Proceedings of the 14th International Congress on Mathematical Education

Volume I

Editor

Jianpan Wang

East China Normal University Press

World Scientific

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The 14th International Congress on Mathematical Education

Shanghai, China, 2021

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East China Normal University, China

East China Normal University Press

World Scientific

Published by

East China Normal University Press
3663 North Zhongshan Road
Shanghai 200062
China

and

W

5 Toh Tuck Link, Singapore 596224
USA 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Control Number: 2024002523

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

PROCEEDINGS OF THE 14TH INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION
(In 2 Volumes)
Volume I
Volume II: Invited Lectures

ISBN 978-981-12-8937-8 (set_hardcover)
ISBN 978-981-12-8716-9 (set_ebook for institutions)
ISBN 978-981-12-8719-0 (set_ebook for individuals)

ISBN 978-981-12-8714-5 (vol. 1_hardcover)
ISBN 978-981-12-8715-2 (vol. 1_ebook for institutions)

ISBN 978-981-12-8717-6 (vol. 2_hardcover)
ISBN 978-981-12-8718-3 (vol. 2_ebook for institutions)

For any available supplementary material, please visit
https://www 142/13700#t=suppl

Printed in Singapore

https://doi.org/10.1142/9789811287152_fmatter

Editor's Notes

This is the first volume of two-volume Proceedings of the 14th International Congress on Mathematical Education, held in Shanghai, China, from July 11−18, 2021.

This volume begins with the opening ceremony and ends with the closing ceremony. They are the procedural contents of the Congress.

Opening Ceremony: My host words introduce and connect the main procedures of the opening ceremony, but we separately include the speeches/addresses of some attenders of the ceremony (government officials and heads of relevant academic groups). The second half of the opening ceremony is the presentation of ICMI Awards (Felix Klein awards and Hans Freudenthal Awards of 2017 and 2019, as well as Emma Castelnuovo Award of 2020) chaired by Frederick Leung, the President of the International Commission on Mathematical Instruction. Although there were no awardees present on site to receive the awards, we still include citations for all awardees and their acknowledgements from the pre-recorded videos.

Closing Ceremony: The Local Organizing Committee introduced the basic situation of the preparation and convening of ICME-14. Then the Executive Committee of ICMI gave reports on ICMI EC's work and changes over the five years since ICME-13. As a routine action, the Convenor of the 15th International Congress on Mathematical Education delivered the welcome speech. Finally, I made closing remarks to thank all the people who offered help and support from bidding to organizing the 14th International Congress on Mathematical Education.

Except for these two procedural contents, all contents in this book are academic in nature. They are

Plenary Lectures: There are four plenary lectures in which speakers share their understanding, practices, and research outcomes with the audience in the field of mathematics and mathematics education. Full texts of these lectures are included, with one transcribed based on the recorded video during the lecture.

Plenary Panels: Three panels involve researchers discussing and debating on topics regarding current challenges facing mathematics educators around the world. Especially in the context of the COVID-19 pandemic, a related panel was designed and added to the Congress program, exploring the challenges, responsibilities, and roles faced by mathematics and mathematics education. These full-length individual reports are included in this volume.

Lectures of Awardees: Five lectures were delivered, representing Felix Klein Award and the Hans Freudental Award of 2017 and 2019, and the Emma Castelnuovo

Award of 2020 respectively. Full texts of these lectures are included, with one transcribed based on the recorded video.

Survey Teams: Four pre-established survey teams conducted their investigation and research on their designated themes before the Congress, and presented their work results brilliantly on the Congress. The full-length survey reports are included in this volume.

Topic Study Groups: The International Program Committee of the Congress designed 62 Topic Study Groups, the largest number in history of ICME's. Participants actively submitted papers/posters. Unfortunately, we do not have enough space to showcase the papers/posters of contributors. Instead, we only include a summary of each TSG provided by the organizers.

Discussion Groups: These are activities designed and organized by the participants themselves, which can to some extent reflect academic concerns. Thirteen discussion groups worked out their succinct reports.

Workshops: These are also participants' self-organized activities. Twenty reports covering a wide range of topics in mathematics education are included.

Thematic Afternoon: This is designed as an event that showcases local mathematics education practices from multiple perspectives. There are a total of thirteen activities presenting mathematics education with Chinese characteristics. An overview of these activities is included.

Early Career Researcher Day: This program was initiated in the 13th International Congress on Mathematical Education. This event is not part of but is attached to an ICME. It aims at providing early career researchers with opportunities to develop their research competencies and establish their contacts with international academic networks. We include in this volume an overview about the Early Career Researcher Day on ICME-14, introducing its general aims, focuses and activities.

The above describes the main contents of this volume.

Invited Lectures of ICME-14 will appear in Volume II of the proceedings, while National Presentations will not be included in the proceedings.

Jianpan Wang
Shanghai
December 2023

https://doi.org/10.1142/9789811287152_fmatter

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https://doi.org/10.1142/9789811287152_0001

The Opening Ceremony

Opening Address and Host Words

Jianpan Wang[1]

Distinguished Guests,
Dear Participants, on-site and online,
Ladies and Gentlemen,

I'm Jianpan Wang, the Congress Chair and the Chair of International Program Committee (IPC) of the 14th International Congress on Mathematical Education (ICME-14). First of all, on behalf of the International Program Committee and the Local Organizing Committee (LOC) of ICME-14, I'd like to extend warm welcome and best regards to all the guests present in the hall and the participants both on-site and online!

Fig. 1. Prof. Jianpan Wang presided over the opening ceremony of the 14th International Congress on Mathematical Education

[1] Professor, East China Normal University; Chair of the 14th International Congress on Mathematical Education. E-mail: jpwang@admin.ecnu.edu.cn.

Please allow me to introduce some distinguished guests to you.

Mr. Qiang Li, Secretary of Communist Party of China (CPC) Shanghai Committee[2],

Ms. Tiehui Weng, Vice Minister of the Ministry of Education of the People's Republic of China,

Mr. Shaoliang Yu, Vice Secretary of CPC Shanghai Committee[3],

Mr. Yujie Zhuge, Secretary General, CPC Shanghai Committee[4],

Mr. Frederick Leung, President of the International Commission on Mathematical Instruction (ICMI),

Mr. Gang Tian, President of Chinese Mathematical Society, and

Mr. Xuhong Qian, President of East China Normal University (ECNU).

Mr. Carlos Kenig, President of International Mathematical Union (IMU) and Mr. Jinpeng Huai, Executive Vice President of China Association of Science and Technology[5], cannot make it to be present for some reason, but they sent us their address for the Congress.

Let's welcome all the distinguished guests with warm applause!

Fig. 2. A glance at the on-site venue

The 14th International Congress on Mathematical Education (ICME-14), originally set to be held in 2020, would have been a perfect chance for fellow scholars of mathematics education all over the world to visit China, an ancient country with vitality, and Shanghai, a city in which East meets West, tradition blends with modernity, and art integrates perfectly with technology.

Unfortunately, as a result of the COVID-19 pandemic, many participants miss the

[2] Mr. Qiang Li is currently the Premier of the State Council of the People's Republic of China.
[3] Mr. Shaoliang Yu is currently the chief editor of the *People's Daily*.
[4] Mr. Yujie Zhuge is currently Vice Secretary of CPC Hubei Committee.
[5] Mr. Jinpeng Huai is currently the Mister of Education of the People's Republic of China.

opportunity to visit this wonderful city, and the Congress, after a whole year's delay, has to adopt a hybrid mode of participation to ensure the health for all. However, we've overcome the challenges of physical distance and time difference, we meet here as scheduled from all over the world at the call of our common dream and mission of mathematics education. I believe, with your strong support and active participation, this Congress will surely become a memorable chapter in the history of international mathematics education regardless of the twists and turns we've left behind.

Now, let's welcome the Secretary of CPC Shanghai Committee Mr. Qiang Li to give a speech.

[Mr. Qiang Li gives his address (Fig. 3).

Fig. 3. Mr. Qiang Li gave an opening address at the Congress

Thank you, Mr. Li. I'd like to extend our sincere thanks to Mr. Qiang Li for his great attention to and support for the Congress during the preparation period, and to Shanghai government, particularly the education and foreign affairs departments, for the consistent help and support.

Now let's welcome the Executive Vice President of China Association of Science and Technology, Mr. Jinpeng Huai to give a speech (Fig. 4). Mr. Huai was supposed to be at the Congress in person, but unfortunately, he was not able to make it due to a business trip. Instead, he sent us a video of his speech.

[Mr. Jinpeng Huai's prerecorded speech is broadcasted (Fig. 4).

Fig. 4. Mr. Jinpeng Huai's video speech at the Congress

Thank you, Mr. Huai.

Vice Minister of the Ministry of Education, Ms. Tiehui Weng, has a close relationship with ICME-14. In 2015, Ms. Weng, then vice mayor of Shanghai, received the investigation group of International Commission on Mathematical Instruction while the group was making an investigation in Shanghai. Now let's welcome Ms. Tiehui Weng to give a speech.

[Ms. Tiehui Weng delivers her speech (Fig. 5).
[Full transcript of Ms. Weng's speech can be found on page 11.

Fig. 5. Ms. Tiehui Weng delivered a congratulatory speech at the Congress

Thank you, Ms. Weng.

Let's welcome the President of International Mathematical Union, Mr. Carlos Kenig to give a speech. International Mathematical Union is the super organization of International Commission on Mathematical Instruction. Mr. Carlos Kenig would have been present today in the Congress if it were not for the COVID-19 pandemic. Now we can only watch the video of his speech.

[Mr. Carlos Kenig's prerecorded speech is broadcasted (Fig. 6).
[Full transcript of Mr. Kenig's speech can be found on page 14.

Fig. 6. Mr. Carlos Kenig's video address at the Congress

Thanks a lot to Mr. Kenig.

Now, let's welcome the President of the International Commission on Mathematical Instruction, Mr. Frederick Leung to give a speech.

[Mr. Frederick Leung delivers his speech (Fig. 7).
[Full transcript of Mr. Leung's speech can be found on page 16.

Fig. 7. Mr. Frederick Leung delivered a welcome address at the Congress

Thank you, Mr. Leung.

The Chinese Mathematical Society represents China in the International Commission on Mathematical Instruction, and it is the main body in the bidding of ICME-14. In the past six years beginning from the bidding of ICME-14, Chinese

Mathematical Society has been through three sessions of councils, and all the three sessions have given us strong leadership and firm support. Now let's welcome President Gang Tian of Chinese Mathematical Society to give a speech.

[Mr. Gang Tian delivers his speech (Fig. 8).

[Full transcript of Mr. Tian's speech can be found on page 18.

Fig. 8. Mr. Gang Tian delivered a welcome address at the Congress

Thank you, Mr. Tian.

East China Normal University (ECNU) is the host of the Congress. It is the tremendous support of ECNU that makes this Congress possible. Let's welcome President Xuhong Qian of ECNU to give a speech.

[Mr. Xuhong Qian delivers his speech (Fig. 9).

[Full transcript of Mr. Qian's speech can be found on page 20.

Fig. 9. Mr. Xuhong Qian gave a welcome address at the Congress

Thank you, Mr. Qian.

Distinguished guests,
Dear participants on-site and online,
Ladies and gentlemen,

Now we're going to solemnly welcome the flag of ICME-14 (Fig. 10).

Fig. 10. Display the ICME-14 flag

After the success of our bidding for ICME-14, the Local Organizing Committee designed the logo and flag of the Congress, and displayed and introduced them at ICME-13 held in Hamburg, Germany in 2016. The flag we're going to welcome today is exactly the one we displayed in Hamburg, on which many mathematics educators left their precious signatures.

Thank you, young ladies.

Now, I'd like to invite

Ms. Tiehui Weng, Vice Minister of the Ministry of Education of the People's Republic of China,
Mr. Shaoliang Yu, Vice Secretary of CPC Shanghai Committee,
Mr. Frederick Leung, President of the International Commission on Mathematical Instruction,
Mr. Gang Tian, President of Chinese Mathematical Society, and
Mr. Xuhong Qian, President of East China Normal University

to come over to the stage. Let's push the rod start countdown, and declare the opening of the 14th International Mathematical Education Congress (Fig. 11).

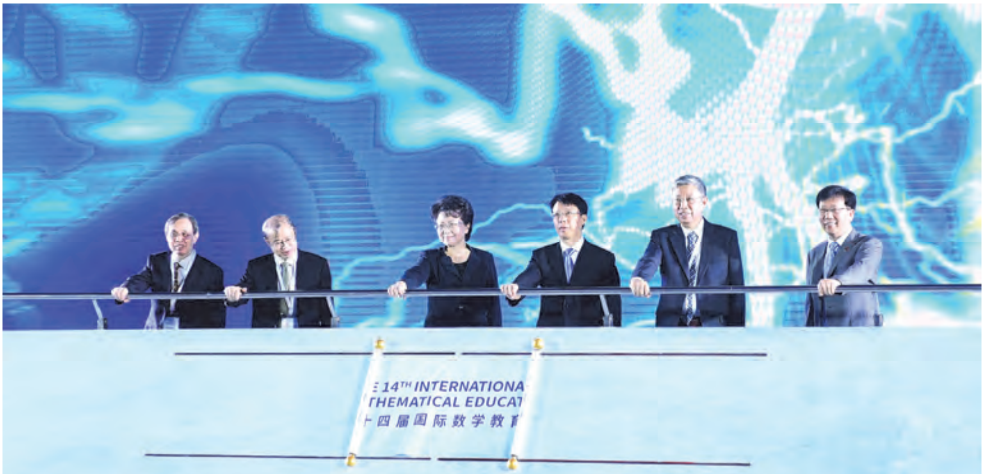

Fig. 11. Pushing the rod to declare the opening of the 14th International Mathematical Education Congress

From left: Jianpan Wang, Frederick Leung, Tiehui Weng, Shaoliang Yu, Gang Tian, Xuhong Qian

https://doi.org/10.1142/9789811287152_0002

The Opening Ceremony

Congratulatory Remarks from Ministry of Education

Tiehui Weng[1]

Honorable Mr. Qiang Li, Secretary of CPC Shanghai Municipal Committee,
Honorable Mr. Carlos Kenig, President of International Mathematical Union,
Professor Fredrick Leung, President of ICMI,
Professor Jianpan Wang, and
Mr. Gang Tian, President of Chinese Mathematical Society,
Dear guests,

In 2015, China succeeded in bidding for the 14th International Congress on Mathematical Education after years of considerable preparations and tiding over the Covid-19 pandemic as a major challenge. Today the Congress is successfully convening at the East China Normal University.

On behalf of the Ministry of Education, People's Republic of China, I would like to take this opportunity to extend warm congratulations at the convening of the Congress and send heart-felt regards to participants both online and on-site.

Recently, China has attached great importance to the international exchanges in the sector of mathematics education. The launching of the Congress serves as an important platform for the International Mathematical Education Committee to have reflections, learnings and plans for the future, which is of great significance. I would like to take this opportunity, based on my pragmatic work and reflections to share with you the ideas from the following four perspectives.

Firstly, to lay a solid foundation and to allow the crystal of mathematics to bloom. Mathematics is the manifestation of human civilization. The famous Chinese mathematician Shiing-Shen Chern once said "No matter how complex the world is, everything can be handled by addition, subtraction, multiplication and division. No matter how huge is universe is, points, lines and planes involve in everything". For teenagers, the significance of learning mathematics well not only lies in the good command of basic knowledge, but also allows them to sufficiently perceive the marvel of number and space, the wonder of the combination of sciences and arts, the secrecy of the combination of simplicity and complexity, to learn to identify clues from the seemingly different sources and to develop vertical thinking and logic reasoning for

[1] Vice Minister of the Ministry of Education of the People's Republic of China

the life benefit. We would sufficiently dig the rational beauty, natural beauty and exploration beauty of mathematics so as to pave the way for more excellent teenagers and inspire top talents to contribute and dedicate to mathematics along their journey to perceive and enjoy the beauty of mathematics.

Secondly, to teach students in accordance to their aptitude and innovate teaching approaches to achieve top-class mathematical education. Children's interest and teaching efficiency are the two key factors. We need to focus on making teaching more entertaining and motivating classroom activation, changing our perception that if a child does not know mathematics at the very beginning, it is impossible to change along the way so as to make mathematics approachable to children and allow them to fall in love with mathematics. Meanwhile, we need to further enhance the transmission efficiency of knowledge, proactively motivate the logic thinking, and avoid simple repetition and mechanical trainings. Taking domestic and overseas practices of mathematics teaching into account, we need to attach more importance to the smart transition from particularity to commonality, simplicity to complexity, proper use of teaching techniques, difficulty in spiral learning, content design for linking theories with practices, more protection of students' personalities and cultivation of diverge thinking and innovative thinking so as to give a full role to play students' personalities, strength, and thinking abilities.

Thirdly, to enhance capability and catch up with and surpass outstanding faculty team. A high-profile faculty team serves as a key to innovate mathematical education. For a long time, China's mathematical education represented by Shanghai has been focusing on pulling senior teachers and young teachers together by means of teaching and research groups and systematically leading teachers to jointly explore problems and cheer spirits by launching collective teaching and research programs, resulting in building up framework of science-oriented teaching and teaching-oriented researches, which has won high evaluation worldwide. However, we are fully aware of the fact that China's education is featured with inequity and insufficiency. We are making efforts to allow rural children and children in mountains to enjoy education the same as the children in Shanghai. China has a total of 618,000 rural mathematics teachers. In order to promote their overall teaching competencies, we've launched a training program with specific responsibilities at states, provinces, cities, counties at school level. The five-tier training program covered most rural teachers, with 42,000 teachers first trained over the 3 years. It is anticipated that participants attending this Congress will give suggestions and advices for the better professional development of mathematics teachers in China and such contributions will boost the modern teaching theories, scientific teaching approaches, and targeted coaching and mentoring.

Fourthly, to intensify cooperation and to expand the path of international teacher learning. Mathematics is the common language of the world. Mathematics education is taking the course for the joint efforts for development since the launch of the exchange program between China and United Kingdom in Shanghai in 2014. There have been over 20,000 British teachers and Shanghai teachers, closely interacting with each other both online and offline to share experiences, export mathematics teaching materials and benefit from such exchanges from mathematics teachers around the world. Based on this interaction, in 2018, United Nations Educational, Scientific and Cultural Organization (UESCO) set up the Teacher Education Center, which has further enhanced international training exchanges. China has given full support to the Center and hope it will play a bigger role. IMU and ICMI are also extremely important platforms to this Congress. We hope to take this Congress as a new starting point to accelerate the building of long-term mechanism of win-win cooperation to make greater contributions to the world scientific advancement, human civilization progress.

Last but not least, I wish the Congress a great success.

https://doi.org/10.1142/9789811287152_0003

The Opening Ceremony

Welcome Address from International Mathematical Union

Carlos Kenig[1]

It is a pleasure and an honor to represent the International Mathematical Union at the opening ceremony of ICME-14.

I would first like to give a warm welcome, on behalf of the IMU, to all the participants, both in person and virtual, in this very important event, which took years of sustained effort to plan and organize.

The IMU is extremely grateful to all the colleagues that have worked so hard and so well, to make this event possible, under the most difficult circumstances imaginable. Our heart-felt gratitude for their heroic efforts goes to our Chinese colleagues involved in the organization of ICME-14, to the Chinese institutions that supported them, to the International Program Committee and the Local Organizing Committee and to the two Executive Committees (EC) of ICMI involved in this enterprise. In particular, I would like to extend our gratitude to Jianpan Wang, the chair of the IPC and of ICME-14, to Yulin Wang, the academic secretary of ICME-14, to Binyan Xu and Jiansheng Bao, the co-chairs of the local organizing committee, to Jill Adler, past president of ICMI and Abraham Arcavi, past Secretary General of ICMI and to Frederick Leung, President of ICMI and Jean-Luc Dorier, Secretary General of ICMI, and to the many others who have worked tirelessly to make this event possible.

I became President of the IMU two and a half years ago, and during this time I have had the privilege of working closely with two Executive Committees of ICMI, as ex-officio member. This has been a very interesting learning experience for me, during which I came to learn something about the crucial work that ICMI and the world community of mathematical educators carries out, dealing with both theoretical research in mathematical education and with the practice of mathematics education, at all levels. All of this is of fundamental importance in the modern world, and particularly so for the developing world, in order for its full potential to be reached. This is one of the areas of direct cooperation between the IMU and ICMI, through the Capacity and Networking Project (CANP). The aim of CANP is to enhance the mathematical education in developing countries, at all levels, by developing the educational capacity of those who educate math teachers (from all levels of instruction). This has a large potential pay-off, since each teacher reaches many students, thus

[1] Professor, University of Chicago; President (2019−2022), International Mathematical Union.

widely propagating the acquired knowledge. We will be hearing more about CANP in this congress.

Even though historically, going back at least to the work of Felix Klein and others, there has been close cooperation between mathematicians and mathematics educators, a gulf seems to exist at the moment in many countries and institutions, between mathematicians and mathematics educators. This seems to me to be very artificial, and very damaging to both communities, since research and education cannot and should not be separated. I hope that both our communities will continue to work together to close this regrettable gap.

I conclude by thanking all the participants for their endurance and grit, and for their attendance, and wish all of you a very successful and memorable congress.

https://doi.org/10.1142/9789811287152_0004

The Opening Ceremony

Welcome Address from International Commission on Mathematical Instruction

Frederick Leung[1]

Honorable Mr. Li,
Professor Wang,
Other honorable guests present here, and
Participants of ICME-14, both in this hall and all over the world,
Ladies and gentlemen:

I echo Professor Wang and Professor Kenig in welcoming you to ICME-14. In the more than 50 years of the history of ICME, this is the first time when ICME is held in China. And being a Chinese myself, I am particularly proud of this.

ICME stands for International Congress on Mathematical Education. The use of the word "congress" means that this is not just an international conference, it is a gathering of all those involved in mathematics education worldwide. We gather together to share our findings in research, and our experiences and best practices. So solidarity and fellowship, and not only exchange of academic ideas, are salient features of ICME. I hope that at least those of us who are present physically here in Shanghai will make use of this opportunity to interact and to socialize, in addition to attending the academic sessions.

ICMI, or International Commission on Mathematical Instruction, is a worldwide organization devoted to research and development in mathematical education and to promote international cooperation. And ICME, the congress, fulfills the most important mission of ICMI in promoting excellence and inclusiveness of research and practices in mathematical education.

There are two important pairs of words here: research and practices, and excellence and inclusiveness. Educational research without relevance in the mathematics classroom is a pure intellectual or philosophical exercise. It may help a university professor to get tenure, but it doesn't contribute to enhancement of mathematics learning of our students. On the other hand, simply sharing of best practices from our classroom experiences will degrade mathematics education to simply a bag of tricks of the trade.

The other pair of words: excellence and inclusiveness, set the standard of our work in research and classroom practices. Our educational research must satisfy the rigorous

[1] Professor, Hong Kong University; President of the International Commission of Mathematical Instruction (ICMI).

criteria of scientific research in order to establish mathematics education as an academic discipline, and our practices must be based on evidence supported by sound research. On the other hand, we need to realize that the worldwide mathematics education community comes from very diverse backgrounds, economically, socially and culturally. So in attaining excellence in educational research and practices, we must reflect different cultural traditions. In fact, our research and our practices will be enriched by such cultural diversity. And ICME is the perfect venue for excellence and diversity of research and practices to be shared and treasured.

COVID-19 has posed a tremendous challenge to the mathematics education community worldwide, and it has posed a great challenge to the organizers of ICME-14 as well. Holding a hybrid congress like this is not a trivial task at all. It is equivalent to holding a physical conference *and* an online conference at the same time. And unfortunately, one plus one is much bigger than two in this case — that is it is even more difficult to hold a hybrid conference than to hold one physical conference and one online conference!

I salute and thank members of the organizing committee for their great efforts and excellent work in the past years which has made this congress possible. I am sure that all the participants of ICME-14, whether physical or online, will benefit tremendously from the congress, and that the hard work of the organizers will pay off.

I wish ICME-14 a great success! Thank you very much.

https://doi.org/10.1142/9789811287152_0005

The Opening Ceremony

Welcome Address from Chinese Mathematical Society

Gang Tian[1]

Your excellency, Mr. Qiang Li,
Mr. Shaoliang Yu,
Ms. Tiehui Weng,
Mr. Calos Kenig,
Mr. Frederick Leung,
Mr. Xuhong Qian,
Distinguished guests,
Ladies and gentlemen,
Friends, both online and in the audience,

Today, in China's metropolitan Shanghai, we usher in the 14th International Congress on Mathematics Education (ICME-14). On behalf of the Chinese Mathematical Society and the organizers of this Congress, I would like to extend a warm welcome to all our guests and friends.

Mathematics and mathematics education have never been separate entities. As called for in the mission of the International Mathematical Union, the Chinese Mathematical Society has dedicated itself to encouraging and supporting various international mathematical activities, including theoretical mathematics, applied mathematics, and mathematics education. The Chinese Mathematical Society sent a delegation, headed by Professor Shisun Ding of Peking University, to the 4th International Congress on Mathematics Education (ICME-4) in 1984, at which the well-known mathematician Hua Luogeng was invited to give a plenary lecture.

Besides international activities, the Chinese Mathematical Society has also been organizing, participating in and supporting various research projects and activities in domestic mathematics education, such as the reform of mathematics curriculum in primary and secondary schools. China's basic mathematics education system is internationally renowned, but not without areas that could be improved.

At the 2002 International Congress of Mathematicians (ICM) held in Beijing, I once said, "Some people find mathematics dry and difficult, but I think mathematics is elegant and clear-cut. Mathematics will reveal its beauty to those who love it. Once you have found an interest in mathematics, it will look completely different in your eyes. Behind different phenomena lies a fundamental connection, once you have found

[1] Professor, Peking University; President of Chinese Mathematical Society

it, you will be drawn towards the call of mathematics.” I hope that, through our joint efforts, not only will our youths achieve excellent results in mathematics, but more so that they will grow to like mathematics, and believe that mathematics is fun and elegant.

Currently, there are two influential international mathematics Congresses organized by the International Union of Mathematics. One is ICM, the other is ICME. In 2002, we held ICM 2002 in Beijing, which provided an excellent opportunity for Chinese mathematicians to interact and exchange ideas, to facilitate discussions with international mathematical luminaries, and also acted as center stage for Chinese mathematicians to showcase their work to their international counterparts.

ICM 2002 and its related events were instrumental in promotion and popularization of mathematics in Chinese society. It focused public attention on mathematics, prompted the application of mathematics in all fields and industries, it also strengthened academic exchanges between Chinese mathematicians and their peers abroad, and propelled global mathematics into a new era. I have no doubt that this current International Congress on Mathematics Education (ICME-14) shall bring our mathematicians a more global perspective, broader and more in-depth international exchanges, and serves to drive many exemplary traditions and innovations of Chinese mathematics education and learning onto a global platform.

The Chinese Mathematical Society was established in Shanghai in 1935. This society has always placed profound significance on mathematics education. Just this afternoon, the Mathematics Education Branch of the Chinese Mathematical Society held its inaugural meeting in Shanghai. This marks the establishment of closer ties between the Chinese Mathematical Association, the International Union of Mathematics (ICM) and the International Commission on Mathematical Instruction (ICMI).

We would like to thank the International Commission on Mathematical Instruction for entrusting the important task of holding ICME-14 to the Chinese Mathematical Society. I believe that we will be presenting to the world a splendid and unique celebration. We would also like to thank the ICME-14 International Program Committee, the local organizing committee and, especially, Professor Jianpan Wang, our Congress chairman. In the process of bidding for the Congress, your perseverance was proven to be formidable. Due to the COVID-19 pandemic, the Congress was forced to be postponed for one year. Not only did the venue have to change, but also the mode of the conference. I can only imagine the challenges and hardship you all overcame.

We are also grateful to the Chinese Association for Science and Technology, the city government of Shanghai, and East China Normal University in extending a helping hand during the difficulties experienced by the organizing committee. Thank you to all those who have contributed to the making of this Congress. Without your contribution, we would not be able to gather here today. Today is a day that will not be forgotten in China’s mathematics education.

I wish the Congress a great success!

Thank you.

https://doi.org/10.1142/9789811287152_0006

The Opening Ceremony

Welcome Address from East China Normal University

Xuhong Qian[1]

Honourable Mr. Qiang Li, Ms. Tiehui Weng,
Dear Professor Kenig, Professor Leung, Professor Gang Tian, Professor Jianpan Wang,
Distinguished guests joining us from around the globe,
Ladies and gentlemen, dear friends, Welcome!

The 14th International Congress on Mathematical Education is being held today at East China Normal University (ECNU), and on behalf of all the teachers and students of the University, I would like to extend a warm welcome to all the participants from all over the world! I would like to express my sincere gratitude to all the leaders, colleagues and friends who have shown their interest in and support for the Congress.

The International Congress on Mathematical Education is the most prestigious event in the field of mathematics education. Seven years have passed since our initial bid to host this international "Olympiad" in mathematics education. The preparation and holding of this event have been fraught with twists and turns, not least because of the impact of the COVID-19 pandemic.

And yet not only have we succeeded in organizing this event, we will also be marking some historic achieving some "firsts":

- It is the first ICME to be held in China;
- The first ICME to be held in hybrid format — both online and offline;
- And this edition will have the largest number of invited lectures and topic study groups ever.

I am sure therefore that this Congress will be remembered for years to come in the history of mathematics education in the world.

Mathematics has played a major role in the development of human civilisation. More recently, in particular, mathematics has become an important foundation for the advancement of various disciplines. Furthermore, mathematical knowledge and thinking are the bedrock of technologies such as artificial intelligence and big data. It is evident that mathematics education is crucial to the development of tomorrow's world and will play a key role in shaping the way we think in the future. This is why it is increasingly important to strengthen mathematics education and expand the new paradigm of mathematics education in the age of artificial intelligence.

As a comprehensive research university with special strengths in education, East China Normal University (ECNU) entered the ranks of national first-class universities

[1] Professor; President of East China Normal University

in 2017. Over the past few years, the university has focused on the mission of "educating, enlightening and developing" students. Our efforts aim to forge the foundations for achieving happiness across society as a whole. The university has made clear that it strives to "achieve excellence through education", which clearly encapsulates the unique approach to education at ECNU. Mathematics education is one of the core components underpinning educational excellence at ECNU. Our aim is to revisit the way we think, in order to unlock the essence and shape of mathematics education for the new era; to remould mathematics education for the future with universal values; and reach new heights in exploiting AI in order to map new methods of mathematics education for the road ahead. Our goal is to open a new chapter in the training of outstanding talent by applying mathematical thinking to teaching and learning. Through mathematical thinking, we can build systems of knowledge which stretch from East to West. By building these bridges, we can contribute to world civilisation, and at the same time harness the potential of mathematics as a new lever for development and source of energy to drive civilisation in the age of artificial intelligence.

Mathematics at ECNU already has a long history, and is developing today faster than ever. Our mathematics education research team, has its roots here in Shanghai. At the same time, this team embraces a broad international vision, and attaches equal importance to research and practice. As such, they have made outstanding contributions to mathematics education research, which has benefited not only Shanghai but China as a whole. The establishment of the Asian Centre for Mathematical Education has given greater international visibility and weight to Chinese mathematics education research and practical pedagogy. The Shanghai maths programme, "One Lesson One Exercise" (English version), edited by ECNU's Professor Lianghuo Fan, is now applied around the world and is used in over 400 schools in the UK. ECNU's score in the European Skills Index (ESI) for maths currently lies within the global top 1%, and is a pillar for cross-disciplinary work in our school.

Today, we have the honour of welcoming researchers and practitioners of mathematics education from all over the world to meet both here on our campus at ECNU and online, to discuss the future of mathematics education together, and to inject new blood into our School's priority, which is to nurture excellence.

Distinguished guests, ladies and gentlemen, in October this year, East China Normal University (ECNU) will celebrate its 70th anniversary. We hope that by virtue of this conference, our school will maintain its momentum and make fresh progress in basic and applied research in mathematics education, and advance in its endeavour to cultivate talented mathematical minds. In doing so, we hope not only to bolster mathematics education in China but also contribute to mathematics education around the world!

Finally, I would like to wish you all fruitful discussions during this congress, and for those here in Shanghai, a wonderful week at East China Normal University!

Thank you!

https://doi.org/10.1142/9789811287152_0007

The Opening Ceremony

ICMI Awards Ceremony

Frederick Leung[1], Jill Adler[2], Anna Sfard[3], Konrad Krainer[4], Deborah Loewenberg Ball[5], Terezinha Nunes[6], Tommy Dreyfus[7], Gert Schubring[8], Nancy Brickhouse[9], and Trena Wilkerson[10]

The ICMI Awards Ceremony was presided by Frederick Leung. Jill Adler gave an introduction of ICMI awards, followed by the award ceremony. Adler presided over the 2017 awards (Fig. 1), Leung presided over the 2019 awards and the 2020 Emma Castelnuovo award.

***Introduction to ICMI Awards by Jill Adler*:**

Greetings to

- Professor Frederick Leung, ICMI President,
- Professor Wang, ICME-14 Congress Chair and all on the Congress LOC,
- Professor Carlos Kenig, President of IMU,

Greetings to all colleagues, friends and our ICMI community world-wide. Ni Hao to all participants in Shanghai, I wish I could be there with you today. I am Jill Adler, immediate past president of ICMI 2017– 2020. Welcome all to the 2021 ICMI Awards Ceremony.

Our ICMI awards — and you can see there are three different awards — medals — are a highlight of the work of ICMI — and we celebrate these at our ICME congress.

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[3] Professor (Emerita), University of Haifa, Israel; Chair of the ICMI Felix Klein and Hans Freudenthal Awards Committee 2016−2020. Email: Annasfard@gmail.com

[4] Professor, Alpen-Adria-Universität Klagenfurt, Austria, Chair of the ICMI Emma Castelnuovo Award Committee 2017−2020.

[5] Professor, University of Michigan, United States of America, awardee of 2017 Felix Klein Award. E-mail: dball@umich.edu

[6] Professor, Oxford University, United Kingdom, awardee of 2017 Hans Freudenthal Award. E-mail: edst0248@nexus.ox.ac.uk

[7] Professor, Tel Aviv University, Israel, awardee of 2019 Felix Klein Award. E-mail: tommyd@tauex.tau.ac.il

[8] Professor, Universität Bielefeld, Germany, and Universidade Federal do Rio de Janeiro, Brazil, awardee of 2019 Hans Freudenthal Award. E-mail: gert.schubring@uni-bielefeld.de

[9] Vice President and Provost, Baylor University, United States. Email: Nancy_Brickhouse@baylor.edu

[10] Professor, Baylor University; President of National Council of Teachers of Mathematics (NCTM). Email: Trena_Wilkerson@baylor.edu

The first two awards were established in 2000 to recognize outstanding achievement in mathematics education research:

Fig. 1. Left: Jill Adler presided over the 2017 awards
Right: Frederick Leung presided over the 2019 and 2020 awards

The Felix Klein Award, named after the first president of ICMI (1908–1920), honours lifetime achievement.

- The Hans Freudenthal Award, named after the eighth president of ICMI (1967–1970), recognizes a major cumulative program of research.

More recently, in 2013, we established a third award that reflects the ICMI principles to promote the reflection, collaboration, exchange and dissemination of ideas on the teaching and learning of mathematics, from primary to university level.

- The Emma Castelnuovo Award, named after the Italian mathematics educator born in 1913 to celebrate her 100th birthday and honour her pioneering work, to recognize outstanding achievements in the practice of mathematics education.

Together these three awards pay tribute to outstanding scholarship in mathematics education. They serve not only to encourage the efforts of others, but also to contribute to the development of high standards for the field through the public recognition of exemplars.

In the ceremony today we present the medals and certificates to the Felix Klein and Hans Freudenthal awardees of 2017 and 2019; and the Emma Castelnuovo award of 2019/2020.

Two committees were charged with making these awards, and this is also an opportunity to thank them and their Chairs, for the work done. As you might know the Chairs are public positions. The committee members, however, are kept confidential to protect the integrity of the process. It is important to add that once the committees are appointed, they work completely independently of ICMI and the EC.

In particular I would like to thank Professor Anna Sfard, Chair of the Felix Klein and Hans Freudenthal Awards committee (Fig.2, left). Anna Chaired the awards

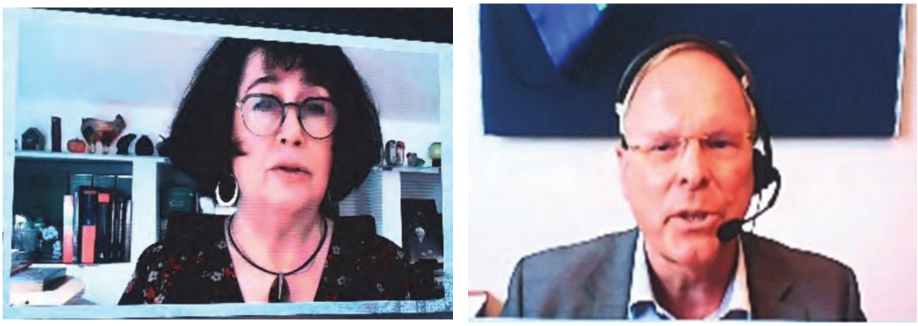

Fig. 2. Chairs of the two award committees

Left: Anna Sfard, the Chair of the Felix Klein and Hans Freudenthal Awards committee (2016−2020), read a citation
Right: Konrad Krainer, the chair of Chair of the Emma Castelnuovo Award committee (2017−2020), read the citation

process for both the 2017 and 2019 awards. Anna, thank you for the integrity and commitment with which you undertook this most important task in ICMI, and for a report on the process that has helped ICMI add refinements needed. Similarly, to Professor Konrad Krainer, Chair of the Emma Castelnuovo committee (Fig. 2, right), thank you too not only for the work done over eight years, but also for contributing to our refining the criteria for this award as well. Both Anna and Konrad have served their terms and new Chairs have been appointed.

Recipients of the ICMI Awards, 2017−2020

- *The awardee of 2017 Felix Klein Award: Deborah Loewenberg Ball, University of Michigan, United States of America*
- *The awardee of 2017 Hans Freudenthal Award: Terezinah Nunes, Oxford University, United Kingdom*
- *The awardee of 2019 Felix Klein Award: Tommy Dreyfus, Tel Aviv University, Israel*
- *The awardee of 2019 Hans Freudenthal Award: Gert Schubring, Universität Bielefeld, Germany, and Universidade Federal do Rio de Janeiro, Brazil*
- *The awardee of 2020 Emma Castelnuovo Award: National Council of Teachers of Mathematics (NCTM)*

2017 Felix Klein Award

The Citation[11] read by Anna Sfard.

The Felix Klein Award, with which ICMI honors the most meritorious scholars within the mathematics education community, is given in 2017 to Deborah Loewenberg Ball,

[11] https://www.mathunion.org/icmi/awards/felix-klein-award/2017-felix-klein-award

the William H. Payne Collegiate Professor in Education and an Arthur F. Thurnau Professor in the University of Michigan, Ann Arbor, MI, US. The Felix Klein Award 2017 is awarded to Professor Ball in recognition of her outstanding contributions and her leadership role in deepening our understanding of the complexities of teaching mathematics and in improving the practice of teaching and of teacher education.

These achievements are grounded in Deborah Ball's firm belief that research and practice of teaching are co-constitutive and must always be developed in tandem. Early in her life, Deborah Ball, at that time an exceptionally talented elementary school mathematics teacher, set out to investigate what was involved in the work of teaching children mathematics "for understanding." Her intention was to uncover the work in order to support the learning of teaching practice. Ever since then, her ambition has been to contribute in a substantial way to the project of improving ways in which mathematics teachers support their students' learning. This goal gave rise to two lines of work, both of them combining research with development in the domain of teacher education. The first strand, in which the research element came first, has been generating studies revolving around the question of what mathematical knowledge is required for teaching learners. In the second line of work, related to the practice of education in a more immediate way, the development of innovative teacher preparation programs has been combined with research, through which Deborah Ball has been trying to gain a better grasp of the moment-to-moment dilemmas with which teachers grapple in the classroom.

The first of these pursuits gave rise to the theory of MKT, Mathematical Knowledge for Teaching, the kind of knowledge that requires competence in both everyday and academic mathematical discourses, but is identical to neither. In her multiple studies, Deborah Ball and her colleagues have been able to identify many unique features of MKT, and then to corroborate the conjecture about a correlation between teachers' competence in this special brand of mathematics and the achievements of their students. With the support of a group of mathematicians, the theory has been translated into an instrument for measuring teachers' knowledge of mathematics for teaching. The MKT project proved highly influential, as evidenced by the widespread use of the term MKT and by the great popularity of Deborah Ball's publications on the topic. Her 2008 paper "Content knowledge for teaching: What makes it special?" co-authored with Mark Hoover and Geoffrey Phelps Thames, which appeared in the *Journal of Teacher Education*, is one example of such a widely read article.

The second, newer strand of Deborah Ball's work is centered in TeachingWorks, a national organization she established at the University of Michigan to help in improving teachers' preparation and to define "a professional threshold for entry to teaching." The mission of the institute is to identify "high-leverage" teaching practices, that is, those recurring elements of teacher's classroom activities that are central to what Deborah Ball terms "the work of teaching." It is also part of the mission to work in partnerships with others to improve the preparation of teachers. To this end, Deborah

Ball has been carrying in-depth analyzes of the ways in which mathematics teachers juggle their multiple classroom tasks, such as interpreting the learner's often idiosyncratic ways of thinking, gradually transforming the children's special understandings into more canonical ones, sustaining equitable learning dialogue and taking care of the emotional well-being of the students. This line of Deborah Ball's research, while relatively new, seems to be an attempt to close the circle that opened with the early reflection on her intuitive efforts, as a teacher, to identify and to bridge the gap between her own mathematics and the mathematics of her students. Indeed, this current research project harks back to Deborah Ball's early publications, such as her now classical 1993 article "With an eye on mathematical horizon: Dilemmas of teaching elementary school mathematics", in which the memorable case of "Sean numbers" helped the author to highlight challenges of classroom teaching.

Deborah Ball has played multiple leadership roles, and not only within community of mathematics education but also within that of education at large, and not only within United States, but internationally. In all these arenas, hers was a systematic effort to build bridges. Her years-long work on bringing together research and practice of mathematics education is just one example of these attempts. Another expresses itself in her striving for a fruitful collaboration between the communities of mathematicians and of mathematics educators. In this later undertaking, she has been acting on her strong belief that certain differences of opinions on mathematics and on teaching that arise occasionally between these two communities, far from being an obstacle, are likely to help in creating a synergetic partnership.

Deborah Ball's achievements as a researcher and a leader have been recognized nationally and internationally. This recognition is signaled, among others, by the unprecedented frequency with which her publications are cited by other authors, by her great popularity as a speaker, by her multiple roles within ICMI and by her membership of numerous policy-making or advisory committees, such as the National Science Board, appointed by President Barack Obama. Whereas her work is firmly grounded in mathematics education, the recognition of its outcomes goes well beyond the community of mathematics education. This is evidenced by Deborah Ball's prestigious membership in the National Academy of Education and by her current roles as the President of the American Educational Research Association and as a member of the American Academy of Arts and Sciences.

Deborah Ball has been an elementary classroom teacher before and during her studies at Michigan State University, which she completed in 1988 with a PhD in mathematics education. Upon graduation, she joined Michigan State University, and in 1996 she was recruited to the University of Michigan to develop the mathematics education group. She has been teaching at the University of Michigan ever since then and also spent over a decade serving as Dean of the School of Education there.

With more than thirty years of outstanding achievements in mathematics education research and development, Deborah Ball is a most distinguished member of mathematics education community and a highly deserving recipient of the 2017 Felix Klein Award.

Acknowledgement from the awardee, Deborah Ball (Fig. 3)

Thank you, Jill. Thank you, Anna. Thank you, Hyman. Thank you, to the International Commission on Mathematical Instruction and the members of the Felix Klein Award Committee, for this honor. I am truly humbled.

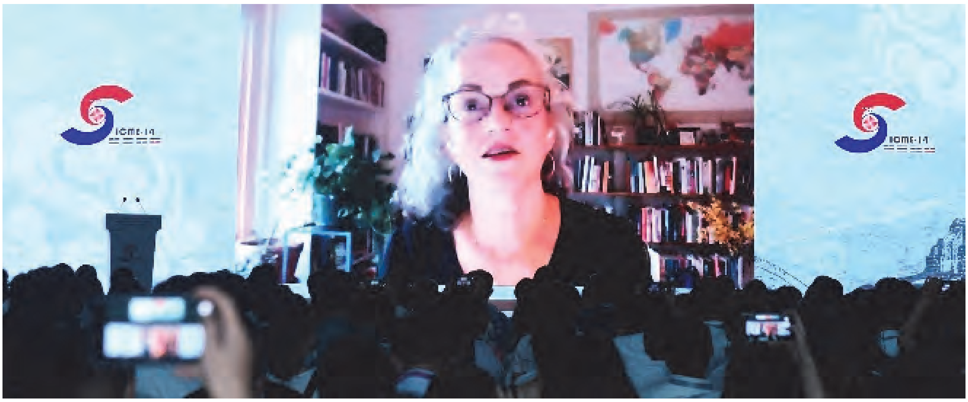

Fig. 3. Deborah Ball gave an awardee speech

Are you going to say that an award in academic world so often singles out individuals for recommendation, as with this award? In fact, it is crucial to say that my work and my learning are so much a product of collaborative work. And for opportunities to work with and learn from so many other scholars, teachers and students, this award actually celebrates those relationships and those opportunities as much as it does my own achievements.

