to nurture them" (p. 10). Choutou also mentions time pressures in facilitating the collaboration, in terms of managing participants' contributions while also achieving the goals of the meetings. Specifically, she noted the difficulty of scheduling enough time for the meetings to allow for productive discussions, and the dilemma as a facilitator, not interrupting and redirecting teacher discussion that is not overly focused on the goal of the PD. It appears that giving due consideration of time is an important aspect of the environment for collaboration. In another Theme C paper, Ribeiro (2020) described the importance of taking time to develop the collaborative learning environment, noting that it took several meetings to establish an environment of trust and mutual respect among participants. Widjaja and Vale (2020) also discussed the important benefits that school-based facilitators noted, when they gave teachers sufficient time to perform activities that were at the core of the project.

From a different perspective, Jütte and Lüken (2020) discussed how the lack of time provided by the organisation impacted the capacity for teachers to collaborate and learn together through a 'forced' collaboration situation. As previously mentioned, in this study two teachers collaborated to co-teach an inclusive classroom where children with special needs were integrated. Initially, the teachers were keen to work together and learn from each other, but, as time progressed, the teachers developed their collaboration in a such a way that they neither prepared lessons together nor taught together. Surprisingly, and much in contrast to what both teachers reported that they wished for, the children with special needs were taught separately from the whole class in mathematics, with the special education teacher responsible for the preparation and teaching of mathematics for the whole class, while the primary teacher separately teaches the children with special needs. As Jütte and Lüken note, this distribution of roles is rather unusual (Friend et al., 2010; Scruggs et al., 2007). One recurring explanation was the amount of time that would be needed in a collaborative preparation of all lessons. In order to save time, the teachers allocated subjects to each other with respect to their professional competence. Neither of the teachers felt competent in mathematics, but as they wanted to divide the labour, and as the special education teacher had recently taken a course in teaching mathematics, he took over the mathematics instruction. The primary teacher felt uncomfortable with the way mathematics lessons were being executed, but, because of her lack of mathematical knowledge, she did not feel in the position to make alternative suggestions. The special education teacher, on the other hand, resented the double requirement of preparing and teaching the whole-class instruction and additionally preparing the tasks for his colleague who was lacking mathematical knowledge as well as special needs expertise. Both teachers were unhappy about their daily teaching and their collaboration, which was reflected in an increasing tension on an interpersonal level. Professional learning from each other was not happening.

This case study shows how the teachers approached the lack of provision of time for collaboration offered by the organisational environment. It also highlights the lack of growth of mathematical knowledge for teaching that possibly could have been promoted, if the special education teacher could have taken a more facilitative role. Yet this did not occur due to lack of time. This case provides an interesting position from which to consider simultaneously the role of the environment with respect to time and the role of facilitation (or the lack of it) in promoting or hindering the development of mathematical knowledge for teaching within teacher collaboration.

4.5.5 Collaboration Without Support of a Facilitator

The role of the facilitator can be placed under the microscope through analysis of teacher collaborations in the absence of a facilitator. Analysis of studies on teacher collaboration without a facilitator can identify situations during collaboration in which external advice is needed, hence pointing to the potential value of a facilitator. Analysis may also indicate the kind of support that is missing with regard to professional learning, and thus further define the required role of a facilitator. Two studies of teacher collaboration without a facilitator are detailed here, to draw implications for the role of the facilitator in teacher collaboration.

It is commonly accepted that collaboration between mathematics teachers can occur without the support of a facilitator, through teachers' informal interactions with colleagues. Significant outcomes of teacher learning can result due to daily conversations, where teachers exchange their ideas and opinions about teaching. These informal talks promote the development of teachers' understanding of knowledge, their creativity in teaching and any changes that may be occurring in the classroom (Cross et al., 2002). Penuel et al. (2009) studied the role that informal teacher interactions play in helping teachers enact instructional changes in practical teaching. They argued that teachers' interactions draw not only on the social context of the school, but also on the expertise and resources that can be exchanged through these interactions. Similarly, Chen and Yang (2013) outlined Chinese teachers' construction of a reform-based teaching strategy, and concluded that teachers within the school context had a shared interpretation system as revealed by their 'native discourse' (their daily language use, concepts and interactions). The significance of teachers' informal collaborations, without facilitators, informed the study by Cao et al. (2020) as further elaborated below.

Cao et al. studied Chinese mathematics teachers' informal interactions within schools. Their study occurred in two phases. In the first phase, 72 middle-school mathematics teachers from three cities in China were surveyed on collaboration with colleagues on teaching-related issues. Responses indicated that teachers collaborated most frequently with colleagues who taught the same subjects with the focus of collaboration on lesson planning and analysing student learning of mathematical knowledge, but less so on sharing and reflecting on student performances during lessons. The results raised questions on how teachers view informal interactions in their daily practice, which led to the second phase of the study that aimed to examine these questions. Individual interviews revealed that informal interactions helped teachers to prepare and revise their lesson plans. Although no facilitators were involved in these informal interactions, teachers were willing to initiate discussions with colleagues about teaching-related issues, through which they also solved problems or learned from each other. The study emphasised the importance of informal interactions in daily teaching, but suggested further research is needed to investigate how informal interactions without a facilitator might contribute to sustaining mathematics teachers' professional learning in teaching practice.

As also outlined earlier, Jütte and Lüken (2020) investigated the first 2 years of a German teaching team collaboratively teaching mathematics lessons in an inclusive classroom. The two teachers had different professional backgrounds, with one being trained to teach whole classes and the other to teach children with individual needs. It was expected that, in this inclusive classroom setting, both teachers potentially would profit from their different expertise through formally interacting with each other on a daily basis. As stated by Hargreaves (2001):

> Teaching together is reputed to be better than teaching apart. Cooperation and collaboration among teachers give access to the new ideas, creative energy and moral support that helps them to be more effective with their student. (p. 503)

In Jütte and Lüken's study, the mathematics lessons were observed and further data was collected by informal talks and semi-structured interviews with both teachers. Results showed similar requisites regarding the teachers' orientation towards inclusion, their didactical–methodical approaches and their idea of working as a team. Both teachers formulated the joint teaching of children with and without special needs in one classroom as their concept of collaborative teaching in an inclusive setting. Both considered frequent exchange about the collaborative work as important. However, it became clear that both teachers had not been trained in mathematics education, resulting in the special education teacher taking responsibility for planning and teaching all mathematics lessons, as detailed earlier. This became a major point of contention in this collaboration. Hargreaves found that collaborating teachers tend to avoid disagreement and conflict. While teachers who teach collaboratively value appreciation and acknowledgement, as well as personal support and acceptance (Hargreaves, 2001), differences in instructional approaches and conflict over strategies need to be actively addressed (Little, 2002), which was not the case here.

This study, where the collaboration occurred without a facilitator, points to the need for a facilitator on three levels. First, a facilitator is needed to initiate professional learning. Second, the facilitator's role is crucial in mediating conflicts if the collaboration struggles. In this case, regular conversations about the collaboration, which would be coached by an external person, might have helped to clear misunderstandings and turn the swelling conflict into a professional exchange about goals for collaboratively teaching an inclusive classroom. Third, this specific team also needed professional support in teaching mathematics, as both teachers felt incompetent to prepare lessons satisfactorily. Thus, the role of a facilitator as an expert in mathematics teaching appears to be essential.

Upon reflection of both studies presented here on teacher collaboration without facilitators, we see contrasting outcomes. We do not aim to resolve this discrepancy. Rather, we view these studies as revealing the complexity of the facilitator–environment relationships, a complexity that calls for further consideration within future research.

In sum, the studies reviewed in this section engage various types of learning environments that enhance or hinder mutual learning of teachers and other participants in collaborative interactions. Although the studies presented are predominantly of a small scale, they provide some useful insights for considering the role of the environment in teacher collaboration. Llinares and Krainer (2006) stated that there is a need for large-scale studies "understanding from a broad perspective how different contexts influence teachers' learning", as it "is essential to investigate their involvement in different relevant environments in which they work, which environments influence them and which are influenced by them through their learning" (p. 451). While not providing definitive answers to these questions, this section of the chapter has highlighted the impact and importance of the role of the environment in teacher collaboration.

4.6 Concluding Comments

This chapter has explored the roles, identities and interactions of various participants in mathematics teacher collaboration at four levels: (1) research methodologies; (2) the role of the facilitator; (3) the professional trajectory of the facilitator; and (4) the role of the environment. We drew upon previous reviews and studies, and also presented an analysis of all 16 papers submitted specifically to Theme C at the ICMI Study 25 conference. It should be noted that the topic of discussion is incipient, as evidenced also in the way in which these papers are distributed by regions of the world. Seventy-five percent of the papers in Theme C are from countries or regions of the world that generally have significant research advances (eight from European countries, two from Israel, one from Australia and one from USA). The other 25% are evenly distributed between Brazil and China. This distribution can also be compared with the general distribution of papers for ICMI Study 25 by region. We therefore see a gap in research aligned to this topic in other countries that are not represented here. We wonder about the research in other countries published in languages other than English, and acknowledge that there is further work to be done to advance this field of inquiry.

In the second section of our chapter, we focused on methodologies and theoretical perspectives of research into different actors within teacher collaboration, to provide an overview of this growing field of research. We found that application of the RATE tool (Krainer et al., Chap. 8, this volume) assisted us to undertake our analysis. Our results aligned with findings of previous reviews, in that research approaches in this field are predominantly qualitative, there is a dominance of self-studies and the majority are small-scale studies. We noted that there is a need for larger studies that utilise quantitative approaches, yet that there is still room for further small studies to continue to build knowledge and theory in this domain. Many studies in the set of the Theme C papers centralised the role of the facilitator. In order to add to our knowledge of various participants in mathematics teacher collaboration, there is a need to extend the focus of reported research to interrogate the role of other stakeholders in the collaboration.

Because of the key role that the facilitator plays in teacher collaboration, the third and fourth sections of our chapter were dedicated to this focus. In the third section of the chapter, we looked specifically at literature around conceptualisation of the role of the facilitator in promoting mathematics teacher collaboration. From the current frameworks and models, we identified the academic knowledge and social and interpersonal skills needed by facilitators, as key to this profession. This section also discussed facilitators as professional experts, and overviewed research around how successful facilitators promote knowledge of teachers. The delicate balance of managing teacher conversations in a PD program, together with propelling teachers forward in building their own knowledge for mathematics teaching, is an important research finding associated with developing facilitator expertise. Our review pointed to the need for further research into how facilitators encourage the collaborative relationships and build a sense of community amongst participating teachers.

In the fourth section of our chapter, we interrogated the developmental trajectory of becoming a facilitator and professionalising in this role, and considered the types of support required for facilitators to develop their expertise. It is not common for research on teacher professional development programs to report on the role of facilitators, their preparation and the support provided to them to orchestrate such programs. Our review here pointed to issues faced by facilitators, particularly how they often must negotiate shifting roles between being a classroom teacher and becoming a facilitator. The importance of ensuring that facilitators are prepared and supported is evident when considering how to scale-up programs, while warranting that program outcomes can be sustained. We pointed to the need for research-based professional development programs for facilitators, aiming that the preparation and support of facilitators will be an integral component of the teacher professional development programs.

The fifth section of our chapter focused on the environment as an integral component of teacher collaboration. In this section, we overviewed selected models that might be useful in analysing the success or otherwise of teacher collaboration.

These models emphasised that environmental factors are at play in any collaboration. In this section, we briefly overviewed the roles that participants other than facilitators may play in teacher collaboration. We also gave due consideration to analysing teacher collaboration with and without facilitators. Such research findings remind us that, while facilitators play a pivotal role in achieving the intended outcomes of a collaboration, facilitators may not be the sole drivers. Despite the expertise and high-level knowledge of a facilitator, the environment may serve to impact the collaboration outcomes. Conversely, there are examples of successful teacher collaboration in the absence of facilitators.

Taken together, these four sections (Sects. 4.2, 4.3, 4.4, and 4.5) in this chapter have drawn upon current research that contributes to our understanding of how various participants take part in, and influence, collaborative endeavours of mathematics teachers. There is further research to be done to build theory around this topic. We need more research to broaden our knowledge of the professional trajectories of Mathematics Teacher Educators (MTEs) and facilitators of mathematics teacher collaboration. We also need more research on the impact of different actors in the environment on teacher collaboration, as well as on facilitators, and how facilitators negotiate the environment of a collaboration.

Another future direction to pursue further concerns addressing the problem of scaling up collaboration programs, going beyond the training of facilitators and the contextual conditions that are required for this to be effectively carried out, and looking into other questions. For instance, how do institutional factors impact the sustainability of mathematics teacher collaboration? What models of scaling-up programs of teacher collaboration exist, and what are the roles of different actors in these models? We also need to know how the topic of actors involved in teacher collaboration is treated in other publications that are not written in English. While there is still much work to be done, we recognise the progress made in recent years in studying different roles in mathematics teacher collaboration, reflected in the considerable body of research that we have drawn upon here to address this important issue. We look forward to seeing this interesting field of research further developing in the future.

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Chapter 5
Tools and Resources Used/Designed for Teacher Collaboration and Resulting from Teacher Collaboration

Ornella Robutti, Luc Trouche, Annalisa Cusi, Giorgos Psycharis, Ruchi Kumar, and D'Anna Pynes

5.1 Introduction

This chapter, corresponding to Theme D in the conference, is dedicated to tools and resources used in, designed for and resulting from teacher collaboration, and is composed of six more sections after this brief one. Section 5.2 introduces the

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essential issues to be addressed. Sections 5.3 through 5.6 are dedicated to tools and resources: designed for teacher collaboration (Sect. 5.3); for learning to improve teaching practice in collaboration (Sect. 5.4); for fostering collaboration (Sect. 5.5); for studying collaboration (Sect. 5.6). Section 5.7 looks to the future, considering not only the discussion at the conference, but also other possible topics of interest for research.

5.2 Resources, Tools and Collaboration: Essential Issues

5.2.1 Presentation of the Chapter, in Continuity with the Past

This chapter describes the past experiences (Sect. 5.2.1)—mainly previous ICMI studies and the ICME 2016 survey on teacher collaboration (Robutti et al., 2016)—and the contemporary experiences—other chapters in the book and plenaries (Sect. 5.2.2)—and progressively guides the reader to the main issues at stake: what tools and resources support teacher collaboration or are derived from teacher collaboration (Sect. 5.2.3). To express this continuity adequately over time and themes, this section aims to get readers' attention to the matter of keywords and their role and contextualisation in theoretical frames and methodological issues (Sect. 5.2.4). Meanings of keywords, paper presentations and the modalities of working in collaboration among the Theme D participants group were discussed in person and remotely before, during and after the conference as described in Sect. 5.2.3.

This sub-section essentially frames this present chapter on two elements: the previous studies that led to the theme of the present ICMI conference, on teachers working collaboratively and the continuity with past ICMI Studies that contain some seeds of this theme, in relation to collaboration and use of tools and resources. Since the ICME survey on mathematics teachers working and learning through collaboration (Jaworski et al., 2017), there has been a continuous increasing "interest in exploring and examining different activities, processes, and the nature of differing collaborations through which mathematics teachers work and learn" (Robutti et al., 2016, p. 652). And the proposal of the present ICMI Study 25 is in continuity with that interest. In particular, the present chapter, on tools and resources for/from teacher collaboration, finds a continuity in previous Studies, such as the following.

1. The 15th ICMI Study: The Professional Education and Development of Teachers of Mathematics, 2005, for the attention on mathematics teachers' knowledge (Liljedahl et al., 2009) as a component of teacher education, and on the methods and forms inside the institutions that support learning in collaboration, for example Lesson Study (da Ponte et al., 2009), with some focus on the tools and resources for teacher collaboration.
2. The 17th ICMI Study: Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain, 2006, for the theoretical frameworks emerged (e.g. the instrumental approach—Guin et al., 2005), and the ways in which

technology can mediate, support and influence the teaching and learning of mathematics, especially for their relation to collaborative practice, as a precursor of the different sections present in this chapter, as we can see in the corresponding book (Hoyles & Lagrange, 2010).

3. The 22nd ICMI Study: Task Design in Mathematics Education, 2013, for its focus on the design, and on the issue of relating the tool-specific discourse representation to mathematical knowledge (Watson & Ohtani, 2015), interpreting tools as physical or virtual artifacts with the potential to mediate between mathematical experience and mathematical understanding (Leung & Bolite-Frant, 2015).

The presentation of the topic of tools and resources for and from this collaboration in the more general context of teachers' collaboration in working and learning, and in continuity with previous Studies, gives us the possibility to enter into the topic of Theme D, in the following sub-section.

5.2.2 This Chapter in the Book

This sub-section identifies the research aims and questions described in the Discussion Document and references the other chapters, or the Themes and the Plenaries, organised within the conference.

This chapter is the result of the work and paper presentations that took place both in person and remotely—before, during and after the conference—and of participants from the Theme D group. This chapter (Theme D) is focused on *tools* and *resources for* teacher collaboration and *from* teacher collaboration: "Resources *for* and *from* teacher collaboration can be considered as two ingredients of continuous processes: *adopting* a resource leads always to *adapting* it, and that is more the case in the context of teacher collaboration" (Borko & Potari, 2019, p. 9; *italics in original*). The research questions addressed in the call for contribution are:

> *What resources are available to support teacher collaboration? With what effects, both on the collaboration and on the resources themselves?*
>
> *What resources are missing for supporting teacher collaboration? How and to what extent can teachers overcome these missing resources?*
>
> *To what extent and under what conditions do digital environments (e.g., mobile devices, platforms, applications) constitute opportunities for teacher collaboration? How have these resources been used to support teacher collaboration?*
>
> *Which resources can be used (and how) to sustain and scale up collaboration over time?*
>
> *How are teachers engaged in the design of resources in collaboration? What are the outcomes of these collaborations?* (Borko & Potari, 2019, p. 9; *italics in original*)

This chapter presents a new way to speak of tools and resources: seeing them as products of teachers' collaborative work or means to support teachers' collaboration and their possible evolution in various settings. The papers submitted examined these issues (and other possible sub-issues) in different ways. We will present and weave different threads over the different sections of this chapter, considering the different topics involved, their interaction, and the theoretical and methodological issues and approaches. In particular, we will look at tools and resources not as static products, but as evolving objects, which can be the products of teachers' collaboration, and can support and mediate collaboration. Since tools/resources are seen in this chapter as two sides (means and products) of a coin, using and designing them are then to be considered as two possibly interrelated processes. In the presentation of the glossary (Sect. 5.2.4), we will describe the shared meaning of the terms used in this chapter.

The chapter is connected not only with past research, but also with the other chapters of this book:

- with Theme A for theoretical frames and approaches for studying collaboration, and for the particular tools, artifacts and resources involved in studying collaboration;
- with Theme B for exploring different forms of collaboration and their outcomes, the different number of collaborators, the subjects involved in collaboration and the timing of collaboration (e.g. synchronous or asynchronous modes);
- with Theme C for the different roles, identities and interactions of the various participants engaged in collaboration (e.g. lead teachers, facilitators, mathematicians, researchers, policy makers).

A continuity can be seen particularly with Chap. 9 (this volume) by Karin Brodie (who made a plenary talk in the Study Conference) and Kara Jackson (her reactor), investigating teachers' collaboration with the use of resources. The authors give a framework for systematically studying professional learning communities and propose various kinds of resources to be studied: knowledge, material/logistical, affective and human.

5.2.3 Participants and Collaboration in Theme D

This sub-section shows the participation to the Theme, in terms of papers, participants, and kind of working—both in-person and remotely—to present, discuss and contrast the different studies. The sub-section also highlights the different collaborative modalities used to organise this work and provides examples.

The participants were: Ornella Robutti and Luc Trouche as leaders of the group, and Karin Brodie and Kara Jackson as plenary speakers (respectively lecturer and reactor). Twenty-two additional scholars presented studies from a number of countries, for a total of 26 participants, who arrived up to about 28–30 for the turnover of observers from the Scientific Committee of the Study and from the plenary speakers. The 26 participants were from the following countries: Canada (one), China (one), Colombia (two), France (two), Greece (one), India (one), Israel (two), Italy (five), Japan (one), South Africa (one), Spain (one), Taiwan (one), Turkey (one) and the

United States (six). They presented a total number of 18 papers, divided in six sessions, to give time in the last two sessions for a discussion in preparation for this chapter.

The collaborative modalities of the Theme D participants included both in-person and virtual formats. During the conference, paper presentations and discussions took place in person. However, the Theme D participants also collaborated virtually before, during and after the conference, via a platform. Participants communicated at all hours and a large amount of contributions and level of synergy was noted. Materials collected in the platform were: papers, presentation slides, reactions, schedules, a common list of references and—last but not least—a shared *discussion sheet* (totalling 73 pages). The discussion sheet was organised by the nine sessions. Each session consisted of the session theme, a session chair, two secretaries, who prepared a brief report of the session, and the paper presentations and reactors that corresponded to the theme.

We must also note that—due to the pandemic—colleagues from China, specifically Shanghai, were unable to attend the conference in person. Therefore, during the conference, our colleagues were invited to present virtually and we paid special consideration to our communication and responses. In Sect. 5.7.2. we will describe how we, as teachers and researchers, are reconsidering the roles of tools and resources as a result of the COVID-19 pandemic.

The participants to Theme D collaborated face-to-face and at distance as a community of practice (Wenger, 1998), and, more specifically, a community of inquiry (Jaworski, 2006), made by researchers who use collaboration to share goals and methods and to study tools/resources in teachers' activities from a theoretical perspective (see Sect. 5.2.4). But there is something more: the discussion sheet was used not only as a passive repository of materials of the discussion, but properly as a resource (see Sect. 5.2.4) enriched by participants in a collective way. Therefore, this discussion sheet had the role of a tool in the collaborative work of participants, and a meta-tool for reflecting on their practices of inquiry on the theme.

5.2.4 Towards a Shared Glossary on Tools and Resources

This sub-section shows and motivates the main terminology choices made in this chapter and contextualises these terms within their theoretical frames, or from a general point of view. The reflections made on the glossary call attention to the fact that using a term with a specific meaning may be contextualised in a theoretical frame and gives sense to how the term is used in research, or the same term can be used in a more generic way, embracing a meaning not directly linked to one frame.

Reporting here we group the terms into four sets:

(a) objects used by subjects (teachers, researchers, students);
(b) modalities of working together as subjects in a community;
(c) interaction among teachers' while collaborating in communities;

(d) work done by teachers, involving processes of teaching and of learning.

(a) The fundamental contrast between *artifact* and *instrument*, *tool*, *resource* and *document* is presented, according to the theory of instrumental approach (Verillon & Rabardel, 1995). An artifact is an object—with its characteristics and affordances—that can be transformed into an instrument, with the introduction of the subject's schemes of use. The process of transformation is called instrumental genesis and has a double side: instrumentation (Trouche, 2020b), in which someone acquires an instrument in order to perform a given activity, and instrumentalization, as "*adapting a tool for adopting it as a support of one's mathematical activity*" (Trouche, 2020a, p. 392; *italics in the original*).

Tool, as well, is an object, but in a broader sense, noticing that "the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computations" (Artigue, 2002, p. 245). The term 'tool' comes principally from Vygotsky, and it is used in the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008). From a Vygotskyan perspective, an activity is composed of a subject and an object and mediated by a tool (material tools as well as mental tools, including culture, ways of thinking and language). While the subject is engaged in an activity, the object is held by the subject and motivates activity, giving it a specific direction (Vygotsky, 1978). The role of instrument-mediated activities (Rabardel & Bourmaud, 2003) can be considered in its different kinds: as *mediation to the object of the activity*, aimed at getting to know the object and also at acting upon it; as *interpersonal mediation*, oriented toward others; and as *reflexive mediation*, towards the subject, in her/his relation with her−/himself, mediated by the instrument (Sect. 5.4).

Resource is intended in the sense of Adler (2000), as something to re-source the teacher's practice. If Adler also includes human resources, the *documentational approach* (evolved by instrumental approach—Trouche et al., 2020a, b) intends objects, with a lesson plan attached (explaining how to use it for teaching, including didactical objectives), that through schemes of use (Vergnaud, 1998) introduced by a subject evolve into documents. The resources can be either textual (e.g. textbooks, curricular guidelines, student worksheets), or digital (e.g. digital textbooks or websites) (Trouche et al., 2020a, b).

Applying a specific theoretical approach, as instrumental or documentational, or the theory of semiotic mediation, these terms are to be intended as specified above. To describe and contrast studies from different theoretical frameworks, in this chapter we prefer to choose a shared meaning for tool and resource, sufficiently framed in literature but not linked to a specific frame, in order to be flexible enough in using them. For this reason, we consider artifacts and resources not synonymous, but one larger than the other: resource is conceived in a larger significance than the notion of artifact, which can be avoided.

The term 'resource' will be used to indicate what is used by the teachers in their teaching activity: a lesson plan, a mathematical problem, a digital animation (in this case digital resource), and so on, of the material, socio-cultural or didactic–methodological type. We may also reference human resources, as determined by the papers we cite. The term 'tool' will be used to indicate something allowing to find and/or manipulate a given resource—a browser, the email, a word processor—or to guide its usage—a theoretical framework, a national curriculum, an assessment system of a school, and so on.

(b) This chapter is relevant to the modalities of working together as subjects engaged in a *community of practice*, conceived as a joint enterprise with a shared repertoire and mutual engagement (Wenger, 1998), eventually *online* (Johnson, 2001). Subjects may also be engaged in a *community of inquiry*, early introduced by Dewey (1902)—as any group of individuals involved in a process of empirical or conceptual inquiry into problematic situations—and used in mathematics education by Jaworski (2006) (see also Sects. 5.3 and 5.4), as a community that brings inquiry into practices of teacher education in mathematics—where inquiry implies questioning and seeking answers to questions and problems.

The *professional learning communities* are centred on shared learning (Jaworski, 2014) and aim "to enhance teacher effectiveness as professionals for students' ultimate benefit" (Stoll et al., 2006, p. 229), and have an organising structure development inside a broader community that acts as a reference point for teachers' professional learning, based on systematic reflection, inquiry into one's own practice and collaboration with colleagues (Brodie & Borko, 2016). In this chapter, we will refer to communities of practice and/or more specific communities, according to the studies mentioned above and to other approaches to communities.

Referring to teachers organised in communities and engaged in professional development (PD), different theoretical frames need attention if we want to describe teachers' knowledge or their learning as a process:

- CK—*Content Knowledge* and PCK—*Pedagogical Content Knowledge* (Shulman, 1986) and its derivations frames, applied to mathematics education, as MKT—*Mathematical Knowledge for Teaching*, defined as "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Hill et al., 2008, p. 499), and particularly the Mathematics Teacher Specialized Knowledge (Carrillo-Yañez et al., 2018) to refer to the knowledge of teachers (Sect. 5.5.3), or TPCK—*Technology, Pedagogy and Content Knowledge* (Koehler & Mishra, 2005; Thomas & Palmer, 2014), more related with the use of technologies;
- MDT—*Meta-Didactical Transposition* (Arzarello et al., 2014; Robutti, 2020) to refer to teachers, professionally engaged in PD process, who are learning in a community of colleagues, in relation to a community of researchers, and are evolving in their meta-didactical praxeologies;
- DAD—*Documentational Approach to Didactics* (Pepin et al., 2013) evidences the dialectic relationship between the development of a community of teachers, and the development of a shared repertoire of resources—giving also a social aspect to the process (Pepin & Gueudet, 2020): the approach has been precisely developed for considering which learning occurs when many people interact with many resources.

(c) Teachers' interactions can be observed by the lens introduced by Akkerman and Bakker (2011), who identified different mechanisms—identification, coordination, reflection and transformation—that sign the boundary crossing across communities (Sects. 5.4 and 5.5). The theme is particularly useful to study evolution in terms of processes and of products (see Sects. 5.4.4 and 5.5.3), intending them as boundary objects (Star & Griesemer, 1989; Star, 2010) upon which communities of researchers and of teachers act on the boundary, and it has been applied to mathematics teacher education (Robutti et al., 2020).

(d) Speaking of the work done by teachers in/for the class, it is important to consider the use of terms such as *learning path*, *lesson plan*, *learning sequence*, *teaching sequence* and *learning trajectory* (Simon, 2014), conceived as proper resources for teachers in their collaborative work. Some examples for researchers and teachers have been identified, for example referring to:

- practical resources for teachers that might become conceptual tools to work collaboratively on at a meta-level;
- tasks that have multiple solutions;
- activities, objects, methods that can be boundary objects;
- evaluating tasks for the purpose of class.

5.2.5 *Structure of the Chapter*

The structure of the different sections of the chapter, presented in the following, came out by the discussions in Theme D sessions during the ICMI Study 25. We could structure the sections according to a categorisation of the different kinds of tools/resources used in the studies, or according to the protagonists: teachers, researchers, teacher educators, . . . However, we present here something more than just a categorisation: we want to give the readers possible ways to interpret the complexity in the using tools/resources in relation to collaboration and with respect to communities of teachers in various contexts. Therefore, we identify a set of threads running throughout the papers that can properly give sense of that complexity. Each one of the next sections describes one of the threads and reflects the ideas that pass across the papers presented and discussed in Theme D.

Starting from the shared meaning of the terms *tools* and *resources*, as declared in Sect. 5.2.4, we accept that they do not only represent material objects, but also represent symbolic abstract objects. Then we intend them as tools/resources that serve *for* teacher collaboration and that come *from* teachers' collaboration. This double sense (*for* and *from* collaboration) of intending tools/resources as part of the collective work of teachers/researchers/educators . . . gives the main idea for starting to find the different threads:

In Sect. 5.3, there are examined studies on resources for teaching mathematics that are particularly designed and developed through collaboration among teachers, researchers and knowledgeable others;

In Sect. 5.4, the focus is on the tools and resources that support teachers' collaborative inquiry into teaching;

In Sect. 5.5 there is a possible classification of different tools and resources for fostering teachers' collaboration, in the sense that may structure and mediate teachers' collaborative activities and support reflection on teaching;

Section 5.6 presents how researchers examined/analysed the teacher collaboration organised by the tools or how teachers interacted with the tools and resources themselves;

Section 5.7 weaves together the themes discussed throughout the chapter to highlight the main research questions that Theme D aimed to address, and examines these themes in the light of the pandemic period, to address issues of equity and inclusion.

The various sections illustrate these perspectives using research from different projects and one project may be discussed in multiple sections. The first time a project is introduced we describe the following characteristics: the country, the teachers involved (pre- or in-service), the number of people involved (if it is known) and the specific context (inside the institution or outside). Other aspects related to the project (e.g. the type of collaboration; the interactions between participants; the resources used) are then discussed in subsequent sections, according to their different foci.

5.3 Resources for Teaching That Develop and Evolve Through Collaboration

This section pertains to resources for mathematics teaching that are specifically developed through collaboration among teachers or/and researchers and knowledgeable others. The sub-sections are organised by the purpose for which these resources are developed: implementing a new curriculum (Sect. 5.3.1); teaching complex mathematics topics (Sect. 5.3.2); supporting teachers to develop teaching (Sect. 5.3.3).

5.3.1 Supporting the Implementation of a New Curriculum

Implementing a curriculum reform is a complex process, requiring a change in teachers' beliefs and practices (Clarke & Hollingsworth, 2002). Remillard (2005) identifies the central role that teachers play in drawing/interpreting/participating with the curricular text and thus "constructing the enacted curriculum" (p. 224). In this sub-section, we discuss the central role that teachers play in creating their own versions of resources, by adapting and adopting them according to their own needs defined as *documents* through the process of documentational genesis which may be individual or collective. We discuss how old and new curricular resources (traditional and digital) are mobilised by teachers in collaboration with knowledgeable others to evolve into new resources either through their implementation or by redesigning post reflection on its implementation. We will rely on three contexts—a Greek one, a French one and an Algerian one—all of which involve the evolution of resources by teachers for classroom use to support this discussion.

The resources considered in all these contexts are a mix of traditional textbooks, drill exercises or new digital repositories like Sesamath or micro-world (eXpresser) created by others but transformed by in-service teachers in their use. The discussion with other collaborators is also considered as a resource in these cases. These collaborators are teachers in the Greek and Algerian cases, and also include researchers in the French case. These are termed *factories*[1] having the responsibility of designing new resources, their implementation and their redesign. The evolution of new resources occurs through the "community documentational genesis" (Gueudet & Trouche, 2012, p. 309), which involves discussing, designing, implementing, reflecting and adapting the design collectively.

Psycharis et al. (2020) highlight how the community documentation in both the Greek and the French PREMaTT[2] community is able to bridge the divide between the primary and secondary grade teacher's approach and thinking towards the teaching of algebra. The Greek case provides an example of a primary and a secondary teacher working together towards developing algebraic thinking using a microworld eXpresser (Noss et al., 2009). The suggestions given by a primary teacher to contextualise tasks, encouraging recursive view of the pattern and verbalising in everyday language led to its redesign by the secondary teacher and results in the hybridisation of the document.

The French case provides evidence of collective documentation through implementation and reflection of predesigned Sesamath resources by *factories* (Psycharis et al., 2020), and reflection on stages of development of algebraic thinking and generalisation in primary and secondary pupils. The primary teachers design a problem to identify different ways to calculate the number of cubes needed to construct the pyramid, making students focus on number properties. Phase 2 involves focusing on pattern generalisation through the reasoning for the stage 100 pyramid structure. The secondary teachers focused the discussion on such tasks provide shared space for both primary and secondary teachers, to make their "perceptions explicit and agree upon common definitions of key terms like modelling, generalizing or patterns" (Psycharis et al., 2020, p. 675). Thus, the evolution of resources (Trouche et al., 2019) involves the process of identification of *boundary objects*, othering, reflection and transformation (Akkerman & Bakker, 2011).

Sayah (2020) presents an Algerian case, connected to the French one through the use of Sesamath resources (https://www.sesamath.net/) created by the Sesamath community to develop textbooks collaboratively and make them freely available online. The new Algerian curriculum is structured on mathematical competencies, problem solving and the usage of information and communication technology (ICT),

[1] Collaborative groups of researchers and teachers to generate resources.

[2] http://ife.ens-lyon.fr/ife/recherche/groupes-de-travail/prematt

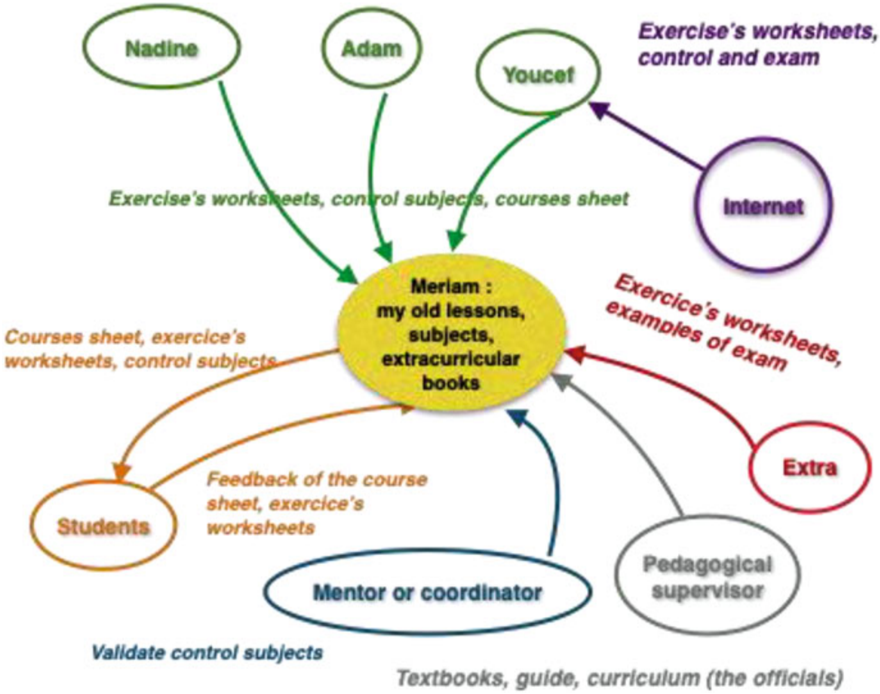

Fig. 5.1 Meriam's schematic representation of her resource system. (Sayah, 2020, p. 649)

and lack of corresponding resources led some teachers to try to appropriate resources from abroad, particularly from the French-speaking countries.[3] Sayah presents a case of a teacher "Meriam", a middle-grade teacher, who uses and adapts Sesamath textbooks from French to Arabic institutional context, while also mobilising other resources and colleagues in her network to create a resource system.

Figure 5.1 represents the drawing of Meriam's *resource system* made at the request of the researcher. It highlights how interactions with her school colleagues (Nadine, Adam and Youcef) lead to the identification of resources and renewing her resource system. Sayah's work illustrates the interactions that develop in the frame of small *communities of practice* (Wenger, 1998), where the participants engage in learning together and have shared purposes and objectives. Youcef and Meriam (as teacher colleagues) constitute a community working mainly for integrating ICT (*GeoGebra* and Interactive whiteboard) in mathematics teaching, and Adam and

[3]French was spoken in Algerian schools from the beginning to the end of the colonial period (1830–1962), leading to the use of a mix of languages. French is still spoken for mathematics teaching in Universities, and most Algerian researchers in the field of mathematics education participate, each 3 years, to the conference "Espace mathématique francophone" (http://emf.unige.ch/), one of the ICMI regional conferences.

Meriam work mainly for the adaptation/translation of Sesamath resources. Thus, the teachers' resources evolve from the interactions of small communities sharing the results of their work leading to a joint evolution of teachers' communities and teachers' resource systems (Gueudet & Trouche, 2012). Sayah proposes a model of teacher resource system based on these evolutions, distinguishing *mother resources* (textbooks, Sesamath resources), *intermediate resources* as results of teachers' collaborative work, and *stabilised resources*, once integrated into teachers' practices.

In all the cases discussed in this sub-section, the need for new resources emerged in the context of the implementation of a new curriculum. This need leads teachers to collaborate for sharing and adapting the pre-existing resources, resources provided by the institutions (as instructions) or from external organisations/platforms. The appropriation of the standardised resources outside the 'designers' circle' is illustrated in Sayah's work through the evolution of a resource system of a teacher in Algeria and by adaptations of Sesamath and eXpresser by *teacher workshops* to address teachers' own needs in France and Greece. The collaborative process of the generation of resources and being available online in open access form plays a critical role in its appropriation and further evolution.

In all cases, the use of language constitutes a critical resource, as a support and a result of teachers' collaboration. In Psycharis et al.'s (2020) study, a shared understanding of terms related to Algebra emerges through the collective documentation work and reflection and transformation of practices across the primary and secondary grades; in Sayah's (2020) study, the interaction between Arabic and French language leads teachers to deepen the mathematical knowledge at stake, while facing the problems of cultural and institutional transposition (Mellone et al., 2019; see also Esteley et al., Chap. 3, this volume). In the next sub-section, we discuss further how the evolution of resources through collaboration is inextricably related to the transformation of teachers' knowledge of mathematical concepts.

5.3.2 Supporting the Deepening of Teachers' Mathematical Content Knowledge

Development of understanding of mathematical concepts can be viewed as a result of a complex and complementary interplay between developing an understanding of the structural aspect as mathematical *objects* and operational aspects as *processes* (Sfard, 2012). In this sub-section, we discuss two cases from Japan (Ohtani et al., 2020) and India (Kumar, 2020), to understand how collaboration supports the evolution of resources for a difficult concept like functions and integers. In both cases, collaborators are in-service teachers (both elementary and secondary teachers in the case from Japan), designers and researchers and additionally, in the case of Japan, ICT specialists and Lesson Study experts engaging in design research.

The intervention is organised and designed by researchers in both cases who are also authors (Ohtani et al., 2020; Kumar, 2020). The teachers play the role of both co-designers as well as implementers. The units developed as a product of collaboration are implemented by the in-service teachers. The tools to promote collaboration are knowledge of collaborators engaged in designing, as well as the awareness of inconsistencies and gaps in the discourse, artefacts like textbooks, representations and variability of meanings of the focused concepts and pedagogical strategies for teaching them. The critical resources generated *from* the collaboration is a unit on functions (Japan) using an ICT-based learning environment in form of *GeoGebra* applets (designed by Nunokawa) and representations, models, contexts and activities included in the individual unit plans developed by teachers through collaborative investigation (India).

The variety of evolving resources that come together to support the discursive practices is discussed with respect to their role in leading to the development of collaborators' teachers' in-depth understanding of these complex topics. The setting for collaboration is outside of the school space where collaborative discussions take place. However, Ohtani et al. also discuss the insights gained from classroom implementation of the designed environment. They designed an ICT-based learning environment that fosters an understanding of functions through reification (Sfard, 2012), by converting an operational procedure into a mathematical object on which subsequent operations can be performed.

Considering the problem of classroom discourse focused only on calculations rather than referring to functions as a 'mathematical object', Ninjas are proposed as a metaphor for functions. The other representations of functions like numbers, tables, graphs and algebraic expressions are then considered as shadows of Ninja, which gives a glimpse of the existence of Ninja and through which several properties of Ninja (function) can be derived to identify how Ninja moves. This distinction between the representation of a function as Ninja and its representations as its shadows is proposed to help students focus on key aspects of functions like co-varying quantities, rate of change and expanding the range of variables in case of direct, indirect, linear and quadratic functions. The students are expected to make conjectures about functions, "saying something like [...] 'the Ninja moves much slower when far from the origin of the coordinate plane'; 'this linear function Ninja moves faster than this linear function Ninja'" (p. 664).

To ensure that classroom discourse is consistent and supports reification, the focus is on developing teaching units and features of *GeoGebra* applets to engage students purposefully to investigate covarying quantities, represent their properties and talk about functions as existing objects. Researchers analysed classroom discourse to identify the use of low-level discourse (focused on calculations) or high-level discourse (treating functions as objects and referring to its property of co-varying quantities). Analysis by researchers indicates the need to maintain consistently the high level of discourse in the classroom as the concept of functions, as "change in variables" become the object of the talk at the beginning of the unit, but not in its latter part. This insight is presented in the Ohtani et al., paper as an implication for the further redesign and implementation of the unit.

In Kumar's (2020) work, the tool for collaboration was the framework of meanings of integers and their operations. The meaning of integers can be interpreted as state, change and relation in different real-life or *realistic* situations (van den Heuvel-Panhuizen & Drijvers, 2020) while the meaning of negative sign as unary, binary and symmetric function is evident in mathematical expressions. The situations depicting combine, change and comparison of quantities are represented by addition and subtraction of integers in the framework (Kumar et al., 2017). The analysis of meanings using contexts, models and symbolic representations for teaching integers lead to discussions of the criteria for selecting, using and designing these resources for teaching.

For example, the teachers from Mumbai (with a tropical climate) felt that it was difficult for students to make sense of negative temperatures as students do not experience it, and therefore selected the negative scores on tests to indicate the negative state. However, the decrease in (negative change) temperature was used to represent the change using negative integers. The relation between the temperature of two different days can be represented by negative or positive integers depending on which day is taken as the reference point. The addition or subtraction of integers can be further represented as change or combination of their states or cumulative changes or as the difference between the two related states in contexts like scores, change in baby's weight, change in height and so on.

The analysis and correspondence of meaning of integers and operations in different contexts, models and numerical expressions led to making the implicit criteria for selecting, using and creating representations explicit in the teachers' collaborative discourse. These criteria are expressed at a surface level or deeper level, depending on the level of concern for meanings and consistency. The 'translatability' criterion refers to mapping in representations like a number line or numerical expressions, when the quantities or their change is represented mathematically through them.

A surface-level concern might be focusing on showing equivalence between numerical expressions (3–4) and (-4 +3) using symbols, while a deeper level of concern is indicated when teachers acknowledge that taking away a larger number from a smaller number may not make sense to students. Not considering the difficulties faced by students due to implicit + (positive sign) in expressions like $3–4 = 3 + (-4)$ indicates a surface-level concern for meaningfulness criterion while being sensitive to students' difficulties indicates a deeper level of concern. When the meaning attributed within contexts and models are consistent with mathematical meanings (e.g. equivalence in numerical expressions through commutativity), the criterion for mathematical consistency is reflected at a deeper level while focus on rules indicates surface-level concern.

Both cases focused on developing resources for concepts considered abstract. There were inconsistencies in the discourse related to both concepts among teachers, in textbooks, language and even in the meanings held by researchers. The collaborative discourse during designing of unit plans for teaching functions and integers engaged members to delve deeper into meanings of concepts, understand difficulties faced by the students and identify inconsistencies in the discourse for teaching them, with the aim of it influencing the movement of discourse at the descriptive level in the classroom to the deductive level (Ohtani et al., 2020). Therefore, the issue of coherence and consistency in the discourse across grades for teaching became an important one for developing an understanding of the concepts among students as was also observed in Psycharis et al.'s (2020) study in Sect. 5.3.1.

To analyse what changed as a result of designing the unit plans collaboratively, Ohtani et al. focused on classroom discourse adopted by the teacher and the students, while Kumar focused on the change in the discourse of the teachers during the collaborative investigation and designing of plans. Ohtani et al. found that discourse that refers to functions as objects was discerned in the classroom discourse when students compared the rate of change of linear function and inverse function dynamically on *GeoGebra* applets. However, students may also construct "idiosyncratic meaning to Ninja movements" (p. 666) using the *GeoGebra* applet and focus on "only surface relationships between those expressions and graphs" (p. 666).

Kumar observed the teachers' discourse within the workshop interactions reflected gradually deeper concerns for translatability, meaningfulness and mathematical consistency, and they reported increased use of contexts and models, and reflected on the importance of using them as representations rather than rules on symbolic expressions to develop the meaning of integers and their operations (Kumar et al. 2017).

Ohtani et al. and Kumar both analyse how the deliberations and interactions between the team, associating researchers and teachers, using the classroom discourse as a tool for and from the collaboration are important for achieving the outcomes in form of teacher's more meaningful use of resources in the classroom. The tasks were collaboratively designed by both researchers as well as teachers while the classroom implementation was led by the teacher and researchers played either supporting or observer roles. Ohtani et al. highlight the need "to establish a transparent context between researcher and practitioner" (p. 667), as necessary for collaborative engagement of all actors.

Kumar highlights the discursive nature of collaboration and discussions about the suitability of the representations as one of the ways that teachers seem to develop consistent discourses for teaching the concept. Thus, in both cases, the collaboration focused on specific mathematical abstract concepts led to the outcome of the development of discourse practices in the workshop and the classroom settings. In the following sub-section, we discuss how these collaboratively designed tools and resources can even support the development of mathematics teaching practices.

5.3.3 Supporting the Development of Mathematics Teaching Practice

The collaboration between teachers and knowledgeable others (Huang, 2020) in professional development settings involves resources that are directly or indirectly related to teaching practice. The resources discussed in the previous two sub-sections are ideas or materials that are integrated with teaching. So, the nature of the tool for collaboration is some form of material resource (like a digital resource) or a cognitive resource (framework of meanings) that can be used for a specific purpose in teaching and therefore gets redefined as a 'tool' (e.g. in the form of lesson plan).

In this sub-section, we discuss the tools for collaboration that are "representations of practice" (Herbst & Kosko, 2014) in the form of scenarios that may occur in the classroom (Cusi et al., 2020) or video-recording of actual teaching (Uzuriaga et al., 2020) that were used in a professional development setting with the purpose to develop new teaching practice itself, such as giving more room to students for participating to the advancement of knowledge. Such tools and resources have been used with the purpose of developing ideas of new ways of teaching, creating substantive learning opportunities for teachers by promoting visualisation of several possibilities and interventions to support student learning in different scenarios (Cusi et al., 2020) or by promoting reflection on teaching through observation and analysis of classroom teaching through identification, analysis, collective discussion, and systematisation (Uzuriaga et al., 2020).

In this sub-section, we take up the case of Cusi et al. from Italy and then Uzuriaga et al. from Colombia to analyse how the evolution of tools or resources, collaboratively driven in the form of representations of practice, contribute to developing mathematics teaching. The case from Italy involves four in-service teachers in six meetings over an eight-month period in a professional development program working on scenario design taking the example of a task aimed at promoting students' exploration of the relationships between elements of a varying figure. The Colombian case illustrates collaborative action research among 15 primary and secondary teachers participating in a Master's-level course in mathematical methods course for 2 years through the design and redesign of didactic, inquiry-based unit plans in workshops, and implementation and analysis of the teaching of didactic unit plans in schools.

In both cases, the teacher educator and/or researchers played a supportive role during discussion and validation, while the task of designing scenarios or didactic plans were done by the teachers. Cusi et al. focus on analysing the teachers' interactions during the workshop and change in praxeologies evident in the transactions, while Uzuriaga et al. focus on analysing the teachers' implementation of didactic plans using tools for observing teaching practice and analysis matrix which was developed and validated by the researchers during the course of 2 years Masters' program.

The work of Cusi et al. involves teachers "not only in designing the tasks for students and the teaching methodology, but also in hypothesising possible students' answers to the tasks and hypothetical excerpts of classroom discussion, containing teachers' interventions" (p. 605). This results in *an ordered set of scenes* called *Scenario*, which includes not only teachers' and students' interventions in a class setting represented in a storyboard format (with characters depicted as teachers and students), but also thought balloons for teachers, in order to express their rationale behind the actions depicted in the scenario.

Every scenario refers to a specific class situation depicting a mathematical task and the teachers discussed the tree of possibilities in response to students' conjectures and teacher actions. In one such discussion, the teacher educator introduced the possibility of all students either agreeing or disagreeing with a conjecture to make teachers rethink and redesign the scenario considering the 'tree of possibilities' and, thus, including different "ramifications" and introducing "thought balloons" to depict teachers' thinking behind her response (Cusi et al., 2020). Thus, teachers' re-designed scenario reflected the evolution of their didactical praxeologies at the classroom level (Chevallard, 1985/1991) by identifying gaps between teachers' intentions and student thinking, adopting ways of questioning and becoming more flexible in their practice.

At the same time, the re-design also reflects the evolution of "meta-didactical praxeologies" (Arzarello et al., 2014) in PD context when facing new educational paradigms and engaging in shared reflection (see Sect. 5.4.3). The meetings in the PD setting involved the development of theoretical ideas and connecting them to practical aspects by playing the role of the learner and then engaging in designing scenarios for using a particular task with students, revising it based on feedback received from researchers and peers. How teachers' reflections on the classroom experiences contributed to the collaboration and the revision of scenarios will be discussed in Sect. 5.5.3.

Uzuriaga et al. used collaborative action research to make teachers question their practice and develop practices related to inquiry methodology in their teaching practice. Both inquiry methodology discussed in the course and the instruments for observation and analysis of teaching practice are used as tools for supporting collaboration, and the outcome from the collaboration is also the development of inquiry methodology as practice. The inquiry methodology had four phases of practical investigation: triggering event, exploration, integration, and resolution (Bustos, 2011), and involved self-evaluation and co-evaluation for regulation of learning. The researchers present the analysed transcribed video records of classroom teaching to identify the extent as well as occurrence and co-occurrence of teachers' adoption of different practices related to inquiry methodology.

Some of the highly appropriated practices included 'connecting the content with the daily life situations' (e.g. making a poster of favourite foods to teach fractions), 'searching for hypotheses for the proposed problem' and 'resolution of solution through teacher evaluation'. The practice of promoting student argumentation co-occurs most during the phase of 'resolution confirmation' during the practical inquiry illustrated with the example of organising the teams for football championship based on average goals scored. Interactivity was promoted by involving students in the construction of meanings illustrated by the example of rewriting repeated addition in the form of multiplication. Teachers developed an understanding of inquiry methodology, as well as reflecting on their teaching practices in the process of engaging in analysis (for further discussion, see Sect. 5.4.3).

Cusi et al. (2020) and Uzuriaga et al. (2020) provided evidence of how both anticipating the classroom events and analysing them collaboratively led to the evolution of teachers' understanding and practices. Thus, the tools for collaboration represented, anticipated and supported practice, as well as became tools for reflection on classroom practices during both pre- and post-implementation. In the case of Cusi et al., the deliberations in the PD setting worked directly on developing teachers' thinking using anticipated students' responses in scenarios. Teachers made the reasons behind their moves explicit, using the thought balloons as well as the different possibilities that may occur as a result of the move. In this way, these deliberations allowed the teacher to make reasoned and explicit choices in terms of interventions and choices on teaching practices and praxeologies, and even get feedback from knowledgeable others about the intervention, thus promoting learning about adopting new practices.

In the case of Uzuriaga et al., the deliberations occurred at multiple levels—first between the peer teachers working together in class and analysing teaching through video records and also with feedback given by faculty on didactic unit plans as well as the tool for analysis of classroom practice. The variability in the appropriation of different aspects of the practice by the teachers points to the possibility that collaboration with the teacher educators and peers in a professional development context impacts teachers' images of practice in different ways. These alternative images of practices get realised in the classroom and reflection on them further revises these images. Thus, the representation of practice that is being used as a tool for collaboration is the old practice adopted by teachers, as well as the idea of inquiry and the practice associated with it. The classroom practice and video records also work here as a tool for the collaboration of teachers as they analyse it.

Both cases presented in this sub-section had used the representations of practice to foster collaboration and to delve deeper into the understanding of teaching practice. Here, representations of practice are revised in several iterations, and therefore are simultaneously resources *for* and *from* collaboration. Further discussion on using tools for collaboration for teacher engagement in activities are discussed in Sect. 5.5.2. Next, we reflect on the three previous sub-sections to identify the transversal issues we have raised.

5.3.4 Transversal Issues and Perspectives Around Living Resources

The three previous sub-sections illustrate the different ways of collaborating (diversity of scale, of agents and of settings) and different purposes for collaborating: facing a curriculum change, facing the teaching of complex topics and developing teaching practices. In this final sub-section, we summarise these findings and identify the transversal strands across the papers. We discuss the diversity of theoretical frameworks, the diversity of resources for/from collaboration and the interconnections between these resources.

The theoretical frameworks used in the papers are diverse, but essentially come from socio-cultural paradigms highlighting an aspect of the collaboration established between different participants including teachers. The documentation approach to didactics (Gueudet & Trouche, 2009), used by Psycharis et al. (2020) and Sayah (2020), emphasises the dialectical relationships between resources and teachers working in collaborative contexts. Ohtani et al. (2020) use the Cultural–Historical Activity Theory (Engeström, 1990) to highlight the dynamicity of the collaboration and the relationship between the participants, while Cusi et al. (2020), using the framework of Meta-Didactical Transposition, focus on how collaborative work fosters teachers' meta-didactical praxeologies through a double dialectic between the didactical (teaching) and meta-didactical (learning in a PD context) level.

Uzuriaga et al. (2020) focus on the appropriate practices by the teachers and the different phases of the inquiry process. Kumar (2020) has teachers engaged in the analysis of representations of integers based on a framework of integer meanings (Vergnaud, 1982), and arrives at an emergent framework of criteria of translatability, meaningfulness and mathematical consistency used for determining representational adequacy for teaching integers through analysis of discourses in PD context. The commonality in all these frameworks is the focus on how interactions between the collaborators bring about a dialectical change in teachers or their community, while what exactly changes might be focused differently.

The nature of the resources *for* and *from* collaboration may vary across the projects.

- The resources *for* collaboration can be classified into two categories: the first one corresponds to the material adaptive resources (Sayah, 2020) or the digital resources (Ohtani et al., 2020; Psycharis et al., 2020) that are adapted or used in teaching; the second one corresponds to resources focusing on planning or reflecting on teaching, highlighting the key ideas to be focused on (Uzuriaga et al., 2020). The collaboration plays a critical role in developing teacher competencies through reinterpretation and recontextualisation within collaborative discussions using either a cognitive resource like a theoretical framework of integer meanings in Kumar (2020) or material tools such as storyboards and resources such as scenarios (Cusi et al., 2020).
- The resources *from* collaboration can be classified into three categories: the first one concerns material resources like digital resources (Ohtani et al., 2020) or didactic plans (Uzuriaga et al., 2020; Kumar, 2020) that can be directly used in classrooms with students; the second one consists of human resources in the form of individual teacher's competencies as mathematical knowledge (Ohtani et al., 2020; Kumar, 2020) or appropriated practices (Uzuriaga et al., 2020); the third one consists in resources which can be considered as community resources in the form of a resource system (Sayah, 2020), collaborative units like factories (Psycharis et al., 2020) or shared criteria among the designers (Kumar, 2020).

Although we have discussed the resources for and from collaboration as distinct categories, both types of resources influence each other as anticipating practice, sharing the experience of practice and reflecting on practice co-occur in interactions in professional development settings. Section 5.3.1 underlines these close interconnections, within collaborative settings, between material (e.g. digital resources), human (e.g. agents and their roles) and cognitive resources (e.g. language) (Psycharis et al., 2020; Sayah, 2020) when addressing a curriculum change. Similarly, Sect. 5.3.2 underlines the connections between the cognitive resources in the form of mathematical meanings and connections between representations in discourse, the interactions with human resources leading to the development of material resources for classrooms for teaching complex topics (Ohtani et al., 2020; Kumar, 2020).

Section 5.3.3 also underlines the connection between the scenarios or didactic units as material resources, the interactions with human resources which led to changes in the material resource using reflection on classroom experiences as cognitive resources (Cusi et al., 2020; Castro Superfine & Pitvorec, 2020). Because of these interconnections among the diverse resources, collaboration and interactions play a major role in establishing and highlighting these connections, incorporating ideas and experience from diverse resources and settings, and bridging the context of professional development with the school context having teachers as collaborators and incorporating their experiences. Thus, collaboration appears as an essential means for producing or making living resources for teaching, in order to support mathematics teaching in various settings.

5.4 Resources and Tools for Inquiring Collaboratively into Teaching

In the context of PD (for both pre-service or in-service teachers, within institutional contexts or outside institutions), involving teachers in collaborative activities could give them the opportunity to learn about teaching and to find out ways for improving teaching practices. While the focus of Sect. 5.3 is mainly on resources (for teaching) as a *product* of these collaborative activities, in this section we focus on those tools and resources that are specifically designed by researchers and teacher educators, with the aim of triggering and supporting a fundamental *process* that characterises teachers' collaborative work—the collaborative inquiry into teaching.

In other words, the focus of this section is on the different ways in which, within PD settings, tools and resources are designed and used to give teachers the opportunity to reflect deeply upon their own teaching, with the aim of promoting their shifts of attention toward constructs, theories and practices that can inform and guide their future choices (Mason, 1998, 2008). We, therefore, refer to those research studies that are focused on PD settings that can be characterised as *communities of inquiry* (Jaworski, 2006—see Sect. 5.2.4).

In this section, we reflect on possible ways of supporting collaborative inquiry into teaching, through the identification of tools and resources to foster and sustain teachers' collaborative work in: designing and redesigning teaching (Sect. 5.4.1); analysing different kinds of data from school practice (Sect. 5.4.2); representing teaching to reflect collectively upon it (Sect. 5.4.3). In order to reflect upon the ways in which these tools and resources are used to support teachers' inquiry into teaching, we will refer to Rabardel and Bourmaud's (2003) categories of orientations that characterise instrument-mediated activities: (a) the *mediation to the object* of the activity, aimed at getting to know the object and also at acting on it; (b) the *interpersonal mediation*, oriented toward others and aimed at both knowing others and acting in interaction with them; (c) the *reflexive mediation*, through which the subject's relation to him−/herself is mediated by the instrument.

Final remarks on the reflections developed in Sects. 5.4.1, 5.4.2 and 5.4.3 are proposed in the final Sect. 5.4.4, which is aimed at discussing how the effects of teachers' collaborative inquiry into teaching, in terms of teachers' learning about teaching, could be investigated and highlighted. Since collaborative reflective inquiry could be seen as a common characteristic of most of the research studies presented within the theme D group at ICMI 25, in this section we will refer to a plurality of voices, discussing the different ways in which teachers' reflective practices are supported within different PD settings.

5.4.1 Supporting Teachers' Collaborative Inquiry into Teaching Through the (Re)Design of Resources

The design and redesign of resources for planning teaching, such as learning trajectories (Simon, 2014) or lesson plans, within communities of inquiry could represent a fundamental opportunity for teachers for collaborative inquiry into teaching. While the focus of Sect. 5.3 is on the product of these processes of design and redesign, here we focus on the ways in which PD settings are organised, around the use of specific tools and resources, with the aim of supporting teachers' collaborative design and redesign of teaching resources.

In this sub-section, the focus is, therefore, on the organisation of PD settings, interpreted as a particular combination of tools and resources aimed at fostering teachers' inquiry into teaching. This combination concerns: (a) the choice of resources upon which to focus teachers' design; (b) the identification of a proper environment within which design and redesign processes are fostered and implemented; (c) the tools provided to teachers to support their design or redesign processes.

The activities within which teachers are involved are initially aimed at fostering the two first categories of orientations that characterise instrument-mediated activities (*mediation to the object* and an *interpersonal mediation*), since teachers, first of all, have to know the objects they are working on (and with) and have to know each other to become able to act together on these objects. Here, the word 'object' refers both to the resources on which teachers' design and redesign processes are focused and to the tools provided to teachers to develop these processes. Moreover, the choice of the environment within which design and redesign processes are realised is fundamental in supporting (or not) the interpersonal mediation.

In the institutional setting introduced by Cusi et al. (2020), for example, in-service teachers have the opportunity to work collaboratively within a face-to-face environment. The object of their design work is the creation of a *scenario*, a specific resource representing different moments within classroom situations and their possible development (see also Sect. 5.3.3). The initial stage of teachers' collaboration is aimed at understanding what scenarios are and what are the main criteria according to which they could be designed.

Another example in which in-service teachers collaboratively work within a face-to-face environment is presented by Chang et al. (2020). These authors provide a PD program (Lin & Chang, 2019) characterised by a mutual collaboration among mathematics teachers, teacher educators and researchers. The resources upon which mathematics teachers' design is focused within this program are what Lin and Chang call mathematics-grounding activities. During the design process, teachers' collaborative inquiry into teaching is fostered by making them reflect upon how the activities they design could be implemented to enhance students' motivation and conceptual understanding in mathematics.

The environment within which teachers' design or redesign is developed could also be online or blended, as in the cases presented by Albano et al. (2020), who introduces a PD setting, characterised both by face-to-face and online interactions, where in-service teachers collaboratively design learning trajectories. In the study presented by Segal et al. (2020), the environment for teachers' design is completely online. These authors present a digital platform (http://RAMZOR.sni.technion.ac.il) designed with the aim of providing teachers with a communal environment where they can collaboratively design, share and preserve their teaching resources (lesson plans, teaching programs, assessment items).

Within the projects presented in the four examples, the step toward the third category of orientations that characterise instrument-mediated activities (the *reflexive mediation*), occurs when teachers are involved in subsequent activities aimed at making them carry out a collaborative reflective inquiry on the resources they have designed and on the process of design itself. During the project presented by Cusi et al. (2020), for example, teachers, after having worked in small groups at the scenario design, are led, during meetings that involve the whole group of teachers and teacher educators, to discuss collectively the effectiveness of their initial scenario design. Within the PD program presented by Chang et al. (2020), teachers' reflections on their design and redesign of mathematics-grounding activities are not developed with the whole group of people involved in the program, but within different groups on different foci (general pedagogical issues, technical aspects related to the construction of tools to be used in the work with students, reflections on students' difficulties).

In the case of the PD program presented by Albano et al. (2020), teachers' collaborative reflexive inquiry is realised through two main steps. Within the first step, teachers have the opportunity to collaborate with researchers, while, during the second step, teachers are engaged in a peer-review process. Also, within RAMZOR (Segal et al., 2020), teachers have the opportunity to reflect upon their development and joint improvement of materials for teaching and learning (Movshovitz-Hadar, 2018), by sharing and discussing their knowledge about teaching practice and daily experience through online and face-to-face meetings between mentors (experienced mathematics teachers) and mentees (groups of teachers of the same school).

Teachers' reflexive inquiry during these kinds of activities is supported by the use of specific tools provided to them. The collective discussions on teachers' scenario design presented by Cusi et al. (2020), for example, are developed not only by means of teachers' spontaneous reflections, but also by referring to specific theoretical tools, which support teachers in their re-design of scenarios. This redesign process is conceived as an on-going process, developed thanks to continuously shared reflections on the most effective ways in which teachers could intervene within classroom discussions to support students' learning processes, to highlight and discuss their difficulties, and to foster their reflections.

Theoretical tools (specific constructs deepened through the course materials) are also used to support teachers' and researchers' reflections, during the face-to-face meetings organised within the program presented by Albano et al. (2020), and to guide teachers' redesign of learning trajectories (Simon, 2014) according to these reflections. In the example presented by Albano et al., a further tool is used to support teachers' reflexive inquiry, in combination with theoretical tools. It is a methodological tool: the peer-review process. Thanks to this process, in which the combination of theoretical reflections and other colleagues' feedback plays a central role, teachers have the opportunity to reflect upon and improve their own teaching, giving rise to a further redesign of learning trajectories.

The collaborative inquiry activities testified in the four examples presented in this sub-section highlight the effectiveness of specific organisations of PD settings, aimed at providing teachers with the opportunity to reflect upon their own teaching practice by designing and redesigning resources and teaching approaches to be developed in their classrooms. The reflections developed within the communities of teachers and researchers that interact within these PD settings could have different foci. Although the products of the design and redesign carried out within these settings could have different characteristics, the four examples highlight the key role played by these shared reflections in determining the on-going evolution of these products. Moreover, the examples highlight the role played by theoretical and methodological tools in fostering teachers' reflexive inquiry into teaching during their design and re-design of teaching resources.

5.4.2 Supporting Teachers' Collaborative Inquiry into Teaching Through the Analysis of Classroom Data

The focus on specific tools (with this term we also refer to theoretical tools, as indicated in Sect. 5.2.4) within PD settings is also aimed at supporting teachers in analysing school practice, by observing, discussing, comparing their own classroom activities and the activities carried out by other teachers, referring to different kinds of data (concerning both students' learning processes and products and classroom interactions). While Sect. 5.4.1 was devoted to the collaborative inquiry work *a priori* developed by teachers, that is before implementing the designed resources in their classes, this sub-section is therefore aimed at discussing the collaborative inquiry work that teachers, within different PD settings, develop *a posteriori*, that is after the implementation of specific resources in their classes or in other educational contexts.

An example of a setting within which in-service teachers have the opportunity to analyse data collected within a wide institutional context could be found in Ferretti et al. (2020). These authors introduce the use of Gestinv, an interactive database with structured information regarding Italian standardised assessment, aimed at creating a PD setting embedded in the institutional context of national assessment (Bolondi et al., 2017) to bridge large-scale assessment to the improvement of teaching and learning of mathematics at the level of the school system (De Lange, 2007). Within this setting, teachers develop, working in groups, a collaborative inquiry work focused on the exploration of Gestinv to find out, in relation to specific mathematical topics, items associated with the lowest percentages of correct answers.

The collaborative work is developed throughout different stages. At the first stage, when the collaborative activity fosters a *mediation to the object* and an *interpersonal mediation*, the focus of the inquiry work is on studying the collected items to start a reflection on typical students' mistakes and difficulties. At the second stage, teachers are provided with resources and tools to deepen the reflection engendered within the previous stage: they are asked to study specific resources, that is research materials regarding historical–epistemological and didactical aspects connected to the identified items and then to prepare a written presentation to be shared during the subsequent general discussion.

At this stage, *reflexive mediation* starts to be developed, since key elements, such as beliefs, convictions, reflections, emotions and agency, are brought to the forefront. The collaborative reflective inquiry is developed at a double level: the level of empirical analysis of typical items that have been difficult for students, aimed at making teachers identify the problem to be studied, and the level of theoretical analysis, focused on epistemological aspects (to support teachers' identification of possible origins of students' mistakes) and didactical aspects (to support teachers subsequent design of activities to help students in overcoming their difficulties). The construction of written presentations represents a key element in making teachers deepen their reflections. In fact, the need to communicate with others fosters teachers' explication of the results of their empirical and theoretical analysis and a consequent deeper awareness about the objects of the analysis itself.

In many PD settings the data that teachers analyse come from their classes or from the classes of other colleagues with whom they are collaborating. Pynes et al. (2020), for example, presented a web-based collaborative setting where in-service teachers, working in school-based teams, have the opportunity to analyse the written work of their own students. Also, in the example presented by Uzuriaga et al. (2020; see Sect. 5.3.3), the data under analysis comes directly from the classrooms. A group of in-service teachers, in fact, is involved, during the third phase of a two-year Master's course, within an activity of observation and analysis of their own implementation of a didactic unit designed according to an inquiry approach to teaching (Wells, 2001). Teachers, who work in pairs, have to observe and analyse their own teaching practice, by focusing on the recordings and transcripts of their interaction with students during the implementation of the didactic unit.

As in the example by Ferretti et al. (2020), to develop the data analysis, also within the projects described in these last two examples, teachers are provided with specific tools that foster the *reflexive mediation* dimension. In particular, the teachers involved in the study presented by Pynes et al. (2020) are supported in self-facilitated collaborative inquiry through the use of a Collaborative Inquiry Tool (Pynes, 2018) aimed at supporting the development of complex skills foundational to noticing effectively children's mathematical thinking (Jacobs et al., 2010). The collaborative dimension of this activity supports teachers' development of deep awareness about the process in which they are involved, since, thanks to group discussions on students' written work, teachers are given the opportunity to share with others what they noticed about students' mathematical thinking. Consequently, they make this noticing explicit to themselves, developing, in this way, reflections on their own ways of noticing.

In the study by Uzuriaga et al. (2020), teachers' observation and analysis of the data collected in their classes is supported by the use of two main tools provided by researchers: a grid for observing the teaching practice according to the three categories of didactic sequence, scientific competence, and interactivity (González-Weil et al., 2012), and an analysis matrix. The authors highlight that, although observing and analysing practice was demanding for teachers, the use of the grid and of the analysis matrix has enabled them to develop a scientific attitude in the observation of their classes.

The three examples presented in this sub-section highlight the effectiveness of supporting teachers' collaborative inquiry work through the use of specific tools (theoretical materials, grids for observation, web-based tools, . . .) that provide them with lenses that direct their attention during the *a posteriori* analysis of different data collected within local or national educational contexts (data from standardised assessment, videos of students work, students' answers or classroom interactions). Independently from the objects of the collaborative analysis carried out within the PD settings presented in this sub-section, the tools that direct this analysis put it from an empirical level to a more theoretical level, effectively supporting teachers' on-going reflections on the teaching and learning behind the data themselves.

5.4.3 Supporting Teachers' Inquiry into Teaching Through the Representation of Mathematics Teaching Practice

Within PD settings aimed at fostering collaborative reflective inquiry into teaching, the design and redesign of lesson plans (Sect. 5.4.1), and the analysis of teaching practices or of other kinds of data from classroom activities (Sect. 5.4.2), are often developed through the use of different representational tools. An investigation of the tools used to support the representation of practice has been discussed also in Sect. 5.3.3, where it has been highlighted how these representations help in bridging the PD context and the school context. In tune with the focus of this section, in this sub-section the ways in which these tools could support a reflective interpretation of teaching are investigated.

The first example we discuss refers to the context of pre-service education. Weingarden and Heyd-Metzuyanim (2020) present a study in which pre-service teachers' analysis of real classroom data is supported through the use of a tool providing them with opportunities for collaboratively discussing and investigating the level of students' authority and the extent to which mathematical objects were treated within real whole-classroom discussions. This tool, the Realization Tree Assessment (in the following, referred to as RTA) (Weingarden et al. 2019), has been inspired by the commognitive theory of mathematical objectification (Sfard, 2008), according to which, since mathematical objects are discursive entities, students have to reify and alienate the different realisations of mathematical objects and to 'same' them. By talking about these realisations as the same thing, students become able to participate exploratively in the discourse about them. The RTA tool (Fig. 5.2) is aimed at visualising the realisations of mathematical objects that arise during classroom discussions and the extent to which students authored the different realisations.

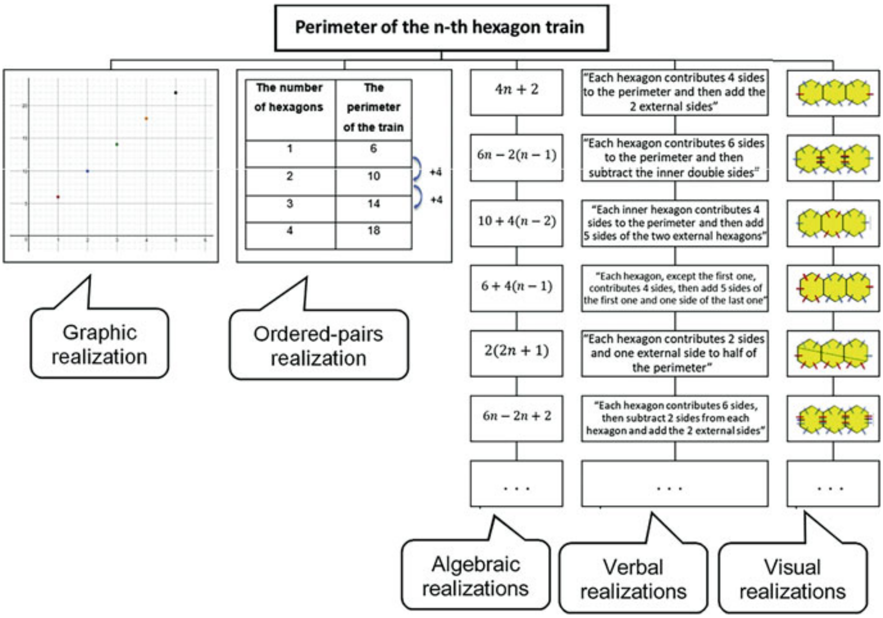

Fig. 5.2 An example of realization tree. (From Weingarden et al., 2019)

In the study presented by Weingarden and Heyd-Metzuyanim (2020), pre-service teachers are asked to work collaboratively at the coding of an empty RTA based on videos of lessons and at comparing and discussing different RTA images. After a phase in which they start exploring this tool (*mediation to the object*) to become able to develop collaboratively the required coding (*interpersonal mediation*), the visualisations realised thanks to RTA support pre-service teachers' in focusing their reflections both on the different types of links that can be made between realisations and on the importance of students' authority in mathematics lessons. Therefore, it is this visualisation that supports a *reflexive mediation*, since it fosters pre-service teachers' observations of the extent to which students authored narratives about the mathematical object and its different realisations during the lesson.

Also, Yuan and Huang (2019) present an approach characterised by a collaborative work focusing on representations of actual teaching, with the aim of making teachers reflect on the ways in which they can activate what the UK National Centre for Excellence in the Teaching of Mathematics (NCETM) defines teaching for mastery (NCETM, 2019). Within this approach, developed within the UK–China Mathematics Teacher Exchange Programme, the teachers are involved in activities of observations of lessons and discussions with other teachers in post-lessons meetings, during which they delineate representations of teaching practices referring to a research-based model, which introduces "five big ideas" behind the construct of teaching for mastery (NCETM, 2019): coherence, representation and structure, mathematical thinking, fluency and variation.

In tune with some of the examples presented in Sect. 5.4.1, this approach is therefore focused on the use of theoretical tools that support teachers' collaborative inquiry into teaching, by making them focus on specific characteristics of the lessons that are examined. Other well-known, research-based models have been developed with the aim of providing teachers with sets of theoretical lenses to observe, represent and reflect on teaching practices. Among them, we mention, for example, Schoenfeld's (2013) model of Teaching for Robust Understanding, Hollingsworth and Clarke's (2017) five-dimensional observational framework, Karsenty and Arcavi's (2017) six-lens framework and Stein et al.'s (2017) quadrants coding scheme.

In other cases, as in some of the examples presented in Sect. 5.4.1, instead of focusing on the analysis of real data from teaching–learning processes, teachers are led to make hypotheses about ways of fostering effective teaching, through the design of specific resources, such as hypothetical lessons, that consists not only in creating classroom activities and their *a priori* analysis, but also in writing down hypothetical transcripts of classroom discussions to foresee the possible interactions between the teacher and the students that could be realised. Also, in these cases, the used representational tools provide teachers with specific lenses that enable them to focus their attention on particular aspects of teaching–learning processes, engendering a collaborative inquiry that makes them develop deep reflections on their own teaching (*reflexive mediation*).

The characteristics of the scenarios presented in Cusi et al. (2020), for example, make them powerful tools that enable teachers, through the representation of hypothetical teaching interactions with their students, to focus their attention on the ways in which students' development of inquiry attitudes could be supported through the activation of specific teachers' interventions within classroom discussions. The shared reflections developed by teachers and researchers, while they carry out the collaborative work on scenario design, therefore boost the evolution of their meta-didactical praxeologies (Arzarello et al., 2014; see also Sect. 5.2.4), that is the specific tasks that teachers have faced in their daily teaching, the techniques used to face these tasks and the justifying discourses through which teachers explain the choices they made in terms of chosen techniques and ways of using them.

Another example in which teachers collaborate by being engaged in cycles of scripting, visualising and arguing about moves within a lesson—Story*Circles*—is presented in Herbst and Milewski (2018, 2020). The product of this collaborative work, which is carried out using specific resources (e.g. a task statement, records of student work), is lesson maps that are represented through partially ordered sets of storyboards. These representations can grow in complexity as practitioners identify new decision points and alternative courses of action in lessons or as facilitators bring in possible contingencies that participants may not have anticipated. When constructing these representations, teachers can deploy not only strategic knowledge (e.g. which problem to be used to lead students to a particular goal), but also tactical knowledge (e.g. how to respond to diverse students' contributions).

The examples presented in this sub-section highlight the effectiveness of using representational tools to support teachers' collaborative interpretation, analysis and reflection on teaching. Moreover, they enable us to highlight different categories of representational tools: (a) from tools aimed at generating visual representations of the ways in which mathematical objects are treated within classroom discussions; (b) to theoretical tools aimed at identifying and observing specific aspects of teaching practices; (c) to digital tools and environments aimed at supporting teachers' construction of representations of teaching episodes and lessons. As Herbst and Milewski emphasise, these tools are more effective in supporting teachers' inquiry into teaching, if compared with other traditional resources, which can leave out much of the tactical problem solving done while teaching (like tasks and lesson plans), or may make it hard to distinguish what is usable elsewhere and what cannot be disentangled from context (like the records of actual instances of lessons).

In summary, representational tools give strengths to the reflective dimension of collaboration (engendering a *reflexive mediation*), since they enable teachers to bring to light what usually is not made explicit, making it a tangible object of reflection.

5.4.4 Transversal Issues and Perspectives Around Inquiring Collaboratively into Teaching

The previous sub-sections enabled us to discuss the use of different resources and tools to foster teachers' collaborative inquiry into teaching, through the *a priori* design and redesign of teaching resources (Sect. 5.4.1), the *a posteriori* analysis of different kinds of data from school practice (Sect. 5.4.2) and the use of various tools to support the construction of representations of teaching to reflect collectively upon it both *a priori* and/or *a posteriori* (Sect. 5.4.3).

Through the different examples we presented, we interpreted teachers' interactions with tools—theoretical, methodological, or technological—when they collaboratively work on specific resources or analyse different data from teaching practices, in terms of kind of mediation (*to the object*, *interpersonal*, *reflexive*) that the work with these resources and tools could engender. Although all the three kinds of mediation are jointly present within each instrument-mediated activity introduced in the previous sub-sections, *reflexive mediation* represents the key aspect that characterises the activities developed when teachers collaboratively inquire into teaching by means of different resources and tools. In fact, in our analysis, we highlighted how the different phases of teachers' collaborative inquiry work gradually foster the shift from the *mediation to the object* to the *reflexive mediation*.

The choice of the resources and tools that mediate the collaborative inquiry into teaching plays a central role in this gradual process. In particular, the combination of theoretical and representational tools (see Sect. 5.4.3) seems to be particularly effective in this sense, since it provides teachers with specific lenses that direct their attention during the *a priori* or *a posteriori* analysis that they develop within the PD programs we have presented in this section. Moreover, the different examples highlight the key role played by the settings within which these processes are realised. The combination of tools and resources that gives birth to these settings is, in fact, critical in fostering teachers' inquiry into teaching.

Another fundamental element in fostering the engendering of *reflexive mediation* is the collaboration between teachers and among groups of teachers and teacher educators or researchers. This collaboration, in fact, fosters the deepening of the reflections that are developed during the inquiry work, since the need of comparing and communicating ideas to others makes teachers bring to light what is usually not made explicit when they work alone. We can therefore observe that the examined studies highlight the influence that *interpersonal mediation* has on *reflexive mediation*.

After having examined the use of tools and resources, within the different examples described in the previous sub-sections, to support teachers in collaboratively learning about teaching, a spontaneous question is: how could the teachers' learning, as an effect of teachers' inquiry into teaching by means of different tools, be highlighted and investigated? The analysis of the studies previously discussed enables us to propose an initial categorisation of the ways in which this investigation can be developed.

A first way of investigating teachers' learning is to *look at the evolution of specific products of their collaborative inquiry work*, such as the resources that teachers collaboratively design and redesign. In the study presented by Ferretti et al. (2020; see Sect. 5.4.2), for example, teachers' learning is investigated by focusing on the evolution of specific components of the learning trajectories designed by them. Also, in Cusi et al. (2020; see Sect. 5.4.3), teachers' learning is investigated by highlighting the evolution of their praxeologies, which is, in turn, highlighted by analysing the corresponding evolution of the products of the teachers' collaborative work, that is the scenarios.

A second category of approach adopted to investigate teachers' learning as an effect of their collaborative inquiry into teaching is *to look at the interactions that characterise the collaborative inquiry work that teachers are developing*. An example of this approach can be found in Psycharis et al. (2020; introduced in Sect. 5.3.1), who investigate teachers' learning while interacting with others for designing and sharing digital and non-digital resources. This learning is investigated not only by looking at the evolution of the products of the collaborative inquiry work, but also by analysing the evolution of processes, in terms of utterances in teachers' interaction, indicating the activation of the four learning mechanisms of boundary-crossing introduced by Akkerman and Bakker (2011; see also Sect. 5.2.4).

In some of the presented studies, teachers' learning about teaching is investigated also by *studying teachers' meta-reflections on their experience within the collaborative inquiry activity in which they are involved*. This third category refers to the idea of involving teachers in what we could call *inquiry on inquiry*. Examples of teachers' involvement into 'inquiry on inquiry' processes are presented in Segal et al. (2020), Uzuriaga et al. (2020) and Cusi et al. (2020). The proposed categorisation is obviously provisional and partial. A wide survey of the research on this field is needed to deepen the fundamental issue of categorising the approaches adopted by researchers to investigate teachers' learning as an effect of teachers' inquiry into teaching by means of different tools.

The remarks shared within this concluding sub-section enable us to stress upon the fundamental role played by the *reflexive mediation* that could be engendered when teachers collaboratively work by interacting with different tools (theoretical, methodological, technological) to inquire into teaching. The studies presented in the previous sub-sections highlight, in fact, that the ways in which collaborative settings are designed to give teachers the opportunity to reflect deeply upon their own teaching certainly foster teachers' learning about the teaching practices that are the object of their reflections. Further studies have to be developed to confirm these results and to deepen the investigation of the ways in which the use of specific tools and resources to inquire into teaching fosters and affects both reflexive mediation and teachers' learning.

5.5 Resources and Tools to Facilitate Teacher Collaboration

In this section, teacher collaboration is considered in itself with a particular focus on the tools and resources designed and/or used to facilitate it. In diverse contexts, such as PD and classroom, teachers, teacher educators and researchers exploit a multiplicity of tools and resources that structure and mediate teachers' collaborative activities and support reflection on—their own or other teachers'—teaching (see Sect. 5.5.4).

In the following sub-sections, we consider the nature of these tools and resources by exploring their different categorisations (Sect. 5.5.1) and how they are used to support teacher collaboration (Sect. 5.5.2). By taking a broader view, we also explore what theoretical tools and professional practices can be oriented towards teacher collaboration and how (Sect. 5.5.3). Finally, we synthesise the findings and draw conclusions (Sect. 5.5.4).

5.5.1 Categorising Tools and Resources for Fostering Teacher Collaboration

The nature of tools and resources fostering teacher collaborative work is an important concern for designing, studying and understanding teacher collaboration. Over the last 20 years, technology has offered a variety of tools and resources that have been used to support teacher collaboration, ranging from specially designed environments to represent practice to online platforms allowing documentation and sharing of materials (Herbst et al., 2016).

Focusing on the types of these tools and resources, we categorise them in two broad categories: (A) tools and resources designed for teacher collaboration, and (B) tools and resources that were not initially conceived for teacher collaboration in an educational context but under professionals' (teachers'/teacher educators') intervention were adapted to operate as formal or informal environments for teacher collaboration. The second category is divided in two sub-categories: (B1) tools and resources that can be considered as designed for collaboration but not necessarily for educational purposes, and (B2) tools and resources designed for educational purposes but not necessarily for collaboration.

In category A, we identify digital environments designed to promote teacher professional development by supporting their collaborative activities. Pynes et al. (2020; see Sect. 5.4.2) present a web-based tool, the Collaborative Inquiry Tool, created to support teachers in self-facilitated conversations with colleagues regarding the mathematical thinking of their students. The tool allows participation of groups of teachers in Collaborative Inquiry sessions to discuss activities such as posing problems to students, analysing students' written work for a common problem type, sharing teaching artifacts and creating new problems based on specific students' understanding.

Similarly, Segal et al. (2020; see Sect. 5.4.1) explore the potential of a digital platform (i.e. RAMZOR) designed to facilitate teacher collaboration around the development and improvement of teaching materials such as lesson plans, teaching programs and assessment items. The environment allows sharing and transformation of materials through feedback comments and joint elaboration and, thus, it can serve as a pillar for the development of teacher communities of practice.

In category B1, we identify tools and resources designed for distance working (and eventually learning), storing/sharing and communicating, that allow the development of activities for subjects in general (not necessarily educational) contexts and can be used by teachers. One strand of tools in this category stems from the area of e-learning platforms (e.g. Moodle) that provide educators with integrated systems to create personalised learning environments. However, it is under question if and to what extent these systems can be used to support virtual collaborative activities among participants in e-learning courses. For instance, working with practicing secondary mathematics teachers in an e-learning PD course, Albano et al. (2020; see Sect. 5.4.1) report that specific tools of Moodle (i.e. Assignment and Workshop) allowing exchange and peer-review of teachers' submitted work (i.e. learning trajectories) can scaffold teachers' collaboration and promote re-design of learning trajectories and reflection on own teaching.

Another strand of tools in this category concerns shared drives of general use that provide access to the same object in a single cloud-based storage facility such as Google Drive and similar drives, clouds, etc. Such tools allow collaborative activities of teachers to take place (e.g. by sharing resources/materials and modifying them according to their needs) in different contexts. For instance, McKie (2020) explores the different ways by which in-service teachers participating in school-based professional learning communities in Canada can collaborate while sharing resources online through Google drives. Also, databases provide another type of tool belonging to category B1. For instance, Ferretti et al. (2020; see Sect. 5.4.2) build a model for designing activities for mathematics teachers' PD based on the use of the interactive national database Gestinv that involves structured information regarding standardised assessment and mathematics tests in Italy (1718 tests in total).

Another strand of tools falling into category B1 concerns social media such as Facebook and other online spaces, obviously not designed for educational purposes. The research interest in how such tools can foster teacher collaboration has been increasing. As teachers find self-directed, online learning opportunities more beneficial than required online experiences (Parsons et al., 2019) elevating teacher-initiated collaboration online is critical. Anderson's (2020) study of a public Facebook group (1738 members, USA) tailored to mathematics education indicates the potential of interactions among group members to promote professional collaboration. The Facebook group members were able to participate in PD through discussions of artifacts from members' practice which generated collaborative learning opportunities.

In category B2, we classify a range of digital tools and technological advances (e.g. video streaming, video-conference software, online forums) that promote the representation of teaching in new ways. These tools can be exploited in diverse educational activities for teachers and can be adapted for teacher collaboration. For instance, the Realization Tree Assessment (RTA) tool (Weingarden & Heyd-Metzuyanim, 2020; see Sect. 5.4.3) was originally designed to assess the extent to which students participate exploratively during the lesson (i.e. identifying different realisations of mathematical objects and authoring narratives about them). The

Table 5.1 Categorisation of tools and resources for fostering teacher collaboration

A. Designed for teacher collaboration	B. Adapted for teacher collaboration	
Web-based tools, (e.g. collaborative inquiry tool), Digital platforms (e.g. RAMZOR)	B1. Designed for collaboration but not necessarily for educational purposes Designed for distance working, storing/sharing and communicating e-learning platforms (e.g. Moodle) Shared drives (e.g. Google drive) Databases (e.g. Gestinv) Social media (e.g. Facebook)	B2. Designed for educational purposes but not necessarily for collaboration Designed for representing teaching in new ways Use of cartoons (e.g. lesson sketch) Web-based storyboarding tools Multiple representations of mathematical objects (e.g. RTA)

authors examine its potential for supporting prospective teachers' learning as participation in explorative pedagogical discourse.

Two other examples of tools and resources from category B2 concern specially designed pieces of software that allow representing classroom interactions through cartoons: Cusi et al. (2020) (see Sect. 5.3.3) engage teachers in scenario design through the use of Lesson Sketch, while Herbst and Milewski (2020; see Sect. 5.4.3) engage teachers in collectively creating a representation of a lesson through a web-based storyboarding tool (i.e. Story*Circles*). The emergence of digital environments supporting representation of teaching in new ways brings to the forefront the need to explore further how these new forms of representational and social/communication infrastructures (Hegedus & Moreno-Armella, 2009) might affect the design/study of classroom practice and teacher collaboration.

The types of tools presented in this sub-section (see Table 5.1) indicate that digital technologies have offered important tools and advances to promote teacher collaboration. Digital tools often allow a wide range of uses—not necessarily anticipated by designers—which provides teachers, teacher educators and researchers with an opportunity to adapt their use to serve the purpose of teacher collaboration in formal and non-formal settings. The potential of social media, online spaces and innovative representations of teaching for teacher collaboration appears to be an emerging field of research.

5.5.2 Designing for Supporting Teacher Collaboration

Even though the proliferation of tools and resources has broadened opportunities for teacher collaboration, there are still open issues about how this could happen. In this sub-section, we describe ways by which different tools and resources from the aforementioned categories (Sect. 5.5.1) are designed and used to support teacher collaboration in recent research studies. While in Sect. 5.5.1 we focused on the nature of these tools and resources, here our focus is on their affordances that shape the design of collaboration, the kinds of activities in which teachers are expected to engage and are actually engaged, and the status/forms of mathematics in teachers' collaborative work. The titles below indicate the different categories/sub-categories presented in Sect. 5.5.1.

5.5.2.1 Tools and Resources Designed for Teacher Collaboration

In this category, we refer to the digital platform RAMZOR (Segal et al., 2020; see Sects. 5.4.1 and 5.5.1) that allows teachers to develop, share and jointly improve teaching materials (e.g. lesson plans). The PD project carried out with this tool in Israel involved engagement of groups of mathematics teachers—supported by mentors (24 teachers, 20 mentors)—in designing, and redesigning collaboratively learning plans and evaluation items in the platform, implementing the learning plans in their classes and participating in periodic meetings with the project staff. Using the platform, the teachers provided feedback to learning plans submitted by their peers and wrote new versions adapted to their own classes.

The analysis indicates that interaction with and employment of other teachers' lesson plans in RAMZOR promoted the development of teachers' mathematical and didactic knowledge, and enhanced their sense of belonging to a community of practice. Elements of the gained knowledge indicating the status of mathematics in teachers' collaborative work include new ways of proving a theorem, visual explanations, focus on mathematical details, a wide range of teaching approaches and different ways of solving mathematical problems.

5.5.2.2 Tools and Resources Designed for Distance Working, Storing/Sharing and Communicating

As regards existing research with tools in this category, we provide an example of a study (mentioned also in Sect. 5.5.1) involving the use of the e-learning platform Moodle. Albano et al. (2020) exploit two specific affordances from Moodle to support teachers' collaboration while carrying out the online activities of a PD course on research-informed mathematics instruction blending face-to-face lectures and an online part: *assignment* and *workshop*. Assignment allows a cyclic interaction between trainers and teachers in the form of 'feedback-responses' around an assigned task (e.g. design of hypothetical learning trajectories) and (re)submission of teachers' work. This affordance allows teachers to prepare their response collaboratively and submit it when they reach an agreement. Workshop allows teachers' engagement in reviewing other teachers' submissions on a task according to criteria given by the trainer. A distinct feature of the kind of collaboration in the two activities is that, in the first case, the teachers create together a product in response to a given task, while, in the latter, each teacher becomes a resource for each other by providing and receiving comments. As regards the status of mathematics in teachers' work, the results indicate that anonymous redistribution of hypothetical learning trajectories and feedback through Workshop strongly influences re-designing and improvement of teachers' activities to enhance students' argumentative competence (i.e. exploring, conjecturing, justifying).

Along the same category, we also note the increasing research interest on how teachers use tools such as Google web-based applications, databases and social media which can be integrated into teacher education for communication and collaboration. McKie (2020; see Sect. 5.5.1) focuses on the collaboration of teachers in the context of their participation in professional learning communities in Canada, while sharing resources online through shared Google drives. She reports on the collaboration within a specific professional learning community focused on selecting pedagogical strategies that would best meet the needs of their students in relation to the Grade 9 mathematics curriculum. Resources promoting teacher collaboration include material resources (i.e. shared computer drives, social media platforms, textbooks, research articles) and human and socio-cultural ones (i.e. verbalisation, communication, time). Human and socio-cultural resources support and facilitate collaboration through sharing of beliefs, enhancing collegiality and evolution of the community of practice.

As regards the potential of social media for teacher professional development, Anderson (2020) investigates how contextually relevant teacher collaboration is mediated through a public Facebook group focused on mathematics education. The group involved 1738 members who interacted asynchronously. The affordances shaping the design of collaboration were Facebook posts indicating questions or requesting in-the-moment support, as well as artifacts (e.g. activities) from members' practice providing all groups members access to real classroom situations. The group's interaction led to four discourse structures: starting from commenters providing desired support (Desired), commenters offering different ideas than requested (Reframe), commenters challenging requested support or previous ideas (Challenge) and commenters working together to build a new understanding of desired support (Generate). The platform allowed for a lengthy collaboration time, permitting individuals to join the conversation at their own pace, to return multiple times and to provide more information by posting, commenting and reacting. The results highlight the potential of informal online spaces in providing diverse collaborative opportunities to teachers and participation in professional development.

The above three examples indicate that tools of this category offer affordances that facilitate sharing of materials and enriched forms of interaction between trainers and teachers and between teachers (e.g. cyclic interaction in the form of 'feedback-responses'). The resources that shape the design of collaboration include material resources (e.g. shared computer drives) and human and socio-cultural ones (e.g. verbalisation, communication at own pace, long collaboration time), while the available records of interaction (e.g. written communication through social media) allow addressing the evolution of collaborative talk and the quality of collaboration. The status of mathematics in teachers' work is related to the everyday teaching practice while feedback and redesigning indicate improvement of designed activities to support student learning.

5.5.2.3 Tools and Resources Designed for Representing Teaching in New Ways

In this category, we refer to three examples of studies in Italy, USA and Israel. These studies have also been discussed in previous sub-sections, with a focus on how the tools representing teaching may help in linking PD context and school context (see Sect. 5.3.3) and how they could support teachers' reflection on teaching (see Sect. 5.4.3).

In the first one, we refer to Cusi et al.'s (2020) study that engages teachers to design, reflect and redesign scenarios through ordered sets of scenes in Lesson Sketch. The tool allows teachers to focus on and discuss the various possibilities in which an interaction might evolve during the classroom activity. The resources shaping the teacher collaboration include scenarios represented as stories with cartoons through the use of the depicted tool and the character set of Lesson Sketch. The final product of the teachers' collaborative work is a net of comic strips. The results indicate that its development is facilitated by two affordances of the tool: (a) 'tree of possibilities' that allows representing in different ways the evolution of a classroom interaction as 'ramifications'; (b) 'thought balloons' that allow to make explicit the reasons behind teacher reactions/interventions. Mathematics in teachers' collaborative work appear interrelated to different aspects of the teaching-learning processes (e.g. teaching practice, teachers' justifications of didactical choices).

The second example concerns Herbst and Milewski's (2020) Story*Circles* (see Sect. 5.5.1), another approach based on the use of Lesson Sketch. Story*Circles* deploy upon two kinds of infrastructure of teacher collaboration: social infrastructure that supports conversations about teaching and representational infrastructure which is used in making teaching an object of negotiation in such conversations. The reported study from the USA involves teacher participants using some resources to script a lesson (e.g. records of students work), visualise classroom interactions by putting together various script moves and offer justifications for alternatives to what is represented. The status of mathematics in teachers' work is dynamic, since Story*Circles* enable viewing a lesson as a multiverse that could be composed of many related but divergent stories.

As regards the third example, Weingarden and Heyd-Metzuyanim (2020; see Sect. 5.5.1) explore the potential of the RTA tool to facilitate prospective teachers' collaborative discussions on explorative teaching by altering the use of its available affordances from assessing teaching to represent teaching. These affordances include mathematical objects (e.g. linear function), their various realisations in classroom teaching (e.g. visual, verbal, algebraic) and the links between them around a common mathematical idea. The tool affords teachers opportunities to focus on the mathematical objects and their emergence in teaching as well as to discuss opportunities for student meaning-making. The results indicate that through these challenges, the status of mathematics becomes more prominent in teacher discourse and it is explicitly linked to the teaching practices that afford students' explorative participation in the lesson.

The above examples show that the main affordances offered by tools for representing teaching are related to two main kinds of infrastructures for teacher collaboration: (1) representational infrastructures, i.e. materialising the diversity of classroom interactions and their underlying dynamics, as well as the emerging mathematical objects (e.g. trees of possibilities, reasons of teachers' decisions); (2) social infrastructures, i.e. enabling teachers' collaborative work/reflection on classroom teaching and students' mathematical understanding (e.g. paths of classroom interactions). Links between mathematics in teachers' collaborative work and classroom practice seem to be enhanced.

In summary, in this sub-section we address the issue of design for teacher collaboration as regards the three categories of tools and resources presented in Sect. 5.4.1. The provided examples of studies bring to the fore the following findings:

- tools of category A offer affordances facilitating joint preparation of materials and self-facilitated collaborative inquiry in PD settings;
- tools of category B1 are used for quite similar activities with the aim to facilitate rich interactions between teachers and trainers (e.g. around a task), as well as storing and sharing of resources in PD settings—with the exception of social media that support non-formal ways of collaboration outside PD settings;
- tools of category B2 offer affordances to visualise aspects of teaching in innovative ways while teachers can be engaged in activities such as scenario design and joint creation of representations of lessons.

5.5.3 Theoretical Tools and Professional Practices Towards Teacher Collaboration

A number of theoretical tools and professional practices have been used to support mathematics teacher collaboration and the communities in which they work. In this sub-section, we refer to such tools by distinguishing two broad categories: those that are shared with teachers and, in this way, become tools to support collaboration, and those that are used only by researchers to frame the design of PD settings or to interpret interactions within these settings. In terms of the categorisation introduced in Sect. 5.5.1, theoretical tools of the first category can be considered as tools 'designed' for teacher collaboration—as they are used operationally by researchers to facilitate teacher collaboration—while theoretical tools of the second category can be considered as tools 'adapted' by researchers to design and study teacher collaboration in PD communities in different (national, institutional, etc.) contexts.

The theoretical tools of the first category are used by researchers/teacher educators in relation to appropriate professional practices and methods (e.g. teacher noticing, Lesson Study) explicitly to orient teacher collaboration and facilitate teacher collaborative work. For instance, *teachers' professional noticing* (i.e. making sense of students' mathematical thinking during instruction and deciding how to respond to that thinking) (Jacobs et al., 2010) is a practice that has recently attracted research interest in professional development contexts where groups of teachers work together. Pynes et al. (2020) use the Collaborative Inquiry Tool designed to support upper-elementary teachers in self-facilitated collaborative inquiry to explore teachers' collective noticing of children's mathematical thinking. In a PD context, three teachers participated in 12 collaborative inquiry sessions to examine and discuss student work for a common story problem they each posed to their own students with a focus on children's thinking of key mathematical relationships. The tool provides access to descriptions of the mathematical thinking of students that are not familiar to teachers, as well as to artifacts from the teachers' classrooms. Teachers could consider the different perspectives and may confirm or extend their own noticing. The results indicate the critical role of the tool in supporting teacher collective noticing by allowing multiple perspectives around the same piece of student thinking to be shared and discussed.

Another example concerns the *Lesson Study* (LS) (Huang et al., 2020) that has been a popular teacher-directed professional development approach in many countries to improve mathematics teaching and learning and strengthen connections between research and practice. Recently teacher collaboration has emerged as a promising research area in studies combining LS with different theoretical and methodological perspectives. Díez-Palomar et al.'s (2020) study taking place in a High Education Spanish institution explores how LS and the *Didactical Suitability Criteria* (DSK) (Font et al., 2010) can complement each other when pre-service teachers collaborate in designing interdisciplinary (mathematics and science) lessons for pre-K and K students. While LS is adopted as a context for engaging teachers in the cycle design-implementation–reflection as regards mathematics teaching, DSK provides a set of observable indicators for different types of criteria/suitability (i.e. epistemic, cognitive, interactional, mediational, emotional, ecological) that may help teachers to design and assess their teaching in terms of different sets of 'mathematics teachers' competencies'. The results indicate that the combination of LS and DSK enriches the available professional tools to support teachers' collaboration and further develop teachers' competencies, such as assessing epistemological aspects of mathematical concepts, addressing their teaching and learning and using appropriate resources.

A third example concerns the qualitative study of Bağdat and Yanik (2020) who investigate changes in question types of two novice mathematics teachers participating in a collaborative PD program in Turkey, focused on designing and implementing cognitively demanding tasks. The program focused on identifying collaboratively factors associated with the decline or maintenance of cognitive demand, modifying mathematical tasks to increase cognitive demand and using the theoretical tool of five practices (*anticipating, monitoring, selecting, sequencing, connecting*) to orchestrate whole-class discussions while maintaining the cognitive demand of the tasks at a high level (Smith & Stein, 2011).

The results indicate that, due to their collaborative PD experience, the teachers after the program maintained the cognitive demand of the task at high level and improved in their questioning and discussion techniques. Thus, the approach of five practices supported the design and actualisation of PD, and allowed describing the evolution of teachers' practices. As regards the use of theoretical tools as resources for teachers to prepare and implement lessons in their classrooms, there are studies using *learning trajectory* (Simon, 2014) (e.g. Albano et al., 2020; Huang, 2020), *scenario design* (Cusi et al., 2020), etc.

As regards the theoretical tools of the second category, our focus here is on how researchers/teacher educators design their PD settings, so as to support teacher collaboration and make sense of the interactions taking place within these settings, teacher knowledge and learning. For instance, Huang (2020) combines LS to *boundary objects* (Akkerman & Bakker, 2011) to study how a mathematics teacher educator and a group of 12 primary and secondary teachers in a Chinese PD setting collaboratively worked to design a research-informed exemplary lesson. He provides an integrated framework to support teacher–researcher negotiation of meanings of effective teaching and learning of mathematics in PD initiatives, where members of the research and teaching communities come together.

Ferretti et al. (2020; see Sects. 5.4.2 and 5.5.1) develop a model to design activities for mathematics teachers' PD by networking Jaworski's (2006) notion of *community of inquiry* and the *Mathematics Teacher's Specialised Knowledge (MTSK)* model (Carrillo-Yañez et al., 2018), that is based on Shulman's (1986) notion of *Pedagogical Content Knowledge* and Ball et al.'s (2008) notion of *Mathematical Knowledge for Teaching*. The model is also based on the affordances of Gestinv database providing information about standardised assessment and mathematics tests in Italy. Teacher collaboration in the model involves interaction with Gestinv's resources and critical reflection on the complexity of standardised assessment in mathematics. Both community of inquiry and MSKT allow describing possible changes in teachers' beliefs about mathematics and mathematics teaching pursued through the inquiry attitude and addressing the formation of mathematics teacher identity.

Other researchers based their approaches to teacher collaboration in PD settings to broader theories of education and learning. As an example, we refer again to Herbst and Milewski's (2018) Story*Circles* in which groups of teachers work together to create a representation of a lesson using a web-based storyboarding tool and cartoon characters collectively. The goal of engaging teachers in making a collective product is inspired from Papert's (1991) *constructionism*, an educational theory of design and learning according to which learning happens best through designing external and shareable artifacts valuing engagement, exposure, bricolage, ownership and discourse.

Summarising, theoretical tools that are shared with teachers to support their collaboration (e.g. DSK, five practices) seem to be used to bring to the fore the complexity of mathematics teaching and the diversity of practices related to it, while theoretical tools that are used by researchers to design their PD activities concentrate on interactions within these settings and how these influence practice and promote teacher learning.

5.5.4 Transversal Issues and Perspectives Around Fostering Teacher Collaboration

A global look at the tools and resources for fostering teachers' collaboration shows a diversity of categorisations in relation to their nature, design purposes, theoretical perspectives and professional practices. As regards the nature of different tools and resources shaping the design of collaboration, our analysis shows that the current landscape in the field is oriented by the following categories of technological tools: (A) designed for teacher collaboration; (B) adapted to operate as formal or informal environments for teacher collaboration, including tools and resources for distance working, storing/sharing and social communication (category B1) and tools promoting new ways of representing teaching (category B2). These tools and resources are based on different kinds of technologies, provide diverse affordances and allow the design of a range of activities for teachers.

Category A includes specially designed digital platforms and online tools (e.g. Collaborative Inquiry Tool, RAMZOR) that allow teachers to develop, share, and jointly improve teaching materials in PD settings. Category B1 includes e-learning platforms (Moodle) and specific tools (Assignment, Workshop), cloud-based storage facilities (Google Drive), interactive databases (Gestinv) and social media (e.g. Facebook). The tools and resources of this type are adapted to be used for quite similar activities with tools of category A involving cyclic interaction (i.e. 'feedback-responses') between trainers and teachers as well as between teachers (i.e. reviewing other teachers' submissions) around an assigned task (Moodle) and sharing online resources (Google Drive) in PD settings.

An additional feature of social media is that they allow teachers to use them to establish groups and collaborate asynchronously outside formal settings (e.g. PD) by posting, commenting and reacting on members' artifacts and practices. Category B2 concerns those technological tools that share the affordance of representing teaching in innovative ways allowing activities, such as designing scenario collaboratively, representing classroom interactions through cartoon stories (e.g. Story*Circles*) and triggering teachers' attention to representations of mathematical objects emerging in a lesson (e.g. the RTA tool).

As regards the theoretical tools oriented towards teacher collaboration, the quoted studies reveal that under broader professional development approaches (e.g. teacher noticing, LS) teacher collaboration is targeted through: (a) theoretical constructs shared with teachers to address the complexity of teaching and the practices related to it (e.g. DSK, five practices); (b) theoretical constructs used by researchers to design PD settings and study the collective part of teachers' work (e.g. community of inquiry), as well as teacher knowledge and learning (e.g. MTSK, constructionism, boundary objects).

Taking a broader look at the research in the field, we can draw some main conclusions. Digital tools and resources seem to have a protagonist role in studies addressing teacher collaboration due to their wide/flexible range of uses such as supporting synchronous/asynchronous interactions around teaching resources, acting as platform and repository for supporting joint work of teachers, and allowing representations and analysis of the finer nuances of teaching practice through digital representations. Online spaces, such as Facebook, Twitter, WhatsApp and Global Math Department Virtual Meetings, constitute an emerging category of tools and resources mediating contextually relevant teacher collaboration outside formal PD settings.

These places have provided teachers with opportunities not only to exchange resources, but also to build learning communities (Larsen & Parrish, 2019) and address individual problems of practice (Risser et al., 2019). Finally, tools and resources providing innovative representations of teaching allow a new look at the social and representational infrastructure of teacher collaboration and which elements of them support teachers in building a broader professional knowledge for teaching (Milewski et al., 2018).

5.6 Tools and Resources for Studying Teachers' Collaboration

The previous sections have addressed: tools and resources collaboratively designed for teaching (Sect. 5.3); tools and resources for collaboratively inquiring about teaching and fostering teacher learning (Sect. 5.4); tools and resources for fostering collaboration (Sect. 5.5). As we consider the tools and resources developed to foster teacher collaboration or evolved from collaboration, we should also consider how these tools and resources can be used to examine the form and the purpose of teacher collaboration. For this section, we consider the methods and theories that are used to examine in what ways, and for whom, the tools and resources developed for and within teacher collaboration are effective, and determine the tools and resources to be developed that will support teachers and teacher educators. This section addresses the tools and resources that are currently available to examine: the impact that teacher collaboration may have on the actors themselves (Sect. 5.6.1); the theoretical and methodological tools that researchers use to examine structure of or interactions within the collaborations (Sect. 5.6.2); the suggestion of potential development for infrastructures to study teacher collaboration (Sect. 5.6.3).

5.6.1 Reflecting on the Impact of Collaboration Tools on the Actors

Researchers and facilitators may examine the tools and resources used in teacher collaboration as a source of data to analyse the impact of the collaboration either directly or indirectly. In using resources directly, researchers may analyse observation notes or recordings of the collaboration, documents for and created from collaboration, observation notes or recordings from the classroom and documents from the classroom (e.g. student work and teacher recordings). Researchers may also use these resources indirectly, examining teachers' reflections on these documents and how their teaching practices have evolved as a result of collaborating with peers. In either level of use, tools and resources can support both teachers and researchers in considering how ideas from teacher collaboration are connected to classroom practice, and viewed as generative and productive for teachers' professionalism.

Albano et al. (2020; see Sect. 5.4.1) analysed a direct resource that was created as part of a professional development activity for secondary teachers. This resource was an instructional plan that the teachers created and revised over a period of time and with the support of professional development. Albano et al. were interested in tracing the impact of the collaboration from both the mathematics teacher experts and the peer teachers in the revisions of the instructional plan. The researchers analysed each iteration of the instructional plan submitted, in order to identify the revisions that the teacher made as a result of the professional development and, in particular, the interactions that may have supported this revision. The final analysis identified a scale of three levels (p. 577):

- Level 0: the teacher made no changes in their instructional plan, or any suggested changes were not properly integrated;
- Level 1: the teacher modified the instructional plan or integrated design details, but there was no evidence these changes were made as a result of the professional development;
- Level 2: the teacher's modifications demonstrated evidence of interactions with content experts and peer teachers.

The researchers found most teachers improved their instructional plans as a result both of the targeted professional development and of peer feedback. Moreover, evidence suggested that, for almost half of the teachers, the feedback from their peers had a greater influence on their task design than the professional development alone. This evidence was supported by analysing several revisions of one instructional plan, therefore future researchers may consider collecting more than two iterations of a document to identify how the collaboration may or may not have supported an individual or set of teachers.

Direct resources may be created for the purpose of supporting collaboration, co-constructed during collaboration or collected from individual teachers before or after collaboration. Through the examination of direct resources, researchers use their own perspectives and theoretical frames to examine the impact of teacher collaboration. However, what could be missing in this analysis is the teachers' voice, or how the teacher identifies the impact of collaboration.

We now turn to indirect resources that researchers use to consider how teachers communicate the impact collaboration has on their practice or beliefs about teaching. Indirect resources include teachers' reflections on the collaboration and can be used to identify the tools and resources that teachers believe are supportive when collaborating with peers. In Sect. 5.4.4, we highlighted how reflexive mediation may impact teachers' learning in collaboration. The following studies demonstrate the indirect resources researchers use to elicit teachers' perceptions of how the tools and resources teachers use in collaboration contribute to their learning about teaching. Some researchers, such as Hollingsworth and Clarke (2017) utilise semi-structured teacher interviews, while others, such as Albano et al. (2020) and Segal et al. (2020), may elicit teachers' reflections on how specific collaborations influenced either their perspectives or beliefs about teaching with follow-up questionnaires.

One form of a semi-structured interview is a stimulated teacher reflection. Hollingsworth and Clarke (2017) created a tool supporting teachers in examining and obtaining feedback about their own teaching practice. The primary purpose of the study was to examine opportunities for teacher learning within a structured stimulation prompting teacher reflection. When developing the observation protocol, the researchers intended the feedback from colleagues to be a conversation about teaching, rather than opportunities to make a critique.

To elicit teachers' perceptions of how the tool supported efforts in improving mathematics instruction, the researchers invited two Australian teachers to participate in the study. These teachers were asked to select the dimensions they were most interested in developing, to video-record one mathematics lesson and to analyse this lesson prior to a video-stimulated feedback conversation with the researchers. After the conversation, teachers reported that the protocol encouraged focused feedback, rather than the more generic feedback they may typically receive outside of structured conversations. The teachers also suggested that the opportunity to observe specific dimensions in their own practice through video was more generative for promoting reflection and informing areas of improvement. Although this particular collaboration was between teacher and researcher, this tool can be used to inform future teacher collaboration protocols that promote teacher agency and self-reflection.

In addition to eliciting teachers' perceptions of collaboration through semi-structured focus groups or interviews, many researchers also elicit teachers' views through written questionnaires or surveys. In this sub-section, we described the analysis tool Albano et al. (2020) created to identify how the comments of both the teacher educators and teacher peers influenced revisions in the teachers' instructional plans. Albano et al. also posed a questionnaire to the teacher participants at the end of the study. The focus question for the questionnaire translated to "What advantages for your teaching profession can you identify in the peer review activity?" (p. 579).

In another study, Segal et al. (2020) discussed how they encouraged collaboration in a digital environment, RAMZOR (see Sects. 5.4.1 and 5.4.4), to support teachers in planning and implementing complex tasks. The researchers developed a questionnaire to determine, via a Likert-type scale, the degree to which the participants believed collaborating in this space with other teachers contributed to both their mathematical and didactic knowledge for teaching and contributed to a sense of belonging to the community (see Sect. 5.2.4). In addition to the scale, the questionnaire included an open-response prompt for teachers to elaborate on the level of agreement selected. Both studies reported teachers appreciated the opportunity to receive constructive feedback from peers. Future researchers may create items to identify teacher perceptions on the use of particular tools and resources used in collaboration.

Indirect resources can inform both researchers and facilitators with evidence for how teachers perceive a collaborative community to support the improvement of teaching practice in mathematics. Eliciting regular feedback from teachers can provide opportunities to revise tools or support structures for teacher collaborations, or create teacher agency to adapt tools and resources to fit better their needs and purposes for collaboration that is generative for their practice.

Both direct and indirect resources can provide researchers and facilitators with a sense of how teachers take up issues from collaboration. The evidence to examine this take up may come from observations of how protocols are implemented or are modified over time; the discourse patterns among the teachers with or without the presence of a facilitator; and observations of classroom practice. Moreover, when evaluating the impact collaborations have on teachers, it is important to include the perceptions of the actors involved in the collaboration.

As we consider the type of resources available to identify or evaluate the impact of the collaboration on the actors themselves, we also find the need to examine the resources available to study the structures in which collaboration takes place. In the next sub-section, we consider methodological and theoretical tools available to researchers to examine the interactions within teacher collaborations at a variety of levels (e.g. school-based, region-based, web-based) and provide examples of how current tools and resources could be used for this purpose.

5.6.2 Studying the Interactions in and Frameworks of Teacher Collaboration

In Sect. 5.2.4, we introduced our shared glossary on tools and resources, and referenced the theoretical frameworks that inform our work as researchers in mathematics education. In this sub-section, we leverage the theoretical frameworks and methodological tools that have been introduced and expanded on: (1) to demonstrate the tools and resources researchers have recently used to study the learning opportunities teachers have in a variety of contexts and structures for teacher collaboration; (2) to introduce new frameworks for studying these opportunities; (3) to consider the learning opportunities that representations of practice afford teacher communities.

5.6.2.1 Learning Opportunities for Teachers in Collaboration

Regarding the various structures of teachers' collaborative work and the learning opportunities that exist within these structures, we examine the work from Chang et al. (2020) and Anderson (2020). From Chang et al., we consider the opportunities one teacher had to revise an instructional task as she moved through three different work groups, or different communities, and from Anderson we consider the learning opportunities for teachers as they discussed problems of practice on a social media platform.

Chang et al. (2020; see Sect. 5.4.1) reported on a lesson-design model that provided teachers the opportunity to create and revise a mathematical task based on feedback. The researchers presented these opportunities through a case study of one teacher, and shared interactions within three distinct communities: the whole group; a small group; a group of mathematics teacher educators. Data sources to capture how the groups supported teachers in learning included revisions to the instructional task, video records of the interactions, and video and written records of the teacher reflecting on the revision process.

Researchers employed a meta-didactical transposition model (Arzarello et al., 2014) to consider how brokers (Wenger, 1998), or the different group settings, supported the mathematics teacher to learn through task revision. Initially, the teacher was resistant to criticise in the whole-group setting. However, as the teacher continued to discuss and receive criticism within the small group and professional group setting, the teacher was more receptive to suggestions in subsequent whole-group discussions. The small group also anticipated how the students might take-up the model, which led to more pedagogical problems that were then discussed with the professional group. These discussions provided the teacher with key questions to consider when revising the task and for what purpose.

Another structure to consider when examining collaboration is the form social media plays in creating spontaneous communities that allow teachers an opportunity to crowd-source for specific ideas. As teachers post problems of interest, they can receive ideas from members with varying experiences and locations. Anderson (2020; see Sect. 5.5.1) examined the discourse structures within one Facebook community to understand better how these collaborative environments could be a generative space for teachers' professional learning and noted four structures of interaction.

Within these discourse structures, made available through social media, teachers not only have access to other's ideas, but also an opportunity to collaborate through exchanges that build on one another's ideas. Therefore, Anderson highlights the ways in which social media platforms can be a generative space for teachers to grow in their professional learning in a more immediate way that not all collaborative communities can provide, and these spaces can provide researchers with a means to identify the current needs communicated by teachers. Next, we review the theoretical tools researchers utilise to identify learning opportunities within teacher collaboration.

5.6.2.2 Emerging Frameworks to Theorise Learning Opportunities for Teachers in Collaboration

A variety of theoretical tools are used for studying teacher collaboration. In Sect. 5.2.4, we introduced a subset of theoretical frames that underlie our understanding of learning opportunities for teachers in collaboration. For the purposes of this chapter, we highlight new research that utilised boundary crossing and boundary objects (Robutti et al., 2020) to theorise how learning is transferred across settings, referencing studies described in previous sections, and also introduce new taxonomies researchers recently created to characterise the learning opportunities teachers have in a variety of collaborative settings.

To theorise how different communities interact to exchange knowledge, some researchers employ boundary crossing (Akkerman & Bakker, 2011; see Sect. 5.2.4). Within these interactions, or boundary encounters, teachers have an opportunity to identify and negotiate new understandings. For example, Psycharis et al. (2020; see Sect. 5.3.1) and Huang (2020; see Sect. 5.5.3) introduced a collaboration that included both primary and secondary teachers to discuss and develop resources that would support students in the development of algebraic thinking. Both frames provided an opportunity for teachers from different schools to collaborate around boundary objects (e.g. lesson plans, student tasks and materials).

To inform their analysis, the researchers employed boundary crossing to consider how the primary and secondary teachers collaborated with one another, given their different instructional contexts and pedagogical and mathematical knowledge. For Psycharis et al., the primary teacher was able to discuss how to promote algebraic thinking and to consider how to support the transition from early algebra to algebra with the secondary teacher. Through analysis, they identified boundaries for teachers including mathematical knowledge (i.e. how the teachers viewed algebraic concepts for their grade level) and pedagogical approaches (e.g. contextualised problems, open-ended tasks, development of generalisations) that were discussed and negotiated as the teachers began to share a view of how to characterise and foster algebraic thinking across the grade levels.

Another theoretical frame comes from the work of Horn et al. (2017) developing a taxonomy to characterise teachers' learning opportunities. When analysing the workgroups, the researchers considered three main questions: (1) the purpose and result of the meeting; (2) the focus of the facilitator; (3) when teachers engaged in dialogue, what was their focus? Through analysis, Horn et al. created six categories of workgroups. The researchers characterised four of the categories as low-depth meetings, suggesting teachers' opportunities for learning within these workgroups were limited (e.g. focusing on pacing and logistics). Horn et al. note these types of workgroups often resulted in one teacher sharing, limiting the opportunities for discussion and/or collaboration. The researchers found that teachers had richer conversations, and thus a greater opportunity to learn, when the workgroup centred on a collective interpretation as they investigated problems of practice.

Similar to Horn et al., Brodie and Chimhande (2020) recently introduced a framework for considering the quality of the content and depth discussed within teacher collaborations. Using this framework, the researchers analysed six activities that the collaboration is centred upon (e.g. analysing assessments, discussions around readings, lesson planning), the content of the talk (e.g. focus on the learner, the mathematics, instructional practice, identified priorities) and the depth in which the teachers engaged with the content. For the analysis of depth, they characterised four levels ranging from no or little engagement with the content (Level 1), to generalising the content or coming to new understandings (Level 3 plus). Similar to Horn et al., Brodie and Chimhande determined that different collaboration activities provided different learning opportunities for the teachers to engage in the content, and the depth of the teachers' conversations did not necessarily shift over time. This analysis suggests researchers still need to develop resources to support the quality of teachers' engagement with the content.

Researchers continue to advance the frameworks we use to examine and characterise the learning opportunities available to teachers to improve their practice, and more research is being done to consider how collaborations are a generative space for teacher improvement. In addition to the theoretical frameworks that conceptualise the mechanisms that facilitate teacher improvement within collaboration, researchers also consider the development and use of representations in collaborative spaces.

5.6.2.3 Theorising How Representations of Practice Provide Learning Opportunities for Teachers in Collaboration

In the previous sections, we discussed the use of representations and the learning opportunities afforded to teachers in two distinct manners. In Sects. 5.3.1 and 5.3.2, representations were developed and/or discussed by the teachers to demonstrate conceptual ideas (such as integer operations or functions), and the section argued how selecting and discussing these representations in collaboration provided generative opportunities for teachers' mathematical knowledge for teaching (Ball et al., 2008). The previous sections also discussed how tools and resources are used to represent teaching practices from two distinct perspectives: the use of representations to support the teaching of mathematics (see Sect. 5.3.3) and encourage reflection (see Sect. 5.4.3); how representations are considered in the design of tools and resources to support mathematics teachers in collaboration (see Sect. 5.5.2). In this sub-section, we consider the theoretical tools or frameworks researchers consider when studying how the learning opportunities' representations of practice afford teacher communities.

To examine records produced through collaboration, Trouche et al. (2019) analysed the collaborative work of two French middle-school teachers planning instruction for a new topic using three theoretical perspectives: Documentational Approach to Didactics (DAD), Anthropological Theory of the Didactic (ATD) and Cultural–Historical Activity Theory (CHAT). Introduced in Sect. 5.2.4, DAD informs the analysis of both how the teachers use resources to create their lesson planning document and how the teachers create this document as a collective. CHAT informs the analysis through the frame of organisational learning, broader than the mathematics education. Trouche et al. argue that, through CHAT, researchers can interpret representation of practice from "rules, artifacts, and division of labour, as well as from community feedback" (p. 55) to theorise how the organisational structure contributed to learning opportunities.

Lastly, the researchers state that, although ATD is a known theoretical frame for mathematics education research, it was not developed to analyse the collective work of teachers. However, they argue that ATD allows researchers to analyse representation of practice through the structure of knowledge and practices, the dynamic work of designing and implementing tasks, and identifying the conditions that afford and/or constrain this work through an ecological perspective.

Many theoretical frames underlie the learning opportunities representations of practice that both afford and constrain teachers' collective work. Teacher communities are often organised around representation of practice, either in the form of considering the classroom practices that already happened (e.g. Pynes et al., 2020; Uzuriaga et al., 2020), but they can also provide teachers with an opportunity to consider decisions from a multitude of perspectives through the work of anticipating student thinking and scripting lessons (e.g. Cusi et al., 2020; Díez-Palomar et al., 2020; Weingarden & Heyd-Metzuyanim, 2020). The theoretical frames that inform our understanding of how teacher communities are generative are not only essential for determining which representations to organise teachers' work, but also in identifying the impact of resources in teacher and student learning.

The theoretical tools and representations of practice described are examples of how researchers are examining the structures of and interactions within teacher collaborations. These tools help researchers to identify the learning opportunities teachers have within these groups and resources to be developed to support teachers in future collaborations. In addition to analysis tools that reflect on the impact of teacher collaborations on mathematics teaching, the Theme D participants also argued the importance of creating analysis tools that consider the affective development of teacher collaborations, that is, how do teachers learn to collaborate and, in particular, what are the differences when this work is either voluntary or obligatory.