My learning has also been the product of the contexts where I have been supported to work, to learn, to grow and to develop. Starting with the public elementary school right top for almost 20 years, to the Michigan State University where I began my academic career, and now the University of Michigan — these have been the places that have supported me and the wonderful places for me to grow and learn with others. As a beginning teacher in that public elementary school, I was mentored by a strong, experienced and loving principal — or head of school — and a usually diverse and experienced set of colleagues who supported me in developing my own ways, learning from others and also breaking out of times. They encouraged me and supported me.

As an early careered scholar as well as across my whole career, I have been nurtured by colleagues and managers who had urged me to probe and pursue the questions that preoccupy me and puzzle me; they helped me build connections to so many kinds of work and learn theories and methods, and also supported me when I had crazy ideas or other ways to go back to work different conceptualizations or different methods. My mentors, colleagues and friends asked me really tough questions. They could see what I was trying to do and encouraged me when I was discouraged. They pushed me, they took my work seriously, they read it, they responded, and they were there for me. And as I worked hard to figure out what it meant to cultivate my experience and my core identity as a teacher, to be a critical resource for my own research, I feel like I was really lucky to have colleagues who had always valued that.

My students have quite been among my most important teachers. I realized when I was reflecting that I have been blessed to teach over 1000 primary grade students by now, and they have really been among my most important teachers. I learned from children just all the time. I also learned incredibly from my undergraduate students who were courageously setting out to become teachers and to learn to do this complex work. I have learned so much from all my undergraduate students with whom I have been incredibly fortunate to work and learn from they themselves are up there now, contributing and adding to the field, changing, and making change. Thank you to all of you from whom I have been so fortunate to work, to learn, and to figure out how to pursue things that really matter and for which our research can actually make a difference, together.

I also want to thank my colleague and friend, Hyman Bass, who has done so much to shape his organization ICMI. As the President of ICMI from 1999 to 2006, Hyman worked deciduously to center on mathematics education research and mathematics educators, and it was really because of his respect for the field that he was determined to conceptualize and inaugurate these awards, like the Felix Klein Medal. He thought it was crucial to foreground mathematics education research, particularly in the context of the International Mathematics Union. Thank you, Hyman, for all of that work that you did for others to make this possible.

I really cannot express how deeply honored I am to be among the awardees of this Felix Klein Award, scholars whom I so deeply respect and admire, and from whom I have learned so much. Guy Brousseau, Ubiratan D'Ambrosio — may his memory be forever blessing, Jeremy Kilpatrick, Gilah Leder, Alan H. Schoenfeld, Michèle Artigue, Alan Bishop, and now, Tommy Dreyfus.

Thank you, everybody. I am so deeply grateful for this honor and I will take that as a continued mandate to support other scholars and to develop our sense that whatever work we do, whatever progress we can make, it is absolutely and fundamentally collective. Thank you.

2017 Hans Freudenthal Award

The Citation[12] read by Anna Sfard

The Freudenthal Award, with which ICMI honors innovative, consistent, highly influential and still ongoing programs of research in mathematics education, is being awarded in 2017 to Professor Terezinha Nunes, University of Oxford, UK, for her outstanding contribution to our understanding of mathematical thinking, its origins and

[12] https://www.mathunion.org/icmi/awards/hans-freudenthal-award/2017-hans-freudenthal-award

development. For more than 35 years now Terezinha Nunes has been researching children's mathematical learning, as it takes place in formal and informal settings. The results of her numerous, exemplarily designed studies combine into an insightful, consistent, and comprehensive story of the emergence and evolution of mathematical thinking. This constantly developing account has been inspiring the work of mathematics education researchers and informing mathematics teachers' practices all over the world. It has had a major impact on both what we know about children's learning of mathematics and on how we know and think about it.

Terezinha Nunes' research has been immensely innovative and influential from its earliest stages. In one of her first studies, she documented the mathematical skills of young Brazilian street vendors who, although almost unschooled and incapable of executing paper-and-pencil arithmetic tasks, proved impressively proficient in complex money transactions. Understandings gained through this research have echoed throughout the mathematics education literature ever since the project's completion, for almost three decades now. It was one of those studies that, in the last quarter of the 20th century, revolutionized our thinking about learning — about its nature, origins and development. Conducted with David Carraher and Analucia Schliemann and summarized in their seminal book *Street Mathematics* (1993), this research made a decisive contribution to what is now known as the "situative turn" in the learning sciences at large, and in mathematics education in particular. Terezinha Nunes' contribution to this conceptual revolution has been evidenced, among others, by the widespread use of the term *street mathematics* and by the large number of cross-situational and cross-cultural studies on mathematics learning inspired by her work.

Terezinha Nunes' later research on the development of mathematical thinking, conducted in Brazil and the UK, spans multiple mathematical topics, from additive and multiplicative reasoning to fractions, variables, randomness and probability. She has studied children's logical reasoning and its role in the learning of mathematics, as well as problem solving and the way mathematics is being used in science. A special place in her work has been reserved for research on the mathematics learning of deaf children and for developing and testing innovative intervention programs based on insights thus gained. In parallel to the work of scrutinizing different types of mathematical thinking and their development, Terezinha Nunes has also systematically constructed a big picture of this development. As research findings have accumulated, she has been adjusting and refining her syntheses. Different versions of these cumulative, integrative accounts have been disseminated, among others, through her 2000 ICME plenary address, her 1996 book written with Peter Bryant *Children Doing Mathematics*, and the 2016 ICME monograph *Teaching and Learning about Numbers in Primary School*, which she co-authored with colleagues.

While forging her stories on children's thinking about numbers, Terezinha Nunes has been transforming her own thinking as a researcher. She has come a long way from being a traditionally trained clinical psychologist, whose research was firmly grounded in Piaget's ideas about human development, to being inspired by cultural psychology and the work of Vygotsky and his followers to at least the same extent. Hers is a special type of synthesis between cognitivist and sociocultural approaches. Today, she speaks about "mathematics learning as the socialization of the mind" and claims the utmost importance of cultural shaping. At the same time, she asserts the existence of cross-cultural invariants in children's mathematical thinking. If these two tenets may sometimes appear incompatible, she argues, it is only because different cultures build on the common elements to produce forms of mathematical competences diverse enough to make the cross-cultural invariants almost invisible. Another basic tenet of her work is that children's quantitative reasoning may and should be developed independently of, and possibly prior to, their numerical skills. These and many other of her research-generated insights on mathematics learning were novel to the mathematics education community when first announced. Careful to notice phenomena that have escaped the attention of investigators wedded to the "deficit model" of research, she portrayed children's mathematics in unprecedented detail and depth.

Terezinha Nunes' tendency for bridging apparent opposites and bringing the separate together finds its expression also in her attempts to improve the practice of teaching mathematics. Not a typical dweller of the ivory tower of academia, she has always made sure that her work finds its way to those for whom it was meant in the first place — educators, parents, and anybody interested in promoting children's learning. She has been consistently translating her research-generated insights into innovative pedagogies.

Trained as a psychologist, Terezinha Nunes began investigating children's mathematical thinking because of her professional interest in human development. Her studies soon began to attract the attention of mathematics education researchers, leading to her membership in the International Committee of PME (1986–1990; in 1989–1990 she served as Vice-President of PME) and on editorial boards of major mathematics education journals, *Educational Studies in Mathematics* (1989–1995) and *For the learning of mathematics* (2000–2004). Since then, she has been one of the most widely recognized members of the community of research in mathematics education. This, however, was not her only professional membership. An interdisciplinary thinker, who has been investigating children's evolving reading and writing skills in parallel to her work on mathematical thinking, Terezinha has enjoyed a prominent status also among developmental and cultural psychologists. Her insights about numeracy and about literacy constantly informed and enriched each other and combined together into a major advancement in our understanding of human development and learning in general.

Terezinha Nunes began her studies in psychology in her native Brazil and earned her masters and PhD degrees at City University of New York (1975, 1976, respectively). She began her academic career in Brazil at the Federal University of Minas Gerais and the University of Pernambuco. Later, she moved to the United Kingdom, where she taught at the Institute of Education, University of London, Oxford Brookes University and, since 2005, at the University of Oxford. She is now Professor Emerita at the University of Oxford and a Fellow of Harris Manchester College, Oxford. Throughout her career, she has completed tens, if not hundreds of studies, most of which were conducted in Brazil and in the UK. An exceptionally prolific writer, she has authored or co-authored more than a dozen books and almost two hundred journal papers, book chapters and encyclopedia entries in English and Portuguese. An ardent team player and highly appreciated teacher, Terezinha Nunes has been an inspiration to her colleagues and to her many students.

As an outstanding researcher driven by an insatiable passion for knowing, one who has made a paramount contribution to mathematics education and is likely to continue adding substantial insights for years to come, Terezinha Nunes is an eminently deserving recipient of the Hans Freudenthal Award for 2017.

Acknowledgement from the awardee, Terenzinha Nunes

I am very grateful to ICMI for this award and I want to stress today that my research is the outcome of the fantastic opportunities to learn about mathematics education that ICMI offered me (Fig. 4). ICME-5 was the first conference organised by mathematicians I ever attended and it radically changed the way I thought about children learning mathematics. My colleagues David Carraher, Analucia Schliemann and I had encountered a phenomenon that we decided to call street mathematics. We knew that many children and unschooled adults in Brazil were competent at using mathematics as they worked in street markets and in trades such as building, fishing, and carpentry. But they were perceived as culturally deprived and did not succeed in school. Psychological theories did not help us to investigate this phenomenon but ICME-5 did. I will mention five events at ICME-5, but these were not the only ones: the three plenary lectures, the PME sessions and the discussions on mathematics for all.

Fig. 4. Terenzinha Nunes gave an acknowledgement speech

Ubiratan D'Ambrosio's unforgettable lecture on the socio-cultural basis of mathematical thinking showed me how I needed to change my view of what mathematics is. I had thought of mathematics as crystallised knowledge to be taught and learned in school. Ubi's demonstration that mathematics is a human activity, which at the same time produces and uses cultural tools, became for me the starting point of a new way of thinking about how children learn mathematics.

Jeremy Kilpatrick's lecture on reflection and recursion was a call to reflection. Many psychologists at the time treated learning as information processing and teaching as programming and debugging to correct the mistakes made in programming. Kilpatrick provided a critical analysis of the computer metaphor and revived the ghost in the machine. His suggestion that consciousness and personal experience cannot be ignored in teaching and learning influenced how I viewed children's learning and the task of investigating teaching and learning.

When the plenary by Renfrey Potts on discrete mathematics was announced, I thought I would go to sleep. The lecture hall was dark, I was jet-lagged, and I had no idea what discrete mathematics was. I thought that mathematicians may be discrete or indiscrete, but I could not see how mathematics could be discrete or not. But I could not go to sleep as Potts questioned whether the floor he stood on and time are discrete or continuous. As he moved back and forth between experience and scientific models, he gave me a glimpse of what it means to think of numbers as mathematical models for quantitative reasoning about the world. This radically changed my thinking about numbers.

The symposium by the study group on the psychology of mathematics education, organised by its president Gérard Vergnaud, proved to be a demonstration of the power of inter-disciplinary research. Researchers moved back and forth between psychology and mathematics, forming conjectures about the students' thinking on the basis of mathematical ideas, and returning to the students' work and explanations to check their conjectures.

Finally, the sessions on mathematics for all, organised by Peter Damerow, changed the way I thought about curriculum. I had naively thought of mathematics as a body of knowledge to be taught to everyone in the same way, and so a curriculum was only a matter of how to organise the mathematical topics sensibly. What I had learned about mathematics at ICME-5 helped me to understand the discussions on mathematics for all.

This is why today I am so deeply grateful for this award and truly indebted to ICME for what I have learned at all the ICME conferences that I attended ever since ICME-5.

2019 Felix Klein Award

The Citation[13] read by Anna Sfard

The Felix Klein Medal, with which ICMI honors the most meritorious members of the mathematics education community, is given in 2019 to Tommy Dreyfus, Professor Emeritus at Tel Aviv University, Israel, in recognition of his life-time achievement.

[13] https://www.mathunion.org/icmi/awards/felix-klein-award/2019-felix-klein-award

This distinction acknowledges Professor Dreyfus's contribution to research as well as his leading role in shaping and consolidating the research community and in fostering communication between researchers.

For four decades, Tommy Dreyfus's research has been systematically deepening our understanding of mathematics learning. Trained as a mathematical physicist, Tommy has been drawing in this work on his deep understanding of mathematics and his first-hand familiarity with ways in which mathematical ideas come into being and evolve. Since the late 1970s and for the next two decades his research has been focusing on students' conceptualization of mathematical objects such as function, and on the role of intuition, visualization and aesthetics in mathematical thinking. With years, his interests have been gradually shifting from the individual student to learning-teaching processes of the classroom. In the last twenty years, his empirical and conceptual work has been devoted to the study of epistemic activities such as proving and abstracting. These efforts resulted in the theory known as AiC — *Abstraction in Context*, which he developed with Baruch Schwarz and Rina Hershkowitz. Conceived in the late 1990s, the AiC framework has become increasingly influential. Since its inception, it has generated much empirical research all over the world. The theory has been found to be useful also to teachers, whom it provides with tools for monitoring student learning.

As impressive in its scope, breadth, depth and impact as Professor Dreyfus's research is, it constitutes only a part of the contribution for which he is honored today with this special distinction. Another outstanding part of his work is his ongoing project of shaping and consolidating the international community of research in mathematics education, a goal that he tries to attain in multiple ways. First and foremost, through his extensive editorial work he has been setting standards and giving directions for research in mathematics education. Particularly influential has been his 30-year long association with *Educational Studies in Mathematics*, which included his three-year long term as the editor-in-chief. Professor Dreyfus has also been serving in, and shaping, numerous professional organizations, with PME (the international group for the Psychology of Mathematics Education) and ERME (the European Society for Research in Mathematics Education) among them.

In addition, he played key roles in numerous professional committees in Israel, Europe and America. His influence on research and on policy directly affecting mathematics teaching is keenly felt over the world. In all these activities, Professor Dreyfus has been consistently promoting cross-discursive dialogues. He has done this by organizing international meetings, establishing trans-continental collaborative research projects, appearing world-wide as an invited speaker and by extensive mentoring in his own country and beyond. Probably the most important and innovative among Professor Dreyfus's consolidating activities have been his multifarious efforts to spur and improve communication among researchers working within differing theoretical frameworks. Being concerned about the fragmentation of the field of mathematics education, Professor Dreyfus has been looking for ways in which community members can engage in a productive dialogue across discursive boundaries.

These attempts began with his own cross-theoretical research collaborations. It continued with his conceptual work on the possibility of "networking theories", the activity of employing multiple theories in the attempt to produce a synergetic, cumulative effect. Through these initiatives, Professor Dreyfus has contributed to changing the dominant narratives about theoretical diversity. With his help, the multiplicity of research discourses is now seen less as a problem to solve than as an opportunity to embrace.

Born in Switzerland and now living in Israel, Tommy is fluent in a number of languages, which makes him particularly well equipped for the project of consolidating the international community. After his 1975 doctorate in mathematical physics from the University of Geneva, endowed with several prestigious fellowships and awards, Tommy began visiting universities all over the world. Since then, he never stopped. In parallel to his work at the Weizmann Institute and at the Center for Technological Education in Holon, and later as a full professor of mathematics education at Tel Aviv University, Tommy served as a visiting professor in 14 universities over the world, including in Canada, Germany, Finland, Israel, New Zealand, Norway, Sweden, Switzerland, and the USA. On all these occasions, he spent much time teaching and working with both young and seasoned researchers. By all accounts, he left an indelible mark in all the places he visited.

This owes, among others, to his ability to communicate fluently and easily, to his sensitivity to other cultures and to his general sense of inclusiveness. His willingness to listen and to share his own insights and his devotion to a common effort of understanding and improving mathematics education have touched everyone with whom he has come into contact. Officially retired since 2015, he remains as active and engaged as ever.

To sum up, over the 40 years of his career, Professor Dreyfus has been contributing to our collective endeavor of promoting mathematics education in great many ways: as a researcher, as an editor, as an organizer and policy adviser, and as a teacher and mentor. So far, he has published more than 120 research papers and book chapters, 9 edited volumes, and diverse teaching materials. His writings continue to be read and cited widely, and research programs he initiated or helped establish continue to thrive and inform the field. Even now in his retirement, he continues to shape the field, to foster young researchers and to influence research and policy, both in his own country and abroad. For all this and his many other contributions to our community, Tommy Dreyfus is an eminently worthy candidate for the Felix Klein Award.

Acknowledgement from the awardee, Tommy Dreyfus (Fig. 5)

I am very much honored by receiving the Felix Klein award. And since many people have contributed to this honor in different ways, I feel that I receive the award in the name of all of us.

I would like to thank ICMI, its presidents, its secretaries general, its committees, and especially the award committee chaired by Anna Sfard.

Fig. 5. Tommy Dreyfus gave an awardee speech

My thanks also go to Angelika Bikner-Ahsbahs, to Michael Fried and to all those other friends and colleagues who invested considerable amounts of thought, time, and energy in the submissions of my candidature. I say submissions because there were repeated submissions over several years. I am aware that the decision-making process is very complex: There are many worthy candidates each year, and the decision depends on how the candidature is presented.

True, my career does have something to do with it, and such a career is also a very complex process, on which very many people have a strong influence, starting from a very young age.

Being aware of this, I would like to thank my parents, who have consistently supported my education without pushing me. I am also grateful to my wife of many years Marianne and to our children, Rafael, Judka and Eytan, who have both profited and suffered from my academic career, for example from its many displacements for sabbaticals and conferences.

I have profited from a supportive education system in Switzerland with teachers who have encouraged my intellectual curiosity and consultants who have directed me towards studying mathematics, physics and teaching.

Once I turned toward mathematics education, four people have had an early and lasting influence on my career:

- I would like to thank Ted Eisenberg who taught me what mathematics education is all about and that mathematics is central to it.
- I would like to thank Rina Hershkowitz who taught me that and how research is closely connected to practice, namely real students learning in actual classrooms.
- I would like to thank Pat Thompson who taught me the importance of original and rigorous analysis of both, mathematical content and students' thinking.
- And I would like to thank Dina Tirosh who somewhat later integrated me into a top-level research team, which offered me the atmosphere and opportunity to focus my activity on research and teaching.

Many others have contributed to my career, including my PhD students. They are too many to name here, but it is important for me to mention David Clarke, whom so many of us are missing at this time. David has shown us by example what it means to do international and intercultural research on a large scale, and how much we can learn from it.

Due to people like David and many others, mathematics education as a scientific domain, has achieved a lot over the past 40+ years, both empirically and theoretically. I think, our research has shown that there is huge potential for improving the mathematical education of students from pre-kindergarten to graduate school everywhere, even if we haven't yet quite learned how to bring the insights from research to bear on a majority of classrooms. I think we live in an exciting period of innovation. This finds its expression, among others, in several new journals which are not simply additions to the literature, but which incorporate new directions in the field: The *International Journal on Research in Undergraduate Mathematics Education*, *Digital Experiences in Mathematics Education*, and most recently *Implementation and Replication Studies in Mathematics Education*. This makes me optimistic about the domain of mathematics education, and I wish a dynamic development to this domain for the coming decades.

2019 Hans Freudenthal Award

The Citation[14] read by Anna Sfard

The Hans Freudenthal Medal, with which ICMI honors innovative, consistent, highly influential and still on-going programs of research in mathematics education, is being awarded in 2019 to Professor Gert Schubring, a long-time member of the *Institut für Didaktik der Mathematik* at Bielefeld University, Germany, and an extended visiting professor at the *Universidade Federal do Rio de Janeiro* in Brazil. This award is being granted to Gert Schubring in recognition of his outstanding contribution to research on the history of mathematics education.

Gert's studies of over four decades has opened new, important avenues of research into the phenomenon of mathematics education. Trained as a mathematician, Gert has been a member of the *Institut für Didaktik der Mathematik* since 1973, when this interdisciplinary research institute for mathematics education was founded. In his doctoral dissertation, defended in 1977, Gert wrote on the genetic principle in approaching historical research in mathematics. Afterwards, he extended his interests, producing wide-ranging writings on the history of mathematics education within and across countries, and publishing on the history of mathematics. One of Schubring's earliest publications came out of the symposium, *Comparative Study of the Development of Mathematical Education as a Professional Discipline in Different Countries*, presented at the Fourth ICME conference in Berkeley in 1980. This set the

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[14] https://www.mathunion.org/icmi/awards/hans-freudenthal-award/2019-hans-freudenthal-award

stage for the mathematics education community's reflection on itself as a discipline, and on how its own social context had framed its objects and methods of study. By inviting us to place ourselves in front of a mirror, Gert also sparked interest in the history of earliest efforts in mathematics education, including the work of Felix Klein, on which Gert has recently co-edited the important book, The *Legacy of Felix Klein* (2019, Springer).

His seminal works have helped to realize the importance of considering the social context in the study of the history of mathematics education. If this field of research is now well acknowledged, it is in large part due to his theoretical and methodological contributions, as well as to his leadership in scientific communication.

Another, related but separate, strand of Gert's pioneering work was the study of textbooks, which he began in his investigations on the evolution of mathematics teaching in Latin America. This is yet another area of research that he helped to recognize as worth attention. In 2017 he also chaired the International Program Committee for the *Second International Conference on Mathematics Textbook Research and Development* held in Rio de Janeiro, Brazil.

Schubring has also laid out the formal structures that helped in turning the study of the history of mathematics education into an academic field. He was the founding co-organiser of *International Conference on the History of Mathematics Education* (ICHME), a forum that since 2009 has already met six times. After leading the Study Group on the 'History of Teaching and Learning Mathematics' at the 10th ICME conference in 2004, Gert became the founding editor of the *International Journal for the History of Mathematics Education*. Gert also co-edited the *Handbook on the History of Mathematics Education* published in 2014, in which he contributed to four of the handbook chapters. He is co-editor of the new book series *International Studies in the History of Mathematics and its Teaching*, which includes the 2019 volume he edited himself, titled *Interfaces Between Mathematical Practices and Mathematical Education*.

An important aspect of Gert Schubring's work was his straddling of the communities of the history of mathematics and of mathematics education. His own book in the former field, *Generalization, Rigor and Intuition*, published in 2005, is a major reference in the history of mathematics focused on 17th–19th–century mathematics. Additionally, several publications in mathematics education journals (such as *For the Learning of Mathematics*) introduced tools and concepts from the history of mathematics, such as methodologies for analyzing historical texts, that greatly enrich mathematics education research. Similarly, Gert brought ideas in mathematics education, such as the notion of "mathematics for all" back into the fold of the history of mathematics, to examine what kind of knowledge mathematics has been taken to be in different cultures and historical periods.

For decades, Gert has been actively promoting the study of the history of the field of mathematics education, while simultaneously conducting significant historical studies of his own. No other researcher has had a greater impact on establishing the social history of mathematics education as a dynamic field of scholarly endeavor. His work has not only made us aware of the past of mathematics education but has also provided important insights into mathematics education as it stands today and sets directions for its future. His work informs current teaching by showing ways in which historical mathematical texts can inspire pedagogy. It makes us aware of future possibilities and of the fact that they do not have to be merely determined by the past, but rather can be moulded by new understandings of past practices, values and ways of thinking. All these important contributions make Professor Gert Schubring an eminently deserving recipient of the Hans Freudenthal Medal for 2019.

Acknowledgement from the awardee, Gert Schubring (Fig. 6)

Fig. 6. Gert Schubring gave an awardee speech

I am deeply impressed, feeling gratitude to be honored by being awarded the Hans Freudenthal medal of ICMI (Fig. 6). I am thanking so much the award committee, and I am thanking, in particular, Anna Sfard for her so even emotionalizing citation, and I am thanking the entire executive committee of ICMI.

As I will explain a bit more in my awardee lecture, I am maybe the one Freudenthal awardee who has been in the most various contexts in contact with Hans Freudenthal. Thus, at first when beginning my mathematical research upon results he had obtained about Lie groups, later on meeting him and being in personal conversations at meetings on mathematics education, then being instigated by his remarks on research about the history of mathematics. And more recently, being in involved in discussions about politics of mathematics education, in particular regarding the impact of the program of 'realistic mathematics education', launched by him. I am most grateful for the analysis and presentation of my research, just exposed in the citation by Anna Sfard, which gives a highly fitting insight to the research program established and developed by me, together with a still growing number of co-workers. Actually, coming from mathematics history, my understanding always was that history of mathematics contributes substantially to the entire field of mathematics education.

But let me make a remark on terminology, namely on English terminology: while one has, in French, in German and in Italian, different terms for the discipline, say Didactique des mathématiques, and Didaktik der Mathematik — to be distinguished from Enseignement des mathématiques and Mathematikunterricht, in English one has just one generic term "mathematics education" — thus, let us say, for theory and practice. Yet, Mogens Niss in 2004, when he created for ICME-10 the first proper Topic Study Group, he chose a more specific term for this research program. It was him who established, by creating this TSG, the research program as an international field and he called it quite appropriately as "history of mathematics teaching and learning", instead of using "mathematics education". In fact, my principal aim with his research always was to approach as much as possible the reality of mathematics teaching, the reality in the mathematics classroom — thus going beyond mere descriptions of, say, administrative decisions. And I will comment more in my awardee lecture upon the methodological challenges facing this approach and upon achievements resulting from this research program.

Regarding the second meaning of 'mathematics education', Didaktik der Mathematik, I have always also researched about the history of mathematics education as a scientific discipline. Thus, besides the workshop organized within ICME 4 in Berkeley, in 1980, about the emergence of mathematics education as a scientific discipline, mentioned already in the citation by Anna Sfard, I might mention that I published even a book about the history of IDM at Bielefeld University (*Institut für Didaktik der Mathematik*), which proved to act so decisively for the development of the discipline. I should also like to mention my lecture at the ICMI Centenary, in Rome in 2008, so amazingly well prepared by the ICMI executive committee to celebrate the hundred years of ICMI, which had been created in 1908 as IMUK, Internationale Mathematische Unterrichtskommission and CIEM, Commission Internationale de l'Enseignement des Mathématiques. And I should like to recall the reaction to my historical lecture about the first period of IMUK by Michèle Artigue, one of the ex-presidents of ICMI who had so well emphasized the challenges for international cooperation, showing the importance of being really careful since one is devoted to international cooperation in education — not becoming instrumentalized for political strategies, as practiced by the victorious Allied Powers after World War I.

Thus, I am thanking you all very much, thanking very much the ICMI executive committee, thanking the award committee, including Lena Koch who is the ICMI administrator, who has done so much for the administrative processes enabling all of this. Finally, I am waiting questions after my awardee lecture. Thank you.

2020 Emma Castelnuovo Award

The Citation[15] read by Konrad Krainer

ICMI is delighted to announce that the 2020 Emma Castelnuovo Award for

[15] https://www.mathunion.org/icmi/ 2020-emma-castelnuovo-award

Outstanding Achievements in the Practice of Mathematics Education goes to NCTM — the National Council of Teachers of Mathematics (USA and Canada) — in recognition of 100 years of development and implementation of exceptionally excellent and influential work in the practice of mathematics education.

Founded in 1920, NCTM is the world's largest mathematics education organization, with 40,000 members and more than 230 state, provincial, and local affiliate organizations and other affiliates whose scope covers the USA and Canada.

The Award Committee found evidence to fulfill all criteria related to the Emma Castelnuovo Award. In the following, some exemplary activities of NCTM's past 30 years are highlighted. These activities fall into a wide range of domains — principles and standards as foundations for policy and practice, publications including research journals, professional development, legislative and policy leadership, and international collaboration.

In 1989, NCTM presented Curriculum and Evaluation Standards for School Mathematics, which turned out to be a highly influential document, not only in North America, but all over the world. This document was followed by a series of further book-length reports aimed at establishing a broad framework to guide reform in school mathematics, Professional Standards for Teaching Mathematics (1991), Assessment Standards for School Mathematics (1995), Principles and Standards for School Mathematics (2000), Curriculum Focal Points (2006), Principles to Actions: Ensuring Mathematical Success for All (2014) and Catalyzing Change in High School Mathematics: Initiating Critical Conversations (2018).

Since its inception in 1920, NCTM has published professional journals for teachers of mathematics. Starting with January 2020, a single journal Mathematics Teacher: Learning and Teaching PK-12, published 12 times a year, will replace what has been for the past 30 years three journals. In 1970, NCTM began publishing the Journal for Research in Mathematics Education, one of the world's first journals devoted to this subject. These periodic publications are supplemented by an extensive publication catalogue for teachers at all levels. Some NCTM publications have been translated into other languages, including Arabic, Chinese, German, Korean, Portuguese, Spanish and Swedish.

For the professional development of teachers, principals, and other stakeholders important for mathematics teaching, NCTM holds an annual meeting and exposition along with three regional meetings each year, with a combined attendance of about 25,000. In addition, NCTM offers multiple professional development activities, professional services, and resources via its webpage. NCTM's Mathematics Education Trust (MET), established in 1976, provides funds directly to classroom teachers, affiliates, and institutions to enhance mathematics education. MET offers 30 grants annually, totaling USD 125,000. In addition, it offers scholarships, award programs, and — usually two — annual lifetime achievement awards.

NCTM is influentially engaged in constructive policy discussions among all stakeholders (in particular in the USA), focusing on improving mathematics teaching for all students. This process is supported by the NCTM Advocacy Toolkit, a collection of materials which provides NCTM members with tools and the guidance they need to advocate for mathematics and education.

For spreading NCTM ideas internationally and for establishing contacts and collaboration worldwide, NCTM founded the International Corresponding Societies, currently with 19 organizations in all continents, and has supported several initiatives with educators in Latin, Central, and South America.

NCTM's work has influenced the efforts by teachers, researchers, administrators, and other stakeholders to foster excellence in the practice of mathematics education. Here are some selected quotations from letters supporting NCTM's nomination for the Emma Castelnuovo Award.

An internationally well-known mathematics educator stresses: "I have never lived or worked in the United States, and yet, as a teacher and as an academic, I was aware of the work of the NCTM. I drew on their resources and publications knowing that I could access a wealth of high quality materials developed by expert practitioners in the field. … (T)he NCTM Principles and Standards and the Curriculum Focal Points are curricular documents that I return to frequently when looking at putting together mathematics teacher education courses for pre- and in-service teachers in ways that ensure breadth and depth, with inclusion of the big ideas in mathematics. I have often passed these documents on to students from many parts of the world to use to think about the relative emphases and absences in their own national and regional curricula. Later, as an academic, I made widespread use of articles published across the raft of NCTM journals. … The NCTM has worked tirelessly to advocate for high quality mathematical access for all children. ... The NCTM is an organization that has succeeded in doing this kind of work at a scale that is bigger than any other organization that I can think of."

An internationally well-known mathematics educator from the USA emphasizes, among other considerations, the important role NCTM plays in supporting ICMI activities, for example by providing grants to NCTM members for attending ICME conferences, and by supporting the writing and distribution of documents about mathematics education in the USA since ICME-9 in 2000.

Finally, here is the voice of a former mathematics teacher in the USA:

> "NCTM has been an integral part of every stage of my nearly 50-year career in mathematics education, from classroom teacher, to school and district supervisor, to state mathematics director, to my varied leadership efforts that continue at the state, local, national, and international levels. … It is clear that the National Council of Teachers of Mathematics has been the voice of mathematics education for at least these past five decades of my personal involvement. More than that, there is no doubt in my mind that the Council has also served as the leader within our profession — articulating a shared vision of professional mathematics educators, supporting and disseminating research behind that vision, and providing resources for the classroom and the board room to make that vision

a reality. NCTM is absolutely indispensable to anyone who cares about or works in any area related to mathematics teaching and learning".

There are many more such quotations that could have been included. It is fully evident that NCTM is an outstanding organization that well deserves the recognition of the Emma Castelnuovo Award for excellence in the practice of mathematics education.

Congratulations from Baylor University, by Nancy Brickhouse

Welcome virtually to Baylor University in Waco, Texas, USA. Our mission as a university is to prepare leaders for worldwide service. So, we are thrilled to take part in this global event held by International Commission on Mathematics Instruction, and honored to present the Emma Castelnuovo Award for excellence in the practice of mathematics education which celebrates international leadership in mathematics education.

Welcome to Baylor's School of Education, where Dr. Trena Wilkerson is a professor of mathematics education and a leader in the university's efforts to train future teachers of mathematics and to conduct impactful research to advance the study of mathematics.

Dr. Wilkerson, because of her outstanding leadership in mathematics education, is serving a two-year term as President of the National Council of Teachers of Mathematics and she accepts this award in that capacity.

Thank you to the International Commission on Mathematics Instruction for your worldwide leadership and for presenting this significant award to NCTM in recognition of NCTM's advocacy, research, policy work and support of mathematics educators at all levels. And thank you to Dr. Trena Wilkerson for representing Baylor University as well as NCTM and all teachers of mathematics in accepting this award. Congratulations to NCTM for this prestigious award in recognition of your leadership in and support for mathematics education for more than 100 years. I am honored to make this presentation of the Emma Castelnuovo Award for excellence and practice of mathematics education to the National Council of Teachers of Mathematics.

Dr. Wilkerson, as you come to receive this award, the first thing I have to give you is a gold medallion of the Emma Castelnuovo Award given again to the National Council of Teachers of Mathematics.

Acknowledgement from Trena Wilkerson*, *in name of the NCTM (Fig. 7)

I appreciate the support and leadership in mathematics education of Dr. Nancy Brickhouse, Provost of Baylor University, who is here with me today to present the award in our virtual environment. I extend gratitude to Dr. Brooke Blevins who is the chair of my department, the Department of Curriculum and Instruction and Dean Shanna Hagen-Burke, Dean of the School of Education for their continued support. The focus of this award aligns well with our Baylor School of Education's vision which is to prepare leaders, impact the world and shape the future.

The National Council of Teachers of Mathematics is honored to receive the International Commission on Mathematical Instruction Emma Castelnuovo Award for

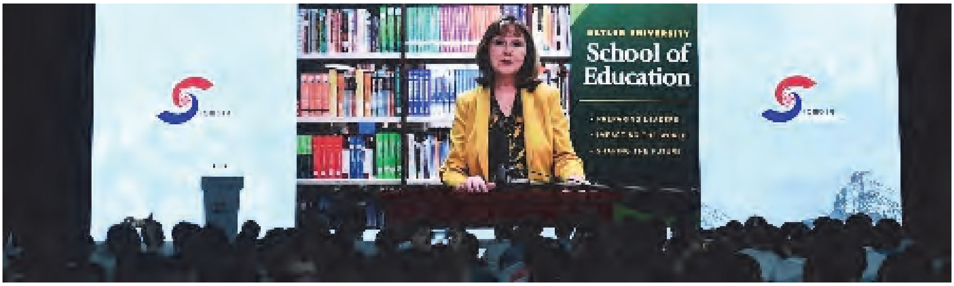

Fig. 7. Trena Wilkerson gave an awardee speech on behalf of NCTM

Excellence in the Practice of Mathematics Education. It is an honor to receive such a prestigious award that was named after Emma Castelnuovo, an Italian mathematics educator, to honor her pioneering work in mathematics education. Her work aimed at a way of teaching that actively engaged students, marked a key point in history for teaching and learning mathematics that fostered a discovery learning environment for all students from elementary through university. NCTM is honored to continue to build on this legacy so that each and every student has an engaging, high-quality experience in learning mathematics.

As President of the National Council of Teachers of Mathematics, I would like to thank The United States Commission on Mathematics Instruction, chaired by John W. Staley for submitting the nomination of NCTM for this award.

The U.S. Commission noted that the National Council of Teachers of Mathematics is the public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for each and every student through vision, leadership, professional development, and research. We also are grateful for the multiple letters of support that were included in the nomination.

Our thanks go to the International Commission on Mathematical Instruction for awarding the 2020 Emma Castelnuovo Award to the National Council of Teachers of Mathematics. I would like to thank Dr. Jill Adler, President of the International Commission on Mathematical Instruction. It was an honor for NCTM to receive notification of the award from Dr. Adler in October of 2019. I also thank Professor Konrad Krainer, Chair of the Emma Castelnuovo Awards Committee and the entire committee for their work in reviewing nominations. We are honored.

NCTM's work in mathematics education is consistent with the International Commission on Mathematical Instruction's principles which are:

- The development of mathematical education at all levels and;
- The promotion of reflection, collaboration, exchange, and dissemination of ideas on the teaching and learning of mathematics from the primary to the university level.

NCTM's mission is to advocate for high-quality mathematics teaching and learning for each and every student from early childhood through secondary school and beyond. NCTM includes mathematics educators from preschool, elementary, middle grades, high school, universities and colleges across the United States and Canada and 171 other countries across the world with over 30,000 members and more than 230 Affiliates. NCTM also established the International Corresponding Societies (currently 19 organizations with representatives from South and Central America, Europe, Asia, Africa and Australia) to build ties with professional associations of mathematics education in other countries.

I would like to take this opportunity to recognize Ken Krehbiel as Executive Director of NCTM. He has provided professional leadership for the organization for over 20 years. He has guided NCTM in multiple efforts to further the mission and vision of the organization. I also want to acknowledge the work and leadership of Dr. Robert Q. Berry, III who was NCTM President at the time of the nomination and continues as Past President and Matt Larson who was serving in the role as Past-president of NCTM at the time of the nomination.

On behalf of the National Council of Teachers of Mathematics, I want to again thank the International Commission on Mathematical Instruction for honoring NCTM with the Emma Castelnuovo Award for Excellence in the Practice of Mathematics Education. It is an honor to accept the award on behalf of NCTM.

https://doi.org/10.1142/9789811287152_0008

The Opening Ceremony

Final Remark by LOC Co-chair

Binyan Xu[1]

Distinguished guests,
ladies and gentlemen on-site and online,

The long-awaited ICME-14 finally opens.

Due to COVID-19 pandemic, ICME-14 has been postponed for a whole year and has to adopt a hybrid mode of participation. Rather, the academic activities of ICME-14 are not affected.

Fig. 1. Ms. Binyan Xu delivered the final remark on the Opening Ceremony

During the Congress, there will be 4 Plenary Lectures, 3 Plenary Panels, 4 Survey Team reports and 5 reports by ICMI Award winners. In addition, more than 60 invited lectures are also in store for you.

ICME-14 sets up 62 Topic Study Groups, As the most active form of academic activities, TSGs will provide opportunities for mathematics educators from all over the world to communicate in different fields, about different themes, concepts, technology, key elements and methods from different perspectives and viewpoints.

Furthermore, the Congress has received more than 300 academic posters, 15 discussion groups and 27 workshops.

There will be 4 national presentations hosted by 4 countries or regions. Besides, ICMI study will report its research results, and ICMI Affiliated Organizations will also

[1] Professor, East China Normal University; one of the Co-chair of the Local Organizing Committee of ICME-14. E-mail: byxu@kcx.ecnu.edu.cn

organize relevant academic activities. ICME-14 sets up “A Special Thematic Activity of Chinese Mathematics Education” for this special thematic afternoon. A total of 13 teams will report their research results, presenting to the world the features of mathematics education in China in a comprehensive way.

Regarding the specific schedule of the Congress, please check out the program brochure. Please don’t hesitate to contact our staff members should you have any questions. Thank you very much for your support for the Congress! I believe, with our joint participation and concerted effort, this congress will be a unique event in the history of international mathematics education conferences.

Today, we feel honored and proud that ICME is held in China for the first time and mathematics educators get together from all over the world. The only pity is that many distinguished guests are not able to be present at the congress because of the pandemic. Here, I’d like to tell you that the invitation from Shanghai is always valid. After we pull through the pandemic, you are welcome to visit Shanghai and China in person to feel the cultural development of Shanghai and China and the latest condition of Chinese mathematics education. We’ll have a communication of thoughts by Liwa River of ECNU — a cradle of mathematics educators.

Finally, I hope all the guests will express your opinions freely at the congress, demonstrate the latest achievements in mathematics education worldwide, communicate about international mathematics education issues, acquire inspirations about the development of mathematics education and promote jointly the vigorous development of mathematics education.

Wish the Congress a great success!

Part II

Plenary Lectures

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https://doi.org/10.1142/9789811287152_0009

Plenary Lecture 1

Mathematics in the Society[1]

Cédric Villani[2]

ABSTRACT Mathematics is an art as old as civilization. Most of the time hidden and respected, sometimes appearing in bright light, mathematicians have always had a privileged role in society, as problem solvers, guardians of an art, deeply attached to values of intellectual freedom and opinion challenge. "The essence of mathematics lies in its freedom", said Georg Cantor. But mathematicians are also accountable to society, which is in need of keeping a link to its most singular and respected science, especially at a time of algorithmic transformation. I was lucky enough to experience the role of mathematician as a public spokesperson, advocating for mathematical sciences as both an art and a technology creator. Later, as a member of Parliament, then head of the Scientific Parliamentary Office, I experienced the intensity and complexity of science in politics, at a time when public action needs to rest on science and when human factors are more challenging than ever.

Keywords: Mathematics; Science; Art; Society; Politics.

1. On Challenging Established Knowledge and Finding Your Way

This is Cédric Villani, speaking from Paris, delighted to be here as part of this 14th International Congress on Mathematical Education.

It's my pleasure to address to such a large audience. All my life has been devoted to sharing and transmitting knowledge, experience, advice. And in particular I'm addressing to young participants of ICME, to talk about mathematics in the society, such an important constantly renewed subject.

One day, I was invited to a very broad audience, French television show. And I knew I had just one minute to talk about the essence of mathematics. And I chose to bring with me three objects to illustrate the nature of mathematical sciences.

The first one was a book — *The Elements* of Euclid (Fig. 1a). The book sounds so familiar to mathematicians. This is the most edited book in the history of mathematics and also one of the most edited books in the whole history of the world. And also for many people like me, as a child, Euclidean mathematics, Euclidean geometry was my first contact with mathematics, with mathematics of reasoning.

[1] This article is the transcript of the video record of Prof. Villany's Lecture on ICME-14, modified and approved by the lecturer.

[2] Institute Henri Poincaré, France.

(a) (b)

Fig. 1. *The Elements* of Euclid and the gömböc

The second object which I brought with me for the television show was this one, the gömböc (Fig. 1b). This Hungarian-discovered solid object, which is homogeneous, which has only one stable equilibrium and one unstable equilibrium. So if I let it, put it whatever position you want on the table, it always come back to the stable equilibrium which is this. And there is just one unstable equilibrium which is the gömböc standing on the tip. You find all the information you want on this object. It has always fascinated the audience whenever I showed it.

First thing was the book of Euclid. The second was the gömböc. And the third one was the very familiar object which is the smartphone.

And I argued that these three objects contain the diversity, the whole narrative of mathematics. Very ancient like Euclid, and very ancient because mathematics is the only science in which discoveries are made for eternity. There is a famous description by Albert Einstein saying that this is the reason why mathematics is above all other sciences. Once you establish a truth in mathematics, it remains true forever. And the theorems of Euclid are still as true nowadays as they were before.

But also Euclid is the symbol of the time when mathematics ceased to be just rules for computation and solving problems, but also became a science of reasoning, induction.

You know the fact that mathematics wants extreme care in the reasoning. Mathematics is the only field of science in which one billion clues all going in the right direction, the same direction is not sufficient to be a proof. That's the only field of mankind's knowledge, that the only field in which you need logical reasoning, and only that to establish truths.

Now on the other hand, mathematics is not just reasoning. It's also very concrete. That's the restriction with smartphones, or with the gömböc. The gömböc is an object. The smart phone is something which allows to do a lot of things. And in one case, the gömböc is something that is just for the art. There is no application which is known of the gömböc. Nobody has made money out of it, except maybe people who make the shape and can sell it to fascinated people like me. It is beautiful but does not serve any purpose except the advancement of science and knowledge.

But on the other hand, you can make fortune on smartphone, and this is a technology which is based on mathematics.

And in math, you cannot distinguish the part which is just for the sake of the art and that part which is for the sake of applications. They go together. You cannot distinguish the concrete realization and abstract reasoning. They go together. And you cannot say that there is old mathematics and the new mathematics. They all go together. And there is a stream from the ancient past to nowadays and to the future. So you have there all the diversity. Also in the problems, the problem you can ask about Euclidean geometry, about geometry of triangles which was so fascinating for me as a student. But somebody else will be fascinated by the mysteries of the complex shapes and gömböc that appear like a challenge. Some other people will be fascinated with the applications of theory of communications and cryptography and whatever. So, the essence of mathematics lies also in the extreme diversity of motivations and applications, and of ways to do this.

And whenever in the conference you insist on the beauty of mathematics, you can be sure, that somebody in the audience will tell you: “But you know, the importance is that mathematics is useful.” And whenever in the conference you insist that mathematics is useful, you can be sure that somebody will tell you that the importance is that it’s also beautiful.

Mathematics is both science and technology, and both science and art. And no other science certainly, is at the same time so deeply an art. Now of course, the status of mathematicians is very much influenced by this stage of mathematics. We have to remember it as a mathematician. The mathematicians are guardians of an art. They keep the recipes, the tricks, the theories. They passed it from one generation to another. They keep alive in their brains, the discovery, the recipes. But also the mathematicians are experts in finding, discovering new ideas, new concepts, new structures, new arguments.

And this goes with the particular mentality which is always on the side of challenging opinions. New mathematics is very often obtained by challenging points of view. And it’s something that I experienced in my career many times as a mathematician to make some progress. If you just follow the rules and advices given by your advisor, you are not making progress. You may obtain new computations, new applications. But you don’t have new ideas. You form the mathematic art, you form the reasoning. These ones come when you challenge what comes from the lessons, coming from the master.