In the Theme D Plenary, Brodie (2020) argued that, "safety and trust are important to be able to learn with others" (p. 40), and therefore collaborative tools should also provide space for teachers to build a community of trust and, as researchers, we should also analyse the development of this trust to support teachers in collaboration. Developing these relationships within collaboratives is important, so that teachers are comfortable sharing perspectives that may not have been introduced to the group and assuming positive intent as differences are discussed. We also wonder how researchers could make more apparent their own role as they study teacher communities. As we could consider the relationship between the researcher and the teacher to be a form of collaboration, we assume the researcher takes on the role of participant in some form.

The next sub-section considers resources that still need to be developed to support researchers developing and studying larger infrastructures that organise teacher collaboration at a variety of levels, including international communities.

5.6.3 Developing and Studying Mathematics Teachers' Collaboration Across Instructional Settings

In this sub-section, we question the tools needed for researchers to study teacher collaborations across instructional settings, including collaboration between: grade levels; locations (e.g. urban, suburban, rural, country), funding sources and space (e.g. in-person, virtual, hybrid). We address: (1) the design considerations for platforms that support collaborative settings in the context of mathematics education; (2) the need for infrastructures to analyse collaborative settings; (3) the importance of supporting collaboration at the international level.

The design of technology for researchers' study of teacher collaboration needs to attend simultaneously to (1) enabling teachers to collaborate on the practices of mathematics instruction and (2) enabling researchers to set up, observe and facilitate such collaboration. The first activity calls for technologies that allow teachers to communicate with colleagues in a context where language is not sufficient. The practice of teaching mathematics in classrooms, just like any teaching, lacks a common technical language for practitioners to communicate. While some have proposed designing such language (Grossman, 2020), others have noted the reductive nature of such project (Horn & Kane, 2019) and yet further others (e.g. Herbst & Kosko, 2014) insisted that practice relies on collective tacit knowledge (e.g. knowledge of instructional norms) that cannot be represented in language. The use of videos, animations, storyboards and classroom artifacts has been useful for teachers to demonstrate what they know.

Particular processes of teacher collaboration through technology, like Story*Circles* (Herbst & Milewski, 2018; see Sect. 5.4.3), have created contexts for teachers to collaborate about practice that accommodate teachers' tacit knowledge in the context of scripting lessons (see also Zazkis & Herbst, 2018). For these technologically mediated collaborations to support the transaction of tacit knowledge of practice, the technology needs to be capable of handling multimodal representations of practice. The design of this technology thus requires not only technological specification (e.g. the capacity to handle rich media) but also semiotic considerations (e.g. the systemic capacity to enable the reading and writing practice-related meanings through the manipulation of multimodal expressive tokens).

For the scripting of lessons in Story*Circles*, the existence of a set of graphic characters and a storyboarding software has been essential. These resources permit the storyboarding multimodality to achieve the same flexibility as writing in language and similar capacity as video for the expression of tacit meanings. The design of this representational infrastructure, in ways that it permits it to be an open system for meaning-making, is an important task for researchers to dedicate time to.

The work of teacher collaboration also requires a social infrastructure for teachers to discuss or exchange representations of practice. Technologies that support the capacity to edit storyboards collaboratively or share them in forums, or that support the collective annotation of media (e.g. Anotemos; www.anotemos.org) are therefore important as well. In particular, considering that instructional practice relies on tacit knowledge, it is important to conceive of this social infrastructure as enabling transactions that are multimodal in nature. Thus, the notion of annotation behind Anotemos is a multimodal one, one can annotate a piece of media by interacting with it graphically (selecting regions or making marks on the screen, attaching images), aurally (attaching an audio file), or in writing (by adding comments to moments in the timeline).

Whereas software that allows for some of those functionalities can be obtained off-the-shelf, their integration in the service of long-term research agendas is an important consideration that researchers need to make. The Lesson Sketch platform (which was operational 2011–2020; see Herbst et al., 2013, 2016) contained annotation and storyboarding tools, and included them in the context of a larger infrastructure. In the platform, researchers could organise experiences for practitioners to interact with practice, and could collect the data that practitioners would generate. The Lesson Sketch platform allowed researchers to create such experiences, assign them to prospective participants and obtain reports. The software would record log data of practitioners' perusal of videos and storyboards, as well as report comments made in annotations. To that end, it is key to enable not only the practitioner collaboration, but also the collection of data that can support research on teacher collaboration about practice.

It leads naturally to the issue of developing specific infrastructures for analysing teachers' collective work. Globally, we must admit that research, regarding teachers' collective work, typically puts more energy for collecting data than for analysing it. And yet, this analysis is complex, due to the amount of data at stake when we consider the nature of teachers' collaborative work, such as: the resources each participant brings to the collective setting; the resources produced by the collective; the variety of contexts collaborations occur within and across; boundary objects (Akkerman & Baker 2011). As part of a national research project (ReVEA, https://www.anr-revea.fr/), the AnA.doc platform (Alturkmani et al., 2019; Trouche, 2019), a prototype developed in France, demonstrates the interest, and the difficulties, of such an infrastructure of analysis. AnA.doc is a platform structured on three levels: data collection; data analysis; a shared glossary of concepts.

At the level of data collection, the platform allows the storage of data related to a variety of *situations* of teachers' individual as well as collective documentation work (e.g. preparing a progression, or a lesson, reflecting on his/her practices). Each situation is described following the same model (i.e. history of the actors; context of the school; context of the curriculum; intentions of the researchers guiding the data collection strategy). Each data related to this situation (e.g. resources used versus produced; videos of teachers' work; self-representations of teachers' resource system; questionnaires) is associated with meta-data facilitating their use.

At the level of data analysis, documents created by the teachers on the Ana.doc platform are utilised, composed of a situation or a set of situations (e.g. two teachers co-constructing a lesson and then implementing it in their own class). As teachers can upload a variety of media, the Ana.doc platform provides an opportunity to analyse a portion of this data (e.g. extracting an excerpt of video). Through the platform, members can conduct initial analyses with a small data set (e.g. the role of textbooks in lesson planning) and communicate the findings on the platform, which could be considered as a draft of a final product. Members of the community are then encouraged to comment on this analysis that can support future revisions, or propose an alternative analysis leading to the generation of a new document.

At the level of glossary, each member of the community had an opportunity to define the concepts that were used in the analysis. Community members may reference the same term or concept, but perhaps have different understandings or perspectives. These instances encouraged members of the community to discuss and come to a shared understanding to define and articulate clearly the different possible meanings of this concept, with respect to different theoretical frameworks.

Ana.doc, as a component of the project ReVEA (2014–2018), provides researchers with a prototype that could feed further projects at an international level. Of course, such a project is quite ambitious and needs important human and technological means to develop. We could imagine, as it was suggested during the Theme D sessions, to use platforms such as RAMZOR (see Sect. 5.4.1) as a single repository of lesson plans around the world. These platforms could include data about individual users for covariate analysis (e.g. location; title or position; years of experience; teaching interests). Repositories could also be dynamic, providing an environment that allows for a variety of actions (e.g. commenting on uploaded documents; creating and attaching supplemental documents; suggesting revisions or modifications) and meta-data could be collected by the platform.

As these platforms collect data, researchers could collaborate to analyse samples from these databases and generate claims around products of teacher collaboration, for example comparing lessons across a set of countries. As we expand our communities and boundaries, we must reconsider our ethical obligations to the communities we work with and learn from to ensure respectful collaborations that meet the needs of each community. Theme D participants also discussed the importance of considering the accessibility, adoptability, adaptability, and sustainability of tools developed to support teacher collaboration within varying contexts and levels.

As we reflect on past research and consider the future of teacher collaboration, we argue the need to create more formalised infrastructures that could allow researchers to draw from the same set of data and provide opportunities for analysis in both novel and collaborative manners. These infrastructures could also open the boundaries and provide accessible opportunities for teachers to collaborate with teachers who work in other countries. In this sense, these structures provide for collaboration at the international level for both teachers and researchers. This sub-section considered potential next steps for creating infrastructures that support mathematics teachers in collaboration and argued for the development of tools that support collaborations among international communities.

5.6.4 *Transversal Issues and Perspectives Around Robust Analysis*

As teacher collaboration, and the tools and resources that support teacher collaboration, continues to evolve, the theoretical frameworks and analysis tools researchers use to characterise these collaborations continue to develop as well. In our current time, geography does not necessarily restrain collaboration and the COVID-19 pandemic demonstrated the potential of web resources to facilitate collaboration. This section of the chapter surveyed the theoretical frames and tools researchers are currently using to study the impact of teacher collaboration (see Sects. 5.6.1 and 5.6.2), and provided suggestions for future development (see Sect. 5.6.3).

While many researchers are studying teachers in collaboration, we thought it was important to highlight the methods and analytical tools researchers have used to study these collaborations and suggest papers that focus on particular methods could be fruitful for the field. The studies presented in the previous three sections highlight the analysis techniques or theoretical frameworks researchers are currently using to study the interactions and opportunities afforded to teachers in these settings. From these studies, we gain new perspectives as we consider collaborations as a part of professional development, a bridge between professional development and the teacher' practice, or the collaborations teachers create for themselves, and identify both direct and indirect resources that are used to analyse the impact of collaborations. Many researchers have provided evidence to demonstrate how teachers take up ideas from professional development into their collaborations or practice, but we also encourage future analysis to consider the impact of collaboration on the learning opportunities for students.

As teacher collaborations require teachers to devote a portion of their time, both teachers and those who support teacher collaboration aim to ensure this investment is productive and generative for teachers. Therefore, it is imperative that as researchers analyse the impact of teacher collaboration, teachers' voices and perspectives are included in this analysis. Especially as we consider the many spaces in which teachers self-organise and collaborate that may not be visible to researchers. This leads to the potential in developing tools that provide the following: a repository for collecting resources developed by teachers both *for* and *from* collaboration to be shared with teachers and researchers and provide opportunity for collaboration; the proposal of analysis tools that can be applied across collaboration contexts; tools for ensuring teacher perspectives are included and valued in the final analysis.

We propose repositories should be dynamic and provide an opportunity for researchers to make connections across the media users (e.g. teachers) upload and conduct both qualitative and/or quantitative analysis. To support this endeavour, researchers should make more explicit the types of data that should be collected, in order to perform robust analysis, and how to identify and make best use of meta-data that digital platforms can generate. In collecting this data from teachers, researchers can monitor the needs of mathematics teachers, the types of resources that they request and share, and the different solutions that are generated in collaboration in relation to the specific areas of mathematics.

5.7 Weaving Threads, Perspectives for Research and Development

This final section weaves together the themes discussed throughout the chapter to highlight the main research questions that Theme D aimed to address (Sect. 5.7.1). The section also discusses to what extent the COVID-19 pandemic affected the accessibility of resources and collaboration (Sect. 5.7.2). For this reason, we address the issue of equity as a transversal issue (Sect. 5.7.3). And, finally, we end this chapter by suggesting necessary perspectives of research to be developed (Sect. 5.7.4).

5.7.1 Weaving Threads, Enlightening Initial Questions of Research

In this sub-section, we summarise the findings provided by Sect. 5.5.3 through 5.6, identify themes that cross these sections and discuss main issues that remain to be addressed.

From the previous sections, we retain some main results, in terms of power and necessity of teacher collaboration.

- From Sect. 5.5.3, we retain the power of collaboration for supporting teachers in developing resources for addressing complex issues: implementing a new curriculum, new topics to teach and new practices to develop. Although the resources *for* and *from* collaboration are presented as distinct categories, both influence each other as anticipating practice, sharing experiences of practice and reflecting on practice co-occur within interactions in each collaborative setting.
- From Sect. 5.5.4, we retain the power of (certain) tools and resources for fostering teachers' collaborative inquiry: dynamic settings rather than static ones; open rather than closed format; oriented towards redesigning rather than transmitting; giving room for reflective analysis. This section demonstrates the fundamental role played by *reflexive mediation* when teachers collaboratively interact with different tools and resources to inquire into teaching.
- From Sect. 5.5, we retain the power of (certain) tools and resources for fostering teacher collaboration. Tools and resources are sorted into two main categories: (A) tools and resources *designed for* teacher collaboration; (B) tools and resources *adapted by* teachers or teacher educators to operate as environments for teacher collaboration. The latter category includes resources designed for distance working and social communication, and resources promoting new

ways of representing mathematics teaching. The critical feature of each of these resources is not in their original purpose, but rather their affordance in allowing teachers to develop teaching materials through feedback, discussion and reflection.
- From Sect. 5.6, we retain the necessity of (certain) tools and resources for studying teachers' collaboration, its forms and effects. Among them: repositories for collecting resources developed by teachers both *for* and *from* collaboration to be shared by teachers and researchers; analysis tools that can be applied across collaborative contexts; tools for ensuring teacher perspectives are included and valued in the final analysis. Some of these tools emerge throughout the developmental projects, but we must still develop more formal research infrastructures that could be shared at an international level.

When we examine the themes across the sections, the following results seems to be critical:

- the necessity of tools and resources explicitly designed to support teachers in collaboration and achieve the aims of collaboration in the specified educational context;
- the power of instrumentalization processes to support teachers in adopting and adapting tools and resources designed: for collaborating: for collaboration outside of educational settings; or not initially designed for collaboration;
- the double aspect of resources as *supports* for achieving a given goal, and as *objects* needing an effort to be appropriated. Adopting a resource leads always to adapting it, and that is particularly the case in the context of teacher collaboration, consisting in several stages: discussing classroom issues; designing for addressing these issues; adapting for his/her own classroom, sharing experiences; revising after a process of negotiation. Using and designing are then to be considered as two intertwined processes;
- the dialectic relationship between the nature of resources and the nature of collaboration: resources shape the collaboration and resources are shaped by the collaboration. The living character of digital resources leads to living interactions between teacher educators and teachers, as well as among teachers (e.g. reviewing each other's work). Reciprocally the quality of collaboration conditions the quality of the resources that are developed;
- The sensitivity of teacher professional development to the resources and the interactions developed within the collaborative settings. We imagine the consideration of resources as a collaborative triangle: developing collaboration, developing resources, and developing teachers' knowledge.

Across the sections the main issues still to be deepened appear to be:

- the issue of *quality* and *coherence* of resources and tools collaboratively designed (Pepin et al., 2015). In some cases (see Psycharis et al., 2020), researchers take care of these essential features. In other cases, the design process itself guarantees quality and coherence due to the continuous improvement of resources used by a large number of teachers (the case of Sesamath, see Pepin et al., 2015);

- the issue of *sustainability* of tools. For example, projects may sponsor a web-based tool and lose future funding for hosting, or the tools themselves may become outdated (i.e. either the content or the program code is no longer supported);
- the issue of *scaling-up*. Resources are often designed for and from teachers involved in small collaborative settings. Under which conditions these resources could benefit teachers beyond these settings? These conditions may be related to the forms of collaboration, the nature of resources, the agents involved in the collaboration or institutions;
- lastly, the issue of *digitalisation* of teaching and learning environments. Under which conditions digitalisation could benefit teacher collaboration and support the improvement of collaboration structures.

5.7.2 Rethinking Resources for/from Collaboration Over the Epidemic Period

As we prepared for the ICMI conference in Lisbon, and as we began to write this chapter after the conference, COVID-19 was recognised as a pandemic. This led to the closing of schools and sheltering-in-place for many teachers and students across the world. Bakker and Wagner (2020) and Engelbrecht et al. (2020) provide evidence of the challenges emerging in such a situation. Under these circumstances, we wondered to what extent could teachers' collaboration constitute a necessary counterpoint against the isolation many experienced. We also wondered which resources and tools teachers used or developed for, as a result of these collaborations. For these reasons, in May 2020, we asked Theme D participants to share their own experiences of teaching mathematics in a time of pandemic.

We received nine responses from: Algeria (Sayah), China (Huang), Colombia (Castro), India (Kumar), Israel (Segal and Movshovitz-Hadar), Italy (Faggiano and Robutti), South Africa (Brodie) and an international team (Aldon et al., 2021). These contributions underlined the enormous amount of work that teachers had to accomplish, in a very short time, when asked to move traditional face-to-face classrooms to a virtual environment. Castro identified the following major issues: curricular changes and adjustments; contextualised activities with less or greater complexity; adaptation of evaluation schemes—formative versus summative; the technological infrastructure of teachers; the technological infrastructure of students; platforms, applications, mobile devices, free software; 'orchestration' between training, evaluation and technology programs; changes in schedules and forms of interaction—synchronous and asynchronous—changes in knowledge' beliefs of teachers and students; parental involvement; institutional support.

What emerges from these contributions is the critical aspect of resources both for and from teachers' collaboration for facing these issues during the pandemic. These issues include: adapting existing resources and/or their uses; designing new resources; and identifying missing resources. These categories are not mutually exclusive and, in our email exchange, our colleagues highlighted a variety of techniques developed by teachers and the discourses that supported their choices.

5.7.2.1 Adapting Existing Resources and/or Their Usages

From Israel, the Mathematics News Snapshots Research & Development project (MNS) was described by Segal et al.:

> According to the project policy, the MNSs have been made available only to teachers who participated in a professional development program [. . .]. As soon as the emergency remote teaching/learning began, the project team made a decision quickly to make the 24 MNSs that were prepared up until that point, openly accessible through RAMZOR [Sect. 5.4.1] a designated Hebrew website, [...]. The openly accessible MNSs enabled teachers who wished to present the MNSs, to do it from the newly created website through Zoom, or even dare to let the students access the website and go through any MNS on their own, then run a flipped-classroom style of discussion.

From Italy, Faggiano described moving a course for the prospective teacher education program online:

> This has offered an unexpected opportunity: an online teaching experiment would have been conducted with students at lower secondary school (Grade 7). Hence, some lessons of the course have been devoted to designing collaboratively the activity to be experimented. The whole group of prospective teachers took part in the online teaching activity. Finally, they collaboratively reflected on it, not only during the lessons, but also in further group meetings that they have autonomously organised alongside the course. An online shared folder became the learning environment by means of which the university students built their storyboard. They annotated every comment to the collaborative design; they uploaded the videos of the teaching activity and their transcriptions; finally, they wrote a collaborative text containing the analysis of the results and their self-reflection on-action. In particular they have been interested in the unexpected changes in the activity and in its analysis that were required by the distance teaching–learning mode.

From Colombia, Castro shared with us that:

> Teachers feel alone facing the challenges imposed by the pandemic, neither the Ministry of Education nor the officials seem to comprehend the harsh time teachers have to face to continue teaching mathematics and complying with the academic standards upon which officials assess teachers' work. Teachers turned to the most experienced colleagues and attended online meetings to share documents and tips to use apps, technology and resources. Once in possession of the resources, the teachers dedicated themselves to sharing suggestions for use and didactic adaptation of the documents, videos and free applications. They also made changes in the management of the courses: for example, they proposed projects that involved the participation of several teachers. In this case, the pandemic leads to new forms of collaboration for sharing resources.

Lastly, we heard from the University of Turin in Italy, regarding the decisions of a mathematics education laboratory for secondary school prospective teachers, directed to designing mathematics activities inside the national curriculum of secondary school, and to deepening didactical concepts. The examination of the laboratory—for these Master's students—usually consists in carrying out the designed activities with a selected group of high-scored secondary school students, engaged in a project called 'Stage di matematica'. Usually, Master's students designed the tasks individually. Due to the pandemic, the professor of the laboratory took new decisions: to hold virtual lessons synchronously via a platform; to ask Master's students to work collaboratively on a common task. All the students, collaborating online as a virtual community of practice (Dubé et al., 2005), were asked to design—for the secondary school students—a mathematical 'escape room', to be implemented experimentally the following school year either remotely or in-person.

This working modality was revealed to be successful, as the students—working together collaboratively at the same task—developed a real community of practice with common tasks, vision, aims and practices. The platform, used previously as a repository, became a virtual room for synchronous classes and included a virtual space for asynchronous interaction. The final product came from the interactions among students, developed along the semester and consisted of a set of virtual escape boxes. Each box contained one or more very challenging mathematical problems, quiz or activities for secondary students. To make a pilot experiment of this product, three teams were remotely involved: secondary teachers; university students; secondary school students. From these three trials, feedback and notes were collected to support the development of the final version of the escape boxes.

In all cases shared, the resources and/or their usages were adapted for a purpose that was not the original intent of the resource. This adaptation may lead teachers to develop new ways of collaborating and reflecting. This was the case for Story*Circles* (Sect. 5.4.3), described by Milewski et al. (2020) for a special issue of a journal dedicated to the pandemic. But, as Sayah emphasised, for a case in Algeria, in the total absence of face-to-face collective work, the possibility and effects of using a variety of ICT resources remain important issues: how has the collective work through ICT contributed to the richness of interactions between teachers during this crisis? How did these interactions contribute to developing other resources for supporting teachers' activities during this crisis?

5.7.2.2 Designing New Resources

We learned that a group of expert teachers in China (Huang et al., 2023) were asked by their institution to develop, with their school colleagues, new resources such as online videos that appear fundamentally important in helping teachers adapt to online teaching, although there were some difficulties in the adoption. The school-based Teaching Research Group played a critical role in helping teachers to develop complementary materials for addressing different student learning needs and various technological constraints. This case demonstrates that the process of appropriation of new resources, even those dedicated to this time of pandemics, fosters teachers' collaboration, and leads them to adjust these resources to their own needs.

In South Africa, Brodie describes a case co-written with a group of mathematics teachers. They collaborated to develop resources to teach mathematics through

COVID-19 in their classrooms. The collaboration took place in the context of a Master's course on pedagogy, which looked at how learners' experiences out of school might be drawn on to teach mathematics and the strengths and limitations of doing this. The case of COVID-19 gives an especially useful example because: (a) it combines everyday and scientific resources for understanding the pandemic, its causes and how we deal with it; (b) it supports integration across contexts differently from how this is usually understood; (c) it shows the strengths and limitations of mathematical knowledge in understanding our environment and experience. The teachers present a newspaper article with data from the first 100 days of COVID-19 in South Africa, together with a set of questions which help learners to understand the mathematics involved and relate it to their on-going, lived experience of the pandemic. The teacher collaboration is not analysed, but this presents a first example of developing a resource through collaboration for the COVID-19 experience.

5.7.2.3 . . . and Being Aware of Missing Resources

The pandemic reveals that using existing resources, and developing new resources based on teacher collaboration is not enough. Castro shares:

> The collaboration between the teachers allowed them to respond to the challenge of teaching entirely online, but according to some teachers now is time for a new form of collaboration among education stakeholders—teachers, students, parents and Ministry of Education officials—in order to redefine the school dynamics and to respond to the challenges bring about by the pandemic. Unfortunately, teachers feel that this type of collaboration is out of their reach.

All the issues linked to the pandemic exacerbated, and were exacerbated by, the inequities between students, particularly regarding social issues (see also the following section), in many countries. For example, in the context of India, Kumar shares:

> In the present context of pandemic, the inequities have become exacerbated because of the issues of access and excessive focus on the digitalised interventions for addressing educational needs, while leaving students without the access of devices and connectivity completely in the lurch [. . .] most of the interventions focus on online teaching using video conferencing apps or a mix of chatting apps with the synchronous interactions and assessments, while few have focused on providing most basic resources like textbooks to students who may not have access to device or internet and may be economically disadvantaged to get this kind of access.

Two strands of literature provide good reasons to be highly concerned about the detrimental effects of the school closure on learning outcomes, in particular to students with a disadvantaged background. There is evidence to suggest that: (a) the time spent in school reduces the learning gap with respect to privileged students, particularly in mathematics (Bovini et al., 2016) and in disadvantaged areas (Battistin & Meroni, 2016); (b) long summer breaks have a negative short-run and long-run effect on educational outcomes and are perhaps one of the major sources of learning inequalities (Alexandre et al., 2007).

In the University of Turin, a team of academics from mathematics education and economics and social statistics joined together to develop a project designed to measure the effects of the COVID-19 school-building closures on the mathematics skills and mathematics learning inequalities in primary school children. This evaluation will be the first done in Italy and will provide a timely assessment of educational impacts (Contini et al., 2021).

The United Nations Development Programme (UNDP, 2020) estimates a steep reduction in human development in 2020 as a consequence of COVID-19, due to the educational and economic losses. These considerations lead us to dedicate a special sub-section to the issues of collaboration, resources and (in)equity.

5.7.3 Rethinking Resources and Collaboration Under the Light of Inequities

In this sub-section, we discuss issues related to equity and a few examples of tools and resources developed collaboratively to address this issue. The word 'equity' appears ten times in the ICMI 25 proceedings, indicating that very few collaborations have directly focused on this issue. We focus here on: collaboration for designing resources to face issues of inequity; resources for supporting teachers' collaboration for facing inequities; collaboration for addressing the issue of *accessing* digital tools; collaboration for addressing the issue of equity when *designing* digital resources.

5.7.3.1 Collaboration for Designing Resources Addressing Inequities Issues

Realising that not only schools, but community and family spaces need to be engaged mathematically, South African Numeracy Chair Project worked in collaboration with teachers to use classrooms after school meetings and community math clubs to promote meaningful and fun-filled engagement with mathematics using take-home resources (Graven & Venkat, 2017). Alternatively, recognising that addressing equity involves not only access to school mathematics, but also taking into account marginalised students, studies have built culturally responsive tools in collaboration with teachers belonging to marginalised communities, such as studying patterns in the Maori art and adopting Maori culture in building relationships (Hāwera & Taylor, 2014). In the context of schools located in slums, for the teaching of proportions Bose and Subramaniam (2019) have highlighted the interest of taking into account the resources resulting from the children's own experience.

5.7.3.2 Resources for Developing Collaboration Among Educational Agents to Address Inequities

Teachers may come together in different forms of collaboration to address their own issues, in rejecting the professional learning communities (Brodie, 2021) due to their working conditions and constraints, for systemic or personal reasons. This may occur in collaborative groups that reflect contrived collegiality or those which come together due to voluntary, but isolated action. Louie (2016) illustrated how even equity-oriented mathematics teachers experienced tensions between inclusive and restrictive discourses about mathematical competence during the collaborative interactions.

This finding highlights the challenge of maintaining the focus on equity- and reform-oriented learning in professional learning communities. Professional learning communities could develop in the classroom themselves, as described by Eden (2020) where co-teaching in the classroom setting proved as a resource for noticing student thinking and "thus expanded access to resources for practice" (p. 300). Professional communities at large may also be a resource: according to Nieman et al. (2020), not only teachers, but school leaders, principals and facilitators can also play a big role in establishing equity-focused collaborations.

However, creating teacher collaborations to address equity issues is not simple, as illustrated by Bottia et al. (2016) stating that, "teacher collaboration will only be effective for Latino/a students who are English language learners if the collaboration is accompanied by both adequate pedagogical and cultural understandings of these students" (p. 527), which includes the lived experiences of Latinos who speak Spanish at home.

Another study conducted by Kokka (2018) focused on social justice in STEM education. This study shares how four STEM teacher activists became involved in grassroots organising, sparked by their own experiences of marginalisation and structural oppression, and how the organisation became a vehicle for their own healing, as well as a means toward addressing the inequities they witness and experience in their communities. Among interventions focusing on developing teacher leadership for developing community of learners (Harris et al., 2017), some involved formation of a series of professional learning communities at different levels ranging from whole staff in a school to particular division to "teacher math buddies", with students across grades to study understanding of one topic of measurement across grades (Lieberman et al., 2016).

5.7.3.3 Collaboration for Addressing the Issue of Accessing Digital Tools

Access to and use of digital technology varies indeed across the countries, depending on socio-economic, cultural and gender factors (Forgaz et al., 2010). This gap is usually referred to as the *digital divide* between the people who have access to and knowledge of using technology and the people who have no access to technology. The issue of access to digital resources can be considered at the following levels:

- issues of access to computing devices for teachers as well as students. Initiatives, like *one laptop per child*, have tried to provide low-cost devices to students aiming to establish a collaborative and constructivist environment in the classroom (Buchele & Owusu-Aning, 2007; Kraemer et al., 2009). However, teachers were often not involved as participants and therefore not provided with adequate resources to support students in making the best use of this technology. Studies have identified the need for adequate teacher education (Pischetola, 2014), because access to digital media is not sufficient to bring about transformational change. In addition, under the present circumstance, the access to devices for teachers at home to support virtual learning environments may depend on either the number of devices at home in relation to the number of users, and/or the gender dynamics in the family for determining who may have priority access to the device for their work;
- the issue of 'where' and 'when' the access to digital devices is provided—at school, at home, as shared community resources or as personal devices. Each type has its implications for access, as well as for collaboration among students, between students and teachers, teachers and parents, and even across schools and regions;
- the issue of lack of connectivity, or internet, required to access the digital resource. The lack of connectivity limits the users access to online resources and prevents devices from receiving the latest updates, which may be critical for its use;
- licensing issues related to the use of proprietary digital resources which may be mitigated by the provision of open education resources (e.g. Sesamath in France https://www.sesamath.net, or Connected Learning Initiative India, see below);
- issues of designing for accessibility using Universal Design Principles, so that the resources are accessible to all;
- issues of cultural norms which constrain access to digital resources (e.g. gender or marginalised students in the classroom) as others fail to share the resources equally.

Over the years there has been a shift from interventions focusing on providing low-cost hardware to developing software that supports teacher collaboration across schools and geographies using hand-held personal devices, such as smartphones which are becoming more and more ubiquitous. The software developed is similar to the ones discussed in the chapter, which might be used as a representational aid for mathematics, a pedagogical aid, or a collaborative aid. However, the issue of access to digital resources for teachers and students, and the quality of experiences of students, remain a complex issue to be addressed, at least in developing countries.

5.7.3.4 Collaboration for Addressing the Issue of Designing Digital Resources

While collaborative research for developing content and pedagogical knowledge among teachers exists in large numbers, there are few studies which explicitly focus on the equity aspects while designing tools and resources. When designing for equity, which needs should we focus on? Here, we discuss a few possibilities.

An important consideration for equity is the language used in the resources designed for teacher collaboration. With the availability of digital resources, how do designers address the language concerns of minorities and make the resource available to those who speak other languages? In many developing countries, there exists a tension between considering English as the preferred language of instruction and utilising the learner's home language to develop conceptual knowledge. The politics of the stakeholders who are often in the position to make decisions about what language is used and by whom underlie this issue of language and equity.

In India, a large-scale intervention, called Connected Learning Initiative (https://clixoer.tiss.edu/home/e-library), released its resources in three different languages (English, Hindi and Telugu). This addition allows students to toggle between and change the displayed language of the resource. Revisions to improve the quality of translations and suggestions of more familiar words, for both teachers and students, were made based on feedback from teachers. Through this addition, the resource is more accessible to learners.

Multiple languages thus become a resource for learning by making resources more accessible. When considering translation from one language into another, an additional issue is to be taken in consideration as a great value: the culture. Language is not only a tool for communication, but also the substantial ground for sharing history, culture and values of a population. So, even if the translation is well done, what we have to save with the translation is this substantial ground, to be effective in supporting learning.

Adler (2017) argues that, in the developing world, providing access to education does not ensure that learning takes place, as there is restricted access to valued knowledge. Giving resources to teachers for their classrooms does not really make sense, if they are not provided with the knowledge of why the specific resource is used or how the resource is best utilised. In this manner, the embedded pedagogy becomes opaque to teachers and fails to empower them to develop the knowledge of the content or pedagogy the resource is meant to support. For this reason, it is imperative that designers include teachers' voices describing how they have adopted and adapted resources and document the modifications teachers have made. The collaborative modes of engaging in research 'with' rather than 'on' teachers to develop tools and resources for classroom use can specifically address this inequity by developing teachers' knowledge of content and pedagogy as discussed in the previous sections (Setati, 2005).

This may further ensure that the use of resources will lead to relevant learning outcomes for students, such as exploratory talk and sense-making, while shifting the research discourse from a deficit narrative of teachers' capacities to a respectful discussion, as reflected in the work of Graven and Venkat (2017). These and other studies highlight issues of equity that need to be considered, not only in the design of the tools for collaboration with the teachers, but also in the tools designed for teaching and for reflections on teaching (see Sects. 5.3 and 5.4).

This sub-section has illustrated how issues of equity crop up in several ways which may determine the selection of and access to resources, the design of resources, the nature of professional development opportunities for teachers and teacher interactions within collaborative groups. It also illuminates a variety of complex factors contributing toward the development of a collaborative community through interactions between agents, and attempts to address the complexity of mathematics teaching and understanding of students' thinking for addressing equity while teaching.

5.7.4 Looking into the Future: New Resources, New Collaboration, New Ways of Teaching Mathematics?

This chapter provides a structured way to look at studies on teacher collaboration that involve the use of resources and tools, first of all distinguishing between resources and tools:

for supporting teachers' collaboration;
(coming) from teachers' collaboration.

For and *from* are two main categories that are actually interrelated (see Sect. 5.2): both types of resources influence each other as anticipating practice, sharing experience of practice and reflecting on practice; they co-occur in interactions in professional development settings. In this final sub-section, we propose a reflection in terms of interactions: interactions between tools and resources; interactions between the agents of mathematics education; interactions between theoretical frameworks.

At the beginning of the chapter (see Sect. 5.2), we have tried to differentiate the concepts of *tool* and *resource*, a *tool* being something allowing us to search for, and/or manipulate a given *resource*, grounding the teacher's work. This distinction has guided the authors in writing their respective sections. But, at the end of this writing, having a retrospective look at these sections, we have to acknowledge the fact that it is not always easy to categorise a 'thing' mediating a teacher's activity as a tool or a resource. Is a national curriculum, or an email coming from a colleague, a 'resource' or a 'tool'? It depends indeed on both the context of the teacher's activity that has to be described, and analysed, and the theoretical lens through which this analysis is performed. The meaning of words are social constructs that develop within communities of practice.

The early stages of studies on teachers were directed to investigate essentially the teachers while teaching in their classes. Recently, the research agenda has become wider and richer, exploring: teachers in communities inside or outside the institutions using resources developed for collaboration or not, for education or not, and—more importantly—teachers working side by side with researchers, teacher educators and generally knowledgeable others. This shift in the research agenda indicates teachers' passage from being the 'object' of research to being the 'subjects' themselves, engaged in educational but also in research processes.

The complex issues of resources and tools for/from collaboration often call for networking existing theoretical frameworks (Trouche et al., 2019) and for rethinking theoretical as well as methodological frames (Arcavi, 2019). The need for analysing many people using/designing a huge number of resources calls for new approaches (e.g. linguistics, Learning Analytics, Big Data, for example). The need for considering the institutions and society grounding the use and design of resources call for socio-cultural or socio-epistemological theories for interactional learning and cognitive theories for learning mathematics.

Finally, considering the wide range of research that has been discussed, regarding resources and tools, during this ICMI conference, we would like to suggest some necessary perspectives of research:

- first of all, due to the numerous technologies involved, and their rapid evolution, their categorisations appear quite fragmented. We need a categorisation of the resources and tools for/from teacher collaboration, their affordances, potential and constraints, for inquiring, teaching and designing, as well as reflecting;
- second, due to the diversity of collaborative occasions of teacher professional development, formal or not, distant or face-to-face, the studies considering them appear often compartmentalised. We need studies taking into account the contributions of this diversity of settings.
- Third, while the effects of collaboration on professional development are often slow, and the appropriation of tools takes time, research in this area develops over relatively short periods. We need research taking into account the long time, and a variety of interactions between teachers and resources. This undoubtedly requires new research infrastructures.

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Part III
Plenary Chapters and Reactions

Chapter 6
Using and Developing Content-Related Theory Elements for Explaining and Promoting Teachers' Professional Growth in Collaborative Groups

Susanne Prediger

6.1 Introduction: Why Content-Related Theorizing?

Teacher collaborative groups form a widespread and promising environment for teachers' professional development (PD), and many different variants have already been explored in research and PD practice, as have been shown by an insightful ICME survey (Robutti et al., 2016; Jaworski et al., 2017). However, the ICME survey also revealed that only 85 out of the 316 analysed papers on teacher collaboration explicitly referred to theoretical frameworks on collaboration, with the rest referring to theoretical frameworks for other aspects. This finding shows the need for *comprehensive* and *systematic theoretical foundations* for explaining and enhancing various aspects of professional development in collaborative groups (PDCG).

The term *theoretical foundation* usually refers to local networks of theory elements that serve a certain purpose (here, explaining and enhancing PDCG). *Theory elements* can be constructs, hypotheses or claims (here, related to mathematics or mathematics education and/or its teaching and learning in classrooms or PD). The local network is usually embedded in a larger (perhaps quite general) theoretical framework.

When starting a research project, researchers usually chose a larger theoretical framework in which to situate the research and design, as well as some already existing theory elements that they can *use* as background for their research. Further theory elements need to be *developed* and *connected* in empirically grounded ways, as an outcome of the research (Mason & Waywood, 1996). The latter process of

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constructing a theoretical foundation by identifying, refining and connecting theory elements (mostly in an empirically grounded way) is called *theorising* (Prediger, 2019a). Whereas other papers have focused on *how* the process of theorising can be conducted (referenced in Prediger, 2019a), this chapter focuses on *what* should be theorised for providing theoretical foundations for PDCG.

This chapter is an extended version of the plenary paper in the study conference proceedings (Prediger, 2020). The main message of this chapter is that, in order to elaborate a theoretical foundation for explaining and promoting teachers' professional growth in collaborative groups, various theory elements should be integrated, at both the classroom and the PD levels.

The discussion of a vignette from a PDCG on the PD content 'differentiating and enhancing access to mathematics for learners with difficulties' (more briefly referred to as 'at-risk students') will exemplify how existing theoretical frameworks (communities of inquiry and models of professional growth) provide a language for explaining the complexities of PDCG, but are mainly *generic search spaces*. In order to inform the concrete analysis, and the concrete PD design and facilitation in particular, they must be refined in content-related ways, referring to both types of content: the mathematical classroom content and the mathematics education PD content. The need for more content-specific theory elements will become visible as the generic theory elements gain their explanative power when being filled in content-specific ways (Prediger et al., 2019b).

After a brief introduction of PDCG and the construct of practices in Sect. 6.2, an introductory vignette in Sect. 6.3 illustrates why we need more theorising. Section 6.4 disentangles the kinds of theory elements required for a theoretical foundation of PDCG by exploiting the analogy between classroom level and PD level, while, on this meta-theoretical base, Sect. 6.5 specifies the relevant theory elements for the introduced vignette and the PD it stems from, which provides the concrete material for the meta-theoretical reflections in the final Sect. 6.6.

6.2 Established Generic Frameworks for Explaining Professional Development in Collaborative Groups

6.2.1 Communities of Practice and Inquiry and the Underlying Sociocultural Theories

According to the ICME survey (Robutti et al., 2016), 80% of the 85 papers analysed that explicitly referred to teacher collaboration in their theory sections addressed the theoretical construct of community, often in the senses of community of practice (Wenger, 1998; Lave & Wenger, 1991) or community of inquiry (Jaworski, 2006). Whereas some articles only referred to community in order to name the PD setting in its practical character, most articles also referred to underlying sociocultural theories of learning that have been generic frameworks and were then specified for mathematics in general, yet not for particular mathematics or PD contents.

Within sociocultural theories, learning is conceptualised as being situated in communities of social practice, where novices are successively drawn into practices, first from a legitimate peripheral position, then to the centre of the experienced practitioners (Wenger, 1998). When these theoretical frameworks are related to teachers' learning, teaching is thereby conceptualised as a set of social practices, and professional growth is described by increasing engagement and alignment (Putnam & Borko, 2000).

Jaworski (2006) has enriched this theoretical framework by emphasising that PD itself is shaped by a set of social practices in which alignment should not be assimilation but critical alignment. This offers the opportunity for collective professional growth within a community of practice, rather than the stability of only enculturating novices. She is widely cited for the enriched construct of communities of inquiry in which, "participants [. . .] align with aspects of practice while critically questioning roles and purposes as a part of their participation for on-going regeneration of the practice" (p. 190).

In both frameworks, practices form the key theoretical construct. Practices in mathematics classrooms can be defined as "ways of acting that have emerged [. . . that make] it possible to characterise mathematics as a complex human activity and [. . . that bring] meaning to the fore by eschewing a focus on socially accepted ways of behaving" (Cobb et al., 2001, p. 120). To apply the analogy, *teaching practices* are ways of acting that have socially emerged to manage typical situational demands at the classroom level, while *practices of inquiry* refer to those at the PD level (see below).

Putnam and Borko (2000) had already advertised lifting sociocultural, situated theoretical frameworks from the classroom level to the PD level. In order effectively to exploit these frameworks, they underlined the need for further considerations on: (1) where to situate teachers' learning experiences; (2) the nature of discourse communities for teacher learning; (3) the importance of tools in teachers' professional learning experiences. In this chapter, I extend their call for further considerations into a call for further theorising, that is not only using additional existing theoretical elements, but also developing new ones in an empirically grounded way.

6.2.2 Adapted Model of Professional Growth

Another well-established model for explaining and promoting professional development is the interconnected model of professional growth by Clarke and Hollingsworth (2002), which has been widely used, not only for describing and explaining, but also designing PD for promoting professional growth in modes of action and reflection. The model includes four analytic domains: the *external domain* (with external sources of information, stimulus, or classroom resources); the *personal domain* (teachers' knowledge or attitudes),;the *domain of practice* (in which classroom inquiries can take place); the *domain of consequence* (with salient outcomes such as students' learning gains).

The model identifies different mechanisms by which change in one domain is associated with change in another. Rather than claiming simple mechanisms of transmission from the *external domain* via changing the *personal domain* to the *domain of practice* and then to the *domain of consequence*, they emphasise an "interconnected, non-linear structure" between these domains and identify different "particular 'change sequences' and 'growth networks', giving recognition to the idiosyncratic and individual nature of teacher professional growth" (p. 947). An example of such an individual change sequence is a teacher experimenting in the domain of practice, monitoring students' thinking in the domain of outcomes, thereby expanding their knowledge about student thinking in the individual domain, which is then the result of the change sequence rather than its start.

This model resonates well with the ideas of communities of inquiry, as they emphasise teachers' active roles in connecting their knowledge with teaching practices and evaluating outcomes. More explicitly than Jaworski (2006), Clarke and Hollingsworth (2002) also conceptualise this domain of outcomes as relevant for steering the decisions in the domain of practice. They include the external domain, relevant when the collaborative groups search for external stimuli, for example, from reading or by facilitators, but also when they choose particular curriculum resources or textbooks to adapt for their own work.

As the model was originally articulated mainly for individual teachers and cognitive constructs, two domains in Fig. 6.1 have been slightly adapted:

- With respect to the underlying sociocultural framework, the sociocultural adaptations modified the *personal domain* (teachers' knowledge or attitudes) into the

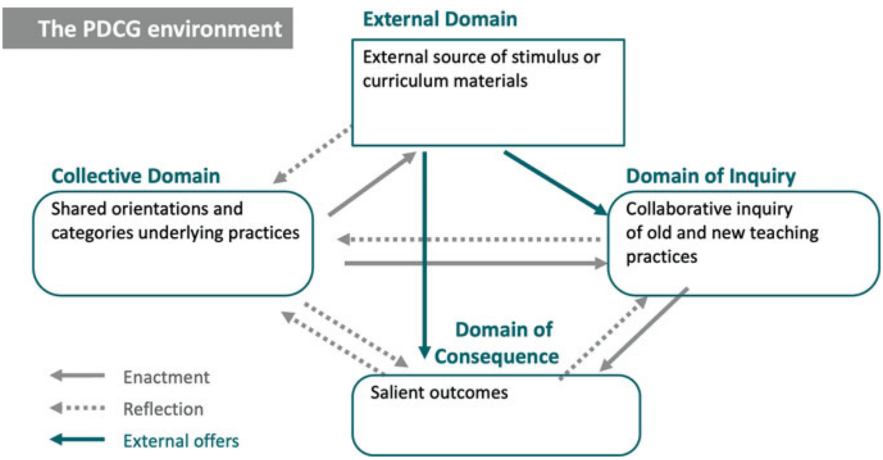

Fig. 6.1 Adapted model for professional development in collaborative groups (generic search space)

collective domain (here conceptualised as including teachers' shared orientations and categories; see below). In particular, this adaptation can extend the analysis to the connection between individual and collective concerns, which is crucial for the conditions of professional growth in collaboration.

- With respect to the particular focus on communities of inquiry, the *domain of practice* is adapted into the *domain of inquiry* (focusing the established practices that teachers reflect on critically and the new practices the teachers experiment with). This adaptation is suitable for PDCG, as it builds upon Putnam and Borko's (2000) call for authentic situated learning opportunities and includes Jaworski's (2006) community of inquiry in the domain of inquiry.

The adapted model is still compatible with the original ideas, as it still connects, "the teacher's professional actions, the inferred consequences of those actions, and the knowledge and beliefs that prompted and responded to those actions" (Clarke & Hollingsworth, 2002, p. 951). In line with Wenger (1998) and Jaworski (2006), teachers' professional practices have been defined as socially established ways of mastering recurrent situational demands in mathematics classrooms (such as differentiating and fostering low achievers). Whereas Clarke and Hollingsworth connect these practices to the underlying knowledge and beliefs, the DZLM research network (led by the author of this chapter) usually refers to Bromme's (1992) situated construct of teacher professional expertise and adapts it to a situated sociocultural framework.

According to Bromme, practices are visibly characterised by shared pedagogical tools (e.g. tasks and teacher moves), but are explainable only by the underlying orientations (i.e. socially shared beliefs about aspects of mathematics and its teaching and learning) and the activated categories for perceiving, thinking and evaluating. These categories are non-propositional knowledge that filter and focus the categorial perception and the thinking of teachers. Within this model, professional growth is characterised not only by changes in practices, but also in the underlying orientations and shared categories for noticing and thinking.

As will be shown in Sect. 6.4, Bromme's general framework gains its explanative power for content-related purposes when filled in content-related ways. This is the key idea of the content-related conceptualisation of teacher expertise (Prediger, 2019b). In order to explain teachers' practices and professional growth for a particular area of PD content, we identify the socially shared, content-related, pedagogical tools, orientations and filtering categories that underlie their utterances and visible behaviour for mastering the self-posed situational demands in the particular area of mathematics education that is the content focus of the PD.

In the following sections, I will try to show why the generic frameworks are highly useful for connecting the different domains (as would also be the general mathematics-specific but not content-specific frameworks such as the Anthropological Theory or Theory of Didactical Situations, which are not dealt with here). But I will also show why the theoretical foundation for a particular PD can substantially profit from a refined framework that allows for more content-specific explorations of particular areas of the mathematical and/or PD content in view.

6.3 Why We Need More Content-Related Theorising: An Introductory Vignette from a Community of Inquiry

To convince the reader that content-related theorising might be useful, I report a vignette from a PD research episode that took place in the beginning of the design research project Mastering Math (Prediger et al., 2019a), aiming to show where the generic frameworks were not sufficient for explaining the vignette. The teacher group collaborated intensively and grew, but did not succeed in achieving their goals.

A Vignette from a PDCG

The very first phase of our PD design research project (Mastering Math) involved researcher facilitators from our Dortmund research team (i.e. PDCG facilitators who are researchers at the same time) visiting active schools in which teachers had started collaborations for developing their teaching practices with the goal of providing better access to mathematics for so-called at-risk students (e.g. students with special needs or students at risk of failing due to limited support at home and/or in earlier school years).

The vignette took place in a school where mathematics and special education teachers worked collectively on differentiating, in order to foster all students' achievement in their grade 5 classroom. The outcome of the first meeting was that the teacher group joined a university–school partnership for 18 months. At the moment of this first meeting, the teachers had already spent 9 months on finding ways to differentiate their teaching material, in order to adapt to students' diverse mathematical abilities. After 9 months of intensive collective work, they were proud to have substantially changed their teaching practices in order to adapt to all students' abilities, mainly in task-based, individualised settings.

When the researcher facilitator first met them, Paul, the mathematics teacher, reported about Suleika, one of their students with learning difficulties, and showed two of her products on multi-digit subtraction (shown in Fig. 6.2):

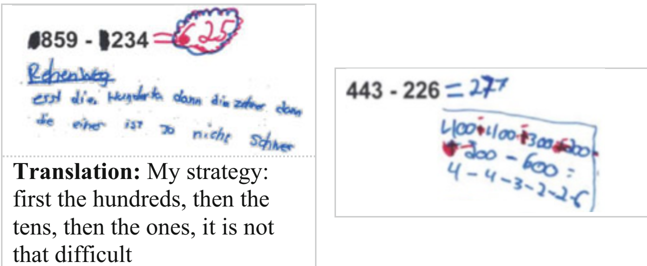

Fig. 6.2 Snapshot from a community—monitoring Suleika's learning

> *Paul*: Suleika can calculate the subtraction well, only the carries pose problems for her. But we can handle this successfully by differentiated tasks: I only give her subtractions without carries.

Although the community agreed on the success of their changes, it took 3 months for one of the members, Maria, a special education teacher, to convince her colleagues to participate in a PDCG program with the university, as she saw an additional problem to be tackled:

> *Maria*: I tried to teach them subtraction with carries several times, but they always forget it.

Analysis Within the Generic Framework

Without doubt, Maria, Paul and their colleagues formed a community of inquiry in Jaworski's (2006) sense. They collaborated intensively, questioned and developed new practices of differentiation (*domain of inquiry* in Fig. 6.1). However, Paul's comment was a characteristic expression of this community's adaptive differentiating practices (that the group articulated repeatedly in different utterances): students receive individualised tasks that are optimised on a level that they can master. Within the frame of these teachers' collective evaluation (*domain of consequence* in Fig. 6.1), the collectively established differentiation practice in their domain of inquiry proved to be successful, as Suleika was able to complete her tasks.

However, in spite of the intensive engagement of these teachers' community of inquiry, they could not develop more productive practices for enhancing Suleika's learning. Although her second product reveals serious struggle with place value understanding (see Fig. 6.2), this was not treated by the teachers. Maria's additional concerns that students "always forget" had also not yet entered the teachers' collective domain and was not part of their shared space of discourse.

What the Analysis Leaves Out

This brief analysis reveals the importance of shared orientations of what the teachers considered relevant in their community of practice. However, the language of analysis provided by the generic theoretical framework from Sect. 6.2 is not yet well-enough elaborated to identify the critical points in the teachers' PDCG in more detail. In this case, it is not the theorising of the mathematics content in view that is lacking, but rather the PD content.

A rough analysis outside the given theoretical framework reveals that, for the teachers, differentiation meant adapting to students' abilities rather than really strengthening their learning, so their shared category for evaluation was reduced to task completion rather than learning progress. The reduction to this evaluation category also reduced the need for critical alignment with their own teaching practices. At the same time, the teacher community did not distinguish between procedural and conceptual knowledge, as they did not problematise whether teaching Suleika the algorithm with carries might miss the conceptual base.

These aspects (procedural and conceptual knowledge and the difference between learning progress and task completion) occur to be crucial for reaching the PD goal that the collaborative group gave itself, namely providing access to mathematics for

Suleika. For the researcher facilitator who started a university–school collaboration with the group, these aspects had to be a substantial part of the reasoning about the PD content of this PDCG.

This indicates that a theoretical foundation which can really inform facilitators' work in supporting the development of this community of inquiry will need to include these aspects of the PD content. The theoretical foundation should also provide a term for explaining why Maria was not able to introduce her 'students forgetting' category into the shared discourse. Again, this has both a generic part and a content-specific part that is tied to the particular PD content in view, which in this case is differentiating and fostering mathematics learning of at-risk students. But what exactly is meant by theoretical foundation? For which functions is it needed?

Section 6.4 will introduce the meta-perspective on generic and content-specific theory elements and its functions for the first steps towards a content-related sociocultural theory of professional growth. Section 6.5 then discusses the concrete elements that can explain the vignette and guide facilitators' actions in supporting a community's professional growth.

6.4 What Kind of Theory Elements Are Required for a Content-Related Theoretical Foundation for PDCG?

This section starts with deepening some structural, meta-theoretical clarifications mentioned in the introduction: what is a theory, and why do I only speak about theory elements? What kind of theory elements are required for an empirically grounded theoretical framework for explaining and promoting teachers' professional growth?

The role of theories for educational research studies is two-fold. On the one hand, theories influence (but do not determine) the design decisions and the methods and perceptions in the empirical investigations of the teaching–learning processes that have been initiated (theories as a framework for research and design). On the other, empirical investigation aims at generating and eventually testing or refining theories (theoretical contributions as outcomes of research). The interplay between theories as frameworks and outcomes of research applies to all kinds of research in mathematics education (Mason & Waywood, 1996; Prediger, 2015). In design research, in particular, it is fuelled by the iterativity and interactivity between theory-generating and theory-guided experimenting.

Whereas the role of theories as frameworks for research have often been discussed (e.g. Cobb et al., 2001; Mason & Waywood, 1996), the role of theoretical foundations for designs includes describing, explaining and predicting what can happen (Prediger, 2019a). The process of *theorising* is worth further methodological and strategical reflections. As already defined, theorising is the process of developing new theory elements in an empirically grounded way, including activities such as identifying an interesting phenomenon and developing constructs for describing and explaining it, refining constructs in order to increase their explanatory power, connecting two descriptive elements to explanatory elements, transforming an explanatory theory element into a normative element or into a conjecture for a predictive theory element, and connecting elements to new explanatory and predictive theory elements (Prediger, 2019a).

From the methodological and technical side, the process has been intensively reflected upon (e.g. Glaser & Strauss, 1967). However, limited explications exist so far on *what* kind of theory elements are to be generated with respect to PDCG. This section therefore makes a suggestion about the kinds of theory elements required, based on distinctions in their *logical structure*, their *size* and, in particular, their *function* (Prediger, 2019a).

Niss (2007) characterises a theory by its *logical structure* as an "organized network of concepts (including ideas, notions, distinctions, terms, etc.) and claims about [. . .] objects, processes, situations, and phenomena" (p. 1308). The claims can be basic hypotheses, statements logically derived from the fundamental claims or empirically grounded propositions about connections and mechanisms. The logical structure of theory elements can therefore entail constructs, basic assumptions and empirically grounded connections.

Theories vary in *size*. Some encompass a well-elaborated theoretical framework with a complex network of constructs and propositions (such as sociocultural theory), while others are reduced to single constructs or claims (such as the communities-of-practice construct). Rather than networking complete theoretical frameworks (Bikner-Ahsbahs & Prediger, 2014), this chapter focuses on the local integration of several constructs and claims. This networking strategy is more suitable for fields that are not yet mature enough for big theories (as Jaworski, 2006, stated for the field of PDCG).

In order to decide which theory elements have to be integrated for a theoretical foundation for PDCG, distinctions based on their *functions* in the design and research process are useful (Prediger, 2019a). On the *classroom level*, there is a long tradition in developing theory elements that can support the design of classroom learning environments and learning trajectories (e.g. Gravemeijer & Cobb, 2006). Theory elements are necessary for at least four main functions (Mason & Waywood, 1996; Prediger, 2019a):

- **C1-content:** theory elements for specifying and structuring the mathematical learning content (e.g. constructs describing relevant parts of the learning content and their relationships);
- **C2-learning:** content-related theory elements for explaining mechanisms of mathematics learning (e.g. a hypothetical content trajectory, hypothesising relevant steps in the learning progress with respect to the specified aspects of the learning content);
- **C3-teaching:** content-related and generic theory elements for explaining the nature and background of mathematics teaching;

- **C4-classroom environment:** content-related and generic theory elements for designing and enacting learning environments (e.g. derived design principles and their justification by C2 and C3).

As the lower part of Fig. 6.3 visualises, these functions refer to different parts of the didactical tetrahedron on the classroom level and have increasing complexity: C1-content refers to the mathematical content alone; C2-learning to the edge between students and content; C3-teaching refers to the faces (e.g. among teacher, students and content); C4-classroom environment to the whole tetrahedron. (Of course, theory elements on other single vertices or edges have also been established insightfully in the mathematics education research community, but to keep the theoretical foundation manageable here, we focus on the main four elements in increasing complexity, where C4 also includes C3 and so on.)

Experience with theory elements and theorising on the classroom level can be lifted to the teacher PD level, as the teacher PD complexities can be grasped with a structurally analogous tetrahedron that relates the teachers (now in the role of learners) to the PD content, the PD resources and the facilitators (Prediger et al., 2019b). Similar to Putnam and Borko (2000), who lifted the generic socio-cultural framework from the classroom level to the PD level, we can lift the assumed need for content-related theory elements to the PD level:

- **PD1-content:** theory elements for specifying and structuring the PD content;
- **PD2-growth:** content-related theory elements for explaining mechanisms of teachers' professional growth;
- **PD3-facilitating:** content-related and generic theory elements for explaining the nature and background of facilitating PDs, if a facilitator exists;
- **PD4-PD environment**: content-related and generic theory elements for designing and enacting PD environments.

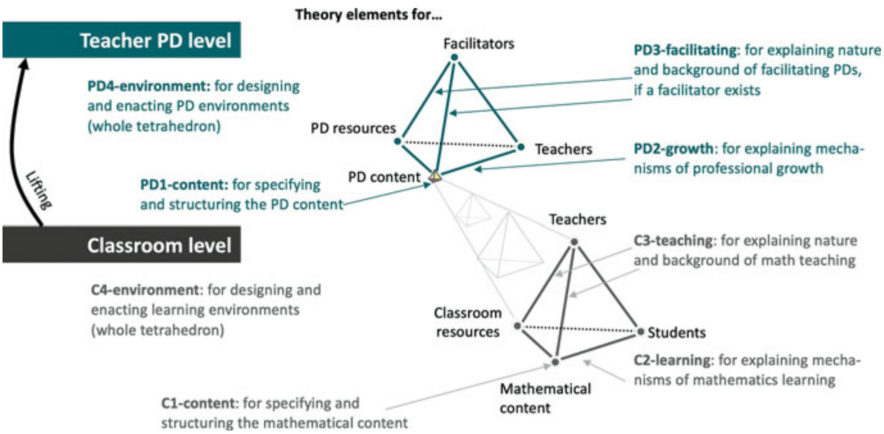

Fig. 6.3 Lifting theory elements from the classroom to the PD tetrahedron. (Prediger et al., 2019b)

The structural analogy between the classroom tetrahedron and the PD tetrahedron has two major limitations:

1. the ways that teachers engage with the PD content is not simply called ‘learning’, but is referred to as ‘professional growth’ to indicate the teachers’ much stronger agency and, in PDCGs, the facilitators’ job is not teaching, but only facilitating in a narrow sense, indicating again the teachers’ higher agency;
2. even more importantly, the nature of the PD content is much more complex than the classroom mathematical content. As the grey lines between the classroom tetrahedron and the PD tetrahedron indicate, the complete classroom tetrahedron is nested in the PD content (Prediger et al., 2019a). This means that all theory elements at the classroom level can potentially be a relevant part of the PD content. For content-related research on teachers’ professional growth, the nested nature of the PD content makes it necessary to unpack PD1-content into C1-content, C2-learning, C3-teaching and C4-environment.

In the next section, I exemplify how these different content-related theory elements can support the understanding of the initial vignette and how they could inform the facilitators’ reasoning and actions.

6.5 Generic and Content-Related Theory Elements for Explaining and Enhancing PDCG

Coming back to the vignette from Sect. 6.3, the researcher facilitator has to obtain a profound understanding of the collaborative group’s practices and challenges before changing their role from observer/analyser to facilitator of the PDCG and before supporting the group’s professional growth. These dual practical goals—the facilitator’s understanding and then intervening—have a counterpart on the theorising side. The repeated (and much more systematic) analysis of these kinds of vignettes can enhance the researchers’ theoretical understanding by processes of empirically grounded theorising. Systematically connecting the theory elements can generate a theoretical underpinning for typical facilitation practices and designs for PDCG. That is how our research group aims to find a theoretical foundation for enhancing teachers’ professional growth. For this theorising purpose, it proved to be highly relevant to unpack the theory elements on both the classroom and PD levels, not only by means of generic theory elements (which apply to all classroom and PD content), but also by means of content-related theory elements. In this case, the classroom content was understanding multi-digit subtractions and the PD content was fostering at-risk students’ access to mathematics.

Although it is not possible to demonstrate the theorising process with all its details here, this section’s intent is to show the power of working with articulated theory elements unpacked down to the level of the mathematical content (C1-content). I will successively introduce theory elements C1–C4, and then PD1–

PD4, and use them for the analysis. The analysis starts at the classroom level to build the ground for analysing the group's teaching practices and the group's processes of professional growth later at the PD level. It ends with a look at how the facilitator reacted and how this experience informed the PD design for future collaborative groups.

6.5.1 Theory Elements on the Classroom Level for Explaining the Vignette

Introducing Established Theory Elements for C1-Content
On the classroom level, the theory elements for C1-content relevant for specifying the classroom content in view are printed in Fig. 6.4. The table entails not only the classical distinction of conceptual understanding and procedural skills (Hiebert & Carpenter, 1992), but also an additional distinction that emerged as necessary when working with at-risk students: the actual learning content and its foundations from previous years (Prediger et al., 2019a).The examples in Fig. 6.4 relate to the classroom content for multi-digit subtraction and its conceptual underpinnings (discussed in Hiebert & Wearne, 1996). Multi-digit subtraction with carries was the actual procedural skill to be learned, with the procedure being based on the conceptual understanding of regrouping units while subtracting. The regrouping of units is based upon the place-value understanding of the meaning of the digits and carries in the subtraction.

Analysing the Vignette with Respect to C1-Content
In the earlier vignette, Suleika mastered the basic skills of subtraction facts up to 10 and used them for multi-digit subtraction without carry. Subtraction with carry, however, is based on the conceptual understanding of decomposing numbers into digits. Suleika could not build on her mastery of multi-digit subtraction without carry due to limited fundamental place-value understanding (which becomes visible in her decomposition of 443 into 400 − 400 − 300 rather than 400 + 40 + 3; see Fig. 6.2).

	Conceptual understanding	**Procedural skills**
Actual content prescribed by syllabus (Grade 5)	Conceptual understanding of actual content, in this case, re-grouping units while subtracting	New procedures, in this case, multi-digit subtraction procedure with carry
Foundations from previous years (Grade 2)	Understanding of basic concepts, in this case, place value understanding (meaning of digits)	Basic skills, in this case, basic subtraction facts up to 10

Fig. 6.4 Theory elements of C1-content for specifying the classroom content, with content trajectory for structuring it (C2-learning)

Introducing Established Theory Elements for C2-Learning and C4-Environment

The generic mechanism of learning (theoretically described in C2-learning) that helps to explain Suleika's challenge in remembering the procedure of subtraction is that sustainable learning always requires connecting to previous knowledge (Hiebert & Carpenter, 1992, p. 67). These connections can best be accomplished with consolidated understanding of the foundations of earlier years. Hence, students who have no understanding of specific basic concepts cannot continue learning along the content trajectory. In other words, the actual conceptual understanding or the procedural skills building upon these basic concepts are not accessible.