And through the centuries, you see so many examples of mathematicians making new discoveries by challenging the theory that was known at that time, by bringing in a new whole point of view to that. There are famous histories about this, famous stories which I often tell about, for instance, about the history of the stability of the solar system. It’s one of these cases in which new theories emerged generation after generation. And the point of view changed so many times. Each time the generation of mathematicians thought with the view point that was contrary to the one of the previous generation.

Let's get started with Kepler. Kepler discovered the famous laws, named after him. He showed perfect, harmonious and simple laws, governing the solar system, an image of perfect stability. Then this was contradicted by Lagrange, Laplace and Gauss using brand new theory of arrays and perturbation theory, a masterpiece showing that stability had to occur over very large scales of time, like millions of years or something like that.

Then later it was again contradicted by Henri Poincaré. Poincaré actually thought he started to think that he could prove the long-term stability. And then he discovered, after receiving the famous prize of King Oscar of Sweden that, in fact, he could prove the possibility of instability, it was one of the famous mistakes in the history of mathematics. And after the work of Poincaré, people believe in the instability of the solar system over a very large period of time.

And then in the middle of the twenty's century, then come Kolmogorov, Arnold and Moser's beautiful work putting together classical mechanics and probability theory and Hamiltonian dynamics and new rules of probability coupled with mechanics, showing that under some assumptions (not satisfied in practice, but you could imagine applying their spirit in real life!), that had to be, with large probability, forever stability. Beautiful piece of work! And then the mood of mathematicians about the stability of the solar system had to swing again.

Then the 70's came people like Laskar and Tremaine using huge computer simulations and very clever mathematical ways of rewriting the equations showing that in practice, there should be some instability after all. So from Kepler to Laskar, you see that the mood of the dominance theory about this particular problem of solar system stability has changed many times.

Each time it comes when somebody challenges previous theory and challenges the point of view in the idea. He does not say that what the previous guys said is wrong. Remember, mathematical truths are eternal! He says "they had one point of view, but I think there is a better point of view which is this one. And this is my set of assumptions. And we try to convince you why it's the better representation of the reality and why it's better for the prediction."

This is one story. There are thousands of such stories. But this one about the stability of solar system may be the most famous because it's one of these problems in which trying to solve the problem has generated entire fields of mathematics, from matrix theory to perturbation theory, to development of probability theory, to the development of numerical computations, numerical analysis. And this is always the interplay of mathematics, the history of problems, the history of concepts. They go hand in hand.

New concepts enable to tackle new problems. New problems force you to discover new concepts. And there you go.

This story is full of adventures, full of unexpected things, full of mistakes. These are unexpected adventures. I would say, as a consequence, the mathematician always has to be on the side of freedom of thought, freedom of speech, freedom of being able to contradict the master because this is so much engraved in our field.

Mathematician is deeply rooted in democracy, and related to democracy. And the idea is that everybody is allowed to contradict and to raise an objection.

I realized this very early, when, as a high school kid, I was confronted with mistakes of my teacher. It did happen. I remembered this course in mathematics, the teacher started to prove some theorem, etc. I saw the theorem. I could not believe it. I was maybe 14 years old, At the end of the course, I went to see the teacher and I told her "Madame, I think there is a problem with the statement that you gave us because I think I can construct a counterexample". And I showed her the counterexample. She said this was strange. I explained to her etc. She said she had to think about it. The next day and at the next course, she told me "Ok, you were right, my proof was bogus". She made a correction for the whole class. So this was an example that when you were a school kid, you are not obliged to believe in what you are told by the master. You can contradict. And if the master is of good faith, good will, the master will recognize that he or she is wrong.

In all other sciences, this is impossible. In physics, they tell you about the existence of atoms, how can you contradict the existence of atoms? You have to believe what you are told about the experiments. In biology, they will tell you about the liver, about the biology of wheat, about the sugar cycle, about the climate… whatever. You are a school kid. How can you contradict that? You have no weapon to contradict. But in mathematics, you do have the power to contradict, because everything is proven by the reasoning. You have a weapon which is your brain, your capacity of reasoning, your free spirit and ability to contradict what you are told. If you give a proof, then you can contradict the authority. I told you the example which I experienced when I was a kid.

But somehow it continued. Look at my PhD thesis. Such emotion when looking at these documents in which I put all my heart. I defended my PhD when I was 25, not so young. And it was about partial differential equations of Boltzmann, and Landau in physical theory, mathematical theory of physics of gas and plasmas. There are many chapters. My PhD started a bit like a joke. I was the president of the students' union in my university at the beginning of my PhD. I was not at all motivated by mathematics. And my advisor was worried about me. And about when I was going to get to work. Once on a small card he wrote me like "You have to, It's time for you to start working." His small document, his small card I reproduced in my PhD. I had no motivation in those days. Then I started to work and work and work. I really started to work when I was promised a good position after my PhD at the École Normale Supérieure.

Anyway, I fell in love with the Boltzmann equations which was a subject in which my PhD advisor was a very well-known master. But in my PhD, the most important parts are not the questions that were asked by my PhD advisor. These were answers to the questions that were found in the course of my PhD. In some cases, questions that my advisor liked. In some other cases, questions that my advisor did not like. Some of

Fig. 2. Villani's book *Optimal Transport, Old and New*

the techniques were radically different from the techniques of my advisor. They were inspired also by other people. My advisor is Pierre-Louis Lions. But I was also very much inspired by Yann Brenier, by Michel Ledoux, by Eric Carlen.

And I developed my own techniques. At the beginning of my PhD, I thought I would do the same spirit and development of what my advisor said. But by the end of the PhD, I had switched. I'd rather use different perspectives, and I had developed some other alternative techniques. And there had to be in this PhD some element of rebellion for it to be a good PhD.

In science, as in the example which I told you, a good will, a good faith, master will always recognize the merit of the student even when the student does not comply to the instructions of the master.

Also I changed subjects. I went into questions that were unorthodox. I developed some old subjects mixing it with new subjects. You know this is a great adventure when you are a young researcher, going everywhere in the world, finding new problems, being invited here, discussing with somebody, finding out there are an interesting problem, going for another collaboration etc etc etc. And this led to a number of productions, a number of new subjects.

This is my favorite, *Optimal Transport, Old and New* (Fig. 2), 1000 pages, thick book which I wrote on the subject which I did not know at the beginning my PhD. And I started to get in the subject as I started to work. By coincidence, by accident, I dipped in the subject. I was invited to some collaborations, then to give a course. Giving a course is the best way to learn your subject. That's also one possibility in which you will grow your spirit, your free spirit, and spirit of contradiction finding for yourself which are the most important things in the subject.

2. On the Art and Science of Mathematics

You need to always remember the sentence of Cantor, the father of the set theory, the essence of mathematics lies in its freedom. It's also freedom which guided me to maybe my most famous work, on Landau damping. The work that let me clear the bar for the Fields Medal. It is quite an accident that I started to work on this problem. Nobody asked me to, of course.

But I wanted to work on another problem regularity of Boltzmann equations. Boltzmann equation is not related to Landau damping. Landau damping is related to Vlasov equations. I wanted to continue my work on the Boltzmann equations. I had a work session with my former student Clément Mouhot — with whom I would wrote the paper eventually. I told him about my ideas. He told me his ideas. He said he remembered a conversation with Yan Guo from Providence. "Maybe there is a relation to Landau damping," he said. I never thought about that. What could it be? Then I remembered another conversation with a mathematician from Princeton. This was the start of our adventure. Then for two years, Clément and I struggled with that theorem. Not knowing at the beginning if we wanted to prove this or that, not knowing which were the tools. Making hundreds of mistakes, some of them were big, some of them were tiny, and going and going on correcting them etc. It was a great story, very fragile in the sense that many times, we thought we were facing a deadend and our proof, our reasoning could not work. In the end, we did, it did work!

Landau damping is a phenomenon of stability of plasmas very important and well-known in plasmas physics and very important in showing applications of plasmas physics which are quite important. And it tells you that in the plasmas, you have some stability, even though the plasmas dynamics is reversible. Reversible in the sense that there are no collisions, no increasing entropy, no arrow of time. So you could think that there is no trend to equilibrium because going to equilibrium in Boltzmann equation is related to the increasing dynamics related to the arrow of time. But in Vlasov equation, there is such damping phenomenon for small perturbations. And genius physicist Lev Landau discovered this in the 30's. It was quite a shock, quite controversial.

Together with Clément, in this huge paper of ours, eventually, we showed, using new theory, new tools that indeed you could set the reasoning of Landau on rigorous mathematical footing provided that you are close enough to a stable equilibrium, provided that your perturbation is smooth enough, and it explains exactly how to compute the expected smoothness needed, making the link with a theory of Kolmogolov, Arnold and Moser, making the link with a famous experiment in plasmas physics called the echo experiment etc, and showing that different various physics phenomenon are related, even though this was not expected by physicists themselves.

So this work was full of adventures, full of wonders, full of miraculous events and so on. I was convinced in my heart that we would tackle and solve all the obstacles. But there was no logic about this belief. Some of my colleagues told me that "It was so difficult so many obstacles. How could you believe that you would solve it?" This is not something rational. It's not something that you learnt at school. It's something about passion that you have inside yourself. I said "Okay, I'm going to solve this mystery and fall in this so deeply that I will become part of the problem." In the sense that you want to solve the problem from within, in the sense that you are dreaming of the problem, with sometime some illuminations when you wake up at night etc. This is the kind of extremely intense state in which you will be.

I always say the three most important characteristics of mathematician are tenacity, imagination, and rigor. Rigor comes third. It's tenacity and imagination will make your mark and your career in the world of mathematics. This what we know when we are researchers in mathematics. But society at large is very much unaware of what it means to be a mathematician and what's the real nature of mathematics. How much passion there is in it. How much surprise there is in it. How much contradiction there can be.

It's very important at a time in which society very much relies on mathematics. It's very important for us to share about the work and the role of mathematics and mathematicians. And this is what I started to do after the Fields Medal in 2010.

I wrote a book to tell about the adventure of Landau damping problem solving. This is the book called "Living theorem" (*Théorème vivant*). The English title is *Birth of a Theorem* (Fig. 3). This is a book which is not written like mathematics book, because there are portraits of mathematicians and there are stories like autobiography a little bit. During a short period of time, during the two years of working on the problem, there are email exchanges, some of them very tormented you know… It's not like usual book explaining about science for broad audience because there are also formulas. There is a portrait of Carlo Cercignani and I explain what was his conjecture. What were the tools that we setup to prove it etc etc. And also I put in the book some of the big formulas that we have to fight with as mathematicians.

Fig. 3. *Théorème vivant* and its English and Chinese versions

And this was, you know, to show in the impressionist way what is the life of the mathematician. Also about cultural aspects like what is it that you do at night, how you work, what kind of the songs you listen to, etc. There is a part of book in which I describe the songs. There is a part of the book in which I describe encounter with various mathematicians etc etc. See, browsing through the pages of the book… This is about the songs… This is about some of the formulas… You know, to show in an impressionist way the life and work of mathematicians, insisting that mathematics is not a deterministic science but also a passionate art.

It's as if your kid wants to know about your work as mathematician. You don't try to simplify things. You bring your kid to work. And the kid will see what you are doing, with whom you are talking. How you are desperate sometimes, happy sometimes. What you are listening to etc. The whole life of a mathematician.

The book, precisely because of the original point of view, had quite a success. It was translated in many languages, including Chinese. An important thing to know about the book is that I was not my idea in the first place. It came by accident by encountering a famous editor in a dinner, several months before the Fields Medal. The editor was attracted by my spider. We talked to engage conversation, what you do, etc. "I was told that you are a famous mathematician."

He wanted to have me write a book for him. But all the examples of books that I could think of were of no interest for him. "I can write you a book on Boltzmann equation", "I don't care", "A book on the Entropy", "I don't care," "I can write you a book on information etc", "No". And he told me "I want to know how you work, how you think as a mathematician. What goes on in your head". I was very much embarrassed. I thought about it. Then the result was the book.

I thought I'm going to write it. It will be what is behind the curtains when you make a discovery of a theorem. And what is the process of being a mathematician. And I will tell it like an adventure. Actually, the editor thought he could even publish it as a novel.

This was the start of many years of communications of mathematics which I undertook in collaboration with various people with always the same idea that to make people understand mathematics, love mathematics. Don't just talk to them about how useful is mathematics. Tell them about the adventure of being a mathematician, about the art.

Tell it like it's a bunch of stories. And stories always mixing three traits, the stories of people, the stories of problems, the stories of concepts. And when you make a lecture, a good lecture, it has to be these three traits together. Maybe the idea of Fourier will come here, in the problem and another problem. Problem that Fourier wanted to solve. And the life of Fourier himself. How he was also in the revolution or after the revolution, the empire etc. And all this makes mathematics an art, a human activity as well as sciences.

I did this for so many projects. For instance, here is a comic strip (Fig. 4a) which I wrote with the great French comics drawer Edmond Baudoin, very famous. It talks about some very important hidden characters involved in science or innovation during

(a) (b) (c)

Fig. 4. Some "non-mathematics" cooperative projects

the Second World War who had a lot of influences through their discoveries, their mistakes, their actions. The kind of spirit they were in, the kind of crazy story that they had to live. And what is in their heads when they are afraid, when they are proud, when they are shamed, when they are full of questions, and so on. So these were Heisenberg, and Turing, and Szilard and Dowding. And telling this as an adventure as well as science in the construction.

Then there's a little bit more crazy project (Fig. 4b) made together with the great photographer Lisa Roze, very particular very special photographs. And she wanted me to write the text corresponding to the photographs. And she was attracted by collaboration with me because she thought highly of my imagination. And she thought I was ready to find some logic behind her photographs, to find some way to order them, to find some text with some kind of boutique twists etc.

This is not mathematics of course. This is contributing to the idea that mathematics comes with imagination as an art.

It's also something which I insisted on the series of standup lectures in the Maison des Métallos, a Culture hall in Paris, very much about bringing quality culture to all kinds of audience, including audience which do not usually get interested in mathematics. Look at the illustrations: You see the gömböc… this is an orrery displaying the motion of planets… the geology… this is a twin surface with negative curvature, whatever.

So in all these lectures, always, it was about the art of it, telling stories, and always stories about bringing together these three straits, story of ideas, story of people, story of concepts. My book being dedicated to a very inspiring mathematical figure that was Maryam Mirzakhani.

So this for a number of years, was the main of my activities, talking about mathematics, making collaborations of mathematicians with artists, finding ways to talk about problems, but also get involved in some projects, some associations, and so on, and being part of society. I strongly believe that we as mathematicians, have to get very much inside the society. This is also something that was one of the main characteristics of the career of my PhD advisor, Pierre-Louis Lions.

3. From Science to Society to Political Decision

Learning to do mathematical communications was not immediate. I had to work hard to find my ways to do mathematical outreach, speak about mathematics to large audience.

It was very important that for many years, I was a professor in Lyon. My neighbor was Étienne Ghys, one of the first masters of mathematical communication. At some point, he was considered the best mathematics lecturer in the world. His lectures in ICM and so on were extraordinary. And he took the job of mathematical outreach extremely seriously and I was very much influenced by that.

In Lyon, there was an atmosphere, both at the École Normale Supérieure in Lyon and in the Lyon I University, there was the atmosphere about sharing to the society, giving account to the society which is legitimate because we are part of the society, we are often funded by the society by public money. But also we need to get from the society the problem, the people, whatever. We are part of the society, we need to find a good way to underline it.

My first interview for a journalist's journal (not a science journal!) was a disaster. The journalist wanted to cancel the interview because it was so obscure, whatever. And then I worked. I went to, you know, learning sessions, master classes whatever and was passionate about that also. I wrote in a newspaper some columns. I was in the radio. I learnt a lot. And there was all these collaborations and the projects that I told you about.

Some of the most important lessons I learnt is that first to attract your audience you need to talk to the heart first, emotions. Something which looks close to you, which will bring a link between the audience and you. Something related to the culture. Culture is what brings people together. Emotion is what brings people together.

Then after you hooked your audience, with these emotions, you can go on with the concepts, with the stories, with the rationality, with the logics, you have to start with the irrational thing to speak. Never separate the concepts from human, human contact, human adventure and so on.

This was somehow in the direction, the continuation of my mathematical work. Then I did more than that and went further on. Becoming a politician and going into politics for real in 2017. Okay actually 2017 was not my first encounter with politics. As soon as 2010, right after the Fields Medal, I started to be engaged in a number of projects, associations, institutes, government, talking with people with all kind of responsibility.

I always had some faithful responsibility. This certainly goes back to the time when I was 21 years old, and I had myself elected as the head of students' office. And I likes it so much, handling the problems, finding how to try to create the cohesion.

After 2010, I was involved in political groups about the construction of Europe, federalist as they are called. I still consider myself as a federalist, name given to those people who believe all nations in Europe have to engage in the strong integration process so that Europe can become a strong unity. A political entity, an economical entity with some cohesion and some political strength. This was back in 2010. But this led me little by little to be more involved with the national politics.

In 2017, with some, you know, some complicated sequence of events, I ran for Parliament. And I was elected. It was a new life that started, new life in which the big problem was how to make sense of my scientific career within the political career. Convinced that each makes sense, to have science and politics, but it's a delicate equation to find the problems and ways in which your scientific experience can be useful. International experiences that goes within research, with the problem solving. That is quite important in politics. Also the fact that, politics nowadays raises a lot of very tricky scientific problems, related to environment, related to pandemics, related to biology, related to energy whatever.

And in these four years, from 2017 to now, on some occasions, I managed to make the better of my scientific experiences. In some other occasions, I did not manage. Whether it was a success or a failure, it was always instructive. Some of the most important things which I think I let to do is the mission on artificial intelligence. I'm showing it to you. This is the journal in which this paper in which there was the interview like "live my life of member of parliament". And it was in those days, when there were many newcomers in politics in 2017, the interviews like "How was it to be in parliament, what do you think". At first in parliament, other members of parliament would look at me as a strange beast, you know. What does this mathematician do, I'm on gas.

There was a time in history in which mathematicians were quite numerous, scientists were quite numerous in the French political life. But this was long ago. There were many examples long ago. Nowadays very few examples.

And one of the first missions that was given to me, for six months, was a mission about artificial intelligence. This is a report which I wrote after six months of work with my team. A report which tries to get a complete picture about the artificial intelligence with consequences for France and for Europe, about data policy, about the ethics of AI, about the job future, about education, about fields of applications that should have priority like health and mobility, environment and defence…

And you know, trying to make the big picture, the whole picture of what AI should be like and why it's the business of everybody. So this report was for writing a technical document, but also for addressing broad audience, going in the TV to talk about AI, going to newspapers to talk about AI, going to debates to talk about AI. Of course, it rested on my experiences of sciences as mathematician.

For instance, this is a book (Fig. 5) which I edited together with medical doctor Professor Bernard Nordlinger, about health and artificial intelligence. I'm writing this on behalf of the academy of sciences, Nordlinger on behalf of the academy of medicine. And we organize conferences, we invite lectures, whatever. So this is the scientific part.

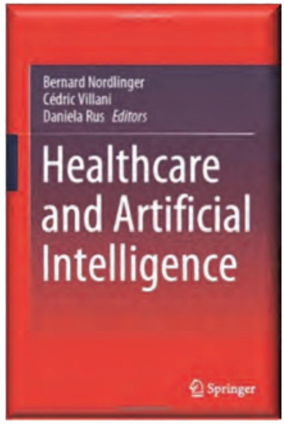


Fig. 5. A book on health and AI edited by Bernard Nordlinger, Cédric Villani et al.

Based on the scientific part, I wrote this report. Nowadays, I still get asked sometimes by members of the government, sometimes by people from the industries, sometimes by just citizens about AI. Asking what I think about this etc. Recently, there was a delegation from people from America to talk and discuss about American report and compare it to the French report and etc. So this is an example in which the scientific knowledge could be brought directly to the heart of the policy and society problem.

The plan which was deduced from this book, from this report has been implemented too, in large part, still a number remains to be done. It became the official plan of AI for the French government.

Another example which was more tricky, was a report on the teaching of mathematics in France (Fig. 6). A work I did with math teacher and inspector Charles

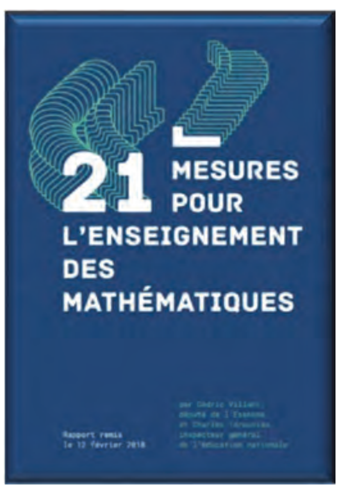


Fig. 6. The report on the teaching of mathematics in France by Villani and Torossian (https://www.vie-publique.fr/sites/default/files/rapport/pdf/184000086.pdf)

Torossian. To show and see what we had to do to repair the French system of mathematics teaching, where were the main difficulties? How to act? Training of teachers on the curriculum, on the tools, on the ways to organize things in high school, in medium high school, in elementary school, etc.

The report went very well. The report was very much praised by the mathematician community, and by the ministry of education and was started to be implemented. And then problems began. The ministry of education did not implement just our report but also a number of other things which had some constraints which were determined by this or that etc. An enormous number of difficulties arose in which I had no role and no part. I did write some warnings for the government and said "okay be careful about this, this and that", but in the end, the situation is extremely confused right now in particular among mathematicians and mathematics teachers.

It's good to write a report. But you always have to remember implementation is much more difficult. I did regret that I was not more involved in the implementation in this case. Maybe some misunderstanding could have been avoided. And currently there's a big number of things to fix still in the mathematics strategy.

Another of the things which I did and which I think was badly needed was the reform of the scientific parliamentary office. More precisely, parliamentary office of evaluation of scientific and technological choices: it is a group made of 18 members of National Assembly and 18 members of Senate. So both chambers of the Parliament working on problems which are of scientific or technological nature, important for enlightening the choice of the politics on it. Say, politicians have to make choices of nuclear energy, but this is a highly technical subject in which you need the opinions of many scientists. We need to organize some contradictory debates and also to hear the point of view from the society etc. That's the role of scientific parliamentary office, organize these debates on problems which are the interfaces of science and politics.

And it was funded in the 80's. It was just funded precisely to work on the subjects related to nuclear energy. So I became the president of this office. And I worked a lot to diversify the themes of the office, to hire scientists within this parliamentary office, to go more in the directions of humans in social sciences, to get more in association with the working groups of the Parliament, to follow more closely the base of the political event, etc. Quite a work.

What subject did we treat recently? Things such as research in polar environment, long-term Covid, or scientific integrity, or viruses used to fight bacterial diseases, or quantum computing and quantum information and quantum cryptography, or open science, or all kinds of therapeutic subjects, you name it. One of the trickiest subjects which we had to handle is the about the new breeding technologies. These new techniques of biotechnology based on editing genome techniques such as CRISPR-Cas9 to edit the genomes of plants. So there are aspects related to biology, related to economics, related to ethics. Lots and lots of debates. In this case for NBT, it was a very hard debate. The two parliament members who were responsible of the subject had conflicting opinions, could not get together. And I, as the president of the office had to step in and organize syntheses with the help of the scientific secretary. We were discussing for a long time on each subject. And we managed to formulate seven propositions, seven recommendations. The two MPs in charge agreed on six of the seven recommendations. And on just one, the seventh one, they had conflicting opinions. And we could exactly pinpoint why and what kind of different values it's corresponding to: Then we have made our job for the benefit of the Parliament. Not taken a decision because it's for the democratic decision to proceed, through the democratic procedure un Parliament. But we studied the subject and analyzed it to the point where a political decision becomes possible by analysing the possible choices. Balancing the various problems and advantages or drawbacks, balancing the natural sciences with the human sciences etc.

A common feature with all these problems that we treated, in fact, is that the most tricky parts are those related to social and human sciences, like opinion issues, bubbles of information, fake news. In science, we are used to the idea that no one a priori has the truth, we have to listen to all the contradictions. Even revered scientists can make mistakes. And there are famous examples in the history of science in which the best scientist of the world makes horrible mistakes, and in fact, some of his opponents was right. We know such examples. Eventually, all these problems get resolved in the right direction. But sometimes it takes time. But one should never a priori believe as a god given truth what somebody or some entity or some institution or some political party says. One always has to think "is this really true? Can I contradict? Where is the logical reasoning?" etc etc. That's what we do at the scientific parliamentary office organizing the political debate, which is the basis of democracy. The heart of democracy is the contradictory, open, sincere debate of ideas.

And I tell you and this would be my conclusion. In a world in which debate of ideas are extremely confused, in a world in which many political problems hugely rely on scientific and technological advances and choices, in a world in which science is hugely used by the power, we, scientists have huge responsibility to get committed in the world and to make sure, but we keep the memory of the past and everything that we can learn from the past, from the history of science, and from history of mankind. And that we get committed to making our vision of the future explicit, saying which are the consequences of the choices that we will make, consequences on our children. What we should transmit to the next generation. Currently, more than ever, and in a world which is more mathematical than ever with the advance of information, technologies, of statistics everywhere, of artificial Intelligence, you name it, all kinds of mathematical application in the world, we, mathematicians have a responsibility of being committed to the society for the future and for the sake of the future generations.

Thank you for listening.

(Transcribed by Miaofen Chen, Guodong Zhou and Zhijie Chen)

https://doi.org/10.1142/9789811287152_0010

Plenary Lecture 2

Forty-five Years: An Experiment on Mathematics Teaching Reform

Lingyuan Gu[1]

ABSTRACT This is an experimental report on mathematics teaching reform conducted in an urban-rural fringe area in the west of Shanghai. From 1977 to 2022, this experiment spanning 45 years has witnessed a change of Chinese society from bringing order out of chaos to Reform and Opening-up and eventually to educational modernization. At the early stage of the experiment, a methodological system of practical research, featuring "investigation – screening – experiment – popularization", was built up, a feasible way to improve the quality of the education under the most common educational conditions was found, and the Chinese experience of "teaching and learning promote each other", specifically, students learning with experiencing variation and teachers developing through teaching reform, was summed up. At the later stage of the experiment, the focus was shifted to the cultivation of students' all-round ability and an empirical method of "abductive reasoning through big data" was developed. Based on big data experiments and long-period sampling analysis, by way of clinical observation and evidential reasoning, the key to promote students' inquiry and innovation ability was found and, with practical effects, it exerted positive influence upon the society. This report is the result of the persistence, perseverance and the collective efforts of three generations, including the life mentors of the older generation, the backbone of the transitional generation and a large number of young talents. The experiment consists of three stages: 1. improve teaching quality generally and dramatically from a low level (1977–1992); 2. comprehend first and break through the bottleneck of high cognition (1992–2007); 3. improve teaching research and promote inquiry and innovation (2007–2022).

Keywords: Qingpu teaching reform experiment; Experiencing variation; Action education; Abduction of creative ability.

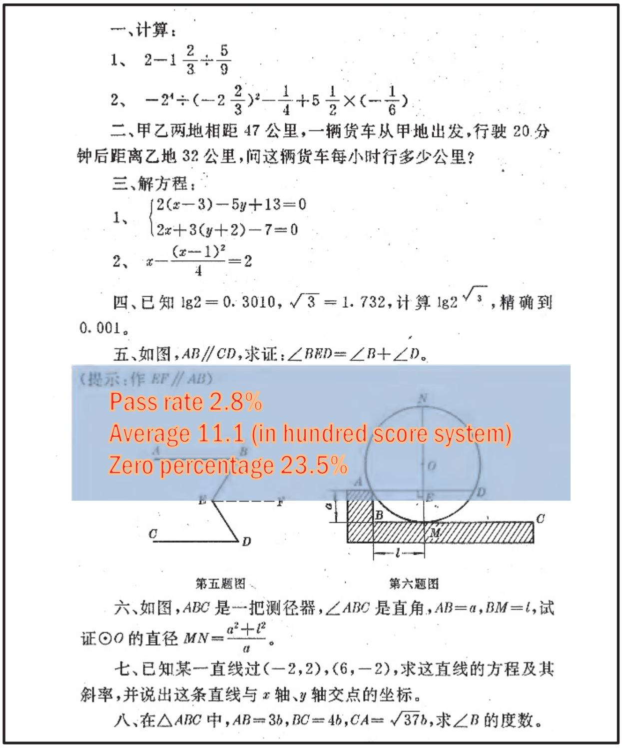
1. Improve Teaching Quality Generally and Dramatically

1.1. A reform experiment initiated by an exam paper

At the end of 1970s, the mathematics teaching quality of the primary and secondary schools in Qingpu County of Shanghai came last in the city. In 1977, we gave a diagnostic examination to 4,373 high school graduates with mathematics problems for

[1]School of Mathematical Sciences, East China Normal University, Shanghai, 200241 and Shanghai Academy of Educational Science, Shanghai, 200032

primary school and junior high school students (Fig. 1). The pass rate was only 2.8%, the average score was 11.1 out of 100, and the zero percentage was 23.5%! How should we deal with the shocking backwardness? The tension between students' low score and modern concepts bred an urgent need for teaching reform.



一、计算：

1、$2-1\frac{2}{3}\div\frac{5}{9}$

2、$-2^4\div(-2\frac{2}{3})^2-\frac{1}{4}+5\frac{1}{2}\times(-\frac{1}{6})$

二、甲乙两地相距 47 公里，一辆货车从甲地出发，行驶 20 分钟后距离乙地 32 公里，问这辆货车每小时行多少公里？

三、解方程：

1、$\begin{cases} 2(x-3)-5y+13=0 \\ 2x+3(y+2)-7=0 \end{cases}$

2、$x-\frac{(x-1)^2}{4}=2$

四、已知 $\lg 2=0.3010$，$\sqrt{3}=1.732$，计算 $\lg 2^{\sqrt{3}}$，精确到 0.001。

五、如图，$AB /\!/ CD$，求证：$\angle BED=\angle B+\angle D$。

（提示：作 $EF /\!/ AB$）

第五题图 第六题图

六、如图，ABC 是一把测径器，$\angle ABC$ 是直角，$AB=a$，$BM=l$，试证⊙O 的直径 $MN=\frac{a^2+l^2}{a}$。

七、已知某一直线过 $(-2,2)$，$(6,-2)$，求这直线的方程及其斜率，并说出这条直线与 x 轴、y 轴交点的坐标。

八、在△ABC 中，$AB=3b$，$BC=4b$，$CA=\sqrt{37}b$，求 $\angle B$ 的度数。

Fig. 1. The examination[2] of 4,373 high school graduates in 1977

The backbone teachers in Qingpu pledged to the then director of the bureau of education that despite whatever hardships, they were determined to change the backwardness completely. With a strong determination to avenge the shame, they drew

[2] The English translation of the examination paper:

1. Calculate (the question items omitted).
2. The distance between Place A and Place B is 47 kilometers. A truck starts from place A, and, after 20 minutes, it is 32 kilometers away from place B. How many kilometers does this truck travel per hour?
3. Solve equation or system of equations (the question items omitted).
4. Given $\lg 2 = 0.3010$ and $\sqrt{3} = 1.732$, calculate $\lg 2^{\sqrt{3}}$, accurate to 0.001.
5. As shown in the figure, $AB /\!/ CD$, verify that $\angle BED = \angle B + \angle D$. (Hint: Draw EF with $EF /\!/ AB$.)
6. As shown in the figure, ABC is an instrument for measuring the diameter of a circle in which $\angle ABC$ is a right angle. If $AB = a$, $BM = l$, show that the diameter of $\odot\ O$ is $MN = \frac{a^2+l^2}{a}$.
7. A straight line passes through points $(-2, 2)$ and $(6, -2)$. Find its equation and slope. Find also the coordinates of the intersections of the line with the x-axis and the y-axis.
8. In ΔABC, $AB = 3b$, $BC = 4b$, and $CA = \sqrt{37}b$. Calculate the degree of $\angle B$.

up a plan consisting of three-year all-round investigation, one-year experiences screening, three-year experimental practice and eight-year promotion and application.

1.2. A reform design — "15 years to sharpen a sword"

Teaching reform was a game between change and inheritance. Our design principles for the reform were: firstly, adhere to openness and inclusiveness and absorb extensively the information from outside, including experience regeneration, theoretical progress and technology update; secondly, base our screening on practice, and, with continuous improvement and repetitive testing of teaching effect, look for appropriate path in view of the current situation.

The teaching practice was made up of four parts:

(1) Teaching investigation (three years). Conduct a comprehensive investigation into the major problems and their crux in local area, and we collected more than 160 specific experiences accordingly.

(2) Experiences screening (one and half years). Applying experiences in classroom teaching, we did 50 circular screenings repetitively, which included experiments, observation, filtering and optimizing. With a focus on the improvement of classroom teaching, we came to four key teaching behaviors: i) facilitate students to learn under urgent demand; ii) organize the sequence of teaching content; iii) guide students to attempt and explore; and iv) provide feedback and adjustment on the basis of learning results.

(3) Contrastive experiments (three years). Select 440 junior high school students from ten classes and, following the four aspects mentioned above, conduct contrastive experiments under strict control, track and compare 50 pairs of matched students closely.

(4) Popularization and reinforcement (eight years). Popularize and reinforce experiences in the entire district, and introduce the concept of communication into popularization so that popularization would lead to internalization and creativity instead of mere imitation or copy. Make a full coverage relying on multi-level teaching research groups. What we want is not tinkering with the problem, but a large-scale improvement.

These four parts make a complete methodological system of the reform experiment (Fig. 2).

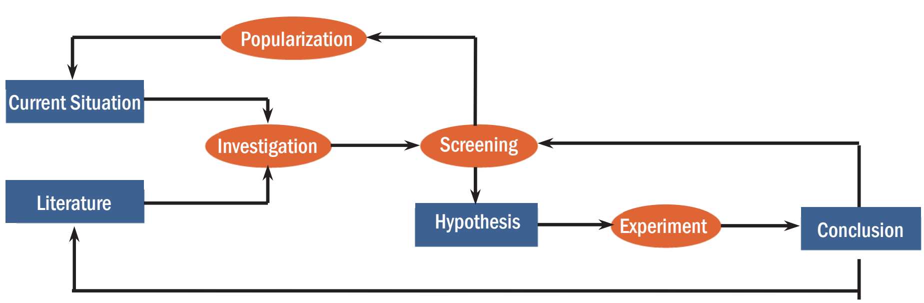


Fig. 2. Method system with circular screening as the core

This experiment always aims at the matching and mutual reflection between full-sample statistics and refined case analysis. We particularly created the method of circular screening, which filled the gap between investigation and hypothesis, and the one between conclusion and popularization & innovation as well. Given our heavy duty of solving real problems, we should avoid the traps of formalization and emptiness, or "scholars revolt" in Chinese idioms.

1.3. Give prompt feedback, make a quick change on the backward situation

The diagnostic examination given in 1977 showed that poor-grade students were all over the county and the total amount was too large to reduce. We took samples from groups with large proportions of poor-grade students, analyzed them individually, observed their homework, and eventually summed up the formation process of their poor performance (1982) (Fig. 3).

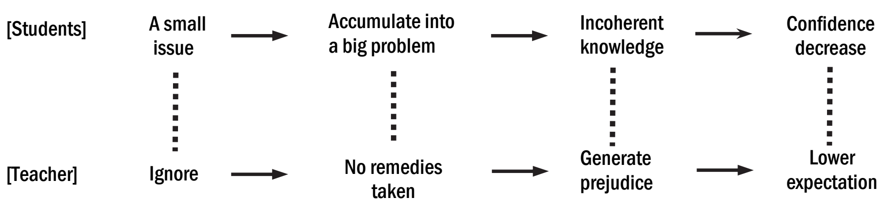


Fig. 3. The formation process of lower-scored students

The observation of students' homework showed that the cause of low score was the accumulation of both teaching and learning problems, which went all the way down to the loss of self-confidence on students' part. We should block this ill cycle promptly. Therefore, we referred to local experiences and gave feedback to students about their assignments within the day, marked their exercises face-to-face and encouraged students generously. Later, we conducted experiments in different groups in 1986, and the result showed that timely and individual guidance with targeted feedback was undoubtedly effective in improving students' score (Fig. 4).

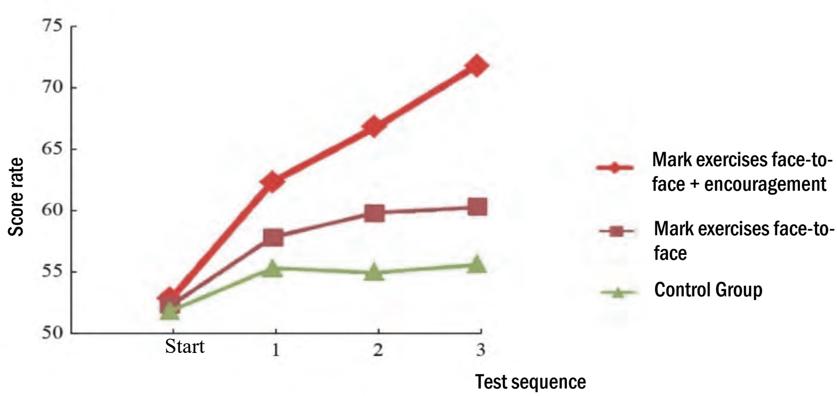


Fig. 4. Experiment results of assignment feedback

As a matter of fact, the *Book of Rites* from ancient China already described a dual-channel feedback model that goes "learn – discovers ignorance – self-reflection" and "teach – discovers difficulty – self-improvement". The module is referred to as "teaching and learning promote each other". It is a comprehensive demonstration of the charm of outstanding ancient Chinese culture (Fig. 5).

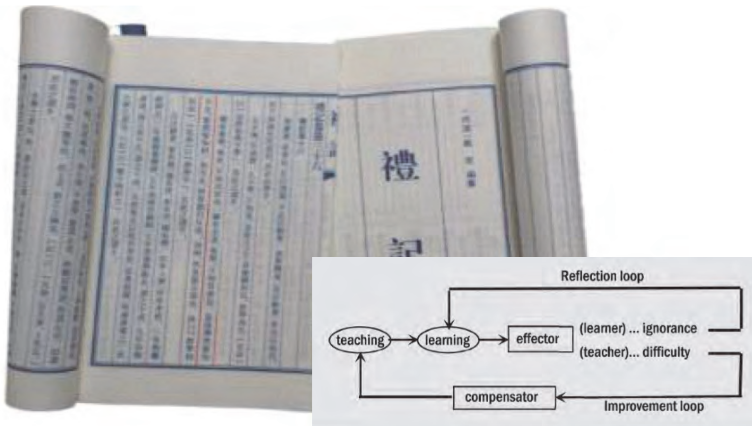


Fig. 5. The dual-channel feedback of "teaching and learning promote each other" in the *Book of Rites*

1.4. Assessment of mathematical thinking, leading to the deepening and innovation of teaching reform

In 1984, we used multi-media audio-visual technologies of "think aloud" to assess 50 pairs of students selected from control group and experimental group about their thinking process in solving a problem (Fig. 6). Individual case studies turned out that "experiencing variation" teaching was effective in improving the accuracy, agility and profundity of students' thinking, and this played a critical role in the innovation and extensive promotion at the second stage.

1.5. From experimental methods to the interpretation of teaching principles

As of 1992, the methodological characteristics of Qingpu practical educational research could be summarized as the follow:

(1) Scientific discovery models including agreement, discrepancy, covariation, deduction and induction were introduced so as to make the improvement of teaching behavior synchronize with critical thinking. As a result, the possibility and expectations of applying practical research in theoretical innovation was increased.

(2) Practice was decided as the screening method for Qingpu Experiment, by which the gap between investigation and experimental hypothesis was filled up. Taking this as the core, a comprehensive system with multiple methods complementing each

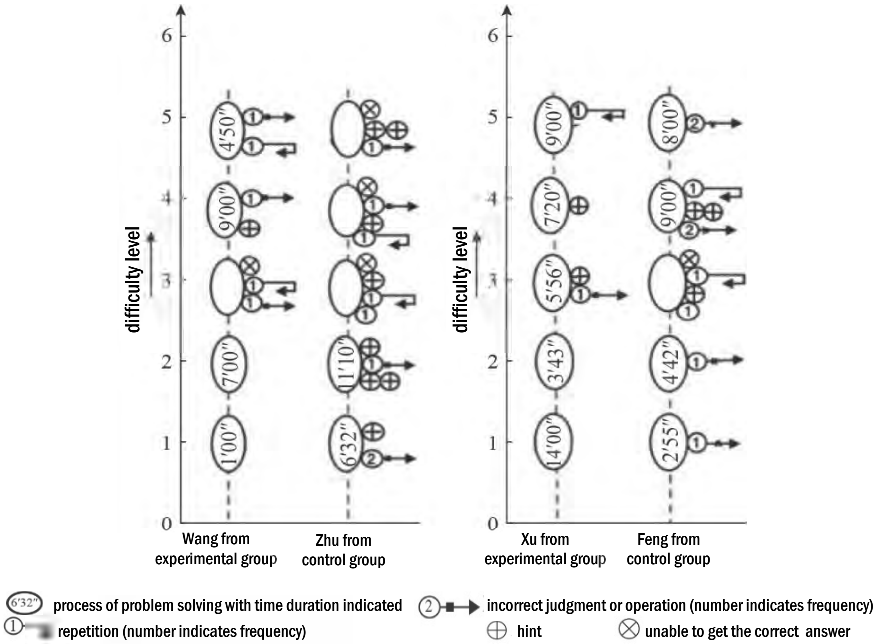


Fig. 6. The comparison of pair students on their problem-solving processes

other was constructed, increasing the ability of practical research in solving real educational problems.

(3) Make an in-depth study on the nature and form of the popularization of teaching experiences and explore the condition, form and effect of the popularization of research results in the hope of providing a reasonable basis for the application of teaching reform results in reality and guaranteeing the effectiveness of it as well.

(4) Establish a scientific research community involving researchers and teachers. Setting this community as the subject of the experiment could bring the research into the deep area of actual teaching process and increase the practical effectiveness and socialization degree of teaching reform experiments.

As a matter of fact, this is a comprehensive research based on teaching practice. It is noteworthy that Qingpu Experiment also provided interpretations for the key behaviors of teaching reform practice by having discussions, theoretically, on the process of mathematical cognition and activity, mathematical thinking and target classification, and gradually came out with a series of fundamental principles including Affection, progression, attempt, and feedback.

1.6. Win back quality

After ten years of efforts, the math score of the 9th graders in the entire Qingpu County had been increasing year by year (Fig. 7), and the pass rate went from 16% in 1979 (the lowest in the city) to 85% in 1986 (the average rate was 68%). In April, 1986,

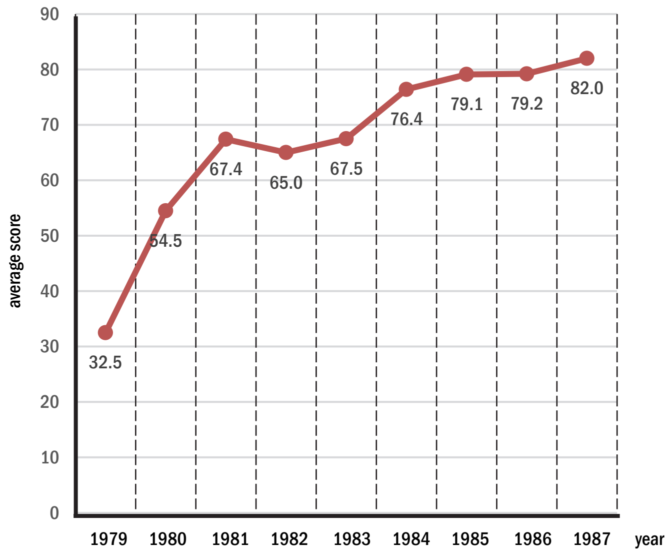


Fig. 7. Scores on the municipal tests of mathematical achievement for junior middle school graduates

Shanghai Education Committee convened a general meeting to popularize Qingpu experiences in the entire city of Shanghai. In October, 1990, the Department of Basic Education of the Ministry of Education dispatched a research group consisting of 18 experts to Qingpu and conducted a 9-day investigation on the experiences, achievements and practical effectiveness of the Experiment. In 1992, the Ministry of Education designated Qingpu experiences as a major achievement of basic education reform, and formally popularize it in the country.

Qingpu Experiment finally worked out a feasible way to improve teaching quality under the most common educational condition. The major research result of this stage are: *Learn to Teach* (1991) and *Theory of Teaching Experiments — A Study of the Methods and Pedagogical Principles of Qingpu Experiment* (1994).

2. Break through the Bottleneck of High Cognition

2.1. Attach importance to the assessment of students' math cognitive ability

The experiment at the second stage is carried out with two themes: first, the assessment of cognitive ability, and second, dig deep for outstanding experiences. With regard to the first theme, Qingpu Experiment had completed three tests and relevant studies on the math cognitive ability of local eighth graders since 1990s, conducted by professional researchers who designed the tests catering to different abilities respectively (last for 28 years with 840,000 standard statistical data).

The first test was conducted in 1990. In light of Bloom and Wilson's framework, cognitive ability objectives were classified into seven categories: computation, knowledge, comprehension, application, analysis, synthesis and evaluation, and thereafter a test with 50 test items involving 106 knowledge points was designed. The test was divided into three sections with a total of 220 minutes. Altogether 3200 8th graders took the test in paper-and-pencil format with 25 students in a group. Factor analysis technique was used for the analysis of the data. The main conclusions were:

(1) Memorizing and understanding were identified as the two most basic latent factors, and all the seven categories can be expressed through various loadings on the two factors. According to the data, the continuity and equidistance among the categories were not reasonable enough; computation and knowledge were mixed together, comprehension and application, analysis and synthesis could be merged and simplified (Fig. 8).

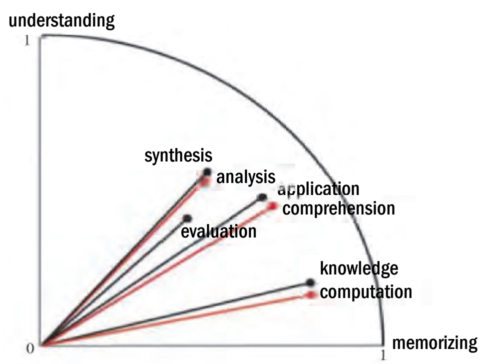


Fig. 8. Factor loadings on memorizing and understanding

(2) The distribution pattern of students' ability tendency was basically clear. Generally speaking, students had a good mastery of computation and concepts, but scored relatively low in comprehension and inquiry (Fig. 9). That is to say, in order to make students "smarter", we must get rid of rote learning and drill practice so as to break the bottleneck of crammed teaching.

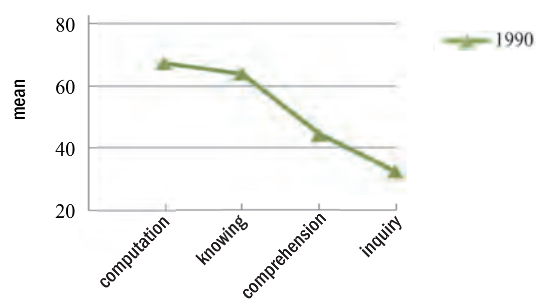


Fig. 9. Students scored relatively low in comprehension and inquiry understanding

2.2. *What if the scores are up while the students are not smart?*

The second theme was, in light of the major goal, digging deep for outstanding experiences. Having reviewed the two-way factorial design of “students attempt under the guidance of teachers + teachers provide timely feedback” (1982–1983), we primarily adopted “experiencing variation” for the practice of “students attempt under the guidance of teachers”. The follow is the result of the experiment (Fig. 10).

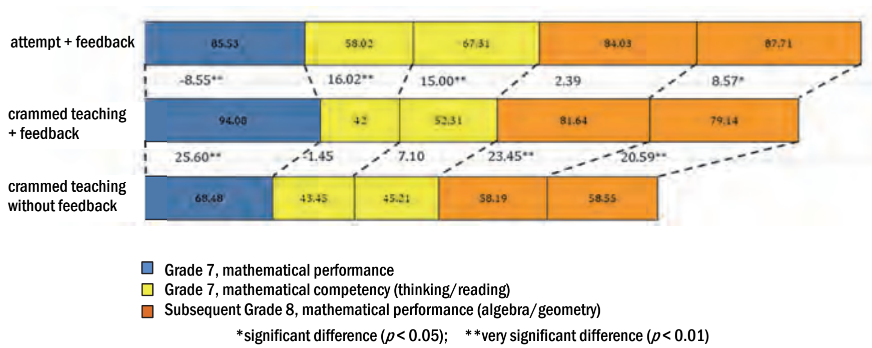


Fig. 10. Experimental results on factorial design by attempt and feedback

(1) “Mastery Learning” with timely feedback accounts for the substantial improvement of grades;

(2) “Crammed teaching + feedback” facilitates neither mathematical thinking nor reading ability and, furthermore, it even affects follow-up learning;

(3) “Attempt + feedback” could make students smarter. “It is more reasonable and helpful for students’ long-term learning than the so-called mastery learning,” Professor Fonian Liu, a former president of ECNU, said in 1985.

Therefore, “experiencing variation” became a hot topic for classroom teachers at that time. In a paper submitted to Shanghai Mathematical Society in 1981 (Fig. 11), it was clearly noted that variation has two types: conceptual variation and procedural variation, and the latter one is exactly an outstanding traditional experience in solving math problems.

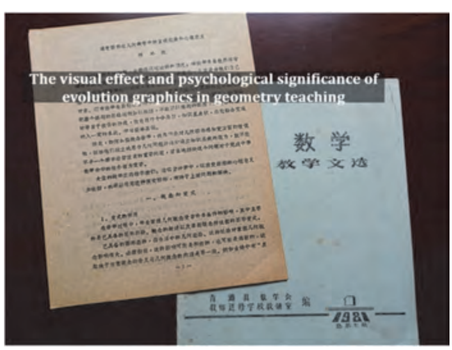
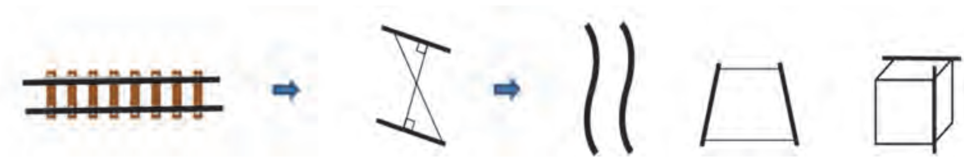


Fig. 11. Copy of a mimeograph paper on variation printed in February 1981

2.3. Deepen understanding in "attempt and experience" (mathematical conceptual variation)

For example, a simple concept – parallel lines. A teacher told the students that "parallel lines are straight lines that do not intersect with each other in a plane (abstract concept)". A student said, "I've memorized it". This is crammed teaching. Another teacher said, "Parallel lines are like two tracks of a train (descriptive definition)". A student asked, "Are they still parallel lines if the train makes a turn?"

Qingpu Experiment adopted another approach – experiencing variation, which includes three steps: step 1 is concrete and intuitive, they are parallel lines; step 2 is abstract and varied, they are also parallel lines; step 3 is plausible, they are not parallel lines (Fig. 12).



"is" (concrete, virsualized) "is also" (abstract, varied) "is not" (apparently "is" but actually "is not")

Fig. 12. Method of experiencing variation

Many experiments showed that: constructing mathematical concepts through the variation of material or form, from concrete to abstract, and via experiences of "is", "is also" and "is not", was able to i) reduce students' cognitive burden significantly; ii) deepen students' understanding of the key properties of mathematical concepts; and iii) improve independent identification ability in deceptive scenes.

2.4. Hold onto "core connection" while solving math problems (procedural variation)

Procedural variation involves meticulously designing "Pudian" （scaffolding） for mathematical problems to go from easy to difficult, or reducing problems from complex to simple, namely "simplification". Hence, students are guided to solve problems by themselves without mechanical practice.

In 2016, we reexamined the experimental materials of "Activity Process Analysis" and studied the connection patterns involved. For example, the application question of "a truck crossing a bridge" for 7th graders usually consists of two parts: i) When does the truck start to drive onto the bridge? ii) When does the truck start to move on the bridge? These are two critical points, which could be expressed through an "external" static line segment diagram and an "internal" one. If we make the truck move from left to right, three more questions could be asked: i) When is the truck not on the bridge yet? ii) When is the truck in the process of going onto the bridge? iii) When is the truck moving on the bridge? Holding onto the "core connection" of "moving line diagram", students could be successfully transferred to the discussion of the five kinds of relationships between two circles for 9th graders (Fig. 13).

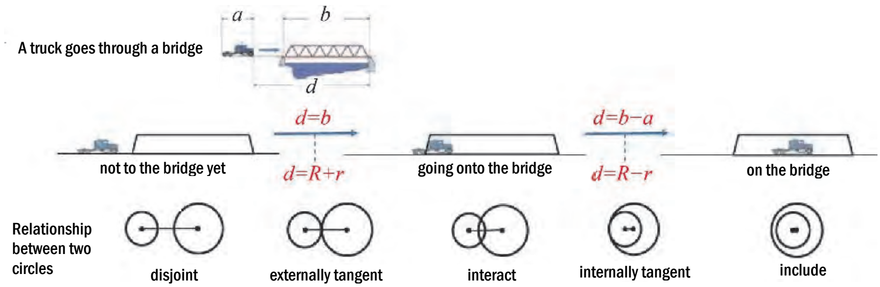


Fig. 13. Connection between truck going through the bridge and the relationship between two circles

The experiment data showed that when students applied what they had already known in solving new problems, it was important for them to find the most essential and transferrable element — the core connection. This connection could i) shorten the cognitive distance between the new problem and the anchoring point of existing knowledge; ii) significantly improve the degree of transfer in learning process; and iii) stimulate students' constructive thinking in solving math problems.

2.5. *Chinese experience of "experiencing variation"*

Teaching via "experiencing variation" focuses on students' conceptual understanding of mathematics and their constructive thinking in solving problems. Early experiments show that it can advance the transition of secondary students' mathematical thinking from visualization-based judgment to logical reasoning at least one year earlier (Fig. 14). After extensive practice, experiments and improvement, it has become a well-known Chinese experience.

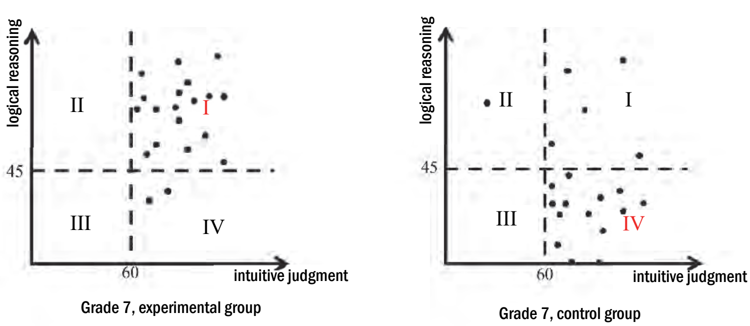


Fig. 14. Comparison of students' mathematical thinking

Variations can be applied in improving teaching via three dimensions: logical reasoning, situational application, and learning psychology. Two issues should be noted: first, that the more variations the better is not true. Rather, variations should be designed in accordance with the goals and needs of various student groups. Second, the key to variations is "experience". They are not crammed teaching in a disguised form. Instead, students should be given more opportunities to participate, attempt and express their ideas.

2.6. The change after more than ten years

The second test was conducted in 2007. With the improvement of classification of learning objectives (such as distinguishing knowledge from cognition, simplifying cognitive objectives, paying attention to creative and inquiry thinking), the test items were adjusted according to the reality of local students. The duration of the test remained unchanged, and 4349 students took the test. This time the accuracy of factor analysis improved significantly. The loadings of each cognitive objectives on the two main factors accounted for 85.15% of the total variance, which was 24% higher than that in 1990. The main conclusions are:

(1) A four-level structure, which was relatively concise and in line with the regional reality, was constructed, aiming to promote in-depth learning and teaching (Fig. 15): computation — rote memorizing; knowing — meaningful memorizing; comprehension — interpretive understanding; inquiry — discovery understanding.

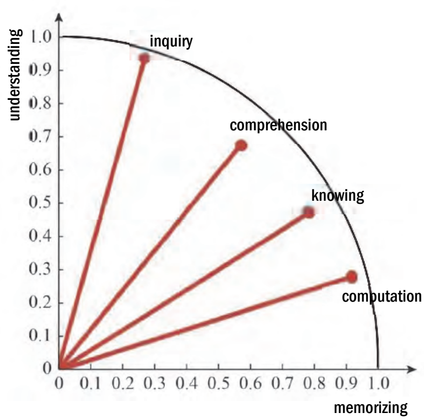


Fig. 15. A relatively simple four-level structure

(2) Compared with 1990, the scores of computation, knowing and comprehension in 2007 had greatly improved. However, the score of inquiry remained unchanged although much effort had been paid. The average score of inquiry was 28.96, which was slightly lower than 32.43 in 1990 (Fig. 16), and it became a new difficult point needed to be dealt with.

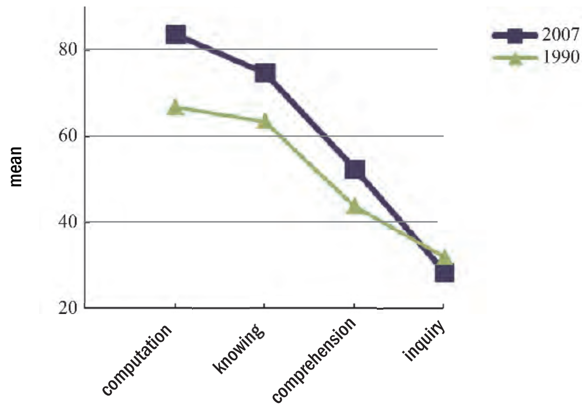


Fig. 16. The score of inquiry remaining unchanged

The major research results of this stage are: *Action and Interpretation of Teaching Reform* (2003); *A Witness of Reform — Lingyu Gu and Thirty-year Qingpu Teaching Experiment* (2008).

3. Promote Inquiry and Innovation

3.1. What if inquiry and innovation abilities remain unchanged?

Before entering the 3rd stage, we'd better take a look at the third test conducted in 2018. This test is exactly the same to the one conducted in 2007 in terms of test items and durations. A total of 3474 students took the test. The main conclusions were as follows.

(1) The average score of comprehension exceeded the passing level, and the score of inquiry was 11.31 percentage points higher. The difficulty was broken through to some extent (Fig. 17).

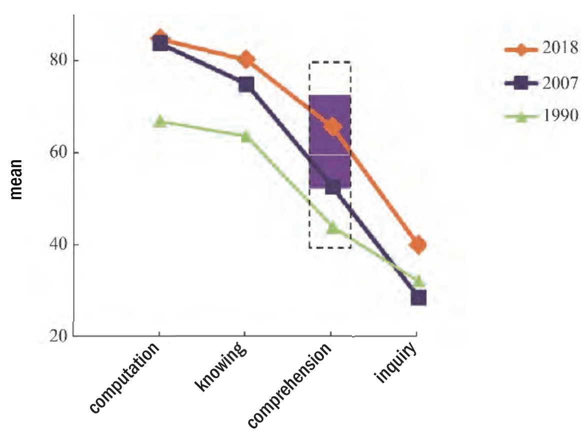


Fig. 17. Results of three tests

(2) The scatter plot showed that there was a positive correlation between comprehension and inquiry. With the improvement of students' comprehension ability, the inquiry level increased exponentially (Fig. 18).

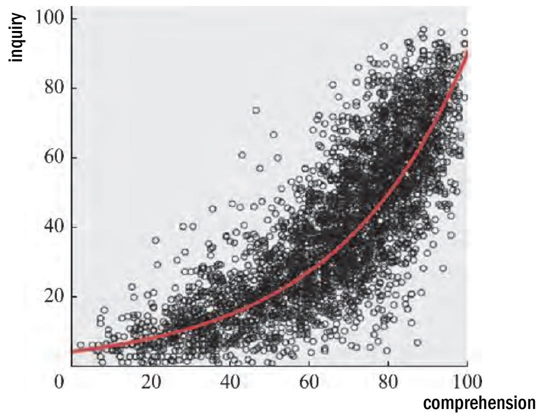


Fig. 18. A positive correlation between comprehension and inquiry

3.2. A turning point

As shown in Fig. 19, we can see the change of the correlation coefficients of comprehension and inquiry. The score of comprehension is between 45 to 95, and the graph is almost two broken lines. The turning point is in the middle. The increasing rate of correlation coefficients on both sides around the turning point shows a cliff style drop, decreasing sharply from 8.4% to 3.2%, indicating that the level of inquiry after the turning point is also affected by other factors besides comprehension.

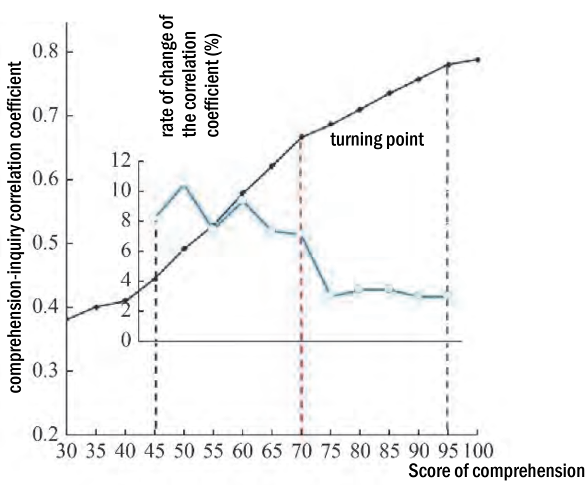


Fig. 19. Turning point of correlation coefficients appearing

Making a 2 by 2 segmentation (Tab. 1) of the average scores of comprehension and inquiry before and after the turning point, two types of students worthy of attention – type A and type B, were isolated for further analysis. The feature of type A students is that they stick to comprehension while type B students incline to inquiry. The data shows that both before and after the turning point, the number of students in groups of "double high", "double low" and "interaction" was about 1/3 whereas the ratio of type A and type B students within "Interaction" changed from 61:39 before the turning point to 46:54 after the point, indicating the significant influence of inquiry.

Tab. 1. 2 × 2 Classification of different student groups

	Low level of inquiry	**High level of inquiry**
High level of comprehension	Type A interaction: stick to comprehension	Double high
Low level of comprehension	Double low	Type B interaction: go toward inquiry

3.3. Cause-tracing empirical research based on big data

In teaching reform practice, in order to look for critical measures to improve students' inquiry ability, we designed an empirical research method named "cause-tracing". Specifically speaking, we selected typical sample groups available for comparison based on big data, and drew our conclusions through natural observations and evidential reasoning. Research flow chart is as shown in (Fig. 20). This method has the following major features: first, it demonstrates the theoretical penetration in the interpretation of phenomena and causal connections, good for the research group to exert its subjective initiative in applying background knowledge; second, it avoids subjective randomness in observations and sampling, and therefore guarantees the objective strictness in discovering logic. As a result, this method significantly increases the actual contribution of natural observations.

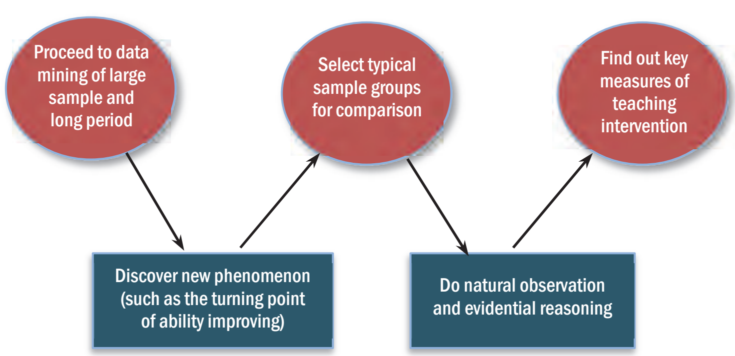


Fig. 20. Research flow chart of cause-tracing

In the research at this stage, we selected a total of 516 students consisting of both type A and B students before and after the turning point for further analysis.

(1) Elaborate data analysis reveals that the cognitive efficiency of type A & B students before and after the turning point is obviously different. They are not of much difference in knowledge understanding and regular application abilities, but type B students demonstrate overwhelming advantage in analysis, judgement, and discovery abilities (Fig. 21).

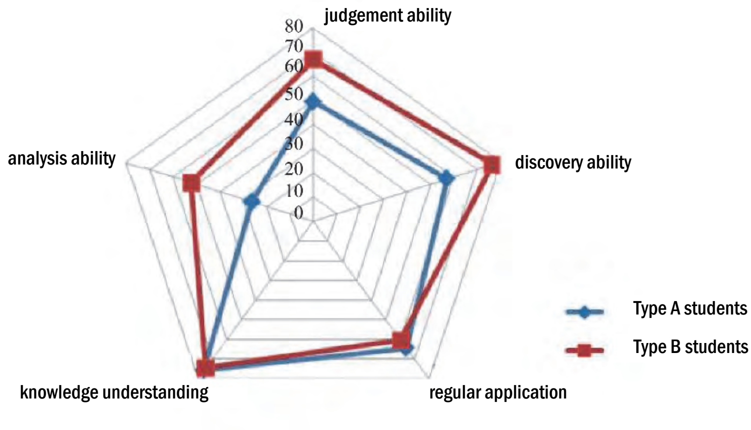


Fig. 21. Different patterns of cognitive ability of the two types of students

(2) Field research made in the schools where these students study reveals that type A students have a strong desire for high scores while type B students tend to spare energy in learning what they are interested in (Tab. 2).

Tab. 2. Characteristics of the two types of students

Type A (Have a strong pursuit for high scores)	Type B (Free up energy for independent study)
Practice common problems repeatedly; familiar with common problems and even reach the degree of “automation”; high accuracy	Dissatisfied with repeated practice; have a strong curiosity; good at asking questions and self-questioning
Good at grasping the requirement of examinations; know their own deficiency in their knowledge and skills, and try their best to make up for them; not interested in inquiry questions	Refuse to follow the majority and have their own opinions; like to solve problems by attempting; have failure experiences
Have a strong pursuit for high scores, sensitive to each single mark	Keep a grade above average; spare energy in solving problems they are interested in
Listen to teachers’ command; care for teachers’ praise	Not easy to get teachers’ attention because they do not obey rules

To conclude, there exist two different types of students, learning methods and results before and after the turning point. It is proved that excessive mechanical training is not advisable. Instead, we must look for key teaching behaviors aimed at improving inquiry ability.

3.4. In-depth clinical observations and comparisons of typical samples

Reasonable math teaching behaviors were not discovered until we went deep into the practical field in which math problems were solved. We wouldn't be able to find key teaching behaviors leading to the improvement of inquiry level until we conducted clinical analysis and comparisons of type A & B students' reaction in the tests and the teaching situations where they are in. The following are some typical teaching cases.

(1) Galileo space amphitheater design (2006)

Experimental schools, where most of type B students could be found, adhere to "activity – development" teaching scenario at a long-term basis. Fig. 22 is an assignment instruction at that time. The design of the space amphitheater is challenging: meet the technical standards and in the meanwhile, accommodate a maximum number of seats. The number of radiation lanes was decided by the designer. The less the lanes, the more the seats, but the number of seats for each row should not exceed thirty. The assignment is not difficult in terms of the knowledge required, but it is very challenging for thinking ability, and students have to deal with dilemmas before working out a solution. Obviously, this is a high-level assignment for students in training their innovative thinking ability.

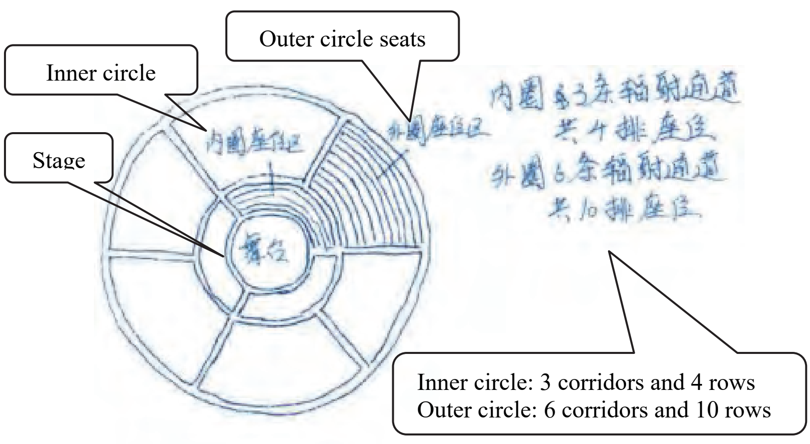


Fig. 22. Students' task: Galileo space amphitheater design

(2) Meticulous thinking ability training for unconventional travel questions (2018)

The follow is a question in inquiry section in one of the three ability tests: Ship A and Ship B set out at the same time from Island I, and shuttle between Island I and Island II. Ship A travels at 10 kilometers per hour and ship B at 8 kilometers per hour, and after 24 hours they return to Island I at the same time. Questions: 1) What is the distance between the two islands? 2) Have the two ships ever reached Island II at the same time?

Though Type A students were familiar with the formula $s = vt$ and the types of questions such as "meet" and "catch up", but they had only arrived an accuracy rate 13.8% when answering to Question 1) due to the demand of careful analysis on "round trip". For Type B students, they thought in this way: Within 24 hours, Ship A travels one more round than Ship B, so

$$s = \frac{24(v_A - v_B)}{2} = \frac{24 \times (10 - 8)}{2} = 24.$$

The accuracy hit 46.2%.

The interview after the test showed that Type B students would question the problem- solving process and further refine it. For example, they would ask, "why one more round for sure?" If we change v_B to 6 kilometers per hour, then $v_A/v_B = 10/6 = 5/3$, Ship A travels 5 rounds and Ship B 3 rounds. They return to Island I at the same time after 24 hours. As a result, ship A travels 2 more rounds than ship B. Generally speaking, the solution is:

$$s = \frac{12(v_A - v_B)}{a - b}, \text{ where } \frac{v_A}{v_B} = \frac{a}{b} \left(\text{in which } \frac{a}{b} \text{ is a reduced fraction}\right).$$

For Question 2), most students worked it out by list method. Some Type B students said, "if the two ships ever reach Island II at the same time, it must be 12 hours later, but ship B is not there at that time". This thinking is generated by counter evidence method, but the students can't explain it clearly. Under the guidance of the teacher, they came to the following result:

The time needed for ship A to reach Island II is $2.4m$ ($m = 1, 3, 5, 7, 9$);
The time needed for ship B to reach Island II is $3n$ ($n = 1, 3, 5, 7$).

Suppose the two ships ever reach Island II at the same time, then $2.4m = 3n$, that is, $4m = 5n$, both m and n are odd numbers. This is impossible. Therefore, the two ships have not reached Island II at the same time.

(3) Interdisciplinary learning and inquiry of the barycenter of a triangle (2018)

This is an in-class inquiry: the barycenter of a triangle — an interdisciplinary problem solving via mechanics and geometry. The mechanics knowledge applied here is primarily using suspension line method to find the barycenter via the balance of particles. The mechanics approach let students understand the consistence of mechanics barycenter with the geometrical conclusion that the three medians of a triangle meet at a point. Students can go further to discuss, via mechanics way, the issue that the three angular bisectors of a triangle or the three heights meet at a point, or more generally to discuss of Ceva Theorem. With closer essential connection between disciplines, there appear many creative methods for problem solving. For example, for an in-class exercise about the barycenter of an isosceles trapezoid, as shown in Fig. 23, $AD = 6$, $BC = 12$, and $EF = 9$, students came up with several solutions:

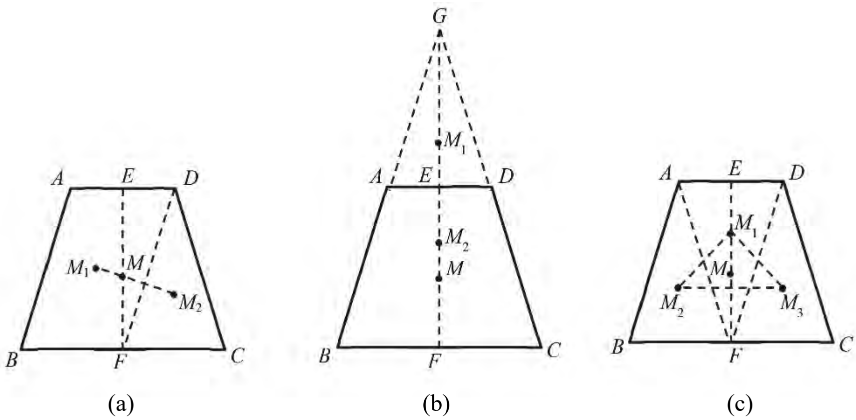


Fig. 23. Barycenter of an isosceles trapezoid

Method 1: intersection plotting

Connect D and F. Denote the barycenter of parallelogram $ABFD$ by M_1 and the barycenter of $\triangle DFC$ by M_2. Line segment M_1M_2 intersects the central axis EF at the barycenter M of the trapezoid, as shown in Fig. 23(a).

Method 2: mechanics equilibrium

Extend BA and CD respectively until they intersect at point G. Let M_1 be the barycenter of $\triangle GAD$ and M_2 the barycenter of $\triangle GBC$. Suppose the barycenter of trapezoid $ABCD$ is M. According to the balance of forces, we get equation $M_1M_2 : M_2M = 3 : 1$, then $M_2M = 2$. The location of M is worked out, as shown in Fig. 23(b).

Method 3: Equivalent reasoning

Divide the trapezoid into three congruent isosceles triangles, and their barycenters are M_1, M_2, and M_3 respectively. The barycenter of $\triangle M_1M_2M_3$ is the barycenter of the trapezoid, as shown in Fig. 23(c).

With regard to the various solutions figured out by the students, the teacher said with great emotion, “the multiple methods the students presented went way beyond my expectation!”

(4) The modeling, evaluation and revision of water hyacinth propagation (2021)

This is a real issue. An aquatic floating plant named water hyacinth grows in the water towns of Qingpu. It reproduces so quickly that it would cover a large area of water in short time and cause trouble to environment, water transportation, drinking water safety for human and livestock and fishery production. In order to discover the reproduction law of water hyacinth, the 8th graders used the knowledge they just learned, namely linear function and line chart, to create a math model of the reproduction of water hyacinth. As a result, they came up with an exponential curve:

$g = 25.01 \times 2.16^t$, where g stands for the quantity of the plant, 25.01 is the initial value, and t refers to the time at 10-day pace. After a comparison between the model data and the measured data, a big error was revealed in the 3rd month. Then they searched references and consulted professionals, trying to find the cause and make many revisions on the model. Surprisingly, some students figured out an error-trial method by themselves and minimized the sum of absolute values of the errors of each test point. This is equivalent to the thinking of so-called "least absolute deviation". With the preliminary experiences in math modeling and revising models, these middle school students acquired a significant foundation for their future study and research in math modeling.

3.5. Finding key teaching behaviors

A review of the backstepping process mentioned before: first, by refining the test data, we found the completely different cognitive abilities between type A students and type B students; through field research, we summarized the distinctive differences of the expressive characteristics between the two groups of students. Second, our research included both the case studies on the classroom teaching scenario in which type B students are in, and analysis and comparison of the two groups of students' reactions in the tests. This is clinical observations and studies under the guidance of data. In this way, we traced back gradually until we found the following key teaching behaviors that were effective for improving students' inquiry ability:

(1) High-level task-driven teaching design

- Organize challenging teaching tasks according to the internal hierarchical and sequential structure of knowledge;
- Pre-design individual targets according to different needs of students;
- Build comfortable and energetic learning environment and encourage learning with high-level concentration.

(2) An independent study process with fine processing of thinking

- With multiple strategies, activate previous experience and connect it with exploratory tasks;
- Through application, experiencing, or and knowledge assimilation, make fine processing of thinking;
- Make constant feedback and revision on reprocessing learning upon the evaluation of thinking effectiveness.

3.6. Teaching research follow up, attach importance to inquiry learning

Going from comprehension to inquiry, teaching research is indispensable. Based on the case studies of a number of master teachers from Jiangsu Province, Zhejiang Province and Shanghai, particularly Madam Yu Yi's experience of "Prepare one lesson three times, keep doing this for three years, and you're bound to be a good teacher", Qingpu Experiment proposed an in-service professional development mode called "Action Education" in 2004 (Fig. 24). It includes three focuses (focus on personal experience accumulation, focus on new ideas and experiences, and focus on the real gains of students), two reflection holders (looking for the gap between oneself and the others, looking for the gap between plan and reality), and one carrier — *Keli* (Exemplary Lesson Development). They make a cooperation platform on which all these elements repeat alternately, indicating the unique organizational culture and action route of teaching research in China.

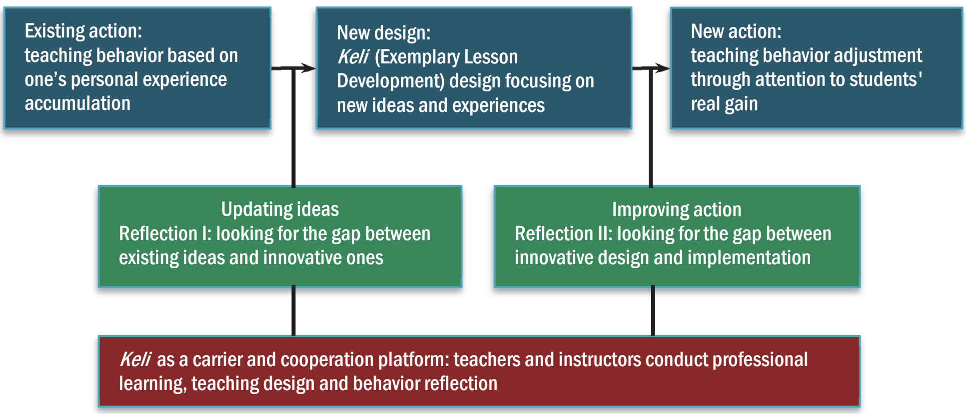


Fig. 24. The model of action research

Some scholars proposed knowledge sharing model of interpersonal learning and developed two skills: first, "expose oneself" (verbal guidance and analysis aimed at exposing real problems); second, "listen and respond" (reflective absorption and response skills). Due to the development of some high-level inquiry learning models such as discovery learning, practice learning and project learning, and so on over the years, teachers usually found themselves in an unknown area called "public issues area". Therefore, besides "knowledge sharing (knowing and not knowing)", the new concept of "create teaching behavior collaboratively (yes or no)" became particularly important. Qingpu Experiment summed up the following two key skills: first, "problem sorting" (sort out the problems about the inquiry target and create "public problems area"); second, "design and improvement" (design inquiry process and make repeated revisions based on evidences). In this way, we came up with two models of interpersonal learning (Fig. 25).

Tab. 3 is a lesson about "introduction of irrational numbers" in 2015. After three times circulation going from design to improvement, it eventually succeeded in making students learn math via "doing math". Students' learning style was successfully changed.

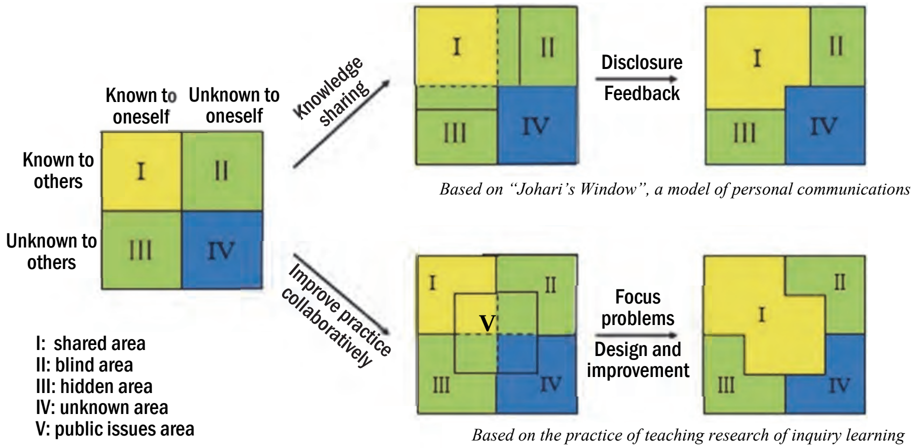


Fig. 25. Two models of interpersonal learning

Tab. 3. Lesson revision based on video analysis

Initial design	Issues during implementation	Revision
The first round — light humanistic purport		
Use historical stories about irrational numbers to arouse students' interest and experience scientific spirit.	Restricted to story plot with the mathematical discovery process left out. Have to return to the old way of passive explanation.	Stories settle in to clarify the cause of the extension of the concept of numbers. Use "$\sqrt{2} =$?" to kick off the replay of the discovery process.
The second round — attend to mathematical process		
Use decimals to approximate, and through deduction, identify the key attribute that irrational numbers cannot be represented as fractions.	Students understood the calculation and deduction. Nevertheless, they still asked: is $\sqrt{2} = 1.4142\ldots$ a definite number while it is growing all the time?	Look for the origin of the issue and choose an appropriate analogy: $\frac{10}{3} = 3.3333$. It grows all the time, but it is a definite number, implying the idea of limit.
The third round — change the learning mode		
To avoid mere deduction and explanation, design worksheets on $\frac{10}{3}$ and $\sqrt{2}$ respectively, requiring students to discuss while doing math.	Deepen the experience of "gradual approximation" in doing and discussing math. The degree of understanding varies from student to student.	The worksheets and discussion questions could be designed into a certain gradient, so as to cater to the need of different students.

"Teachers become practitioners with research ability" is the pursuit and expectation of Qingpu Experiment in educating outstanding teachers.

(1) At early stage, we required backbone teachers to sit in on the class and make observations. Not a single class should be missed. At that exact moment and inspired by what they saw, teachers thought about the advantages and disadvantages of the lesson and worked out methods for improvement. This strategy was very effective in

training teachers' teaching sensitivity. But this kind of observation was restricted by incompleteness of information and the limitation of memory.

(2) Since the end of 1990s, the introduction of information technologies, including video tapes at the beginning and CD and other video facilities later on, has witnessed breakthroughs one after another in our research. Multi-angle holographic recording made it possible to collect a great deal of data for analysis; playback, freeze and "microscopic" made it possible to study fleeting subtle behavior (including verbal expression, action and facial expression). Among these, the change of teachers' role — teachers became "both the actors and audience", was the centerpiece that provided a convenient way to educate a great quantity of good teachers.

(3) Nowadays, many teachers are becoming "cycle improvers" because the diversity of evidences has been greatly expanded by video analysis, the resource of data statistics is unprecedentedly abundant, and the materials for case analysis are much more refined. In the process of teaching research, these serve as a foundation for some people to make empirical interpretation, and for theorists to make rational deduction. Cycle improvement based on multiple evidences that prove each other pushes "Teachers become practitioners with research ability" to a higher step.

The major research results of this stage are: *Action Education – A Paradigm Innovation on Teachers' In-service Education* (2007); and *Oral Teaching Reform — Regional Experiments or Research Chronicle* (2014).

4. Concluding Remarks

Based on classroom experiments, our reform started from a low point in the hope of improving teaching quality, and gradually developed into a goal-oriented practice with different stages. We broke cramming via "experiencing variation" and promoted inquiry-based mathematics learning by designing "high-level tasks". Nevertheless, we're still on the way of reform given the big change in the current era, and therefore, negligence and mistakes are unavoidable. Criticism and suggestions are welcome.

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(English version translated by Honghong Li and Yingkang Wu)

https://doi.org/10.1142/9789811287152_0011

Plenary Lecture 3

Equity in Mathematics: What Does It Mean? What Might It Look Like?

Robyn Jorgensen[1]

ABSTRACT In this plenary lecture I draw on the findings and subsequent analysis of a large 5-year project that was conducted in some of the most disadvantaged contexts in the Australian educational landscape — remote and very remote Indigenous communities. The intent of the project was to develop an understanding of the successful practices adopted in these schools that were creating success for Indigenous learners. It was a strength-based project and intentionally moved from deficit models to one that sought to document what was working in these schools. It was not interventionist but rather drew on the knowledge of those working in the field, those who experienced the contexts, the learners, the communities and sought ways to build success. It was grounded, and ethnographic in its design. This plenary, and paper, shared the outcomes of the study. The implications of the learnings from this research have application to other equity contexts.

This plenary lecture was based on the findings from the "Success in Remote Indigenous Communities" project[2]. The project sought to investigate, document and celebrate the numeracy successes of schools working in remote and very remote Indigenous communities. These communities are seen as the most disadvantaged communities in the Australian educational landscape. The project focused on the practices of those schools. Using an ethnographic case study approach, project was not evaluative since a fundamental premise of such an approach is to document what, and how, practices are contributing to the success of Indigenous learners in these contexts. The project adopted a strength-based approach, so underpinning the ethos of the project was a celebration of the work being undertaken in selected remote schools. Such successes would documented and shared with others. A website, hosted by Stem Educational Research Centre at the University of Canberra, acted as repository for the case studies and could be freely accessed by any educator or other persons interested in learning of the success of these schools. It was intended that such case studies could support others working in similarly challenging contexts and provide practical ideas to support their work A final, summative report was developed to augment the case studies so that an overarching sense of the schools, the participants and the practices could be developed. This Plenary Lecture is a synopsis of that project.

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[2] This project was funded through the Australian Research Council scheme (DP130103585).

1. Equity in Mathematics Education

Foundational to this Plenary Lecture is the notion of equity as it applies to mathematics education. Many terms are used in this domain and many are rooted in particular paradigms and ideologies. When discussing equity, the domain from which I operate is critical sociology. This domain of thinking shapes the actions and patterns of thought of the researcher such that there is an assumption that in some ways schools are set up to reproduce social inequalities and the task of educators and researchers is to challenge the reproductionist agenda to bring about change. It is structuralist in its orientation so that practices that are adopted in mathematics education often reproduce social inequalities, often at a very subconscious level.

Often in mathematics education we see researchers and systems adopt practices of the past with the intent that they will bring about positive outcomes for learners. An example of such an approach is the adoption of Direct Instruction (DI) (Kinder and Carnine, 1991) in remote Indigenous contexts in Northern Australia. This is a model of pedagogy formed in the 1960s in the USA to work with disadvantaged learners. Since the 1960s there have been significant advances in research and pedagogy as to what constitutes quality learning and pedagogy yet the education systems in Australia have adopted an antiquated model of pedagogy for implementation in remote Indigenous schools. Some critical educators have been quite scathing of the rollout (and costs) of DI in remote Indigenous schools (Guenther and Osborne, 2020) while other conservative educators advocate the success of the program (Pearson, 2020). The difference between the two perspectives was the data upon which they drew to mount their cases. What is clear is that the expense of implementing DI (in excess of $20 million) in these communities has been very high while the outcomes have been limited and their sustainability has been questioned (Luke, 2014).

From an equity perspective, there is a need to do things differently as the practices of the past have reified social, cultural and linguistic differences so that such differences have been normalised and seen as part of the natural order, thus preserving the status quo. From a critical equity perspective, those who have been marginalised in their access and success in mathematics need to be treated differently so as to reduce the gap in outcomes. Treating learners the same as other learners will only help to reify social and educational differences. That is, not only in terms of mathematics scores, but also in terms of attitudes towards mathematics, continuation in the study of mathematics, application of mathematics into contexts beyond schools.

2. Method

The project was funded to undertake 32 case studies but for a number of reasons, the final project involved 39 schools. Some of the case studies, however, were not published. This was due to a range of issues including the lack of triangulation between the views of the leadership teams and the practices described or observed of the teachers, or lack of coherence in the data sets or the timelines not being met for approvals for inclusion of the case studies.

Schools were initially selected on their performance on National Assessment Plan for Literacy and Numeracy (NAPLAN). This is a national testing scheme for Years 3,5,7 and 9 across Australian schools in literacy and numeracy. Trends in school data for numeracy were used for the initial selection of schools so schools could be seen to be performing above similar schools; increasing in their performance over time suggesting that improvements were being made at the school; or through recommendation by the systems as schools that were performing well but this was not reflected on the test scores. At the outset, it is acknowledged that national testing scores are not a fair representation of success but they were the only 'objective' measure of success in numeracy. Hence, personal recommendation by systems of particular schools was a way to circumvent the narrow definition of success in NAPLAN results. Schools were required to justify their success on other measures including but not limited to other testing schemes or other measures.

The case studies were ethnographic in form and were developed through site visits to each school. Data were collected via interviews with members of the leadership team, teachers and local workers at the school; observations of classrooms; profiling of lessons, and collection of school artefacts. Collectively these were used to develop individual case studies for each site. A positive, strength-based report was generated in consultation with the school, and once approved by the leadership teams at the schools, were uploaded to a website for sharing (and celebrating) the successes of the schools. The Remote Numeracy website was hosted at the University of Canberra3 .

A meta-analysis across the schools was undertaken at the completion of data collection and coding. Trends across the data were undertaken through the application of a software package — NVivo — into which all interviews were coded and analysed using grounded theory. This enabled the identification of key trends across the data set. Two further analyses were conducted using Leximancer and a separate NVivo of the published case studies, to confirm the trends reported here were valid. Across the three analyses there was a very strong confirmation of the themes/coding. A further statistical analysis was undertaken of the classroom observations. This analysis was undertaken by a statistical expert outside the project to ensure validity of the claims being made.4 The data from the classroom observations is not included in this paper.

Schools Permissions were gained from State Government and Catholic sectors in Queensland, Western Australia (WA), New South Wales (NSW) and South Australia (SA). The Northern Territory (NT) Department of Education and Training (DET) denied access to government schools. There was only one Northern Territory Catholic school that met the criteria for inclusion but elected not to participate in the study. One NT Independent school approved access for the study. This is not to say that there is an absence of good practice in NT, as quite clearly there is some outstanding practice. Rather, it is a factor of the regulatory requirements to access schools through permission from the DET.

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[3] https://serc.edu.au/remote-numeracy/

[4] The views expressed in this report are those of the author and not the funding authority

The distribution of schools across the states varied and were contingent upon success as defined previously. There was no attempt to seek inclusion of schools in terms of ensuring what would be typically described as a representative sample of the schools. The schools were included solely on their performance so I was incongruous to pursue a selection of schools that proportionately represented the diversity of schools in this field. However, the sample of schools did represent the diversity of schools, states and systems as a collective.

Tab. 1. Distribution of school — published case studies

	Government	Catholic	Independent	**Total**
WA	11	3	7	**21**
QLD	4			**4**
SA	4			**4**
NSW	5			**5**
NT			1	**1**
Total	**24**	**3**	**8**	**35**

The schools include the range of schools that could be expected across Australia — including primary, secondary, schools to year 10, schools to Year 12, Vocational Education and Training (VET) schools, and boarding schools. There was considerably diversity in the structure of the schools. The schools also range in size from 'one-teacher' through to schools with 50 teaching staff. Some of the schools are located in the community, while others (such as boarding) are outside the immediate community. Some schools serve the community in which they are located, while others draw students from surrounding communities. Some schools are single campus, while others are multi-campus. Some schools are boarding schools including a senior vocational college, while most are day schools. One case study is based on a system-level approach so spans many schools within that system. The project's method has endeavoured to capture the diversity of schools operating across Australia. Two schools were visited and data collected, but at the completion of the site visit there was no coherent story to be written, so no case study was developed. A further two schools were completed towards the end of the project but the principals relocated and hence the stories could not be confirmed/approved. In total, four sites were visited without case studies being published from those sites. Accordingly, of the 39 sites visited, 35 case studies were published.