This proposition about the generic structure of content trajectories has been empirically proven for the case of arithmetic in long-term assessment studies (e.g. by Moser Opitz, 2007) and resonates with the disentangled content aspects for multi-digit subtraction (Hiebert & Wearne, 1996). In Fig. 6.4, the resulting proposition for a content trajectory is indicated by the arrow. Understanding of basic concepts often underpins basic skills, and these are necessary for building further conceptual understanding, which then underpins the new procedures.

Based on these empirical findings on typical content trajectories in arithmetic and the high relevance of the basic concepts, particularly for students at risk of missing access to mathematics, our design research group has designed learning environments in Mastering Math that enable teachers formatively to assess and foster students' understanding of basic concepts (Prediger, Fischer et al., 2019a).

The theory elements of C4-environment underlying these learning environments include three design principles (for their empirical and theoretical justification, see Moser Opitz et al., 2017):

(DP1) focusing on conceptual understanding;
(DP2) monitoring students' learning progress using diagnostic tasks;
(DP3) promoting discourse.

This learning environment was shown to be effective for giving students safe access to the understanding of basic concepts, with significantly higher learning gains than the control group (Prediger et al., 2019a). However, the learning gains varied substantially and relied heavily on the teachers, so further PD research is required for optimising support for all collaborative groups.

Analysing the Vignette with Respect to C2-Learning and C4-Environment

The teachers in the vignette did not know about the learning environments designed and evaluated by the university at the time of the first meeting, and were not aware of the relevance of place-value understanding for multi-digit subtraction with carries. In the vignette, they reported that they had chosen other curriculum resources that provided learning opportunities for Suleika that were not aligned to the trajectory (C2-learning) in Fig. 6.4. The fact that the teachers tried to teach her later stages of the content trajectory, without taking into account the earlier stages, might explain why she always forgot the content. In this way, the theory element of the content trajectory entails the generic mechanisms of learning (procedural skills are acquired sustainably when they are connected to the underlying understanding of basic concepts) and their content-specific substantiation (multi-digit subtraction with carry should be connected to decomposing numbers and the meanings of the digits in place value understanding).

However, the teachers' practices for differentiating relied neither on the categories from C1-content (Fig. 6.4), nor on the propositions about the learning trajectory (C2-learning) and design principles in C4-environment. Instead, they were based on the practices that the community of inquiry collectively developed, independently from the design research team. Rather than blaming the teachers for not using this unknown approach, theory elements of C3-teaching enabled the researcher facilitator to explain the teachers' practices and their logical consistency with the approach of differentiated tasks and to describe their forms of inquiry, acknowledging the enormous efforts and achievements in their community.

Introducing New Theory Elements for C3-Teaching

Teachers' professional practices are socially established ways of mastering recurrent situational demands in mathematics classrooms; in our case, the situational demands are differentiating and fostering at-risk students' access to mathematics. Qualitative analysis of the teaching practices for specific situational demands is done with respect to the shared pedagogical tools (e.g. curriculum resources, tasks and teacher moves), the underlying orientations and the activated categories for perceiving, thinking and evaluating (Prediger & Buró, 2021, 2024). Empirically identified categories are usually marked by ||...|| and orientations by <...>. In these studies, we found that teachers' differentiation practices are not always guided by the idea of enhancing students' ||learning progress||, but sometimes simply by the category of ||task completion||, a category that Gravemeijer et al. (2016) have also identified and called 'task propensity'. Rather than focusing on how they can leverage their students' understanding to the next zone of proximal development along the content trajectory, some teachers optimise their differentiating practices in a way that all students can succeed to complete the task, even if no ||learning progress|| is initiated.

Analysing the Vignette with Respect to C3-Teaching

In our vignette, the practices to be analysed using the theory elements of C3-teaching are Paul's and Maria's (and their colleagues') differentiating practices dedicated for at-risk students such as Suleika. The teacher community was driven by the shared inclusive orientation <a good inclusive classroom is adaptive to students' abilities>, and they realised it using the pedagogical tools of differentiated tasks and activity settings of individualised learning. In line with design principle DP2 (monitoring students' learning progress using diagnostic tasks), they used diagnostic information, but only about Suleika's procedural skills, not about the underlying conceptual understanding. The design principle reflects a <diagnostic orientation> to initiate adaptively students' ||learning progress|| that entails going back in the content trajectory first to ensure she gets access to earlier steps in the content trajectory and then can make progress along the trajectory.

This was not relevant for the teacher group, because they chose their practices guided by the shared category of ||task completion||. Applying this category, the teachers evaluated the short-term success by assessing whether a student was able to complete the task with the given support and simplifications. In this case, the teachers were successful in creating classrooms in which all students worked and completed tasks. And, indeed, simplifying Suleika's mathematical demands to digit-bound subtractions without carries proved to be efficient for fulfilling their evaluation category ||task completion||. However, this category is guided by a <short-term orientation>, whereas a <long-term orientation> would refer to ||learning progress||, in other words, amplifying Suleika's skill and understanding. The case of Paul in the vignette resonates with empirical results found for many teachers (Prediger & Buró, 2021, 2024).

6.5.2 *Theory Elements on the PD Level for Explaining the Vignette and Taking Actions of Facilitation*

Deriving Theory Elements for PD1-Content

The content-related analysis in Sect. 6.5.1 reveals not only *that* the teachers held certain orientations and categories that guided their practices of differentiating, but also *which* orientations and categories were relevant in the case of Suleika and the mathematical content of multi-digit subtraction. What was exemplified in this vignette was also found more systematically in many other vignettes, and revealed a specification of further content-related categories and orientations that could bring the teachers' collective inquiry forward. In a process of empirically grounded theorising, researchers who observed several teacher groups finally structured the relevant categories and orientations as shown in Fig. 6.5, which includes, as a nested core, the categories for specifying and structuring the classroom content.

For facilitators, Fig. 6.5 later turned out to be a useful tool for communication with teachers (although it did not exist during the vignette itself). The theory elements C1-content and C2-learning, namely the categories from Fig. 6.4 and their structuring in a content trajectory, can help teachers make decisions about learning goals and assess students' learning pathways along the content trajectory. This also requires a general <conceptual orientation>, in order not only to focus skills in a <procedural orientation>. However, the choice between <short-term

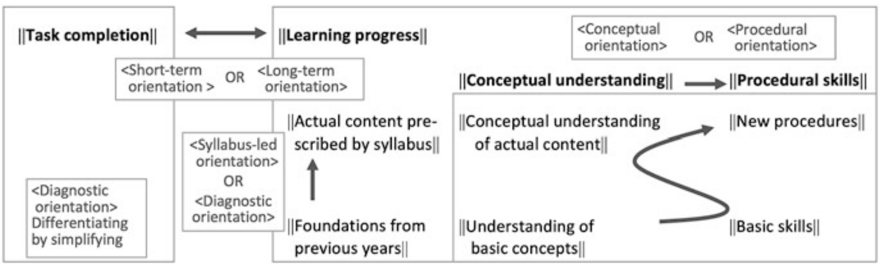

Fig. 6.5 Specifying the PD content—Teachers' categories and orientations (PD1-content)

orientation> and <long-term orientation> determines whether teachers focus on ‖ learning progress‖ or on ‖task completion‖ as their major category of thinking, perceiving and evaluating their practices. Only when the category ‖learning progress‖ is involved can the content trajectory become relevant.

Introducing Theory Elements for PD2-Growth

To explain the mechanisms of teachers' professional growth, the adapted model of PDCG from Fig. 6.1 is used and systematically substantiated with the unpacked PD content (in Fig. 6.6).

Analysing the Vignette with Respect to PD2 and PD1

Until the facilitator came in and provided external sources, the vignette can be analysed in the three lower domains without the *external domain*: the collective domain, the domain of inquiry and the domain of consequences. Based on the status quo in the *collective domain* (as presented in the analysis by C3-teaching), the community of inquiry was working hard on new differentiation practices (their *domain of inquiry*). However, their current focus on the evaluation category ‖task completion‖ had substantial impact on the way they perceived their success in the *domain of consequence*.

With the theory elements of the specified PD content in mind (see Fig. 6.5), the researcher facilitator noticed that the teachers were not concerned about Suleika's place-value understanding, although her written product in Fig. 6.2 provided strong evidence of its peculiarity. Thus, the researcher facilitator noticed that the teachers did not activate the category ‖understanding of basic concepts‖ for assessing Suleika's work. The researcher facilitator assessed that the community of inquiry was driven neither by the <conceptual orientation> nor by a <long-term orientation>, which would have led them to focus on her understanding of basic concepts, rather than trying to teach her multi-digit subtraction without any place-value understanding. A first rough approximation of this analysis allowed the researcher facilitator to explain why the teachers' enormous efforts had not yet led to satisfying long-term results (see Fig. 6.6).

At the same time, outside the shared collective domain, Maria (the special education teacher) put a second evaluation category on the table by saying, "I tried to teach them subtraction with carries several times, but they always forget it". This category of ‖forgetting‖ in Maria's utterance is a remarkable one. While it refers to the category identified as crucial in PD1-content, ‖learning progress‖, it does not relate students' ‖learning progress‖ to teachers' practices, but explains the failure of ‖ learning progress‖ solely within the students. In this way, it had not yet initiated a focused reflection in the *domain of inquiry* for several months in the collaborative group, but a call for external support. Indeed, Paul and other colleagues articulated that they were not interested in ‖forgetting‖, as this could not guide their teaching and assumed too much responsibility by the students (Jackson et al., 2017). Here, the incompatibility of teachers' orientations explains why the community of inquiry could not adopt Maria's concerns.

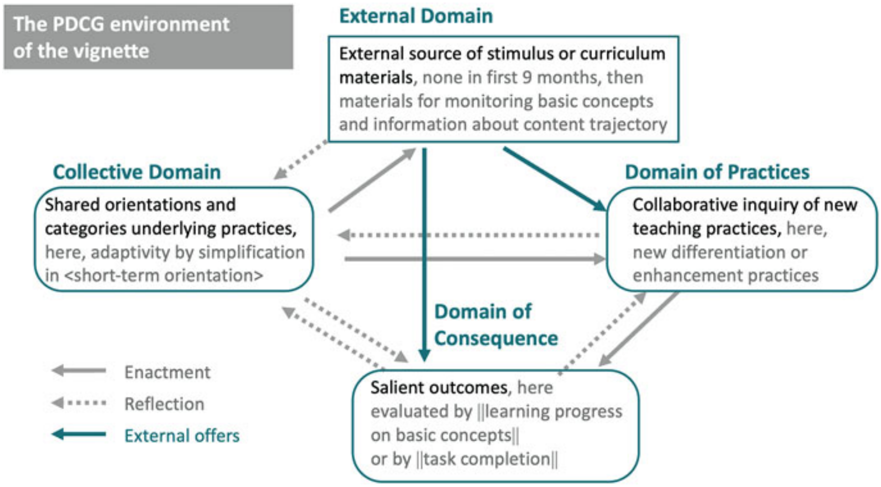

Fig. 6.6 Content-related substantiation of adapted model for PDCG (PD2-growth and PD4-environment)

This analysis in line with Fig. 6.6 gave the researcher facilitator an idea about how to enter the discourse with the collaborative group, in order to turn Maria's concern into a collective and productive concern.

Consequences for PD3-Facilitation and PD4-Environment in the Continued Vignette

In order to draw consequences for PD3-facilitation and PD4-environment, the content-related substantiation of the adapted model for PDCG provided a helpful framework. The facilitator researcher's practice for enhancing professional growth started with listening to the teachers and analysing their collective efforts in all three domains, by means of the theory elements PD1-content (Fig. 6.5) and PD2-growth (Fig. 6.6).

Following the principle of building upon teachers' collective starting points, she realised that the <conceptual orientation> was not the ideal starting point to discuss in this community, as it was too far from their actual collective concern. Instead, she chose <short-term versus long-term orientation>, in order to relate Paul's and Maria's points and build upon Paul's intention to consider aspects that they could influence in order to offer a new orientation. The following utterances are synthesised from a longer conversation:

> *Facilitator*: Paul says you can handle Suleika's difficulties successfully by giving her only subtractions without carries. However, Maria does not seem to be satisfied with the learning outcome. What is the problem with Suleika always forgetting the procedure, Maria?
>
> . . .
>
> *Facilitator*: You also seem to be interested in the long-term learning. Can we go back some steps and check what Suleika can master on her learning pathway towards multi-digit

> subtraction? I see how she decomposed the numbers (443 = 400 – 400 – 300); do you think this could have any impact on her ability to remember the procedure of managing carries?

After 30 min of discussion, Paul, Maria and their colleagues collectively decided that they needed to go back in the content trajectory, in order to stabilise Suleika's learning pathway in a more sustainable way. It took them much longer to realise that they needed to provide learning opportunities for the understanding of basic concepts, and that this might also be the more productive practice of differentiation. Thus, the first external offer provided by the facilitator could strengthen Maria's implicit <long-term orientation>, which also opened the teachers to other evaluation categories for their teaching success. However, to enact teaching practices towards the new evaluation category ‖learning progress‖ in ‖understanding of basic concepts‖, they required further external offers, namely pedagogical tools for formatively assessing students' ‖understanding of basic concepts‖ and teaching material for enhancing them.

The externally offered curriculum materials for formative assessment and remediating sessions provided them not only with the required pedagogical tools, which they could now integrate into their practices, but also with access to the detailed pedagogical content knowledge on understanding of basic concepts for other mathematical topics, such as place-value understanding on the number line and meanings of multiplication and division (see Prediger et al., 2019a).

Once the teachers had incorporated these categories and orientations into their collective domain, their inquiries resulted in bigger changes of their practices and a closer approximation to the newly set goals: enhancing all students' ‖understanding of basic concepts‖ in order to assure adaptation practices to ‖individual learning progress‖. Based on these experiences, it took another year before a <conceptual orientation> really started to guide their work instead of a purely <procedural orientation>.

Interestingly, it entered their *collective domain* via the *domain of inquiry*, when experimenting with the curriculum materials for all students and experiencing "lovely aha moments, when students say 'now, I really got it!'" In this way, the teachers' pathways of long-term collective professional growth reflected an interesting interplay among the four domains, with a growth pathway that was not at all linear.

Tentative Content-Related Theorising on PD2-Growth and PD4-Environment
As these considerations illustrate, the adapted model for PDCG can serve as a theoretical framework for explaining teachers' professional growth (PD2-growth) and for offering external sources to the PDCG environment to promote the professional growth (PD4-environment). Looking back on the vignette and its illustrative analysis, the term *generic search space* can be further unfolded. The general model only provides the framework for necessary content-related theorising. The analysis of this vignette and many further cases with the same PD content (e.g. Prediger & Buró, 2021, 2024) resulted in the first tentative theorising about teacher communities' content-related growth pathways towards striving for differentiating practices

(PD2-growth) and the roles of external resources such as classroom material in supporting the process (PD4-environments).

When communities of inquiry work on innovative practices (in this case, on their differentiating practices), their evaluation categories in the domain of consequence might be the most crucial to develop as it has important impact of all practices. As an interview study on self-reported practices revealed (Prediger & Buró, 2021), teachers' leading evaluation categories influence their monitoring practices because they have an impact on what teachers intend to notice in students' work. For fostering students' learning in differentiated ways, the leading evaluation category influences whether teachers adopt compensation practices (while aiming at circumventing weaknesses by supporting students to work around limited abilities in the leading category ||task completion||) or enhancement practices (starting with setting differentiated learning goals along the content trajectory and guided by the category of adaptive ||learning progress|| along the content trajectory). The difference between compensation and enhancement practices is also visible in a video observation study on classroom practices (Prediger & Buró, 2024).

As the leading evaluation categories are revealed to be so crucial across various case studies from PD research projects, they are therefore an important focus for theorising. In the project, we hypothesise that potential growth trajectories of teacher communities' learning of the PD content 'fostering at-risk students' access to mathematics' might be characterised as a successive extension of evaluation categories in four stages (see Fig. 6.7). Teacher communities often start with evaluating only ||work intensity|| (no matter on what), but quickly turn to ||task completion|| to incorporate a first mathematics education perspective. ||Task completion|| stays an important category for the evaluation in <short-term orientation>, but should be complemented in <long-term orientation> by ||learning progress||. Often, teacher communities first focus on the evaluation category ||learning progress|| only in <procedural orientation>, which can later also be extended to including both || procedural and conceptual learning progress|| when the <conceptual orientation> can be established.

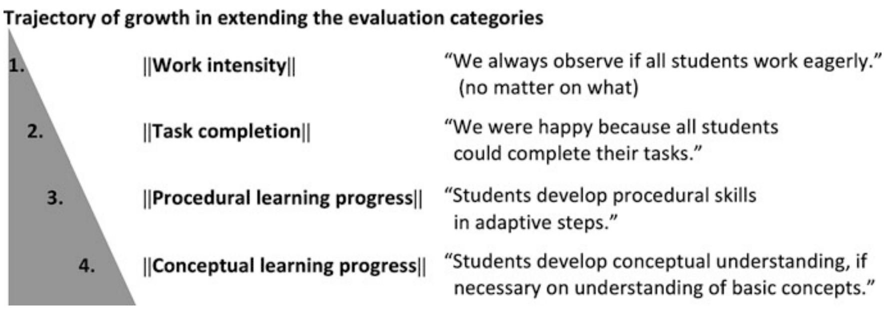

Fig. 6.7 Content-specific theory element for PD2-growth—hypothesis on trajectory of growth by successively extending the repertoire of evaluation categories for fostering practices

Hence, initiating shifts in the evaluation categories might be the most crucial external input required to allow teacher communities to continue their independent inquiries. However, orientations and evaluation categories cannot simply be 'taught', so facilitators need to find entry points for their successive extension. Offering curriculum materials for monitoring (formative assessment) and enhancement sessions may not only provide a pedagogical tool for developing teaching practices, but might also be a key external source for indirectly influencing the collective domain when the teacher group starts to adapt and appropriate the curriculum materials for their purposes. In our situation, the curriculum materials seem to have strengthened the shared pedagogical content focus on further understanding of basic concepts in different mathematical content (see Fig. 6.6). Although both hypotheses (the hypothesised trajectory of growth and the hypothesis on the effects of the curriculum material) emerged from the qualitative analysis of various case studies, they will require further systematic investigation before they can count as stable empirically grounded theory elements.

Zooming out from the specific PD content 'fostering at-risk students' access to mathematics' to a more generic perspective, three main lessons learned might be derived for PDCG in general, with respect to Putnam's and Borko's (2000) three areas of consideration: (1) where to situate teachers' learning experiences; (2) the nature of discourse communities for teacher learning; (3) the importance of tools.

1. It is worth situating teachers' learning experiences in communities of inquiry with emphasis on the content-specific domain of inquiry.
2. The nature of the discourse in PDCG is heavily influenced by shared orientations and categories, specifically by evaluation categories. Extending these categories in the domain of consequence seems to be an important step in the trajectory of the community's growth, to be taken into account by the facilitator.
3. Classroom materials are relevant tools that can support the teachers' monitoring and enhancement practices and, at the same time, serve as tools to extend the communities' shared orientations and categories specific for the mathematical content in view.

6.6 Meta-theoretical Reflections on the Necessary Topic-Specific Theory Elements

The intent of this chapter is to contribute to developing theoretical foundations for explaining and promoting teachers' professional growth in collaborative groups. Building upon the general sociocultural framework on teachers' practices (Wenger, 1998) and the construct of communities of inquiry (Jaworski, 2006), as well as an adapted model of professional growth (adapted from Clarke & Hollingsworth, 2002), it used the exemplification in one vignette in the community of inquiry to show the following.

- Theory elements of PD content (PD1), teacher growth (PD2), facilitating (PD3) and PD environment (PD4) are all useful and necessary for explaining and promoting professional growth in collaborative teacher groups. Of course, other vertices and edges of the tetrahedron also serve theorising, but these four may be considered a minimal set.
- The PD content comprises the classroom mathematics content (in our case, multi-digit subtraction, for which a substantial research background exists), but also the specific teaching practices that the communities of inquiry have chosen to work on—in our case, differentiating and fostering at-risk students' access to mathematics. Even if an area of PD content is usually not sharply defined in communities of inquiry but successively emerges, unpacking it with respect to the underlying orientations and categories that the teacher community implicitly or explicitly refers to is crucial.
- Although PD practices are always content-specific, research papers and particularly theorising processes tend to abstract from these contents; however, we should talk more about content-related theorising.
- The structures of the big theoretical frameworks (communities of inquiry and models of professional growth) are helpful in understanding the complexities and intertwinement of different domains. However, they mainly provide a *generic search space*. Informing the concrete analysis and especially the concrete PD design and facilitation, they must be elaborated in content-related ways for different areas of PD content (Prediger et al., 2019b). Of course, the large body of mathematics education research on the mathematical content and on mathematics teaching and learning has revealed many candidates for theory elements, most importantly for C1–C3, but also for C4.
- The generic theory elements from the PD level gain their explanative power when being filled in content-related ways, and this also requires the nesting of corresponding theory elements (C1–C4) from the classroom level into the PD level. The more this nested structure is unpacked, the more we learn in content-related ways about the PD content.

Finally, let me briefly respond to two questions. Is the meta-theoretical reflection here (a) specific to mathematics and (b) specific to PDCG, or could it have been suggested with respect to any form of PD for any subject matter?

(a) Whereas other colleagues have emphasised the need to be specific to mathematics, I emphasise that we need even further substantiation in *content-specific* ways. At the same time, the model itself is not mathematics specific and might be applied also for other subjects, with their particular content.
(b) Whereas the above meta-theoretical arguments are not specific to the form of PD in view here (PDCG), the main result of the current case study might be characteristic for the collaborative setting and relevant for supporting collaborative groups. The shared evaluation categories seem to be the crucial point, more than the shared knowledge or orientations as a whole. As long as individual evaluation categories have not really entered the collective domain (as Maria's || forgetting||), they cannot exert their influence, and this can also hinder the

professional growth of the individual within the collaborative group. This interplay of individual and collective learning in particular will require substantial further empirically grounded theorising.

However, both claims of potential specificity or generalisability will require further PD research, in order to be explored in depth by comparing across content, across subjects and across particular PD settings.

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Chapter 7
The Art of Being Specific While Theorising for and from Practice of Mathematics Teachers' Collaboration

Boris Koichu

7.1 Introduction

Theorising is one of the most endorsed practices in contemporary mathematics education research. At the same time, such questions as "what counts for useful theorising?" and "how can theory inform practice?" are still subjects of debate. Arguably, this is in part because of the theoretical diversity characterising our field (e.g. Sriraman & English, 2010), and in part because of the diversity of approaches to handling the research-practice relationship (e.g. Schoenfeld, 2020).

In her plenary talk at the IMCI Study 25 conference, and in the corresponding chapter of this book (Chap. 6), Susanne Prediger makes an important contribution to the debate, by substantiating the following ideas:

> Although PD practices are always content-specific, research papers and particularly theorising processes tend to abstract from these contents; however, we should talk more about content-related theorising. [. . .] Informing the concrete analysis and especially the concrete PD design and facilitation, they [theoretical frameworks] must be elaborated in content-related ways for different PD content. (p. 18)

Prediger uses strong modal words 'should' and 'must' in the above sentences, while mainly reflecting on her own line of research. However, her central suggestion—more content-specific theorising is needed—is far from being obvious, at least in light of the theoretical landscape reflected in contributions to Theme A of the conference (Chap. 2, this volume). Playing for a moment the role of devil's advocate, we can imagine a colleague proposing the following line of questions. If the existing, and somewhat overwhelming, diversity of large, intermediate-level and local theories is still not enough in order to provide us with insights for explaining and

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facilitating mathematics teacher collaboration, perhaps we need less theorising? After all, many decisions that researchers, PD facilitators and teachers make are experience-based.

A possible alternative to 'classic' theorising might be what Mason (2002) introduced as a call to produce brief-and-vivid recollections of our observations in ways that would trigger others' own recollections. These recollections, if properly disseminated, can inform and influence practice (Begg et al., 2003). And if we do need more theorising—for instance, for enhancing our explanatory, design and predictive powers or as a safeguard against policy based on over-generalised anecdotal evidence (Niss, 2007)—perhaps we should move towards the development of unifying meta-theory, as Steen (1999) and Dahl (2010) have reasoned, rather than towards elaboration of the existing theories by developing content-specific elements, as Prediger suggests?

A winning argument is rare in our field, and a counter-argument to our devil's advocate argument can readily be constructed. To this end, a structural analogy between research on mathematics teacher collaboration and research on mathematical problem solving can be instrumental. (After all, Lampert, 2001, considered teaching as a kind of problem solving.) Let us recall that, under the great influence of Pólya's (1945/1973) work, mathematics education research protractedly experimented with an idea to teach *general* problem-solving heuristic strategies to teachers and students. When this idea was found not to be as useful as hoped (Schoenfeld, 1992), the idea to unpack general heuristics into *content-specific* ones and promote those via appropriate practices was put forward (Schoenfeld, 1985, 1992). By analogy, unpacking teacher collaboration by closely attending to its specific content can usefully complement general characterisations of collaborations in terms offered by big theories.

In addition, the proposal for developing content-specific theoretical elements is well-aligned with Lester's (2005) suggestion that mathematics education researchers should be equipped with various theoretical tools in the pursuit of solutions to problems at hand. This sort of argument, however, is also disputable. For instance, Lester uses the metaphor of a researcher as a *bricoleur*, which presumes certain theoretical eclecticism, whereas Prediger strives for theoretical coherence when talking about developing and *connecting* content-specific theory elements as a worthy goal for current and future research. To me, though the vignette used in Prediger's chapter for illustrative purposes is truly convincing, it still needs to be elaborated if and how content-specific theorising can be undertaken coherently in different contexts pertinent to different situations of mathematics teacher collaboration.

In the rest of this commentary, I focus on Prediger's suggestion that the proposed line of content-specific theorising can provide theoretical foundations for "typical facilitation practices and designs" (p. 10). I examine this suggestion by first attempting to characterise Prediger's research strategy that affords and includes content-specific theorising. I then engage with some of theoretical elements suggested by Prediger, and check their applicability by attempting to use them to analyse a situation of mathematics teachers' collaboration that is very different from

Prediger's example, both in its content and in its context. This commentary is concluded with remarks on connecting between content-specificity and content-generality in theorising mathematics teachers' collaboration for the sake of further accumulation of knowledge and supporting practice. In sum, I address in this commentary the following three questions:

> What characterises the research strategy presented by Prediger that affords and includes content-specific theorising?
>
> How can the suggested content-specific theoretical elements be applied to a situation of teacher collaboration that is different in its mathematical, epistemological and PD content?
>
> How can content-specificity and generality of theorising be coherently connected in future research on teacher collaboration and beyond?

7.2 Research Strategy that Affords and Includes Content-Specific Theorising

7.2.1 About Content-Specificity and Design Research Practices

In Prediger's argument, *content* is a multi-faceted term. As the tetrahedron model proposes (see Fig. 3 in Prediger's chapter), it includes mathematical, epistemological and PD aspects, and attends to students' and teachers' past, current and desirable knowledge and orientations. In the illustrative vignette, mathematical content refers to subtraction algorithms and to underlying mathematical principles (e.g. 'place value'). Epistemological content consists of teachers' knowledge about obstacles for performing the algorithm in the case of subtraction with carries.

The PD content is related to the teachers' apparent lack of knowledge on how to teach subtraction with carries, so that the students would master (and remember) the algorithm in all cases. In addition, PD content relates to the teachers' inferred belief that short-term success of all students is of primary importance, whereas their long-term learning progress is of secondary importance. Admittedly, all these contents are specific in different meanings: from a specific mathematical topic to a specific teacher belief and knowledge for teaching.

Prediger's further analysis of 'where to go from there' puts forward the purpose for the teachers to recognise the primary importance of the students' long-term learning progress, which should become at least as important as the teachers' devotion to the students' short-term success associated with task completion. This purpose is unpacked into suggestions as to what PD facilitators may work on with the teachers, so that the teachers would work differently with their students in the future. The central suggestion is gradually to help the teachers appreciate the distinction between procedural skills and conceptual understanding, and to equip them with pedagogical tools for promoting conceptual understanding in the context of the subtraction algorithm (i.e. place-value understanding).

This suggestion connects specific contents of the vignette to a big theory of learning via conceptual understanding, among other big theories used in theoretical elements at the levels of teaching and facilitating. Indeed, the above suggestion is essentially informed by research on learning and teaching with understanding (Hiebert & Carpenter, 1992) and on conceptual understanding in the realms of arithmetic operations (Hiebert & Wearne, 1996). In this way, *content* is not only multi-faceted in Prediger's argument, but also theoretically-laden.

In addition, Prediger explicitly adheres to practices of theorising that originate in the Design Research perspective (see also Prediger, 2019). In her words, this perspective includes, "iterativity and interactivity between theory-generating and theory-guided experimenting" (Chap. 6, this volume, p. 284). This description is compatible with cross-cutting features of design experimentation. According to Cobb et al. (2003), the purpose of design experimentation is "to develop a class of theories about both the process of learning and the means that are designed to support that learning" (pp. 9–10). The second feature is the use of the interventionist methodology; the third is adhering to prospective and reflective aspects of data analysis; the fourth is iterative design that features cycles of invention and revision; the fifth is accountability of theories developed by means of design experimentation to the activity of design presuming that, "the theory must do real work" (p. 10).

All five cross-cutting features of design experimentation seem to be present in Prediger's extended example and argument. Furthermore, the very focus on functions of the theoretical elements suggested by Prediger (in her words, the functions are *specifying*, *structuring*, *explaining*, *designing* and *enacting*) is well-aligned with the principles of design research. Therefore, it seems to be reasonable that her argument in support of content-specific theorising should be considered first of all in relation to problematics within the reach of design research tradition and methodology.

7.2.2 About Context-Specificity

The illustration is not only content-specific but also context-dependent, and thus it is indicative of additional interesting elements of the chosen research strategy. Without attempting to re-analyse the vignette, I would like briefly to stop on its two specific features which are not explicitly reflected upon in Prediger's analysis, but presented as contextual information. First, in contrast to many PDs initiated by university researchers, the teacher collaboration under consideration began as a local initiative having an agenda formulated by teachers (i.e. to afford at-risk students better access to mathematics). The researcher facilitator came to assist and enhance a collaboration, which had already been appreciated as viable by the teacher-participants, rather than being imposed on them 'from above'. Accordingly, the teachers were from the outset in a position to act as fully fledged stakeholders in the university–school partnership (Krainer, 2014).

Second, the vignette represents an episode that occurred at the beginning of an 18-month university–school partnership. Such a long collaboration period provided the researcher facilitators with an opportunity to manifest a great deal of patience and sensitivity while striving, along with the teachers, for sustainable change in teaching practices. It is notable in the vignette that the researcher refrained from immediate actions, but preferred to study the situation carefully. In Prediger's words, "the researcher facilitator has to obtain a profound understanding of the collaborative group's practices and challenges before changing the role from observer/analyser to facilitator of the PDCG and before supporting the group's professional growth" (Chap. 6, this volume, p. 287).

This intention conforms with Swan's (2011) notion that working with teachers should rely on recognising their current values, beliefs and practices. (Notably, this idea is well-aligned with the above-mentioned principles of design research.) Also, this researcher facilitator intention is indicative of *not* falling into the trap of a deficient discourse on teachers, as frequently happens when the researchers strive for educational change while perceiving the teachers as 'lacking something' (Adler & Sfard, 2016). It is worthy of attention that Prediger first decided to attend to what the teachers were proud of (adaptation to all students' abilities) and to what was disturbing for them (i.e. 'the students forget . . .'), rather than deciding to reveal quickly to the teachers what was disturbing for her (i.e. the teachers' overly strong commitment to students' short-term success associated with task completion). These decisions, which perhaps were natural for Prediger as a researcher facilitator having a design-research orientation, are far from obvious for many of us.

7.2.3 Summary of Prediger's Research Strategy

In this sub-section, the above comments are summarised by means of a list of characteristic features of a research strategy that affords and includes content-specific theorising of teacher collaboration, as it is reflected, at least for me, in Prediger's example and argument (but see Prediger, 2019, for the first-handily produced list of theorising steps in the context of design research).

- Adhere to theorising practices of design research as an overarching perspective.
- Recognise teachers as fully-fledged stakeholders in school–university partnership, respect their current achievements, beliefs, aspirations and agenda for professional growth.
- Observe existing practices patiently and interpret them without falling into the trap of deficiency-discourse on teachers. Produce accounts and pause on events that are disturbing for the teachers or for yourself.
- Do not act upon what disturbs you too quickly. Begin from what disturbs the teachers and introduce additional goals gradually, in order to nurture a school–university partnership towards sustainable educational change in the long run.

- First analyse the accounts simply while trying to make sense of what is going on and why. Then analyse them more systematically, by means of hierarchically organised theoretical elements attending to different but interrelated types of content.
- While analysing cumulative data in depth, use structural analogies, in order to connect phenomena at the levels of students, teachers and facilitators.
- Be functional while theorising at different levels: specify, structure, explain, design activities, support enactment.
- Be explicit about and continuously reflect upon your own theoretical premises, preferences, beliefs and results of theorising.
- Look for the general in the particular, but do not over-theorise.

7.3 Same Theoretical Elements: Different Situation

In this section, I engage with some of the theoretical elements suggested by Prediger in the context of an episode from the project *Raising the Bar in Mathematics Classrooms* (RBMC; Cooper & Koichu, 2021). This project is very different from Prediger's *Mastering Math* project in its key characteristics. The project's goal, which was predefined by a research university and a philanthropic foundation, is to motivate and support middle-school teachers of so-called 'excellence' classes in Israel to incorporate problem solving systematically in classrooms while focusing on developing students' autonomy and high-level mathematical competences (OECD, 2018). The teachers are recruited to the project's communities and to collaborate towards creating and enacting opportunities, in order to challenge their students with increasingly demanding problem-solving activities at an increasing frequency. Thirty teachers were recruited in 2021; the project seeks to reach two hundred teachers over 4 years.

RBMC develops around the core community comprising six experienced mathematic teachers, three teacher facilitators and two researchers. One of the roles of the core community is to design problems and ways of discussing them for use in general communities and for subsequent enactment in the teacher participants' classrooms. The episode in question occurred in the core community 3 months after RBMC had been launched.

7.3.1 An Episode

In preparation for one of the core-community meetings, the teachers were asked to solve several candidate problems, including the Elevator Problem in Fig. 7.1.

Three teachers (Maria, Peter and Rachel) and a researcher (Baruch) discussed this problem and possible scenarios of its use with students during 40 min in a Zoom breakout room. The discussion began from considering the teachers' own solutions

Epsilon Tower in Zedland is famous among tourists for its wonderful view from the top floor. During the pandemic, it is permitted to use the elevators ONLY on condition that every visitor wears a protective mask AND keeps the distance of at least $1m$ from the other visitors. Each elevator in Epsilon Tower is designed as a parallelepiped having a square floor with a side of $1.4m$

a. How many visitors may simultaneously use one elevator?
b. Prove that the number you found in (a) is the maximum number, that is, it is impossible to place any more visitors in the elevator without violating the rules.

Fig. 7.1 The Elevator Problem. (Adapted from Andžāns & Johannesson, 2005)

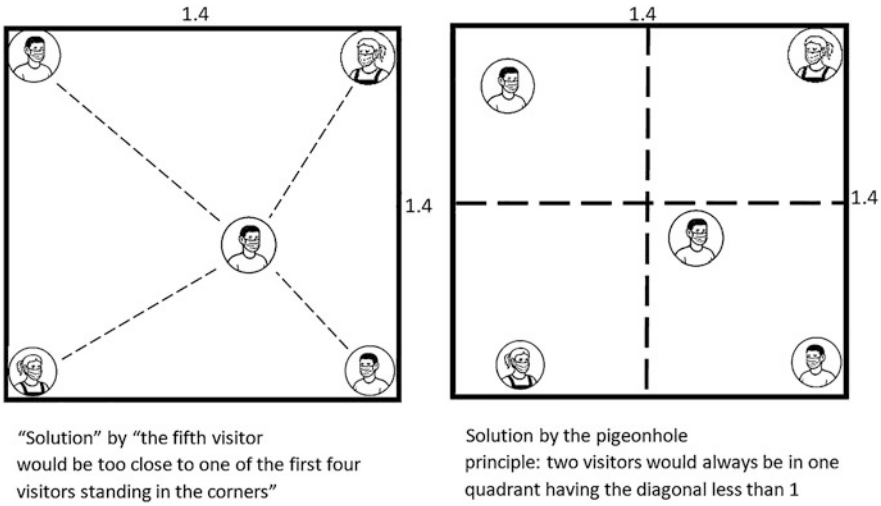

Fig. 7.2 Approaches to solving question (**a**) Synopsis of Maria's solution (**b**) Synopsis of Peter's solution

to the problem. They confidently answered question (a)—the maximum number of visitors is four—and vividly discussed question (b) (see Fig. 7.1). Maria shared with the group that she approached the problem "wearing students' shoes" and suggested a 'solution' presented in Fig. 7.2a. Peter observed that this 'solution' is constrained by an assumption that four visitors are placed in the corners of the square whereas there are endless placement options. He then presented his solution based on the pigeonhole principle (also tagged as the Dirichlet Principle, e.g. Andžāns & Johannesson, 2005), a synopsis of which is presented in Fig. 7.2b.

Rachel approached the problem by placing the visitors in the corners and drawing areas forbidden for the other visitors (i.e. circles of radius 1). Both Maria and Rachel endorsed and appreciated Peter's solution. Then the following dialogue took place.

Peter Actually, this is one of those questions that when you know the right method, it takes 2 min to solve. I have encountered 10–15 problems like this [the Elevator Problem] in the past, so I just saw that the pigeonhole principle would work [...] Therefore, it was quite a technical question for me, like an exercise.

Maria And if you were not familiar with the pigeonhole principle, how would you approach the problem, say, as a ninth-grade student?

Peter Apparently, I'd do what you did [referring to the 'solution' in Fig. 7.2a] [. . .] And then I'd begin moving the first four [visitors] from the corners somehow.

Maria So, this would not be a technical question.

Peter Not at all! I think that it would be an impossible question [. . .] This is why I don't know what to do with it [in a classroom . . .] Question (a) is OK, it would be some trial-and-error, but question (b) would be a real obstacle.

Maria Baruch, do you have another solution to (b), without the pigeonhole principle?

Baruch Actually, I don't. We posed this problem having the pigeonhole principle in mind. However, I think that the pigeonhole principle is just a nice name, but the logic of the principle is quite intuitive.

Peter [nods in agreement]

Baruch Peter, I understand that it is difficult for you to depart from your knowledge of the pigeonhole principle, but how did you solve the first problem of this type? Had somebody just told you this principle? Or did you succeed to somehow discover the idea behind it and later somebody just told you the right name?

Peter I don't remember [. . .] It was many years ago. [. . .]

Baruch Let's assume that the intended solution [to the Elevator Problem] is by the pigeonhole principle, and that the students don't know it. So, with all our pedagogical wits, how can we help students without telling them that there is some mathematical principle that solves the problem? . . .

Maria I don't know [. . .] Maybe, we can decompose the problem? To show them a square divided into small squares as a hint? I am not sure where they would take it.

Baruch I think that a serious obstacle with this problem for students is: how can one prove that something is impossible?

Peter [nods enthusiastically in agreement] Yes! Yes! That's it! If I'd have an opportunity to teach this [. . .] Now I really wish to use this problem in my classroom! First, I'd hear their solutions, and I would somehow convince them that they are wrong [. . .] Then I'd focus on the most problematic aspect of this problem, which indeed is how to prove that something is impossible. They know how to prove [that something is possible], but they don't know how to prove that something is impossible. Are there such things in the curriculum?

Maria Well, there are proofs that begin from 'assume that' and arrive at a contradiction.

Peter Proofs by contradiction!

The discussion continued, and eventually the teachers developed a lesson plan consisting of three stages that would circumvent the need to reveal the pigeonhole principle. The first stage consisted of a guideline for how to help the students to appreciate the difficulty of the problem. Here, Rachel's idea about "forbidden circles" was recalled and implemented by means of a *GeoGebra* applet for exploring the problem. The next stage—in Peter's words, "a meta-mathematical stage"—was devoted to proofs by contradiction in general. For the last stage, the teachers composed specific hints that were intended gradually to lead the students to construct a contradiction for the Elevator Problem by partitioning the square (see Fig. 7.2b). After the meeting, all three teachers successfully implemented the problem with their students, though the actual lessons deviated essentially from the plan. However, this is another story.

7.3.2 *Analysis by Means of Prediger's Content-Specific Theoretical Elements*

In the spirit of Prediger's ideas, the goal of the forthcoming analysis is to explain the above-episode in terms that would inform the PD design towards enactment of challenging problems in a classroom. A simple account might be that the change in teachers' attitude occurred due to the intervention of the researcher aimed at encouraging the teachers to move beyond the telling/not-telling dilemma. Let us now see what the (inevitably concise within the space constraints of this commentary) use of Prediger's theoretical elements may bring.

7.3.2.1 C1 (Classroom Content)

The episode does not contain actual student data (cf. Suleika, in Prediger's example). However, the teachers plausibly suggested that their students would approach the problem by focusing on its particular case (visitors in the corners), while being unaware that there are additional cases to consider. This suggestion can be theoretically backed, for example, by Buchbinder and Zaslavsky's (2019) framework for characterising logical structure of mathematical statements and, in particular, by their research on the role of examples in students' proving. In addition, the content at C1 level includes the pigeonhole principle and the associated question of how the problem can be solved without explicitly referring to the principle. Of note is that the teachers did not initially attend to mathematical ideas behind the principle.

7.3.2.2 C2 (Learning/Problem Solving)

The mechanisms underlying the students' hypothetical response to the Elevator Problem may be unfolded in terms of an educational perspective on intuition

(Fischbein, 1987), by conception of a *problem situation image* evoking in students' tentative solution starts (Selden et al., 2000), or by a discursively-oriented conceptualisation of problem solving that puts forward the interplay of students' existing and emerging discursive resources (Koichu, 2019). Of note is that elaborated theorising in terms of these perspectives would lead us to consideration of the relationships between mathematical knowledge and competences needed to solve the problem and the actual students' knowledge and competences as developed in their past problem-solving experiences. These relationships might have been presented in a way structurally similar to the mention of learning trajectory (Prediger, Chap. 6, this volume, p. 290).

7.3.2.3 C3 (Teaching)

Given that all the teachers in the core community of RBMC had strong positive orientation towards problem solving, how can their initial difficulty with didactical handing of the Elevator Problem be explained? Two possibilities come to mind: the teachers' close attention to their own experiences with the problem ('technical' versus 'impossible') and the perceived gap between the knowledge needed to solve the problem and their students' actual knowledge. In alignment with Prediger's analysis of C3, the teachers' initial perception of the problem was not based on a deep analysis of its mathematical content. We can also borrow from Prediger's analysis a suggestion that a successful scenario of using the problem in a classroom for all three teachers was initially associated with completion of the task. An alternative that emerged later was to use the Elevator Problem as an opportunity to discuss important mathematical ideas (see "meta-mathematical stage" offered by Peter). Theoretically speaking, these suggestions bring us to a classic distinction between teaching for problem solving versus teaching via problem solving (Schroeder & Lester, 1989) as a big theory (paralleling teaching for conceptual understanding in Prediger's case).

7.3.2.4 C4 (Environment)

Though it is not explicitly evident in the presented excerpt, classroom environments promoted in the RBMC project are designed bearing in mind the following principles: (1) the problem-solving environments should be emotionally safe for students, and in particular detached from formal evaluation; (2) students can fruitfully engage with atypically challenging mathematics when feeling praised for effort and for sharing incomplete solution ideas; (3) student discourse and dialogue are not only means but also a goal of problem-solving activity (see Goldin, 2009; Koichu, 2017; Schwarz & Baker, 2016; for theoretical justification of principles 1–3, respectively).

7.3.2.5 PD1 (Content) and PD2 (Growth)

As mentioned, the PD content of RBMC is oriented towards motivating and supporting teachers to incorporate challenging problems in their regular teaching. As follows from the preceding analysis, this process requires a gradual change from valuing teaching *for* problem solving to valuing teaching *via* problem solving (cf. a desirable shift from *task-completion orientation* to *learning-progression orientation* in Prediger's analysis). Furthermore, in RBMC, we adopt a perspective on educational change worded by Swan (2011) as follows: "We do not seek to change teachers' beliefs so that they behave differently, but rather offer opportunities to behave differently so that their experiences may give them cause to reflect on and modify their beliefs" (p. 57).

In the above episode, this perspective is manifested by effort invested in noticing deep mathematics involved in the problem (i.e. existential versus universal statements), explicit formulation of learning opportunities for the students (i.e. deciphering the pigeonhole principle as a case of proof by contradiction without necessarily telling the principle) and in creating an imaginary scenario of a successful lesson, which begins from didactical handling of the students' anticipated responses (i.e. a *GeoGebra* applet for 'forbidden areas').

7.3.2.6 PD3 (Facilitation) and PD4 (Environment)

In Prediger's example, the teachers were those who formulated the content and agenda for collaboration, and the researcher facilitator acted as a silent observer in preparation for the future facilitation. In RBMC, the content and agenda of teacher collaboration were offered by the project team and thus can be seen as (arguably) fragile. In addition, the researcher (Baruch) intervened in the teacher discussion in the episode in question. Despite these differences, teacher collaboration is vivid in both examples. To this end, we can observe that Baruch first listened carefully to the teachers' discussion and entered the conversation only when asked by one of the teachers (Maria). Furthermore, his first intervention was shaped as a question to another teacher (Peter) aimed to deepen his reflection on his past experience.

The second intervention (about the main difficulty of the problem) was prepared by Maria's previous assertions about "the students' shoes". Similar facilitating behaviour can be seen in many additional episodes of the project. Therefore, I believe that I am in position to argue that PD3 and PD4 in RBMC can be analysed by means of the same framework that Prediger offers for analysing PD3 and PD4 in her *Mastering Math* project. Namely, the focus on the interplay of external domain, collective domain, inquiry domain and the domain of consequences (Figure 7.1, p. 307) seems to be relevant also for RBMC.

To conclude this section with a personal note, I would like to acknowledge that writing the above two pages deepened my understanding of Prediger's ideas, as well as my understanding of the presented RBMC episode. I have now clearer ideas about how further to run RBMC and to explore additional episodes while preparing myself for future theorising the content-specific phenomena identified. Therefore, thank you, Susanne!

7.4 Remarks on Connecting Content-Specific and General Modes of Theorising

Prediger argues for content-specific theorising as a way to enrich and complement insights that may stem from the use of big theories. Indeed, several big theories are referred to in the theoretical elements of unfolding student learning, teaching and professional growth. So, in what way can content-specific and general modes of theorising be coherently connected? To contemplate this question, let me revert to a thought experiment.

One of the big theories used in the analysis of Prediger's vignette is a theory of learning via conceptual understanding. Let me now imagine that some other research group might have considered alternative theoretical approaches for dealing with the same content. For example, some would prefer to focus on the development of appropriate discursive routines where the ritual performance precedes explorative participation (Nachlieli & Tabach, 2019), as informed by the commognitive theory (Sfard, 2008). Some would consider helping teachers design sequences of exercises highlighting similarities and differences between different cases of the use of the same algorithm (Watson & Mason, 2006), as informed by the variation theory of learning (Marton & Booth, 1997). And some might choose to work on gradual change of classroom norms and patterns of students' engagement (Goldin et al., 2011), as informed by socio-cultural theories relying on Vygotsky's legacy. However, in all these hypothetical cases, which would apparently lead to different explanatory and PD-facilitation decisions, the researchers might benefit from the multi-level and multi-faceted, functionally oriented scheme of analysis in terms of C1–PD4. (At least, this worked for me in the context of an episode from another project.)

Therefore, I suggest that a functionally oriented scheme of analysis, which is informed by design research as an overarching perspective, is one feature that makes Prediger's ideas transferable across different mathematical, epistemological and PD contents, though, as mentioned, specific PD-facilitation decisions may depend on the big theory being used. Another feature is the explicit bottom-up approach that begins with in-depth engagement with mathematical content, and gradually unfolds by theorising questions on learning, teaching and PD-facilitating.

Aligned with a design research paradigm, connection between the content-specific and the general is achieved by considering particular content as a case of something more general (e.g. a specific teaching decision for Suleika is seen as an instantiation of a specific teacher belief, which, in turn, is seen as a basis for long-term facilitation in a specific direction). The scopes and time-lines of the observed phenomena are different, but theorising remains content-specific. In turn, general characteristics of teacher collaboration (e.g. a repeatedly made observation that the collaborating teachers formed a community of inquiry) are mentioned as conditions that can enhance or hinder the work towards *specific* changes in practice.

To summarise, for me it is Prediger's decision to focus on *functions* of theoretical elements that connects the specific and the general. As to coherence, of special note is a systematic implementation of the proposed functionally oriented scheme of analysis for contemplating and acting upon phenomena at the levels of student learning, teaching and PD-facilitating. This is without declaring the use of one big theory as an umbrella, which relates nicely to Prediger's past work on networking theories (e.g. Prediger et al. 2008). Elegantly done!

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Chapter 8
Capturing Collaboration in Mathematics Teacher Education, in Terms of Relevant Actors, Targets and Environments

Konrad Krainer, Bettina Roesken-Winter, and Carina Spreitzer

8.1 Introduction

The discussion document *Teachers of mathematics working and learning in collaborative groups* (IPC, 2019) identifies "Roles, identities and interactions of various participants in mathematics teacher education" as one of four major themes needing further elaboration. In regard to this Theme C, the document indicates, among others, that collaborative groups can include different 'actors' in various combinations. These actors can have a variety of roles, which can shift over time. In collaborative interactions, the learning of all participants is also important. The document indicates six 'actors' in an exemplary way.

It is easy to increase this list by including, besides mathematics teachers themselves, other individuals such as teachers of other subjects, lead teachers or teacher leaders, department heads, principals, parents, teacher students, students, critical friends, facilitators, coaches, mentors, mediators, designers, multipliers, mathematics teacher educators, mathematicians, researchers, administrators, superintendents or policy makers—and even this extensive list is not exhaustive at all. In addition, also organisational entities like departments, schools, school boards, districts, committees, ministries or enterprises can be environments relevant to a collaboration: for example, by influencing the goals, processes and results of collaborative activities which take place in projects, programs, teams, communities of practice, networks,

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study groups, etc. (see, for example, Krainer, 2008). The word *initiative* sums up different forms of collaborations and is used in this chapter as an umbrella term.

This diversity makes it difficult not only to get an insightful overview, but also to compare initiatives or to grasp their specificity. In preparation for this contribution, we decided to look for selected articles and to analyse them along the following three dimensions:

- the **relevant actors** of the collaboration (e.g. teachers, teacher educators, researchers, etc.);
- the **relevant targets** (e.g. aims, goals, etc.) of the collaboration (e.g. improving the knowledge or beliefs of students and/or teachers in geometry, writing a new algebra curriculum, establishing or further developing a school's emphasis on mathematics teaching, etc.);
- the **relevant environments** of the collaboration (e.g. a school or a district, a mathematical association, a university, policy makers, curriculum makers, etc.).

Since we focus on **R**elevant **A**ctors, **T**argets and **E**nvironments, we call our tool RATE. In order to facilitate potential users' grasping of essential features of each initiative, we visualise selected exemplary initiatives (see the next section, "The selection of articles") in a diagram (triangle), indicating the number of actors. We then highlight the key intervention of each initiative, its type and duration, and the specificity of the collaboration. Additionally, we provide selected findings on different forms of teacher collaboration presented in the selected articles.

In system theory (Willke, 1999, p. 12; see also Krainer, 2005), *noticing a relevant difference* (observation) and *producing a relevant difference* (intervention) are important terms. In teacher education, one kind of difference is of particular interest, namely the *status quo* of teaching, which often is regarded as unsatisfied, versus the desired target situation ("good" or "high quality" teaching, etc.), marking a possible need for improvement. For example, if nobody (teachers themselves, researchers, educational policy makers, etc.) would see a need to produce a difference (improving teaching), then reforms, professional development courses, etc. would rarely be initiated. However, it is interesting to ask: what is the relevant difference that should be produced? Who defines this? By what means should the relevant difference be achieved? Who are the actors? Who is learning, changing, implementing, improving, etc.? Who supports, who documents, who evaluates, who takes decisions, who controls, etc.? Is collaboration between teachers fostered, are knowledgeable others involved and, if so, in which roles? Such reflections are used to work out the *specificity of collaboration* in these initiatives and the type of intervention.

The **RATE diagram** has the form of a triangle with the corners *Teachers*, *Knowledgeable others* and *Relevant environments*. These three corners stand for social entities relevant for the respective initiative.

1. **Teachers**

   This corner represents all participants of the initiative who will become, are or have been members of the teaching profession at a school or a kindergarten (prospective teachers, teachers and retired teachers), aiming at improving teaching (e.g. through planning, implementing, observing, evaluating and investigating teaching, developing curricula or material, etc.). For example, all teachers taking part in an Action Research project or in a Lesson Study group at one school are 'teachers', including the department head or lead teacher. Also, a teacher who serves as a teacher educator, multiplier or researcher in another initiative, defining another context, but is a common member in the initiative at hand, is regarded as a 'teacher' in this initiative.

2. **Knowledgeable others**

   These participants of the initiative, in most cases, come from outside the teaching profession at a school or a kindergarten, for example, university staff, teacher educators, school developers, researchers and people from school administration or economy. In a few cases, knowledgeable others might be teachers, eventually part-time, if they have gained expertise beyond teaching, for example, in providing teacher education, counselling schools and doing research, and in case they bring in this special expertise in this initiative going beyond their teaching expertise.

3. **Relevant environments**

   These social entities, such as individuals, groups, institutions, etc., are not participants in the initiative, but they have a relevant direct or indirect influence on the initiative. This could be a principal supporting a professional development course and/or the whole school, a superintendent and/or a whole school district, a mathematician and/or a whole mathematical association, a researcher and/or a whole research group, a policy maker or a whole ministry, curriculum makers, etc.

The heading of the diagram indicates the intervention focus and the context of the initiative (in parentheses and in bold), while (uni- or bi-directional) arrows indicate interconnections between the corners, eventually characterised by specific wording. The labelling of the arrows highlights some of the specificities of the collaboration, such as teacher educators bringing in theoretical frameworks for helping teachers better to frame their students' learning or teachers bringing in their teaching experiences to help teacher educators better to understand the practical needs and constraints. Double arrows indicate, for instance, relevant issues of the different actors' co-learning. Circle arrows highlight the effect on the self-development of the different actors.

In order to get additional information about communalities and differences between the seven initiatives regarding the *specificity of collaboration*, we also counted key words across all articles, excluding the references. The key words helped to distinguish the individual collaborations and to identify differences. The results of the searching for key words is part of the description of each initiative. The search comprised key words like activity/ies, belief/s, broker/s, boundary/ies, club/s, collaboration/s, colleague/s, community/ies, decision making, evaluation/s, group/s, intervention/s, learning, Lesson Study, mathematics, mathematician/s, member/s, ministry/ies, parent/s, participant/s, participation, partner/s, project/s, reflection/s, researcher/s, school/s, share/ing, student/s, teacher/s, team/s and university/ies.

8.2 The Selection of Articles

Acknowledging existing large surveys on mathematics teacher education (Adler et al., 2005; Gellert et al., 2012; Robutti et al., 2016, with a focus on teachers working and learning through collaboration), we focused on a small number of recent publications on teacher collaborations in high-quality journals. Since school systems are very different around the world, entailing different forms of collaboration between stakeholders, our goal was to analyse one or two articles from each continent (Africa, Asia, Europe, North America, Australia and Oceania, and South America), using the same dimensions (aspects of collaboration) for each article to allow a structured comparison. The selection of articles followed three criteria:

- searching for articles with a clear focus on the topic: keywords were linked by logical operators 'mathematics' AND 'collaboration' OR 'teacher collaboration' OR 'collaborative lesson research' OR 'community of practice' OR 'community of inquiry' OR 'teacher interaction' OR 'co-operation';
- searching for recently published articles (in 2018 and 2019);
- searching in high-quality journals (according to Williams & Leatham, 2017, the focus was directed on the following two 'very high' and five 'high' quality journals: ESM & JRME; FLM, JMB, JMTE, MTL & ZDM). In order to fulfil the all-continent goal, a hand search for additional quality articles was connected, starting from literature in the discussion document.

When focusing only on titles and abstracts, the systematic search identified 20 articles. The reading of the full texts led to five articles, focusing clearly on the topic: Asia 1, Europe 2, North America 1 and South America 1. Therefore, a hand search was needed, focusing on Africa and Oceania. This led to eight further articles in journals and one article in an anthology. The selected African article is cited as 'in press' in the discussion document and is published now, the Australian article is published in a journal very close to 'high' ranked, according to Williams & Leatham (2017), and refers to authors cited several times in the discussion document.

A first overview of the seven articles is presented in Table 8.1, indicating the initiatives' continent, the studied collaboration, the intervention focus and the context in which the implementation occurs, with the research focus including the used method and the research results.

8.3 Description of the Seven Initiatives

In the following, the seven selected articles are described and visualised using the RATE tool. Additionally, we provide information on the authors of the article, the type of the initiative, the specificity of the collaboration and we present selected findings.

Table 8.1 Overview initiatives

No.	Authors	Continent	Studied collaboration	Intervention focus (context)	Research focus (method)	Research results
1	Adler and Alshwaikh (2019)	Africa	Four teachers and four teacher educators (both regarded as 'co-learners').	Improvement of teaching (professional development, Lesson Study).	What changes in lesson plans do occur and how? (qualitative)	Changes initiated through 'example change moments' as 'collective accomplishment'.
2	Cooper (2019)	Asia	Twenty teachers and two mathematicians; in addition, a mathematics educator (as a participant observer, broker).	Improvement of communication and collaboration between teachers and mathematicians (professional development).	What and how do the two communities learn from and with each other? (qualitative)	Three domains in which the two communities explicated and sometimes changed their own perspectives on teaching and learning mathematics.
3	Goos and Bennison (2018)	Australia and Oceania	Six project teams of mathematicians and mathematics educators (having prospective and practicing teachers as target group).	Improvement in the quality of mathematics and science teachers (variety of teacher education strategies).	What boundary practices emerged? What conditions influenced interdisciplinary collaboration? How did learning occur? (qualitative)	Examples of successful boundary practices, classification of conditions for success and first steps for conceptualising learning mechanisms at the boundary between communities.
4	Maass and Engeln (2018)	Europe	Large number of continuous professional development (CPD) course leaders, educating 326 teachers (involving 3505 students).	Improvement of teaching on modelling (scaled-up CPD).	What impact has the CPD course on teachers' perception of implementing modelling and on students' perception of possible changes in teaching? To what extent do these perceptions differ? (quantitative)	Teachers' and students' perceptions of modelling differ, in particular regarding authentic connections to students' life, indicating the importance of sharing what is relevant to our lives, leading to questions of collaboration.
5	Potari et al. (2019)	Europe	Eleven teachers, fifteen researchers and eight policy-makers.	Improvement of a curriculum: Development of a reform-oriented national	Which emerging contradictions occurred during the interaction of team members	Four main contradictions were identified and these came up while defining the

(continued)

Table 8.1 (continued)

No.	Authors	Continent	Studied collaboration	Intervention focus (context)	Research focus (method)	Research results
				mathematics curriculum for compulsory education (design team commissioned by the ministry).	and how was these dealt with? (qualitative)	curriculum objectives and taking specific content into account. For all these contradictions, the involved activity systems, the boundary objects and the boundary crossing were identified.
6	Wallin and Amador (2019)	North America	Three teachers and one researcher.	Improvement of teachers' capacity regarding 'noticing' (of student thinking) and pedagogical design (professional support through video club).	What impact has the participation in a video club for rural secondary teachers' development of noticing? How does their noticing influence their pedagogical design capacity? (qualitative)	During participation, all three teachers reached a higher level of noticing and shifted their beliefs regarding curriculum usage. The co-participation among the teachers and the process of joint reflection and discussion were relevant factors for teachers' growth.
7	Estrella et al. (2019)	South America	Four teachers and three researchers.	Improvement of mathematics learning through using high cognitive demand tasks (PD with Lesson Study).	What factors are associated with the maintenance or decline of high cognitive demand tasks and how does the Lesson Study group aid to maintain such tasks? (qualitative)	High relevance of the Lesson Study group's discussions and reflection processes for the identification of aspects for improvement (high cognitive demand, repertoire of ideas for improving interactions).

8.3.1 Initiative 1 (Africa): A Case of Lesson Study in South Africa (Adler & Alshwaikh, 2019)

Relevant Actors Four secondary mathematics teachers (‘Lesson Study group’) and four researchers (‘project members’, ‘project team’ from university) participated in the project.

Relevant Targets The targets refer to the two target domains, teachers’ teaching and researchers’ co-learning, thus, “improving the learning and teaching of mathematics in previously disadvantaged secondary schools” (p. 318) and “to systematically research our co-learning” (p. 327). The key intervention, producing a relevant difference, is directed to teachers’ improvement of teaching.

Relevant Environments One university and three schools from one province in South Africa were involved in the project. Additional partners were the provincial departments of education, while the collaborative work between schools was organised in school clusters.

Authors The two authors were part of the project team consisting of four researchers. The first author had the role of the project director; the second author, who was relatively new to the project and learning about Lesson Study, was supported by the project director.

Type of Initiative (Duration) The initiative was a one-year, research-linked professional development, with a Lesson Study group at its core. The key processes of ‘Planning–Teaching–Reflection’ were supported by using an analytic framework (Mathematical Discourse in Instruction and Mathematical Teaching Framework), which serves as a boundary object, moving between being a research tool and a tool for teaching.

Specificity of Collaboration Small collaborating groups of teachers came from different schools. The authors aimed at “opportunities for teachers and researchers together to learn about teaching and how the tensions and dilemmas we [the researchers] faced were simultaneously opportunities for strengthening the coherence of the community” (p. 326). Although the initiative was regarded as a professional development, those carrying this out did not name themselves as teacher educators or something similar (role), but rather described themselves as researchers (identity). Thus, the ‘actor system’ showed with ‘teachers’ and ‘researchers’ a clear difference in identity and goals (learning in order to teach better; learning in order to generate new scientific insights); however, both had the role of ‘co-learners’. Compared with the six other initiatives, the statistics of key words in the text, excluding references, regarding this initiative shows the highest occurrence of ‘learning’, ‘reflection/s’, ‘group/s’ and ‘Lesson Study’. In combination with the high frequency of use of ‘teacher/s’ and ‘researcher/s’, this confirms the claim of co-learning.