3. Analysis

Two levels of analysis were undertaken in this study. At the first level was the ethnographic case study approach. Each school has had a case study produced. These draw on the key themes of the school following the site visits. The case studies were negotiated with the school so that the stories presented in the case study were validated by the school and were seen to be a fair representation of the school. The case studies were published on the project website. The second level of analysis was undertaken and will be ongoing given the enormity and breadth of the data. All interview data were entered into a qualitative database (NVivo) and coded using a grounded theory approach. This enabled trends across the data sets to be identified. The data presented in this plenary provided a summary of the key findings of the project.

A quantitative analysis was undertaken on the pedagogical profiling of the lessons and the complete dataset in NVivo which is not part of this plenary. A further macro analysis of the data sets was undertaken with Leximancer and a NVivo analysis of the case studies. These analyses were used to verify the macro trends in the data sets and to triangulate the subjective coding used in NVivo with the mechanistic (objective) coding made through Leximancer. There was strong congruence in the three systems of coding.

4. Key Findings

Unsurprisingly, there is no unifying approach across the states, or schools. However, there are some features that appear in many cases that are noteworthy. While there are examples of practices that would appear to be diametrically opposed such as problem based/investigative group work with the highly structured worksheets of 'direct instruction', there is a unifying philosophy behind the teachers' intent with the adoption of these practices. First is that they sought to identify the entry level of the students (through assessment for learning practices) and then to develop targeted strategies to meet the needs of the individual students (differentiation). Much can be said of the practices observed in schools, and these are contained in the case study reports. As the project data set was very large, some model was needed to make sense of the data across the project.

4.1. Masking Sense of Surprises in the Data

Initially the project was developed with the intent that there would be reporting on teacher practices, however as the project unfolded, there were times when teacher practice was not the key practice of the schools. Many issues beyond the practices of the teachers became critical in rethinking the initial assumptions that underpinned the project. IN the next two sections, I will share two examples (that were among many) that resulted in rethinking the macro analysis of the project.

4.1.1. Attendance

Attendance is a key issue in education, and more so in remote Indigenous education where attendance rates are notoriously problematic for teachers and systems. There are often good reasons for non-attendance from the perspectives of the families and learners.

> *Principal: I am not sure why you are here. We have not focused on maths at all so I don't know how we can help you.*

At this school, there had been a very strong focus on student welfare. Absentism was high. The principal had created a position of student welfare officer whose role was to follow up with students and why they did not come to school. Issues such as no food in the home, or no shoes for the learners so they could not come to school were often found to be the cause for non-attendance. The school then developed strategies — such a breakfast club, free lunches, a clothes stores — so that the issues being confronted by the families and learners were addressed and students could come to school without shame. Increases in attendance converted to increases in performance.

4.1.2. Attitudes Towards Schools

> *Principal: When we arrived at the school, the students (and families) would stand on the other side of the school fence and taunt us. There was clearly a strong divide between the school and community so we had to do something to redress this issue.*

At this school, the whole of school worked on developing a culture at the school where the learners were valued and welcomed to the school. Over time, more children and families started coming to school, feeling that they were valued and important. Teachers and administration staff would welcome children when they came through the gate and farewelled them when they left. Any child who had problems during the day, were especially encouraged to return the following day. The school staff conducted event outside the school gates and in the broader community, bringing the school into the community. Again, attendance translated to success.

These two examples were among many where schools had not primarily focused on teaching mathematics, but had used other strategies to bring about changes that impacted indirectly on performance in mathematics. To this end, the project needed to develop a model that accounted for these learnings.

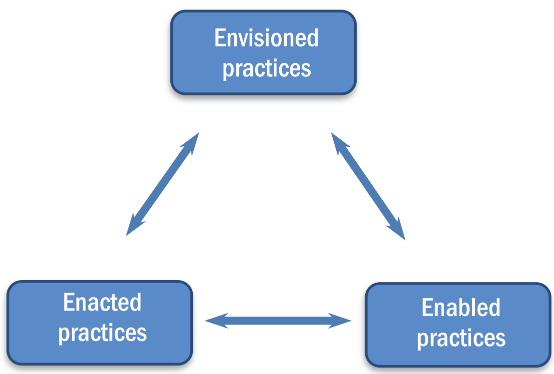
4.2. Identifying Norms

Rather than focusing on describing practices per se, the project has identified the norms that appear to underpin the practices. To make sense of the multiple levels of practice observed across the study, three levels were developed — envisioned, enabled and enacted. Schools need to have a strong and well-articulated vision. They then put practices in place to enable the vision to be enacted by the staff at the school. Different schools had different emphases in their case studies. Each of these levels of analysis and examples are provided in this report. While this is represented in a nested manner, it is the case that each of the levels of practices interact with the other, thus suggesting a much more dynamic model.

5. The Three Levels of Practice

As a result of the macro analysis and to make sense of the 'surprises' in the data, a three level model was developed. The levels are not meant to be hierarchical but rather cyclical and dynamic with each informing the other.

- ***Envisioned Practice*** — this level referred to those practices that created a vision for the school and how the leadership team worked with the school and wider communities to share and enact the vision of the school
- ***Enabled Practices*** — these were an intermediary practice that worked between the enactment of the vision and the work at the level of classroom. Two standout practices at this level were the inclusion of a numeracy (middle) leader who would support the work of the teachers to ensure that they were able to enact the vision of the school. The second type of practice was the employment (and upskilling) of local Indigenous people who could work with teachers in the area of numeracy but also around issues of culture and language as they impacted on learning.
- ***Enacted Practices*** — these were the actions of the teachers and support staff in the classroom level.



5.1. Envisioned Practices

Many of the schools in the study were very clear about the culture of the school that they sought to develop (or had developed and sought to maintain and sustain beyond the principal's time at the school). Features of these included:

• Articulating and leading the rollout of a school-wide approach to the desired culture and vision for the school.

• A supportive leadership team to work with staff to enable the effective management of the school culture — both in terms of the culture of the school, and the mathematics learning culture.

• Working relationships with community to share the visions of both the community and the school.

• Being prepared to evolve a positive culture over an extended period of time, and to ensure that the culture is embedded so that it endures changes in staff. Change needs to be slow if it is to be effective. Communities and families are often change-weary.

They were wary of new leaders coming in to make their personal mark for personal gain, rather than for the gains of students and community.

- Sharing vision and working with staff and community was an important factor for success.
- Middle leadership was a strong theme emerging from the school data — this level of leadership mediates the vision of the school and supports teachers to enact the vision of the school. Being able to see the value of middle leadership was a vision in many of the schools.

5.2. *Enabled Practices*

To ensure staff and students meet the goals of the school and thrive in the classrooms, schools employed a wide range of practices to enable teachers and support staff to enact the vision of the schools. These practices sought to implement the vision of the school and to ensure that teachers were given quality opportunities to develop as teachers while aligning with the values and approaches of the school. Some enabling practices observed included:

- Employment of quality local (Indigenous) staff to work alongside teachers. Investment of time and resources were evident. Local people took a strong role in the classroom and were an invaluable resource within the school. In some schools, their title was a co-teacher to signify their status within the class and wider community. Quality learning opportunities were provided to upskill the local people so that they felt strong and comfortable in these roles.
- Quality professional learning for teachers — most of the schools were staffed by graduate teachers who were often in their first remote position. Considerable support was made available to induct these teachers into remote education, and to provide ongoing support in their development as teachers of numeracy/mathematics.
- Numeracy Coaches were employed at some of the schools. These individual roles varied depending on the context and needs of the teachers, particularly in relation to funding the role. The role included sharing the vision of the school and supporting teachers to enact the vision of the leadership team. It also included providing in-class support for teachers, from planning lessons to providing feedback (middle leadership). Many terms were used to describe this role depending on the school.

5.3. *Enacted Practices*

At the level of the classroom, there was an extensive range of quality practices that were articulated and observed. While this was the original intent of the project, it was clear that for this role to be successful, the other levels of practice also needed to be considered. Enacted practices included:

- Being explicit about the intent of learning, how lessons were organised and what was expected of the students. There was no "hidden" agenda of lessons. Students knew teacher expectations.

- Differentiating learning to enable identification of students' learning needs through assessment for learning practices and then to build quality learning experiences that met and extended the needs of each learner.
- Recognising language as a key variable in learning, providing appropriate scaffolding in language (both home and SAE) to build bridges between the home and school, and provide entry into school mathematics.
- High expectations — of both students and staff — across social and mathematical norms. Students were provided with age-appropriate learning outcomes (e.g. algebra for secondary students) and then quality teaching practices to scaffold learners to achieve those learning intents. There was no space for deficit approaches to pedagogy and learning.
- Focus on mathematics — mathematics was a priority for learning. The mathematics that was being taught was age-appropriate so that students were being exposed to levels of mathematics that could be expected in regional settings. It became the task of the teachers to provide appropriate scaffolding for students to enable them to reach these levels of learning. High mathematical expectations were reinforced.
- Culturally responsive pedagogy was evident. Many strategies were developed to cater for culture of the students. Most obvious were strategies used to build language (of mathematics and the home language as well) and to have strategies that were cognisant of issues of "shame" within the classroom (Robyn. Jorgensen (Zevenbergen), 2019). There has only been one class to date that incorporated the more overt aspects of culture (e.g. art) but other teachers had sought to draw on the everyday activities that the students undertake (e.g. fishing, trips to town).
- Creating a sense of numeracy for life. Most communities had limited numeracy practices synonymous with urban living. Teachers have developed many strategies to create opportunities for students to see the purpose of mathematics/numeracy in their lives.
- Pacing of lessons, or parts of lessons, was often quick so as to engage learners, and prepare them emotionally as well as mathematically for the mathematics lessons. Using a quick pace engaged the learners. Humour was often part of the lesson as well, again to engage learners in a non-threatening manner.

6. Norms within the Levels of Practice

While categorising practices into these three levels provided a way of clustering the practices, a further analysis of the data produced a theorisation of these practices around the notion of norms what were aligned with the three levels. These norms provide an overarching model that describes the principles that underpin each of the levels of practice. The model gives coherence to a collection of practices at each level. Having norms to underpin and guide practice provides principles for success.

In Tab. 2 below, I have intentionally reversed the order of the levels to reinforce the non-hierarchical model of these practices. The norms provide principles to underpin the practices at each of the levels of practice.

Tab. 2. Summary of the norms identified from the project

Enacted Mathematical Norms	Enabled Mathematical Norms	Envisioned Mathematical Norms
• All students can learn mathematics — to high levels • Embedding mathematics is critical for understanding — embedding in the brain as well as embedding in contexts • Mathematics is as much about language as it is mathematical concepts • Transparency in learning and teaching mathematics enables students to access the "secret knowledge" of school mathematics • Mathematics lessons should engage learners at their levels of understanding, and then extend learning into new levels	• Teacher quality is essential for quality learning • Recruitment, development, retention of staff • Teacher support is integral to developing quality mathematical environments • A key person for mathematical support across the school enables quality teaching and environments — numeracy coach • Indigenous people are a key resource in teaching and the classroom	• Leadership is critical for developing a positive mathematics culture, supporting teachers and supporting community • Establishing a whole school approach to teaching mathematics ensures consistency and transparency — for students, teachers, and community

There is considerable literature in mathematics education on the notion of norms as they pertain to the mathematics classroom. Most notably the work of Yackel and colleagues (2000) draws on the notion of sociomathematical norms that relate to the normative understandings of the mathematical realities in classrooms. While these norms relate to the mathematical activities that are undertaken in a classroom, they are different from the mathematical content. The sociomathematical norms in a classroom provide the framework for the interactions through which mathematical meanings can be negotiated by the learners and teachers. Norms act as a means for mediating learning. Many terms are used to describe the practices that come to constitute norms in the classroom, including discourses, discursive practices, practices, and/or culture. For this project, the term "norms" has been chosen as it is well established in the mainstream mathematics education literature and dates back to the very early work of Cobb and colleagues (Yackel and Cobb, 1996) which still endures in the current context (Campbell and Yeo, 2021; Hofmann and Ruthven, 2018).

Norms can be seen as principles that underpin the practices. They provide the implicit and explicit guidelines through which the practices will evolve, and reflect what is valued within the classroom, school and/or community. They shift thinking from deficit models to strength-based models of practice. Under each of the norms, there are many sub principles and practices that need to be developed in order to embed and enact that particular norm. For example, if a school were to opt for a program that spanned across the whole school, as it was recognised that this would not only help students and community to see a common approach across the school, but there was also a need for a commitment from the staff to develop strategies that would enable this to happen. This would include staff development so that they would be familiar with the program. There may need to be consideration as to how best enable staff to undertake that staff development given that the schools are often unable to release teachers during teaching time due to the issues around isolation and inability to access to teacher relief. Considerations would also need to be made of how to ensure that the program is sustainable in the immediate context, but also how it may continue after the current staff leave the school. As such, each norm reflects various strategies (or practices) that have been shared through this research project.

7. Conclusion

While this study was conducted in some of the most disadvantaged communities in Australia, the findings have wider application. In concluding, I would like to make two key points. The first is with regard to the three levels of practice and the norms associated with those levels. The three levels of practice that arose from the study suggest that as a community, mathematics education research needs to consider the interaction of these levels. It may not be sufficient to have an interventionist program where it is not sustained by other levels of practice. While a program aimed at improving mathematics teaching and/or learning may have merit, it becomes necessary to consider the other levels of practice if success is to be enduring and long term. It became clear throughout the conduct of this research that each of the individual levels may impact on mathematics learning in and of itself, but a richer, more holistic way of working in the research domain may be at the intersections and inclusion of these levels. As argued elsewhere (Jorgensen (Zevenbergen) and Lowrie, 2015), there has been little achieved in the systemic redressing of inequities in outcomes in mathematics education. Perhaps, we have been barking up the wrong tree and may need to consider expanding research projects to include multiple levels of practice if real and deep change is to be affected.

The second point that I would like to conclude with is the focus of this study and its expansion into other equity target areas. While this study was focused on, arguably, the most disadvantaged learners in the Australian educational landscape, it is possible that the learnings from this study can be applied to other equity target areas. The levels of practices and the norms within those three levels may offer insights to support other equity target areas. This study should not be considered as relevant to only remote and very remote Indigenous learners.

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Plenary Lecture 4

Mathematical Work of Teaching in Multilingual Context

Mercy Kazima[1]

ABSTRACT It is well acknowledged that teaching mathematics in multilingual classrooms where the language of teaching and learning is not the students' home language presents challenges. It is also well acknowledged that there are various knowledge demands on teaching mathematics, and that teachers face different tasks that constitute mathematical work of teaching. In this paper I explore the link between the two by conceptualising the mathematical work of teaching in multilingual contexts. I do this by drawing on lessons from studies on mathematical knowledge for teaching and mathematical work of teaching, from studies on teaching mathematics in multilingual contexts, and from my work in Malawi. My conceptualisation yields four categories, and I illustrate these using some examples from the context of Malawi.

Keywords: Multilingual contexts; mathematical work of teaching; Malawi.

1. Introduction

In this paper I discuss the mathematical work of teaching in multilingual contexts. "Mathematical work of teaching" are the tasks that teachers are regularly faced with as they teach mathematics (Ball, Thames, and Phelps, 2008). In the discussion I make reference to mathematical knowledge for teaching, and by this I use the definition by Ball et al. (2008, p. 395) to mean the "the mathematical knowledge needed to carry out the work of teaching mathematics". I will use the terms 'multilingual classroom' and 'multilingual context' to refer to all classrooms and contexts that have one or more languages besides the language of teaching and learning. While I acknowledge the differences between bilingual and multilingual that have been highlighted by many researchers, the differences are not crucial for this paper.

Discussing mathematical work of teaching in multilingual context is complex because it involves two large fields of research in mathematics education: (i) teaching mathematics in multilingual contexts and (ii) mathematical work of teaching and mathematical knowledge for teaching. Language and the teaching and learning of mathematics has been researched in different parts of the world. The emphasis and focus of the research has varied depending on the contexts and issues relevant to the situations at hand. Across the various contexts and research studies, the issues appear to be similar in general ways but also different in specific ways. For example, one

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general similarity is that the various studies uncover some challenges in teaching and learning mathematics that come about because of language, while the specific nature of the challenges vary across the contexts.

When we look at mathematical work of teaching and Mathematical Knowledge for Teaching (MKT), research studies in this field generally agree that teaching mathematics requires different types of knowledge in addition to the subject matter knowledge. There are also some variations in this field in the way researchers conceptualise and name the various types of teacher knowledge. In comparison to the research on language and mathematics, in the research on MKT, there seem to be more similarities than differences. This could be because the focus is on mathematics and the mathematical knowledge demands on teaching, which can be explored in a similar manner across various contexts despite the different languages or language contexts. However, the demands on teaching might not be similar.

My work in mathematics education over the years has been mainly in these two fields and my research has mostly been in Malawi, my home country, and occasionally in other parts of south and east Africa through collaborations with other researchers. My experience, therefore, is that of multilingual classrooms where the language of teaching and learning is English, but not the home language of the students and the teachers. Malawi uses English as the language of teaching and learning from the fifth year of primary school onwards. In the first four years of primary school, local languages are used, however, textbooks for these first four years are in Chichewa only, the national language. Chichewa is also taught as a subject from the first year and throughout primary and secondary schools. Therefore the case of Malawi is often bilingual for students and teachers that have Chichewa as their home language, or trilingual (Phakeng et al., 2018) for those that have other home languages. While the classrooms as a whole can have multiple languages, thus multilingual.

My recent work in Malawi in collaboration with colleagues from University of Stavanger aims at improving the quality of mathematics education in schools through professional development of teachers and teacher educators. What has been evident in all my work and through my post graduate students' studies is that language is always an issue in such contexts even if we attempt to focus on different areas of research and not foreground language. The same is likely to be the case for others in the different multilingual contexts. Therefore it is important to discuss the work of teaching mathematics in multilingual contexts.

Researchers that have studies mathematical work of teaching include Ball and her colleagues at University of Michigan, and Adler and her colleagues at University of the Witwatersrand. Studying the mathematical work of teaching is important because "strengthening our understanding of the mathematical work of teaching … is a critical dimension of enhancing its teaching and learning" (Adler, 2010: p. 122). Ball et al. (2008) suggested 16 tasks that teachers are faced with when teaching mathematics, and Adler (2010) elaborated 4 tasks of teaching mathematics. All these tasks describe the mathematical work of teaching that is universal and apply to various contexts around the world. While these universal tasks strengthen our understanding of the mathematical work of teaching, they do not describe fully the work of teaching in specific contexts such as multilingual classrooms. It is therefore important to study the specific contexts and also strengthen our understanding of the work of teaching in such contexts.

I build from previous research to conceptualise the mathematical work of teaching in multilingual contexts. The main question I am exploring is: what is the mathematical work of teaching in multilingual contexts? I answer this question through the following three guiding questions and I relate to the context of Malawi and my work over the years.

(1) What lessons can we draw from studies on teaching and learning mathematics in multilingual contexts?

(2) What lessons can we draw from studies on mathematical work of teaching and mathematical knowledge for teaching?

(3) How can these inform conceptualisation of mathematical work of teaching in multilingual context?

My discussion is presented in four sections. First, I discuss the lessons we learn from studies on teaching in multilingual classrooms. Second, I discuss lessons from studies on mathematical knowledge for teaching. Third, I relate the discussion to the context of Malawi, the research studies in Malawi and lessons we learn. Finally, I discuss how the lessons can inform conceptualisation of mathematical tasks of teaching in multilingual classrooms.

2. Lessons from Studies on Language and Mathematics Teaching

Research on language and mathematics teaching can be grouped in a number of categories. I consider the three main categories as (a) those that focus on the mathematical language and terminology, which concerns all students even first language speakers of the language of instruction; (b) those that focus on students from minority language groups such as immigrant families into a country or community where the language of instruction is not their home language, for example Spanish language immigrants in the United States; (c) those that focus on students in countries where the language of instruction is a foreign language and not the students home language, for example previously colonised countries that use the colonial language. The third category is the case for many African countries.

I acknowledge the work by many other scholars who have focused on language and communication in mathematics and those that have discussed theoretical resources that can be used to study and understand use of language in mathematics (e.g. Sfard, 2008; Barton, 2008; Barwell, 2007). Morgan, Craig, Schutte and Wagner (2014) highlight the importance of theorising when studying language and communication in mathematics education. They point out that while there are many studies that focus on language use and how it contributes to students' learning of specific mathematics concepts, "little attention is given to the more general issue of the acquisition of mathematical ways of speaking or writing that may be applicable and acceptable in a wide range of areas of mathematics" (p. 852). Morgan et al. (2014) identify three areas of concern in this regard, one of which is "what knowledge and skills might teachers need to use in order to support the development of students' linguistic mathematical competence". This is an important concern which is relevant to this paper, and relates to questions raised by other researchers discussed later in the paper.

2.1. Lessons on mathematical language and terminology

As early as the work of Pimm (1987) and Orton (1992), we learn that some mathematical vocabulary pose problems for students, even for first language speakers. For example, words that have everyday meanings that are different from the mathematical meanings. This is difficult for all students to learn the precise mathematical meanings and use appropriately. Research and studies in this category inform us of the need to understand the complexity of mathematical vocabulary, to pay attention to words that might be difficult for students, and to plan carefully how to teach the meanings and use of such words in mathematics lessons.

2.2. Lessons from studies on classrooms with minority language groups

Examples of this work are that of Moschkovich in the US, Planas in Catalonia, and Barwell in the UK and Canada. The lessons we learn from these include that students from minority languages who are not fluent in the language of instruction need special attention to make the mathematics accessible to them. Without deliberate attempts to include these students in mathematics lessons, they will not access the mathematics and thus we will not achieve equity in mathematics education. Planas (e.g. Planas, 2019; Planas and Setati-Phakeng, 2014; Planas and Civil, 2013) and Moschkovich (e.g. Moschkovich, 2012; 2018) also caution against perceiving learners as deficit, and they suggest instead to view the learners' home languages as resources.

2.3. Lessons from studies on classrooms with foreign language as the language of teaching and learning

The context of teaching and learning in a foreign language that is not the students' home language is common in most of Sub-Sahara Africa. The foreign language is usually the colonial language, which after independence has continued to be used as official and school language. For example, English in Malawi, French in Cameroon and Portuguese in Mozambique. My review of research in this category is limited to English as a foreign language. Most of the research in sub-Sahara Africa on teaching and learning in multilingual contexts has been done in South Africa (e.g. Adler, 2001; Setati and Adler, 2000; Setati, 2008; Setati, Molefe, and Langa, 2008; Essien, 2010; Webb and Webb, 2008). There are at least seven general and related lessons that we learn from the studies:

(1) Students and parents prefer use of English as the language of teaching and learning because English is the language of power and brings "social goods" (Setati, 2008).

(2) Use of English or home language in teaching and learning should not be taken as a dichotomy where use of one completely excludes use of the other (Setati, 2008).

(3) Students' home languages should be seen as resources and not problems (Setati, 2008).

(4) Code switching can be used effectively where teachers and students share a common language (Webb and Webb, 2008).

(5) Code switching presents teachers with challenges and dilemmas (Setati and Adler, 2000; Adler, 2001; Setati, 2001)

(6) Bilingual approach where use of home language alongside English is planned and used proactively can be an effective way of teaching mathematics (Setati et al., 2008)

(7) Teacher education does not prepare teachers adequately for teaching in multilingual classrooms (Chitera, 2009; Essien, 2010)

(8) Setati, Chitera and Essien (2009) reviewed research of mathematics education in multilingual classrooms in South Africa, and raised three important questions, one of which is: "What do all teachers need to know, and what skills do they need to develop in order to be able to teach mathematics effectively in multilingual classrooms?" (p. 76). This is an important question that I will address in this paper, but with a focus on Malawi context.

3. Lessons from Studies on Mathematical Knowledge for Teaching

Since Shulman's seminal papers (Shulman, 1986, 1987) on teacher knowledge and his conceptualisation of subject matter knowledge (SMK) and pedagogical content knowledge (PCK), many researchers have based their studies on Shulman's work and elaborated on the categories of knowledge for teaching mathematics (e.g. Ball et al., 2008; Rowland, Huckstep, and Thwaites, 2005). I will focus on the work of Ball et al. (2008) that suggest six domains of teacher knowledge — three under SMK and three under PCK, which they illustrate in a figure shown in Fig. 1.

One of the lessons we learn from this work is that there are different types of knowledge that are needed for teaching mathematics. We also learn from their further work that these can be measured through carefully designed items. Review of MKT research and publications (Hoover, Mosvold, Ball, and Lai, 2016; p.9) reveals that although there have been many studies on the nature and composition of knowledge for teaching mathematics, it is difficult to draw lessons because most of these studies "do not build on each other in obvious ways and clear lessons are hard to identify". Hoover et al. (2016; p. 9) acknowledge that "the one avenue of work that represents progress for the field is the development of instruments." According to the review, there are many studies that have teacher education as a priority; these include studies

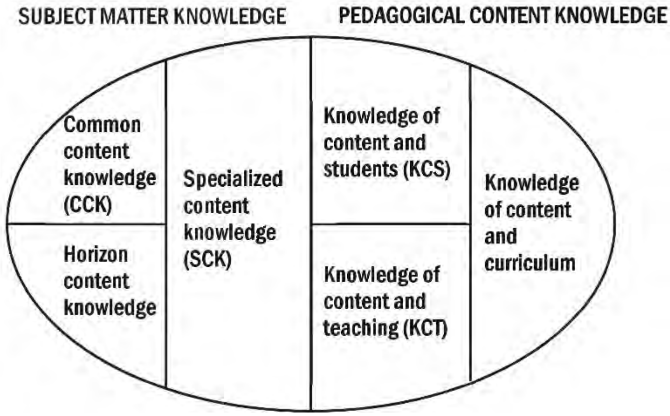

Fig. 1. Domains of MKT (Ball et al., 2008, p. 403)

on the design and evaluation of teacher education programmes. Hoover et al. (2016; p.11–12) suggest the following as what they observe as related emerging lessons from the several decades of research:

- Teaching teachers additional standard disciplinary mathematics beyond a basic threshold does not increase their knowledge in ways that impact teaching and learning.
- Providing teachers with opportunities to learn mathematics that is intertwined with teaching increases their mathematical knowledge for teaching.
- The focus of the content, tasks, and pedagogy for teaching such knowledge requires thoughtful attention to ways of maintaining a coordination of content and teaching without slipping exclusively into one domain or the other.

These are important lessons, because the main goal of studying MKT is that we understand it in ways that inform our work with teachers – both pre-service and in-service. However, I add that each of these requires careful consideration depending on the context, school curriculum and other issues surrounding teaching of mathematics in the specific context. For example, the basic threshold might vary across different school curricula, and identifying these might not be easy. Similarly designing and planning mathematics that is intertwined with teaching depends on the context: the curriculum, resources, language, and other characteristics of the context.

Hoover, Mosvold and Fauskanger (2014) shift the focus from knowledge for teaching to tasks of teaching, pointing out that there seem to be no discussion among the researchers of mathematical knowledge for teaching, about identifying common tasks of teaching that would form an international body of knowledge. They suggest that "the idea of common tasks of teaching – that represent a decomposition of the work of teaching into professionally recognizable components – constitutes a potential foundation for an internationally useful practice-based theory of MKT" (p. 8). Hoover et al. (2014; 7) call for "increased efforts to identify professionally defensible mathematical tasks of teaching that can serve as a common foundation for conceptualizing and measuring mathematical knowledge for teaching internationally"

What Hoover et al. (2014) are calling for is similar to what Adler (2010) did earlier when she elaborated on some tasks of teaching mathematics drawn from studies of mathematics classrooms in South Africa. She identified 'designing, adapting and selecting tasks', 'processes and objects', 'valuing and evaluating diverse learner productions' as interrelated mathematical tasks of teaching. These seem to be examples of "common tasks of teaching" Hoover et al. (2014) are calling for. As Adler explains, the identified tasks of teaching illustrate some of the mathematical knowledge for teaching discussed by Ball et al. (2008), thus building onto the body of knowledge.

Ball (2017; 29) also shifts focus from knowledge domains to specific mathematical work of teaching and argues that "the quest to answer the perennial question of what mathematical 'knowledge' teachers need should be based on a deep and nuanced understanding of what teachers actually do".

I respond to this shift in focus and conceptualise the mathematical work of teaching in multilingual contexts and not knowledge domains. I do this by drawing on the professional knowledge that I extract from the lessons learnt from the previous studies.

3.1. MKT studies in multilingual contexts

Studies of MKT in Africa are not many (Jakobsen and Mosvold, 2015). The available studies use the theories from University of Michigan; the earlier ones (e.g. Adler, 2005; Kazima and Adler, 2006; Kazima, Pillay and Adler, 2008) use Ball, Bass and Hill (2004) aspects of mathematical problem solving that teachers do, while the later ones (e.g. Adler, 2010; Mwadzaangati, 2018; Mamba, 2018) use Ball et al. (2008) domains of MKT. Lessons from these studies in African multilingual contexts include that the tasks of teaching as identified in the US are also identified in these contexts, furthermore there are some additions to the work of teaching that include paying attention to language; for example what is said and how it is said (Kazima and Adler, 2006). This suggests that teachers in this context are faced with additional specific tasks of teaching linked to language.

Studies of MKT in multilingual contexts outside Africa where the students are English language learners in an English-speaking country are also limited. I will focus on the work of Sorto, Wilson, and White (2018) and Wilson (2016) at Texas State University in the US. They studied specifically MKT for teaching English language learners and developed from Ball et al. (2008). Lessons we get from their studies are that there is a special kind of extra knowledge that teachers of English language learners require and that this knowledge fits within the Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) domains of MKT. They classify this knowledge as knowledge of obstacles, knowledge of resources, and knowledge of strategies. They represent these as in Fig. 2 which they call pedagogical content knowledge for teaching mathematics to English language learners (PCK-MELL).

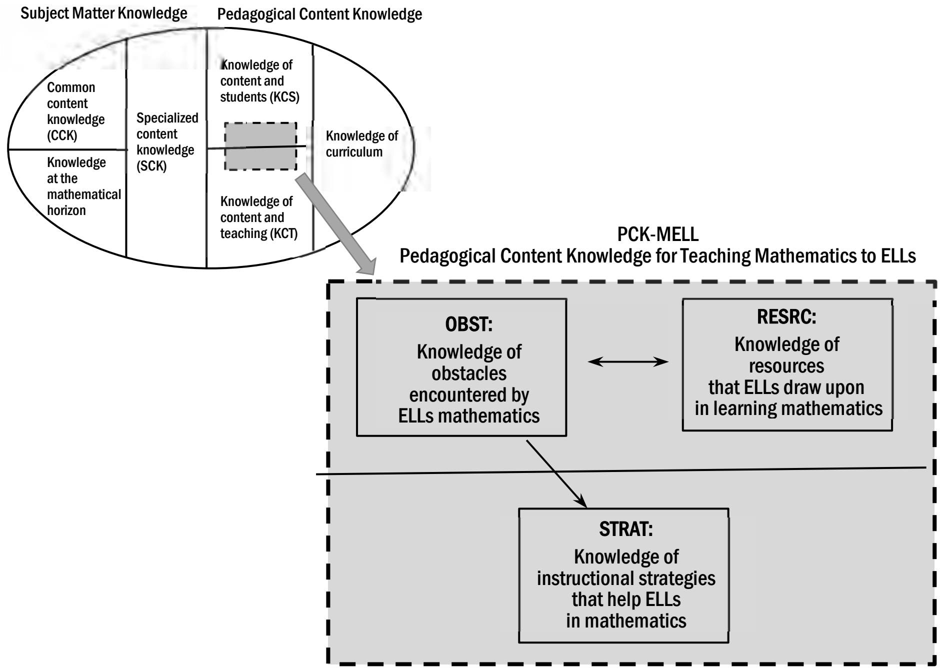

Fig. 2. PCK-MELL (Sorto et al., 2018, p. 222)

Another lesson we learn from these studies is that this knowledge can be measured using carefully designed items, similar to measuring the other knowledge of MKT domains. Sorto et al. (2018) aim was to identify domains of knowledge. My emphasis differs in that I aim to identify the mathematical work of teaching.

4. Lessons from My Work in Malawi

My work in Malawi over two decades has been about teaching mathematics in multilingual contexts, mathematical knowledge for teaching, teacher education and professional development of teacher educators and teachers. In my earlier studies I focused on language and mathematics, then later and until now on mathematical knowledge for teaching. Since 2013, my work has mostly been both research and development projects on teacher education and professional development of teachers. I am privileged to work in collaboration with colleagues from University of Stavanger. In this collaboration we have conducted two large projects, the first was from 2014 to 2018 and titled: Improving quality and capacity of mathematics teacher education in Malawi. The second project is titled: Strengthening numeracy in early years of primary education through professional development of teachers, currently in progress, started in 2017 and will end after 2021. I also work with colleagues from Malawi some of whom started as PhD students in the projects.

In Tab. 1 below, I present a summary of selected and relevant studies that I have participated in over the years.

Tab. 1. Relevant studies in Malawi

Year	Study	Researchers
2000–2002	Students understanding of probability vocabulary	Kazima
2010	Code switching in primary mathematics classrooms	Kazima and Pwele
2011	Using bilingual approach in standard 6 mathematics	Kazima, Pwele, and Kasakula,
2014–2015	Language and students' conceptions of logic in undergraduate mathematics	Levis Eneya, Mercy Kazima, Patrick Sawerengera
2014–2015	Exploring mathematical tasks of teaching in Malawi schools	Kazima and Jakobsen
2014–2018	Improving quality and capacity of mathematics teacher education in Malawi	Kazima, Jakobsen, Mosvold, Bjuland, Fauskanger, Eneya, Mwadzaangati, and Mamba
2015–2018	Measuring development of MKT in prospective primary school teachers	Kazima, Jakobsen, and Kasoka
2015–2018	Investigating MKT for teaching equations	Mamba
2015–2018	Exploring MKT for teaching geometric proofs	Mwadzaangati
2017–2021	Strengthening numeracy through professional development of mathematics teachers in Malawi	Kazima, Jakobsen, Fauskanger, Hedgevold, Mosvold, Bjuland, Eneya, Mwadzaangati, Mwale, Gobede, and Longwe
2017–2020	Investigating mediation strategies used by early years mathematics teachers in Malawi	Gobede
2017–2020	Making sense of MKT: insights from primary preservice teachers in Malawi	Jacinto
2017–2020	Exploring how primary teacher education prepares pre-service teachers to teach number concepts and operations	Longwe

4.1. Lessons from the studies on language in Malawi

Findings from these studies in Malawi supported findings reported in literature, and the lessons drawn are similar. For these studies specifically the lessons are:

(1) Students' meanings for mathematical terms can be different from the mathematical meanings, and these are influenced by their home languages (Kazima, 2007; Kazima, Eneya, and Sawerengera, 2015)

(2) Code switching can be used effectively to make mathematics accessible to students (Kazima, Pwele, and Kasakula, 2011)

(3) Bilingual approach where use of home language is planned and proactive can be effective in making mathematics accessible to learners (Kazima, Pwele, and Kasakula, 2011)

There are at least two other researchers that have studied language and mathematics in Malawi: Chitera (2009a, 2009b) and Kaphesi (2002). Chitera (2009a, 2009b) studied teacher education and the discourse practices in preservice mathematics education classrooms. She found that English is used in all lessons and there is no reference to home languages. The lesson we draw from Chitera's work is that teacher education does not prepare teachers for the teaching in multilingual contexts. Our ongoing project work with mathematics teacher educators in Malawi confirms this. Kaphesi's (2002) work analysed language practices of primary school teachers of mathematics. His findings confirm that code switching is the common practice that teachers use in teaching mathematics in Malawi. The lesson drawn from this is therefore similar to earlier lessons, that code switching between English and home language can be an effective strategy in teaching mathematics to learners that are not fluent in English.

4.2. Lessons from the studies on MKT in Malawi

It has been interesting to use theories developed in the US which is a context different from Malawi. Our use of the MKT framework was informed by many other researchers that used the framework outside the US. For example in South Africa where some of the school contexts are similar to Malawi. General findings from these studies on MKT in Malawi include:

(1) All the tasks of teaching suggested in the US were viewed as relevant by Malawi teachers but at varying levels of importance (Kazima, Jakobsen, and Kasoka, 2016).

(2) All MKT domains were observed in Malawi teaching demands (Mwadzaangati, 2018; Mamba 2018).

(3) Adapted MKT measures were appropriate to use in Malawi context (Jakobsen, Kazima and Kasoka, 2018).

Lessons from these findings are that the general tasks of teaching mathematics are the same in Malawi as elsewhere. However, language always surfaced as an issue, indicating that there are other specific demands on teaching related to language.

5. Conceptualising the Work of Teaching in Multilingual Contexts

Learning from all the discussion so far, it appears that teaching in multilingual contexts makes additional demands on teachers. I will focus on the context of Malawi and draw from the previous studies to conceptualise the mathematical work of teaching in this context. I do this by first considering the lessons drawn from studies on teaching mathematics in multilingual contexts. I take these as professional knowledge of teaching mathematics in multilingual contexts. Professional knowledge because they inform the mathematics education field and profession. From the professional knowledge, I identify the specific knowledge demands on teaching, then further identify the mathematical work of teaching that teachers face. I present this in form of a table as shown below.

As seen in Tab. 2, I propose four categories of mathematical work of teaching that teachers face: (1) identifying obstacles in home language, (2) identifying obstacles in English, (3) identifying resources in home language, and (4) identifying strategies; strategies to address the obstacles and strategies to draw from the resources in home language. I give some examples below to illustrate each of the proposed mathematical work of teaching

Tab. 2. Professional knowledge, specific knowledge demands and mathematical work of teaching in multilingual contexts

	Professional knowledge of teaching mathematics in multilingual contexts	**Specific knowledge demands on teaching**	**Mathematical work of teaching**
1	Students' meanings for mathematical terms can be different from the mathematical meanings, and these are influenced by their home languages	Knowledge of words in home language that are equivalent or closest equivalent to mathematical terms Knowledge of different meanings of the home language words that can cause difficulty. Knowledge of words in home language that students draw on to make meaning of the mathematical terms Knowledge of how to use the home language words in mathematics	Identifying resources in home language Identifying obstacles in home language Identifying strategies to draw from the resources Identifying strategies to address the obstacles
2	Code switching can be used effectively to make mathematics accessible to students	Knowledge of words in English that students find difficult Knowledge of words in home language that are equivalent or closest equivalent to the English words Knowledge of different meanings of the home language words that can cause difficulty. Knowledge of how to use the home language words in mathematics	Identifying obstacles in English Identifying resources in home language Identifying obstacles in home language Identifying strategies to draw from the resources Identifying strategies to address the obstacles
3	Code switching presents challenges and dilemmas for teachers		
4	Bilingual approach where use of home language is planned and proactive can be effective in making mathematics accessible to learners		
5	Home language should be viewed as a resource rather than a problem	Knowledge of what students can draw on in the home language to make sense of the mathematics being taught Knowledge of how to use these in mathematics	Identifying resources in home language Identifying strategies to draw from the resources

5.1. *Identifying obstacles in home language*

In drawing from home languages there are at least two types of obstacles (i) equivalent words in home language not as precise as the mathematical words and (ii) non-existence of equivalent words in home language.

5.1.1. Example of equivalent words in home language not as precise as the mathematical words

One example in Chichewa is the concept and operation of multiplication. Multiplication is translated as kuchulukitsa which literary means to increase or to make more. Being able to identifying the obstacle of using kuchulukitsa requires understanding the limitation of describing multiplication as to increase or make more. Specifically it requires understanding that it is true only for positive numbers multiplied by numbers more than 1; and that it is not only multiplication that can result in an increase, other operations on numbers can also result in an increase. For example, adding a positive number or dividing by a positive fraction. The work of teaching is to identify such obstacles and find ways of addressing them. This is mathematical work and is what teachers face for effective teaching of multiplication in this context.

Another example is the equal sign (=). In Chichewa it is translated as zikhala, which literally means will become. Thus the meaning assigned to the equal sign is that of getting a result after performing an operation. The limitation of zikhala is that it does not convey the equivalence meaning of the equal sign. For instance, showing equivalent fractions, such as $2/3 = 4/6 = 6/9 = 8/12 = \dots$, requires the equivalence meaning of the equal sign.

5.1.2. Example of non-existence of equivalent words in home language

There are some concepts such as similarity which do not exist in Chichewa and are difficult to explain. Similar is translated as kufanana literary meaning look alike, there is no word for proportional and is very difficult to explain proportionality in Chichewa. The work faced by teachers is to identify such obstacles in the home language and find other ways of explaining, such as using many examples of what are and what are not similar in mathematics, as well as the limitation of thinking of similar as only *kufanana.*

5.2. Identifying obstacles in English

There are many obstacles in English for learners that are not fluent in the language. The work for teachers is to observe closely and identify these. For example, sound alike words such as *size*, *side* and *sight* have been found to be used by learners interchangeably (Adler, 2001) and the words *probability*, *disability* and *ability* were thought to mean the same by some learners in a secondary school mathematics lesson on probability (Kazima and Adler, 2006; 53). Kazima and Adler call this “hearing disconnects” and argue that it adds to the description of mathematical knowledge for teaching. I emphasise the argument and add that it is a recurrent task that teachers face in this context. Hearing what students say and being careful in how to pronounce the words, as well as anticipating what students might hear and might mean mathematically are part of the mathematical work of teaching.

5.3. Identifying resources in home language

Resources in home language include (i) the words that teachers chose to use to provide mathematical explanations in the home language; and (ii) mathematical concepts in the home language that can be deliberately sought for use in teaching. The first case of identifying words to use mostly happens during lessons as teachers teach and code switch between English and the home language. Teachers are challenged to think fast on the spot while moving the lesson forward. For instance, looking at the two examples of multiplication and equal sign discussed above, after identifying the obstacles in home language, further work for the teacher is to identify words in home language that can be used to explain the concepts in a way that avoids the obstacles. Identifying words in home language also happens during planning where teachers have time to think and plan carefully the words to use in teaching. This is a major part of the bilingual approach where the use of home language is done proactively rather than reactively (Setati et al., 2008). The planning for this approach requires a lot of time and mathematical reasoning for teachers to provide mathematically appropriate and accurate versions of written materials for the lessons (Kazima et al., 2011).

For the case of mathematical concepts in the home language, one example is that of the number system in Chichewa. The number system has combination of base 5 and base 10, which can be drawn upon when teaching number bonds, number bases and other number concepts and operations. The counting in words is as in Tab. 3:

Tab. 3. Counting in Chichewa

Numeral	Number words in chichewa	Literal meaning in numerals
1	Chimodzi	1
2	Ziwiri	2
3	Zitatu	3
4	Zinayi	4
5	Zisanu	5
6	Zisanu ndi chimodzi	$5+1$
7	Zisanu ndi ziwiri	$5+2$
…		
10	Khumi	10
11	Khumi ndi chimodzi	$10+1$
…		
19	Khumi ndi zisanu ndi zinayi	$10+5+4$
20	Makumi awiri	10×2
…		
28	Makumi awiri ndi zisanu ndi zitatu	$(10\times 2)+5+3$
…		
70	Makumi asanu ndi awiri	$10\times(5+2)$
…		
99	Makumi asanu ndi anayi, ndi zisanu ndi zinayi	$(10\times(5+4))+5+4$
100	Zana	100
864	Mazana asanu ndi atatu makumi asanu ndi awiri ndi zinayi	$(100\times(5+3))+(10\times(5+2))+4$

The mathematical work for teachers is to identify such mathematical resources and make decisions of what and when to use in teaching.

5.4. Identifying strategies

Identifying strategies is in two parts: identifying strategies to address the obstacles and identifying strategies to draw from the resources in home language. The mathematical work of identifying strategies is linked to all the other three mathematical work discussed above. For example, if a teacher decides to teach division by using some activities of equal sharing, then he or she would need to draw upon the words that the children use when sharing and find ways of using those words in teaching that would convey the mathematical concept of division. Thus as the teacher is identifying strategies to draw from the resources in home language, they are also identifying resources in home language and identifying obstacles in home language as well as identifying obstacles in English and identifying strategies to address the obstacles. Another example, if a teacher decides to use the strategy of word origins to explain meanings of mathematical terms such as polygon, triangle, or isosceles, then the teacher would need translations of the word origins into the home language. The mathematical work is therefore linked to identifying resources in home language, identifying obstacles in home language and identifying obstacles in English. One can see that the mathematical work of identifying strategies is not independent of the other mathematical work, however, it is important to acknowledge as its own category of mathematical work of teaching because it makes specific knowledge demands on teaching and requires special mathematical problem solving by the teachers to decide on the best strategy to use for effective teaching of the specific mathematics content in the multilingual context.

Comparing these identified mathematical work of teaching to the PCK-MELL knowledge domains (Sorto et al., 2018), it can be noted that the categories of mathematical work of teaching and the categories of knowledge domains are similar: they both have categories of obstacles, resources and strategy. I take this similarity as evidence of support for the findings of Sorto et al. (2018), although my focus is not knowledge domains. While there is this similarity in categories, there are some differences in the details, which appears to suggest that there is mathematical work of teaching that is common among multilingual contexts, as well as specific mathematical work of teaching for specific multilingual contexts, such as Malawi.

6. Conclusion

In this paper I have conceptualised mathematical work of teaching in multilingual contexts. I have done this by drawing from the professional knowledge of teaching in multilingual contexts that we learn from studies of teaching mathematics in these contexts. Considering the professional knowledge, I first identify specific knowledge demands on teaching then from these further identify the mathematical work of teaching entailed. I identify four categories: identifying obstacles in home language, identifying obstacles in English, identifying resources in home language, and identifying strategies. My approach is different from earlier studies on mathematical work of teaching (e.g. Ball et al. 2008; Adler, 2010) where they start from classroom practice and analyse the work faced by teachers as they teach mathematics. Some might argue that my approach, that is almost the other way round, could miss some details of what teachers are faced with when teaching mathematics. While I appreciate the limitation my approach might have, I would like to highlight that the professional knowledge I have presented is drawn from findings of classroom studies and therefore captured what goes on in practice in the multilingual context.

This identification of mathematical work of teaching in multilingual contexts is important because it informs us that there is specificity to teaching multilingual contexts. Looking at the specific case of Malawi, it appears that the specific nature of the mathematical work of teaching depends on the nature of the multilingualism. The Malawi multilingual context is similar to some that have national languages or common home languages between the students and the teachers. The context is different from others that do not have such common languages. Nevertheless, the conceptualisation of the mathematical work of teaching that I suggest can also inform those contexts.

Acknowledgement

I acknowledge Reidar Mosvold with great appreciation for insightful comments during preparation of this paper.

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Part III

Plenary Panels

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https://doi.org/10.1142/9789811287152_0013

Plenary Panel 1

Actors for Math Teacher Education: Joint Actions versus Conflicts

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Angel Ruiz[4], and Mikhail Sluch[5]

ABSTRACT We focus on the interaction of mathematicians and mathematics educators as they relate to the preparation of prospecting teachers and professional development of practicing teachers. We emphasize collaborative experiences and show how much can be gained with close collaborations. For this, we describe some examples and point out various factors that have made collaboration possible as well as potential conflicts that existed in certain institutional, cultural political and social-economic environments, and thus draw emerging issues. It is not intended to state rigid conclusions applicable in all contexts, however we consider two general perspectives and suggest some questions to guide research on this area. In general, although we did not find systematic strong research on the collaboration of mathematicians and mathematics educators in the context of teacher professional development, this does not necessarily mean that such efforts do not exist in different countries.

Keywords: Actors; Communities of practice; Conflicts; Cooperation; Mathematics educators; Mathematics teacher education; Mathematicians.

1. Introduction: Joint Actions versus Conflict — Some Key Characteristics

Various actors, with different roles and impact, interact as they participate in mathematics teacher education. These actors can be, for example: mathematicians, mathematics educator researchers, pedagogues, mathematics teacher educators, teachers, mentors, policy makers, curriculum developers, heads of schools, administrators, parents, or students. The actions of these actors and their impact on

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mathematics teacher education depend on several factors such as for example the institutional and social contexts where the interaction takes place or the availability of resources.

The role of these actors and these factors have varied over time. They were not the same for example during -let's say- the "New Math" and "Going to Basics" during the sixties or seventies of the previous century, or during the recent decades where "Competencies" and "XXIst Century Skills" are promoted.

If we look at these roles today, the conditions and the actors involved in mathematics teacher education have evolved in many settings. Some of these changes are due to research in mathematics education (during the last two decades) that has focused on the mathematics teacher and has contributed to the development of teacher education practices that seem to be crucial in the education of both prospective and practicing teachers. For example, the focus on the special nature of teacher knowledge has an impact on the courses offered to prospective teachers and to the professional development of teachers (Cooper and Karsenty, 2008) where mathematics teacher educators and mathematicians play the central role. The emergence of practice-based pedagogies emphasized the importance of field experiences in teacher preparation and professional development (Solomon et al., 2017) that involves mainly mathematics teacher educators, mentors, teachers, administrators. Currently, research on large-scale studies on professional development has considered the important role of policymakers and has also developed strategies that involve a large number of teachers (Maass et al., 2019). In this setting, besides the policymakers, mathematics teacher educators, mathematicians, teachers are also key actors. This research has played an important role and attempts have been made to become available to other actors such as mathematicians (an example of this is the papers on solid findings written by the Education Committee of the European Mathematical Society (EMS) in the newsletter of EMS, see e.g. Hoyles, 2014), to teachers (teacher journals, professional development activities, conferences), to policy makers (through conferences and policy makers workshops, e.g. in the conference Educating the Educators III organised by the International Centre for STEM Education in Freiburg, Germany, https://icse.eu/educating-the-educators-iii/). Moreover, the emergence of international comparison and ranking of nations in math performance for students (PISA, TIMSS) and the last years for teachers (TEDS study) is also one aspect of the international context which has an impact on how the mathematical preparation of mathematics teachers is perceived.

The actors that we have mentioned above who play a crucial role in mathematics teacher education, as well as the underlying factors that drive their actions, have intervened and intervene in teacher education in different ways, in diverse cultural contexts (e.g. European, Asian, African) and socio-economic contexts (e.g. developed world, developing world, countries in transition).

Although each community of actors usually has its own goals and perspectives for the preparation and professional development of mathematics teachers, the collaboration between them is essential for promoting an effective way of intervention.

The attention in section 2 is given to the interaction of mathematicians and mathematics educators in connection to prospective teachers' preparation and, in section 3, to the professional development of practicing teachers. The fourth section is devoted to collaboration as a Community of Practice.

2. Mathematicians, Mathematics Educators, and Teachers in the Initial Teacher Education

2.1. Collaboration and conflicts between mathematicians and mathematics educators

The initial teacher education for prospective secondary school mathematics teachers includes several actors, and often varies according to the targeted level (grades K-12) of teaching and the presence of specialist mathematics teachers (as opposed to generalists). The level of involvement of mathematicians and mathematics educators and the potential for joint action, collaboration and conflict is dependent on context and varies greatly between countries.

In some settings, the potential for joint action appears to be lower, as mathematicians and mathematics teacher educators (MTE) participate, essentially separately. This may be the case when prospective secondary school teachers spend most of their university education in mathematics courses taught by mathematicians. In some such settings, a broad body of research has documented a disconnection between the mathematics taught and practiced at university and the mathematics required for school teaching (Zazkis and Leikin, 2010). While some mathematicians have looked at the differences between these type of mathematics in a positive light and many have acted to address this, there are tensions which can lead to conflict, as illustrated by the Math Wars (Ralston, 2004). To develop connected knowledge, numerous collaborative efforts between mathematicians and MTEs have taken place (Bass, 2005; CBMS, 2001, 2012; Ferrini-Mundy and Findell, 2001; McCallum, 2003; Wu, 2006). As a result, mathematics courses, which can be dramatically different from regular advanced courses, have been developed.

The inherent difficulties in working across institutions, faculties or departments can be a strong deterrent to joint actions. In some cases, mathematicians and mathematics teacher educators work in the same department, enabling collaborations and the emergence of educators who are experts in both domains. This creates opportunities, which are possibly reminiscent of the situation lived by earlier researchers in mathematics education. An example of research in this direction comes from Greece (Petropoulou et al., 2011; Karavi et al., 2020), where a strong expertise in mathematics and mathematics education enabled an individual to develop courses with a clearer expectation about students' difficulties, choosing representations enabling students to build a stronger understanding of advanced concepts. In this case, the MTE was both a mathematician and a mathematics education specialist.

However, it is a challenge in many cases to find how mathematics educators can support mathematicians in a study of their teaching and its impact on students' learning. As Bass points out, a fruitful collaboration with mathematics educators may not be practical for all mathematicians who wish to contribute to teacher education (Bass, 2005, p. 418). It also remains true that an important barrier is what some mathematicians expect from mathematics education research, such as the search for the effective teaching strategies, is very different from the views of mathematics education researchers (Sierpinska and Kilpatrick, 1998; Schoenfeld, 2000).

On a very positive side, in other settings, team-teaching between mathematicians and MTEs has taken place (Grassl and Mingus, 2007; Heaton and Lewis, 2011; Sultan and Artzt, 2005; Thompson et al., 2012). Research into the conditions enabling interdisciplinary collaboration in the way of team-teaching and joint work showed that shared goals, mutual trust, and open-mindedness (Goos and Bennison, 2018; Ponte et al. 2003) were key issues. It also showed that there was an initial fear of being judged by the other: mathematicians on their teaching, and mathematics educators on their mathematics.

Conditions hindering such work include cultural differences, grounded in epistemological differences between disciplines, as well as the lack of recognition, in both communities, of the value of such work (Goos and Bennison, 2018). There is also a difficulty for those working at the boundary between disciplines who can feel "like they belong to both one world and the other, or to neither one world nor the other" (Goos and Bennison, 2018, p. 272).

The emergence and the importance of brokers in such collaborative work is one of the important aspects considered in Goos and Bennison (2018): here, the collaborative work is not entirely in mathematics education or entirely in mathematics, and is done in such a way that actors from these distinct communities are engaged towards a common goal. When initiating collaborations, the presence and the emergence of brokers play a key role. They can be crucial in promoting and sustaining further work, and thus are both a product of successful collaborations and an ingredient for their ongoing success.

The presence of such brokers can also help in avoiding conflict or dealing more constructively when conflict arises. They play an important role in shaping the way one community views the other, and can help fight against ignorance and judgment, stereotypes and narrow views. This can be at the local, national and international levels, sometimes simply by sharing views about the complexity of the work done by "the other side".

As actors in different countries and institutions, many of us are in a situation where we can work towards enabling joint work by recognizing the importance of work done in collaboration in our respective communities. Despite the numerous challenges and obstacles, positive experiences show how much is to be gained from close collaborations.

2.2. Cultural aspects and their influence on the development of collaborations

Building joint actions among the actors involved in the education of prospective secondary school teachers is a long-term process and is framed by institutional, cultural, and political factors. What are some of the factors that have made this possible in different settings?

On the international level, and particularly at ICMI, several outstanding individuals have been recognized as genuine members of both communities. They were, generally, established research mathematicians who developed a very strong interest in mathematics education, in some cases leading to a career as a researcher in mathematics education: Hyman Bass and Michèle Artigue, two recent former presidents of ICMI, are prime examples of such individuals, and many other examples are found in Karp and Roberts' book (Karp and Roberts, 2014). These individuals have contributed to the development of the perspective that mathematics education is a genuine scientific endeavour, albeit very different from mathematics.

One important example motivated by ICMI has been the Capacity and Networking Project (CANP) which has conveyed the participation of mathematicians and mathematics educators in workshops held in different developing regions: Francophone Sub-Saharan African (Mali, 2011), Central America and the Caribbean (Costa Rica, 2012), Southeast Asia (Cambodia, 2013), East Africa (Tanzania, 2014), Andean Region and Paraguay (Perú, 2016).

In Norway, joint projects of mathematicians and mathematics educators have been established. The Erasmus+ European project PLATINUM (http://platinum.kubg.edu.ua/en/), which consists of seven European countries partnership between mathematicians and mathematics educators, aims to improve the teaching and learning of mathematics at the university level developing resources promoting inquiry-based learning. Mathematics education researchers and mathematicians contributed in different ways in the development of these resources and in their enactment.

In Canada, the establishment in 1978 of the Canadian Mathematics Education Study Group (CMESG) has led to annual meetings of mathematicians and mathematics educators in a highly collaborative work setting, around issues of mathematics education, particularly in mathematics teacher education. A Canadian community of mathematicians and mathematics educators was gradually formed, and numerous personal relationships forged. This enabled the development of mathematics courses for teachers influenced by mathematicians and mathematics educators (Hodgson, 2016). Initial teacher education has been an ongoing theme of working groups at these meetings, some examples are given in series proceedings of the annual meeting (Marynowski, Dufour and Liljedahl, 2017; Gourdeau and Nolan, 2016; Gourdeau, Oesterle and Stordy, 2014). The collaboration and joint involvement of mathematicians and mathematics educators in mathematics education discussion (through CMESG as well as in the Canadian Mathematics Education Fora of the Canadian Mathematical Society), and in the initial teacher education, may help explain why the Math Wars, which have affected the USA, have been much less intense in Canada.

3. Mathematicians, Mathematics Educators, Teachers and Other Actors in the Professional Development of Practicing Teachers

3.1. Joint actions in teacher professional development

Although in the initial teacher education there are research studies where mathematicians and mathematics educators collaborate, especially in designing and even team-teaching common courses (e.g. Bleiler, 2015), the research on collaboration between mathematicians, mathematics educators, and practicing mathematics teachers is rather rare. The discussion document of the ICMI-25 Study on "Teachers of mathematics working and learning in collaborative groups" (International Program Committee for ICMI-25 Study, 2019) addresses also as an important question the role of the different actors in teacher collaboration, including teachers, leaders, mathematicians, researchers in mathematics education. However, in the conference related to this Study that took place in Lisbon from the 3rd to 7th of February 2020 (http://icmistudy25.ie.ulisboa.pt/) there were no submissions reporting research in this area. Nevertheless, mathematicians and mathematics teacher educators are involved in supporting practicing mathematics secondary school teachers to develop their teaching. Mathematics educators usually organize practice-based professional development programs for teachers or act as facilitators in teacher collaborative groups (e.g. Cooper, Olsher and Yerushalmy, 2019). Mathematicians support teachers mainly by designing resources such as curriculum documents, textbooks, teacher's guides (e.g. Potari, Psycharis, Sakonidis and Zachariades, 2019).

Concerning the professional development of mathematics teachers, there is an increasing research interest on the collaboration between mathematics educators and mathematics teachers with a particular focus both on the process and the outcomes of collaboration (see the ICME international survey in (Robbuti et al., 2016)). In addition to the ICME international survey, we see several papers reporting collaborative efforts between mathematics teacher educators and mathematics teachers (Arbaugh, 2003; van Es, 2009) and a special issue in ZDM (issue 46) focusing on the collaboration addressing the importance of the joint actions, see (Jaworski and Huang, 2014). Collaboration between mathematicians/mathematics educators and teachers in the context of professional development has also been seen in offering professional development programs to teachers for revisiting advanced mathematics content that they had met during their university studies. An example of such a program concerns the teaching of linear algebra for ten practicing teachers in the US (Harel, 2017). Another example of collaboration among mathematicians, mathematics educators, prospective and practicing mathematics teachers is also reported in the study of McGraw, Lynch, Koc, Budak and Brown 2007) focusing of the use of multimedia cases as tools for teacher professional development. Through the analysis of online and face-to-face discussions, the authors show that the different backgrounds and experiences of the participants can blend in such a way that it promotes rich discussions about mathematics, teaching and learning.

In designing resources for teachers, we also see examples of collaboration between mathematicians, mathematics educators and teachers. In the study of Potari et al. (2019) that took place in Greece, different actors participated in the design of a new mathematics curriculum for the compulsory education. In that setting, tensions emerged between the different communities of participants while persons that participated in these communities (e.g. teachers who had also been involved in research in mathematics education), acted as boundary persons and facilitated the overcoming of the tensions. A similar example from China in improving teachers' teaching, university mathematics educators collaborated closely with mathematics teachers in designing and implementing lessons, and it was found that the identity of the participating teachers changed from "problem posers and solution receivers" to "collaborative problem solvers" in negotiating and finding solutions to practical problems with mathematics educators (Qi et al., 2021).

3.2. Mathematicians' efforts on teacher professional development with collaboration of other actors

The fact that there is no systematic research on the collaboration of mathematics educators and mathematicians in the context of professional development of practicing teachers does not necessarily mean that there are no such efforts in different countries. The members of the panel address collaborations that have taken place in their countries in the context of conferences and workshops, and annual conferences of the mathematics teacher associations.

More systematic collaborations are seen in China and the Russian Federation. In China, mathematicians have participated in many professional development programs for primary and middle school practicing teachers. Currently, one program is organized by the Ministry of Education (MOE) (named "Guopei" Project). It aims to improve teachers' professional skills, especially in rural areas. From the total of 45 mathematics expertise trainees in the first issued name list by MOE, 8 are mathematicians, which reveals the emphasis on the role of mathematicians from the government. In addition, mathematicians are involved in designing and planning mathematics curriculum standards and textbooks with other actors such as mathematics teacher educators and mathematics teachers. The two leaders of the current two mathematics curriculum standards (Ministry of Education [MOE], 2011, 2017) are all mathematicians. Among the six versions of current high school mathematics textbooks (Sujiao, Shanghai, Renjiao, Xiangjiao, Ejiao, BNU), two-thirds of chief editors are mathematicians who work in in-depth collaboration with other authors/actors (e.g., mathematics educators, teachers, Jiaoyanyuan, etc.).

In the Russian Federation, school textbooks and curriculum materials have been developed by working groups headed by leading mathematicians, while both mathematics educators and teachers participate in these groups. In this context, mathematicians are mainly responsible for the mathematics content while mathematics educators and teachers for the ways that the content can become accessible to the students. Moreover, mathematicians often participate in the professional development of practicing teachers, and they pay much attention to the popularization of mathematics both among students and among mathematics teachers. Mathematicians design and develop online courses for raising mathematics content knowledge of practicing teachers. For example, on http://ptlab.mccme.ru/node/5107 teachers can find a course on combinatorics and probability. Another example is the summer schools for teachers (in particular, for teachers of mathematics), organized by the top Russian universities such as the Moscow State University and the Higher School of Economics, where working scientists and educators give lectures and workshops for in-service teachers.

Another example of joint action between mathematicians and mathematics educators, in which some teachers are involved, is given by Mathematics Olympiads for secondary education in Latin America. [For example, the Brazilian Math Olympiad for Public School Students (18 million yearly participants), a nationwide educational project, includes teacher training programs.] Another such collaboration has been around mathematics modelling in Latin America and China. There is also a rich and long-time tradition of joint action of mathematicians, future math teachers and math educators in the Mathematics Olympiads for school children in Russia.

One very particular experience was developed in Costa Rica, following an unusual political decision made by a minister of education. A group of mathematics education researchers from public universities (whose initial training was in mathematics) and some practicing teachers worked as a team (Mathematics Education Reform in Costa Rica Project, https://www.reformamatematica.net) to design a new mathematics curriculum for all Primary and Secondary education (approved in 2012).

The same team with the inclusion of technology specialists have participated in the implementation of the new curriculum with a special emphasis on virtual instruments: designing, developing blended courses (with face-to-face and online dimensions) for primary and secondary teachers and pedagogical advisors (2012–2017), avant-garde MOOC and Mini-MOOC courses (since 2014) for teachers, and high school students and many other innovative virtual resources since 2019 (Ruiz, 2018, 2020). Since 2012, this team has had the support of several ministers of public education during three different national governments (Ruiz, 2020).

Here two things can be emphasized: There was a cooperation between political actors, researchers, teachers, technology specialists, pedagogical advisors within a scenario of curricular design and development. And second, during the pandemic since 2020, when virtual educational strategies gained extraordinary relevance, the multiple materials produced by this team associated with the Ministry of Public Education have constituted a non-improvised base of pedagogical support for the student population.

4. Collaboration as a Community of Practice

In the previous sections, some effective examples of collaboration between mathematicians and mathematics educators have been reported. Some of these examples share characteristics of a Community of Practice CoP (Wenger, 1998) in which mathematicians and mathematics educators are mutually engaged in an activity (e.g., co-designing, co-teaching courses), are held together by a joint enterprise (e.g., MOOC for teachers in the pandemic period), and have a shared repertoire of customs of practice (e.g., sharing views and experiences about the related resource). In this way, we can see the collaboration as forming a CoP. With consideration of seven principles of CoPs proposed by Wenger et al. (2002), we conclude three aspects:

- Achieving agreement on understanding the common objects
- Promoting equality in communication and mutual respect, and
- Facilitating transformation in identity.

To conduct the planned activities smoothly and efficiently in the collaboration, merely identifying common objects (e.g., co-designing course, co-developing curriculum, or textbooks) is usually not adequate given that mathematicians and mathematics educators/other actors may have different interpretations of them. Thus, one key factor is to ensure that every actor in the group has a common understanding of objects.

Based on the understanding of objects, promoting equality in communication and mutual respect is very crucial. A supportive atmosphere builds trust and enables mathematicians and other actors to express ideas and concerns openly (Henrick et al., 2017) without fear of others' judgment. In mathematicians' or mathematics educators' individual activity, most of them have only one fixed identity which guides their daily activities. However, during the collaboration, the objects require every member in the group to co-design the course for teachers, so that all members become co-learners and co-designers. Through their participation and collaboration, the mathematicians, mathematics educators, and other actors share and absorb each other's wisdom and sometimes act as "brokering" (Wenger, 1998) to facilitate transforming their old identities into the new ones of co-learners and co-designers. This type of benefit-sharing mechanism enables all the actors to work well with a clear understanding of the participants' identities in collaboration, performing their "delegations" from each community (mathematics content and education content respectively) and undertaking joint efforts for common development (Wenger, 1998).

Forming communities of practice in which mathematicians, mathematics educator researchers, teachers and other actors collaborate for contributing to initial teacher education and teachers' professional development is not an easy task. However, it seems that it is a promising way to offer prospective and practicing teachers learning opportunities that can have a positive impact on the mathematics education of students in schools.

5. Closing Remarks

Based on the aforementioned statements, we come to two main conclusions and propose four questions for further consideration.

Conclusion 1: *The relationships between the social agents involved in the preparation of teachers are not identical in various countries and regions due to diverse cultural or socio-economical contexts or individual or group experiences.*

For example, mathematicians may or may not participate directly in the design of curriculum materials for the school actors. Active teachers may or may not have a “say” in some teacher preparation programs.

Mathematicians can publish books on the history of mathematics or collaborate in publishing textbooks for pre-university education. In some countries there is no participation of mathematicians or even math educators in such activities.

Conflicts or tensions are not the same in all latitudes. The "Math Wars" in the USA were not a worldwide phenomenon, and in other countries the nature of conflicts may have been different.

So, it seems that it is not possible to offer a prescription for all settings.

However, we can enunciate

Conclusion 2: *It is always possible to identify internationally good practices that promote collaboration between educational agents and to manage conflict appropriately, but always with careful calibration of specific contexts.*

Even if conflict can be a problem for collaboration, they can also be an opportunity to calibrate the complexities of collaboration and to further develop these collaborations in fruitful ways. To conclude, we suggest some questions that could support research in this area as well as the emergence and development of such collaborations:

1. How to promote trust, mutual respect, and shared beliefs, values and goals, and stimulate joint practices among the several actors involved in mathematics teacher education? How can a community of practice can be developed and sustained?
2. What are the main features of institutional environments that facilitate collaborative work between the different actors?
3. What practices can help achieve convergence between the priorities and practices of universities and those of schools?
4. How to strengthen the participation of teachers in communities as a context for their professional learning? What is the role of the several actors in this process?

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https://doi.org/10.1142/9789811287152_0014

Plenary Panel 2

Mathematics Education Reform post 2020: Conversations towards Building Back Better

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ABSTRACT Our plenary panel focused on issues of mathematics education reform that have gained increasing significance since the start of the pandemic in 2020. Our panelists contributions provoked conversation about directions for mathematics education reform post 2020 with a view towards building towards ‘better’ quality and equitable mathematics teaching and learning for the future. Our topic cohered well with the 2021 International Day of Mathematics theme which was “Mathematics for a Better World”. The panel engaged with equity issues relating to two interrelated aspects of ‘building back better’ as we emerge from the crippling challenges of the pandemic. The first issue related to transitions towards remote and online teaching and learning in mathematics education which gained speed due to the massive closure of, and disruption to, schools due to Covid across the globe. The second related to the 2015 United Nations Sustainable Development Goals (SDGs), aimed at building ‘a better future’. We shared insights into emerging issues and research from across the globe and raised questions about the role of mathematics education in providing a better footing to achieve the SDGs and so build towards a more equitable future and a better world. Our paper is based on our panel contributions with voices and perspectives from diverse geographical backgrounds, research interests and expertise. The aim of the paper is to further provoke conversation about emerging and increasingly urgent issues our mathematics education community must grapple with.

Keywords: Reform; Equity; Sustainable development goals; On-line learning; Build back better.

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1. Introduction (by Panel Chair Mellony Graven)

In 2020 education systems across the globe faced major challenges in finding ways to enable continued teaching and learning in the context of a global pandemic that challenged opportunities for face-to-face classroom-based learning. In our panel, with the brief of focusing on key emergent mathematics education reform issues needing attention for 'building back better' in this post 2020 period, we identified two key issues to focus and frame our panel discussion. The first issue relates to transitions towards remote and online teaching and learning in mathematics education. This transition has recently gained speed due to the massive closure of, and disruption to, schools due to Covid across the globe. While online learning opportunities bring the possibility for increased access to quality mathematics education resources there is enormous uneveness in systemic and individual preparedness to optimise these opportunities. This plays out across countries and within countries and there is concern that the digital divide could exacerbate inequities across gender/social-class/race. Recent research evidence points to this concern playing out in real time across our home countries and the globe (Oxfam India, 2021; Borba, 2021; Martin, 2021; Jablonka and Bergsten, 2021; Vale and Graven, 2021). The second relates to the United Nations Sustainable Development Goals which were established in 2015 and aim to 'build a better future' by 2030 (UN General Assembly, 2015). While most countries are signatories on the SDGs, of which the fourth focuses on quality education for all, five years on these goals are far behind in meeting these targets. The SDGs include, for example, goals such as: no poverty; zero hunger; good health and well-being; quality education; gender equality; reduced inequalities; climate action; and peace, justice and strong institutions (among others). Covid is noted to have exacerbated challenges to progress in these goals yet some argue that emerging from the pandemic provides an opportunity for using the SDGs as a 'roadmap' to 'building back better'.

The phrase 'Build Back Better' initially emerged in the context of the Sri Lankan 2004 tsunami recovery efforts with the phrase looking to capture a comprehensive approach to recovery in which building back would not recreate or exacerbate existing vulnerabilities (Khasalamwa, 2009). The phrase was taken up in the UN conference on Disaster risk reduction in 2015 (UNISDR, 2015), in the same year the SDGs were adopted. In 2020 the world became increasingly aware of other 'disasters' beyond the pandemic. Some of this increasing visibility is a result of increasing access to digital news and social media. For example, the increasing exposure of extreme systemic racial inequality gave rise to the Black Lives Matter movement that began in the United States and gained momentum across the globe. 2020 and 2021 witnessed several climate related 'natural' disasters that disproportionately impacted those with fewer resources. The increasing proliferation of fake news highlighted the urgent need for people to be able to make informed decisions about how to respond to the many disasters (health, social, political, climate, economic challenges etc.) confronting their lives. While many mathematics educators have long argued the urgency of this need, 2020 was a pivotal year in placing a spotlight on the consequences of the failure of education in general, and mathematics education in particular, in preparing learners for critical citizenship.

In the following sections our panelists focus on raising questions about how mathematics education should respond to a range of issues in the context of the rapid transitions towards remote and online teaching and learning and supporting the SDGs. We acknowledge that these are not the only issues needing urgent grappling post 2020, but we considered these important, timely, and of global relevance for providing foci for our panel.

2. What Is the Role of Media and Things in Mathematics Education Post-2020? (by Panelist Marcelo C. Borba)

The goal of my panel input was to show how the role of media and things in mathematics education. Ideas presented below can be found in a more profound way in Borba (2021), Engelbrecht et al. (2020a, 2020b) and Borba, Souto e Canedo Jr. (2022) for those that Portuguese is easier to understand than English. Artefacts have always had a role in the development of mathematics, whether we talk about writing in paper, writing in tablets in Mesopotamia, the use of compass in geometry and more recently the use of digital technology (Villarreal and Borba, 2010). In the last decades different governments have tried to implement different programs to increase the use of digital technology in (mathematics) education. However, none of these programs, when we consider countries such as Brazil, have been as effective as the pandemic in stimulating the move to digital technology, illustrating the agency of the virus SarsCov-2. Questions worth asking are: Is this true in your country? If so to what extent and how has the pandemic influenced the move towards digital technologies for mathematical learning? What are some of the consequences of this for learners, teachers, the nature of mathematics taught and the pedagogies of teaching mathematics?

Reducing inequality is one of the goals for sustainable development, a major theme of this panel. Social inequality and economic disparities are a problem around the world. The pandemic has brought the need to incorporate digital technology in mathematics education, but teachers in schools in Brazil, and I expect elsewhere, had widely differing levels of readiness for their use of digital technologies – and these unequal levels or readiness link with societal inequalities of socio-economic status.

In the references mentioned above the notion of domestication of media was developed. Domestication means using a new media without exploring its potential, using it with rules and practices of an older media. In the case of Brazil, a clear example of this is that, many textbooks were just copied and transferred to digital media in the emergency remote teaching response. Many teachers, understandably, were teaching online “in the same way”, with practices from regular classroom dominating. In this respect, most teachers, students and parents were not taking advantage of the emergence of a new medium. Instead, they simply reproduced existing teaching practices and text resources. So for example, teachers used the internet to simply record their usual chalk on board regular class instruction, followed by provision of standard school practice exercises for learners to do at home (as they would in the class). However, there were others using PowerPoint, and the camera for recording images of key ideas or representations to share with learners, and thus it is possible to think of a rainbow of domestication of digital media.

In the papers by Borba and colleagues mentioned above it is argued that teachers could (and should wherever possible) be using animation in pieces of software such as GeoGebra to do dynamic mathematics. This then would allow one to explain to a much larger public, for example, that the sigmoid is important to model the way that the pandemic is unfolding. So, at a secondary school and beyond level, teachers could teach students about the sigmoid, its derivative, and how different sigmoids, lead to different peaks of the derivative. The notion of flattening the curve then may be understood by many more citizens and in more complex ways than simply 'seeing' the shape of the curve. An illustration of the sigmoid and its derivative can be visualized at this link https://igce.rc.unesp.br/#!/pesquisa/gpimem---pesq-em-informatica-outras-midias-e-educacao-matematica/animacoes/curva-epidemica-no-geogebra/. For each red curve (sigmoid) a different peak of the derivative, the rate of change curve, that can put the health system of a given community under stress can be seen.

In all the media presentations there is little discussion of why is it important to flatten the curve? Understanding mathematically the meaning of flattening the curve and how this connects with saving lives and allowing health systems to cope with cases is important. Saving lives can be demonstrated mathematically but students need to be able to access the mathematical meaning. I have argued with my students and shown them that the sigmoid in its derivative is true of exponential functions. In this way mathematics around exponential function may become meaningful in concrete rather than only abstract ways. Similarly, exponential functions are also important to understand social inequality to understand that the rich are getting exponentially richer and the poor are getting exponentially poorer and that the social disparity is growing exponentially. During the pandemic, the large multi-national companies that dominate the lives of many (and many more in these recent pandemic times), such as Facebook, Amazon and Google made their owners exponentially richer at the same time that populations in general were becoming exponentially poorer with many more people slipping into poverty. Therefore, it might be important to teach children how to count 1-2-3-4-5-6 but also 1-2-4-8-16-32-64… and backwards 64-32-16-8....

The use of technology in mathematics education can help with the tasks mentioned above whether we use concrete material, paper and pencil and/or digital technology. Digital technology, as other types of technology, can reduce or can increase inequality. Knowledge is seen as a product of collectives of humans-with-media. Different humans, different media (orality, writing, computing, for example) different knowledge, different ways of attaching meaning. Humans-with-Geogebra in the mathematics shown in the link above is a product of humans-with-media and once it is used in someone's formal or informal teaching, the animation presented in the link is part of such a collective.

In the pandemic another non-human-actor became very prominent in a collective that constructs mathematical knowledge. The agency of the corona virus resulted in emergency remote mathematics teaching. This meant children and young adults at home became important agents in their own and their families mathematics education. Quality of the home, availability of rooms for studying, internet connection and possibility of parents, other adults or older siblings in the home as helpers and facilitators of learning became paramount for mathematics education. This was irrespective of whether technology was used in a domesticated way or not.

Collectives of humans-with-media-homes exposed the increasing social inequality compared to those without. One cannot have education for all, or thinking about “building back better” without simultaneously addressing such social inequality and inequality in access to digital technologies and learning opportunities. We must face and eradicate social inequality to open the door to mathematics education to all. We must struggle to promote change in pervasive societal and economic inequality that gives rise to extreme differences in access to digital resources, access and learning opportunities to make mathematics education for all a possibility.

Different cell phones, different video, different connection influence different knowledge construction. We cannot have extreme social difference and simultaneously have equal opportunities for mathematics education. Schools may have shadowed the differences in home, as the picture in Borba (2021) shows. The capital to have different home and the new cultural capital - that includes not only the education of the parents and the parents and the community but also includes now the quality of the internet and homes - make mathematics education different for different students. If we are back to schools, we should never be able to forget the difference of schools itself depending on the neighborhood but also on the difference of homes. These differences the corona virus has made even more visible to us.

We have used more and more videos in mathematics education during the pandemic. Mathematics videos cannot hide social issues of homes and neighborhood. With the use of subtitles you may see a sample of videos produced by students-teachers-with-mobile-phones (https://www.festivalvideomat.com/). Digital technology can contribute to make visible what may be invisible for students: different social realities of their peers. So yes, while everyone needs clean water and functioning services in today’s world they also need internet and homes that enable effective use of digital resources to be part of our collective of humans with media and things!

I end with drawing on Paulo Freire, a Brazilian educator. He has said If education alone does not transform society, neither does society change without education? We can paraphrase such a sentence! Without mathematics education for all, without understanding of exponential functions linked to the mathematics of the pandemic, can we overcome inequality? Can a collective of students and teachers with media be prepared for the next pandemic?

3. The Use of Mathematics in Communicating an Urgent Need for Action and Questions for Education (by Panelist Eva Jablonka)

3.1. Mathematics in contexts of crises

In approaching the theme of the panel on the ground of my expertise I shall focus on the role of mathematics in communicating an urgent need for action. It has been emphasised that mathematics education should address the public use of numbers, quantities, metrics or indicators – henceforth all denoted as ‘numbers’ (e.g., Fischer, 1993; Jablonka, 2003; Dowling 2009, Chronaki, 2017). In contexts of crises numbers are often used (i) to communicate the ‘reality’ behind something that happens, of which we see only some symptoms, (ii) to project what will continue to happen if no action is taken, and (iii) to predict what would happen if particular actions were taken.

Before proceeding to examine the public use of numbers, I would like to look at a fictitious situation of crisis in the story of an epidemic in a quarantined city: In the beginning of the epidemic, when regular broadcasting of the official statistics started, “[…] the reaction of the public was slower than might have been expected. Thus, the bare statement that three hundred and two deaths had taken place in the third week of plague failed to strike their imagination.” Subsequently, “[…] a new phase of the epidemic was ushered in when the radio announced no longer weekly totals, but […] deaths in a day.” When observing this practice, the fictitious chronicler comments, “The newspapers and the authorities are playing ball with the plague. They fancy they're scoring off it […].” (Camus, 1991/1947, p. 113)

Numbers still do not strike audiences’ imagination but listening to numbers via radio has become rare in 2021. The ubiquity of screens on TV, computers and mobile devices clearly has increased engagement with the visual; black-boxed mathematics in the form of visualisations of complex data and interactive tools are designed to inform non-experts (with stark differences in the quality between the so called “prestigious” and “popular” media). There has been an intensification in the use of mathematical models and speedy computations with massive data and an extended use of experts in political decision making (whose role is contingent upon the type of political system). There is a need to examine the implications for mathematics education.

3.2. An example of numbers in public discourse in Germany

The public discourse in Germany during the first epidemic wave of COVID-19 in 2020 is a useful example. Numbers featured prominently in the (changing) construction of what constitutes the crisis and in communicating an urgent need for action. The numbers were generated in a range of academic fields, such as epidemiology, infectiology, virology, demography and public health care resource statistics. Numbers included: cases, new cases, person-time incidence rate, deaths, case fatality rate, excess mortality, recovered, doubling time and effective reproduction number. In a systematic study we identified different strategies of using these numbers to defend the regulations directed to contain the spread of the virus (cf. Jablonka and Bergsten, 2021):

Rationalisation: This is the strategy one expects in a rational policy discourse that relies on experts from academic fields. It consists in the use of numbers in factual statements for providing evidence as well as in arguments and explanations, that is, elements of scientific discourse from the relevant fields. A residual strategy of rationalisation creates the impression of fact, probability and verisimilitude by means of mere repetition and compilation of numbers.

Contrast: This strategy of using numbers consists in establishing a stark contrast between numbers qualified as "high" and others described as "low", and at the same time invoking emotions on one side of the comparison, as, for example, a looming menace associated with "high" numbers. This strategy is highly adaptable in terms of supporting or opposing a particular need for action, as both the context of comparison and the selected numbers can be chosen accordingly.

Association: This strategy also appeals to imagination and fantasy, but here numbers are not compared across or within contexts; instead, particular numbers are selected and then combined with representations of concrete events, objects or behaviours seen as equivalent with overcoming the crisis or hindering this. For example, some metrics were combined with images of clubbing young people to create the association that their behaviour is responsible for "high" numbers.

Recharging: This strategy consists in presenting concrete examples of first-hand experiences, personal narratives, testimonies or individual fates "behind" the numbers. Thereby the numbers become recharged with subjectivity and materiality that have been stripped off through the mathematisation. It can be interpreted as a move towards overcoming the effect of "ethical filtration" (Skovsmose, 2006) resulting from the transformation of the handling of a complex situation into a technical problem by which moral considerations or ethical dilemmas apparently are cleared away.

These strategies of overcoming the apparent neutrality of depicting the problem by means of numbers (the kernel of rationalisation) appeared as an integral element of the public policy discourse. The discourse also included the daily presentation and production of numbers on liveblogs, infographics and interactive maps. In addition, there appeared some black-boxed mathematics, such as simulators for different epidemic scenarios or interactive tools for calculating the individual risk of infection in different environments; these fool the user into believing to be in a position of attaining definitive solutions to complex problems (cf. Gellert and Jablonka, 2009).

3.3. Examples from other contexts

Similar use of numbers that emanate into public discourse from mathematical models or simulations produced in diverse academic fields are particularly found in policy fields that relate to the United Nations Sustainable Development Goals. These often rely on complex data that can only be communicated via sophisticated visualisations designed for non-expert audiences. Space does not permit more than two examples.

The example from the German Aerospace Center (DLR) is a good illustration of a visualisation (Fig. 1) of complex data used for providing evidence (rationalisation). It relates to DLR's programme theme "earth observation" tailored "to the current and future scientific, societal, political and economic challenges posed by global processes and changes" (DLR, n.d). It shows the reduction in air pollution as an outcome of lockdown measures in 2020. Data have been adjusted for effects due to weather; one still can "see" the positive effect on air quality through comparing the images.

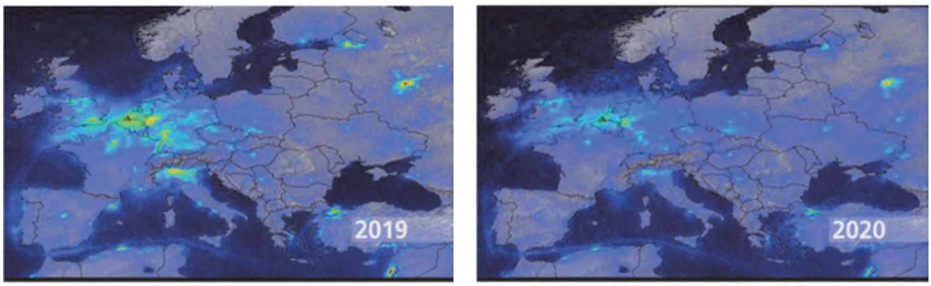

Fig. 1. DLR (2020, May 5). Comparison of nitrogen dioxide emissions over Europe between March/April 2019 and 2020, image 1(4)
Credit: DLR (CC BY-NC-ND 3.0)

The second example illustrates the use of numbers in strategies that involve appeal to affect, emotion or fantasy. In the yearly reports "Global Trends" of the United Nation's Refugee Agency with statistics on forced displacement of people that aim at enhancing public understanding, the strategy of recharging is commonly used. The reports regularly include photographs and short descriptions of individual fates. Examples in the latest report (UNHCR, 2021) include the fate of a Syrian refugee and his grandson (p. 13), an internally displaced woman in Kongoussi (p. 23); displaced people in canoes on flood water around Pibor (p. 26); the internally displaced young footballer Maria Romanchenko in Odessa (p. 28); an asylum-seeking family who fled from Honduras to Guatemala (p. 31); asylum-seekers in Congo Rive village (p. 37).

3.4. Questions for mathematics education

To summarise, in many public reports of supranational institutions, non-governmental organisations and local campaigns of policy actors who communicate the urgency of some political action we find black-boxed mathematics in form of sophisticated visualisations and strategies of communication similar to those found in the context of the pandemic. These are necessary when the data are too complex and also because numbers never speak for themselves. How should the use of numbers in communicating political priorities enter mathematics education?
Resulting from this unavoidably incomplete and inconclusive sketch of various issues related to the use of numbers, I would like to propose three questions to be asked in mathematics education as research and as practice:

(1) Where is the space in the curriculum for dealing with black-boxed mathematics in the form of visualisations?

(2) If more locally relevant topics related to Sustainable Development Goals are to be included in the teaching of mathematical modelling and statistics, how can a balance be found between simplifying the socio-political complexity of the problems and focusing on mathematically important issues?

(3) Shall mathematics still be taught as neutral tool and mathematics educators try to avoid appealing to affect, emotion and fantasy or shall it include discussions of strategies and the art of using numbers for getting a political message across?

4. Is the Future of Mathematics Education Black? (by Panelist Danny Bernard Martin)

4.1. Motivational context

My decision to frame the future of mathematics education in the United States around the question in the title of this paper is partially shaped by my positionality as a Black man in America. In this country, I belong to a group whose bottom-level status in the social hierarchy has been fortified in law, policy, practice, and reform for more than 400 years. Recurring instances of extrajudicial killings of Black people by police, the disproportionate effects of COVID-19, and efforts to disenfranchise Black voters across America serve as visible reminders of this bottom-level status. While many people across the globe may have been shocked by the recent killings of Black people by American police, or by the disproportionate impact of Covid-19 on Black people, the year 2020 was in many respects completely normal for Black people. It was not an aberration. Going back 10, 20, or even 100 years would reveal similar outcomes in many different areas of Black life.

Therefore, my thoughts about the future of mathematics education cannot ignore the realities of the past and present, and they cannot be confined to issues of teaching, learning, curriculum, and assessment. Against the backdrop of these observations, let me say a bit more about the question, Is the future of mathematics education Black? I claim that: there is no future for mathematics education in the United States if a reimagined (not reformed) mathematics education does not value the lives and humanity of Black people, contribute to their collective liberation and flourishing, and stand in opposition to white supremacy, antiblackness, and racial capitalism. if not, the future of mathematics education will continue to reflect the present and the past.

4.2. White supremacy, antiblackness, and racial capitalism

In America, the past and present reflect the fact that despite decades of curriculum and teaching reform, many Black children continue to experience dehumanizing and violent forms of mathematics education. These dehumanizing and violent experiences are rooted in systems of white supremacy, antiblackness, and racial capitalism. Invoking white supremacy, antiblackness, and racial capitalism should not be viewed as unnecessary provocation. Historical analysis shows that they are foundational to the birth of America, and they continue to undergird almost every American institution. In many ways, white supremacy is the thread that holds America together, stitching together the past, the present, and the future. Invoking antiblackness provides the necessary language to explain the particularities of Black oppression, and the willingness of American institutions to dehumanize Black people and inflict gratuitous violence. Racial capitalism helps to explain how Black pain, Black suffering, and fabrications of Black pathology continue to serve as commodities for capital accumulation. During chattel slavery, for example, the bodies of Black people were considered property and their free labor generated much of America's wealth. More than one hundred fifty years later, the marginalization of Black people in mathematics has resulted in a proliferation of 'intervention economies' focused on fixing Black learners and increasing their achievement and participation in mathematics (Martin, 2021).

The persistence of Black racial oppression in America over the past 400 years serves as a reminder that white supremacy is a self-correcting system that has resisted every attempt to dismantle it. As noted by the writer Ta'Nehisi Coates (2014), "white supremacy is… a force so fundamental to America that it is difficult to imagine the country without it. And so, we must imagine a new country." In my opinion if America cannot be reformed away from white supremacy, then the same is true for mathematics education. I can modify the Ta'Nehisi Coates from earlier to say: white supremacy is … a force so fundamental to [mathematics education] in America that it is difficult to imagine [mathematics education] without it. And so, we must imagine a new [mathematics education].

4.3. Myth of Black inclusion

Confronting the realities of white supremacy, antiblackness, and racial capitalism in mathematics education also means confronting the myths and realities of Black inclusion and the myths that are perpetuated about reform. Recent data show that after steady increases from the late 1980s to the early 1990s, the percentage of Black mathematics majors has decreased during the past 25 years and stabilized to around 5% (Bressoud, 2018). This time frame coincides with several mathematics education reforms in the United States, with many of these reforms focused on diversity and inclusion (Martin, 2019). Over the same time frame, the number of Black mathematics majors at US universities, represented by the blue curve, has remained relatively flat (Bressoud, 2018). Across these contexts, at least, Black inclusion into mathematics is a myth.

My own research and the work of several colleagues has documented other realities and material consequences of white supremacy, antiblackness, and racial capitalism for Black learners in mathematics education (Martin, 2013, 2019; Davis and Jett, 2019). In very recent work, colleagues and I discuss how Black learners encounter and negotiate various forms of violence, including epistemological, violence, systemic violence, and symbolic violence (Martin, Price, and Moore, 2019). For example, we show how knowledge production often reifies the idea of Black inferiority in mathematics by using statistics as a proxy for truth. Drawing on the work of Thomas

Teo (2010), we frame this as epistemological violence. We also show how school-based mathematical practices limit Black students' access to identities as creators and doers of mathematics, reserving these identities for white and Asian learners. We link these practices to symbolic violence.