Research Results The article aims at responding to two questions: “What changes occur in the example set across the lesson plans over a cycle? How do these changes occur?” (p. 326). With respect to the first question, the authors stressed the relevance of a specific framework, theoretically informing change processes. However, regarding the focus of this article on collaboration, the second question was the more important one: “How did the change in plans evolve?” (p. 331). This question involves the issue of interaction within a Lesson Study group where four researchers and four teachers collaborated. The authors show that the changes to the example set were initiated through “example change moments” (p. 335), and that these changes were a ‘collective accomplishment; of both teachers and teacher educators/ researchers in a double role. However, within this interaction, teachers’ and teacher educators’ contributions to the change were different: while teacher educators initiated an explicit and theoretically-grounded focus on example sets, teachers brought in their specific concerns about their students’ learning as experienced in the teaching of the lessons and the reflections upon it. The authors stress both the importance of discussing and reflecting as a decisive part of the ‘collective enterprise’ of a Lesson Study group, and the critical role of ‘knowledgeable others’, in particular fostering “structured and theoretically informed observation and reflection” (p. 338) (Diagram 8.1).

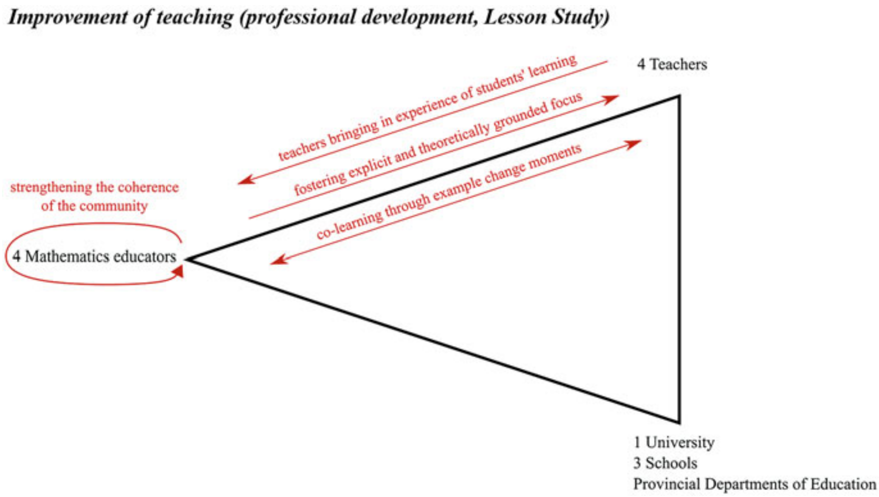

Diagram 8.1 RATE 1—Africa

8.3.2 *Initiative 2 (Asia): Mathematicians and Teachers Sharing Perspectives on Teaching Whole Number Arithmetic: Boundary-Crossing in Professional Development (Cooper, 2019)*

Relevant Actors Approximately 20 primary mathematics teachers, two research mathematicians and one broker (“participant-observer researcher”, “I”, “author”, “PhD in mathematics education”) from one university were the stakeholders in the project.

Relevant Targets The requirement by the Israeli Ministry of Education that primary-school teachers need “to enrol in mathematics PD courses in order to specialise in mathematics” (p. 71) sets the context for the initiative. The author stresses that the project enabled a setting for the primary teachers and the research mathematicians in which, “the two communities could share their perspectives with each other, not only allowing the teachers to benefit from the mathematicians’ perspective, but also providing an opportunity for the mathematicians to attain the sensitive understanding” (p. 70). This indicates that sharing perspectives is the goal of the initiative. The key intervention (producing a relevant difference) is fostering communication and collaboration between all relevant stakeholders regarding mathematics teaching, in particular including mathematicians, thus overcoming “conflicts between the communities of mathematicians and mathematics educators” (p. 69).

Relevant Environments The article draws on contributions in the realm of ICMI as a powerful international association, which encourages “a link between educational researchers, curriculum designers, educational policy makers, teachers of mathematics, mathematicians, mathematics educators and others interested” (quoted in Cooper, p. 70)—see International Mathematical Union (IMU, 2019). Further involved in the project were the Israeli Ministry of Education and one university.

Authors The single author of the article was the mathematics teacher educator acting as a ‘participant-observer researcher’ between the participating mathematicians and primary mathematics teachers.

Type of Initiative (Duration) The initiative reports on a professional development course, co-taught by a Ph.D. student of mathematics and a Master’s student of computer science, lasting one academic year.

Specificity of Collaboration Two mathematicians provide a professional development course for primary teachers, whereby a mathematics educator serves as a boundary-broker to mediate between the two “communities” or “parties” and pays attention to their sociocultural difference: “I also highlight the role of a participant-observer researcher as a broker in this process, supporting events of boundary-crossing in which the parties came to explicate, and sometimes change, their own perspectives on teaching and learning mathematics with respect to the perspectives of others” (p. 69). A main relevant difference refers to the goals which are mixed in a delicate way. Teachers need to upgrade their mathematical knowledge; at the same time, they are expected to be co-learners with the mathematicians:

> Together, these two lesson segments represent two types of PD activity—one “content based”—designed and led by mathematicians and dealing with particular mathematical content—and the other “problem based”—led by teachers and dealing with authentic issues of classroom teaching. In the first, episodes were selected and analyzed in detail to showcase opportunities for learning through “boundary-crossing”. (p. 72)

The frequency of key words like ‘boundary/ies’, ‘sharing’, ‘mathematician/s’ and ‘community/ies’ underline the descriptions above.

Research Results The article investigates “mechanisms of perspective-sharing” (p. 69) among 2 mathematicians and about 20 primary teachers in a professional development context. In particular, the article discusses what and how the two communities learned from and with each other, “drawing on the notion of boundary as sociocultural differences between communities” (p. 69) and regarding the researcher as a broker in this process, supporting events of boundary crossing. The article highlights three domains in which the two communities “came to explicate, and sometimes change, their own perspectives on teaching and learning mathematics with respect to the perspectives of others” (p. 69):

(a) when the researcher began to share his analysis of feedback (to teachers) with mathematicians, “they came to appreciate the relevance of this data for their planning and teaching” (p. 78);
(b) the researchers’ explicit attention (‘listening’) to teachers’ ideas lead to (self-reported) changes by some mathematicians in their university teaching: they dedicate “more attention to students’ ideas” (p. 78), by giving them more time to discuss;
(c) the primary teachers, by experiencing their “contribution to the mathematical discourse” (p. 79), in particular, “their expertise in the domain of mathematics-for-teaching” (p. 79), were able to build more mathematical confidence.

In all three cases, shared reflections between the two communities—stimulated by the researcher as broker—were the ground for a significant intervention (Diagram 8.2).

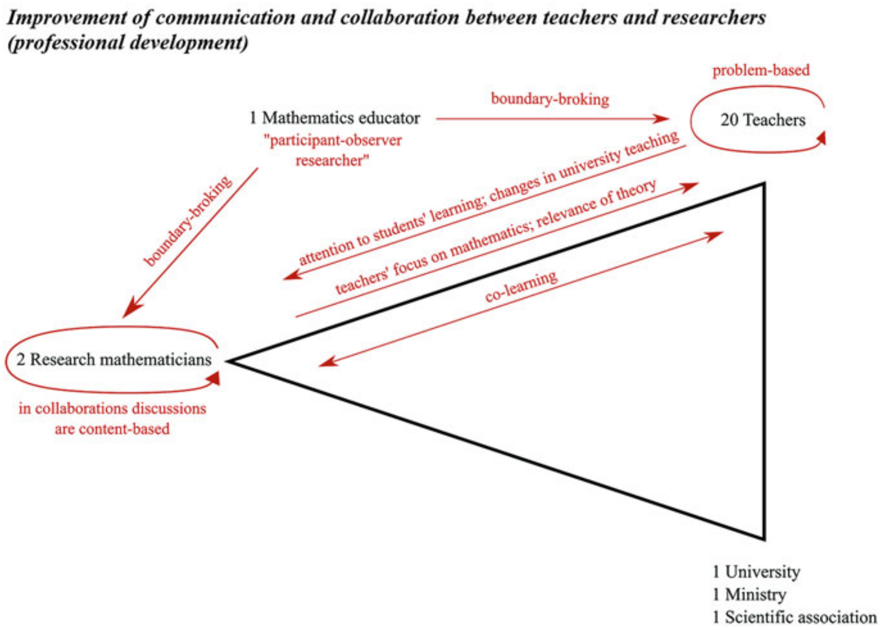

Diagram 8.2 RATE 2—Asia

8.3.3 *Initiative 3 (Australia): Boundary Crossing and Brokering Between Disciplines in Pre-service Mathematics Teacher Education (Goos & Bennison, 2018)*

Relevant Actors Six project teams, "comprising at least one discipline academic and one education academic" (p. 256), "mathematicians and mathematics educators" (p. 258) from six universities were involved in the project. Involved were also 23 investigators, that is, "the participants in the research were the mathematicians and mathematics educators who comprised the IMSITE project teams" (p. 261) and six lead investigators.

Relevant Targets The *Inspiring Mathematics and Science in Teacher Education* (IMSTE) project contributes to the "improvement in the quality of mathematics and science teachers" (p. 256). This main goal is pursued by "(1) fostering genuine, lasting collaboration between mathematicians, scientists, and mathematics and science educators who prepare future teachers, and (2) identifying and institutionalising new ways of integrating the content expertise of mathematicians and scientists [. . .] with the pedagogical expertise of mathematics and science educators [. . .]" (p. 256). This includes, "identification of principles for fostering new forms of collaboration between discipline academics and education academics" (p. 258). This goal and its ways of fostering it indicate that the key intervention, producing a relevant difference, is a two-fold one: through fostering collaboration between relevant stakeholders in mathematics and science education, the quality of mathematics and science teachers should be improved, assuming that this will improve mathematics and science teaching.

Relevant Environments The project involves six Australian universities, each working with a "cascade university", in establishing different "teacher education strategies" (p. 257), funded by an Australian ministry.

Authors The first author was a mathematics educator and had the role of a project co-leader—the other co-leader was a mathematician. The second author was a project officer and interviewed a larger sample of mathematicians and mathematics educators, including the two project co-leaders.

Type of Initiative (Duration) Various mainly pre-service, but also in-service, teacher education strategies such as "design courses" in order to improve recruitment and retention, or to initiate innovative curriculum arrangements or "conduct a mathematics pre-service teacher education alumni conference to connect current students, graduates, teachers, teacher educators, and mathematicians in order to promote continuing professional learning" (p. 257), were implemented over 3 years.

Specificity of Collaboration The collaboration encompasses both initiating and sustaining interdisciplinary collaboration on mathematics and science as subjects, content and pedagogy/education as fields of expertise and inter-university collaboration between universities and between universities and their cascade universities. The collaboration is thought to enable boundary encounters that, "give people a sense of how meaning is negotiated with another practice" (p. 259). Such boundary practices are facilitated by brokering. In this initiative, the boundary broker was the mathematics educator, who mediated expertise of the two fields of mathematics and mathematics instruction.

This complexity is even amplified by integrated foci, that is, primary and secondary schooling, prospective and practicing teachers. Besides the already mentioned goals, the project has also the strategic long-term goal, "to promote strategic change in teaching and learning in the Australian higher education sector" (p. 258), and the ambitious research goal to investigate conditions that enable or hinder sustained interdisciplinary collaboration. This initiative has the highest occurrence of the key words 'collaboration/s' and 'project/s', indicating the clear focus on bridge building through joint activities. In addition, also the word 'mathematics' is used most in this initiative, mirroring the involvement of mathematicians in the project.

Research Results First, the project initiated collaboration in terms of boundary practices among different communities of practice, involving discipline academics and education academics that focused on curriculum development and community-building activities in teacher education courses. Second, the project contributed an

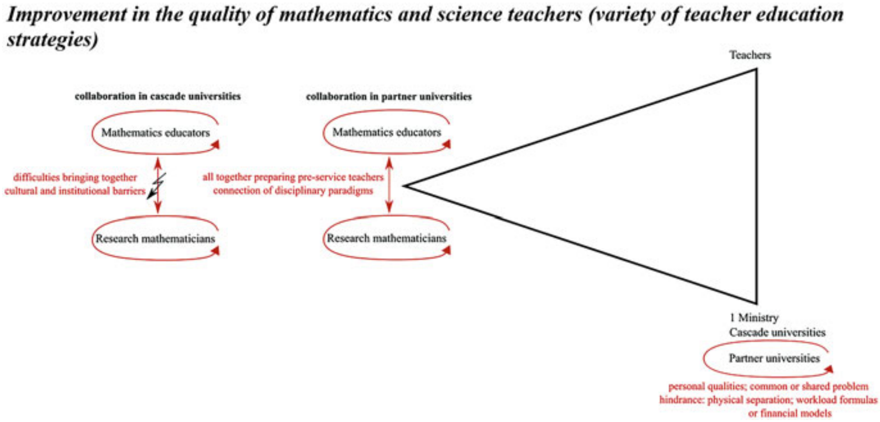

Diagram 8.3 RATE 3—Australia

evidence-based classification of aspects hindering or fostering collaboration. For the collaboration within the partner universities, several personal qualities were identified to help building up collaboration between mathematics educators and mathematicians, among them mutual trust, open-mindedness and recognising common or shared problems of preparing pre-service teachers for their future role.

The situation was different for the collaboration between the partner universities and at the cascade universities. Here, collaborating just emerged between the groups of mathematics educators and mathematicians solely. Institutional and cultural barriers were high and made it even difficult for the brokers to bring the two worlds together. Third, the project provided empirical grounding for understanding boundary practices regarding the learning mechanism of transformation at both interpersonal and intra-personal levels, including identifying together shared problems, and developing broking positions to connect disciplinary paradigms (Diagram 8.3).

8.3.4 Initiative 4 (Europe 1): Impact of Professional Development Involving Modelling on Teachers and Their Teaching (Maass & Engeln, 2018)

Relevant Actors Participants were 326 secondary mathematics teachers. In the overall project, more than 1000 primary and secondary teachers for mathematics and science were involved. The 'course leaders' from 12 countries, namely teachers, pre-service educators, persons from CPD institutions, were selected and educated by the 'project partners' in each country. The 'we' in the article refers to the whole project, including all project partners.

Relevant Targets The goal of the project was to achieve an impact "on teachers and their teaching" (p. 273), in the direction of "implementing innovative teaching" (p. 273) through "inquiry-based learning" (p. 275). Regarding mathematics in secondary school, for example, the target was attaining, "a significant change regarding the implementation of modelling in mathematics teaching" (p. 274). For improving teaching, the project also had the goal of implementing it at a large scale (double-goal). Thus, also the key intervention was doubled: the relevant difference aimed at is innovative teaching, as opposed to current teaching, and this should be implemented and scaled up at many places as opposed to some individual places.

Relevant Environments The project involved 14 universities ('project partners') from 12 countries, participating in a large EU-project (seventh Framework program of the EU which defines the context of the project).

Authors The first author was a mathematics educator and had the role of the leader of the EU-project and of the international centre for STEM, being primarily responsible for co-ordinating the project. The second author was a science educator from a collaborating research institute and was co-responsible for project-related evaluation and research.

Type of Initiative (Duration) Continuous professional development (CPD) courses were provided in 12 countries, running, "within a timeframe of 2 years in each country", from "several weeks" to "the duration of 1 year". With "about 100 teachers in each country" taking part, the project is relatively large. It is also complex, as reflected in the target groups of both mathematics and science teachers, the diverse cultural contexts of the 12 countries, the heterogeneity of the course leaders' education and competencies, and their role as brokers, navigating between the goals of the project and the needs of the teachers.

Specificity of Collaboration In the courses, 7 CPD principles, in the sense of quality criteria, were implemented to "stimulate cooperation between teachers so as to support teachers in the learning-on-job phases" (p. 277). In order to ensure quality across all 12 countries, the project partners "discussed the overall CDP principles and their implementation in the PD course at the biannual project meetings" (p. 278). The research questions reported in the article refer to the secondary level, particularly the mathematics teachers participating in the project, and their students, focusing on the impact of a mathematics course on modelling.

The collaboration among teachers, course leaders and project partners and between them is not a major focus of the article. Given the size of the project and the various contexts of the participating countries, it is likely that the collaboration is manifold and takes very different forms. The size of the project and the large number of teachers dealt with in the research article (macro-perspective) seem to shift the view on concrete collaborations (micro-perspective) to the background. It is not surprising that the occurrences of key words like 'collaboration/s' and 'community/ies' were low, although those of 'teaching' and 'student/s in regard to the other cases were relatively high.

Research Results The study, designed as a quantitative pre-post-evaluation using questionnaires, investigates which impact the CPD courses on modelling in 12 countries have on the participating mathematics teachers and their teaching. Two research questions examine to what extent significant changes in teachers' classroom practice occurred, as perceived by the teachers themselves and by their students. The results, focusing on teachers' perceptions, show statistically significant pre−/post-differences regarding all three scales of mathematical modelling used—investigative teaching, student-centeredness and authentic connections to students' life. The results regarding students' perceptions are quite different: the scales of investigative teaching and student-centeredness show an increase, but not statistically significant. Only the scale of authentic connections to students' life shows a statistically significant difference, however, as a decrease.

The third research question investigates to what extent students' and teachers' perceptions of classroom practices are in agreement. Regarding investigative teaching and student-centeredness, data show a relevant correlation between teachers' and students' perception. However, teachers' perception of the frequency of authentic connections to students' life is significantly higher than students' perception. By referring to other research suggesting that authenticity is a social construct that needs to be agreed on in different communities, the authors indicate the importance of sharing what is relevant to students' (and teachers') lives, in classrooms and surely also in teacher education and in collaboration among teachers (Diagram 8.4).

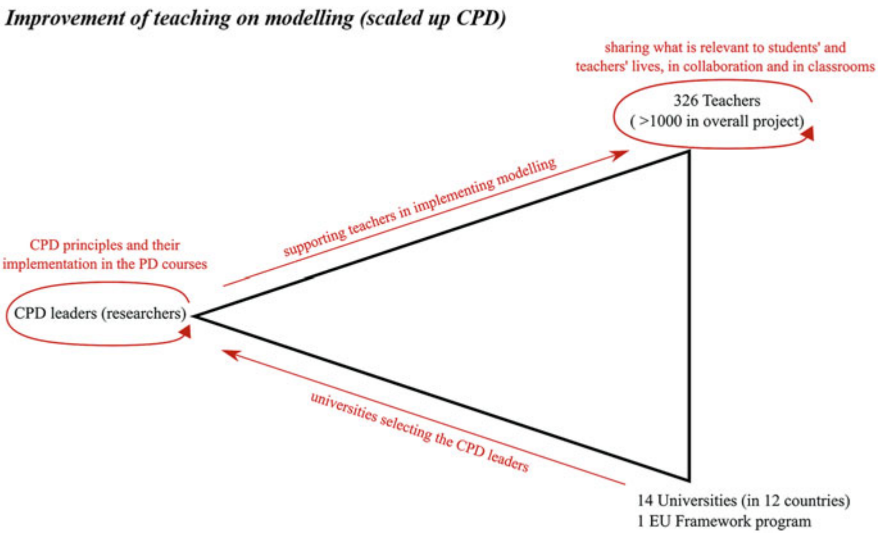

Diagram 8.4 RATE 4—Europe

8.3.5 Initiative 5 (Europe 2): Collaborative Design of a Reform-Oriented Mathematics Curriculum: Contradictions and Boundaries Across Teaching, Research, and Policy (Potari et al., 2019)

Relevant Actors In sum, thirty-four members participated in a ‘design team’ for developing a reform-oriented mathematics curriculum. Among them were eleven ‘classroom’ teachers from kindergarten to secondary school, fifteen ‘academic researchers’ (two mathematicians and thirteen mathematics educators), and eight ‘policy makers’ (two ministry and six school advisors).

Relevant Targets The goal is to develop “reform-oriented national mathematics curriculum [that] concerned compulsory education in Greece” (p. 418), in order to establish “the improvement of students’ learning as a common goal” (p. 432). Particularly, “the quality of students’ mathematical thinking and their future citizenship” (p. 430) is stressed. The key intervention is shifting from ‘traditional teaching’ and corresponding resources to ‘research-informed teaching’ and adequate resources. The ministry plays an important role, which “initiated a curriculum reform through the *New School* act [. . . focusing on] active engagement of students, openness of the education to society, [. . .] and new roles for teachers as active agents of the curriculum” (p. 421). The commissioned ‘design team’ amplifies this orientation on active engagement by giving key members a voice when generating interview data about the design process.

Relevant Environments One Ministry of Education commissioned a design team, based on a new policy document (“*New School* act”, p. 421).

Authors All authors were mathematics educators and members of the ‘design team’ for the new curriculum. They all carried out interviews and did the analysis. The co-ordinator of the team served as the first author.

Type of Initiative (Duration) A design team produced a final version of a curriculum (9 months). The article reflects and analyses the design process, grounding it within a specific theoretical framework and on empirical data (e.g. based on interviews with 11 “key design team members”, p. 423).

Specificity of Collaboration The ministry appointed a co-ordinator, a researcher from a university of a national mathematics curriculum and further members of the design team “based on the coordinator’s recommendations” (p. 422). During the curriculum design process, “the coordinator acted as a broker between the educational policy activity and the designing activity” (p. 421). Thus, this collaboration does not directly take place in a teacher education context, but, in a curriculum context, it sets essential general conditions for future teacher education plans and activities. The social dimension of designing a curriculum and the use of an activity theory perspective is mirrored in a relatively high occurrence of words such as ‘team/s’, ‘community/ies’, ‘member/s’, ‘colleague/s’, ‘broker/s’ and ‘ministry/ies’. Also, words like 'teaching', 'activity/ies', 'boundary/ies' and 'mathematics' are used (relatively) often.

Research Results The research questions centred on the interaction of the design team members, stemming from different communities. Particular attention was paid to emerging contradictions against the background of the three activity systems, that is, within and across the three communities of practice, comprising mathematics teachers, mathematics education researchers and policy-makers. The authors could identify four main contradictions and coded them as: (1) educational innovation versus teaching reality; (2) theoretical versus practice-oriented ideas; (3) research-informed teaching resources versus traditional teaching resources; (4) arithmetic versus algebra in primary school teaching. The first three relate to defining the curriculum objectives, while the fourth one came up as the design team was working on specifying the algebra content and the intended learning outcomes for different educational levels—thus, while taking specific content into account. Additionally, the authors specified for each contradiction the activity systems involved, the boundary objects and the boundary crossings that were found. Regarding the latter, the curriculum structure of algebra was identified as a boundary object. As for boundary crossing, establishing the improvement of students' learning as a common goal was deemed essential. Also, the role of educational materials and resources for facilitating the boundary crossing was underlined (Diagram 8.5).

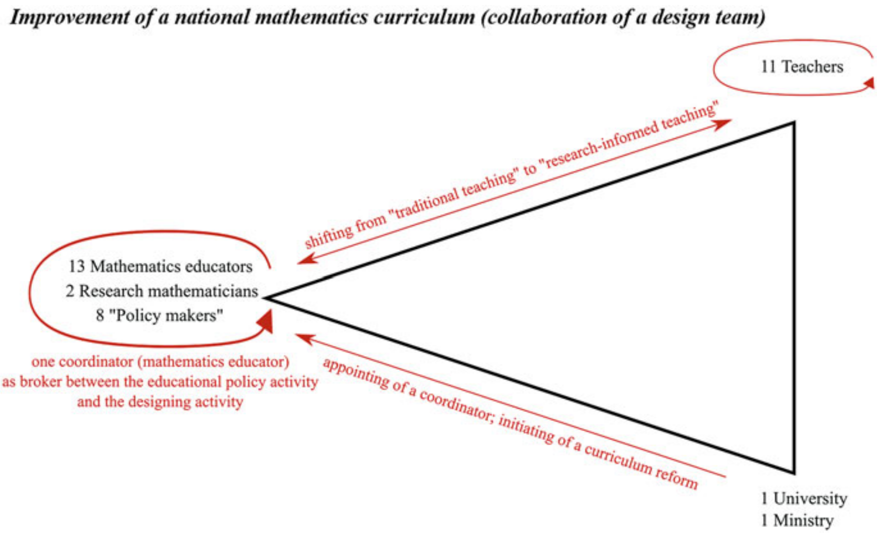

Diagram 8.5 RATE 5—Europe

8.3.6 Initiative 6 (North America): Supporting Secondary Rural Teachers' Development of Noticing and Pedagogical Design Capacity Through Video Clubs (Wallin & Amador, 2019)

Relevant Actors Three mathematics teachers "who comprised the entire mathematics department of one secondary school" (p. 515) and one researcher formed a "video club".

Relevant Targets The goal is a high level of teachers' capacity regarding "noticing" of student thinking and "pedagogical design" (p. 515). This means that a key relevant difference (intervention) is teachers' improved competence. Among others, the research focused on the question whether teachers' participation in the video club influenced "their view of collaboration" (p. 515). The quotation, "Furthermore it is likely that without the video component of the collaboration process and the coparticipation among the teachers [. . .] of these teachers [. . .] would not have made the degree of growth they were able to accomplish due to their initial beliefs" (p. 534), shows that another relevant difference is seen between teacher collaboration and no teacher collaboration.

Relevant Environments One school with its entire mathematics department: during the first phase "Introduction to school setting", a "Meeting with administration" is also mentioned in a rural area and one university participated in the project—the teachers attended university PD courses, etc. led by the first author.

Authors The first author was a mathematics educator who designed the "video club", worked with the participating teachers and collected all data. The second author, also a mathematics educator, supported him in analysing the data and writing the article.

Type of Initiative (Duration) The teachers attended five video club meetings over 1 year. The article reflects and analyses the design process, grounding it in a situated perspective and on empirical data, based on interviews (p. 521).

Specificity of Collaboration A major part of the collaboration was reflecting on lessons, based on lesson plans and videos. The video club aimed at fostering a "culture of supportive constructive feedback and discourse" (p. 521). "The researcher intentionally selected the video clips for the video clubs himself, as opposed to having the teachers select clips" (p. 523), in order to focus specifically on students' mathematical thinking: recognising "the value of co-participation, these conversations were informal and mostly directed by the participants, but moderated by the researcher" (p. 523). The school-external member of the video club had a variety of roles, at least comprising researcher, author, teacher educator, club designer, moderator and collaborator. Compared with other cases, the relatively high occurrence of key words like 'participation/s', 'participant/s', 'colleague/s' and 'school/s' indicates the collaborative nature of the video club. Often, the used key words, such as instructional ‘decision making’ and teachers’ ‘beliefs’ regarding curriculum, mathematics, tasks, etc. mirror the work on concrete instructional activities.

Research Results The single-case study on three mathematics teachers aims at answering two research questions (p. 517). (a) How does participation in a video club structure for rural secondary teachers support the development of noticing? (b) How does what rural teachers notice from this experience influence their pedagogical design capacity? Regarding the first question, the study examines—by using a specific framework—the development of teachers’ level of noticing. The findings show that all three teachers, starting from different levels, reached in the fifth and last video club meeting a higher level than in their second meeting. In addition, all three teachers shifted their beliefs regarding curriculum usage and became more comfortable rethinking their current curricular materials. A vignette of one teacher’s growth indicates the process of slowly valuing both student thinking during her lessons and the instruction which promoted it. Thus, this links the two constructs of noticing and pedagogical design capacity.

The findings support other research claiming that teacher noticing is a skill which can be learned. Taking a situated perspective, the authors reflect how the context of the teachers’ interactions may have influenced their growth. They indicate the importance of the video component of the collaboration process and the co-participation among the teachers, being able to reflect on decision-making both of themselves and of their peers. Although none of the teachers taught the same mathematics topics as it is common in urban schools, the process of joint reflection and discussion led to teachers’ growth. Thus, the authors argue that the findings provide evidence of the viability of video clubs to support teachers in rural contexts. The video club experience began to erode professional isolation within the group, fostered their collaboration and led to an increase of the frequency of meetings (Diagram 8.6).

8.3.7 Initiative 7 (South America): How Teachers Learn to Maintain the Cognitive Demand of Tasks Through Lesson Study (Estrella et al., 2019)

Relevant Actors Four primary school teachers (‘them’; teachers “with training in mathematics education and who had more than 5 years of experience” (p. 297)) and three researchers (‘we’; “with experience in Lesson Study and teacher training” (p. 297)) worked together in a “Lesson Study group”. There was also a research team, involving six researchers—three of whom had worked in the Lesson Study group, who analysed the implementations.

Relevant Targets The overall goal was “the improvement of mathematics learning” (p. 293). The specific goal of this study was to investigate “how primary school

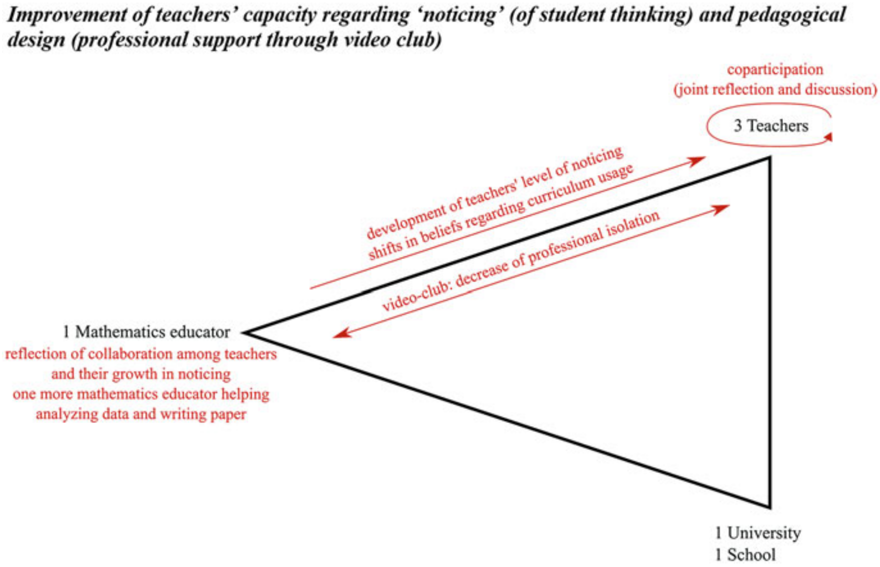

Diagram 8.6 RATE 6—North America

teachers implement high-level cognitive demand tasks in a data analysis lesson in the context of Lesson Study" (p. 297). Implementing and maintaining a high level of cognitive demand is a special indicator for students' high quality. The key intervention is directed towards producing a relevant difference in students' learning, intending a shift from low to high level of cognitive demand. Thereby, Lesson Study is seen as a teaching method "for transforming teaching" (p. 295), by overcoming "the teacher-centered teaching model" which "remains dominant in most schools" (p. 297). The rather general conclusion of the study is that, through "collaborative work among working teachers and researchers in the context of Lesson Study", it is "possible to design and implement tasks that maintain high cognitive demand in primary school" (p. 305). This indicates that 'collaboration' in the context of a Lesson Study is regarded as a key intervention leading to success.

Relevant Environments Four "Chilean public schools", where the four teachers work (p. 297) and one university.

Type of Initiative (Duration) The Lesson Study group had eight two-hour Lesson Study sessions (weekly). During these sessions, "the group prepared the lesson plan and material and discussed the implementation of the lesson and how to improve it" (p. 297). The research team for the analysis of the implementation "met weekly for 2h for 6 months" (p. 298).

Authors The first three authors were mathematics educators who worked in the Lesson Study group and, being also members of the research team, analysed the data.

The fourth author had a master in mathematics education and supported the other authors.

Specificity of Collaboration One of the researchers' most important working fields during the Lesson Study group was providing support: "With the support of the researchers, the teachers designed an open-ended task with consideration of the presentation of the data and the context and elements of high cognitive demand tasks for the grade, such as representing and arguing" (p. 297). Another collaborative working field was the professional development of the teachers during the sessions—"The eighth session was dedicated to the teachers' self-evaluation and reflection on the experience of the Lesson Study cycle, the statistical knowledge acquired and the impact on their professional development" (p. 298). The researchers had also other roles, at least including author, teacher educator, support provider and collaborator. Not surprisingly, this initiative, like initiative 1, has the highest occurrences of the key terms 'Lesson Study' and 'group/s'. In this case, also 'evaluation/s', 'intervention/s' and 'reflection/s' are more-often used words, in contrast to other initiatives with exception of 'reflection/s' in initiative 1. This mirrors the teachers' active and self-critical stance as fostered by the researchers, for example: "The [...] session was dedicated to the teachers' self-evaluation and reflection on the experience of the Lesson Study cycle" (p. 298).

Research Results For the research presented in this article, the implementations of two out of four teachers were scrutinised. Two research questions were pursued to reveal how these two teachers implemented open-ended tasks for third graders within the context of the Lesson Study project. Particularly, the authors investigated how the teachers maintained the cognitive demand and how it declined, also with the help of the Lesson Study group in view. For the two primary teachers, the results reveal different scores on factors associated with the maintenance or decline of high cognitive demand during lessons. The authors particularly highlight the relevance of the Lesson Study group's discussion and reviewing of the teachers' lessons to recognise deficient aspects and to take responsibility for improvements. That is, the collaboration with the researchers helped the teachers to become explicit with respect to obtaining cognitive demand and thus challenging high quality student productions. Also, the reflections initiated in the discussion sessions of the Lesson Study group, helped teachers to understand and improve their deficient actions in the classroom and to build a repertoire of ideas for improving these interactions. The main source for teachers to develop towards the intended goal of the project was the benefit of the co-operation with respect both to the researchers and among the teachers (Diagram 8.7).

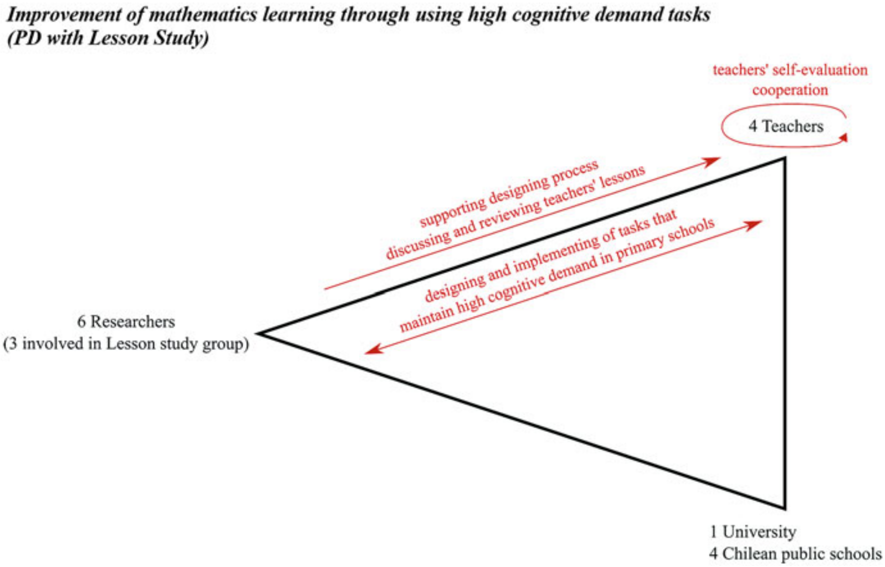

Diagram 8.7 RATE 7—South America

8.4 Comparing the Cases

Comparing seven cases can only be a first step in grasping phenomena. However, it is possible to generate observations (noticing relevant differences), possible hot issues and blind spots. Also, we might create some ideas for developing a tool that can be refined in a larger study, aiming at representation. Before we focus on commonalities and differences of the seven cases, we briefly discuss the seven cases against the background of a survey on research of mathematics teacher education from 1999 to 2003 (Adler et al., 2005). The following two of three claims of this survey are relevant for our comparison.

Claim 1, "Small-scale qualitative research predominates", can be substantiated with respect to our sample. Since more than the half of our cases (4 out of 7) can be counted as small-scale (N < 20), we have nearly the same picture as in Adler et al. (2005) who found 98 out of 145 empirical articles based on small-scale research. In their study, 135 of 145 empirical articles involved studies with fewer than 100 teachers, in our case the figures are five out of seven. Based on Adler et al., Gellert et al. (2012) conducted a similar survey for the period 2005–2010. Focusing on research methods, methodology and techniques in studies on mathematics teacher education, they reported that Claim 1 is still legitimate (e.g. 89% of studies involved fewer than 100 teachers). A survey for the period 2005–2015 by Robutti et al. (2016;

see also Jaworski et al., 2017) focusing on teacher collaboration showed similar results (90% of studies involved fewer than 100 teachers).

Claim 2, "Most teacher education research is conducted by teacher educators studying the teachers with whom they are working" (Adler et al., 2005; similar results can be found in Robutti et al., 2016), gets a very strong confirmation. All seven articles were written by people being involved in the activities of the projects—in six cases directly, in one case as a broker. Presumably, the extreme result is connected to the choice of journals with the scientific community as the major target group. Although, for example, research initiatives with teachers and teacher educators as collaborators (e.g. participatory action research; see Gellert et al., 2012) exist and provide interesting insights, it seems to need time for such initiatives to be presented in the above-mentioned journals.

Another, more general explanation is based on the fact that research on teachers working and learning in collaborative groups is a specific domain within the broad domain of research on teacher education in general, where a focus can be directed on teachers' actions, beliefs or knowledge without being involved into an intervening activity (besides collecting data). Working with teachers and doing research on their professional growth combines the goals of contributing to the further development of teachers and of the scientific community, thereby also bringing research closer to teaching (see, for example, Cai et al., 2018). Of course, when dealing with research on 'teacher collaboration', a research project just might investigate it without intervening. However, how can researchers deeply grasp the phenomenon of 'collaboration', when they are not part of the collaboration? More generally, experts in system theory claim that, in order to understand a social system deeply (namely the ways, routines, patterns, hidden rules of collaboration and resistance, etc.), researchers need to intervene in it. This happens automatically when they work with teachers and, thus, when they try to improve something within the system.

Interventions into a social system such as a group, a course, a department, etc. can cause a lot of reactions, in particular towards those intervening, namely the educators, facilitators, designers, etc. The reactions can range from open enthusiasm and collaboration within the social system and with those intervening to open or hidden resistance and tensions, with stark impact on the collaboration, internally and externally. All articles we surveyed reported success, for instance, that teachers' collaboration improved and that the teacher–researcher collaboration was regarded as a powerful means. No single case reported activities that failed, at least partially. Also, the seven cases do not provide much information on critical aspects of collaborating.

However, collaboration would be an ideal topic to reflect critically on interactions at different levels. For example: (a) the teachers might explicitly be asked to share critical aspects of the collaboration among teachers and with the educators from their point of view; (b) the teachers might explicitly be invited to comment on the results of the research on collaboration—one could even think about inviting one or more teachers to be a co-author of a article, eventually as an additional article to a pure research article; (c) the authors (researchers) could—in addition to presenting their research results on collaboration—also integrate critical reflections on their collaboration as a team and on their activities of fostering teachers‘collaboration. Thus, the above-mentioned claim 2 seems to be particularly relevant when studying teacher collaboration.

In the following, we sketch some observations, using the RATE scheme, in relation to the relevant actors of the collaboration, the relevant targets, the relevant environments and the specificity of the collaboration.

Relevant Actors

The seven initiatives show—apart from mathematics and science teachers in all seven cases—a variety of actors, including educators (six initiatives), researchers (six), heads/principals (five), mathematicians (four), brokers (three), facilitators (three), heads (three), administrators (two), policy makers (two), multipliers (one) and teacher leaders (one). In all initiatives, the key words ‘student/s’ appear, but not ‘parent/s’. As social entities, we find (video) ‘clubs’, different ‘communities’, (Lesson Study) ‘groups’, (project) ‘partners’ and (design and project) ‘teams’. Regarding the number of involved teachers, the seven initiatives include three small ones fewer than ten collaborators (including three or four teachers), two medium ones (20–40 collaborators, including 11 resp. 20 teachers) and two large ones (> 100 collaborators, including 326 teachers resp. not specified).

The relevant actor ‘broker’ needs to be more discussed. We see a ‘broker’ as a particular type of actor. An actor, who is in the role of a broker, could be also a researcher, a principal, etc. In some initiatives, e.g. initiative 2, the mathematics educator did not act as a mathematics educator, but rather as a broker. That is, a broker mediates between actors coming from different communities and supports events of boundary crossing. In this way, an actor in a collaboration can step out of his or her original role and take on that of a broker.

Relevant Targets

In all initiatives, although using different expressions and stating the goal explicitly or more implicitly, one target is the learning of teachers and the improvement of teaching. The improvement of teaching is connected to quite different meanings like ‘innovative teaching’, overcoming ‘teacher-centred teaching’, ‘research-informed teaching’ or supporting ‘inquiry-based learning’, hardly combined with a clear definition of the intended shift in teaching, marking a relevant difference between the *status quo* and the desired situation. In some cases, other adult learners like mathematicians and researchers are mentioned as co-learners.

Relevant Environments

In all seven analysed articles, universities as working places of researchers play a role. In five cases, also policy-related entities (three Ministries, an EU-program, a scientific association and provincial departments of education) are relevant. In three cases, all related to small initiatives, schools as working places of teachers and places of researchers’ intervention were involved. In the case of one of the seven initiatives, the article indicates that a requirement by a ministry established a distinct context for the initiative. Also regarding other initiatives, it is assumed that it would be interesting to read more about the context, in particular the goals and roles of the stakeholders having an impact on the initiative.

Authors

The number of authors ranges from one to four, with two authors in four cases as the most frequent case. All seventeen authors were involved in the initiative as educator and/or as researcher. Nearly all authors (fourteen) were mathematics educators, one was a mathematician and the two others were a project officer and a science educator. Thus, no teacher or another person of a relevant environment took the role of an author. All first authors had a pivotal role in the initiative, e.g. being a broker, co-ordinator, club designer, director or leader.

Specificity of Collaboration

The ways of collaboration are highly diverse, largely depending on the context of the initiative. For example, the three small and local initiatives (1, 6 and 7, each fewer than ten collaborators) refer to small communities: two Lesson Study groups and one video club where a few researchers collaborated with a few teachers; the two medium initiatives (2 and 5, each 20–40 collaborators) refer to a professional development course and a curriculum design team where mathematics educators, mathematicians and mathematics teachers collaborated; the two large initiatives (3 and 4, each more than 100 collaborators) refer to a national and an international program where researchers and teachers—both from mathematics and science—collaborated, in the context of institutionalised collaborations between universities.

It is not surprising that in the case of the two larger and more complex initiatives, the importance of the cultural context is stressed. For example, due to the collaboration of educators from different countries, it was necessary in initiative five to define common CPD principles. The three small initiatives have in common that they all deal with the impact of researchers' initiative, finding out that collaboration and reflection are decisive factors for bringing about change. Most initiatives describe extensively their particular approach. In many cases, it would be interesting to read more about similar approaches and what the initiatives have in common or how they differ.

Research Results

As the ways of collaboration reported in the seven articles are diverse, unsurprisingly the research questions and the yielded findings paint a broad picture too. The majority of articles report on generic issues of collaboration, such as the importance of discussions and reflections, the critical role of knowledgeable others, institutional and cultural barriers and the role of brokers, and teachers' classroom practices seen through the lens of noticing and the design capacity. Some articles integrate aspects specific to teaching and learning mathematics as they, for instance, report on contradictions regarding teaching arithmetic and algebra occurring for the different groups collaborating or on mechanisms of perspective-sharing between mathematicians and primary teachers. Thus, what is specific when researching collaboration in the field of mathematics is not that strongly emphasised in the contributions.

Additional Observations

All seven initiatives report about interventions with a focus on improving mathematics teaching and learning. However, the foci on improvements differ. Three cases (2, 3 and 7) relate to collaborative processes (learning from and with each other; emergence of boundary practices; factors maintaining high cognitive demand tasks in a group); two cases (1 and 6) relate to teachers' changes in lesson plans and levels of noticing; one case (4) relates to teachers' and students' perceptions of implementing modelling or changes in teaching in different countries; one case (5) relates to writing and thus preparing the legislative act for a new curriculum in order to improve students' learning.

However, the research investigates emerging contradictions occurring during the interaction of team members. The fact that only one case (4) uses data from students as indicators is not surprising, because the corresponding research focuses on collaboration among and with teachers. Like initiative 4, when focusing on students' perceptions of classroom practices regarding modelling, initiative 7 also refers to the student level in elaborating on high cognitive demand tasks for students. However, in both cases, the focus is not on students' achievements. In all these three cases, the inexplicitly stated impact chain seems to be as follows: collaborating with a small community leads to changes at teachers' level, which leads to certain changes at the students' level (cognitive demand, perceptions), finally leading to a higher quality of students' learning.

8.5 Summary

Finally, based on the analysis in this chapter, we formulate some observations related to research on relevant actors, targets and environments of the collaboration.

Observation 1: Small-scale qualitative research predominates (see Adler et al., 2005). Similar results are reported by Gellert et al. (2012) and Robutti et al. (2016).

Observation 2: Most research is conducted by teacher educators studying the teachers with whom they are working (see Adler et al., 2005). Similar results can be found in the survey by Robutti et al. (2016).

Observation 3: Most research focuses on improvements and success: (critical) reflections on teacher educators' (co-)learning—although focusing on collaboration—are rare. Increasing literature on teacher educators' growth (see, for example, Beswick & Chapman, 2020; Krainer et al., 2021) might stimulate more publications on teacher educators' learning through collaboration.

Observation 4: Most initiatives focus on the learning of teachers and the improvement of teaching. However, the intended shifts in teaching, that is, the marking of a relevant difference between the *status quo* and the desired goal of the intervention, is rarely defined in a clear way. Also, relating teacher learning explicitly to the collaboration within the projects is rarely in the focus (see findings by Robutti et al., 2016, in their international survey on teachers working and learning through collaboration).

Observation 5: Only a few initiatives describe the context and relevant environments having a potential impact on the initiative.

Observation 6: Most initiatives describe their particular approach extensively. In many cases, it would be interesting to get comparisons with similar approaches.

Observation 7: Most initiatives stress the importance of sharing reflections as a crucial factor for working with teachers and, possibly, with other groups. A similar observation stems from a literature review of PME articles by Llinares and Krainer (2006).

This chapter does not allow the space to discuss about RATE as a tool for reflecting on research on teacher collaboration, for example, regarding its advantages, potentials, challenges and limitations. It is hoped that the description and the comparison of the seven cases and the formulated observations will serve as starting points for discussion and future research. In this chapter, RATE helped to grasp heterogeneity of the collaborations in the seven initiatives. RATE has also provided a framework through which it has been possible to describe the seven initiatives comparatively and to work out the differences using a uniform presentation.

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Chapter 9
Resources for and from Collaboration: A Conceptual Framework

Karin Brodie and Kara Jackson

9.1 Introduction

Resources are important for all joint work and learning, and particularly for collaborative professional work and learning among teachers. They provide a means for the design of learning environments, for joint work in pursuit of learning goals, and can support links between teacher collaborative learning and teaching practice. They can also be a focus of analysis for research and explanatory mechanisms that illuminate the complexity of relationships in collaborative work.

In this chapter, we provide a framework for conceptualising different kinds of resources and their functions in supporting productive teacher collaboration. Drawing on our work with teacher collaborative groups in two different contexts, we argue that resources can be conceptualised more broadly than they are currently, and that a broader conceptualisation supports us to see and design for aspects of collaboration previously hidden. In particular, we draw on two sets of questions to guide our work:

> What resources are available to support teacher collaboration? With what effects, both on the collaboration and the resources themselves?
>
> What resources are missing for supporting teacher collaboration? How and to what extent can teachers overcome these missing resources?

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These questions were an important part of the work of Theme D at the ICMI 25 Study Conference and our discussions there suggested that resources are dynamic and living, that they evolve in relation to the contexts of collaboration and the classroom, and they go beyond material objects to include human, social and cultural resources (Chap. 5, this volume). We discuss our use of the term 'resources' in more detail later in this chapter.

In developing a framework of resources somewhat broader than conventionally conceptualised, we hope to draw attention to missing resources, in two senses. First, some resources are missing from current conceptualisations of resources, and this absence might limit our view of what can and does happen in teacher collaboration. Second, some resources that are available in some contexts, for example online resources, may not be available in others, because of the inequitable distribution of resources. We therefore hope that our framework provides ways for designers and researchers of collaboration to focus on inequities among contexts and not to assume that middle-class contexts in the 'developed' world are the norm.

We develop the framework in relation to our work with collaborative teacher professional development in two geographic contexts—South Africa and the United States—which are different in many ways and remarkably similar in others. Looking across our similarities and differences has supported us to see how aspects of our framework are sensitive to context and, at the same time, may be applicable in various other contexts—supporting others to describe their current work and to see possibilities for new design and research.

9.2 Conceptualising Collaboration

The ICME-13 survey on teachers working and learning through collaboration defined collaboration as:

> co-working (working together) and can also imply co-learning (learning together). It involves teachers in joint activity, common purpose, critical dialogue and inquiry, and mutual support in addressing issues that challenge them professionally. (Robutti et al., 2016, p. 652)

and teachers working together as:

> collaborating for some specific aims, which could be directed towards: improving students' learning; improving their professional role in the school; learning to use new resources (e.g. technological tools); creating a professional network within the school or region; and discussing institutional reforms and demands around the curriculum, the national evaluations system, etc. (p. 653)

These definitions suggest that joint work produces learning and that collaborative learning requires the intent to work together and a purpose for this work. Collaboration can be organised, as in professional development sessions, or occur spontaneously, as teachers talk to each other about their practice; it can be long-term, as in on-going professional learning communities, or *ad hoc*, as the need is felt by teachers. Collaboration can take place in face-to-face settings, or virtually, and can be local, taking place within a school or among nearby schools, or more global, among teacher associations or groups constituted online. Organised collaboration is usually designed by professional development providers, by teachers or by teacher educators and teachers working together.

What collaborative groups look like, and how they work, have been described differently by different researchers. Wenger (1998) defines communities of practice (COPs) as groups of people engaged with each other, focused on a joint enterprise, and creating a shared repertoire—a set of resources which support their enagement in relation to the joint enterprise. Communities learn together and are always sociohistorically situated in webs of social relations, as are the resources that support communities and their learning. Gueudet and Trouche (2012a) refer to ‘collectives’ as, “not necessarily implying cohesion or involvement in a common project” (pp. 305–306), thus including a broader range of potential collaborative work.

Brodie and Borko (2016) have defined a professional learning community (PLC) as a special kind of community of practice, with the distinguishing feature of *professional learning*, where professional learning entails becoming confident with and competent in the knowledge base of the profession, and using it to make and justify professional decisions (see also Chauraya & Brodie, 2017). Professional learning involves regular and sustained inquiry into various aspects of local practice, as they might relate to more global concerns (Jackson & Temperley, 2007; Jaworski, 2008). Louis and Marks (1998) argue that school-based PLCs allow teachers to “coalesce around a shared vision of what counts for high-quality teaching and learning and begin to take collective responsibility for the students they teach” (p. 535).

COPs and PLCs provide opportunities for systematic teacher collaboration and learning, usually facilitated, and it has been argued that thinking and practice develop more powerfully when worked on systematically. The studies reviewed by the ICME-13 survey (Robutti et al., 2016) showed a variety of teacher learning from collaborations, including teacher learning of mathematics, mathematics teaching practices, attending to student thinking, supporting student articulation of their thinking and an increased valuing of collaborative work. Stronger professional collaboration can also be an outcome of COPs and PLCs, i.e. professional collaboration itself is both a means for and and an outcome of teacher learning.

However, the ICME-13 survey found that only about 20% of the studies they looked at explicitly considered whether, how and why collaborations were effective, that “very few studies have revealed unsuccessful collaborations” (p. 680), and that barriers to successful collaboration were largely attributed to teachers’ ownership of the process, to time and to institutional constraints. In addition, it has been shown that the focus of teachers’ interactions—broad enough to allow for disagreement, but focused enough to develop shared understandings—and how teachers interact with each other—with respect, challenge and trust—is important for successful collaboration (Brodie & Shalem, 2011; Katz et al., 2009). While many communities develop these attributes and support growth among their members, others may become unproductive, if some of the important processes break down (Maloney & Konza, 2011; Schaap et al., 2019).

In this chapter, we focus on the critical role that resources play in teacher professional collaboration, noting that which resources are available and how they are used may play an important role in the success of collaborations and teacher learning from them. We focus on collaborations in which the purpose is to engage in extended inquiry into teaching and learning, but suggest that our framework can be used more broadly to think about other forms of collectives and collaborative work.

9.3 Conceptualising Resources in Teacher Collaboration

The nature of resources that support teacher professional collaboration is an important theoretical and practical concern for designing and researching teacher collaboration. A number of different theoretical perspectives have been used to research teacher learning through professional collaborations, each with associated terms and concepts for resources (Theme D, Chap. 5, this volume). Terms such as 'tool', 'artifact', 'instrument', 'document' and 'resource' are all used and point to particular theoretical positions. In this section, we review some of the important theoretical perspectives that have informed our work and argue for our use of the term 'resource'. We also look at the extent to which the 'materiality' of resources is important.

Vygotsky initially conceptualised tools as mediating between subject and object, or between the learner and what is to be learned. Tools act externally on the object to be learned, and/or internally on the subject, with a tool acting internally called a psychological tool or a sign (Vygotsky, 1978). For him, the notion of tool comes from material tools, which we use to achieve material ends and gain control over our physical environment. However, as people, especially when learning, we both imbue material tools with symbolic meaning and create symbolic tools, which mediate the objects of our learning semiotically (Bartolini Bussi & Mariotti, 2008). Tools that mediate and support learning have a psychological function, i.e. they change cognitive functioning, and tools that look similar can nonetheless function differently, depending on how they are used. The most powerful tool, which acts both internally on our minds and externally on other people, is language. So, for Vygotsky, tools both are material and go beyond the material—what is important is where and how they act. Crucially, objects and tools do not exist outside of social relations and meaning-making.

Building on Vygotsky's germinal work, Activity Theory broadened the notion of tool use to include artifacts which draw their meaning from and contribute to activity, mediating between subject and object in the context of a community (Engeström, 1999). Activity Theory also broadens our understanding of social contexts, by considering the division of labour and the rules, or norms, of practice in the community, as important interactors with the semiotic mediation between subject and object using tools or artifacts. In this way, Activity Theory supports an institutional focus, necessary for understanding learning in schools. It distinguishes between object and outcome, where object is the focus of or motivation for the activity, while outcome is a product of the activity. In the case of learning, focusing on the same object of learning with different artifacts can produce different outcomes for different learners. Researchers working with Activity Theory use the terms tool, artifact and instrument somewhat interchangeably, and these are both produced in and organise various aspects of activity systems and how different activity systems work together to produce learning (Akkerman & Bakker, 2011; Perks & Prestage, 2008).

Vygotsky and Activity Theory provide us with strongly elaborated theoretical perspectives on which to base our understanding of the use of tools in collaborative teacher learning. Some important principles underlie these perspectives:

1. we need to distinguish between the tool/artifact and the object of learning it mediates, and to see distinctions and relations among tools, objects and outcomes;
2. as learning happens, objects and outcomes of learning shift and new tools are employed to mediate new objects;
3. prior objects and outcomes can become tools for further learning and so we talk about the evolving use of tools as lived and living supports for learning;
4. the use of tools and artifacts to mediate learning relies on people assigning meaning to them—they only function in relation to their meanings and salience for the learner—and these meanings are shaped by the broader contexts in which tools and artifacts are used;
5. tools and artifacts mediate learning because they support us to see both the objects of learning and ourselves in new ways.

There are two important additional points. First, the distinction among task, object and outcome is not always realised in mathematics teaching, nor in mathematics teacher collaboration. Many teachers and learners see the completion of the task as the outcome of the activity itself, rather than as an object mediating the outcome which is, for example, developing mathematics concepts (in mathematics classrooms) or deepening the practice of inquiry (in teacher collaboration). Success is often seen as the completion of the task, which might constrain the depth of learning in collaboration (Prediger, 2020). Second, we note a further elaboration of the notion of mediation, into three kinds: mediation of the object of the activity, aimed at getting to know the object and also acting upon it; interpersonal mediation, oriented toward others and aimed at both knowing others and acting in interaction with them; and reflexive mediation, through which the subject's relation to themself is mediated by the instrument (Theme D, Chap. 5, this volume).

Vygotsky and Activity Theory illuminate the use of tools and artifacts in learning, from the perspective of learning. A more recent theorisation—the documentational approach to didactics (DAD)—starts with the notion of tool or instrument, and presents a framework for understanding how teachers use resources, individually and collectively (Gueudet, 2019; Gueudet & Trouche, 2012a, b). Focusing on online resources, they argue that resources made available to teachers are combined and transformed by teachers as they use them. These transformed resources are called documents, which bear teachers' schema of use, including their intentions and inferences. The range of documents developed by teachers are organised into document systems. Two key processes are studied: instrumentation—how resources shape teachers' activity—and instrumentalism—how teachers shape resources—in document systems. So resources are living documents that act in various ways in relation to those who use them, and support and extend meaning-making, making this approach consistent with Vygotsky and Activity Theory. What DAD adds is the idea that resources act as systems, although Gueudet (2019) notes that these systems are difficult to identify and suggests a focus on 'pivotal' resources.

Drawing from the discussion above, we are ready to clarify our use of the term 'resources'. Writing about resources in the context of school instruction, Cohen et al. (2003) argue that research has shown weak links between the presence of material resources and student outcomes. They argue for the study of resources to be embedded in the study of instruction, taking into account a broader range of attributes of classroom instruction, including the knowledge and orientations to learning and teaching that teachers and students bring to the classroom. So, together with them and Adler (2000), we prefer the term 'resources' to 'tools' or 'artifacts', because resources go beyond material resources, as tools and artifacts have often been thought of, to include social and human resources. We believe that we use resources in similar ways to how the DAD uses documents, in that resources both shape and are shaped by teachers as they use them. We consider this meaning of resources as central for designing and researching teacher professional collaboration and have gone further in our framework to talk about knowledge, affective and institutional resources, including time and space, as key resources for teacher professional collaboration. We work with the notion of lived resources shaping and shaped by their users in practice.

We do not distinguish between material and non-material resources, because the materiality of resources refers to their form, rather than their function, and most material resources, as well as many non-material resources, serve to represent practice in various ways, so we refer to these as representational resources which can be supported in different formats. We do not distinguish between online and other resources, again because it is the function of the resource that is important for us. However, we note that assumptions about access to digital and online resources may perpetuate inequalities. As noted in Theme D (Chap. 5, this volume), when working globally, we cannot assume universal access to digital resources, particularly in the two contexts of the authors of this chapter: South Africa and the United States.

Our final theoretical perspective brings together how we use resources in relation to teacher collaboration. Wenger's (1998) theory of situated learning defines learning as participation in communities of practice, producing and constituted by meaning and identity. Wenger's work requires understanding of the use of resources in communities of practice, through mutual engagement in relation to a joint enterprise and creating a shared repertoire—a set of resources which support enagement in relation to the joint enterprise. Communities and their resources are socially and politically situated.

Wenger introduces the notions of participation and reification. Participation refers to the active, living aspect of interactions and collaborations, as people engage with each other and learn. Reification denotes the products of such participation, which might be ideas, language, processes or material products, which then inform future participation. Participation and reification continuously revivify each other and therefore create living and evolving resources and learning practices. For Wenger, identity is key to learning: learning is as much a becoming as a coming-to-know, and teachers' identities are key in how they shape and are shaped by different resources and how they learn through professional collaboration.

A situative view of professional learning (Putnam & Borko, 2000) is informed by: how resources move and change across the different sites of professional learning—predominantly the community and the classroom (Akkerman & Bakker, 2011; Kazemi & Hubbard, 2008), i.e. how they are used for collaboration and from collaboration; and how they function for teachers in particular contexts because of the meanings ascribed to them (Cobb & Jackson, 2012). For example, if teachers tend to review student work to determine whether students have 'correct' or 'incorrect' understanding, they are likely to orient similarly to student work that is introduced in a collaboration context. So, if a goal of collaboration is to orient differently to student work, for example to gain insight into how students are thinking rather than simply evaluate their thinking as correct or incorrect, then careful attention needs to be given to the different orientations that teachers bring to the collaboration and how to work with them.

9.4 A Framework for Resources in Collaboration

Drawing on the above, the following principles underlie our framework: resources are socially and historically situated; they mediate the object of inquiry and therefore lead to learning to produce different outcomes; their meaning and functioning depend on the contexts of their use; they emerge from and support activity systems and practice; they are shaped by and shape participation in practice, and support reifications as products of learning; and prior reifications can become resources for further collaborations, learning and practice.

In line with these principles, we distinguish five kinds of resources that support teacher professional collaboration: representational resources; knowledge resources; affective resources; human resources; and institutional resources. These resources support teachers' interactions and learning with each other. We describe these in more detail below, but we note here that some have argued that our conceptualising of resources may be somewhat broad (L. Trouche, personal communication, February 4, 2020), with institutions providing affordances and constraints, rather than resources, and knowledge being an outcome of collaboration, rather than a resource.

The designation of people as resources also may not sit comfortably with some, as people interacting with each other are seen to use, rather than be, resources (Trouche, 2020). We prefer the broader use, together with Adler (2000), who indicates both knowledge and human resources as important. Given the stark inequalities across contexts, we argue that what institutions provide should be seen as resources—indeed, they are resources. Access to other people, for example teacher educators or researchers, may be constrained in some contexts and these people bring resources to, and are themselves resources for, teacher development. We also argue that affective resources are crucially important in learning, and have been under-theorised in prior research on teacher collaboration, so we want to put this squarely on the map here.

9.4.1 Five Kinds of Resources

9.4.1.1 Representational Resources

This category refers to what are usually thought of when referring to resources for collaboration, namely various representations of practice, which support teachers to inquire into teaching and learning, for example: learners' work; lesson plans; textbooks; assessment items; learners' errors; classroom scenarios or actual lessons. These may come from actual classrooms, including teachers' own classrooms, or be designed specifically for teacher collaborative work. They may come in different formats, such as print and digital. Specially designed digital forums for teacher collaboration are included here. Existing online platforms, such as Moodle, Google Docs or Facebook, are included if used for collaborative purposes (e.g. Anderson, 2020). Chapter 5 in this volume, representing the work of Theme D at the Study Conference (see also papers from Theme D in Borko & Potari, 2020), has many creative and inspiring representational resources, and the authors discuss how these have and might work to support teacher collaboration, learning and practice.