Consider recent research by Faulkner et. al. (2014). Using data from the Early Childhood Longitudinal Study–Kindergarten Class of 1998-1999 (ECLS-K) data set, they analyzed the mathematics placement profiles of Black students and White students from late elementary school through 8th grade. In particular, the authors analyzed the impact of teacher evaluation of student performance versus student demonstrated performance on the odds of being placed into algebra in the 8th grade. Please note that in the United States, Algebra serves as an important gateway and gatekeeper course to more advanced mathematics and other prized educational opportunities. Results of the study revealed that Black students had reduced odds of being placed in algebra by the time they entered 8th grade even after controlling for performance in mathematics. The odds of placement in algebra by the eighth grade for Black students were reduced by two-thirds to two-fifths compared to their White peers. The authors concluded:

> Black students confront an untenable impediment in that their Blackness (or, as we suggest here, the teachers' implicit responses to these students' Blackness) … is an invisible… obstacle to gaining access to higher level mathematics courses, irrespective of their demonstrated performance.

In other words, meeting and exceeding standards are not enough counterbalances to antiblackness and white supremacy. Let me be clear that I am not making an argument for inclusion. If inclusion means inclusion into a system that is fundamentally anti-Black, I cannot support that. Building back better must mean more than inclusion into mathematics as it is. Rather, the focus should be on building a humane, anti-racist mathematics education free of white supremacy, antiblackness, and racial capitalism.

4.4. Racial projects and mathematics education

Why has mathematics education in the United States not stood in opposition to white supremacy, antiblackness, and racial capitalism? Let me make another claim: mathematics education reforms in the United States have always been aligned with political projects promoting white supremacy, antiblackness, and racial capitalism (along with nationalism, xenophobia, militarism, neoliberalism, etc.).

What evidence do we have for this claim? In a recent paper (Martin, 2019), I discuss how various math education reform movements in the United States have co-evolved with the prevailing political and racial projects in the larger society. For example, the new math reforms of the 1950s and 1960s unfolded while America was maintaining legalized segregation. Those new math reforms were not intended for Black Americans, and there is no historical evidence that mathematics education stood in opposition to white supremacy, antiblackness, and racial capitalism. Thirty years later, beginning in the 1980s, mathematics education was enlisted to support Reagan-Clinton-era neoliberalism and Bush-era neoconservatism focused on national security and social welfare reform. Currently, U.S. mathematics education is in the Common Core reform era and America finds itself entangled in the racial politics that led to the election of Donald Trump, the Black Lives Matter movement, COVID, white supremacist insurrection, and the transition to the Biden presidency.

Given the ongoing entanglement of racial and political projects with the project of mathematics education, what is the future of mathematics education? Rather than asking, Is the future of mathematics education Black? I suggest that the future of mathematics education in the United States must be Black. Mathematics education in the United States has no future if it does not value the lives and humanity of Black people and contribute to their collective liberation and flourishing. There must be an agenda to build Black futures and forms of mathematics education that stand in opposition to white supremacy, antiblackness, and racial capitalism.

4.5. Beyond the United States: Race and mathematics education in global contexts

Before closing this paper, I want to acknowledge that the realities of white supremacy, antiblackness, and racial capitalism in America do not map neatly onto other locations around the globe, even in those contexts where race is socially and politically significant such as South Africa and Brazil. The meanings, processes, and material consequences of race and racism are different in these three contexts. However, white supremacy, antiblackness, and racial capitalism are still very salient in all three contexts. Let me also be clear that even in those contexts where it is believed that race is not an issue because 'we have no Black people here,' critical questions can, and should, be raised. What are the implications of the existence of a far-right, conservative racial project for mathematics education in Denmark and for immigrant families and their children? How do experiences with everyday racism by Malays and Indians in Singapore, groups who occupy very different positions in the social hierarchy, play out in the context of mathematics education? How do the manifestations of caste shape mathematics education in India? What are the racialized conditions of mathematics education for Indigenous people of Australia (post White Australia policy) or the Māori in New Zealand?

5. Can Mathematics Education Help Reduce Inequality? (by Panelist K. Subramaniam)

The pandemic has caused a disruption of scholarly work and exchange that is only a shadow of the devastation it has caused in the lives of the less privileged. I will use the occasion not so much to look back on my previous work, but to interrupt it with questions that our collective experience of this disruption throws up. The question that I focus on is “Can mathematics education help reduce inequality?”. At the back of my mind is the question “What conversations do we as mathematics educators currently prioritize what conversations do we need to prioritize?" One of the sustainable development goals, SDG 10, is to reduce inequalities. The pandemic has sharpened the inequality across the world. Countries like India with deeply entrenched inequality are particularly hard hit. About 90% of the Indian workforce is in the informal economy. Many millions of Indian workers lost their jobs when the pandemic related lockdown was imposed and had to walk hundreds of kilometres to return to their home states. About 400 million Indians risk falling deeper into poverty due to the lockdown imposed in the wake of the pandemic (Oxfam India, 2021). In shocking contrast, in the period of the disaster, the wealthiest in the world leap frogged into even greater wealth. During the pandemic, the top ten billionaires in the world increased their wealth by nearly 50% over ten months (Oxfam India, 2021). In India, the top billionaires increased their wealth by 80%. Some increased their wealth multiple fold, even in comparison to their wealth before the pandemic began. Our political economic arrangements ensure that this huge suddenly acquired wealth for a rich minority bring no relief at all to those who have been impoverished by the disaster. Wealth tax is taboo and the preferred route by governments is to increase the tax on fuel further burdening those at the bottom of the pyramid. My co-panelists have already spoken about educational and social inequality. Clearly inequality, whether economic or social, will not really be overcome without democratic political struggle by those who are marginalized.

The period of the pandemic has also seen large scale protests in several countries. For example, the Black Lives Matter movement in the US, the farmers' movement in India, the movement to restore democracy in Myanmar, and other movements in several countries. Most of these movements were not related to the pandemic, but they were thrown into relief because of the extraordinary times we were living through. They served as a reminder that political movements are great opportunities for education, indeed they are opportunities for us educators to be educated. We forget easily that such movements can be occasions for mass education. How can we restore the connection between social movements and the education curriculum? What does this mean for mathematics education? In India, hundreds of thousands of protesting farmers demanded the repeal of laws that were thought to favor corporates and put farmers' futures at risk. The movement shone the torch not only on issues related to farm income, but also on issues like ecological degradation, and the control of the public discourse by media manipulation. These are important issues for education that is aimed at transforming society.

In developing countries with widespread inequality and poverty, the agenda of social transformation becomes one of the primary aims of education. Ambedkar, who came from the oppressed Dalit caste and is regarded as the architect of the modern Indian constitution, emphasized the role of education in emancipation from caste oppression: "Coming as I do from the lowest order of the Hindu society, I know what is the value of education. The problem of raising the lower order is deemed to be economic. This is a great mistake…. The problem of the lower order is… to create in them [a] consciousness of the significance of their lives for themselves, and for the country, of which they have been cruelly robbed by the existing social order" (Quoted in Velaskar, 2012).

Education, as Ambedkar says, has a great role in social transformation. But what about mathematics education? How can mathematics education help meet the goal of reducing inequality and of social transformation? Of course, mathematics opens up the pathways to highly valued jobs. To break out of the rigid occupational structures is an important aspect of social transformation and mathematics education can create pathways for such occupational mobility. Ambedkar emphasized not only material gain from education, but also gaining self-respect, and self-understanding. Critical mathematics education researchers from many different places have shown how mathematics can help sharpen our perception and understanding of inequality and the structures that underpin it (Gutstein, 2016; Rampal, 2015). However, in the existing curriculum in India and many countries, there is very little of mathematics that can illuminate social reality.

In the last round of major curriculum reform in India in 2005, efforts were made to introduce critical mathematical perspectives in the curriculum. For example, a "problem" from the Grade 5 textbook, discusses the wages paid to a couple who are farm labourers. It is mentioned that the legal minimum wage is 71 rupees per day. However, the man is only paid Rs 58 and the woman Rs 55 per day by the landlord. The problem asks students to find how much money the couple will earn for a certain number of days. More importantly for our purposes, there are two supplementary "discussion" questions in the problem. One points out that the landlord pays less than the minimum wage. Another points out that the woman is paid less than the man and asks students to discuss these observations. Firstly, even the inclusion of such questions that take a critical look at society in a mathematics textbook is remarkable and rare in the Indian context. Second, Shikha Takker found that teachers often omit discussion of such questions for various reasons (Takker, 2015). On one occasion, a teacher told Takker that she would like to focus on the mathematics and avoid the distraction that such questions entail. On another occasion, the teacher said that she did not feel equipped to deal with the discussion that might ensue if these questions were brought up. We can also imagine that there is resistance to raise questions with political undertones, which in certain situations may carry a risk for the teacher. But as mathematics education researchers, we need to ask if we can we continue to ignore these dimensions if we are serious about reducing inequality.

The mathematics involved in the examples discussed above about wealth and incomes is simple: at the most, finding and comparing percentages and ratios. Their application to understanding social reality can however be powerful. And even though the mathematics is simple, the socio economic and social political concepts involved may be sophisticated. Sometimes, the emphasis on "important mathematics", which is echoed in many curriculum reform efforts, can lead to giving less importance to the applications of mathematics which can bring about a critical focus on social reality (Noronha and Soni, 2019).

Finally, let me summarize some questions that these reflections have given rise to. Do we need to revisit our notion of what constitutes mathematics to allow for socially meaningful questions to be raised? Why does the use of mathematics to understand social reality appear as a distraction? Do our notions of "powerful mathematics" conflict with powerful uses of mathematics? Why does our curriculum not make any connection to social movements? Finally, while the mathematics curriculum typically makes at least some connections with the science curriculum, why does it make so little connection with the social science curriculum?

6. Concluding Remarks (by Panel Chair Mellony Graven)

The aim of this paper, emanating from our panel, is to continue to provoke conversation about critical issues in our current world that require urgent attention by mathematics educators. As panelists we are committed to contributing towards a more equitable world and grappling with the way in which mathematics education can contribute meaningfully to this goal.

Marcelo Borba raised the importance of placing the spotlight on the role of media ('and things') and the dangers that come with the domestication of media without maximizing the full potential of different media. Existing unequal access of different groups to increasingly essential media and technological devices need to be urgently addressed to avoid compounding inequalities. He shared a range of ways in which he and other colleagues have drawn on digital media and animations in ways that allow students to understand more deeply the emerging pandemic data (particularly exponential data) and the meaning of emerging phrases such as 'flattening the curve'. He urges mathematics educators to consider how we might prepare students and teachers with media for the next pandemic.

Eva Jablonka focused on the role mathematics plays in communicating the need for policy measures. She reveals different strategies used in public discourse to overcome the seeming neutrality of depicting with numbers and communicate the urgency for action. Since some black-boxed mathematics occurs to simplify complex data and is then used to communicate and sway public views and actions she argues that we need to ask how the use of numbers in communicating political priorities should be included in mathematics education, how we should manage simplification of socio-political complexity of problems and the mathematics involved and whether we should discuss the ways in which strategies push for action or continue to present the myth of mathematical neutrality.

Danny Martin drew on his work and experiences, positioned as a Black man in America, to argue that mathematics education cannot ignore the racist realities of the past and present that are endemic across institutions. He argues that there can be no future for mathematics education in the United States without a reimagined mathematics education that values the lives of Black people and opposes white supremacy, antiblackness and racial capitalism. He distinguishes reimagined from reformed, drawing on research to show that reforms in mathematics education have themselves contributed to entrenching inequalities and injustices in opportunities for

Black Americans. He emphasises this is not limited to the United States, arguing the myth of Black inclusion is relevant and important across first world and developing countries.

K. Subramaniam highlighted the pandemic as an opportunity to interrupt earlier assumptions about priorities in mathematics education and ask what must be prioritised now for reducing rapidly growing inequality. He notes that since the pandemic 400 million Indians are in deeper poverty while the top billionaires increased their wealth significantly (as did the top billionaires in the world). He reflects on the way in which 2005 curriculum reform in India highlighted developing a critical mathematics perspective and yet even in the few resources that include problems with data pointing to social injustices teachers tend to avoid such discussions. He argues that we cannot continue to ignore developing a critical perspective through engaging with mathematical and other data that highlights the need for prioritizing a more equitable society and world.

We hope this paper has stimulated thinking about how the issues raised relate to the contexts in which you are working. Our wish is that the questions we have asked will provoke further conversation and action in responding to the challenges to build a better future.

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https://doi.org/10.1142/9789811287152_0015

Plenary Panel 3

Pandemic Times: Challenges, Responsibilities and Roles for Mathematics and Mathematics Education Communities

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Nelly León Gómez[4], Jean Lubuma[5], and David Wagner[6]

ABSTRACT The Plenary Panel 3 at ICME-14 was especially devoted to the COVID-19 pandemic. Its goal was to review and reflect on the challenges raised by the pandemic, the responsibilities and roles for mathematicians and mathematics educators in this context, and to draw some lessons for the future. This text begins with the presentation of the panel and the four invited panelists. Then each panelist synthesizes her/his contribution to the panel, and we end by some lessons drawn from these contributions and the exchanges between the panelists and with the audience.

Keywords: COVID-19 pandemic; Mathematics; Mathematics Education.

1. Introduction

Our lives, our educational systems and our societies have been turned upside down by the COVID-19 pandemic. The goal of Plenary Panel 3, late addition to the congress scientific activities, was to review and reflect on the challenges, responsibilities and roles for mathematicians and mathematics educators in these pandemic times, to identify possible synergies between communities that can assist them in this context, and to draw some lessons for the future. Its co-chairs, Michèle Artigue and Ingrid Daubechies, respectively former president of ICMI and IMU, invited the contributions of four panelists, two mathematicians and two mathematics educators, living and working in very different environments, with a wide range of expertise and experiences.

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The mathematicians were Timothy Gowers from the Collège de France in Paris and from the University of Cambridge in the UK, and Jean Lubuma, then working at the University of Pretoria in South Africa. Gowers is a well-known mathematician, specialist of combinatorics, who was awarded the Fields Medal in 1998. He has long been interested in societal problems and popularization of mathematics, at many levels, and with respect to the Covid-19 pandemic, his writings influenced UK government decisions. Lubuma is an applied mathematician, fellow of the African Academy of Sciences and member of the Academy of Science of South Africa, who has done extensive epidemiological research in Africa, to provide a sound mathematical analysis and elaborate realistic solutions to (re-)emerging diseases such as COVID-19, Ebola, HIV/AIDS, Malaria, etc. that pose a threat to development of the continent.

The mathematics educators were Nelly León Gómez from the Universidad Pedagógica Experimental Libertador (UPEL) in Maturín, Venezuela, and David Wagner, from the University of New Brunswick in Canada. León worked for more than 40 years at UPEL as a teacher educator and researcher in mathematics education. Her responsibilities in the Inter-American Committee on Mathematics Education (IACME) and the Mathematical Education Network of Central America and the Caribbean (REDUMATE) make her especially knowledgeable about the problems generated by the pandemic in Latin America. Wagner is a researcher in mathematics education whose research especially addresses the cultural nature of mathematics and the impact of mathematics teaching practices on individuals and society. Co-editor-in-chief of Educational Studies in Mathematics, he was co-editor of the special issue "Mathematics education in a time of crisis – a viral pandemic", in preparation at the time of ICME-14.

Taking into account the diversity and complementarity of expertise and experience of the four panelists, the co-chairs prepared a specific set of questions for each of them, the exact formulation of which was discussed with them (see sections below). A collective question was added regarding the synergies observed between mathematics and mathematics education in these pandemic times. During the panel session, the co-chairs first presented the panel organization and introduced the panelists; this was followed by the videos recorded by each of the panelists in response to the specific questions asked to them. Next came an interactive phase in which the panelists exchanged views among themselves, with the co-chairs and with the participants attending the session in Shanghai.

Sections 2 to 5 are devoted to the panelists' contributions. We follow the order of their presentation at ICME-14 and each section starts with the questions especially addressed by the panelist. Section 6 draws some main lessons from this panel.

2. Teaching, Learning and Research in Mathematical Sciences in Pandemic Times. *Panelist: Jean Lubuma*

Part of your mathematical research has been focused on questions related to epidemiology. How would you describe the role of mathematicians in this field? Does the current pandemic present particular challenges? If so, how *are your colleagues and yourself dealing with them? Which of these do you find most difficult?*

We describe the role of mathematicians in disease epidemiology and highlight some challenges presented by the COVID-19 pandemic. We outline how we dealt with them at the University of Pretoria (UP), and we make a call to the international community to invest in teaching and research in mathematics in developing countries, particularly in Africa that counts the top 10 poorest countries in the world.

2.1. The role of mathematicians in disease epidemiology

The role of mathematicians in the epidemiology of human infectious diseases is to develop, analyze and simulate realistic mathematical models for gaining insight into the disease transmission dynamics and control. The use of mathematics in epidemiology goes far back to the middle of the eighteenth century, with the pioneering work of Bernouilli (1760) on modelling the effectiveness of immunization against smallpox. Two centuries later, Ross (1911), who received the Nobel Prize of Medicine in 1902, presented the first mathematical model for malaria transmission, and showed that the disease could be effectively controlled or eliminated if the malaria vector population is reduced below a certain critical threshold. In a series of seminal papers, Kermack and McKendrick (1927) and Macdonald (1957) further developed and formalized Ross' work into the theory of epidemics and laid the foundation for compartmental modelling. In recent years, there has been a strong focus on the modelling of emerging and re-emerging infectious diseases of public health significance; see Castillo-Chavez et al. (2002).

The threshold theory is expressed in terms of the basic reproduction number, $\mathcal{R}_0$, an important epidemiological threshold quantity, defined as the average number of secondary infections generated by a single infectious individual (during his or her entire infectious period) if introduced into a wholly susceptible population. The mathematical definition and the methodology for practically computing $\mathcal{R}_0$ are due to Diekmann et al. (1990) and van den Driessche and Watmough (2002): $\mathcal{R}_0$ is the spectral radius of the associated next generation matrix, K, of the model being studied. For typical models, the threshold theory is stated in the next theorem that is illustrated in Fig. 1 for $\mathcal{R}_0 = 2$; see Gumel (2021).

Theorem 1: *The disease-free equilibrium is locally asymptotically stable if* $\mathcal{R}_0 < 1$, *and unstable if* $\mathcal{R}_0 > 1$.

It follows that, for a vaccine-preventable disease, a fraction $p > 1 - 1/\mathcal{R}_0$ of susceptible individuals should be immunized (assuming a perfect vaccine) against the disease, to achieve herd immunity.

2.2. Challenges and COVID-19 modelling

Despite the rapid effort made by scientists to isolate the SARS-CoV-2 virus (the causative agent of the COVID-19 pandemic), to sequence it, to develop a diagnostic

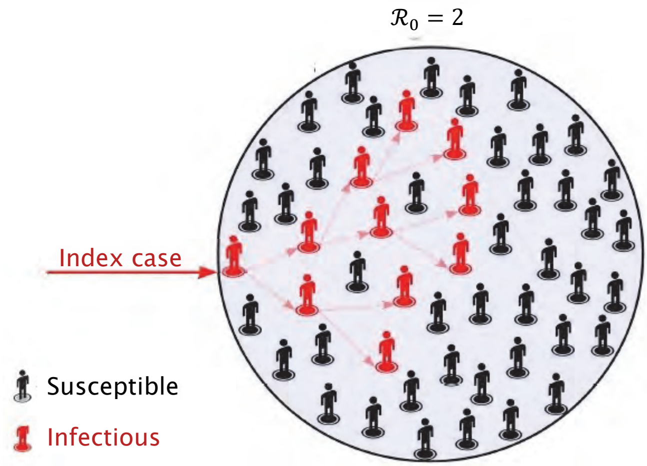

Fig. 1. If $I_{n+1} = K^n I_0$ with I the infective variables and K the (typically positive) next generation matrix, the number of secondary infectious per generation, n, is $\mathcal{R}_0$, the largest eigenvalue of $K \geq 0$. For $\mathcal{R}_0 > 1$, the final state/size of I is $I_\infty = 0$ (epidemic) or $I_\infty > 0$ (endemic).

test and to produce vaccines, numerous unknown and open questions and challenges linger around the COVID-19 pandemic. Some of the challenges are linked to the following points: social-cultural dynamics of transmission, comorbidity *vs* forgotten diseases, new variants of the virus, concerns around vaccines, and lockdown *vs* economy recovery. Regarding the lockdown, the suspension of face-to-face teaching in Africa practically means no teaching and learning since the continent is not well digitalized to offer online teaching, which is taken as granted in Europe, North America, etc. In addition to the challenge of funding research in African countries, the pandemic paralyzed collaborations between dedicated individual researchers/groups and overseas partners. Nevertheless, the pandemic brought some interesting points of contact, interactions or synergies at UP. There are ongoing weekly town hall meetings where science and mathematical communities share their experiences on issues such as staff/student mental health, online teaching, assessment, invigilation, student success, research and postgraduate supervision.

Given these challenges and the complexity of the societal problem at hand, mathematicians have adopted the transdisciplinary research approach. We developed a model for the spread of COVID-19 in South Africa, see Garba et al. (2020). It is an extension of the SEIR model, modified by adding *A, J* and *P* respective classes of asymptomatic, isolated individuals and contaminated environment, thereby considering direct and indirect transmissions. The flow diagram of the model is given in Fig. 2.

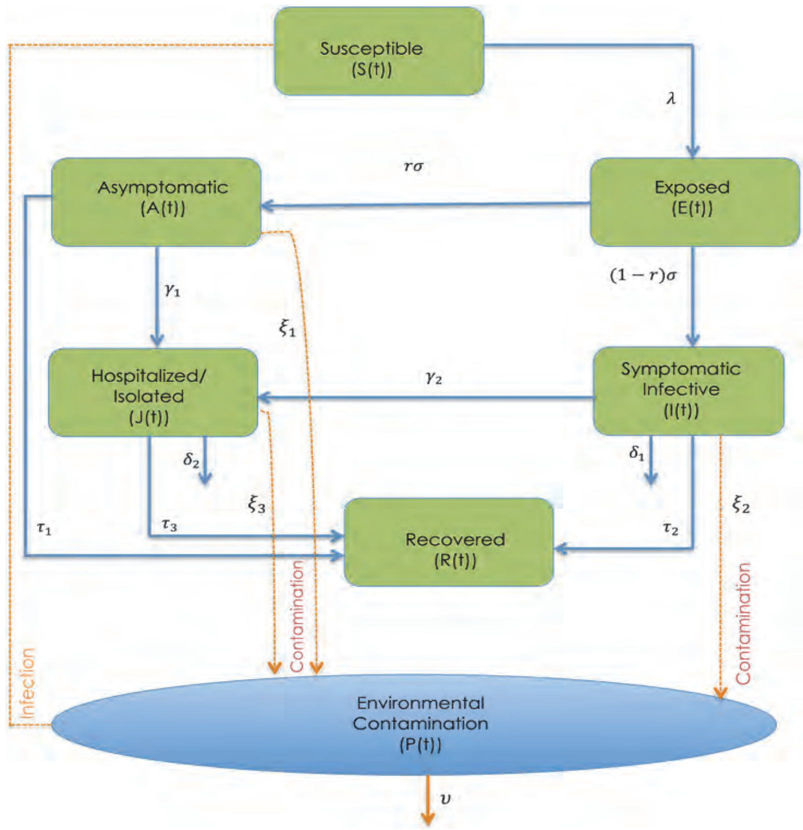

Fig. 2. Susceptible-exposed-asymptomatic-symptomatic infective-isolated-recovered & contaminated environment model

Our findings are summarized below. The model fitting with the number of deaths for the first six months of 2020 is excellent. We obtained reliable predictions in terms of peak times, numbers of cases and deaths. The continuum of disease-free equilibria is globally asymptotically stable when $\mathcal{R}_0 < 1$. Thus, the disease will eventually die out, particularly if Non-Pharmaceutical Interventions (NPIs) are implemented early and for a sustainable period of time. Further, the control reproduction number was estimated to 2.8 (3). Hence, 64 (67) % of the population should be vaccinated to achieve *herd immunity*, which is consistent with South Africa government predictions.

2.3. Conclusion

We initiated this work at the University of Pretoria. Since its emergence two years ago, COVID-19 has posed serious challenges, which led to some synergies between different mathematical communities as well as to transdisciplinary research (e.g., Google captures 36 700 000 mathematical articles). However, the challenges in teaching and research in Africa have been exacerbated by the lack of sufficient investment in this sector. These facts were echoed by the Heads of Governments at the

Summit on Financing African Economies that was held in Paris on 18 May 2021. Therefore, mathematicians should speak out, advocate and make a call to the international community to invest in education and research in mathematics in Africa. Our ongoing research includes COVID-19 models with focus on multiple strains of the virus, comorbidity issues and vaccination intervention.

3. Challenges and Lessons Concerning Mathematics Education Brought by the Pandemic in the Latin American Context. *Panelist:* Nelly León Gómez

The pandemic has forced most teachers to make a very sudden transition to remote teaching; at present, mathematics teaching is still done remotely, or at best in some hybrid mode in many countries.

What are the principal challenges this situation has brought in your country, and more generally your region? How have these challenges been met? Which lessons for the future would you distill from this experience?

3.1. Introduction

The COVID-19 pandemic has plunged the entire world into a spiral of complex unprecedented situations that have affected the lives of human beings in all their dimensions: personal, family, social, emotional, economic. In particular, education has been greatly affected, especially in the less favored regions, as Latin America.

Below I will refer to some challenges faced and lessons distilled from this crisis in the Latin American region. These are supported by reports required from the national representatives of REDUMATE (Mathematics Education Network of Central America and the Caribbean), on the impact of the epidemic in schooling in our context; the policies, limitations and innovations to face this situation and to what extent families have been able to collaborate. I also followed the results of research and experiences published in the special issue of *Journal on Research and Teachers Preparation in Mathematics Education* entitled “Mathematics Education and the pandemic in the Americas” https://revistas.ucr.ac.cr/index.php/cifem/issue/view/3099

3.2. Principal challenges and how these challenges have been met

When the COVID-19 pandemic began in March 2020, one of the first actions taken in Latin America, as in the rest of the world, was to close schools and move to a distance education modality as a way to contain the spread of the virus. This measure had to be accompanied by others to guarantee school continuation, such as supporting food, health and the biopsychosocial well-being of children and young people. By abruptly switching to distance learning, I believe the main challenge was *how to deliver the teaching of Mathematics to every student guaranteeing quality and equity of mathematics learning in non-presence contexts*. No one was prepared to deal with this challenging situation. Although each country implemented action plans to face the challenge, each according to its own social, political and economic reality, I would say that it has not been fully met in Latin America. Multiple factors have had an impact on the development of these plans. These include the following:

3.2.1. Availability of technological resources

The conditions of accessibility to technology have meant a serious limitation for remote education in Latin America. A significant number of students and teachers do not have access to a computer, smartphone and internet connection, and many of them do not even have a TV set. According to the World Bank (2021) at the beginning of the pandemic, less than 43% of primary schools and less than 62% of secondary schools in Latin America had access to the internet for educational purposes. Among the online distance learning modalities, virtual asynchronous learning platforms were the most prevalent in the region of Latin America and the Caribbean; only 4 of the 29 countries offered live classes. (CEPAL – UNESCO, 2021) In addition, other distance learning solutions were deployed to bridge the gaps between schools and learners, such as broadcast educational programs by more traditional mass media such as radio or television, audios and videos (WhatsApp, e-mail) and printed materials.

3.2.2. Teacher's preparation and their willingness to take on the challenges of non-face-to-face mathematics education

There is evidence of a lack of teacher preparation for an efficient application of technological tools in distance learning (beyond the use of these tools to maintain communication between students and teachers); in addition, artifacts and methods were used that were not entirely appropriate to the didactic transposition and the assessment of mathematics in this setting. Besides, some teachers feel that preparing a non-face-to-face Math class requires extra time, which they do not have because, in addition to teaching, they must do other tasks to supplement their low wage income.

3.2.3. Curricular adjustments

The adoption of distance or blended teaching modalities has created the need to identify key points of the curriculum on which to focus the attention of educational action. Content prioritization was necessary; consequently, the coverage of the actual mathematics curriculum is far from the expected standards, especially in public schools. This will cause the gap in mathematical skills between children from lower and higher socioeconomic background to get wider in Latin America.

3.2.4. Engagement of parents and families

Family support has been key to guaranteeing the continuity of education during the pandemic. In a depressed socio-economic context, numerous parents have faced difficulties in terms of their abilities and availability to support their children in learning mathematics and in the use of technology. Apart from family support, in some cases the children themselves have had to do informal work to help with the household economy leaving aside their studies, increasing the risk of dropout (León, 2021)

3.2.5. Motivation and emotional issues

The lack of direct interaction between students and teachers has increased the risk of disengagement especially for students who do not have access to online education. The stress of confinement, living conditions, restrictions on entertainment activities with friends, and the increase in physical abuse have greatly affected the emotional health of students. For this reason, regional government plans, such as "Every Family a School" in Venezuela, include a component of psycho-emotional care for families (MPPE, 2020).

3.3. Lessons distilled from the pandemic

The experience of facing the COVID-19 crisis has left important lessons to take into account in the new post-pandemic educational reality.

Lesson 1: The educational world has been digitized to large extent; as a result better versus worse access to digital communication channels translates into more versus less educational quality, inclusion and equity. Therefore, a strong investment in education is required in Latin America to address the technological gap. Such investment must be aimed not only at the equipment itself and at the improvement in connectivity, but also at the initial and on-service preparation of teachers to face the challenge of designing online tasks to engage students with mathematical content, while making use of such technological resources.

Lesson 2: It is important to capitalize on the technological push in education and the alternative forms of remote education developed during the pandemic. The use of digital technologies has generated new ways of thinking and representing mathematics, its teaching and assessment. The pandemic has also left us with a wide variety of innovative virtual resources created to enhance the teaching and learning of mathematics, focused not only on content but also on skills and competencies.

Lesson 3: Clear guidelines need to be established for the review of school contents and approaches in Mathematics in order to identify the mathematical knowledge and skills that will actually be required in the post-pandemic reality and to cover the knowledge gaps stemming from school closure, taking into account both the depth and variation in Math learning loss. This will take time and it will be necessary to act with prudence taking into account issues of equity and social justice in making quality mathematics education available to all.

Lesson 4: The classroom is not the only place for learning Mathematics. The closure of schools has generated interesting changes in the interaction model and has opened a range of possibilities that should be exploited to supporting students' independent and significant learning of Mathematics. Homeschooling is an experience not to be dismissed, since the new forms of relationship between school, family and community have led them to assume shared responsibilities regarding children's learning, and this must continue going on.

Lesson 5: No matter the form (in-person, remote or hybrid), teachers have to find the ways of teaching Mathematics that take into account the social and emotional needs of students, in order to maintain or to raise the expectation in their students, before launching into mathematical content.

4. Roles and Responsibilities of Mathematics Educators. *Panelist:* David Wagner

You are a researcher in mathematical education. How would you describe the role and the responsibilities of the mathematical education research community in the context of the current pandemic?

You are co-editor of a special issue of the journal Educational Studies in Mathematics centered on the pandemic and its challenges. What are the main messages you have taken away from this experience?

This pandemic has taught me that the responsibilities of mathematics educators in times of crisis should consider our complex arrays of responsibilities, from local, immediate challenges to the big questions. We need to devote time to love the people in our family and communities. We need to recognize the physical suffering, the isolation, and the compelling demands to care for others in new ways. In our teaching roles, we help students achieve their immediate needs, even when we think those needs are crazy—crazy demands from a crazy society. In our research, we study the local, immediate needs but also look at the big questions, and examine the structure beneath the crisis—the invariant things.

The coronavirus is probably not the most significant destructive force of our era. The social fabric of our world, the power structures we humans have erected and maintained, and our deep manipulation of our physical environments have been more destructive. These forces probably set the stage for the virus to be born, and certainly for the virus to multiply as it has and for the social chaos that ensued. The pandemic is an entanglement of the virus, the socio-political vectors and the environmental landscapes, all acting together. Thus, to answer the question about the responsibilities of mathematics educators in this pandemic, I step back to generalize and consider our responsibilities in crisis. We knew crisis was upon us before the pandemic. In 2013, crisis theorists Topper and Lagadec pointed to the human environmental footprint and the increasingly interconnected world and concluded: “major events are not new, but they have got denser” (p. 6). We live in a volatile world. Mandelbrot applied fractal geometry to volatile financial markets (Topper and Lagadec, 2013). Using this approach in the current crisis I see that we have to look through the massive changes and upheaval to examine what lies beneath—structures that have not changed.

When I consider the 161 articles received from our call for papers for the special issue on the pandemic in *Educational Studies in Mathematics* (Chan et al., 2021), and the thoughtful reviews of these papers, I see mathematics educators in action, responding to crisis. My co-editors, Man Ching Esther Chan and Cristina Sabena, and I saw that the pandemic has challenged usual patterns of research in our field. Many scholars were pressed with other demands and unable to focus on their research plans or on their reviewing commitments. Others found themselves with more time than before. Not surprisingly, the disparities appear to align with and magnify existing disparities such as gender disparities. This is an example of something that stays the same while we feel like everything is changing.

We saw scholars looking for accessible data that would help the field understand the pandemic. This was not easy. The pandemic makes it hard to start new studies involving participants because human interaction has been restricted. This is a major problem because we know that studies of social structures really need researchers to listen to the people most impacted by the structures. Over time, I expect that we will see more research that uses data that is harder to access, with deep engagement with the people most impacted by the crises.

Some researchers contributing to the special issue saw the pandemic as a prompt for questioning school curriculum. To identify what mathematics is needed by citizens to make sense of the crisis, researchers looked to public dialogue for analysis. We have seen governments, citizens and special interest groups disseminating graphs and statistics to explain pandemic events and inform action. Some of the research groups asked what mathematics is needed to make sense of these statistics (e.g., Kwon et al., 2021). We see that this does not exactly answer the question of what curriculum is needed. This is because the forms of statistics representation chosen by governments and others is guided by what they think citizens will understand. There is circularity when people ask what mathematics to teach based on what mathematics is being used. We need our field to identify new priorities for school mathematics based on analysis of the major challenges faced by society and individuals in the current age.

Some mathematics teachers have seen the pandemic as a prompt to re-examine their teaching. Surely, students should not accept a focus on the usual skills and knowledge when the world's habitual ways have clearly spelled catastrophe. One should expect that supposedly powerful mathematics would be used in class to address the most obvious disruption of our era. I would expect a call from students and from society, echoing the decades of injunction from Ubiratàn D'Ambrosio to examine the complicity of mathematics in the structures that allowed the virus to thrive in addition to the possibilities for using mathematics for justice in these times. However, speaking from my own experience and conversations with teachers, I see people distracted from asking deep questions—distracted by our social systems and the immediate needs of disrupted networks. Students and teachers focus on their compelling, immediate, local needs. While many are distracted, some educators are asking the bigger questions and trying to take the crisis seriously with their mathematics teaching. The special issue received articles that tell of ways mathematics teachers have addressed disease spread by studying different mathematics (Maciejewski, 2021), mathematics teachers' changing conceptualizations of their teaching due to isolation restrictions (e.g., Albano et al., 2021), and analysis of historical conversations about vaccination (Gosztonyi, 2021).

Many of the manuscripts we received studied distance teaching via digital and other technologies. This has been studied in our field for decades but now more scholars are interested. Borba (2021) reminds us that technologies of distance learning need to be seen as entangled with their contexts. One thing that is immediately clear in pandemic teaching is the inequities—unequal access to internet, unequal access to computers and tablets, unequal home infrastructure for uninterrupted time, unequal competing demands for time. Even while teachers and school systems work very hard at combatting them, these inequities persist. This is another example of something that is invariant in this time of massive change. We see that rural families have greater challenges (Yılmaz et al., 2021), and the needs of Indigenous students (Allen and Trinick, 2021), students of colour (Matthews et al., 20221) and students who have recently migrated are ignored in this crisis. The effects of poverty are magnified.

So I ask again, what are our responsibilities as mathematics educators? To answer this question, we need to answer other questions. What should every citizen know? Surely the answer is different than it was thirty years ago, considering the massive changes in interconnectivity in our world. To particularize this question, we need to identify the human and social problems of our time: What mathematics is necessary to understand interconnectivity? What mathematics is needed to understand climate? What mathematics is needed to understand biodiversity? What mathematics is needed to understand wealth distribution? With such questions I see deeper question: Should school mathematics focus on learning the useful algorithms when they can be performed instantaneously on handheld devices that are ubiquitous, or on applications to actual human problems? I know that this is an open question for many people, but I suggest that the public's evidently poor understanding of the science and mathematics of the pandemic may lead us to question the value of focusing school mathematics as we have in the past on procedural skills.

5. What Challenges Has the Pandemic Raised for Mathematics Education? *Panelist: Timothy Gowers*

The pandemic has put mathematics in the spotlight, and the media have solicited mathematicians, even those not experts in epidemiology. You have long been committed to communicating mathematical concepts to a wide audience.

Can you comment on challenges that are specific to the pandemic context?

What can we learn from initiatives realized by the mathematical community?

It is obvious that mathematics and the pandemic are closely intertwined. There is an entire academic discipline, epidemiology, devoted to the study of how diseases spread, and a significant component of that discipline is mathematical.

It is perhaps slightly less obvious that the pandemic raises challenges for mathematics education, but after a moment's reflection one can identify two important ones, at least if one takes "education" in a broad sense that moves beyond the confines of the classroom and takes in more general dissemination of mathematical ideas. The first is to improve mathematical understanding in society at large, so that people are better able to judge the reasons for and likely effects of the painful restrictions that have been imposed. The second is to improve the mathematical understanding of the principal decision makers, so that the decisions they make, which can have huge consequences, are made as rationally as possible. These two challenges are closely related, since it is much easier that politicians will make the right decisions if they can count on public support and understanding.

5.1. Public understanding of pandemic-related mathematics

Over the last two years, many people have made extraordinary sacrifices. Some have argued that these sacrifices were largely unnecessary, or at least that the benefits were outweighed by the costs. And while it is certainly right to weigh up the costs and benefits, to do so properly requires an appreciation of a few basic mathematical principles.

5.1.1. Exponential growth

To many non-mathematicians, the word "exponentially" is little more than a synonym for "quickly". So if they are told at the early stages of a pandemic that the case numbers are growing exponentially, but after two or three weeks the numbers are still small, they may wonder whether they have been misled. A more serious problem is that it is very hard to persuade people to accept significant restrictions while numbers are still small. The argument for doing so is that the earlier one imposes restrictions, the less time it takes to reduce case numbers to a level where the restrictions can be relaxed again, and the less illness and death there will be as well. But to understand this properly, one needs mathematics – not the sophisticated mathematics of a professional epidemiologist, but just the basics of exponential growth. Without such an understanding, one may be swayed by arguments such as "More people were killed by lockdowns than by Covid." In many countries that is clearly untrue, but in a country such as New Zealand, which had several lockdowns and very few COVID-19 deaths, it is almost certainly true. Does that mean that New Zealand made a big mistake? No, because the relevant comparison is between the number of deaths caused by lockdown and the number of COVID-19 deaths that there would have been without lockdown.

The point I am making here is not so much the arguments for and against lockdowns and other restrictions. Rather, I am arguing that greater public understanding of mathematics can lead to a greater willingness to accept policies that are for the good of everybody, but which are somewhat counterintuitive.

5.1.2. The role of data and modelling

Mathematical modelling is of central importance in epidemiology, but the role of modelling is not well understood by the general public. If a simulation is run on a computer, for example, it will be based on assumptions, which themselves will be based on data that is usually uncertain and incomplete, especially in the early days of the pandemic. Thus, predictions are typically conditional, but they are often presented as unconditional by journalists. Furthermore, models can be self-defeating in the following sense: if a model makes an alarming prediction, politicians may well impose restrictions that stop the alarming prediction from coming to pass.

These phenomena, which are absolutely normal and expected, have led in the UK to considerable public distrust of modelling as a discipline. This is a problem not just for managing the pandemic, but also for other areas where modelling meets public policy, an obvious example being climate change.

5.1.3. Probability and risk

Another aspect of policy making that was underappreciated by the general public was the role of uncertainty. In the early stages of a rapidly developing pandemic, decisions had to be made quickly when many facts were unknown, of which the most important was how COVID-19 spread. When one is weighing up the costs and benefits of a possible decision under these circumstances, they necessarily come with a probability attached, or more precisely a probability distribution.

A simple example of this was the question of whether mask mandates were a good idea. Early on in the pandemic, the evidence for beneficial effects of mask wearing (mainly in protecting others from the wearer) was weak. To a non-mathematician, it might seem an obvious consequence of this that there was only a weak case for encouraging the wearing of masks. However, because of the nature of exponential growth, a small reduction in the growth rate is hugely beneficial, whereas the cost of widespread mask wearing is small. So even if the reduction in the growth rate was not certain to occur, the expected net benefit of mask wearing was large.

5.2. Trust in science

One of the great potential benefits of better understanding of mathematics among the general public and politicians would be a healthier relationship with scientists. I have given several examples already of how a lack of understanding of mathematics can lead people to lose trust in science, and this has been a serious problem.

Another problem, which I have not yet mentioned, concerns the relationship between politicians and scientific advisors. A politician with a good mathematical understanding can understand not just the advice but the justification for the advice; a politician without it simply has to accept or reject the conclusion. It was extraordinary as a British person to see Angela Merkel, who has a scientific background, giving a beautifully clear explanation of exponential growth and how it informed her decisions, and to contrast that with Boris Johnson, who does not have a scientific background, simply saying in a vague way, "We are following the science," (though he said that less as the pandemic proceeded).

5.3. What can we do?

While one can see the benefits that improved mathematical understanding would have, it is less clear how any improvement can be achieved. The raising of standards of mathematical literacy is of course a central aim of mathematical education, so to call for it may seem a little pointless. However, my conclusion is a little more specific, since some parts of mathematics have a much bigger effect on the quality of public decision making than others. The basics of probability and statistics, for example, are clearly important for conducting risk-benefit analyses, whereas an understanding of polynomials, while essential for any technical uses of mathematics, is less necessary for public appreciation of political decisions.

So a potential way forward is to identify those areas of mathematics that would be most helpful for improving public discourse and decision making, and to think of creative ways of explaining them to non-expert audiences. That is still a big challenge, and maybe for many people it is simply too late to get them interested. But at the very least one could think about how best to bring up a new generation to be better educated in these aspects of mathematics than the current generations are. A natural idea to try is to design school mathematics courses that are principally aimed at people who will not be specializing in STEM subjects. Such courses could analyze current events from a mathematical perspective, giving people the tools to think about them more effectively. A well designed course of this kind would have the potential to demonstrate that mathematics is, to use words of Jordan Ellenberg (2014) "like an atomic powered prosthesis that you attach to your common sense, vastly multiplying its reach and strength". In that way, it might appeal to people who would otherwise remain unaware of its benefits to individuals and to society.

6. Reflections and Lessons

We often complain that despite their crucial role in our technological societies, mathematics remains invisible. This is no longer the case. The COVID-19 pandemic has shown the importance of mathematical models for understanding the course of the pandemic, anticipating and weighing possible consequences of different policy decisions, or managing and analyzing the very large volumes of data collected. It has shown the potential but also the limits of these models, the need to constantly adjust them due to the emergence of variants, the effect of decisions taken, etc. It has confronted the general public with science in the making, made up of questions and doubts, as opposed to the image of certainty that many had formed during their schooling. It has shown the need for a solid and shared mathematical and scientific culture within our societies, a culture that is largely lacking as shown by the permeability of the population to the incredible fake news stories that have multiplied. Furthermore, the sudden shift to distance and, at best, hybrid teaching has profoundly destabilized education systems that were totally unprepared for it. Educational inequalities have been exacerbated between countries and between pupils within the same country. The challenges are immense!

The panel has made clear that, faced with these challenges, the mathematics and mathematics education communities have mobilized strongly, each with its own expertise, means and fields of action. They have mobilized at the level of research and practical action. In addition to epidemiological mathematicians, such as Jean Lubuma, who are directly involved in research on this pandemic, many mathematicians, as Timothy Gowers has shown, have helped the general public and the media to make sense of epidemiological models, of the growth processes and probabilistic reasoning modes involved. The world of mathematics education has also reoriented its research to meet the challenges encountered, as David Wagner has shown, and beyond research, it has invested heavily in the production and sharing of online resources for teaching and training. Solidarities have been strengthened or created, as Nelly León has shown.

From this point of view, the panel also carries a message of hope. This message is all the more necessary as the current pandemic is not an isolated crisis. We will face, and are already facing, other crises, undoubtedly even more serious and lasting, such as those associated with climate change. The need for quality mathematical and scientific education for all is essential, and taking up this challenge requires the synergy of the strengths of the mathematical community at large.