A key resource often used in collaboration, different from the others in this category, is protocols that guide the collaboration. These were not discussed in Theme D in the conference. We discuss them briefly later, in relation to how they were used by facilitators as to human resources, but we do believe they are deserving of more study as representations of the collaboration itself (Andrews-Larson et al., 2017; Segal et al., 2015).

9.4.1.2 Knowledge Resources

The knowledge resources for mathematics teachers include mathematical knowledge and pedagogical content knowledge (Ball et al., 2008), as well knowledge of their and their learners' families and communities. We also include teachers' orientations to teaching and learning, and their visions of high-quality instruction as part of their professional knowledge (Munter, 2014).

Kazemi and Hubbard (2008) distinguish between knowledge and knowing, where knowing is seen as knowledge-in-use, namely how our actions and practices are related to what and how we know. Teachers come to their collaborations with knowledge and ways of knowing developed in practice, further develop these through working with representional resources in collaboration, and their knowledge and knowing further inform future practice and collaboration. Knowledge resources shape and transform representational resources, which in turn shape and transform knowledge resources. Developing professional knowledge requires on-going, sustained inquiry, with a shared object of learning.

9.4.1.3 Affective Resources

All learning involves emotions. For teachers to teach well, they need emotional sensitivity to themselves and their learners. For teachers to work together productively, they need to be able to challenge each other's thinking and practices in generative ways, going beyond "contrived collegiality" (Hargreaves, 1991). Safety and trust are important to be able to learn with others, and emotions such as fear and anxiety might work against collaborative learning, while emotions such as empathy might support it.

9.4.1.4 Human Resources

These include all the people involved in collaborative groups and supporting professional learning. People form an overarching category, since they make meaning of and draw together the other key resources. Teachers often interact with facilitators in their collaboration. School principals and district advisors can also be included here, as they support collaboration, linking human and institutional resources. We include learners in this group and will elaborate further below. Teachers' identities and dispositions are key to their learning and collaboration, and are included in this category, as is their ownership of the process of collaboration (Robutti et al., 2016). While this key resource is the focus of Theme C (Chap. 4, this volume), it is also central to our framework.

9.4.1.5 Institutional Resources

Here we refer to the resources that schools and districts make available to teachers to collaborate. These include the provision of time and space for professional collaboration, whether and how the time is scheduled into teachers' work, whether appropriate space is (made) available and whether the importance of collaborative teacher work is acknowledged and valued. Institutional resources include the expectations of school and district leaders for such work, the extent to which collaborative work is seen a priority for teachers and the hierarchies that exist among teachers in schools and in the collaborative context. Institutional resources also refer to how teachers perceive their obligations to the institution (Chazan et al., 2016) and how their views play out in their collaborations. Institutional resources refer both to the very real material resource constraints experienced in many contexts, and to the ways in which limited or adequate resources are made available for collaborative teacher work. This category draws attention to the deeply cultural, political and historical contexts in which teachers' collaboration occurs across different contexts, and the many inequalities that still pervade our school systems.

In summary, institutional resources frame the work that happens in collaborative contexts and the relationships among actors and resources. All of this work is situated in social, cultural and political contexts, mediated through the institutional contexts, and these contexts are fundamental to which resources can be and are used and how they are used. The human resources that we are directly concerned with in the collaborative context are the teachers and facilitators, but other human resources such as school and district leaders are key and, as noted earlier, their values, priorities and expectations contribute to the institutional context of the collaboration.

Representational, knowledge and affective resources form the core of the actual collaboration among teachers, facilitators and the content. While many of Theme D papers discussed at the ICMI 25 Study Conference (Borko & Potari, 2020) do relate the use of representational resources to knowledge resources, there is little focus on institutional and affective resources. We discuss each of the resources categories further below, with examples from our two research contexts.

9.4.2 Resources in Collaboration and Practice

We now develop our framework in more detail, noting two key points made earlier. First, resources travel between teacher collaborative groups and classroom practice and, as they travel, they are transformed for use in the different contexts. This is particularly the case for representational and knowledge resources but, we will argue, it is also the case for the other resources in our framework. The processes of adopting, adapting or redesigning resources across contexts are not simple, and are deserving of our attention and study. Second, in many cases, missing resources, particularly but not only institutional and representational resources, contribute to unequal opportunities for teacher development and for classroom learning and, ultimately, to inequities in society. This is most often observed in lack of access to online and material resources, but may apply to some of the other resources as well, as we will show below.

Kazemi and Hubbard (2008) make a distinction between unidirectional and multidirectional analyses of teacher learning across professional development and classroom settings. From a unidirectional perspective, researchers assume that the intent of professional learning is to 'learn', i.e. accrue knowledge, within the professional learning setting, and to then 'apply' this learning in the classroom. Current design and research takes less account of how teachers bring their ideas and ways of knowing from the classroom to professional development. From a multi-directional perspective, this direction is equally important—it is assumed that teachers' participation both in the professional learning and in the classroom settings is transformed, as they are "engaged in knowing in both contexts" (p. 432) and bring their knowledge and knowing across contexts.

A multi-directional perspective does not mean that all participants orient and learn in the same way in professional learning. Rather, teachers participate differently in the different contexts, for good reasons. In our terms, the varied *resources* that teachers bring to bear on their participation, and the *resources* associated with the professional learning and classrooms settings, shape participation.

As Kazemi and Hubbard argue, a key challenge both for professional learning facilitators and for researchers is to learn about the meanings that teachers generate in relation to representational resources, like lesson plans, student work or curriculum materials, in and across contexts. We expand this to take account of all the resources in our framework, so that we can leverage what is known about teachers' sense-making across contexts, as they design new resources and forms of activity.

9.5 Our Research Projects and Contexts

We develop our framework further by drawing on specific projects that we have each worked in, Karin in South Africa and Kara in the United States. The two contexts enable us to highlight important contextual similarities and differences in teachers' use of resources for collaboration. In what follows, we first briefly introduce each of the projects, and then elaborate as we apply the framework to each of the contexts.

9.5.1 Data-Informed Practice Improvement Project (DIPIP), South Africa

The Data-Informed Practice Improvement project (DIPIP) (2011-2014) was located in Johannesburg, the major urban area in South Africa. South Africa is the most unequal country in the world, as measured by the GINI coefficient, and our education system reflects this inequality, with the vast majority of learners located in schools that struggle to support their learning (Motala et al., 2012). Poor achievement in mathematics is widespread and strongly correlated with race and socio-economic status (Spaull & Kotze, 2015). The teaching profession is not well respected, teachers are not well paid and there is substantial 'teacher-bashing' in the press. Teachers, particularly those in low socio-economic status schools, where the DIPIP project worked, often have high teaching loads and teach large classes, and teacher morale is low. There are often strong hierarchies and little trust among various levels of the system, with government, principals, teachers and parents often blaming and judging each other for the widespread low achievement of learners. As with many other countries, there is little tradition of collaborative teacher work in South Africa.

One of the aims of the DIPIP project was to understand what is possible for mathematics teachers' professional learning in PLCs in such a context. A key characteristic for successful collaboration is a focus, or object of inquiry, that is both narrow and broad enough to allow for substantive discussion and sustained learning (Katz et al., 2009). In the DIPIP project, we chose the object of inquiry to be the reasoning underlying learners' mathematical errors. The assumption, based on the substantial errors and misconceptions research, was that systematic errors are built on partially valid mathematical reasoning and that making that reasoning explicit for teachers and learners can help both to value learners' current mathematical thinking and to develop more robust mathematical thinking (Smith et al., 1993). The focus on errors was a mechanism to access three important dimensions of teaching and learning mathematics: learners' thinking, which makes sense to them and can be worked with, even (and especially) when partially correct; teaching practice, which can work with learners' errors and thinking; and teachers' own knowledge, both content and pedagogical content knowledge. Errors can be seen as absences, as they often are viewed by many teachers, or as presences, as a resource for future learning—and in this way they can become a knowledge resource for teachers.

Teachers were supported by facilitators to work on a set of developmental activities to develop their knowledge of the reasoning behind learner errors in the various sites of teaching. Although the activities were set up before the project started, we built in areas of choice and flexibility for PLCs. A key area of flexibility built in from the start was to choose the mathematics content they would work with, based on analyses of their learners' errors. As they became more familiar with the project, they chose to repeat some activities, to leave out some or to change the order, in consultation with the project team. The project can therefore be seen as somewhat adaptive (Koellner & Jacobs, 2015), since the model specified some key parameters, but also allowed for flexibility in relation to local contexts.

Teachers analysed tests, which provided an overview of strengths and weaknesses in learners' mathematical knowledge in a particular school, grade or class. Based on the test analysis, they chose learners who had made interesting errors that they wanted to understand more deeply, and interviewed them. They then took the results of these two analyses and mapped them against the curriculum, working out when and how the key concepts were taught and what curricular issues might have contributed to the errors.

Based on these three activities, teachers chose a leverage concept, which is a concept that underlies many of the errors that learners make in a topic, for example: the meaning and use of the equal sign. Once a concept was chosen, the DIPIP project facilitator found literature on that concept, including learner errors on it. The PLCs read and discussed these papers and drew on these discussions to plan lessons together. The lessons aimed to surface learner errors in regard to the concept and to find ways to engage the errors, rather than avoid them. These lessons were taught and videotaped and the community then reflected on episodes in each teacher's lessons, in order to understand their strengths and challenges in dealing with learner errors in class.

Our research team, consisting of graduate students and post-doctoral fellows, researched the teachers' collaborations by analysing their conversations in the communities and their teaching practices across the four years of the project. We will draw on the various analyses and what we have learned to exemplify parts of the framework (Brodie, 2013b, 2014, 2021; Brodie & Chimhande, 2020; Brodie et al., 2018; Chauraya & Brodie, 2017).

9.5.2 *Middle-School Mathematics and the Institutional Setting of Teaching (MIST) Project, United States*

The Middle-School Mathematics and the Institutional Setting of Teaching (MIST) project was a design-based research project (2007–2015) in which mathematics education and education leadership and policy researchers partnered with leaders in each of four, large educational jurisdictions ('districts') located in US cities, with the goal of investigating and supporting instructional improvement in middle-grades mathematics at scale (Cobb et al., 2018).

As described in detail elsewhere (Cobb et al., 2013), the team purposefully recruited districts which were typical of large, urban US districts in many respects; they faced a number of challenges including limited funding, high rates of teacher turnover, high numbers of novice teachers in any given year and significant numbers of students identified as 'low-performing'. However, the partner districts were atypical in one important respect—their response to high-stakes accountability pressures to increase the performance of students, especially students from historically under-served communities (e.g. students of colour, students living in poverty, students receiving special education services or students for whom English is not their native language), on standardised assessments.

Rather than 'teach to the test', which targeted low-level understandings, these leaders aspired to support all students to develop conceptual understanding of key mathematical ideas, procedural fluency and proficiency in problem solving. And they recognised that supporting students to meet rigorous learning goals would require both high-quality curriculum materials and professional learning for most teachers. District leaders provided a number of professional learning supports for teachers, including one-on-one coaching sessions and district-wide professional development sessions, while—especially relevant to the focus of this ICMI study—in all schools and districts, teachers were provided with regular time to collaborate. The amount and regularity of the time to collaborate varied across schools and districts. For example, in one district, grade-level teachers were expected to collaborate each day for 45–60 min, while, in others, teams were provided 45–60 min to meet on a weekly basis. In some schools whole departments met, while in most schools teams met in grade-levels.

The MIST research team engaged in annual cycles of data collection, analysis and feedback. At the beginning of each school year, the team interviewed central office leaders to document their intended strategies for the year. They then collected and analysed data on how the various strategies were playing out in schools. Between 30 and 60 teachers in each district were interviewed each year, as well as coaches and principals, while video-recordings of classroom instruction and audio- and/or video-recordings of professional learning, including teacher collaborative time, were collected. At the end of each academic year, the team shared findings with the district leaders and made recommendations regarding how to improve the strategies for the coming year.

Different from the DIPIP project described above, members of the research team did not directly facilitate teacher collaboration, although in the second half of the project, team members did co-design and co-lead professional development for facilitators. Within the MIST project, mathematics educator Ilana Horn led the study of teacher collaborative time across districts. In what follows, we especially draw on findings that she, postdoctoral fellows and graduate students produced (e.g. Andrews-Larson et al., 2017; Horn et al., 2017, 2015, 2018).

9.6 The Resource Framework

We now indicate how the five kinds of resources supported or constrained the teachers' collaborative work in the two projects. We show how the different resources work in concert with each other and, where applicable, their co-evolution across classroom and collaborative contexts (Kazemi & Hubbard, 2008), and we suggest how this framework may inform design and research more generally.

9.6.1 Representational Resources

Although collaborating with colleagues about teaching has become more common in recent years, by-and-large classroom instruction remains a private endeavour. Representations of practice, for instance, accounts of teaching such as verbal 'replays' of what transpired in a given lesson (Horn, 2010), student work (Kazemi & Franke, 2004), lesson plans (Ball & Cohen, 1999; Lampert, 2001) and/or video-recordings of teaching (Borko et al., 2008; van Es & Sherin, 2008), are therefore a critical resource for teacher professional collaboration. Representations of teaching comprise the shared text of collaborative inquiry (Grossman et al., 2009). As Grossman and colleagues argue, representations of practice "can vary significantly, both in terms of comprehensiveness and authenticity" (p. 2065)—and the nature of the representations shapes community members' opportunities to learn.

Engaging with a representation of practice is an interpretive, meaning-making act, and is influenced by the other resources we discuss in this chapter. For example, what particular members see, or notice, in a representation of practice is shaped by their current mathematical knowledge for teaching, as well as their perspectives on teaching and learning (Jackson et al., 2019). And whether community members treat a representation of practice (for example, student work) as an opportunity to investigate students' sense-making, as opposed to an opportunity to triage students for further tutoring, may be shaped by principals' expectations and broader institutional discourses (Rigby et al., 2020).

Not surprisingly, all of the papers at the ICMI 25 Study Conference in Theme D (Borko & Potari, 2020) provide some form of representational resources. Many of these used some form of digital and online forums. Of the 18 papers presented, only five had no digital input (not counting classroom videotapes as digital). The benefits of digital formats are that they more easily support the collaborative design and redesign of resources by teachers, and may support access to resources not possible in other ways (Karima, 2020). However, Chap. 5 (this volume) argues that access to digital resources cannot be taken for granted in many contexts and, if we are not able to support access for all teachers and learners, inequities may be exacerbated.

In the South African project, the teachers analysed tests, learners' responses to tests, interviews with learners and videotapes of their own lessons, and created tasks and lesson plans, which were also subsequently analysed and improved upon. Taken together, these resources focused teachers on the key representation of practice that we used in our project, learners' errors and the reasoning underlying them. Conversations in the communities varied in content and depth (Brodie & Chimhande, 2020), and teachers' use of learner errors as a resource for teaching also varied (Chauraya & Brodie, 2017). The focus on errors was useful for some teachers and not so for others (Brodie, 2021), as can be seen in the following quotations.

> Being able to get the reasons behind learners' answers, I can now at least try to ask them [. . .] to keep on probing the learners until they realise their mistakes.

> We analysed question by question, concept by concept, that's where I saw that maybe, somehow, we are short-changing our learners. That was quite a rude awakening, *ja*, and I hope that I could use that even in the other subjects that I teach. Because I think it would have a far-reaching positive impact.

> Where, what, how do I benefit from this [...] it was nice arguing, identifying some of those errors made by learners, trying to think why they made these mistakes, how, why, you know, those different views from learners, people justifying those wrong answers. It was fun, it was fun, but you know, in as much as it was fun [. . .] I couldn't link what we were doing with what we are doing in the classrooms.

The first two quotations suggest that the focus on errors supported teachers to see their learners differently and possibly become more responsive to them. The third quotation suggests that, for this teacher, a focus on more immediately applicable resources for teaching may have been more useful than our plans for a longer-term developmental sequence of activities.

In the US project, the focus of the groups varied somewhat. Common foci included lesson planning and analysing student data, and a small number of groups focused on investigating and developing specific forms of teaching practice, e.g. how to introduce rigorous mathematical tasks. As such, representational resources tended to include lesson plans and curricular materials, as well as students' responses to interim district- and school-based assessments. Whereas these representations were common across groups, *how* teachers engaged with these resources varied in crucial ways.

In productive teacher collaboration, as Horn et al. (2017) argue, teacher groups engaged in conversation about the 'how' and 'why' of instruction, in relation to the representational resources. For example, when discussing student assessment data, they examined the details of *how* students responded, like noticing patterns in students' incorrect responses. And, crucially, they then engaged in an analysis of *why* those patterns might exist. They connected an analysis of the patterns of student thinking with an analysis of what was emphasised, or not, in instruction. These connections laid the groundwork for discussions of *future* work; that is, what teachers might do differently in future instruction to support students to develop desired understandings.

Similarly, when discussing curricular materials in light of lesson planning, in productive collaborations, teachers identified the key mathematical ideas and practices they wanted to target in a given lesson, and they anticipated student responses. They dug into *how* particular instructional decisions might shape what students would learn, and they made public their *whys*, that is, their rationales for why they might choose to make a particular decision. This airing of rationales supported teachers to weigh alternatives, and to make principled choices about what they would do in the future.

Unfortunately, conversations in which teachers examined the 'how' and 'why' of instruction were rare. Horn et al. (2017) studied the conversations of 24 work groups across 16 schools (a total of 77 meetings). Instructional leaders nominated these work groups as 'best-case scenarios', yet still Horn and colleagues documented rich learning opportunities in only about one-third of the meetings. When analysing standardised assessment data, by-and-large conversations tended to focus on determining if students 'got it' or not, as well as identifying students for tutoring or remediation, apart from an inquiry into why students might not have 'gotten it' to begin with (Horn et al., 2015). And conversations about lesson planning tended to focus on pacing or logistics, rather than discussion of mathematical content or students' ideas.

The findings from the two projects suggest that the focus and quality of representational resources shape teachers' learning opportunities within collaborative contexts. Across both contexts, we see the importance of representations that have the potential to support teachers' conversations about learners' reasoning, and to connect insight into learners' reasoning to instructional decisions. In the DIPIP project case, the specific representations of learners' errors (assessment data paired with interviews about students' thinking) provided teachers with access to learners' reasoning; and video-records of instruction and lesson plans supported teachers to connect learners' reasoning to specific instructional decisions they made. Similarly, in the MIST case, teachers identified patterns in students' responses and then conjectured what about instruction might have contributed to those patterns.

The design of both projects supported multi-directional movement of resources between classroom and collaboration. In the DIPIP project, teachers engaged in cycles of work in which they planned lessons in the collaboration, tried them out in their classrooms, returned to collaboration with resources and, on the basis of their analyses, planned future lessons. Similarly, in the productive examples in the MIST project, teachers engaged in cycles of inquiry, in which analyses of records of practice in collaboration informed their future lessons. While the two cases illustrate the centrality of representational resources in teacher collaboration, how teachers engaged with representational resources was very much shaped by the other categories of resources discussed below.

9.6.2 Knowledge Resources

Professional learning builds on, challenges and produces knowledge. This knowledge can be local, such as data or experiences from practice and knowledge of learners and their communities, and global, as in research findings and ideas for best practice. Successful professional learning relates local and global knowledge (Jackson & Temperley, 2007; Katz et al., 2009). Mathematics teachers' knowledge is a key resource in collaborative learning—what teachers bring to their collaborations will inform the collaboration and its outcomes, and knowledge is obviously a key outcome of collaborative learning. Thus, knowledge can be both resource and object, depending on the situation. Teachers' mathematical knowledge for teaching is best described as a combination of mathematical content knowledge (CK) and pedagogical content knowledge (PCK) (Ball et al., 2008).

In addition to mathematical knowledge for teaching, we also include in this category teachers' professional visions (Hammerness, 2001; Munter, 2014), including what they aspire to accomplish in their practice, and their views of their students' current capabilities (Jackson et al., 2017). Both shape teachers' instructional choices (Horn, 2007), including how they select and interpret representational resources, and how they enact instruction (Wilhelm, 2014; Wilhelm et al., 2017). Moreover, teacher collaboration is often intended to support teachers to develop a shared instructional vision—one that is attuned to broadening access to all learners, especially in the context of a reform effort. Thus, just as with mathematical knowledge for teaching, teachers' instructional visions and views of their students travel between class and collaboration, and may be transformed through this travel.

The DIPIP project design took a particular view of the development of teacher knowledge and practice. The assumption was that teachers tend to be most focused on what they do every day in their classrooms, i.e. their practice and their PCK. Therefore, the best way to draw on and develop teachers' knowledge in an integrated way is to start with their practice and PCK, and to develop CK in relation to them (Brodie & Sanni, 2014). This is different from many PD programmes that start with CK and then move on to related PCK. One of our key principles was that, in coming to understand learner needs, teachers can come to understand their own learning needs: what mathematics they need to learn and how to use this new knowledge to improve their practice (Brodie, 2014). All teachers notice learner errors; however, very few teachers see them as based on valid reasoning and as opportunities for deepening mathematical knowledge. We explicitly positioned teachers both as experts and as learners. As experts, they contributed their knowledge of teaching mathematics, their contexts and their learners, while, as learners, they deepened this knowledge.

We have some evidence that the focus on PCK supported conversations both about PCK and about CK. In one community, we found that 34% of conversations were on CK and 66% on PCK, depending on the particular activity (Brodie et al., 2018). Of the 58 CK conversations across the 17 meetings, 30 (52%) were triggered by PCK conversations. Of the 161 PCK conversations, 23 (14%) were triggered by CK conversations. Thus, PCK conversations did lead to CK conversations, while the converse was less visible (Brodie, 2014; Marchant & Brodie, 2016).

We also found that the knowledge content of the conversations in the PLCs varied in relation to the different activities. For example, the lesson planning sessions supported conversations on mathematical content, as teachers tried out the tasks, and the videotaped reflections supported conversations on practice as teachers discussed responses to learners' errors (Chimhande & Brodie, 2016; Marchant & Brodie, 2016). Taken together, the different activities supported conversations on mathematics (CK), as well as on learner thinking and teacher practice (PCK).

In the MIST project, similarly to the DIPIP project, there was evidence that, while rare, teachers did occasionally deeply engage with the underlying mathematics, thereby providing opportunities to develop mathematical content knowledge. For example, Horn et al. (2018) describe how a sixth-grade team at Magnolia Middle School engaged in a three-week assessment cycle. In week one, teachers used interim student assessment data to identify ideas with which students were struggling, and then investigated the underlying mathematics, and developed a common formative assessment to give to students. In week two, the team analysed the resulting student work. Aided by a coach and assistant principal, both of whom had substantial mathematics teaching expertise, they investigated students' ideas deeply and designed instructional responses. In one case, the workgroup identified that students were having difficulty making sense of the concept of a unit rate.

In light of student thinking, the facilitators "introduced a new instructional strategy—in this case, a double number line—that teachers might use to support students in making sense of the difference between additive and multiplicative problems" (p. 102). In week three, teachers reported on the enactment of their instructional responses, and again reviewed student work to note progress and to identify next steps. Throughout, the facilitators supported and pressed teachers to position students as sense-makers, to treat mathematics as a sense-making activity and to deepen their mathematical knowledge for teaching.

Describing work groups like the Magnolia sixth-grade team, Horn et al. (2020) write, "As teachers identified, elaborated, and addressed instructional challenges, their understandings of students' learning difficulties and students' mathematical thinking [their views of their students' capabilities and mathematical knowledge for teaching] were conveyed to colleagues" (p. 11). In fact, they quantitatively investigated relationships between engagement in meetings like those in Magnolia (which they called 'high-depth' meetings) and teachers' mathematical knowledge for teaching and views of their students' capabilities. They found that teachers developed more productive views and deepend their mathematical knowledge for teaching through high-depth meetings. However, as noted earlier, high-depth meetings were unusual, even in the best-case sampling of teacher workgroup meetings in the MIST project.

Across both projects, we see that knowledge is a key resource in teacher collaboration. Teachers come to their collaborations with professional knowledge and this knowledge is transformed during the collaboration to produce new knowledge. Professional learning activities can support teachers to talk about their knowledge with each other, to build explicitly on their own and others' knowledge, to develop ways of seeing differently and more deeply, and to develop different relationships between different forms of knowledge and between knowledge and practice. In both the DIPIP and MIST project examples, the activity of examining students' thinking and reasoning on its own, and in relation to instruction, supported the deepening of pedagogical content knowledge and content knowledge.

In both cases, making visible the 'hows' and 'whys' of their teaching with colleagues also supported teachers' deepening of professional vision; that is, of what it is that teachers were aspiring and attending to in their own instruction, and in coming to view more of their students as capable of making sense of mathematics. As with all resources, how knowledge is transformed in collaborative conversations depends on how it is appropriated and developed in the collaboration, as well as how it is shaped by and shapes the collaborators. It is possible for conversations to become unproductive, when current knowledge, visions of practice and views of learners do not support growth and become solidified in the community, rather than flexible and open to change.

9.6.3 Affective Resources

While it may seem strange to think of emotions as resources, they are an important resource for collaborative learning. There is a strong history of research on emotions associated with learning mathematics (e.g. Black et al., 2009; Hannula, 2012), but a definite absence of the role of emotions in mathematics teacher learning (Breen, 2009; Vedder-Weiss et al., 2020). There are no emotion words in the frequency cloud of words in the titles of the survey sources in the ICME survey on mathematics teachers' collaboration (Fig. 9.1) (Robutti et al., 2016, p. 662), and a look through the volume *Tools and processes in mathematics teacher education* (Tirosh & Wood, 2008) does not show a focus on emotions. So emotions are a missing resource in the research, and may be important when thinking about building equitable participation in collaborative learning.

Fig. 9.1 Wordcloud from ICME survey. (Robutti et al., 2016, p. 662)

A supportive environment for collaborative learning requires space for professional disagreement and conflict among ideas, so that there can be generative conversations and space for growth (Katz et al., 2009). At the same time, if conflict becomes personal, possibilities for collaboration and learning are reduced. So two key features for collaboration to be productive are safety and trust. Safety to challenge and be challenged, to agree and disagree, and trust that the process will support everyone's learning, and that contributions can both be given and received. Research suggests that, where there are strong hierarchical relationships within schools and where teacher morale is low, it is difficult to sustain engagement in PLCs (Schechter, 2012; Wong, 2010). The following quotations from the DIPIP project show a range of trust that teachers experienced in their PLCs.

> We can talk to the other community members freely without, how can I say, stage-fright. We are confident because we are talking with colleagues, knowing that no-one is judging you.

> I don't know about videorising [*sic*]. I think we're coming from an era where people were critical about you, they were looking at all the bad things that you were doing [...] Now, we understand you're videorising[1] it so that we can see ourselves developing. But, somehow, at the back of our minds, it's like, is it true? Are they not hiding something from us?

> Because, at the end of the day, we would see it as a research which benefits someone and not us really.

A real fear about working with others is being judged, explicitly or without your knowledge. In some school systems, 'poor' performance can affect job security, and so being judged can have material consequences for teachers. However, it can also have emotional consequences, creating or reproducing doubts about not being good enough. Trusting that teachers and learners are the ultimate beneficiaries of the work

[1] She means videotaping.

may also be difficult for teachers working in schools and systems where their needs are not always taken seriously.

At the same time, communities that work well can support positive emotions associated with learning. A key role for faciliators of collaborative learning groups is to create and support safety and trust. In the DIPIP project, we articulated the following facilitator moves for building community: validate all ideas as useful contributions to the conversation; articulate positive aspects of negative situations; identify with teachers' issues and concerns; notice and counter disengagement and exclusion (Brodie, 2016).

One reason that we chose a focus on learner errors in the DIPIP project is that we wanted to support teachers to see learners as mathematical thinkers, to hear and interact with their thinking. As teachers began to interpret and explain learners' errors, and to see errors as reasoned and reasonable, they began to blame themselves, or other teachers, for learners' errors, finding the reasons for learners' errors in how they were taught previously. The PLC facilitators worked hard to counter this tendency to blame, but we saw it recurring in our data and believe that it is difficult to work against, given the widespread blame of everyone by everyone in our system, and the emotions associated with not suceeding in mathematics, which are made even more salient when focusing on errors. As one teacher told us:

> Because now it was somehow it was a bit painful. If you find that learners who were doing grade eleven, and learners who were doing grade eight, they were given almost the same test. But now, when you check the errors that were done by grade-eleven learners, they were the same as the errors that were done by grade eight.

Many of the teachers expressed similar pain when confronted by the fact that learners made the same errors as they progressed in their school careers. One teacher told us (half-jokingly) that, when you see learners making errors in grade eight, you can blame it on the primary school teachers, but when the errors persist when taught by herself and her colleagues, it becomes difficult to decide where to apportion blame. For some teachers, this was a learning and growth point—they realised that there is no blame for errors; they are a normal part of learning mathematics and we can develop strategies for working with them to develop learners' mathematical thinking. But, for other teachers, the explicit focus on learner errors was demoralising and contributed to their feeling overwhelmed. At the same time, many teachers talked about positive emotions that supported their engagement in the project. The first teacher quoted above declared how she felt confident in sharing ideas without being judged. Many teachers enjoyed that their communities provided safe spaces for sharing ideas and gaining new insights.

So, emotions have both positive and negative consequences for collaboration and learning. Emotions are also both a resource for and an object of inquiry, because an important part of teacher learning must be to focus on their own and their learners' emotions, particularly when dealing with errors. Emotions are transformed through collaboration and learning—understanding why and how is an important task for researchers.

Emotions travel everywhere with human beings, so they will manifest both in collaboration and in classrooms. In thinking about emotions as a resource, it may be important to focus on them more explicitly in the design of collaboration. For example, teachers' fears about being judged in the collaboration can be used as a resource for them to think about how learners fear being judged in the classroom. They may then be able to develop some classroom scenarios, based on classroom events, where they identify and respond to learners' emotions. They can also bring actual cases from their classrooms for discussion in the groups.

9.6.4 Human Resources

People are both actors and resources in teacher collaborative work and, as they bring together the other sets of resources, they function at an overarching level. As ICMI 25 Theme C notes, there can be a number of participants in collaborative teacher learning: teachers, facilitators, researchers and principals, among others, and they bring different strengths to the collaboration. From the perspective of Theme D, how people revivify resources for learning is important, as well as how they may become key resources for and from collaborative learning. We focus on teachers and facilitators, and also on students, as a potentially missing resource.

How teachers orient towards their collaborations is important in how they work and learn together, particularly given their customary privatised practice (Little, 1990). In the DIPIP project, Chauraya (2016) showed how teachers' views of collaboration shifted over time in a community. Initially, teachers saw collaboration as asking other teachers for help, particularly more 'senior' colleagues, reflecting the hierarchies in the schools. After some time in the community, the teachers began to see it as a forum for joint development, with everyone able both to give and to receive support, and, as this happened, they came to see themselves in new ways. A related finding occured when some project teachers wanted to know why the better-resourced school down the road did not come to share their knowledge in the community. While this can be interpreted as teachers assuming that better material resources means better knowledge resources, it can also be seen as understanding that extending the community beyond one school can be helpful, and that other teachers would bring in different resources (Brodie, 2013a).

Robutti et al. (2016) refer to teachers' ownership of their collaborations, where teachers feel that the leadership and responsibility of the collaboration lies with them, and that their needs and interests are being served. In collaborative groups initiated and guided by teachers themselves, such ownership might be easier to achieve, although it may be seen differently by different participants. In collaborations guided by teacher educators, we have to work carefully to develop and support teacher ownership of the process, the resources and their learning. Related to ownership is voice (Robutti et al., 2016), where teachers feel that they have the space to articulate their ideas and are confident that they will be heard and engaged with. For teachers to experience ownership and voice in communities, there needs to be some level of trust, among teachers and among teachers, facilitators and the project team. Trust is an important affective resource as discussed above. In the DIPIP project, some teachers did feel trust, and thus ownership and voice, while others did not, and withdrew from the project (Brodie, 2021).

Not all collaborative groups include a person tasked with being a facilitator. However, there is research to suggest that such leadership is important, particularly for systematic, regular inquiry (Katz & Dack, 2013). Across our two contexts, we found that a facilitator can be an especially important resource in teacher collaboration. In the MIST project, most teacher groups were allocated a facilitator. The faciliator was sometimes a 'teacher leader', e.g. a teacher who was assigned specific responsibilities by the school administration; a 'coach', e.g. a person designated to support teachers' learning, either by the school or district; or an administrator. The faciliator's expertise in teaching mathematics and in supporting teachers' learning, as well as their relationship with the teachers, mattered greatly for how they were perceived by the teachers.

The MIST project analyses of teacher collaboration indicated that a facilator could shape opportunities for teachers' learning in important ways. For example, Andrews-Larson et al. (2017) analysed audio-recordings of grade-level collaboration meetings in two schools, in which there was evidence that teachers exhibited growth in instructional quality. They observed that, especially in the context of co-planning for instruction, facilitators routinely pressed teachers to elaborate their verbal representations of their classrooms, to articulate evidence or reasoning behind their claims about students, to treat students as sense-makers and to surface the instructional principles behind their decisions.

Importantly, Andrews-Larson and colleagues found that, while protocols for organising a meeting appeared helpful in structuring the *questions* a facilitator might ask in collaborative time, a protocol did not guarantee that a facilitator would *press* teachers to elaborate their rationales and principles. As another example, in the Magnolia sixth-grade team example described above (Horn et al., 2018), we see the important role that the facilators—in that instance, a coach and an assistant principal, both with extensive expertise in mathematics teaching—played in supporting teachers to engage deeply with mathematics content and the ideas of their students.

While much of the research focuses on teachers and facilitators, and their use of resources, learners might be thought of as a missing resource in teachers' collaborative learning. Although learners are often represented in teacher collaboration through their work, in videotaped lessons and other representations of practice, teachers often do not see learners' perspectives when thinking about learner work. The DIPIP project focused on learner errors and, as we have discussed above, this supported some teachers to see learners differently, although with some pain and blame. But we did not find out about learners' views of their errors or about emotions that are evoked among learners when teachers work with learners' errors.

There are some examples of work with learner perspectives in relation to teacher learning. Vogler and Prediger (2017) captured students' views of a teaching situation on videotape and then showed them to the teachers, reporting how seeing students' perspectives helped the teachers to think about the consequences of their interactions with students and proposing how they might interact differently in future. In work subsequent to the DIPIP project, we have shown that different ways of valuing and working with learner errors are important for learners' mathematical identities (Gardee, 2019). Also interesting is a study by Sherman and Catapano (2011), where students participated in a mathematics club and in-service and pre-service teachers participated as mentors in the club. From interacting with their own students in a different context, the teachers came to see them as productive mathematical thinkers. So introducing actual learners as real people, with thoughts and feelings about their learning, may be an important missing resource for teacher collaboration and might support stronger collaborations and learning.

9.6.5 Institutional Resources

Institutions are key mediators of social and power relations and frame much of what happens in collaborative learning for teachers. How resources are allocated, the nature of school hierarchies, the extent to which collaboration is valued in the school and district, and school leaders' expectations for collaboration all form part of this framing. In arguing for a view of mathematics teaching that takes relationships between teacher agency and social and institutional structure seriously, Chazan et al. (2016) argue:

> It has been customary in our field's literature to look at resource use as an individual issue only, one that depends on individual teachers' access to resources or personal knowledge or beliefs about the use of resources. We contend that, in our efforts to consider the institutional context of the work of the teacher, we should also consider that the use of resources in a teacher's work also depends on the role that those resources play in framing the position of the mathematics teacher in its institutional context. (p. 1078)

Here, we focus especially on the institutional resources of time and space to collaborate, and school leaders' values and expectations.

9.6.5.1 Time and Space

Time is probably the scarcest resource for busy professionals. In the MIST project, time to collaborate was built into teachers' contracts. However, in the DIPIP project, this was not possible and all the teachers we interviewed stated that time was a key challenge to participation: finding time to meet as a group and finding time to do the work of the project outside of meetings, for example the project readings or personal reflections on videotaped lessons (Brodie, 2021). Given the high teaching loads in the schools, there was no time during the school day for teachers to meet. All of the communities met after school, which often clashed with teachers' personal commitments, such as picking up their own children from school. Some teachers spoke about making time for their collaborative work, for example:

> Because what I've also learnt is that here on earth there's no time, but one has to make time for anything after all.
>
> But as for me I consider it as pressure that I can't do anything about, other than finding a better way to deal with [. . .] I must just find myself time.

However, a number of teachers said that they could not find time and, when there was a clash between school commitments and project commitments, they had to prioritise their school commitments. For teachers with many, big classes, there was little time for other activities.

> You really find no time to give to this other project, because you know I have at least to give also my time to this because it's my contractual obligation [. . .] with the workload that we had, it's overwhelming, you know really, really it's overwhelming.

Teachers at wealthier schools are more likely to have smaller and fewer classes, and therefore more time for joint work and professional development. At some wealthier schools, time can be found within the school timetable for collaborative work, but in most schools in South Africa, teachers have to make use of their own time and personally prioritise their own development above other commitments.

Space is also related to resource inequalities. Most government schools in South Africa have little space that can be dedicated to teacher collaboration. Classrooms, libraries, staffrooms, laboratories and small offices were variously used as spaces in the DIPIP project, which meant that records of past meetings were difficult to display and all equipment had to be brought in for each meeting (equipment was also vulnerable to theft).

Not all collaborative groups need to meet face-to-face and online collaborations allow for teachers from different schools to meet and learn together. However, there are also inequalities between rich and poor in relation to access to the internet, for example data is very expensive in South Africa. It is not yet known whether virtual collaboration can support the kind of systematic learning that PLCs do, particularly when trying to design resources for multidirectional influences between classroom and collaboration. Supporting on-going, sustained inquiry among teachers in the same school, who can then make strong collective changes to their practices, and bring their issues from practice into the collaboration, is central to PLC work and requires appropriate spaces in which to work.

9.6.5.2 School Leaders' Values and Expectations

Another critical institutional resource concerns whether and how school leaders value collaboration and their expectations for what happens in collaboration. We saw a range of leadership support and consequences of this support in the DIPIP project (Brodie, 2021). Some principals actively supported the project while others spoke about trying to encourage their colleagues to participate, but they did not know much about the details of the project. For the most part in South Africa, principals' roles are conceived of in terms of management rather than academic leadership, leaving many principals themselves with few resources to support academic learning among their colleagues.

A key element of how leadership support played out was in the relationships between teachers and school leadership. Some teachers said that while the school leadership supported the project because it reflected well on the school, the teachers experienced challenges. The comments below suggest some conflict with the school administration.

> As an administrator, the picture you are giving, whether it's nice or not nice, you would want things to work for your school. So you would say, let's try it. But the people on the ground who were supposed to do it, they said, we can't. Because I still remember we had a meeting and it was a push and pull.

> Because, really, we were overwhelmed we were coming up with reasons, you know, sometimes you don't have to give your real reason, you try to look and source a polite way of saying.

These comments reflect the hierarchical nature of South African schools. Teachers are unlikely to express critical views, to explain their needs or to ask for explicit support from principals; rather, they might accept requests on the surface, but then reject and refuse them in other ways. Relationships between teachers and school leadership also relate stongly to trust and other affective resources as discussed above.

In the MIST project, the school leader's expectations for what happened in teacher collaborative time very much shaped the nature of conversations, and thus teachers' opportunities to learn (Horn, 2018; Horn et al., 2018; Rigby et al., 2020). Recall that time to meet was built into teachers' contracts in these districts. This meant that school leaders had some influence over the focus of teacher collaboration. And, especially given the context of high-stakes accountability, it was common for school leaders to press for teachers to use collaborative time to identify students in need of tutoring or remediation, with the hope that if those students received auxiliary support, they would perform better on standardised assessments. Futher, in light of the skills-oriented nature of the assessments at the time, a focus on improving student performance did not call for deep investigation of mathematics or student thinking.

In a case study of one school, Rigby et al. (2020) found that, over the course of three years, teachers' discussions across the three grade-level groups dramatically shifted; whereas 60% of their initial conversations focused on unpacking mathematical concepts and procedures, by the third year, 60% of their discussions focused on naming topics and standards to be targeted. Concurrently, researchers documented that teachers shifted from initially framing their work as about helping students learn mathematics to focusing on helping students to pass tests. And, not surprisingly, whereas in the first year, the majority of their conversations focused on how they might adjust their instruction to support students, by the third year, it was common to hear teachers attribute students' performance to factors that they perceived to be not in their purview (e.g. students' behaviour). Notably, the researchers explain these shifts through a focus on the role of the school leader. Administrators attended teacher collaborations frequently in the second and third year, as compared with the first one. Regardless of whether admistrators were the official facilitators, they consistently shaped the focus of the meetings when they attended, directing the conversation to focus on what standards students were 'low' in and identifying students for remediation in those standards.

There were exceptions. Recall the assistant principal at Magnolia Middle School described above, who, along with an expert coach, supported and pressed teachers to focus on the 'how' and 'why' of instruction. Summarising findings about school leaders' influence on teacher collaboration in the MIST project study, Horn et al. (2018) identified two critical roles that school leaders played in support of productive collaboration. One key role concerns furnishing teacher teams with facilitators with instructional expertise, as well as expertise in supporting teachers to collaborate. In some cases, this required leaders to seek a faciliator outside of the school community. A second key role concerns "protecting teachers' time during appointed meetings" (p. 106). In productive communities, school leaders ensured teachers were able to use collaborative time to inquire into instruction and plan for future instruction, as opposed to fulfilling administrative tasks.

9.7 Conclusion

Teacher collaboration holds great promise for supporting teachers' professional learning, as well as their intellectual and emotional well-being, and, in turn, students' learning and well-being. However, as has been elaborated in this ICMI Study, realising positive outcomes in the context of teacher collaboration is non-trivial. In this chapter, we have elaborated a framework that centres the *resources* that both shape and are shaped through teacher collaboration. We have intentionally taken a broad perspective on resources—considering representational, knowledge, affective, human and institutional resources—in an effort to support others to design, prospectively, for productive forms of collaboration, and to account for, retrospectively, what might have supported and/or constrained the possibilities of a particular collaboration.

As we have illustrated above, how representational resources are used productively (or not) in the context of collaboration is shaped by knowledge, affective, human *and* institutional resources. Thus, as we have argued, a limited focus on representational resources is likely to fall short in supporting design of good teacher collaboration, as well as in explaining why teacher collaboration resulted in particular outcomes.

We elaborated the framework in relation to two contexts, the DIPIP project in South Africa and the MIST project in the United States. As we argue above, collaborative efforts are necessarily socio-culturally situated. Representational resources are given meaning by people in communities, and they will be used differently in different contexts and will support different kinds of learning. Similarly, what counts as a knowledge resource in a given setting, or the kinds of human and institutional resources that are available, will vary across contexts. However, while the nuances and details of each of the categories of resources that we elaborate may vary across contexts, we argue that the categories, writ large, are important to consider when designing for or researching structured teacher collaboration across contexts. Here, we step back to consider key takeaways.

These two cases suggest the importance of ensuring that representational resources include representations of teaching, and that those representations of teaching provide access to learners' thinking and/or experiences, as well as to the ability to connect learner outcomes to the specifics of instruction. Representations of students' learning need to be textured enough (e.g. to include not just learners' answers, but also learners' reasoning) such that they can support teachers to generate and investigate conjectures about *why* learners might be thinking in particular ways and may have developed more or less useful forms of reasoning. And representations of instruction need to be detailed enough such that teachers can investigate together how instructional decisions may have shaped learner outcomes.

However, well-designed representations of teaching do not ensure productive collaboration. Teachers' current forms of knowledge and perspectives on teaching and learning matter for the kinds of questions they surface and explore in relation to their instruction. In designing for teacher collaboration, it is critical to attend to the forms of expertise that exist in the community, to consider carefully the role of a facilitator both in pressing and in supporting teachers to deepen the kinds of questions they are asking of their instruction, and to deepen their content knowledge and pedagogical content knowledge. It may be that an insider can take on the role of facilitator, but, especially in communities early in their development, an outside faciliator may be needed. As suggested above, protocols can support facilitation, but they cannot substitute for the judgment of a facilitator, the know-how of when and how to encourage new voices, when to press teachers to elaborate their reasoning, and when to insert new ideas. Although not discussed in depth here, learning to facilitate teacher collaboration is a key consideration in the establishment of generative teacher communities.

What we have termed 'affective resources' are central both in designing for and in researching teacher collaboration. Creating a setting in which teachers lay bare their instructional challenges and genuinely collaborate in making sense of them and in proposing alternatives, requires that they trust and value one another. Unfortunately, as was true in both of these cases, teaching often takes place in contexts that are marked by scrutiny and mistrust. Therefore, there has to be intentional work done to establish and sustain a community that explicitly works towards ways of relating with one another that are likely counter-cultural.

The extent to which it is possible to realise sustained, productive collaboration depends largely on institutional resources, including time and space, and instructional leaders' expectations for what happens in collaboration. As much as is possible, attention to these resources at the start of a project is adviseable, and consideration of the institutional resources is crucial in researching collaboration, in order for others to consider the applicability of findings in new contexts.

The DIPIP and MIST projects were both focused on formal teacher collaboration, and much if not all of the literature on which we drew focused on formal teacher collaboration as well. Teachers' joint work, or collaboration, can happen in a number of ways: from informal conversations to asking for help, to marking or planning together, through various forms of systematic learning in groups or PLCs, like those described in this chapter. In each of these learning contexts, resources will work differently and mediate the work and learning in different ways. How this framework may apply or need to be adjusted, in less structured forms of collaboration, is an open question.

Lastly, we return to the issue of resources 'for' and 'from' collaboration. Resources are explicitly designed *for* teacher collaboration, for example protocols to guide the work of a group, specific representations of teaching that the group might analyse, the appointment and training of a facilitator, and the allocation of time and space for the group to meet. And teachers transform and construct resources as a product *from* collaboration, for example new lesson plans and assessments, knowledge of student thinking and of mathematics, an elaboration of an instructional vision, a sense of affinity and purpose among math teachers, and so forth. Drawing on Kazemi and Hubbard's (2008) call for multidirectional analyses of teachers' participation in professional learning and in classrooms, we have identified attention to the travel and transformation of resources 'back-and-forth' between teacher collaboration and classroom as one area for future research.

This seems especially important in studies of teacher collaboration, given that the shared text of the group is typically a representation of classroom practice. How do the resources (e.g. knowledge and ways of knowing, emotions, understanding of institutional norms and expectations) that figure in classroom teaching relate to the resources prioritised in the collaboration? How can collaboration be designed to take teachers' classroom-based resources and lived realities into account, while also supporting the transformation of those resources? Careful attention to this back-and-forth would help in elaborating the relations between teacher collaboration and classroom practices, both of which are embedded in political contexts, and in which teachers are often asked, implicitly or explicitly, to reconcile competing expectations.

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Chapter 10
Working and Learning in Collaborative Groups: What's Key to Mathematics Teachers?

Hilary Hollingsworth, Yiyi Chen, Christelle Fitamant, Lameck Dition Sandram, and Shelli Temple

10.1 Introduction

Research suggests that collaborative professional development has the potential for impactful effects on teaching practices and student achievement (Borko, 2004; Jensen, 2014; Opfer, 2016). Yet, the OECD's most recent Teaching and Learning International Survey (TALIS) data show a comparatively low percentage of teachers participating in collaborative professional learning activities (OECD, 2019).

This chapter examines the involvement of four mathematics teachers in collaborative groups, with the aim of: (i) understanding the kinds of opportunities for learning that these groups provided and the impact that they had; (ii) synthesising insights from the teachers' participation in these groups to inform the ways that mathematics teachers might work and learn in collaborative groups in the future. The four teachers were invited to participate in a Plenary Panel at the ICMI Study 25 Conference, where they shared accounts of their collaborative group work and learning. Each teacher was asked to report on three areas, including: the context, purpose and design of the collaboration; outcomes of the collaboration; particular

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lessons learned (for example, in terms of: any factors that supported or limited the collaboration; any surprises or challenges encountered by participants; specific components of the collaboration that they believe provoked sustained changes in their own approach to teaching mathematics; ways they might reimagine the collaboration to make it more effective).

The four teachers work in different continents (Africa, Asia, Europe, North America), teach at different levels (primary, secondary, tertiary) and were involved in collaborative groups that have different forms (a pre-service, Lesson Study group, an international teacher exchange program, a teacher/researcher network, an online teacher network). Table 10.1 provides an overview of the different collaborative groups, including their location, form, participants and foci.

Interestingly, despite the diverse collaborative group locations, forms, participants and foci, there were several elements in common to the design of the teachers' collaborative group experiences, including: a consistent overarching purpose for the collaborative work; the involvement of group members in setting goals for the work; a focus on increasing teachers' technical capability; the provision of access to specialist expertise. Some observations related to each of these design elements are presented next.

10.2 An Overarching Purpose

An overarching purpose for the work of the collaborative groups is consistently articulated across the teachers' accounts. Each group has a strong focus on the improvement of teaching practice with the subsequent outcome of improved student learning outcomes. The teacher from Malawi, for example, reports:

> The overall aim was to improve the quality of mathematics teaching in Malawi. As such, Lesson Study was introduced as a way to achieving the aim. It was thought that Lesson Study will be a scaffold for student teachers and teacher educators learning in Malawi.

In the Malawi context, the collaborative group Lesson Study focused on improving the quality of teaching of mathematics educators working in the important area of initial teacher preparation.

The teacher from China reports that a strong focus of their teacher exchange program is on improving UK teachers' mathematics knowledge and pedagogy:

> The goal of the programme is to raise curriculum standards in the UK in mathematics by improving teachers' pedagogical and subject knowledge, and to refine the curriculum to ensure that all pupils achieve their full potential in mathematics without anyone 'left behind'.

The teacher from France reports that their collaborative group is part of the network of Institutes for Research on Mathematics Teaching (IREM). These institutes provide collaboration between mathematics teachers from schools to universities, focusing their work on issues related to improving mathematics teaching, disseminating outcomes and conducting professional development for teachers.

Table 10.1 Overview of the four teachers' collaborative groups

Form	Participants and role of the teacher	Work and learning foci
Location: Shanghai, China and England		
Teacher exchange program Program funded by UK federal grant Main activities: Reciprocal school visits, collaborative workshops	More than five hundred year 1 to year 9 exchange teachers led by experts from a national Centre for teaching in England and a university in Shanghai *Teacher's role*: Teacher in one of the participating Shanghai schools	Collaborative planning, sharing and research in preparation for UK teacher visits to Shanghai; mathematics topics for lesson study research lessons purposefully selected
Location: Brest, France		
Teacher/researcher network Part of a network of Institutes for Research on mathematics teaching (IREM) Main activities: Monthly meetings, regular email communications	Three secondary school mathematics teachers Two university mathematics teachers One French teacher working both at a secondary school and a teacher training institute *Teacher's role*: Secondary mathematics teacher	The teaching of logic in secondary school; specific related foci negotiated based on the needs of the group
Location: Malawi, Africa		
Pre-service lesson study Part of national initiative to improve mathematics teacher education Main activities: Create, test and refine a lesson study research lesson	Nine teacher educators from Machinga teacher training college *Teacher's role*: Leader of the lesson study group	Planning, teaching and analysing a lesson study research lesson for pre-service student teachers
Location: Oklahoma, United States of America (USA)		
Online professional learning networks Main activities: Informal collaboration using social media; organised efforts including book studies, professional learning events, face-to-face conferences, websites, blogs	Thousands of mathematics teachers from around the globe networking online *Teacher's role*: Participant, virtual mentor, online partner	Flexible content and foci developed by participating teachers

The teacher from the USA reports that thousands of mathematics teachers from around the globe are engaging in an online Professional Learning Network (PLN), "actively working to promote quality mathematics instruction, mentorships for new teachers, and curriculum development". She notes that, although activities associated with the network have diversified over time:

> Throughout these efforts, the main goal has remained the same—a grassroots 'for teachers, by teachers' professional learning network to improve the quality of mathematics instruction for our students.

The different groups' focus on developing teaching quality is not surprising, given that this is a priority consistently articulated in education policies, guidelines and initiatives globally and, as such, it frames the profession's improvement agenda. As suggested by Emeritus Professor Dylan Wiliam during a keynote presented in 2012, "every teacher needs to improve, not because they are not good enough, but because they can be even better" (Wiliam, 2012). The teachers' accounts clearly show their commitment to continuous professional learning and improvement of their teaching practice.

10.3 Participation in Goal Setting

A design feature common to the collaborative groups is the involvement of participants in determining the focus of, and explicit goals for, their work together. Each group used particular approaches to identify their foci and goals. In the context of the mathematics educators' Lesson Study in Malawi, Lesson Study members worked with one another to set a long-term goal to increase the pass rate of mathematics student teachers by 2021. In the teacher exchange program between Shanghai and England, different core content topics were purposefully selected by group members for use in each round of Lesson Study, with guidance from an expert from Shanghai Normal University.

The teacher–researcher network in France first collectively defined their main objectives, which were to focus on "developing classroom settings and tools", related to logic and logical reasoning, and "disseminating these tools to teachers". Then, over a period of 6 months, "each participant worked freely choosing his line of work", presenting their work, thoughts and questions at group meetings where they then identified agreed areas of work, "based on the points of convergence and divergence". And, the online teacher network in the USA formed different goals according to the particular work and learning tasks they engaged in, for example: virtual mentoring; book studies; sharing research articles; sharing of mathematics problems on websites; inviting the public into classrooms virtually.

Potential benefits of the approaches used to set goals in these collaborative groups include authentic engagement of teachers and a sense of ownership associated with the process and outcomes of the work. In addition, as noted by the teachers from Malawi, France and the USA, participants were able to define and receive differentiated support specific to their individual needs.

10.4 A Focus on Increasing Technical Capability

There is a consistent focus across all four collaborative groups on improving mathematics teachers' technical teaching knowledge and skills. As mentioned earlier, in the case of the French teacher–researcher network, the particular focus was on supporting increased capability related to the teaching of logic and logical reasoning, and this focus targeted not only secondary mathematics teachers, but also a teacher of French language and university teachers. In Malawi, teacher educators had the opportunity to plan in great detail a research lesson, and then engage in observing, analysing and reflecting on the teaching of the lesson, as well as refining the lesson design. The teacher from Malawi emphasises the importance of improving his teaching practice through a fine-grained examination of teaching:

> Planning for a research lesson involves putting forward teachers' content knowledge of a particular topic and the best teaching practices that could be used. There is inclusion of critical thinking approaches, probing questions and challenging tasks. This has become my common practice when planning my lessons. More time is spent figuring out how students will think and learn a particular concept than the teaching itself.

Shanghai teachers involved in the China–UK teacher exchange were similarly very focused on increasing their technical ability when they explored how to teach selected topics in reform-oriented ways, in order to share with UK teachers. And the teacher from the USA describes various ways that teachers collaborate in their online network to increase their technical capability, including engaging in discussions about content and pedagogy, supporting one another with learning activity design and sharing effective classroom management techniques.

This focus on increasing technical knowledge and skills suggests that the collaborative groups that the four teachers were involved in are, by design, examples of what Jensen (2014), drawing on the research of Clement and Vandenberghe (2000) and Rosenholtz (1989), refers to as 'active collaboration'.

> Active collaboration, in which teachers learn from each other through team teaching, joint research projects and classroom observation and feedback, has a positive impact on students. Collaboration that concentrates on administrative issues does not. (p. 7)

10.5 Access to Specialist Expertise

All four teachers express that they greatly value the way that their collaborative groups enabled them to access specialist expertise. This includes the expertise of other teachers, researchers, mathematics educators and other education professionals. The mathematics teacher from France reports the tremendous opportunities provided to their teacher–researcher network by working with a specialist teacher of French language, as well as university teachers. The French teacher, for example, "helped the mathematics teachers at the secondary school to define words that did not have the same meaning in mathematics and French" and supported the analysis of language used in teaching sessions. The teacher from Shanghai involved in the teacher exchange program reports that the importance of the collaborative partnership between Shanghai and UK teachers unfolded over time as the program developed, and teachers from both locations stated that they learned much from one another.

The teacher in Malawi notes, “I have learned a lot of teaching techniques through observing fellow educators teach”. And, the secondary mathematics teacher from the USA strongly emphasises the ways that the online teacher network she is a part of creates opportunities to share and learn with others. She reports:

> Due to the rural nature of Oklahoma, there are minimal opportunities for professional development as it relates to mathematics education. [...] when teaching a specialised curriculum such as AP [Advanced Placement] Statistics, the opportunities for traditional professional learning workshops are limited, often making it necessary to look for non-traditional methods of collaboration and networking [...] One of the major benefits of collaboration via social media is the 24/7 access to teachers around the globe. With a single Facebook post or Twitter tweet, you can easily receive responses from teachers with a variety of teaching experience, backgrounds and geographic locations in a matter of minutes.

In addition to some shared design features, there were also some common elements related to the outcomes of the teachers’ collaborative group work and learning. A synthesis of lessons learned from the experience of the teachers and some reflections on future directions related to mathematics teachers working and learning in collaborative groups are provided after the presentation of the teachers’ stories that follow.

10.6 China–England Mathematics Teacher Exchange (Yiyi Chen)

10.6.1 Context, Purpose and Design of the Collaboration

The outstanding performance of Shanghai students in mathematics and science on the 2012 PISA assessment has attracted United Kingdom (UK) educators’ attention to the Shanghai mathematics curriculum (called Shanghai Maths), as well as to mathematics teaching and learning in Shanghai (called Teaching for Mastering). Shanghai students achieved a mean score of 613 (119 points above the OECD average) on the PISA in mathematics with an excellence rate of 55.4%. Since 2012, education authorities in China and the UK have had frequent contacts to explore mechanisms of collaboration. In February 2014, a delegation, including a representative from Ofsted (Office for Standards in Education) and other UK educational experts, visited Shanghai. During this visit, the Department of Education in the UK and the Education Commission in Shanghai agreed to launch a teacher exchange program, called the ‘China–England Mathematics Teacher Exchange Programme (MTE)’.

The goal of the programme is to raise curriculum standards in the UK in mathematics by improving teachers’ pedagogical and subject knowledge, and to refine the curriculum to ensure that all pupils achieve their full potential in mathematics without anyone ‘left behind’. The programme is funded by a UK federal grant and is jointly led by the United Kingdom’s Ministry of Education and the Shanghai Municipal Education Commission. Shanghai Normal University, UK National College for Teaching and Leadership (NCTL) and UK National Centre for Excellence in the Teaching of Mathematics (NCETM) are responsible for carrying out the programme.

By 2018, there have been three rounds of exchanges between Chinese and British mathematics teachers with more than 500 teachers from Shanghai and the UK directly participating. Additionally, approximately 12,000 British teachers have participated in Teaching for Mastering which disseminates what has been learned from the exchange programme. The programme will continue until 2023, with a goal of benefiting teachers from at least 9300 primary and 1700 secondary schools in the UK (Boylan et al., 2019). Major activities include reciprocal school visits and collaborative workshops. Throughout the programme, groups of UK teachers (Year 1–Year 9) led by experts from NCETM visit Shanghai schools. During these two-week visits, they observe classroom teaching, participate in school-based teaching research activities and attend workshops on developing Lesson Study.

Similarly, groups of Shanghai teachers, led by Dr. Xingfeng Huang from Shanghai Normal University, visit England. During the two-week stay in England, the Shanghai teachers teach lessons in local schools to explore how Shanghai teaching methods could be adapted and implemented in England. The details of the implementation have been evolving according to the goals in each round of exchange. My school (Cao Guangbiao Primary School) is one of the programme base schools that has served as the platform for the exchange programme in Shanghai over the past 4 years.

Before the exchange visit, the Shanghai participating teachers are divided into collaboration groups to prepare for the UK teachers' visit, to learn about the UK education system and to update their knowledge about mathematics teaching in Shanghai. Some teachers who taught in the UK on their previous visit share lesson plans used, as well as their observations. When collaboratively planning lessons, teachers often simulate situations which may arise in UK classes. In addition, a Lesson Study group with ten to twelve teachers from programme-based schools explores how to teach a purposefully selected topic in a reform-oriented way, in order to share with UK teachers. With the leadership of Dr. Huang from Shanghai Normal University, each Lesson Study focuses on a different core content topic, such as equivalent fractions or fractions on a number line.

The collaboration not only takes place among teachers in Shanghai, but also between Chinese and British teachers. When the English teachers from the UK maths hub visit Shanghai, they are invited to teach a math lesson to the Shanghai pupils with the UK teacher and a partnering Shanghai teacher working together to develop the lesson plan. Similarly, when a Chinese teacher teaches in England, the partnering English teacher works with the Chinese teacher to develop the plan.

10.6.2 Overview of the Collaboration Outcomes

The programme has had a substantial impact on both British and Shanghai teachers and schools. In the UK, the Sheffield Institute of Education was commissioned by the Department of Education in December 2014 to undertake a longitudinal evaluation of the programme. Mixed methods have been used to analyse data collected over three academic years. Data from student testing has shown that in schools most directly involved in the exchange program, there has been an increase in pupils' KS1 mathematics attainment. (The UK national curriculum is organised into blocks of years called 'key stages' (KS). KS1 refers to pupils at age 5–6, including Year 1 and Year 2.)

Moreover, survey and interview data has revealed that cohort I teachers (exchange participants in 2014–2015) improved in their beliefs about mathematics teaching and commitment to learning from Chinese mastery teaching methods. Observing the mastery teaching in Shanghai classrooms was perceived to have been particularly impactful. Cohort II teachers (2016–2017) particularly appreciated their visit to Shanghai, which deepened or challenged their previous understanding of Chinese mastery teaching methods. The visit by the Chinese teachers to England supported UK teachers' implementation of teaching for mastering mathematics (Boylan et al., 2019).

To reflect on Shanghai teachers' observation and understanding of mathematics teaching in Britain, Professor Huang edited a book called *I Teach Mathematics in Britain*, which includes chapters by programme participants from Shanghai. The book includes four sections and highlights the differences between Chinese and English mathematics teaching as observed by the teacher authors. One salient point noticed is that homogeneous grouping is prevalent in English primary school, which does not enable weaker students to progress as desired. Some teacher authors used vivid examples to describe differences in language and culture, and how best to address and learn from these instances. Additionally, some teacher authors highlighted their development as mathematics teachers, particularly in their use of hands-on activities.

10.6.3 Description of What Was Learned

Through participating in this project over the past 3 years, I have learned so much. I would like to highlight what I have learned: (1) learning to collaborate with UK teachers; (2) learning about cultural differences in defining mathematics concepts; (3) learning about differences in learning progressions; (4) learning to use research-based teaching practices.

10.6.3.1 Learning to Collaborate with UK Teachers

At the beginning of the collaboration, we thought it would be easy for us to know the learning situation in UK classrooms, simply by having the British teachers tell us what pupils had previously learned from the UK curriculum. But, we found it was often hard for us to design lessons based just on this information, and pupils having already been taught something does not mean that they have mastered it. Teachers from both countries realised that it is necessary to have deeper and more extensive discussions about students' readiness to have effective collaboration. Now, teachers from both sides jointly select the teaching content and share more extensively about school culture and student learning. For example, my partner teacher, Mrs. Louis, gave me information about her school, Caroline Haslett Primary School, through videos and photos. These artefacts helped me to understand her classroom environment, as well as student homework and exercises, so that I could understand their learning situation in advance.

10.6.3.2 Learning About Cultural Differences in Defining Mathematics Concepts

Different cultural and educational backgrounds increase the difficulty of teaching in another country. Through collaboration, Chinese and British teachers are learning how to minimise the negative impact of these differences. For example, students recognise a rectangle and square at an early age, but when exploring the relationship between these shapes conceptually, UK pupils struggled to realise that a square is a specific rectangle. I was surprised because this had never happened in the Shanghai classrooms where I taught. My British partner teacher shared that, in British primary schools, attention is paid to the visual characteristics of the shapes of square and rectangle separately, without establishing conceptual connections between them. Furthermore, we checked the English National Curriculum which says that pupils should be taught to recognise rectangles (including squares) in Year 1, which is the age of five. This means that students in the class which I will teach were taught the concept 3 years ago. Moreover, the curriculum does not give a clear definition of rectangle, which surprised the Shanghai teachers.

After asking several teachers and pupils in the UK school, we found that the problem largely stemmed from ambiguous definitions. In the UK, most people think that the rectangle is a shape with four right-angles and two pairs of opposite sides equal but different in length. In Shanghai textbooks, a rectangle is defined as a quadrilateral with four right angles and two pairs of opposite sides equal. To understand further how a rectangle is defined in British textbooks, we consulted the textbook *Maths—No Problem*, which is recommended by the government. It says that the opposite sides of a rectangle are always parallel and equal. Based on this, a rectangle should include square. Thus, we believe that the student learning difficulty could be solved by appropriate practice.

After some deliberate practice, we were delighted to see that pupils could understand the concept very well. This makes me feel deeply that the difference between the pupils' mathematical achievement in the two countries is not due to pupils' different learning abilities, but to cultural differences. Reflecting on my own teaching, I now realise how important it is to focus on the knowledge that students have already learned and the context in which they learned it. This new idea leads me to think further about lesson planning.

10.6.3.3 Learning About Differences in Learning Progressions

Through the China–UK collaboration, we have discovered substantial differences in learning progressions in mathematics content between Shanghai and the UK. For example, when I was teaching addition and subtraction within 100, I found that UK pupils had a weak foundation of addition and subtraction within 20, an issue which will likely result in them having difficulty learning addition and subtraction with larger numbers. This weak foundation may be caused by the school teaching plan which showed they have spent almost half of the term learning multiplication involving 2 s, 5 s and 10 s, without any review of addition or subtraction. Nevertheless, British teachers insisted on continuing to teach this content.

To seek a solution to this dilemma, I posed the following in the WeChat group of the programme team: how to design the lessons about addition and subtraction within 100 when pupils are not fluent with operations within 20. (WeChat is the most popular social media platform in China, and is routinely used by Shanghai teachers to share teaching resources and ideas, and to discuss problems in teaching.) Teachers in the group provided various suggestions. For example, some teachers suggested using songs to help pupils remember the number bonds of 10. These ideas prompted me to study further the teaching content of addition and subtraction within 100 and clarify the relevant content in the Shanghai textbook. After comparing the presentations in Shanghai and British textbooks, it was found that the initial operation relies on the fluency of decomposition of number. As a result, I added a warm-up activity of reciting Make 10 songs in the teaching design. While students were excited and interested, they also naturally made sense of the decomposition. Based on this design, my teaching in the UK went well.

10.6.3.4 Learning to Use Research-Based Teaching Practices

Teachers in Shanghai conduct self-evaluations and reflect on the reform of mathematics teaching in China–UK workshops. I will illustrate with the example of an on-going Lesson Study of fractions on the number line which occurs in two stages. First, the learning trajectory of the content (fractions on number line) across grades in the Shanghai textbook was examined and the ways of presenting the content in different textbooks were compared. To understand student readiness, we gave a pre-test. To broaden our understanding of the presentation of the topic, we also consulted textbooks from other countries.

Finally, learning goals were set: (1) to find the position of the proper fraction; (2) to use the number line as a learning tool to compare the size of the fraction; (3) to have a preliminary experience in the integration of numerical and pictorial representations. The second stage includes the cycles of design, implementation, reflection and revision. After teaching the first class, teachers shared their thoughts about this lesson one by one, such as the large capacity of the whole lesson and insufficient teaching time. For language sentence patterns, such as two one-thirds is two-thirds, the meaning of fraction can be emphasised.

Drawing on the feedback from other teachers, and my self-reflection on the lesson, I changed my lesson plan mainly to focus on understanding how to locate fractions on a number line while using comparison of fractions to help students build links between fractions and integers on the number line. The lesson was taught a second time using the revised lesson plan. Pupils had more time for discussion and communication, and naturally established the relationship between the previous knowledge of fractions and number lines. For example, I asked pupils to find more fractions on the number line after they had found the fractions with denominators 2, 3 and 4. Students discussed in groups and then shared their group's ideas in the whole class. They realised that all fractions can be found on the number line, because they could go on forever. Furthermore, with the help of language sentence patterns like four-thirds means four one-thirds, they could locate improper fractions on the number line.

10.7 Collaborative Work Within an IREM (Christelle Fitamant)

10.7.1 Context, Purpose and Design of the Collaboration

The IREM of Brest is part of the network of Institutes for Research on Mathematics Teaching. These institutes provide a collaborative organisation between mathematics teachers from schools to universities. They can work together on issues of mathematics teaching, disseminate their outcomes and conduct professional development for teachers. The meetings take place on the premises of the IREM at Brest University.

10.7.1.1 Birth of the Group

In 2009, new mathematics curricula were set up. During the annual conference planned by the Brest IREM, there were informal discussions about the teaching of logic in secondary school. These discussions led to an observation: on the one hand, the students receive little teaching of logic, while, on the other, the teachers feel a lack of resources to develop a practical teaching of logic as described in the curricula. Subsequently, a working group was formed to focus on this subject in September 2010.

The group consisted of six people: two mathematics teachers from Brest University, three mathematics teachers from secondary schools and a French teacher, working both at a secondary school and at a teacher-training university institute. In a rather traditional way, the role of university teachers is to bring theoretical content and analyse the activities proposed to the students. Teachers at the secondary school design class sessions, test them in their classrooms and analyse them. The French teacher designs class sessions for French courses and points out the different language elements used in mathematics and in French.

10.7.1.2 The Collaboration Purpose and Design

During the first meeting, we collectively defined our main objectives: at first, developing classroom settings and tools (like exercises, test, mind mapping, ...) wherein students can practice logic and logical reasoning without theoretical courses and, secondly, disseminating these tools to teachers.

One of the university teachers proposed to lead the group. We decided to meet once a month, Friday afternoon, at Brest University and we communicated regularly by e-mail. One of the participants (often the same) took care of the minutes of the meeting and reminded the work to be done by each participant with an e-mail just after the meeting and just before the next meeting. During the first 6 months, each participant worked freely, choosing his line of work. At each meeting, the participants presented their work, their thoughts and their questions. This allowed the group's leader to identify areas of work based on the points of convergence and divergence upon which we agreed to work.

For the secondary teachers, the goal is not to teach theoretical logic, but the university teachers use theoretical logic with their students from the first year. So, we needed to design practice activities for secondary school that prepare students for the academic logic of the university and design additional practice activities for the university students that connect with the secondary school.

One of the objectives of our collaboration was to create class activities that can be used in mathematics courses, French courses in secondary school and university. Together, we created the same written tests for each of these levels. We asked, in our school, teachers who did not participate in the group to take the test to their students (fewer university teachers were involved in this phase).

With the data from the results of the test, we were able to assess the needs of our students. In order to prepare students for the teaching of logic at the university, the secondary teachers suggested that theoretical logic professional development be set up for all mathematics teachers by the university teachers.

During our meetings, we found that the new mathematics curricula and schoolbooks did not provide any progression in logic learning. Consequently, we had to plan a teaching of logic from the first year to the last year of secondary school. We have designed practice activities fitted to the level of our students, which are sequenced to progress from simple to more complex logic concepts over time. The French teacher helped the mathematics teachers at the secondary school to define words that did not have the same meaning in mathematics and French, and the secondary mathematics teachers recorded their course sessions so the whole group could analyse the sessions.

10.7.2 Overview of the Collaboration Outcomes

Written tests were conducted in all educational levels from first year of high school to first year of university. The activities for secondary school students were tested by secondary teachers and analysed by the whole group. The statistical results from the test and the analysis of the class sessions were published in the *logique au fil de l'eau* brochure.

The needs of secondary teachers regarding the notions of logic were identified and gave rise to a professional development. It was set up with the help of the Brest IREM Director during the annual conference. This professional development lasted one afternoon and could not address all of the secondary teachers' questions. Some logic concepts (the most significant for secondary teachers) were reviewed and a list of resources was given to the participating secondary teachers, in order to supplement their learning from the day.

Our work dissemination was made thanks to the *logique au fil de l'eau*, a brochure published by the French APMEP (*Association des professeurs de mathématiques de l'enseignement publique*). In this brochure, the mathematics secondary school teachers described the sessions and their analysis. A university teacher wrote a logic course for the secondary school mathematics teachers. The French teacher and university teacher wrote a text on the links between logic and language. Moreover, secondary mathematics teachers conducted logic workshops at conferences organised either by several IREMs or the APMEP.

The Inter-IREM Committee of secondary schools is a commission which includes secondary teachers and university teachers from several IREMs. This commission has meetings in Paris five times per year. Several IREMs have also worked on logic teaching, so a group named 'logic' was set up within the commission. A broader collaborative work has been put in place with several IREMs. Our group chose a representative, a secondary teacher, who goes to Paris and has exchanges with the representatives of other IREMs about logic teaching during the Inter-IREM Commission meetings in Paris. Currently, members of the commission are writing a brochure summarising work on logic teaching from several IREMs.

10.7.3 Description of What Was Learned

10.7.3.1 Factors that Supported or Limited the Collaboration

The existence of an IREM was the key element in setting up this collaborative work. The IREM is a place of discussions about mathematics teaching, known to all the teachers from primary school to the university and easily accessible. The annual conference provides an opportunity to discuss without the institutional hierarchy. Additionally, an IREM has a library of teaching resources to which all teachers have access, so that when teachers want to improve their practice or have a question on mathematics teaching, it is natural to look to an IREM for collaborative work. Furthermore, the institution recognises IREMs, so the schedule of teachers who work at an IREM can be arranged so they can attend the meetings.

An issue was the organisation of remote work: often, we want to present a perfect document to the group, but it takes a lot of times to prepare and, even then, our document was not perfect for the group. It is difficult to present an unfinished work to the group, but the discussions are more open. Furthermore, collaborative tools have not been fully explored. Some of us did not use these tools and the others used different tools (Google Drive, Dropbox, email, ...). This situation has limited our ability to exchange our work between the meetings. One of the lessons learned was that the group members should take time to choose and learn how to use the same digital tools.

10.7.3.2 Challenges Encountered by the Participants

Another secondary teacher and I were to lead professional development for teachers for the first time. We were apprehensive about facilitating this experience for other teachers. The teachers who attended the professional development did not have a lot of knowledge about the teaching of logic, but they did teach the same level that we do in mathematics. Some participants did not want to teach logic without theoretical logic. We needed to convince them that our practical activities have a good impact on students' progress and can prepare secondary school students for the academic logic at university.

All the participants of our group prepared this first professional development. We chose to prepare a debate and presentation of our work rather than a course of logic. During the session, one participant of our group was present with the teachers attending the professional development, in order to feed the debate between the attending teachers and the leaders of the professional development, if necessary. Finally, teachers who participated in the professional development worked well during the sessions with us, but it is difficult to determine how they use this training in their classroom. After this first experience, we continue to lead additional trainings with less apprehension.

During our work, we discussed with other IREM groups outside of ours at Brest. I was appointed to be the reporter of our work during the Inter-IREM Commission meetings in Paris. On this occasion, I met specialists in mathematical logic who work together in a committee with secondary teachers. They needed the point of view of several secondary teachers, so I accepted to join this committee. Since then, I continue to meet them regularly, and now I plan the committee meetings with another secondary teacher regarding mathematics learning in secondary school.

10.7.3.3 Changes in My Approach to Teaching Mathematics

This collaborative work provoked some changes in my mathematics teaching practice. It forced me to have more perspectives on my teaching. In a collaborative work, there are different points of view of the participants, sometimes contradictory. Each participant has to argue, defend his or her point of view and be able to make it evolve through the others. For example, during our work, I had to explain to the other participants why I chose some activities. I learned how to analyse the activities to be convincing, and now I continue to analyse the activities that I propose to my students. We also recorded course sessions and we listened to the recordings together to analyse the students' reactions. Since the study of these recordings, I believe I am more attentive to the reactions of my students.

Teachers are alone in class in front of their students; collaborative work is an interesting way to improve teaching practice.

10.8 The Improving Quality and Capacity of Mathematics Teacher Education in Malawi Project: A Norwegian and Malawian Collaboration (Lameck Dition Sandram)

10.8.1 Context, Purpose and Design of the Collaboration

Collaboration in mathematics takes different forms and is practised in different contexts. This section, discusses experiences of collaboration in Lesson Study. This is a case of Malawi, a Southern African country of an area of 118,484 square kilometres and with a population of about 18.7 million people. The collaboration is actually taking place at Machinga Teacher Training College, one of the eight public teacher training colleges for primary school teachers in Malawi.

It all started at a national level as both a professional development program and a network of teacher educators from three Teacher Training Colleges. The program was under the Improving Quality and Capacity of Mathematics Teacher Education in Malawi Project, with funding from the Norwegian Program for Capacity Building in Higher Education and Research for Development (NORHED). The project was a collaboration between the University of Malawi and the University of Stavanger, and the overall aim was to improve the quality of mathematics teaching in Malawi. As such, Lesson Study was introduced as a way to achieving their aim. It was thought that Lesson Study will be a scaffold for student teachers and teacher educators learning in Malawi. Forty-six participants were involved in the program.

From the national level, Lesson Study activities trickled down to college level. At Machinga Teacher Training College, nine mathematics teacher educators were involved. A Lesson Study group was formulated and, in the group, there were five male educators and four female educators. I took the leading role of the group and, with the other educators as members, a goal was set. A long-term goal was to increase students' achievement by improving the pass rate from the current 40 percent to 95 percent by 2021. Generally, students' performance in mathematics was not impressive and, as a mathematics section of the department of mathematics and science at the college, we thought of putting in place strategies that could assist to rectify the problem, so we welcomed Lesson Study as one strategy.

Eight mathematics educators collaborated in the planning of the research lesson by developing a lesson plan, identifying teaching and learning resources, and observation tools to be used during the teaching of the research lesson. One of the eight mathematics educators taught the research lesson, while the other seven observed and collected data. Thereafter, a reflection process was initiated and the teacher educators discussed the research lesson and shared their experiences. The experiences were also shared at a national level during another professional development workshop, with teacher educators from two other colleges and experts from the project.

10.8.2 Overview of the Collaboration Outcomes

The first outcome of the collaboration is change of attitude. As indicated in one of the limitations to collaboration, the attitude of some educators who were not ready to have their lessons observed by fellow educators greatly changed. Educators are increasing their flexibility, accommodating presence of observers and taking part in the sharing of teaching experiences.

The second outcome is upon improvement in instruction. Lewis and Hurd (2011) argue that, "If you want to improve instruction, what could be more obvious than collaborating with fellow teachers to plan instruction and examine its impact on students?" (p. 3). Indeed, through collaboration, educators are able to develop lessons rich in critical and problem-solving strategies. This is helpful both to students and to teachers. Students are challenged with activities that keep them active throughout the lesson. For instance, in one of the research lessons, students were asked to model the addition of 45 and 16. The students came up with different ideas like using place-value boxes and tins or using an abacus and counters. Of interest was the use of stones in the place-value box instead of sticks. At the same time, student teachers are also developing teaching skills and chances for them to use critical thinking approaches. CORD (1999) argues that many teachers tend to interpret the learning environment according to their own experience as students – that is, they teach the way they have been taught. So, the likelihood that student teachers will use ideas of Lesson Study in their teaching, using critical thinking and problem-solving approaches to be specific, after having experienced it themselves, is very likely.

10.8.3 Description of What Was Learned

10.8.3.1 Factors That Supported Collaboration

There were some factors that supported collaboration in the teaching of mathematics. The first one was environmental in nature: that is, the context in which the collaboration was taking place. The college had everything the teacher educators needed to carry out Lesson Study. Rooms and curriculum materials were available, and students were also in college. The college administration gave the group a go-ahead and made teaching resources available.

Culturally, teachers are lifelong learners. It is in their tradition to seek knowledge. They would always want to learn to update their knowledge base and, when such a chance unveils itself, they go for it. Any initiative that proves to be productive in improving achievement of students is often taken seriously by teachers. Lesson Study came at a time when it was needed most. There was a need to understand a reviewed teacher curriculum. This called for a collective effort of teacher educators to understand its contents. Lesson Study was the timely solution and motivator to that cause.

10.8.3.2 Factors That Limited Collaboration

On limitations, the size of the class involved was big. There were about 40 student teachers involved during the research lesson. That affected mobility of the teacher educator, as well as the students during the lesson, because of limited space in the classroom. It also became difficult for the teacher educator to reach every student and give individual assistance. Resources were also inadequate for every student to be in contact with them. For an effective follow-up on each and every student during a lesson, it could be good to have not more than 20 students in one class. A Lesson Study lesson requires full understanding of how instructions are influencing learning in each and every learner, and this is only possible where the size of the class is small.

Practically, most schools in Malawi have large classes and that will take some years to be solved. I see this as the greatest challenge, and it cannot be overlooked when planning for lesson study. However, there are a number of aspects with research lessons that can be accomplished and improve learning other than focusing on the learning of individuals: for example, team planning, collective reflection and use of critical-thinking approaches can enhance learning. Hence, I feel modifying some areas of the Lesson Study process to suit the Malawian context can help teachers to carry out Lesson Study in highly populated classrooms.

Another limiting factor, which came as a surprise, was the unwillingness of some teacher educators to participate in some stages of the Lesson Study. They could neither make themselves available during planning, nor accept the role of teaching the research lesson. The thinking that they were the best teacher educators kept some participants away from the collaboration process. The solution to this has been geared towards attitude change. This is being coupled by allocating two educators to one class, so that they can plan and teach as a pair. By working in pairs, people will see the relevance of sharing ideas and working as a team.

It was also not all that easy for the leader of the collaboration group to lead through a model which was new to all of them. This demanded more time for the leader to study and search for more information, so that a right track could be followed. This helped the leader to become more knowledgeable about the Lesson Study.

10.8.3.3 Components of the Collaboration That Provoked Sustainable Change

The following paragraphs describe three components of Lesson Study that provoked sustainable change in my approach to the teaching of mathematics. The first component is planning. Lesson planning is a daily activity that teachers do as they prepare for their lessons. It is in its natural context to see a teacher planning for lessons. Success and failure of a lesson depends heavily on its planning. However, planning a research lesson collaboratively becomes more rewarding than planning it individually. Planning for a research lesson involves putting forward teachers' content knowledge of a particular topic and the best teaching practices that could be used. There is an inclusion of critical thinking approaches, probing questions and challenging tasks. This has become my common practice when planning my lessons. More time is spent figuring out how students will think and learn a particular concept than the teaching itself.

Furthermore, I have learned a lot of teaching techniques by observing fellow educators teach. The way they approach their lessons and taking it through developmental steps is an important practice: for example, starting a lesson by asking students a challenging question, then building the lesson on students' responses, until the objectives of the lesson are met. This was also the case with other educators who were involved in the collaboration.

The third component is about conducting a research lesson. In this stage, one member of the lesson study group teaches a research lesson, while the rest of the group members observe and collect data. A data-collection tool is used where experts observe the lesson and collect relevant data. This is the data that inform instruction and bring improvement. When one teaches a lesson individually, without colleagues monitoring the proceedings, very little data is obtained. However, the practice of collecting data when teaching is what is very important. I now treat my lessons as sources for data collection for my learning about my students' learning.

I am able to identify gaps in my teaching and learning of students. For instance, I was teaching about subtraction of mixed numbers, e.g. $7\,\frac{1}{3} - 3\,\frac{2}{5}$. I asked students to explain how they could solve the problem. One student explained, "First subtract three from seven and get four, then subtract two-fifths from one-third". The student proceeded up to this stage: $4\,\frac{5-6}{15}$. And then the student said, "We take one from four, the whole number, and add to five (minuend) to make fifteen and then subtract six from the fifteen, which means we have now $3\,\frac{15-6}{15}$". This assisted me very much because I was able to understand the student's thinking on the problem. The gap was identified and ways of handling the problem were shared with other educators.

Reflection forms an integral component of the Lesson Study circle. This is the stage where the teacher and members of the Lesson Study group share data from a research lesson. Members share what they feel are the successes and the challenges of the lesson they observed, as well as what they learn about the students' learning. Collectively, they once again plan the lesson, fusing in new ideas and approaches and eradicating elements of the lesson that are not significant in realising the objectives of the lesson. There is power in reflection and every time a lesson is being reflected upon, new insights are realised. No wonder reflection has become part of the Teacher Education Philosophy of the reviewed Initial Primary Teacher Education, which states, "to produce a reflective, autonomous, lifelong learning teacher, able to display moral values and embrace learners' diversity" (Malawi Institute of Education, 2017, p. ix), and is being implemented now.

It is my wish that, 1 day, in-service primary school teachers be introduced to Lesson Study as a form of collaboration. This will greatly assist to improve instruction and the performance of learners in mathematics. That might take a long time, but it will be a good undertaking. The challenge I anticipate is a lack of research skills in the primary school teachers. Lesson Study lessons are research lessons and research skills are very crucial to the Lesson Study process.

10.9 Professional Learning and Collaboration Via Social Networks (Shelli Temple)

10.9.1 Context, Purpose and Design of the Collaboration

Jenks Public Schools is a suburban school district serving approximately 12,000 students in grades Pre-K to 12 in Northeast Oklahoma. Oklahoma is a mostly rural state, located in the Central Plains of the United States, with two main metropolitan areas, Tulsa and Oklahoma City. Jenks is a southwestern suburb of Tulsa, and the school boundaries cover the city of Jenks, as well as a section of the southern city limits of Tulsa. Jenks High School, serving grades 10–12, has a graduating class of approximately 750 students, with a mathematics department of 15 regular education and five special education teachers, teaching classes ranging from Algebra 1 to Calculus 3. During my 20-year tenure at Jenks High School, I have taught a variety of classes, with a current teaching assignment of Advanced Placement (AP) Statistics, Geometry, and Forensic Science and Data Analysis. In addition, I serve as our site Professional Development Co-ordinator, as well as on our Leadership Team.

Due to the rural nature of Oklahoma, there are minimal opportunities for professional development, as it relates to mathematics education. In many districts, there may only be one or two mathematics teachers, so Jenks is fortunate to have a large department of educators. However, when teaching a specialised curriculum such as AP Statistics, the opportunities for traditional professional learning workshops are limited, often making it necessary to look for non-traditional methods of collaboration and networking.

In the late 1990s, teacher message boards and email listservs were vital elements to online teacher collaboration, but in the mid-2000s, online teacher journals, called blogs, started to become more popular, followed soon by the use of social media, such as Twitter, Facebook and Instagram, as a way to connect these teachers together and create real-time collaborative conversations revolving around lesson ideas and pedagogy. There are now thousands of mathematics teachers around the globe who are active participants in an online Professional Learning Network (PLN) called the #MTBoS, or the Math Twitter Blog-o-Sphere. Through these online connections and social networks, the members of the #MTBoS are actively working to promote quality mathematics instruction, mentorships for new teachers and curriculum development.

In general, the collaborative nature of the #MTBoS is fairly informal, using social media hashtags and Facebook groups to connect subject-area teachers. However, there have been organised efforts regarding book studies, outreach at national professional learning events and even a face-to-face, multi-day, math teacher conference, called Twitter Math Camp (TMC), during the summers of 2012 through 2018. Throughout these efforts, the main goal has remained the same—a grassroots 'for teachers, by teachers' professional learning network to improve the quality of mathematics instruction for our students.

10.9.2 Overview of the Collaboration Outcomes

The nature of social media as a medium for collaboration lends itself to opportunities for discussions with a wide reach, both geographically and longitudinally. A single tweet can create a multi-hour or even a multi-day discussion with contributors around the globe, all sharing their input and guidance on an activity, lesson plan or classroom management advice. Collaborations via the #MTBoS have resulted in pedagogical books being written, open-source software and curriculum, free sharing of lessons and Desmos activities, and even public outreach programs such as 'Math on a Stick' at the Minnesota State Fair.

The impact of the online teacher collaboration is, in many ways, difficult to measure, but the effects are far-reaching. One example of this can be seen with the success of Twitter Math Camp (TMC), drawing both presenters and participants from the greater #MTBoS community, as well as from the local hosting region. In 2012, the original TMC hosted approximately 40 teachers from a variety of teaching experiences and backgrounds for a three-and-a-half-day workshop. In 2014, Jenks High School hosted TMC and the workshop had grown close to 150 teachers. At its end, in 2018, TMC had impacted close to 600 teachers and classrooms through in-person attendance plus an additional unknown number through virtual interactions.

While the physical TMC conference lasted three-and-a-half days each year, the virtual portion of the conference lasted year-round. In the weeks and months preceding each TMC, the conference presenters were hard at work preparing for their sessions. Since many of these presenters were not in geographic proximity, they organised their presentations using online collaboration tools, such as Google Docs and Skype calls to hash out the details. During the actual conference, the whole-group sessions, such as the keynote speakers and the 'My Favorites' portions, were videoed and shared via the YouTube channel, plus participants were encouraged to 'live-tweet' from each session using a social media hashtag, so that people not in attendance could follow along. In the days and weeks following the conference, the conversations continued as teachers shared their learning experience through blog posts and Twitter discussions.

10.9.3 Description of What Was Learned

One of the major benefits of collaboration via social media is the 24/7 access to teachers around the globe. With a single Facebook post or Twitter tweet, you can easily receive responses from teachers with a variety of teaching experience, backgrounds and geographic locations in a matter of minutes. The exposure to teachers from different cultures and teaching environments enriches the personal professional learning experience, which can lead to richer experiences for students, from both a pedagogical and a social justice aspect. With traditional professional learning opportunities, teachers tend to be limited due to geographic proximity and, as a result, the participants generally come from very similar backgrounds and teaching experiences. In contrast, developing a PLN via social media allows for a diversity of perspective, which in turn creates a robust and responsive professional learning experience as classrooms and social environments evolve.

Within the AP Statistics community, one limiting factor of traditional collaboration is isolation, with most AP Statistics teachers being the only person in their district and surrounding area that teaches the course. Through the power of social media, these teachers, including myself, are no longer alone. By reaching out through a Facebook post or via Twitter, new AP Statistics teachers have ready access to experienced teachers to help guide and mentor them through the course and how best to teach challenging content. Around 15 years ago, a young teacher from Hattiesburg, Mississippi, reached out on a then-active teacher message board looking for another AP Statistics teacher to discuss course content and share teaching ideas.

My response to that post and the virtual mentorship that resulted is a key reason why I am so invested in the power of social media for teacher collaboration. This commitment to helping new AP Statistics teachers has continued throughout the years, including the development of a Facebook group in 2015 called the 'AP Stat Teachers Support Group' and through the online AP Statistics community on Twitter. While every teaching context is unique, these online partnerships are very empowering to teachers, as they seek to best prepare their students for the standardised AP exam given each year in May—a test that can earn students with college credit for specific scores.

Throughout the history of the #MTBoS, educational trends can be seen, often before they show up in traditional professional learning opportunities. One of the most powerful movements that I have been involved in was in the area of student assessment. Approximately 10 years ago, several prominent teacher bloggers started implementing Standards Based Grading (SBG) in the mathematics classroom, based on works by Robert Marzano, Dylan Wiliam, Ken O'Connor and others. During this same time frame, I had become disillusioned with traditional grading methods and the inability of the grading system to communicate clearly what my students knew. The desire to read the works of these authors and discuss thoughts with my virtual colleagues led to the creation of an online book club via Twitter, with weekly group chats to support the use of formative and summative assessment in the classroom using hashtags of #sbarbook and #eduread for easy curation. The change from traditional grading systems and appropriate use of formative assessment tools is one that has been slow to take off in mainstream educational circles, but is quite common within the online teacher community.

A more recent collaboration of the #MTBoS is the use of instructional strategies that truly inform and transform student learning. Through the use of rich mathematical tasks, teachers and students alike are growing as mathematical learners and thinkers. The online teacher community regularly shares these 'low floor—high ceiling' or 'open middle' problems with each other, presenting them freely for feedback and use by teachers around the globe. Several websites have been developed and crowd-sourced by the #MTBoS, including Visual Patterns (www.visualpatterns.org), Which One Doesn't Belong (https://wodb.ca) and Open Middle (www.openmiddle.com).

Within my own classroom, these rich tasks have been vital in helping students see themselves as mathematical knowers and doers. In the past, there has been a disconnect between the mathematics classroom and what mathematicians actually do—look for patterns, explore curiosities and enjoy challenging problems. By utilising these tasks, students are able to showcase their thinking and reasoning skills and truly to see the joy and beauty of mathematics. While lengthy conversations with distant colleagues and websites full of tasks can definitely have a positive impact on the classroom and student learning, another powerful influence can be found through the collaborative efforts of the '180 blog'.

In the United States, an average school year consists of 180 days, so, several years ago, a few teachers decided to use social media platforms as a way to invite the public into their classrooms virtually to observe the day-to-day learning that takes place. Originally, the '180 blog' utilised online blogging platforms, such as WordPress or Blogger, to journal these daily activities, but, over time, this idea has morphed to the micro-blogging platforms of Twitter and Instagram. By using the social media hashtag of #teach180, teachers are able easily to share photos each day of student work and learning activities with their parents and local stakeholders, as well as with the greater #MTBoS community. This initiative, whether through a traditional blog or through Twitter or Instagram, is an excellent way for teachers to receive a daily dose of inspiration and to spark new ideas for the classroom.

All of the above initiatives are important to the improvement of mathematical instruction, but, by far, the most powerful outcome of the #MTBoS is the relationships formed by teachers who would otherwise not know each other. The exposure to teachers from a variety of teaching environments, with diverse student populations, the ability to get teaching and learning advice from experienced educators and the real-time feedback for lesson development are the most valuable aspects of the #MTBoS community. By forming friendships across time zones and geographic boundaries, teachers are no longer limited by the size of their physical mathematics department within their district or surrounding area; they now have infinite opportunities for learning and collaboration within the virtual world.

10.10 A Synthesis of Lessons Learned

As noted in the introduction, and evidenced in the teachers' accounts, the four collaborative groups were very different from one another, and there is much that can be learned about the nature, design and implementation of collaborative groups from the teachers' experiences in these groups. Following is a synthesis of some observations related to lessons learned from the teachers' experiences, together with questions to provoke consideration of how these might inform mathematics teachers' participation in collaborative groups in the future.

10.10.1 Factors Supporting Collaboration

The four teachers identified and described a number of factors that supported the work and learning that took place in their collaborative groups. These included cultural, social, environmental and physical factors. Some of the key supporting factors were:

10.10.1.1 A Culture of Learning

The teacher from Malawi proposed that there is a strong tradition among teachers to see themselves as life-long learners and this positively influences their participation in professional learning:

> Culturally, teachers are life-long learners. It is in their tradition to seek knowledge. They would always want to learn to update their knowledge base, and when such a chance unveils itself, they go for it. Any initiative that proves to be productive in improving achievement of students is often taken seriously by teachers.

In the French example, the connection to an existing organisation, the Institutes for Research on Mathematics Teaching (IREM), was considered to be a key element in setting up their collaborative group. The IREM is familiar and accessible to teachers, respected by them, well-resourced, and "when teachers want to improve their practice or have a question on mathematics teaching, it is natural to look to an IREM for collaborative work". Participation of teachers in the collaborative group was encouraged and the scheduling of their attendance supported. Implicit support of this kind can be invaluable to the success of collaborative activities.

10.10.1.2 Motivation and Timing

For participants in the Malawi Lesson Study collaborative group, there was high motivation to be involved as the mathematics educators needed to understand a revised curriculum:

> Lesson Study came at a time when it was needed most. There was a need to understand a reviewed teacher curriculum. This called for a collective effort of teacher educators to understand its contents. Lesson Study was the timely solution and motivator to that cause.

The teacher from the USA reported that her motivation for collaborating with others online was sparked by a young teacher reaching out for support:

> Around fifteen years ago, a young teacher from Hattiesburg, Mississippi, reached out on a then-active teacher message board looking for another AP Statistics teacher to discuss course content and share teaching ideas. My response to that post and the virtual mentorship that resulted is a key reason why I am so invested in the power of social media for teacher collaboration. This commitment to helping new AP Statistics teachers has continued throughout the years, . . .

She also described how, at various points across her teaching career, she has had opportunities to source information and discuss ideas with virtual colleagues and these occasions have motivated her to reflect on and transform different aspects of her teaching.

10.10.1.3 Available Resources

The availability of needed resources, including physical space and materials, student-participants' time and approval from administrators, was seen as key to implementing the collaborative group Lesson Study in Malawi:

> The first one was environmental in nature. That is the context in which the collaboration was taking place. The college had everything the teacher educators needed to carry out a Lesson Study. Rooms and curriculum materials were available, and students were also in college. The college administration gave the group a go-ahead and made teaching resources available.

10.10.1.4 Partnership

The teacher involved in the China–UK exchange highlighted the importance of a genuine partnership in their collaborative group. She noted that, as their exchange program unfolded, the teachers from both locations realised they needed to work closely together to ensure deeper understanding about teaching and learning in the two countries, and she reported that now, "both sides jointly select the teaching content and share more extensively about school culture and student learning".

10.10.1.5 Connection to Others

The teacher from the USA reported that a positive aspect of online teacher collaboration is that it connects teachers together to "create real-time collaborative conversations revolving around lesson ideas and pedagogy". She noted:

> The nature of social media as a medium for collaboration lends itself to opportunities for discussions with a wide reach, both geographically and longitudinally. A single tweet can create a multi-hour or even multi-day discussion with contributors around the globe, all sharing their input . . .

She described benefits of the exposure to teachers from different cultures and teaching environments through online platforms, contrasting these with traditional professional learning opportunities:

> With traditional professional learning opportunities, teachers tend to be limited due to geographic proximity and, as a result, the participants generally come from very similar backgrounds and teaching experiences. In contrast, developing a PLN [Professional Learning Network] via social media allows for a diversity of perspective, which in turn creates a robust and responsive professional learning experience as classrooms and social environments evolve.

She also highlighted the important role that online collaboration can play for mathematics teachers who specialise in less common courses, or who may be working in less-populous areas:

> one limiting factor of traditional collaboration is isolation, with most AP [Advanced Placement] Statistics teachers being the only person in their district and surrounding area that teaches the course. Through the power of social media, these teachers, including myself, are no longer alone.

10.10.2 Factors Limiting Collaboration

The teachers also identified and described factors that placed limitations on their collaborative group outcomes. These included:

1. *Participation avoidance*

   The teacher from Malawi reported that some mathematics educators were initially unwilling to participate in components of the Lesson Study process, requiring the group to implement strategies to provoke attitude change. A practical strategy that they applied involved pairing educators to work together, so that those who were reluctant would see the relevance of sharing ideas and working as a team.
2. *Resource constraints*

   In Malawi, when teaching the Lesson Study research lessons, student class sizes were not conducive to the particular instructional approaches they were trying to implement, and lesson materials for students were limited. The teacher from Malawi has signalled that there may need to be some modifications to Lesson Study approaches for the Malawi context, because most schools in Malawi have large classes.
3. *Leading 'new' ideas and approaches*

   Leading change in areas and approaches that are new is challenging. The teacher in Malawi noted that it was not easy for him "to lead through a model that was new to all of them", and he needed additional time to study and research relevant information to support the group. Similarly, the French teacher reported that, when she was to lead professional development for teachers together with a colleague for the first time, they were apprehensive. She noted the pressure she felt when working with peers, and the need "to convince them that our practical activities have a good impact on students' progress and prepare secondary school students for the academic logic of university". She also reported, however, that she continues to provide training with less apprehension and, additionally, she contributes to a committee working with mathematics specialists.
4. *Communication protocols and tools*

   The French teacher reported that one issue arising in their context related to the ways that their group members organised and shared their work remotely. She noted that they were reluctant to share documents that were not 'perfect' with one another, and the preparation of such documents requires time. She also noted that there was a lack of consistency in members' use of collaborative tools, limiting the ability of the group to exchange work between meetings. A lesson learned, she suggested, is that, "group members should take time to choose and learn how to use the same digital tools".

10.10.3 Provoking and Sustaining Personal Professional Learning and Growth

In their accounts of their collaborative group work and learning, each of the four teachers mentioned aspects of personal professional learning and growth that were sustained over time. The teacher from Malawi reported three particular components of Lesson Study that "provoked sustainable change" in his approach to the teaching of mathematics. The first involves collaborative planning. He noted that, "planning a research lesson collaboratively becomes more rewarding than planning it individually", because you need to contribute and scrutinise knowledge and ideas, and you learn much from what others contribute. The second component involves conducting the research lesson. He suggested that, because colleagues monitor the lesson proceedings, there is an opportunity for collecting data about one's teaching:

> When one teaches a lesson individually without colleagues monitoring the proceedings, very little data is obtained. However, the practice of collecting data when teaching is what is very important. I now treat my lessons as sources for data collection for my learning about my students' learning.

The third component involves reflecting on the teaching of the research lesson and refining the lesson. The teacher noted, "There is power in reflection and every time a lesson is being reflected upon, new insights are realised".

The teacher from Shanghai noted that the following key aspects influenced her professional learning over time: close collaboration with exchange partners; the detailed examination of mathematics curriculum and teaching approaches; thorough planning of Lesson Study research lessons; close examination of lessons taught (including receiving feedback about the lesson, reflecting on the lesson and refining the lesson). The exchange program that she is involved in has been in place for several years and, as she noted, "the details of the implementation have been evolving according to the goals in each round of exchange". The program duration and its evolving nature (facilitating program relevance and currency) appear to have contributed to her sustained learning over time.

The teacher from France suggested that her involvement in the collaborative group provoked some lasting change in her mathematics teaching practice. She believes it forced her to consider more perspectives about teaching:

> In a collaborative work, there are different points of view of the participants, sometimes contradictory. Each participant has to argue, defend his point of view and be able to make it evolve through the others. For example, during our work, I had to explain to the other participants why I chose some activities. I learned how to analyse the activities to be convincing, and now I continue to analyse the activities that I propose to my students.

She also noted the impact that analysing recorded teaching sessions had on her:

> We also recorded course sessions; we listened to the recordings together to analyse the students' reactions. Since the study of these recordings, I believe I am more attentive to the reactions of my students.

The teacher from the USA reported a variety of ways that different online opportunities have stimulated her collaborative activity and improved her mathematics instruction. However, she suggested that the most powerful outcome of her involvement in her collaborative online network is the relationships formed with other teachers. It is these, she suggests, that provoke and sustain her commitment to on-going learning and improved teaching practice:

> The exposure to teachers from a variety of teaching environments, with diverse student populations, the ability to get teaching and learning advice from experienced educators, and the real-time feedback for lesson development are the most valuable aspects of the #MTBoS community. By forming friendships across time zones and geographic boundaries, teachers are no longer limited by the size of their physical mathematics department within their district or surrounding area, they now have infinite opportunities for learning and collaboration within the virtual world.

10.10.4 Reflection and Questions

The stories of the four teachers included in this chapter provide evidence of considerable work and learning in different collaborative contexts around the globe. Their experiences provide some insights into what is key to mathematics teachers working and learning in collaborative groups, and the kinds of professional learning and growth that can be provoked by, and sustained following, participation in collaborative groups.

A question of interest arising from the examination of the four teachers' experiences is: how can the professional growth of teachers within and across collaborative groups be described, understood and compared? Teacher learning is very complex and any model selected to understand and describe the process must acknowledge this complexity. One model potentially useful for thinking about the sustained learning and growth of teachers working in collaborative groups is the *Interconnected Model of Professional Growth* (Clarke & Hollingsworth, 2002).

The Interconnected Model recognises both professional growth as an inevitable and continuing process of learning and the complexity of professional growth through the identification of multiple growth pathways between four domains in which 'change' might be located: the External Domain (change in external sources of information or stimuli); the Personal Domain (change in professional knowledge, beliefs and attitudes); the Domain of Practice (change in practice through professional experimentation); the Domain of Consequence (change in perceived salient outcomes related to classroom practice). Change in one domain is translated into another through the mediating processes of reflection and enactment as shown in Fig. 10.1.

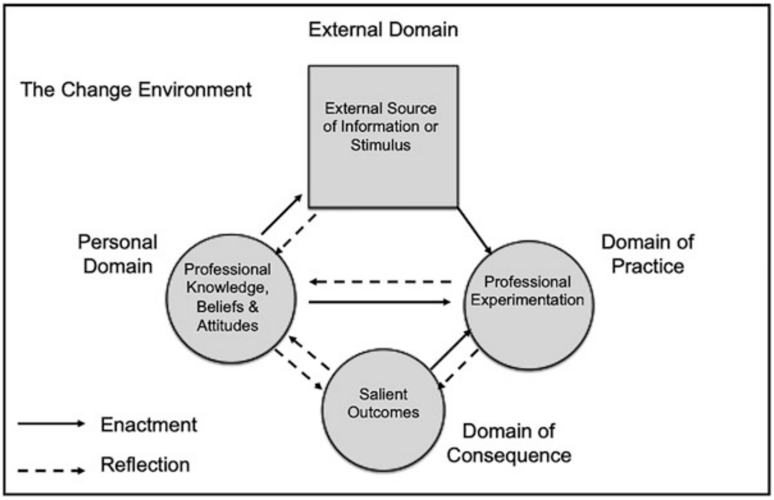

Fig. 10.1 The Interconnected Model of Professional Growth. (Clarke & Hollingsworth, 2002, p. 951)

The Interconnected Model asserts that any processes of professional growth occur within the constraints and affordances of the enveloping change environment. The four teachers in this chapter had change environments particular to their locations and their collaborative groups. The teachers were provided with 'external sources of information or stimulus', as they experienced new things in their collaborative groups. Then, as they engaged in their collaborative group activities, they individually reflected upon and experimented with different aspects of their mathematics teaching practice, sometimes building new knowledge or adjusting beliefs and attitudes about their teaching.

The teacher from Malawi, for example, reported that one aspect of his professional growth that has been sustained following his participation in the collaborative Lesson Study at his teacher training college relates to his use of 'new' teaching techniques. Figure 10.2 displays the pathway—or *growth network*—that represents his learning using the *Interconnected Model of Professional Growth*. The teacher reported that he: observed colleagues using teaching techniques that were new to him (External Domain); applied some of these in his classroom (enactment—arrow 1); reflected on the implementation of these (reflection—arrow 2); continued experimenting with some techniques in his classroom (enactment—arrow 3); reflected on the outcomes associated with his use of the new techniques (reflection—arrow 4); reflected on the value of the techniques, establishing new beliefs about their efficacy (reflection—arrow 5).

While the learning journey of each of the four teachers was undoubtedly unique, it is possible that some of the pathways that led to their professional growth—as can be represented by the *Interconnected Model of Professional Growth*—might be

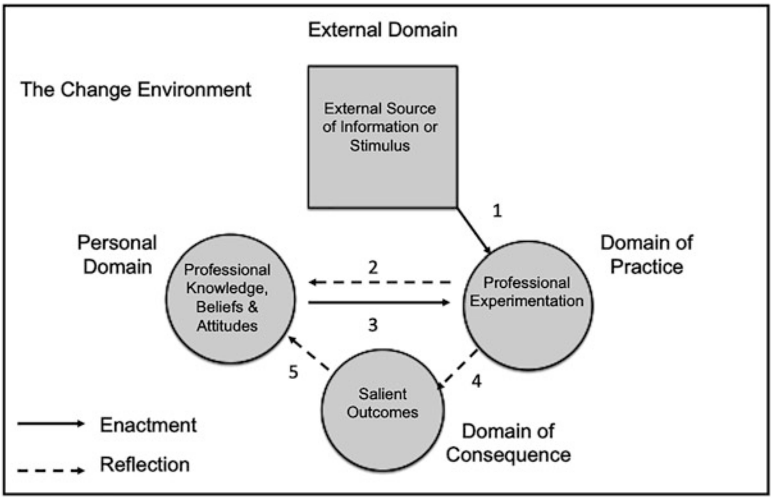

Fig. 10.2 An example growth network

similar. Further exploration of these pathways (for these teachers and others) could provide opportunities to develop our understanding of teacher professional growth within and across collaborative group contexts, in particular the different growth pathways experienced by participating teachers and the factors that might promote or inhibit professional growth.

The experiences of the four teachers also provoke other questions about mathematics teachers' work and learning in collaborative groups. Such questions include:

Which support elements are absolutely critical to prove effective for collaborative groups?

What kinds of processes facilitate authentic partnership roles in collaborative groups?

How might flexibility and responsiveness be effectively incorporated in collaborative group work and learning?

How might competing professional learning needs of collaborative group members be effectively managed?

How might the processes and products of collaborative group work be effectively shared?

How might effective collaborative group activities and outcomes 'reach' more mathematics teachers?

How might cross-cultural insights related to mathematics teachers working and learning in collaborative groups be effectively shared?

Depending on how each of these questions is interpreted, they could relate to one or more of the four study themes examined in this volume: *Theoretical Perspectives on Studying Mathematics Teacher Collaboration; Contexts, Forms and Outcomes of*

Mathematics Teacher Collaboration; Roles, Identities and Interactions of Various Participants in Mathematics Teacher Collaboration; Tools and Resources Used/ Designed for Mathematics Teacher Collaboration.

For example, the question 'What kinds of processes facilitate authentic partnership roles in collaborative groups?' could be investigated in relation to the theme Contexts, Forms and Outcomes of Mathematics Teacher Collaboration, with a particular focus on the processes and form of the collaborative groups, or in relation to the theme Roles, Identities and Interactions of Various Participants in Mathematics Teacher Collaboration, with a focus on partnership roles. The association between the different study themes was observed during the ICMI Study 25 Conference, with several participants noting that, although their papers were located in one study theme group, they were also relevant to other themes.

It is anticipated that consideration of questions such as those listed above, as well as further exploration of teachers' growth pathways using the *Interconnected Model of Professional Growth*, might usefully inform directions for mathematics teachers working and learning in collaborative groups in the future.

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Part IV
Commentary Chapters

Chapter 11
Advances and Challenges of Collaboration as a Learning and Research Field for Mathematics Teachers

Dario Fiorentini and Ana Leticia Losano

11.1 Introduction

After receiving the invitation from the editors of this volume to write this chapter, we decided to organise our discussion in four perspectives that, in our view, systematise relevant issues highlighted by the papers presented at ICMI Study 25. Such perspectives are:

1. different forms and meanings for collaboration;
2. what do we investigate, how do we investigate, and who investigates collaboration?:
3. the complex relations between collaborative groups and classroom practice;
4. possibilities for scaling up collaborative professional development initiatives.

Through the discussion of such perspectives, on the one hand, we seek to reflect on the advances and possibilities of collaboration considered as a field of investigation and as a process that promotes professional development (PD). On the other hand, we aim to identify some of the challenges that collaboration confronts nowadays.

For this purpose, we build our arguments by drawing on two references. Firstly, the chapters included in this book, the papers presented at the ICMI Study 25 conference mentioned in those chapters, and other complementary literature resources. Secondly, we draw on our experiences of collaboration with teachers in Brazil. To do so, we refer to our participation in the *Grupo de Sábado* [Saturday Group] (GdS), a Brazilian collaborative group operating for more than 20 years. The GdS gathers

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schoolteachers, teacher educators, graduate students and pre-service teachers interested in learning about and researching on mathematics teaching practice. As we will show in this chapter, the GdS differs from many groups and collaborative initiatives presented in this book in several ways. Thus, it provides a counterweight that reveals and interrogates some of the ideas assumed by diverse research endeavours focused on mathematics teacher collaboration.

11.2 Different Forms and Meanings of Collaboration

In England and the U.S., the movement of collaboration among teachers gained visibility and recognition in the 1970s, when Stenhouse (1975) systematised a type of action research or design research aimed at improving learning and teaching through inquiry. Such a framework involved a cycle of four steps: examining current practice, making decisions, planning optimal learning environments; implementing the findings in the classrooms with reflection. This movement gained strength in the U.S. in the 1980s, mainly in the context of in-service teacher education. At that time, it was adopted stressing its potential contributions to teachers' PD. Considering that not all PD initiatives are collaborative, the term 'collaborative PD' has come to be used by the pertinent literature to differentiate those that are.

By the end of the twentieth century, according to Hargreaves (1994), collaboration had already become a "meta-paradigm of educational and organizational change in the postmodern age" especially because it made possible the articulation and integration of "action, planning, culture, development, organization, and research" (p. 244). Collaboration was recognised as a creative and a "productive response to a world in which problems are unpredictable, solutions are unclear, and demands and expectations are intensifying. In this kind of context, the promise of collaboration is extensive and diverse" (p. 244). Since then, collaborative work and collaborative research among professionals from different institutions and levels of education have emerged worldwide as a response to the social, political, cultural, and technological changes that are taking place on a global scale and that jeopardise the traditional ways of organising PD initiatives (Fiorentini, 2004).

This movement gave rise to several models and conceptualisations of collaboration and collaborative research. Besides, models were recovered and adapted according to different sociocultural realities: that was the case of the Japanese Lesson Study (LS). It emerged at the end of the nineteenth century, but only came to the attention of educators outside Japan in the late 1990s, due to the systematisation conducted by Yoshida (1999). Since 2000, educational researchers have tried to use the LS model, adapting it to different cultural realities (Isoda, 2020). We suggest that the study of these processes of modification and adaptation of models of collaboration is a great opportunity for our research field. On the one hand, it would enable us to systematise and discuss diverse theoretical perspectives of collaboration, revealing different meanings assigned to collaboration. On the other hand, it would allow us to identify and problematise each model's educational possibilities, contributions, and limitations.

da Ponte et al. (Chap. 2, this volume), drawing on Robutti et al. (2016), conceptualise collaboration as "a group of participants, who work together pursuing a common aim, by establishing some joint working processes in which active involvement, balanced roles and caring relationships are central features" (p. 21). In our view, such conceptualisation is relevant, since it includes PD initiatives that promote horizontal or dialogical relations among teachers and teachers' educators. However, we consider that this conceptualisation does not enable us to distinguish between superficial and deeper or sustainable forms of collaboration (Azorín & Fullan, 2022). In addition, this broad conceptualisation makes it difficult to analyse diverse collaborative initiatives in relation to their impact and appropriateness in different circumstances and sociocultural contexts (Hargreaves, 2019).

In what follows, we discuss briefly these different conceptualisations of collaboration, especially since it seems to have been an issue little discussed during the ICMI Study 25 conference. To "work together pursuing a common aim" may happen, for instance, in situations of contrived collegiality (Hargreaves, 1994, 2019), especially when subjects are requested to participate in groups without having the possibility of negotiating the group's goals. Besides, it could happen in a group that lacks a common culture of sharing and negotiating practices and meanings. In a similar direction, both Azorín and Fullan (2022) and Hargreaves (2019) stress that a group becomes effectively collaborative and sustainable when it develops *collaborative professionalism*.

Such a notion refers to a learning community whose members, even having different knowledge, develop joint practices, negotiate their goals and carry out collaborative research. Therefore, the development of collaborative professionalism demands time: a time that cannot be pre-established by administrators or teacher educators, since it depends upon the disposition and upon the previous experiences of the members. Similarly, our years of participation in the GdS and other collaborative groups, as well as the research we conducted (Fiorentini, 2004, 2013; Losano et al., 2021), point to the fact that groups are not born collaboratives. Instead, they became collaboratives over time. For example, when new members join the GdS, their participation during the first 6 months involves experiencing and reflecting on practices historically produced by the group over the years—practices oriented at planning, implementing and analysing investigative classroom tasks.

To understand better what we mean, let us consider, for example, the study by Cooper (2019) that was analysed by Krainer, Roesken-Winter and Spreitzer (Chap. 8, this volume), using the RATE tool to highlight the relationships among Actors, Goals and Relevant Environments. It is an Israeli PD initiative that gathered twenty primary schoolteachers, two mathematicians and a facilitator with a Ph.D. in mathematics education, who was the study's author. The goal of the initiative was to bring together two groups that could share their perspectives on teaching integers, crossing boundaries between the world of primary school and the world of academic mathematics.

Such a goal emerged from the requirement, external to both groups, of the Israeli Ministry of Education, establishing that, "primary-school teachers need 'to enroll in mathematics PD initiatives to specialize in mathematics' (p. 71)" (cited by Krainer et al., Chap. 8, this volume, p. 8). Such collaborative arrangement gathered representatives from two fields of study, enabling sharing perspectives about how to teach integers. The group facilitator also gave her feedback, valuing the contributions of the two communities. The author concluded that the initiative allowed teachers to benefit from the mathematicians' perspective. In addition, it enabled mathematicians to achieve sensitive understanding. According to Cooper, collaboration contributed to mathematicians paying more attention to their students' ideas, opening spaces for discussion in their university math classes. Primary teachers, on the other hand, were able to build more mathematical confidence.

There is no doubt that there was learning involved in this PD initiative. Each subgroup learned something from the dialogue with the other. Each participant mobilised a *surplus of vision* and knowledge (Bakhtin, 2003) in relation to the other. Notwithstanding, is it possible to state that such learnings are sustainable and have the potential of transforming the participants' teaching practices? We suggest that this short PD initiative opened up the possibility of negotiating and developing a joint project around a common goal in the future. From this experience, mathematicians and primary teachers could engage in a sustainable project focused on teaching and learning mathematics at school (Azorín & Fullan, 2022). Thus, they could establish a "genuine collaborative work" according to the expression used by Esteley et al. (Chap. 3, this volume). Our point is to reinforce the etymological meaning of the verb 'to collaborate', which means to work together (*collaborare* from Latin) around a common objective, defined or negotiated jointly by all the participants.

Voluntariness is another relevant condition for participation in a collaborative group (Esteley et al., this volume). Although participation can be initiated on a mandatory basis, a PD initiative may become effectively collaborative if the participants have opportunities to jointly define their goals and actions. Several articles presented at the ICMI Study 25 Conference show that this scenario is feasible within a school (Collura & Di Paola, 2020) or an educational system (Canavarro & Serrazina, 2020; Soto et al., 2020). However, those authors stressed that such success depends on the way in which the leaders build, in collaboration with the participants, the group design and dynamics.

Hollingsworth et al. (Chap. 10, this volume) shed light on this process. They report on a teacher who participated in a collaborative group that was part of the network of Research Institutes in Mathematics Teaching (IREM—France). IREM usually promotes collaboration between the university and the school, improving mathematics teaching and teachers' PD. Thus, the collaborative IREM group sought to address a problem that had emerged during the annual conference of this network. Teachers lacked didactical resources to teach logic effectively in secondary education, as recommended by the new curriculum.

The group members were two university teachers, three high-school mathematics teachers, and a French language teacher. During the first meeting, they collectively defined the group goals and work schedule. Thus, they agreed the group would produce classroom tasks to develop the students' logical reasoning. The tasks were tested in their classrooms before being analysed and disseminated to other teachers. The two university teachers initially assumed a traditional role, providing mainly theoretical content and perspectives. Over time, they also assumed a more collaborative role, analysing and discussing the teachers' proposals. One of them actually ended up leading the group.

To sum up, the analysis of the papers presented at the ICMI Study 25 conference reveals diverse meanings for collaboration, and different and rich ways of promoting it among mathematics teachers. Such analysis also brings to the forefront two relevant features of our conceptualisation of this notion: collaboration requires time and demands shared negotiation of goals and actions. We identified such features by drawing on the literature focused on collaboration as well as on our years of experience collaborating with teachers.

Considering such diversity of meanings for collaboration, we suggest some future directions for research. How do diverse ways of promoting collaboration contribute to transforming teaching practice? What are their contributions to teachers' learning? How do mathematicians, mathematics educators and teachers negotiate mathematical meanings and reconstruct their professional knowledge? How does that knowledge differ from the school and/or academic tradition?

11.3 What Do We Investigate, How Do We Investigate and Who Investigates Collaboration?

The works presented at the ICMI Study 25 Conference indicate the presence of two privileged research perspectives. The first concerns the study of ***collaboration, its resources and its theoretical–methodological bases***. This perspective was the focus of three themes covered in this event, highlighting the following descriptive aspects: the theoretical, epistemological, and methodological bases to promote and investigate the collaboration of mathematics teachers (Chap. 2, Theme A, this volume); the design and dynamics of collaboration, with an emphasis on its goals, its environments and the different collaborative actors and their roles, interactions, and identities (Chap. 4, Theme C, this volume); the tools and resources mobilised and produced to support and organise the collaboration (Chap. 5, Theme D, this volume). The second research perspective focused on the **effects of collaboration**. Such perspective had a dual emphasis: (1) on participants' PD and learning as well as on the growth of the collaborative community; (2) on the improvement of curriculum and teaching linked to the collaborative process (Chap. 3, Theme B, this volume).

The discussion of the **theoretical bases of collaboration** is relevant, since it is an emerging field of study. Furthermore, the discussion of this issue during the ICMI Study 25 addressed a limitation noted by Jaworski et al. (2017): only one-third of the papers analysed by Robutti et al. (2016) explicitly stated their theoretical bases.

Although the theoretical aspects in collaboration were the main focus of Theme A, they were also discussed by all themes and were present in all parallel plenary sessions.

Regarding the studies focused on the **resources and tools** produced in/for collaboration, the authors who contributed to Theme A argued that theory can be seen as an important tool for designing and developing relevant and sustainable collaborative projects. Concretely, such theories may be useful for analysing the contributions of collaborative groups to teachers' learning, curriculum development, or teaching improvement. This point was also made by authors who contributed to Theme D, albeit placing greater emphasis on the resources of collaboration. Brodie and Jackson (Chap. 9, this volume), for instance, considered knowledge and representations of professional practices as resources. Differently, Robutti et al. (Chap. 5, Theme D, this volume) highlighted the dialectical nature of resources. Thus, they argue that, on the one hand, resources are essential for challenging teachers' thinking and practices, producing the desired results of collaboration. On the other, such results become resources that can lead to new cycles of collaborative learning.

We value the emphasis given to resources in/for collaboration during the ICMI 25 Study Conference. However, we would like to call attention to one resource that, in our view, has great generative power and was not discussed in this volume: the narratives written by participants of collaborative groups. According to our experiences in the GdS (Fiorentini & Carvalho, 2015; Fiorentini et al., 2018; Losano et al., 2021), narratives written by teachers are relevant resources to represent their histories of participation and learning processes. Therefore, narratives increase collaborative work and provide rich material for analysing teachers' learning. In addition, teachers' narratives are loaded with affections, meanings and perceptions of the support (or lack of) provided by their schools to introduce innovations in teaching practices. Hence, they also reveal forms of teacher resistance and the strategies they develop to mobilise their agency, implementing in their classrooms aspects of what they learn in the collaborative group.

Narratives written by teachers who participate in collaborative groups are "means and products" (Robutti et al., Chap. 5, this volume, p. 3) of collaboration. From their authors' point of view, these narratives are not only the means (semiotic mediation), but also a way in which they develop their collaborative professionalism and identity in dialogue with other professionals and members of the group. Thus, by becoming the authors of published narratives disseminated to a wide audience, they also become agents of change in the school culture and productive members of a broader educational community (Hargreaves, 2019; Fiorentini, 2013). From the point of view of teachers' educators, once published, narratives become relevant resources to support pre-service and in-service PD initiatives (Fiorentini & Carvalho, 2015).

In collaboration, all voices have value and need to be heard, as each member has a *surplus of vision* and knowledge (Bakhtin, 2003) about the practices of teaching and learning mathematics at school. In this sense, the organisers' decision to give teachers a platform to share their experiences while participating in collaborative projects was quite pertinent to the purposes of ICMI Study 25 (Hollingsworth et al., Chap. 10, this volume). We suggest that encouraging participating teachers to write narratives about their learning in this context is another way to give them voice and authorship and to value their perspectives.

The discussions presented in this volume, as well as our experiences of collaboration, reveal that the design and the resources have strong impacts on the ***effects of collaboration***, which lead us to the second research perspective on collaboration mentioned previously. Such effects may be analysed in terms of teachers' learning, community development and/or institutional improvement. These benefits highlight the multifaceted and complex nature of collaboration, given its different purposes and modes of organisation, as shown in Borko and Potari (2020).

The attempt to understand and theorise the learning and the development of teacher' knowledge for-in-of-practice (Cochran-Smith & Lytle, 1999) from participation in collaborative groups makes us return to the heading of this section. Considering who investigates collaboration, when we examined the papers presented at the ICMI Study 25 Conference, we verified that its authors are mainly mathematics educators, especially teacher educators or graduate students who participated in collaborative groups. Many of them also assumed the role of facilitators.

The focuses of such investigations include the two perspectives described above—that is, the study of collaboration and its effects. There is a clear trend toward developing studies *about* teachers. Such trend explores teachers' learning, PD, professional knowledge or their roles in collaboration. These results indicate that there are still few collaborative investigations, that is research carried out collaboratively by university academics with schoolteachers. Of the eighty papers reported in ICMI Study 25, only two are of this nature.

In this sense, we stress that collaboration is also a good opportunity for both parties to investigate together. University academics and schoolteachers engaged in collaborative groups can negotiate the focus of the research and develop joint interpretations about the participants' knowledge, actions and discourses, revealing knowledge situated in the collaborative practice (Lave & Wenger, 1991). Faced with the challenges of the current school context and the unfavourable conditions for schoolteachers to carry out research, they are left with the possibility of participating in collaborative inquiry groups, as underlined by some studies presented in ICMI Study 25 (Castro Superfine & Pitvorec, 2020; Uzuriaga et al., 2020).

Many of the Brazilian studies that assumed this perspective adopted the Relational Narrative Investigation (RNI) as a research methodology (Clandinin & Connelly, 2000). Cristovão and Fiorentini (2021) consider the RNI suitable for academics to develop investigations ***with*** schoolteachers, focusing mainly on teachers' professional learning and the PD. In this investigative process, "teachers are also encouraged to investigate their practices, narratively, with the collaboration of teacher educators, especially when both are committed to discuss and understand what and how they learn in this context" (p. 35).

Thus, schoolteachers generally explore their professional work. They may, for instance, analyse their students' or their own learning during a cycle of planning–implementation–reflection–evaluation of lessons. By sharing these investigations and findings in the collaborative group, teachers may problematize their practice, developing an inquiring and critical attitude towards their work and public policies in the educational field (Jaworski, 2008; Fiorentini, 2013).

Collaboration between schoolteachers and university academics is a powerful context for PD and for producing knowledge about school practice from a non-colonising perspective. In addition, we suggest that it is also a rich field of research for both, namely academics at the university and schoolteachers. Thus, we argue in favour of carrying out collaborative research ***with*** teachers, instead of conducting exclusively research ***about*** them. In recent years, this research perspective has flourished in Brazil and Latin America, fuelled by the expansion of the Lesson Study process. To map these investigative experiences and analyse their findings and contributions seems to be a relevant topic for research and discussion in an upcoming ICMI Study.

11.4 The Complex Relations Between Collaborative Groups and Classroom Practice

The chapters included in this volume show the wide variety of ways in which mathematics teachers can work and learn in collaborative groups. Chapter 3 (Theme B, this volume) reveals that each one of these ways of collaborating establishes different relations with regular classroom practice. For example, the initiative studied by Kooloos et al. (2020) connects the collaborative setting with teaching practice through the analysis of classroom videos to develop teachers' noticing of students' thinking. Soto et al. (2020) employed problems as a linking resource: teachers engaged in a community of practice were invited to solve problems, implement them in their classrooms and discuss such experiences in the community. Also, there are social contexts that developed powerful forms of school-based collaborative professional development. This is the case of Lesson Study in Japan or China, where the cycles of planning, implementing and analysis are job-embedded tasks with a long tradition. On the other extreme, the work of Heck et al. (2020) analysed a PD program based on the mathematics immersion of secondary teachers. The authors admit having trouble attending some of the program's goals since, "discussions about the connections between what they experienced in mathematical immersion and teaching were infrequent or lacked depth" (Esteley et al., Chap. 3, this volume, p. 33).

These examples highlight that PD in collaborative groups and regular teaching practice are different social, cultural and historical situations. Even in the cases in which the PD is strongly connected to classroom practice, the participants, the activities, the positionings, the times and the spaces specifics of collaboratives groups are not the same as the ones of regular classroom practice. Such understanding is evident in Brodie and Jackson's words, in this volume, when they state that, "although collaborating with colleagues about teaching has become more common in recent years, by and large, classroom instruction remains a private endeavour"

(Chap. 9, this volume, p. 13). In our view, further work analysing the complex relationships among collaborative groups and regular classroom practice would be highly beneficial to the field. To pursue such a research interest encompasses theoretical and methodological challenges.

Considering theory as a way of understanding—i.e. theory as a means "to understand the educational phenomena related to teacher collaboration, by providing conceptual and/or methodological tools to analyse and understand phenomena from different perspectives", in da Ponte et al.'s words (Chap. 2, this volume, p. 15)—the challenge concerns how to conceptualise the relations between PD in collaborative groups and regular classroom practice. One possibility is to frame the problem in terms of 'impacts': we need to study how participation in collaborative groups impacts teachers' classroom practice. This is a perspective frequently adopted and mentioned several times in this volume. Although we agree with the point being made, we would like to problematize the cause–effect metaphor underneath the notion of 'impact'. Theoretically, this perspective assumes that teachers learn within the collaborative group and then apply such learning in their classroom.

Such an assumption is strongly questioned by socio-cultural perspectives—extensively employed in our research field—that stress the mutual relations among people, activity, and the social world. According to Lave and Wenger (1991), what teachers learn while participating in collaborative groups is situated in the practices and social arrangements developed by the group. Therefore, we cannot assume such knowledge will be directly transferred into the classroom setting without consideration of the different activities, goals, circumstances, and social positions (Lave, 1996).

The perspective of Kazemi and Hubbard (2008), squarely brought in Brodie and Jackson's chapter, brings another important point in this regard: the relations between PD and classroom settings are not unidirectional, but multidirectional. Teachers' participation in PD and classroom practice co-evolves, since they are engaged in knowing in both contexts and bring their knowledge across contexts.

The two premises presented previously—the one that states that there is no direct learning transference between different contexts, and the one that assumes that the relations between collaborative groups and regular teaching practice are multidirectional—bring to the forefront the challenges involved in theorising about the complex relationships between PD in collaborative groups and classroom practice.

In terms of methodology, we identify two main issues. The first one is related to the temporal dimension. It is possible to adopt a short-term perspective, considering only the period in which the teacher participates in the group. Otherwise, it is possible to employ a long-term perspective, addressing the question of the sustainability of outcomes, that is, to analyse if changes made in the context of collaboration sustained long-term changes in the classroom setting (Esteley et al., Chap. 3, this volume). The second issue concerns the analytical procedures mobilised. In our view, it is necessary to develop methodological strategies to establish relations between data coming from the PD setting and data coming from the classroom and school settings. This would require a careful and creative endeavour. Further exploration of this issue might usefully inform directions for advancing research in this area.

Considering the theoretical and methodological challenges involved in researching the relations between collaborative groups and regular teaching practice, we believe that the chapters of this volume, as well as the papers presented at the ICMI Study 25, point to two promising directions to further our understanding of the topic. The first one is the study of the resources and how they are transformed while travelling between the collaborative group and the classroom setting. Considering Brodie and Jackson's framework in this volume, we refer specifically to the representational resources created for and from collaboration. Frequently, participation in collaborative groups involves creating and/or adapting tasks, lesson plans, curricular material, websites, etc., as well as analysing students' errors, assessment items or classroom videos. As stated in Chap. 9 (Brodie & Jackson) and Chap. 3 (Esterley et al., Theme B), resources are important products of collaboration and, in our view, can support important links between classroom practice and collaborative groups.

We believe that the situated perspective proposed by Brodie and Jackson, in this volume, is a fruitful approach to the problem of analysing how resources are transformed as they travel back and forth between the collaborative group and the classroom. Such a perspective assumes that, "the use of tools and artifacts to mediate learning relies on people assigning meaning to them [. . .] and these meanings are shaped by the broader contexts in which tools and artifacts are used" (Brodie & Jackson, Chap. 9, p. 4). This assumption is evident in our experiences in the GdS when a task or a lesson plan, carefully planned during several meetings, is subtlety—or sometimes substantially—transformed when implemented by the teacher in her classroom.

For instance, while interacting with her students, the teacher modifies the duration of the task or emphasises one aspect of the task over others. How and why do these transformations take place? What can we, and the teachers, learn from them? In addition, when we consider a long-term perspective, the evolving nature of resources comes to the forefront. Prior resources developed inside collaborative groups are often retrieved and adapted by teachers to be employed in their teaching at present. How are these resources shaped by the users over time? When, why and how are they recovered and adapted? Such questions highlight the complexity of the transformation operated over the resources when they travel from the collaborative group to the classroom, and *vice versa*. We believe that research focused on such issues deserves our attention and study.

The second promising direction for studying the relations between collaborative groups and regular teaching practice is to focus on the teachers. They regularly cross the boundaries between the collaborative group and the school context, introducing elements of practice and ways of knowing and of being from one context into the other. In this process, they become boundary brokers (Wenger, 1998). To co-ordinate their affiliation to both communities is a delicate endeavour, since it often requires reconciling, implicitly or explicitly, competing expectations. How do teachers manage to develop a sense of themselves among such conflicting practices and discourses? What conflicts do they experience in this process? How do teachers solve them? What do they learn in the process?

Another possibility is to adopt the notions of professional identity and agency. While participating in the GdS, teachers come to know other ways of understanding mathematics teaching and learning. In addition, they engage in reflexive processes that often problematise the implicit rules and identities fostered by the school world. Teachers also implement classroom activities inspired by these new perspectives and, later, they share and analyse such experiences in the group. Thus, participation in the GdS enables teachers to take a stance in front of the demands and expectations of the school world, gradually expanding their room for manoeuvre to make decisions and choices concerning their work (Vähäsantanen, 2015).

The collaborative group allows its members to experience new ways of being teachers, engaging themselves in an evolving process of identity development. Over time, teachers also expand their agency. They become actively involved in conceiving and directing their teaching practice according to their purposes, principles, and interests, as well as to the requirements and possibilities set by the school context (Losano & Fiorentini, 2021). How do teachers develop new positions and roles in the school and the collaborative group? How do they recover practices and discourses coming from one context to develop senses of themselves as mathematics teachers in the other? In our field, research on teachers' identity and agency has flourished over the last decades. We suggest that further exploration of the process of identity and agency development of teachers who participate in collaborative groups could provide opportunities to develop our understanding of teachers' professional growth across contexts.

11.5 Possibilities for Scaling-Up Collaborative Professional Development

In our view, collaboration is a way of transforming the colonising relationship commonly established between the university and the school. In this way, collaborative groups, such as the GdS, are an opportunity for university teachers and schoolteachers to engage in joint learning processes and imagine together ways of facing the current challenges involved in teaching and learning mathematics.

The great potential of these groups is underlined throughout this volume. Thus, the question posed by Hollingsworth et al. (Chap. 10, this volume), is particularly relevant: "How might effective collaborative group activities and outcomes 'reach' more mathematics teachers?" (p. 23). To respond to such a question is a complex endeavour, because, in our perspective, collaboration cannot be imposed in small-scale projects and much less in large-scale initiatives.

A partial answer to this issue can be found in our experience with collaborative groups in Brazil. The sustained work of some of those groups allowed many schoolteachers and university teachers to have relevant experiences of participation.

Many of these members moved to other regions of Brazil due to personal or career opportunities—a common thing in a country with continental dimensions such as ours—and decided to promote and cultivate collaborative groups in the new institutions in which they began working. In addition, collaborative groups also developed practices oriented at disseminating their work. Thus, they created and organised diverse events—congresses, seminars, etc.—and journals devoted to presenting and discussing teachers' reflexive work developed from their participation in collaborative groups.

In addition, books containing narratives written by teachers who participate in collaborative groups began to be published. In such narratives, teachers problematised and analysed classroom situations, producing knowledge-*of*-practice (Cochran-Smith & Lytle, 1999). These dissemination processes inspired many school and university teachers to constitute collaborative groups over the country. In a snowball effect, collaborative groups expanded and gained legitimacy inside the mathematics education community in Brazil (Carvalho, 2014).

The expansion of collaborative groups in different regions of the country happened informally and spontaneously. In this scenario, the participation of schoolteachers, pre-service teachers, mathematics educators and researchers in such groups is still little or not recognised by public policies (Gonçalves et al., 2014). The members of many collaborative groups devote their own time to participating in the group. In this way, they prioritise their PD over other responsibilities. Moreover, several of the members that abandon the GdS make that decision based on their difficulties to balance group participation with work or familiar commitments—to take care of their children or attend to the demands of a new job.

As stated by Brodie and Jackson in this volume, the issue of the resources made available for collaboration by public policies, "draws attention to the deeply cultural, political and historical contexts in which teachers' collaboration occurs across different contexts, and the many inequalities that still pervade our school systems" (Chap. 9, this volume, p. 8). To expand opportunities for collaborative PD public policies that explicitly support collaboration are needed. Such policies should provide time and spaces for professional collaboration, as well as value and acknowledge teacher collaborative work.

Some of the papers presented at ICMI 25 suggest a promising possibility of scaling up collaborative PD: the development of blended professional networks that gather schoolteachers, pre-service teachers, mathematics educators and researchers. Two examples of these networks are described in this volume. The first one is mentioned in Chap. 3 (Theme B, this volume) and refers to the research conducted by Heck et al. (2020). In this project, groups of teachers from different cities in the United States were engaged in PD involving both synchronous and asynchronous activities. In such a network, teachers were immersed in mathematical activities connected to their teaching practice. The second example is described in Chap. 10 (Hollingsworth et al.) and concerns Shelly, a teacher who participated in an online professional learning network directed at promoting "quality mathematics instruction, mentorships for new teachers, and curriculum development" (p. 3) via social media.

Both examples highlight the potentialities of online or blended collaboration. They enable access to "many participants from different geographically distant regions and from a variety of contexts [...] bringing together a myriad of perspectives" (Esteley et al., Chap. 3, this volume, p. 48). In this way, such networks have the potential of connecting teachers in distant or isolated contexts. They also make possible to gather professionals who work with different student populations, enriching the personal learning experience. In addition, social media enable teachers to receive real-time feedback for lesson development, as Hollingsworth et al. stated in this volume. Social media open up new possibilities of collaboration, reaching, in seconds, a vast public and allowing extended discussions in which people from all around the world can contribute. Finally, online communities are often flexible, embracing new educational tends more rapidly than mainstream educational circles.

Inspired by such experiences, as well as by our last research projects, we suggest that blended collaborative networks would be a powerful way for scaling up collaboration. Considering the demands and interests coming from classroom teaching practice, the members of such networks would organise themselves in small groups. Such small groups would congregate teachers, pre-service teachers, facilitators and researchers interested in discussing one topic related to teaching practice considered particularly problematic. Each group would negotiate its goal and the activities it would develop—for example, to plan and implement classroom tasks oriented at teaching a specific mathematical topic or analysing textbooks or classroom material. The small groups would gather periodically face-to-face or online. The network would act as a support space. Thus, all their members would meet more sporadically to share and discuss the work of the small groups.

In addition, the members of the network would be able to help and support each other through interactions via digital technologies. Once the small groups achieve their goals and complete their activities, the members of the network would reorganise themselves, forming new small teams. This kind of collaborative network would be organised according to a bottom-up model, since it would have the autonomy for establishing its agenda through the negotiation of aims and topics explored by the small groups.

Several works discussed in this volume showed that the Covid-19 pandemic challenged many aspects of teaching and teacher education. In the present context, it is unlikely that teacher education would return to its previous traditions. Thus, developing effective online or blended opportunities in which teachers could work and learn through collaboration, such as the ones we are suggesting, is an urgent endeavour.

Despite their potential, we cannot be naïve about the constitution of this kind of network. Thus, we anticipate some challenges to be faced. As Esteley et al. stated in Chap. 3, "collaboration is essentially about relationships, about finding a common ground to have support for the possible changes" (p. 4). Gathering together people who work in different places, with diverse publics and resources could produce rich exchanges. But it also demands establishing shared understandings among the members. In addition, collaboration frequently requires adopting an open attitude and sharing uncertainties, problems or ambiguous situations of teaching practice.

Therefore, each member should feel safe and embraced, trusting that the interactions inside the network would be oriented toward seeking alternatives in a non-judgmental way.

The division of responsibilities and roles inside each small group and in the network also requires fine-tuned negotiation processes. Also, it is necessary to set up carefully the processes through which each group delineates its topic and the resources—technological or not—that are best suited for each one of them. If collaboration is not established instantaneously in groups interacting face-to-face, this would neither happen in blended networks.

Considering that many agents would participate in the network, careful scheduling for interactions among the members can also be challenging. Brodie and Jackson (Chap. 9, this volume) point out another challenge located in a different dimension—nowadays, there are still "inequalities between rich and poor in relation to access to the internet" (p. 22). We had already co-ordinated online teacher education initiatives, in which teachers should take a one-hour boat trip on a Saturday to reach their school, since only there do they access a stable internet connection. Thus, when establishing blended collaborative networks, we cannot assume that all the teachers in the region or the country would have the same online accessibility.

How to promote trustful relationships among members of blended networks? What practices and strategies support transparent negotiation processes inside the network? How can technological tools contribute to these processes? How to cope with the inequalities regarding access to technological devices? We suggest that the development of studies centred on these issues would provide a more accurate vision on the advantages and disadvantages of blended collaboration.

11.6 Final Remarks

In this chapter, we have made an effort to highlight the advances resulting from the ICMI Study 25, drawing mainly from the systematisation carried out in the chapters of this book. The contributions of this systematisation allowed us to understand better the possibilities and potentialities of collaboration for the PD of teachers who participate in collaborative projects.

We believe that this progress would help us—and the rest of the members of the mathematics education community— to design better opportunities of collaboration, as well as to gain understanding about the development of collaborative communities. On the other hand, the studies in this volume also showed that collaboration is a complex and multifaceted undertaking (Theme B), since it depends on the conditions and dispositions of the participants and on the support of the institutions of which they are part.

In this direction, we argued that the possibility of collaboration does not entirely depend on the institutions' desire to promote it, nor on the willingness of participants who want to work together. Collaboration is a cultural practice that needs time to be developed. In our view, teachers' communities are not born collaboratively, even if that is the initial intention of their members. Therefore, no ideal model designed to foster collaboration can be applied without adapting it to local conditions and cultures, as Isoda (2020) has shown about the international diffusion of Lesson Study. The chapters of this book also acknowledge the importance of material and theoretical resources to support the design of collaborative PD and to conduct research in this context (Themes A and D).

In this regard, we argued in favour of using teachers' written narratives (Fiorentini, 2013; Fiorentini & Carvalho, 2015). We believe they provide rich opportunities for teachers to investigate their own practice and reveal their learning processes. We also highlighted the methodological potential of Relational Narrative Investigation as a framework that enables developing joint research *with* teachers. In this direction, our analysis of the papers presented at ICMI Study 25 suggests that a great challenge in our research field is to describe and characterise the knowledge-of-professional-practice produced inside collaborative groups and how it is co-produced by its members.

In addition, the chapters of this volume also delineate issues concerning the way collaborative groups relate to other settings. In particular, the study of the relations between collaboration and regular classroom practice emerges as a relevant and exciting topic for further research. We suggest that the notions of resources, professional identity and agency may be key concepts to develop nuanced and rich analysis in this direction (Themes C and D). Such theoretical constructs have the potential of moving forward simplistic perspectives based on cause–effect relations to measure the 'impact' of collaboration.

Finally, another challenge faced by teachers and researchers interested in fostering and investigating collaboration is how to scale up opportunities of collaborative professional development. Such challenge becomes more urgent since the work of the ICMI Study 25—starting with the literature review presented at ICME 13 (Robutti et al., 2016) and finishing with the synthesis of the papers presented at the conference in this volume—had shown the immense potentialities of collaborative groups. Considering the uncertainties of the global context, the possibility of cultivating blended collaborative networks seems to be highly promising.

The chapters of this volume stress that teacher collaboration, especially among teachers with different knowledge and views on practice, is multi-faceted and takes diverse forms in different parts of the world. However, in each of these forms of collaboration, what is learned and how it is learned has its own singularities and nuances that are different from other traditional learning and PD processes. This is one of our challenges as researchers in the field of collaboration: namely, to systematise and theorise these epistemological processes of co-learning and co-production of knowledge from practice. Collaboration, therefore, is a fertile and still little explored field that demands continuity of studies and socialisation, discussion and systematisation in events, as was the case of ICMI Study 25.

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Chapter 12
Mathematics Teachers in Collaboration: A Commentary

Rina Zazkis

12.1 Introduction

Having read all the chapters in the volume "Teachers of mathematics working and learning in collaborative groups" based on ICMI Study 25, I can summarise my reaction in a few words: *I am overwhelmed.* I am overwhelmed with the broad range of professional development initiatives for mathematics teachers. I am overwhelmed with the described diversity of the kinds and forms of collaborative groups of mathematics teachers. I am overwhelmed with the richness and variety of engagements of mathematics teachers. I am overwhelmed with the variety of rationales and outcomes, as well as of theoretical approaches used to investigate teacher collaboration.

A commentary? As chapters that report the activity of the working groups already summarise and analyse the studies in the submissions, I find it useless to position my commentary as a 'summary of summaries' or an 'analysis of analyses'. As such, I interpret 'commentary' as a set of comments, as my personal reflection on what I noticed, what I wondered about and what attracted my attention when reading the chapters.

12.2 On a Teacher's Work

Imagine a teacher.

Actually, imagine a mathematics teacher. Has your image changed?

Now imagine a mathematics teacher in an act of teaching.

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I suspect that a traditional image puts the teacher in front of a board, with students, if they are part of the image, sitting at their desks and taking notes. In a more ‘progressive’ image, the students are in groups, working on some task, either sitting at their tables or standing by a board, while the teacher is engaged with one of the groups. If the image is dynamic, the teacher is circulating among the groups. Teaching is the main work, or ‘job description’, of a teacher. However, in all these images, while the number of students may vary, there is only one teacher. The teacher teaches alone. And teaching, in its limited interpretation as ‘delivering a lesson’, is an individual activity.

As such, to consider teachers of mathematics working in collaborative groups, we need to reconsider the work of a teacher, or to interpret teaching broadly. In fact, it has been suggested that we consider as teaching all the work that surrounds presenting a face-to-face or online lesson, including planning, communication with students about the taught material and assessing student work (Zazkis & Leikin, 2010). This broad interpretation of teaching practice was suggested when examining the relevance of knowledge acquired in the study of mathematics at the tertiary level for school teaching.

While the suggested breadth would have been applicable for the publications in this volume, the chapters consider teachers’ work even more broadly. In addition to what surrounds the act of teaching, attention is given to teacher learning and professional development, with the explicit or implicit goal of informing and improving teaching. In turn, this goal is geared towards the improvement of student learning outcomes.

12.3 On Broad Applicability

In reading Chap. 9, by Brodie and Jackson (this volume), based on the plenary presentation, I note that the authors included the WordCloud from the ICME 13 survey (Robutti et al., 2016). So, I considered a WordCloud based on the text of Brodie and Jackson’s chapter. The main featured word, not surprisingly, is ‘resources’, followed by ‘collaboration’, ‘teacher(s)’ and ‘learning’. The word ‘mathematics’ (also ‘mathematical’) is featured, but at a lower level than ‘teachers’ and ‘collaboration’. The authors mention mathematics teaching practices, mathematics concepts, mathematics classrooms, mathematics content, mathematical thinking, mathematical ideas, etc. I wondered, however, what would happen if the word ‘Mathematics’ was replaced with ‘Biology’ or ‘History’ or even ‘special education’. In most places, such a replacement would make sense and the argument would be sustained. Consider, for example, making such a replacement in the following:

> Mathematics teachers’ knowledge is a key resource in collaborative learning—what teachers bring to their collaborations will inform the collaboration and its outcomes, and knowledge is obviously a key outcome of collaborative learning.

> The facilitator’s expertise in teaching mathematics and in supporting teachers’ learning, as well as their relationship with the teachers, mattered greatly for how they were perceived by the teachers.

I invite you to verify my claim by either omitting ‘Mathematics’ or replacing it with a different subject. Was the intention altered?

In my view, the suggested framework of resources—detailed as representational, knowledge, affective, human and institutional resources—does not explicitly focus on Mathematics. Neither do the two questions that guided the authors:

- What resources are available to support teacher collaboration? With what effects, both on the collaboration and the resources themselves?
- What resources are missing for supporting teacher collaboration? How and to what extent can teachers overcome these missing resources?

A harsh critic may point to the lack of specificity in the described research. I would like to adopt a positive tone, focusing on the scope of applicability. A broad applicability of presented claims to teacher community at large, regardless of the subject matter in focus, is where I see the main contribution of the authors. In fact, this breadth exemplifies the leadership of mathematics education community within subject-focused education at large. It could be the case that students’ difficulties with mathematics are those that resulted in a plethora of research in mathematics education, whereas research related to other subjects followed suit.

The initial research initiatives in mathematics education attended to curriculum and student learning, with the development of a more recent focus on teachers—the ‘era of teacher’, according to Sfard (2005)—including teachers’ knowledge, teachers’ pedagogy and . . . teachers working and learning in collaborative groups. I believe that framework of resources is applicable broadly and could be adopted and modified when attending to teachers of other subjects.

A related claim of broad applicability can be made with respect to the RATE framework—Relevant Actors, Targets and Environments—developed and used by Krainer, Roesken-Winter and Spreitzer (Chap. 8, this volume). The authors described professional development initiatives in seven recent studies chosen from different continents. While the described projects concerned teachers of mathematics, mathematics teacher educators and mathematics classrooms, I suggest that the same dimensions (that is, relevant actors, targets and environments) can be used for comparative analysis of any professional development program, that is, not specific to teachers of mathematics, and possibly applicable to other professionals, not only teachers.

Similar claims can be made about other chapters. In particular, continuing the theme of ‘broad applicability’, I find the discussion of ‘theory’ in Chap. 2 by da Ponte et al. (this volume), to be a valuable resource for any researcher in education. The chapter presents an importance “conceptualising the notion of theory” (p. 7). The authors’ notes on the “heterogeneity in what is called a theory by different researchers and different scholarly traditions” (p. 6) is a valuable summary which highlights the main issues in educational research. While the summary is tailored for research on mathematics teachers’ collaboration, it is applicable to mathematics education and teacher education broadly. The particular section on “Generalities of theories” has been added to the reading list in my doctoral seminar. I trust that it will help novice researchers situate themselves within the web of theoretical constructs and plethora of theoretical models and frameworks.

12.4 On Mathematics

My appreciation of generality and the applicability of ideas and frameworks beyond the professional development of mathematics teachers also came with a search for specificity. While the word 'mathematics' is mentioned multiple times, I explicitly sought specific mathematical concepts or problems. I found the mention of topics, such as 'logic', 'modelling' or 'whole number arithmetic', but the particulars were omitted. This search was based on my selfish approach to the chapters. As I spent long hours on reading the dense material, I wanted to find something 'useful' for my work with teachers. I recognised in this selfish approach my frequent critique of teachers: while we (that is, researchers, mathematics educators) attempt to extend teachers' horizons of knowledge, we are often disappointed that teachers are more appreciative of tricks and whistles that could be 'used in their classroom on Monday'.

Teachers collaborate on learning mathematics. What mathematics are they learning? Teachers engage in collaborative discussion after watching a video of a mathematics classroom. What was the video about? What was addressed in discussion? Teachers collaborate on reading and discussing research. What was read? What issues were discussed? Teachers are collaboratively working on mathematics problems and sharing their teaching of the problems. What are the problems? What teaching strategies or approaches are shared? I realised that in order to find specificity, one has to examine papers discussed in ICMI Study 25 and published in the conference proceedings (Borko & Potari, 2020), rather than chapters summarising the submissions and discussions.

There were, however, a few specific examples that extended my personal repertoire of tasks and experiences that I intend to bring to my class (more on this in the next section). One such example is found in Chap. 10 by Hollingsworth et al. (this volume), where a teacher described the following:

> I was teaching about subtraction of mixed numbers, e.g. $7\frac{1}{3} - 3\frac{2}{5}$. I asked students to explain how they could solve the problem. One student explained, "First subtract three from seven and get four, then subtract two-fifths from one-third." The student proceeded up to this stage: $4\frac{5-6}{15}$. And then the student said, "We take one from four, the whole number, and add to five ...". (p. 14)

I suspect that my students (prospective teachers) may attempt to correct the student and insist on the 'correct way' rather than to engage with student thinking, but I intend to test and hopefully revise my expectation.

12.5 On Content-Related Theorising

Prediger (Chap. 6, this volume) makes a case for content-related theorising. She describes a vignette related to a student difficulty with subtraction, and demonstrates how content-related theory elements may go beyond what is afforded by 'general' theories. I appreciate the particular mathematical content, and, when reading the chapter, I wondered whether the notion of 'content-related' refers to mathematics-related or particular-mathematical-topic-related. In a way, in his reaction to Prediger's chapter, Koichu (Chap. 7, this volume) addressed my query. He implemented several theory-elements suggested by Prediger in analysing teachers' engagement with a particular mathematical problem. Here is the problem:

> Epsilon Tower in Zedland is famous among tourists for its wonderful view from the top floor. During the pandemic, it is permitted to use the elevators ONLY on condition that every visitor wears a protective mask AND keeps the distance of at least 1*m* from the other visitors. Each elevator in Epsilon Tower is designed as a parallelepiped having a square floor with a side of 1.4*m*.
>
> a. How many visitors may simultaneously use one elevator?
> b. Prove that the number you found in (a) is the maximum number, that is, it is impossible to place any more visitors in the elevator without violating the rules.

This problem satisfied my thirst for the particulars, as I immediately engaged with the problem.

While this problem is embedded in the context of the current pandemic (hopefully just an endemic by the time someone reads this), it reminded me of the '4-Trees' problem, which is to plant four trees such that every pair is exactly one metre apart. Of course, the '4-Trees' are an attempt to create a story line, but the mathematical problem modelled by the 4-Trees is that of finding four points, any two of which are equidistant. The presumed immediate (for most solvers) impossibility of a solution is defeated when the imposed condition of finding 'Four points on the same plane' is challenged. The solution is offered by a regular tetrahedron, in which the distance between points is constant for each pair of the four vertices. Bringing the model back to the 4-Trees task, a solution involves planting one of the four trees on a hill.

Before attending to the solution suggested in Koichu's chapter, a beautiful solution that relies on the pigeonhole principle, I immediately thought of the 4-Trees problem. I imagined the four people in the corners to be kneeling or sitting on the floor, and placed the fifth person in the middle, choosing the tallest person or placing him on a stool, just in case. Note that the elevator in the Epsilon Tower is designed as a parallelepiped having a square floor (so I assume this is a rectangular prism), but that the height of the elevator was not given and restricting that height could influence the solution. Intuitively, having respiratory transmission in mind, I interpreted the distance between visitors as a distance between peoples' heads. Note that my suggested solution is represented by a square pyramid rather than a tetrahedron, because the requirement is not for equidistant pairs, but for a minimal distance, that is, for each distance to be at least one metre. So, the apex of my pyramid should attend to the distance requirement.

I shared the problem with a colleague, who was not familiar with the 4-Trees problem. She noted, based on the height of the people, and attending to the choice of 1.4m as the size of the base-square side, that there may not be any need for kneeling and stool. That is, if the distance between the mid-point of a square, and the vertex of a square (taken as hypothenuse of a right-angle isosceles triangle with each leg of

0.7) is about is about 0.9899 m, then the middle person has to be 'just a bit' taller for the distance between the heads to be over 1m. But this 'works' when people or their heads are viewed as points in space. Thinking of people, rather than points, there could be a need to consider the 'space' that each person occupies. Turning this thinking to modelling, what if people are represented by circles or spheres? How does this change the problem? In Koichu's solution based on the pigeonhole principle, what are the assumptions about such representations necessary for the suggested solution to hold? How does changing the given side length of 1.4m effect the solution?

I wonder what insights can be added to my collaborative engagements with a colleague by using the theory-elements suggested by Prediger and used by Koichu. In particular, what content-related theory-elements are applicable in a non-facilitated collaborative work on a problem?

12.6 On Effects or Products of Collaboration

What are the products or outcomes of teacher collaboration? These are related to the goals of collaboration or of professional development. Most outcomes can be described as teachers' 'professional growth', but such growth can take different forms. The particular outcomes, acknowledged across various chapters in this volume, are related to:

- enhancement of mathematics teachers' knowledge of mathematics (for personal enrichment);
- further development of mathematics teachers' knowledge for teaching;
- development of interdisciplinary knowledge needed for STEM education;
- improvement/change of teaching practice that promotes student learning;
- extended availability resources, such as particular problems, task sequences, assessment activities or lesson plans;
- extended familiarity with technological resources, such as particular platforms or websites;
- extended knowledge of a new curriculum;
- further knowledge in interpreting student work, student solutions or errors;
- changing views of the discipline of mathematics;
- development of teacher professional learning community.

In considering this partial list of possible outcomes, I recall John Mason's comments on the products of research:

> What are the significant products of research in mathematics education? I propose two simple answers: 1. The most significant products are the transformations in the being of the researchers. 2. The second most significant products are stimuli to other researchers and teachers to test out conjectures for themselves in their own context. (Mason, 1998, p. 357)

Here, I paraphrase and extend: what are the main products of teacher collaboration? They are the change in the teacher-collaborator and the stimulus to implement what is learned in their own context. In particular, it is an extended repertoire of mathematical tasks or instructional approaches. It is an extended appreciation of students' approaches or difficulties. It is an extended awareness of 'others'—other teachers, other curricular sequences and other professional environments. At least initially, these transformations in 'being a teacher' are not directly measurable.

12.7 The Main Issue

Some background is necessary before I get to the 'main issue', so I seek the reader's indulgence. Simon Fraser University—which is my affiliation for over 30 years—offers a Master's program in Secondary Mathematics Education (SME). The program, that attracts secondary mathematics teachers, was designed as a collaboration between the Faculty of Education and the Department of Mathematics several years before I joined SFU; the core coursework in the program is from both Education and Mathematics. However, the distinctive feature of the Mathematics courses is that they were designed specifically for teachers. That is, unlike other graduate level courses in mathematics, these courses do not assume fluency with undergraduate content. It is a 'cohort program'—meaning that the courses are offered in a 2-year sequence and the students take all the courses together.

The first course in the program is titled *Foundations of Mathematics* and is intended to focus on 'big ideas' and 'great theorems'. For several initial offerings this course was taught by a Professor of Mathematics, the late Dr. Harvey Gerber. However, following Harvey's retirement, the course has become part of my regular teaching assignment. That is where we get to the 'main issue'.

Preparing for teaching *Foundations of Mathematics* for the first time (this was in 2001), I set an appointment with Harvey, expecting to learn his perspective on what 'foundations' are essential for secondary school teachers of mathematics and how teachers should be introduced to foundational ideas. I also had my own list prepared, intending to seek feedback from an experienced colleague. Harvey's response surprised me at the time, and it still resonates with me today. He said: "The choice of a particular mathematics topic is not important. What is important is that students work together. They have to take courses together for two years. Your main goal in the first course is to build community." I was astonished, not only by the response, but by the fact that this was a response from a mathematician! And it happened before the constructs like 'professional learning community' or 'community of practice' entered my lexicon.

Meeting and teaching every new cohort of teachers I remind myself of Harvey's advice. While teacher collaboration is not one of the explicitly stated program goals, it has become an extremely valuable feature and outcome of our graduate SME program. The courses engage students in collaborative problem solving, collaborative task design and collaborative learning of new (for them) mathematics. Using the framework of resources (Brodie & Jackson, Chap. 9, this volume), I suggest that the coursework curriculum attends mostly to knowledge (both mathematical and pedagogical) and representational resources (the latter may include lesson plans, videos of teaching and excerpts from student work). However, as teachers progress in the program, they develop their assembly of human resources, which often compensate for insufficient institutional resources.

As years pass, we learn that many of our graduates continue to collaborate after their graduation. They meet in pairs or small groups to share experiences and ideas, as well as problems and solutions. I wish I could share with Harvey this success.

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Afterword

The mathematician is often visualized as a grey-hair man working alone with paper and pencil in front of his desk. Such stereotype may convey to the student of mathematics that this is the standard way of learning mathematics. Research in mathematics education in the past decades has however refuted such a model of learning mathematics. On the contrary, in line with a social constructivistic view of learning, collaboration in the process of mathematics learning is favoured.

In the same vein, research in the past decades has promulgated teachers working together in collaborative groups in preparing for and implementing their mathematics teaching. This is partly due to the Japanese model of Lesson Study being widely disseminated internationally. At ICME-13, a Survey Team was devoted to the issue of Mathematics Teachers Working and Learning through Collaboration, and the findings of the Survey Team provided the basis on which this ICMI Study on Teachers of Mathematics Working and Learning in Collaborative Groups is built.

Collaboration among mathematics teachers enhances their teaching (which in turn benefits student learning), furthermore, the collaboration among teachers itself enriches their personal growth. Likewise, collaboration among researchers and other educators, as exemplified by the IPC members, authors and participants of the Study Conference during the process of working on this ICMI Study, leads to their own professional learning and personal growth through working together on this long collaborative journey.

As we reach the conclusion of ICMI Study 25, let us take a moment to reflect on this remarkable collaborative journey that the group has embarked upon. This study has brought together a diverse mix of educators, researchers, and experts from around the globe to explore the myriad ways in which collaboration can enhance the professional development of mathematics teachers and, in turn, the learning outcomes of their students. Throughout this Volume, we have reaffirmed the theories underpinning collaborative learning of teachers, which help us to better conceptualize such collaboration for deeper understanding of what teachers do as they work collaboratively in different cultural contexts. We have witnessed a wealth of

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innovative ideas, best practices, and research findings that have broadened our understanding of the power of collaboration in mathematics education, and its wide applications in future for professional enhancement of teachers. The contributors to this book have delved into various facets of teacher collaboration, such as the role of technology, the impact of cultural context, the challenges and opportunities that arise from working in collaborative groups, and the potential for collaborative efforts in transforming teaching practices and curricula.

One of the key insights gleaned from this Study is the importance of fostering a culture of collaboration among mathematics educators. By engaging in ongoing dialogues, sharing experiences, and reflecting on their practices, teachers can deepen their understanding of the subject matter, improve their instructional strategies, and better meet the diverse needs of their students. Another significant takeaway from the Study is the recognition that collaboration is not a one-size-fits-all approach. Successful collaborative efforts require flexibility, adaptability, and an openness to learning from one another. The diverse experiences and perspectives shared in this Volume serve as a testimony to the value of embracing a wide range of collaborative approaches in mathematics education.

As we move forward, we can build upon the findings of this Study to continue advancing the field of mathematics education through collaborative efforts. We hope that this Volume will inspire researchers, policy makers, teacher educators, and school teachers to further explore the potential of teacher collaboration as a means of enhancing professional development and improving student outcomes in mathematics.

In closing, on behalf of ICMI, I would like to express my deepest gratitude to Co-Chairs Hilda Borko and Despina Potari, the rest of the IPC members, and all contributors, reviewers, and editors who have been part of this extraordinary journey. Your dedication, expertise, and passion for mathematics education have made this Study a success, and you should be proud to have been partaking in this collaborative effort. Thank you for your invaluable contributions to the field and for your commitment to shaping a brighter future for mathematics educators and their students worldwide.

Frederick Leung
ICMI President (2021–2024)

Appendix: Discussion Document

ICMI-25 Study

Teachers of Mathematics Working and Learning in Collaborative Groups

Discussion Document

Contents

Prepared by the International Program Committee
13 February 2019

The Need for the Study

Collaboration implies a careful negotiation, joint decision-making, effective communication and learning in a venture that focuses on the promotion of professional dialogue (Boavida & da Ponte, 2002). Across education systems, and at all educational levels, mathematics teachers work and learn through various forms of collaboration. Such collaborative work of teachers has a long tradition in mathematics education as it is critical as a way to bring educational innovation into the everyday practice of teaching. For example, just after the first ICME congress (1968) in Lyon, Freudenthal founded the Institute for the Development of Mathematical Education at Utrecht University and the IREM network (Institute of Research on Mathematics

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Teaching) was created in France. Both institutions were based on the collaboration of teachers from different educational levels (see Trouche, 2016).

In mathematics education research, teacher collaboration is gaining increasing attention, particularly since the report on Lesson Study in Japan from the TIMSS classroom video study (Stigler et al., 1999). This attention to teachers learning through collaboration is especially relevant as countries around the world strive to improve educational experiences for all children and to see these improvements reflected on international assessments such as PISA and TIMSS (Schleicher, 2015). Indeed, Schleicher's OECD report includes a policy recommendation to "Encourage collaboration among teachers, either through professional development activities or classroom practices" (p. 56). It cites research indicating that collaborative professional development is related to a positive impact on teachers' instructional strategies; their self-esteem and self-efficacy; and student learning processes, motivation and outcomes.

Efforts to understand what teachers do as they work in collaborative groups, and how these experiences lead to improvement in their expertise and teaching practice, has led to increasing interest in examining the different activities, processes, contexts, and outcomes for teacher collaboration around the world. The work completed by the ICME-13 Survey Team on this theme is further evidence of the considerable international interest in research on teachers working and learning through collaboration (Jaworski et al., 2017; Robutti et al., 2016). However, the ICME-13 Survey also identified several gaps and limitations, not only in the existing research base but also in the coverage of relevant topics related to teacher collaboration. For example, Jaworski et al. reported that their research questions about learning outcomes were the most difficult to address. They did not have consistent clarity on the specific mathematics knowledge and pedagogy that were learned, the ways in which learning occurred, or the relationship between learning and collaboration. As they also noted, there were issues such as sustainability, scaling up, the role of digital technology in teachers' collaborative learning, working with teachers of different educational levels, and making teachers' voice more evident for which the survey showed that research is not extensive and further studies are needed.

These gaps and limitations highlight the need for the ICMI Study 25. We hope that this Study will help us to better understand and address these challenges in the study of the processes and outcomes of mathematics teacher collaboration.

Aims and Rationale

The Study's theme of teachers working and learning in collaborative groups implies a focus on teachers as they work within teams, communities, schools and other educational institutions, teacher education classes, professional development courses, local or national networks—that is, in any formal or informal groupings. Teachers' collaborative work might also include people who support their learning and development such as teacher educators, coaches, mentors, or university academics. Collaboration can extend over different periods of time, and take place in face-to-face settings or at a distance. The role of online platforms and technology-enabled social networks is an additional focus in supporting 'virtual' collaboration.

We encourage reporting on promising forms of collaborative work among different groups of participants (e.g. teachers/researchers, teachers/curriculum designers, teachers from different disciplines) and collaboration that addresses different goals (e.g. design of tasks, lessons and curriculum materials; improvement of teaching; development of mathematical and pedagogical understanding). The Study will acknowledge that learning is mutual; that is, those who work collaboratively with teachers to develop their practice are also learning from these interactions.

The primary aims of the study are to report the state of the art in the area of mathematics teacher collaboration with respect to theory, research, practice and policy; and to suggest new directions of research that take into account contextual, cultural, national and political dimensions. Because there are different ways of understanding teacher collaboration and its characteristics, enablers, and consequences, the Study will include multiple theoretical perspectives and methodological approaches. We encourage contributions that report research using a variety of methodological approaches including large-scale experimental and descriptive studies, case studies, and research approaches characterised by iterative or cyclical processes such as design research and action research. We also solicit contributions from teachers as well as researchers, to ensure that teachers' voices are given prominence in accounts of their learning.

Themes and Questions

The areas and questions that the Study will investigate are outlined below, organised into four themes. These areas are not independent, and some questions can reasonably be placed in more than one area.

(A) Theoretical perspectives on studying mathematics teacher collaboration

A number of theoretical and methodological perspectives have been used to study teacher collaboration, illuminating the dynamics of teachers' collaborative working and the communities in which they work. Below we discuss some of these perspectives. This list is not meant to be exclusive; papers that address other theoretical and methodological perspectives are welcome.

Several theoretical perspectives have focused on the nature of the communities in which teachers collaborate. In studying teacher learning communities one must be aware that the word "community" is polysemic (Crecci & Fiorentini, 2018), encompassing different meanings. Wenger's theory of communities of practice (Wenger, 1998) has been used to study the process of teacher collaboration, focusing on the negotiation of meaning, the formation of common goals and the building of a teaching identity (e.g. Goos & Bennison, 2008). Adaptations of this theory focused on teacher collaboration include communities of inquiry (Jaworski, 2006) where the teachers align critically to the practice of the community; that is, they do not accept this practice as it is but instead question some of its characteristics. An example of how this perspective has been used to study the impact of the collaboration between upper secondary mathematics teachers and academic researchers in a national project in Norway is reported in Goodchild (2014). The construct of "critical alignment" has been used to describe tensions that the teachers faced to adopt the inquiry teaching approach that the project promoted.

The idea of community also has been conceptualised using the perspective of Activity Theory where the activity and its object—for example, the teaching of mathematics and the learning of mathematics—have been achieved collaboratively through the mediation of tools and framed by the communities' rules and division of labor (Jaworski & Potari, 2009). In Activity Theory contradictions are central in the transformation of the activity (Engeström, 2001) and have been used as a way to study tensions emerging in the context of teacher collaboration and the process of overcoming them as an indication of professional learning (Stouraitis et al., 2017).

Professional learning in these perspectives has been seen as shifts of teachers' participation in a community of practice or as expansive learning in relation to the transformation of the teaching activity at the boundaries of different practices (Akkerman & Bakker, 2011). Goos et al. (2007) use Valsiner's (1987) theory of zones and its application to mathematics teacher education by Goos (2005) to examine how the Zone of Proximal Development (ZPD), the Zone of Free Movement (ZFM) and the Zone of Promoted Action (ZPA) can be interrelated in the suggested professional development program and what interrelationships of these zones indicate for teachers' professional learning.

Some theoretical frameworks developed within mathematics education research allow us to investigate different aspects of teacher collaboration. For instance, the Documentational approach to didactics studies teacher collaboration in focusing on their interactions, as users as well as designers, with resources (Pepin et al., 2013). Based on the Anthropological theory of the didactic (Chevallard, 1985), some concepts have been elaborated to describe mathematics teachers working in collaboration in different settings. This theory characterizes mathematical knowledge and its teaching and learning in terms of *didactic transposition* and *praxeologies.* The concept of *meta-didactical transposition* (Arzarello et al., 2014; Robutti, 2018) takes into account the different dimensions of collaboration and the differemt actors involved. The concept of *paradidactic infrastructure* (Miyakawa & Winsløw, 2019) characterises the different settings for teacher collaboration inside and/or outside school. Other theoretical perspectives have also focused on conceptions of teacher learning.

Although coming from a different starting point, the themes and questions they illuminate overlap with those identified by theoretical perspectives on the nature of community. For example, a situational perspective posits that knowing and learning are situated in particular physical and social contexts; social in nature; and distributed across the individual, other persons, and tools (e.g. Greeno, 1997). Putnam and Borko (2000) identified three issues to consider when applying these themes to teacher learning: where to situate teachers' learning experiences, the nature of discourse communities for teacher learning, and the importance of tools in teachers' work. The practice-based theory of professional education introduced by Ball and Cohen (1999) also addresses several of these issues, and in addition considers the mechanisms underlying teacher learning. Ball and Cohen suggest that professional development programs should situate teacher learning in the types of practice they wish to encourage.

Teacher collaboration is studied with different methodological approaches, in connection with the underlying theoretical framework, both from within the collaborative group and from the outside. Designs and approaches used to study collaboration include case studies, action research, design-based research and others. Data sources can include participants' journals, narratives, questionnaires and surveys, interviews, and audio and video recordings of the collaborative activities or of the activities that the collaborative group carries out with other participants. There is no single way to document collaboration—the important issue is that data is thorough, systematic, reliable and authentic regarding the perspectives and practices of participants.

These theoretical and methodological perspectives suggest several questions to be explored in this ICMI Study:

- *How do the different theoretical perspectives or networks of theories enhance understanding of the processes of teacher collaboration?*
- *How do they enhance understanding of the outcomes of teacher collaboration?*
- *What is illuminated by the different perspectives and methodologies and what needs further investigation?*
- *What are promising research designs and data colection and analysis methods to study teacher collaboration?*

(B) Contexts, forms and outcomes of mathematics teacher collaboration

The assumption underlying this Study is that teachers *learn* through collaboration; however, it can be challenging to investigate and explain the *processes* through which this learning occurs and to gather evidence of *what* teachers learn. The goals of teacher collaboration are multi-faceted and might be related to the mathematics content, to the learning experience of students, to the development of mathematics teaching that promotes students' learning (e.g. to implement new curriculum materials), to the design of resources such as classroom and assessment tasks, to the creation of a community in which ongoing professional learning is supported, or even to day-to-day teaching (e.g. lesson preparation, team teaching). Similarly, the outcomes of the collaboration also vary. For example, within the context of Lesson Study, researchers have identified changes in teachers' beliefs or disposition for working and learning, their mathematics knowledge for teaching, and their teaching practice (Huang & Shimizu, 2016; Xu & Pedder, 2015). This theme focuses on outcomes related to teachers, teaching and students. Outcomes related to teachers' and teacher educators' interactions are addressed in Theme C and those related to instructional materials are addressed in Theme D.

Various forms of collaboration have been used to support mathematics teacher learning. One central form is Lesson Study, a highly structured practice-based approach, originated in Asia (Chen & Yang, 2013; Lewis & Tsuchida, 1998), that has spread globally (Huang et al., 2018). Variations of lesson study have been used in different national contexts, such as the United States (Murata et al., 2012), the United Kingdom (Dudley, 2015), Italy (Bartolini Bussi et al., 2017), Thailand (Inprasitha, 2011) and South Africa (Adler & Alshwaikh, 2019); and in pre-service as well as in-service teacher education (Rasmussen, 2016). Although there are some core elements common to all variations of lesson study, these enactments have different design features and have been associated with differences in teacher learning outcomes (Akiba et al., 2019). This is due, in part, to the different institutional, cultural and social environments in which they have been implemented (Mellone et al., 2019). The study of Miyakawa and Winsløw (2019) shows that even in Japan lesson study functions in different ways, depending on the institutional conditions and motives that the teachers have in the context of their practice.

There are several other types of professional learning opportunities for teachers in which collaborative work plays a central role. Learning study is a combination of lesson study and design research, initiated in Sweden and Hong Kong, driven by the theory of variation (Marton, 2014), where the goal for students' learning is more explicit and the way that this goal can be achieved is very clearly defined (Pang & Marton, 2003). In the context of professional development initiatives in Zimbabwe (Mtetwa et al., 2015) teacher collaboration can be found in workshops where teachers meet on their own initiative to organize common instructional goals in designing curriculum and in networking, for example, between teacher associations and government authorities. The different professional development providers and the social demands in this country seem to pose different constraints to mathematics teachers and teacher educators than in other developed countries. Similarly, Cristovão and Fiorentini (2018) documented teachers' learnings through reviewing mathematical instructional task sequences developed by teachers and researchers within a collaborative community.

In professional learning communities the creation of a culture of collaboration and formation of common goals become central. Successful professional learning communities are characterized by a systematic process in which the group of teachers engage to explore mathematics learning and teaching (DuFour, 2004). In addition, teachers work collaboratively in numerous formal mathematics professional development programs and courses for mathematics teachers, which also vary in their design features, goals and outcomes (Sztajn et al., 2017).

Many of these forms of mathematics teacher collaboration are offered traditionally in face-to-face settings, although online mathematics teacher collaborative approaches have become more and more popular (Community for Advancing Discovery Research in Education, 2017). Some are offered by university researchers, others by private vendors or by professional development leaders within schools or school systems. The Study will address the various forms of teacher collaboration, their outcomes related to teaching and learning, and the contexts in which they are offered.

- *What models of teacher collaboration have been developed? What are the design features, goals, and outcomes of the different models?*
- *How effective are various models for promoting different outcomes?*
- *Which forms of collaboration are appropriate in different contexts?*
- *What are the affordances and limitations of each form of teacher collaboration?*
- *What are the benefits and the challenges that online teacher collaboration poses to the teachers?*

(C) Roles, identities and interactions of various participants in mathematics teacher collaboration (e.g. lead teachers, facilitators, mathematicians, researchers, policy makers)

Collaborative groups can include different "actors", such as teachers, facilitators, mathematicians, researchers, administrators, policy makers or other professionals, in various combinations. These participants can assume a variety of roles in collaborative activities, including learners, leaders, designers, researchers, and more. The literature indicates that different roles can support productive interactions. Robutti et al. (2016) highlighted the value of diversity of roles amongst group members: For example, university academics' perspectives help teachers and others to see and interpret local practices in new ways (Redmond et al., 2011). Olsen and Kirtley (2005) reported that "interaction between high school teachers and elementary teachers with their different expertise was critical" (p. 31). Within teacher collaborative groups, the participating teachers may assume different roles (van Es, 2009). Also, in many collaborations, the roles of participants shift over time (Jaworski, 2006).

In collaborative interactions, the learning of *all* participants is important. For instance, Cooper studied the mutual learning of mathematicians and primary mathematics teachers in a professional development program (Cooper, 2018; Cooper & Karsenty, 2018). Bleiler (2015) focused on the process of collaboration between a mathematician and a mathematics educator and indicated that the collaboration resulted in professional development by both participants.

The nature of roles that people play can vary in different countries and cultural contexts. For example, in lesson study, the role of the "knowledgeable other" varies across and within cultural contexts (e.g. Adler & Alshwaikh, 2019; Gu & Gu, 2016; Lewis, 2016; Takahashi, 2014). In some places, established relations between policy makers, researchers, facilitators and teachers can support the process of collaboration (e.g. Bobis, 2009; Higgins & Parsons, 2009), while in other places this might not be the case (e.g. Santagata et al., 2011). Since unsuccessful collaborations are not frequently reported in the literature, we encourage submissions that explore less successful cases and analyse the challenges they faced.

A variety of research-informed approaches for supporting teachers to work collaboratively and also for developing teachers as leaders have emerged around the world. In these studies, the role of the facilitator and the nature of interactions between the facilitator and the teachers are important topics to explore (van Es et al., 2014). Challenges faced by those taking on the role of facilitating teacher collaborations, can include on the one hand supporting teachers to develop their teaching and on the other hand valuing and promoting their own goals and perspectives. This and other facilitating challenges are often reported in the research. It is also agreed upon that a critical component of a sustainable and scalable model of collaboration is the preparation of facilitators who can adapt the model to various contexts while maintaining integrity to its original goals and agenda (Borko et al., 2014). The non-trivial move from being a good mathematics teacher to becoming a successful facilitator is increasingly studied in recent years (e.g. Borko et al. 2015; Even, 2008; Kuzle & Biehler, 2015). However, empirical studies on the professionalization process that facilitators undergo are still relatively scarce. Specifically, facilitators may hold multiple identities (Gee, 2000) regarding their role in the collaboration; for instance, lead teachers may experience dual identities, as teachers and as facilitators. This has not been sufficiently reported in the literature, and we particularly encourage contributions to this theme by lead teachers.

We invite contributions focusing on these issues, as reflected in the following questions:

- *What is the role of lead teachers, facilitators, mentors and teacher educators in supporting teacher collaboration?*
- *How are different roles and identities shaped and developed among various "actors" (teachers, leaders, mathematicians, researchers, etc.) within a collaborative group? How do lead teachers negotiate their dual roles and identities as both teachers and facilitators of peer-collaboration?*
- *What are characteristics of a good facilitator of teacher collaboration? How can these facilitators be prepared and supported?*
- *How can different stakeholders impact teacher collaboration?*
- *What types of learning environments enhance or hinder mutual learning of teachers and other participants in collaborative interactions?*

(D) Tools and resources used/designed for teacher collaboration and resulting from teacher collaboration

This theme focuses on the role of tools and resources in facilitating and supporting teacher collaboration. Tools, as well as resources, are understood in a broad sense that go "beyond material objects and include human and cultural resources" (Adler, 2000, p. 207). Taking into account their diversity, we are interested here in tools and resources with respect to teachers' collaboration: tools and resources *for* teacher collaboration and tools and resources *from* teacher collaboration.

1. Resources *for* teacher collaboration

Drawing from activity theory (Wertsch, 1981), Grossman et al. (1999, p. 14) make a distinction between *conceptual* tools and *practical* tools. Conceptual tools are "principles, frameworks, and ideas about teaching [and] learning . . . that teachers use as heuristics to guide decisions about teaching and learning". Practical tools—classroom practices, strategies, and resources such as daily and unit plans, textbooks, and instructional materials—in contrast, "do not serve as broad conceptions to guide an array of decisions but, instead, have more local and immediate utility". A variety of conceptual and practical tools have been used to support mathematics teacher collaboration: for example, frameworks of student mathematical thinking (Carpenter et al., 2014) or teacher noticing (Jacobs et al., 2010) in the case of conceptual tools; mathematical tasks (Kaur, 2011), students' mathematical work (Brodie, 2014; Kazemi & Franke, 2004), video of mathematics learning and teaching (Jacobs et al., 2009; Karsenty & Arcavi, 2017) or animated representations of teaching (Chieu et al., 2011) in the case of practical tools.

There is also a wide range of "new" tools, arising in the digital era that have the potential to resource collaboration between teachers. For example, researchers have studied spontaneous and supported teacher collaboration in MOOCs (Panero et al., 2017; Taranto et al., 2017) and the use of digital curricular resources in teacher education (Pepin et al., 2017). These resources also include software packages that support the annotation of video records of lessons as a means of stimulating collaborative reflection (e.g. Angles®, fulcrumtech.com; Interact®, mangold-international.com), and portable devices such as mobile phones and tablets that facilitate social networking on dedicated platforms.

A further resource for teacher collaboration comprises professional, institutional or governmental support for forming teacher associations or school clusters, which may lead to creation of larger regional or national networks of teachers. This resource will be explored across contexts (curricular, cultural and social), in which teacher collaboration can be supported and/or constrained in different ways. Considering resources for teacher collaboration raises issues of *quality* (for example, the affordances/potential of a given resource for fostering teacher collaboration) and *equity* (for example, the *missing* resources for supporting teacher collaboration in a given context).

2. Resources *from* teacher collaboration

Resources as outcomes of teacher collaboration are addressed in Theme D as *concrete* evidence of building a community in the sense of Wenger (1998): developing teacher collaboration and developing a shared set of resources go together (Gueudet & Trouche 2012). For example, the intertwined relationship between the process of collaboration and the development of resources is illustrated in lesson study. It is reinforced in the digital era, with its new means for collaborative design: teacher collaboration may lead to the development of large repositories of resources, as in the case of the French association Sésamath (Gueudet et al., 2016), designing a complex set of resources, including textbooks, software, and a platform for teacher collaboration.

This resource approach to teacher collaboration raises a number of issues, such as the effects of interactions between the teachers' individual resources and the resources emerging from teacher collaboration; the interactions between different sets of resources coming from different collectives (Akkerman & Bakker, 2011; Kynigos & Kalogeria, 2015; Robutti et al., 2019); and the *coherence* and *quality* of resources resulting from teacher collaboration (Gueudet et al., 2016).

Resources *for* and *from* teacher collaboration can be considered as two ingredients of continuous processes: *adopting* a resource leads always to *adapting* it, and that is more the case in the context of teacher collaboration. Using and designing are then to be considered as two intertwined processes. Taking into account this dialectical point of view, the Study will investigate the roles of resources in facilitating teachers' collaboration, and how those roles differ in different contexts. It will focus on the following questions:

- *What resources are available to support teacher collaboration? With what effects, both on the collaboration and on the resources themselves*
- *What resources are missing for supporting teacher collaboration? How and to what extent can teachers overcome these missing resources?*
- *To what extent and under what conditions do digital environments (e.g. mobile devices, platforms, applications) constitute opportunities for teacher collaboration? How have these resources been used to support teacher collaboration?*
- *Which resources can be used (and how) to sustain and scale up collaboration over time?*
- *How are teachers engaged in the design of resources in collaboration? What are the outcomes of these collaborations?*

The Study Conference

ICMI Study 25 is planned to provide a platform for teachers, researchers, teacher educators and policy makers around the world to share theoretical perspective, research, policy, and professional experiences related to mathematics teacher collaboration in small and large scale settings. The Study is built around an International Study Conference and directed towards the preparation of a published volume. The conference will encourage collective work on significant issues related to the topic of teacher collaboration that will form parts of the study volume.

The Study Conference will be organized around working groups based on the four themes described in Part III. These groups will meet in parallel during the conference. It is the work of these groups that is captured as chapters in the ICMI Study 25 volume.

Location and Dates

The Study Conference will take place in the Institute of Education of the University of Lisbon from 3rd to 7th of February 2020, with a reception on the evening of Monday the 3rd of February.

Participation

Participation in the Study Conference will be by invitation only, for one author of each submitted contribution that is accepted. Proposed papers will be reviewed and a selection will be made on the basis of their quality, their potential to contribute to the advancement of the Study, their links to the themes described in the Discussion Document, and their contribution to a diversity of perspectives. The number of the invited participants will be limited to approximately 100 delegates.

Unfortunately, an invitation to participate in the conference does not imply financial support from the organisers, so the participants should plan to finance their own attendance. Some partial support to enable participation from non-affluent countries can be offered, however we anticipate that only a few such grants will be available.

Outcomes of the ICMI Study 25 Conference

The accepted papers will be be published in an electronic volume of conference proceedings that will first be available on the conference website and later on the ICMI website. The proceedings will have an ISBN number, which can be cited as a refereed publication.

An ICMI Study 25 volume will also be developed on the basis of the papers and the discussion in the working groups. This volume will be published by Springer as part of the new ICME Study Series. The International Programme Committee (IPC) will be responsible for editing this volume. It is expected that the Study volume will be structured around the themes included in the Discussion Document, as they are developed further during the Study Conference. Therefore, the chapters will integrate the outcomes of the working groups of the conference, as well as contributions from the plenary addresses and panels. Options for authorship of the chapters in a Study volume are outlined in the Study guidelines (https://www.mathunion.org/icmi/publications/icmi-studies/guidelines-conducting-icmi-study). Authorship of the working group chapters of this Study volume will be decided in the context of the groups.

Call for Contributions

The IPC for ICMI Study 25 invites submissions of several types including: reports of research studies, syntheses and meta-analyses of empirical studies, discussions of theoretical and methodological issues, and examinations of the ways that teacher collaboration has taken place in local or national contexts. Studies from different cultural, political, and educational contexts and submissions by researchers, teachers, and policy makers are encouraged so that mathematics teacher collaboration can be addressed in its complexity.

The papers should be clearly related to the themes that are discussed in Part III and address the questions associated with the themes. Authors must select one of the themes to which their paper will be submitted.

Submission

The papers should be submitted through the ICMI Study 25 online system. A template for submission of papers is available on the Study website (see below). Papers must be maximum of eight pages and not have been submitted or published elsewhere. The working title of the paper must contain the author(s) name(s) and the theme letter to which it is submitted: for example—James Theme B.

Conference Presentations and Proceedings

Verbal presentations at the conference will be brief, at most five minutes, with the expectation that participants will have read the papers in advance. Presenters will focus on posing questions and issues raised by their paper and its relation to other papers presented in the working group. As explained above, accepted papers will be published in online proceedings. Accepted papers will also form the basis for discussions in the working groups at the Study Conference and, eventually, for the chapters in the ICMI Study 25 volume.

Deadlines

30th of June, 2019 Submissions must be made online through the ICMI Study website no later than the 30th of June but earlier if possible.

30th of September, 2019 Decisions from the reviewing process will be sent to the corresponding author by the 30th of September.

Information about registration, costs and details of accommodation may be found on the ICME Study 25 website: *icmistudy25.ie.ulisboa.pt*

[The website is currently under development.]

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