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Part IV

Lectures of Awardees

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https://doi.org/10.1142/9789811287152_0016

Lecture of Awardee 1

Understanding the Power of Teaching and Its Role (in) Justice[1]

Deborah Loewenberg Ball[2]

1. Background

Before I begin the lecture, I'd like to say just a bit more about where I am. As I mentioned a moment ago, I am in Michigan. And here on the slide, you'll see where Michigan located in the United States as well as a map of Michigan itself. I also included a few photographs. You could get a glimpse of this diverse and beautiful land. What I really want to say is that I want to acknowledge that Michigan occupies the traditional and contemporary homelands of the three files of peoples: the Ojibwa, the Ottawa and the Potawatomi peoples. And I understand that as I stand on this land, I am part of the history of this United States that store lands from the indigenous people who are living here and remain occupied to this land to this day. I also appreciate that as I've learnt more about our history, I've come to appreciate the way in which the indigenous people on whose land I am standing, the use of the land as a teacher, from whom they've been learning since the beginning of time. The indigenous people on whose lands we stand remind us the power of the teaching from the lessons they have learnt from the land and from the water that surrounded. These powerfully shaped their past, their present and their future. I offer this landing acknowledgment here, even as I acknowledge my complexity and my white privilege. And I commit to not only give such landing acknowledgement which could be seen simply as performances but to link those to my own efforts that continue to in target to my own actions on on-going basis, learning from my mistakes, and find it useful to both my work and my personal life to contribute to dismantling oppression rather than contributing to its perpetuation. So we've often talked about the land we are standing and acknowledge whose land actually is. We also sometimes feel to acknowledge the world such as I am receiving today is actually the product of not just one person but of collective.

And I wanna pause to thank the many people who have been on this journey that I've been talking about today with me. They include former students, colleagues, mentors, teachers, my current doctoral and master students and the more than 1000 children who I have been so deeply privileged to teach over 45 years. Thank you all for your contributions to what I've been learning.

[1] This paper presents the transcription of Deborah Loewenberg Ball's Felix Klein Award Lecture at ICME-14 Congress.

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So, what we would begin? As I said couple of times, I am deeply honored and grateful at being awarded the Felix Klein Award for the International Commission Mathematics Instruction. The surprise I am hearing it and the humility that I was filled with when I thought about it. Let me to look back across the kind of unexpected arch of learning across the time in my professional and personal life. This lecture that I designed then is reflected to my continuing efforts to try to understand the part of society that I landed in as a brand-new teacher in an elementary school in an unusually racially culturally and linguistically diverse U.S school. I, identified as a white woman and as a Jew and I've been a daughter of a father who together with his family escaped from Nazi Germany. I later attended primary school in that country. And my identities have intersected over time in my every evolving perspective in the country I am living in the United States. Its history of enslavement of African peoples and the massacre of indigenous lands people, and the ways in which those stories are not only in the past but continue in our present in the various forms of oppression that characterized our country. This last year and half have only further highlighted the legacy of slavery and oppression that shaped my nation. My deep engagement with and wonder about teaching and how it is fundamentally tied to the role that plays in the context of systemic anti-black racism, sexism and other forms of structure of oppression characterized this arch of my learning on which I am still continuing and is expected to continue as long as I live. I took the occasion to try to look back and now and forward as I talk to you today. On this slide, here you can some pictures from my earliest years of teaching children, including a class of children who I taught at my second and third year as an elementary teacher. And some of the children I taught more recently. Give you a glimpse of this woman growing up through time through this amazing career of teaching.

And I chose to frame today's lecture to continue the journey I represented at ICME13 in Hamburg, Germany in which I was honored to give one of these invited lectures. At that Congress, I talked about something I called the Special Mathematics Work of Teaching and I reflected on how the history of my efforts to try to understand the role that teachers' mathematical understanding plays in their actual work had been both one of which I learnt a lot with my colleagues but also frustrated me in the ways in which despite of my effort to understand mathematics is dynamically part of the work of teaching. I found myself often just talking about the knowledge again or something static, and not as part of practice. And I tried to give lectures to engage all of you thinking the role of teaching, taking social cultural perspectives of that work, and trying to think that mathematics as a verb, as part of the work of teaching. Yet when I look back on that talk, I realized that there were still thing missing in my efforts to try to capture and explain how mathematics comes together with other aspects of the work in practice.

So, when I served this president of the American Education Research Association two years later, in 2018, I was still on the same journey that I had begun years earlier. I gave a lecture that year called *Just Dreams and Imperatives, the power of teaching in the struggle for public education*. And in this talk, I tried to surface the ways which teaching is powerful, both for harm and for good. And I tried to think more about this notion and its power. I tried to think about how the larger system of oppression, the systematic oppression of racism, the structures of the inequity, how those actually found their way into the everyday micro moments of teaching. I used a video to try to show the connection between larger societal pattern of racism and everyday moments of teaching. And I used particular moments in video clip to think about the ways of which marginalization of black girls easily finds its way to perpetuate in classroom teaching but also the opportunities that exist in teaching to stand and disrupt those in the moment.

In my lecture today, I am continuing this journey to try to understand what I called the "work of teaching" and today to foreground as I think about this mathematical work of teaching, how do I understand and how it intersects the power that the teaching has in the context of the society with its history of enslavement, of oppression and of racism. So, I ask three questions today. I continue my question of the work of teaching. What it is to *do* the 'work of teaching'? What does it mean to foreground the 'power' of that work and why does it even matter? And finally I will ask what are continuing challenges in trying to understand this work of teaching and why should we care? So let's begin.

2. Work of Teaching

What is it to do the 'work of teaching'? It is worth appreciating that teaching is both incredibly common and also supremely complex. Here you see images of a couple of teachers. What you can see here, although you cannot hear anything or feel anything, as you can see mathematical content, you can see body, you can see looking at one another, you can see relationships, you can see gestures, you can see mathematical ideas, you can see representations, you can see space. There are so many things even it was still a photo that help you and remind you of the commonness of teaching, the things that all of us would teach and deal with all the time, and also the complexity of it.

So you would think about how common it is. I gathered a few data to show us and remind us about how common teaching is. Here are a few countries from around the world represented at this ICME Congress. And the numbers of teachers there are in each of your lands. You can see that there are many people who work in the role of teacher. And that in fact in all of these countries, teaching is the largest occupation. There are approximately 72.5 million teachers worldwide. So the part when I said it is common, we take it for granted that in every country we have adults who are willing to commit themselves to the next generation by fulfilling this occupation we called teaching.

It's common, but it is also incredibly complex. This is the word that surfaces all the time in the literature. I think we do not always ask ourselves what does it mean when we are saying is complex. So let's take a moment to pause and try to think when any of us says that, what do we actually mean? We will take a moment to watch this short video segment from a classroom. I am showing you all this image of a classroom here. And I will just briefly explain what the children and the teacher are talking about. They are working on this mathematics problem (Fig. 1) which might be seen incredibly obvious to you. But it's worth understanding that as children begin to understand the representation we call Area Models that it is not all that simple to interpret them. Here the children are comparing two rectangles that made in fact to them look entirely the same. On the left you can see a rectangle divided into three parts and one of them is shaded to grey. And on the right, you see a rectangle that is also divided into three parts and one of them is shaded to grey. And the question asked them what fraction of each rectangle is shaded grey. If you set aside that you don't understanding, then you might realize that for children, their answers to both of these might be one third. And indeed the problem constructed to surface what it is to look at Area Models, and the importance of equal areas and the notion of the whole. As you are watching this short video clip, I would like you to notice what do you pay attention to, and what do you think of the 'complexity of teaching'. What are your eyes drawn to? What are your ears drawn to? What do you notice? Try to ask yourself — to you, from your perspective, from your expertise and experiences, what seems to you to be complex here? So I show this short clip and these two questions that I would like you to be thinking about while I play it.

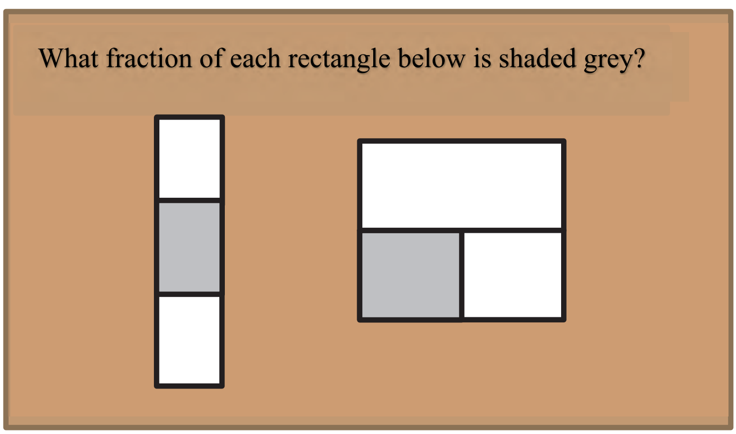


Fig. 1. The mathematics problem

(Video playing)

So take a moment and think about what is meant by "complex". What were you noticing? What were you thinking about? What were your eyes drawn? What were you hearing? What did you notice about space, about bodies, about language, about the mathematics? And what did you think about what it is meant to be called even in that very short segment "complex"? Let's think about that for a moment. Put yourself in the role of the person doing the work and being the teacher there. And you think about all the things that are to see, to understand, to sense, to think about the question like this "What does Antar mean by 'it's not a fraction'?" You might have to ask yourself

"What is the mathematical point of this of what I'm doing?" You might be worrying about "How is Antar being positioned in front of the class?" You might be watching other children saying and wonder "Are those two children over on the side following this?" You might be wondering "Whom to call on?" or "How is Antar feeling about his contribution?" or many other questions that easily can be on your mind in that moment as you make the next move. You are wondering about whether to keep the class all altogether right now. You are wondering about whether you have to do something to position both Antar and Gabi. You might wonder whether giving her the sticky line is actually a good idea. You might be thinking about your own body and where you should be. You might be thinking about what do you say and do next. So many things could be in the space of work that you are doing. And all of these have to do with try to feel, see and hear what the children are doing and what the children are thinking, because the work you are doing are with them. It is not to them. It is with them. It's about this mathematics. So keeping in mind the math, who the children are, how they positioned, their identities, what they might be feeling, all of these flowing around. And I think if that does illustrate what it means for something to be complex. I don't know what would.

So when I keep using this phrase the "work of teaching"", what do I mean by? what am I actually trying to use it for, why do I use this phrase? As reflected back, it came to me that I've been thinking about this for a very long time and struggling with what it means to talk about the work that I and so many millions of other people do every day, take it for granted and yet so complex. I found small parts of different things that I had written even 20 or 30 years ago when I saw myself beginning to think about what does it mean and try to understand that work. I see that in an article I wrote in 1996 and later some work I did with my colleague David Cohen when we talked about the role of curriculum materials. I continued to try to zoom in as I thought about what role mathematics play is in the work of teaching and what kinds of mathematical reasoning are inside and are required in that work and later with my colleagues Mark Thames and Jail Phelps, we asked ourselves questions about what makes that kind of content knowledge special when we thought again about the work for dynamic. And with my colleague Frank Forzani, we thought about how we could name some of the aspects of that work. So why I am insisting on using this phrase of "work of teaching"? I asked myself that question at my ICME-13 lecture. What I said then was that I thought it was important to focus our attention to what teachers are actually doing and to distinguish that from other features of classrooms, like different instructional formats such as small group work, or classroom culture and norms, what students are doing, how the curriculum is designed. I don't quite think about that way now, but I understand what I was asking at that point. And I do still feel that it's crucial that we honor the effortful and deliberate nature of teaching, its complexity, it's taken-for-grantedness, and not be invisible, so implicit, and taken for granted. But I am thinking I am still trying to figure out what does it mean to talk about the work and I no wonder think it is separated from what students are doing.

And so when I ask myself what do I mean by the "work of teaching", I am continuing to revisit and revise the definition. In some ways, it might be disappointing to you that I think that trying to figure out what do I mean by that is part of the ongoing inquiry of my work. I think it rests on trying to understand what is involved for teachers in their interactions with learners in context, and with more explicit learners of broader sociopolitical and historical environments in which that work takes place even in micro moments. I see this is the fundamentally both a deeply theoretical and practical question. And what I want to try to do together today is how can we better understand what I am calling the power of that work of teaching and the ways in which it either perpetuates or in fact can be used to disrupt injustice, racism, and oppression. So I am keeping in play this notion of mathematical work in the dynamic relational aspect, but I am trying to foreground more our ability to understand its power in the broader sociopolitical context of the work. And it is a hopeful question as well because in the end my dream is to begin to leverage its possibilities of that work of teaching is common, is on the presence to build a better world, a world that is more just.

3. The Power of the Work of Teaching

So my second question of this lecture is “what does it mean to foreground the ‘power’ of the work of teaching and why does that matter?” The first you could say that commonness and the complexity of teaching have powerful consequences for patterns of racism and oppression in our society.

I know that I don’t need to remind you of the many forms this takes as it taking in our current world and has taken place over time in our nation, in multiple world places around globe. I include a few images just to remind us of this larger systems of oppression, of the mass cultivation in United States and around the globe, of homeless people living in without safe shelters, the deep persistent ever brought economic inequity, the starve of indigenous lands from the people who occupied those land and occupied now by people who dominate them through oppressive ways of being, the battle of law enforcement and policing, unemployment and what the COVID-19 pandemic has reminded us this unbelievably vast deep disparities in health care and the health around the world. There are many other images that I couldn’t have given. But these seem perhaps to be large macro issues and you might be saying why we are talking about this right now? We are talking about it because I want to connect the dots between the systems of horror and hate and inequity and injustice and racism. I want to connect those to the commonness and complexity of this work we called teaching.

And to do that, I’ll try to take you zoom in from the picture I just showed to you, back into the classrooms. I’ll show you images of some adults who are themselves the people who perpetuate in the systems. It’s people who make systems. Systems are not some abstract structure out there. They are made by people and they are perpetuated by people and they could be dismantled by people. Systems and people are connected. Here is a picture of real estate agent someone who sells homes, and there is nothing could be said about how the ways in which those people’s work contributes to the ongoing and incredible racial segregation in the United States. Here is the image of a police officer. Here is a physician. Here is somebody who works in voting counting ballots in registration voters. And if you have seen some of the news from our country in the efforts, to try to deny voting rights to people who are so much deserved to be part of our electoral system. You could see the roles of these people playing in the electoral system is fundamentally important. Here is an image from the terrible day we saw in the Capital in the United States where rioters destroyed and attacked the Capital and the members of the government. And here you see a classroom. Let's pause for a moment and just remind ourselves that these adults in their various contributions to the systems in our country and in the countries all around the world. All of those adults were one time children. They were once third Graders or fourth Graders. They were in classrooms in all of our country. And in fact, even the people who teach in our countries were at one time children. So, if nothing else, if we will remember this, then when we notice hate, oppression and systems of racism, if we can remind ourselves that all the people who uphold and maintain those the systems were one time children, it might begin to illuminate how teaching is powerful.

So I make some claims here. "Teaching is powerful" I argue. And what I mean by that is when it is done carefully and sensitively, students can thrive, learners can grow, they can learn mathematics, they can develop positive identities, they can learn to value other people and work across different and work collectively. The second thing I am going to assort is that teaching involves enormous discretion. And from there, I go on to say that how that discretion is exercised can either reinforce racialized and oppressive patterns of social, personal and epistemic injustice and harm, or it can disrupt those patterns.

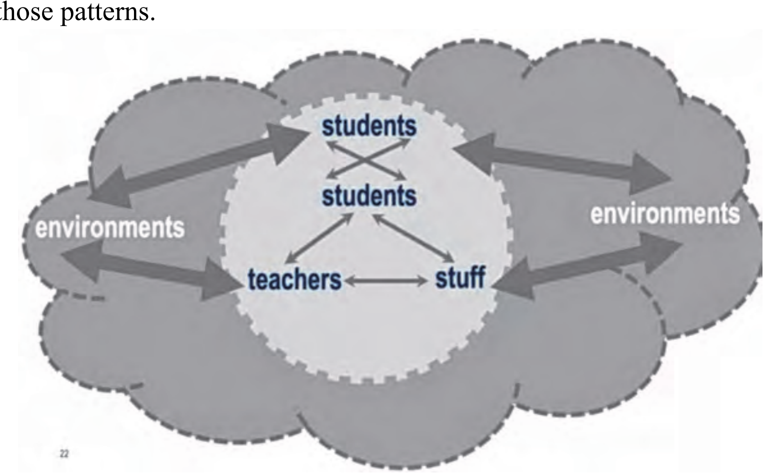


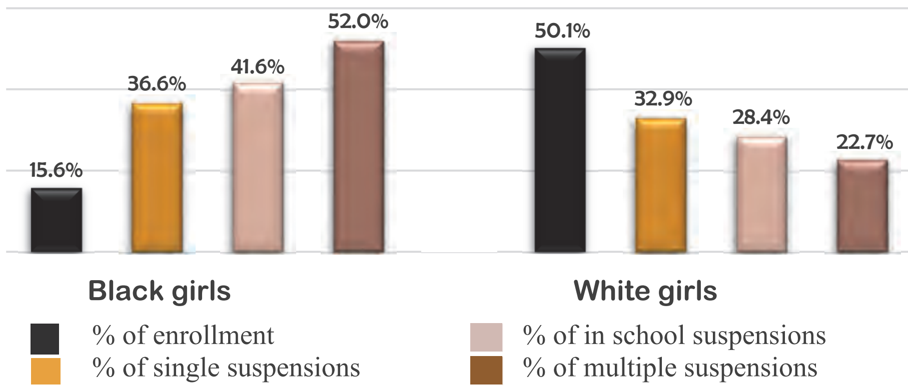
Fig. 2. The instructional diagram

Let me explain. With this diagram (Fig. 2) that's an adaption of something that David Cohen and I and Steve Rodin originally called this instructional triangle, I modified that for my 2018 AERA Presidential address, to make some changes highlighted some features of the work of teaching and its position in environments that I thought merited highlighting. So I wanna start by looking at the environments in this picture. And remember what I am trying to represent is the analysis inside the classroom represents the inter-circle. So when I say environments, I am talking about some of the things we've already been discussing: anti-black racism, colonialism, the legacy of enslavement, Whites supremacy, housing policies of insurrection, school structure, teaching as an occupation, what families want from schools, the enormous health and wealth disparities, the curriculum, what particular committees think and believe about schooling, so many things are around the work of teaching and schools. The environment is also the arrow towards students remind us that students bring in enormous resources from their own lived experiences, from their families, from their cultures and from their communities. But they also bring bias from living and inherited from races to class and society. I am thinking about a very moving piece of research that I read over 25 years ago, by Timothy Grim Mill. He studies a classroom in which children were being helped to learn to write. And one of the things that was important in the classroom is that children could write about whatever they care about. It was deliberately in gender. They were invited to write about their ideas, their feeling to be creative. But what had not been taken account of by the teacher was all the ways in which the classroom is permeable from those outside environments. And these children lived in a various socially economically racially diverse community. What happened was the children wrote stories about other children in the class, reflecting the broad societal bias, prejudices and history of marginalization. The teacher had not realized by asking children to do what they could bring to the school, could bring inside the classroom, these forms of hate and bias. On the other hand, the ideas of cultural environment, culturally sustaining pedagogy advanced by so many scholars Gloria Ladson-Billings, Django Paris and many others required the class to be purist. So, if we want to take advantage of resources that children bring, from their own lived experiences that we don't want the classroom to be sealed off from the broader environments. So, it's a dilemma. So, the environments represent histories and present, family, community, culture and other ways of being interpreted. What were inside the classroom remind ourselves with this diagram that the students were interpreting and interacting with one another; they were interpreting and interacting with their teachers; their teachers were interpreting and interacting with them. And all of this happening around particular staff, it could be mathematics, it could be discussing something in current events, it could be discussing a world issue. But teachers and students are interacting together. And as they work, they are bringing in with them their experiences in their broader environment. The curriculum itself is influenced by the larger environment. In this country right now, there is enormous debates about the teaching of the U.S. history and the representation of White supremacy that has been raced in our history and has been targeted in our schools. So even the stuff is deeply influenced by the environments for better or for worse. So, what the diagram intends to show is that teaching, with students and teachers, is inside the classrooms that are deeply poised to the broader environments. And those broader environments include the histories, the patterns of racism, the patterns of oppression marginalization. And they also include the resources and restrains of communities and people as they come together inside the school. There is no simple answer to this, because on one hand you could argue let us constrain the classroom so that those larger patterns can't sit it inside the dynamics between teachers and students, but obviously if we want children to do things that bring forth their expertise, their experiences, their communities, their language, we cannot seal classroom out from the broader environment.

So here is where I move to talk about discretion. In the video clip that I showed you a few minutes ago, we saw a few minutes of interactions between two children and their classmates over particular set of rectangles, involving area model diagrams. I don't expect you to read what's on the slide. But basically, what I've got here is a recording in writing of the things that were set in some of the body movements that were going on in the video. And what I am arguing based on my 2018 AERA lecture is that every one of those separate lines in this representation represents some moments with the teacher has discretion to do one thing or another. It could be a decision the teacher is making, it could be the way the teacher moves, all of these are discretionary. What do I mean by discretionary, I will show you what I mean in a moment. They cannot be dictated by some outside authority. They are things happen in the complex dynamic of teaching. So, for example, at the beginning of the video, we heard the teacher say "Who'd like to answer what you think about the second rectangle? We're now only going to be able to talk about this briefly. We probably won't finish it." There are so many ways a teacher could begin that part of the lesson, and neither the textbook, nor the school leader, nor the teacher education program can tell a teacher what to say exactly in that moment. There are many different ways a teacher could begin. A teacher could have said "what is the answer to the second rectangle." which would signal something very different about the work that children are being engaged and doing together. We also see a moment when the teacher sees that many children have their hands up and responds to the question what it is that Antar just said. As a teacher has to face a moment like who is going to be called on next? Who is going to speak next? That cannot be governed or dictated by some outside force. It is a discretionary space. When Antar is done with explaining, the teacher said "Antar, do you want to stay there or do you want to sit down?" He indicated like to sit down. She says, "Okay, Thank you very much. You did a good job of explaining your thinking." How to help a child exit from the front of a room has a lot of going on in it. How Antar might feel? How might he seem to his classmates? What different things could be said and done? So many possibilities exist. It's another discretionary moment. And what I am trying to demonstrate with this diagram is that in this two-minute and twenty-one seconds of video that I showed you, there were twenty-five such moments or discretionary spaces. In order to do the work of teaching, the teacher exercises judgement, acts from patterns that she already has, says things, does things, moves, all of these are products of how she works in this discretionary space. And I hope you could see from this diagram that my argument is that teaching as I've argued elsewhere is dense with discretionary spaces. Almost like the real line is dense with numbers. Teaching is dense with discretionary spaces.

And how was this related to my argument about the power of teaching? Well, when you think about this diagram in the density of discretionary spaces, what I am saying is a discretionary space is where the interpretations, next moves, the comments or questions are not necessarily determined by the teacher-and not by a policy or a curriculum. These interpretations and actions that the teacher takes are learned by that teacher through her own firsthand experience as a child in school, through her professional training and through her experiences as a teacher in school. So, these things become absorbed by living in the broader society. They are not all highly ideally syncretic, yet they are also syncretic. They are also not all rational and planned. They are often habituated. They are often matters of habit of pattern because teaching is so complex that of course a great deal of work become routinized and it indeed a must. But what does it mean is that the use of the discretionary spaces has a lot of opening for either the dismantling in interruption of patterns of oppression that carry with us, in our bodies, in what we come to consume, or in the opportunity to disrupt these patterns of bias and oppression.

So, the question next has to do with how can we harness that power, that power of discretion in the work of teaching. So, to give you a concrete example of the way the larger systems and the micro moments interact, this is a brief diagram that I won't talk about it at length, but represents some very important research in the United States, about the disproportionate punishment of black girls compared to white girls in a nation school (Fig. 3). What the diagram basically shows is the large patterns of differential and much harsher punishment for the same infractions, committing the same offenses, or doing the same thing. You can see the black girls account for about half of the multiple suspension of the school whereas the white girls account for only one fifth for the same behaviors. So, what the researches are showing us is that these differential outcomes are related to teachers' judgements. They are the products to teachers' discretion and subjective judgements. So, you can say, so what we are going to do about that? We see these patterns. We do not want them to continue. Teacher education could work to disrupt the habits that the teachers come into teaching with, from their observations in classrooms going up, from their experiences, doing clinical works in schools. These patterns are embedded in normalized and oppressive patterns of practice. So, what could Teacher Education do? Well, here is an image of teacher screaming at the black girl. A white teacher is screaming at the black girl in a math classroom. This video showed the teacher behaving in one of the ways we see very unfortunately in the classroom. So, we can look at the Teacher Education to surface these patterns and to wipe them out through professional training. But teachers would need more than these patterns exist and more than commitment to be people who can work to combat racism and patterns of oppression in normalized practice. Teachers to do something about this



R. Epstein, J. Blake and T. Gonzalez (2017). *Girlhood Interrupted: The Erasure of Black Girls' Childhood*. Washington, DC: Georgetown Law Center on Poverty and Inequality

Fig. 3. About the disproportionate punishment of black girls compared to white girls in a nation school

would also have new habits. They would need knowledge, repertoires of possible ways of practicing and judgement. So, it's more than beliefs and knowledge, it's also changing habits of practice.

This what I am going to do today is the work of teaching: to learn and to manage that complexity and to be able to understand the broad and powerful role that it plays in justice in order to carry out that work the ways don't reproduce and perpetuate patterns of racism in oppression, patterns of anti-black racism, patterns of sexism, but to recognize these normalized practices, to see the relationships to the larger patterns in our societies and to find ways to do something different, for that we research among other things. But doing that kind of research is challenging: challenging to study the work of teaching, challenging to identify the ways that could be useful to this project of disrupting normalized patterns of racism, embedded in everyday common practice.

4. The Challenges for the Work of Teaching

So, the third question for my talk today is what are some of the continuing challenges for those of us who do research or for those of us who are practitioners in trying to understand the work of teaching? What are the challenges of trying to do that because indeed these are challenges in it? And why should we care? So, let's return to the classroom and revisit the video clip that I showed you earlier. Let's think about what are the challenges of trying to study the work that is going on in these minutes in the classroom?

(Video playing)

So this point we have one small segment of the lesson and the teacher and students are together beginning to try to figure out how best to distinguish between the first and second rectangle. Again, I am asking you this question about what are the challenges of trying to study the work of teaching here. The work of teaching that involves the mathematics, the teacher, the students, the broader environments, and the ways which children bring different identities, histories and experiences from broader societies. How all those things come together? What does it mean to study the work of that complexity? So, I wanna show you one more thing goes on a few minutes later to continue our questioning about what is it the challenging about studying the work of teaching, what it is challenging by teaching, and what is it challenging by studying it? So just as the lesson is about the end, the children are about to leave the classroom and go to lunch. A girl named Kassie raises her hand and says "Antar is right. It's not equal". And she brings upon an idea that might be something that you are not expecting. Let's watch.

(Video playing)

So, we stop there and again the class is almost over. Here is this new idea that's coming up. We have almost 30 children in this class. I think by looking at them you can see that there are overwhelming black children. You could also see it is a diverse class of people with different backgrounds and undoubtedly you could understand they bring in different intersectionality and identity. This is a math class in which they were working on this particular concept. Time is running short. And we see the teacher say many different things, for example, asking Kassie to name the child whose idea was – that was Antar; asking her to elaborate idea; not having her to come to the board at that moment. So many different things are going on. And earlier the clips we saw twice, we see the ways in which the teacher moved from Antar's contribution through Gabriella, to Gabi, and what went on. It's all those different places. We have to watch many times to appreciate the different things that might be to see, to name, to study, to analyze.

These moments that I am showing you are filled with discretionary spaces. Each of these is related to reinforcing or disrupting patterns of racism and harm. So, what are some of discretionary spaces and risks? We could name many of them in the video that I showed you. But let's just consider a few. We have Antar and things around Antar's position. We have Gabi and what she has done. And we have Kassie now as well. And we have all the other children in the class. There was the question that we want to try to understand that the researchers bring in enormous theoretical power and empirical power in studying. For example, how were these three different black children, Antar, Gabi and Kassie, positioned in front of their classmates? How were they being positioned as contributing or not as understanding or lacking understanding? Are their brilliance and their humanities be seen? Is it a hard question that yet being central? So there are especially spaces of work on the way which children experience and the work of teaching play out. Another thing we could ask and try to study is what's being signals about being and doing mathematics, both what and who? And what kinds of things are those signals? What are other children in the class besides those who are the main players in the video we saw? What are they learning about the black children about who gets to be smart in math and what does it mean to be smart? What's happening with math? What's happening to children's understanding of equal areas? There are so many different things being managed in this work of teaching and in each of these different questions I am naming and many more, there are discretionary spaces whatever gets said or done has consequences. It has matters for questions of reproducing and perpetuating patterns of how particular children get positioned, how Antar as black boy gets positioned; how Kassie gets positioned as black girl, or Gabi or the rest of the class see what's happening with the math. These are all spaces of discretion. And through them, what is the enormous power to either reproduce or to disrupt patterns of racism.

So, when we are thinking about practicing justice and practicing in justice, we have to understand more about the work of teaching which is challenging. I name now particular moments in the video clip in which these spaces arise and I invite you to think about the many kinds of questions we talked about earlier that are flowing through the work of teaching in any given moment about what to do next, where to be. Many of these are unconscious. They are habitual. They are part of striving, complex demanding interaction between the teacher and students in mathematics and in these larger settings. There is so much research that helps to drive and situate this work of teaching in the discretionary spaces. There are racial narratives about the ability and struggling learners. There are patterns of black girls. Lots of research on this. There is research about area models in fractions. There is research on what patterns are about being and do mathematics while black. I could name many other kinds of research, including misconception research, research on errors, all of these things are part of what can be studied about and is studies about the classrooms, still they could help us think about exactly what to do as you with your body do this work of interacting with children in real time. And here are lies in the power of discretionary spaces. As we begin to learn how to elaborate these discretionary spaces, they are moments and places to bring and bear the knowledge together with the practices in learning habits of action that can be powerful, for not perpetuate patterns of oppression.

So, standing back for a moment. What are the challenges in trying to do this kind of research and this kind of study of the work of teaching in justice? I will name just five and make a few comments about each of these. I hope it begins a conversation for us as a community interested in challenging, patterns of racism and oppression, in understanding the work of teaching, in finding ways to honor its complexity, and yet make its commonness able to be used in powerfully good ways for children in our countries.

So, the first challenge is combing the embodied nature of the work and the relational nature of the work with the cognitive and knowledge entailments. Some of my earlier work, I and my colleagues tried to study mathematics work, and it's super important and matters. But there's also the meaning towards the child, the use of tone and voice, the ways in which the teacher stands in the room or where children are positioned. We see children at the board and we see the teacher off camera. These are part of the body of work of teaching. And one example of our challenges our researchers, and I know many of you are working on this is finding ways to represent that work, with video, with transcript, and with so many other kinds of presentations that don't in vertically cut off parts of the work of teaching. I found myself in recent years more and more frustrated with even very imaginative use of transcript for the way in which it either goes so far away from the work or seems to translate so much about the work into language. And language is really important in the work of teaching. But so many things are also not about language, about relationships. How do you transcribe relationships? How do we move about the spaces of the room? So, there is a challenge for us: how we bring together the multi-model nature of the work of teaching and find ways to represent it and analyze it.

The second challenge is the one I won't talk about it at length, but about the importance of building theoretical and more general knowledge about teaching, and building inside while contextualizing the work and centering identities. So, for an example, I have not told you anything about the community where the classroom is, I have not told you very much about the children's identities, have not told you what time of the year it is, and yet there were some general things we've been able to talk about and think about. What does it look like to do this work with integrity where we are centered that very important contextual dimensions of the work, because it is contingent and decontextualized. We are also building theoretical knowledge. What are the responsibilities that we have to be honest about where we do this work? How are we accounting for differences? We are also looking for theories.

A third challenge is how we bring together the work that intends to be systematic, creative work on structure, racism, on macro-structures together with micro-interactions. Too often in our history on the work of teaching, we look at micro-interactions or we look at cognition, or we look at larger structures, or we look at patterns. But in fact, these things are interplaying in the work of teaching, and how can we do better to link together larger patterns of microsystems, macrosystems together with the moments and the spaces. What are the ways to do with that integrated?

A fourth challenge is how to distinguish more responsibilities between prescription and detail. There's been a kind of allergy in our field about breaking down the work of teaching in the ways that many colleagues have named decomposing practices. And it means that it has been critiqued for foreseeing to describe teaching and prescriptive ways as though one did these technical prescriptive moves that teaching would work. And yet, when we believe teaching to be vague, without ways of actually breaking it down and understanding it, we leave it to be at mercy of people doing this out of their old idiosyncrasy, their experiences of patterns in our society. We don't help with the work. It is so complex, begin to have texture, begin to have detail. How do we learn to distinguish between the should(s) and the description? A few years ago, at ICME-13, Ester Enright and I, together with other colleagues, worked on the work of questioning and teaching. And one of the things we argued is questioning is one of the most important things that teachers learn to do and yet we lack so much detail about the different ways that questions to get asked, awarded and acted. There are so many things to understand, yet we leave question asking to be very vaguely understood idea. How do we learn to look in detail and yet understand the contingent use of that detail? Analysis still remains something that has to be deliberated about with discretion.

And finally, how do we learn to represent the work of teaching in a usable discourse of practice? Some very fine work attends and is devoted to building theories and yet won't help those of us who work with teachers to help them learn to do the work. What are some of the language demands of learning to name and identify teaching in ways that are theoretically deep that are the product of analysis and yet are usable for practice? Is that an impossible goal or one to which we could aspire? And these are just some of the possible ways that we need to continue to confront challenges of work of teaching. Because when I look back across the arch of my work as a teacher, a scholar and a practitioner of teaching, I find that it is been a struggle in our field to honor the work of teaching and to represent it is not displacing work on learners or learning, or work on structures, or bias or oppression or on classroom environments, but to find ways to understand that the work of teaching is done in the context of all of that and to honor that is one form of important research that we so much need if we want to be able to leverage the work of teaching, to disrupt rather than let it simply perpetuate patterns of oppression.

And there are a few other things we need to contend more deliberately as we face challenges of studying teaching. One of those is that too often we neglect to say more about where we do this work and we need to vary that where we the work matters. We need to talk more about time of day, and time of year. All of us who have taught to know that there is a big difference between Thursday afternoon at four o'clock and Monday morning at eight. There is big difference between October and June. And these differences might look different in different parts of the world that they matter. So, in addition to thinking about whom we studying and where we are doing it, and how we blend together, some of the things I discussed in last slide, we need to think about how we don't leave out these very important parts of context of the work of teaching. Things that teachers are thinking about all the time and so rarely part of our research accounts. In fact, we very rarely, too often neglect to say anything about the identities of the children except sort of to describe them. It's not part of our analysis. And finally, and maybe most important to all, we need to diversify the research community if what we want to do is to understand how schooling is experienced, how the work is done. We need the perspectives and experiences, and expertise of broader range of people. The experience on our doctoral programs, the experience on our conferences, the experiences on so many spaces we understand that it's a matter of justice and the study of justice, to have a much more diverse research community. And all the work we can do together to understand how that would improve, not only who gets to do the work, but also what we collectively come to understand.

Because this in fact, this project is really to understand the work of teaching and its role in justice. It's a collective work. It's not a work of any one person, it not a work of somebody getting the Felix Klein Award. It's an agenda that matters for our future and the world. Teaching is powerful. It demands diversity and who 'we' is when we say 'we'. And this will necessarily broaden what we as scholars come to consider evidence, what we think of the objects of studying are, and what we value. And this would mean confronting some of the patterns of epistemology, ontology, and axiology that are shaped our research community.

(Transcribed by Bo Yang, Xiaoli Lu and Yan Zhu)

https://doi.org/10.1142/9789811287152_0017

Lecture of Awardee 2

The Interplay between Construction of Knowledge by Individuals and Collective Mathematical Progress in Inquiry-Oriented Classrooms[1]

Tommy Dreyfus[2]

ABSTRACT In a long-term research collaboration, the author and colleagues analyzed learning processes in inquiry-oriented classrooms in a comprehensive manner. In such classrooms, mathematical progress typically occurs in complex interplay between individuals, small groups, and the whole class. We analyzed this interplay by coordinating two theoretical frameworks, Abstraction in Context and Documenting Collective Activity, and their respective methodologies. In the present chapter, I briefly review our research.

Keywords: Classroom-based research; Inquiry-oriented classroom; Abstraction in Context; Documenting Collective Activity; Coordination.

1. Introduction

Learning in inquiry-oriented classrooms (Laursen and Rasmussen, 2019) is becoming more and more common at all levels of schooling. There is evidence from meta-analyses that student-centered approaches have advantages for learning outcomes (e.g., Theobald et al., 2020). It is important to understand why and how these advantages come about. Yet, qualitative research on classroom-based mathematics learning in inquiry-oriented classrooms is scarce. One reason for this may be the complex interplay between learning processes occurring at the different scales of social settings in such classrooms: individuals, small groups, and the class as a whole, learning processes that may strongly depend on each other. This complexity brings with it methodological questions of how to deal with learning processes at different scales. The need arises to use different theoretical frameworks, and to link between them.

I present an overview of a collaboration between Chris Rasmussen, Michal Tabach, Rina Hershkowitz, Naneh Apkarian and myself, whose aim it is to develop a methodology based on two theoretical frameworks that is designed to deal with the above complexity by networking the two previously independent frameworks. I will

[1] The research reported in this lecture has been partially supported by the Israel Science foundation under grants 1057/12 and 438/15.

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give an overview over the collaboration and some of its achievement. I will refer to other publications for detailed reports on specific research studies.

2. Background

The aim of our collaboration is to investigate learning in intact inquiry-oriented classrooms. All of us had earlier experience researching aspects of mathematical progress in such classrooms, but without obtaining a comprehensive picture of such progress. Some of us have researched the construction of knowledge by individuals and small groups learning in such classrooms (see Dreyfus et al., 2015, and references therein). This research has been based on the Abstraction in Context (AiC) theoretical framework (Hershkowitz et al., 2001). This methodology did not allow for the investigation of mathematical progress of the classroom as a whole. Others have researched knowledge becoming normative and functioning-as-if-shared in the classroom as a whole (Rasmussen and Stephan, 2008). This research has been based on the Documenting Collective Activity (DCA) framework (Stephan and Rasmussen, 2002). This methodology did not allow for investigations of the constructing of mathematical knowledge at the individual and small group level. The question thus arose whether the two frameworks and their associated methodologies could be combined so as to provide a comprehensive picture of the learning processes at the different scales in a classroom and the interactions between these learning processes. In parallel, the question arose how to network the two frameworks in order to achieve this aim, and with which networking strategy (Bikner-Ahsbahs and Prediger, 2014).

After collaborating for more than a decade, we are in a position to give positive answers to both these questions. The two frameworks, AiC and DCA, can be networked, specifically they can be coordinated, and their methodologies can be used in tandem to achieve a detailed qualitative analysis of learning in inquiry-oriented classrooms. In this chapter, we give an overview of several research studies to substantiate this claim.

3. The Studies

In this main section of the chapter, I present brief reviews of four research studies. The first three studies are empirical and illustrate the coordination of the AiC and DCA frameworks to study mathematical progress in intact inquiry-oriented classrooms; the fourth study is theoretical and explains why the coordination between AiC and DCA has succeeded.

3.1. Study 1 — knowledge shifts

We analyzed mathematical progress in an early lesson of an inquiry-oriented differential equations course (Rasmussen at al., 2018). We used Abstraction in Context for analyzing the construction of knowledge by individuals and small groups; we used the Documenting Collective Activity approach for analyzing whole class discussions. We found that students in different groups seemed to be going through similar but not identical processes of knowledge construction. Indeed, it is on this basis that participants can communicate across small groups yet still have differences to debate. Our analysis provides further evidence of the importance of whole class discussions, which partially emerged from the small group work, but also were occasions for participants to develop ideas beyond what was constructed in the small groups.

In order to coordinate between the analysis of knowledge constructed by individual students and small groups of students (using the AiC methodology) and the analysis of the development of mathematical classroom knowledge and reasoning for the whole classroom community (using the DCA methodology), we looked at mathematical progress in the classroom through the two lenses in parallel. The combination of the two methodologies allowed us to follow the evolution of ideas as they flow between individuals, small groups, and the whole class. We identified the links between the constructs that emerged and the ways of reasoning that became normative. These links revealed shifts of knowledge in the classroom: up-loading ideas from a small group to the whole class, and down-loading ideas raised at the whole class level into the work of a small group.

The links also focused our attention on the students who initiated the up- and down-loading and thus assumed the role of knowledge agents. A knowledge agent is a student who initiates an idea which is later taken up by others within the classroom community. We found knowledge agents initiating up-loading of ideas, others initiating down-loading of ideas, and still others initiating new ideas within the whole class discussion itself. The instructor has a double role with respect to knowledge shifts and knowledge agents. One role is to create opportunities which afford the activity of knowledge agents in the class. The second is to help other students benefit from the actions of knowledge agents for constructing their own knowledge. We consider the notions of knowledge agent, up-loading and down-loading to be a crucial product of the methodological coordination between AiC and DCA. A detailed report on this research study has been published elsewhere (Tabach et al., 2014).

3.2. Study 2 — teacher role

In this study, we analyzed a sequence of two lessons of a probability course for eighth grade students. The students worked on purposefully designed sequences of tasks intended to afford the emergence of abstract mathematical thinking in discussion. Our overall goal was to illuminate the role played by individuals and groups in the class as well as by the class as a whole and by the teacher in the knowledge constructing process, and to learn more about shifts of knowledge between the different social settings in a mathematics classroom during the knowledge constructing process.

We used the same approach as in Study 1 — DCA, for analyzing whole class discussions, and AiC, for analyzing group work — and we found that this approach is significant in that it offers a novel methodological tool by which to document the evolution and constitution of mathematical ideas in the classroom and the processes by which these ideas move between individuals, small groups, and the whole class under the facilitation of the teacher.

The two theoretical frameworks and associated methodologies describe different but closely related aspects of the classroom learning process. The AiC analysis has a particular focus on cognitive constructing while the DCA analysis examines how ideas function at the collective classroom level. These are complementary foci and their coordination allows for tracing the growth of ideas from small groups to classroom community and vice versa. The coordination of the findings emerging from the analyses according to the two methodologies allowed us to study knowledge shifts in the classroom as one continuum, and to trace students who have a crucial role in knowledge shifts in the classroom.

The analysis also showed that the teacher adopted the role of an orchestrator by balancing between the whole class and group work in terms of time and tasks. She kept an equilibrium between the need to teach certain content on one hand, and the strategy of affording opportunities for students to construct their knowledge on the other. She assumed responsibility to provide a learning environment that affords argumentation and interaction. This enables normative ways of reasoning to be established and enables students to be active and become knowledge agents. A detailed report on this research has been published by elsewhere (Hershkowitz et al., 2014).

3.3. Study 3 — complexity

The setting for this research is a lesson from a mathematics education master's level course on Chaos and Fractals; the topic of the lesson was the area and perimeter of the Sierpiński triangle and the apparent paradox stemming from an infinite perimeter enclosing a region without area. Our focus in this study is on an enhancement of our methodology and on the insights into the complexity of mathematical progress in inquiry-oriented classrooms that the methodological enhancement reveals. The methodology of coordinating the AiC and DCA theoretical frameworks used in studies 1 and 2 was enhanced in this study as follows: not only did we analyze small group work using AiC and whole class discussions using DCA as in the previous two studies, but we also carried out DCA analyses of the small group work and AiC analyses of the whole class discussions. This enhancement allowed us to layer the explanations for the collective and individual mathematical progress in a manner that fully integrates the collective with the individual mathematical progress, thus exhibiting the complexity of the interplay between collective and individual mathematical progress in inquiry-oriented classrooms. Thus, it enabled us to make sense of the lesson as a whole.

Our analysis showed a multiplicity of ways in which knowledge developed and mathematical progress was achieved. We showed that knowledge was not only constructed by a few students in a small group but also by a large group of students collaborating and arguing together in a whole class setting. In parallel, ideas were shown to function-as-if-shared in different situations: some without a preceding constructing process, others in proximity to a relevant, prior or almost simultaneous, constructing process. In other words, our enhanced methodology allowed us to show that in an inquiry-oriented classroom, knowledge that is new to the students may be constructed not only in small groups but also in whole class discussions. This conclusion should be seen in tandem with the conclusion that ideas may begin to function-as-if-shared not only in whole class discussions but also in small group work. We believe that this complexity is not exceptional but that similarly complex interactions between processes of knowledge construction and processes of ideas becoming normative are typical for inquiry-oriented classrooms. A detailed report on this research is available from the authors (Dreyfus et al., 2022).

3.4. Study 4 — argumentative grammar

From the previous three studies, we conclude that to make sense of learning processes in inquiry-oriented classrooms, networking two or more methodologies with somewhat different foci and grain sizes is insightful. On the other hand, researchers' experience with networking (Bikner-Ahsbahs and Prediger, 2014) has shown that there are conditions for networking to succeed. In the fourth study to be summarized here, we asked ourselves what underlies the success of the coordination between AiC and DCA in the previous three studies.

In this study we identified commonalities between the two frameworks that contribute to the productivity of their networking. To start with, there are environmental commonalities: Both frameworks require classrooms in which students are routinely explaining their thinking, listening to and indicating agreement or disagreement with each other's reasoning; such classrooms typically interweave collaborative work in both small group work and whole class discussions, where the teacher adopts a role that encourages argumentation and inquiry. Both frameworks also require the intentional use of a coherent sequence of tasks that is purposefully designed to offer students opportunities for constructing new knowledge by engaging them in problem solving and reflective activities. The tasks should be designed to afford inquiry and the emergence of new constructs by vertical mathematization (Treffers, 1987) from previous constructs. While vertical mathematization appears here as component of an environmental commonality, it is also a theoretical commonality. The methodologies associated with both frameworks are based on the premise that vertical mathematization is core to mathematical progress.

The theoretical relationship between the frameworks goes much beyond vertical mathematization. The analyses of empirical data allowed us to establish a net of internal-theoretical commonalities in the form of a correspondence between the analytical constructs of AiC and those of DCA. For example, a constructing action in AiC corresponds to an argument as a whole in DCA. Another theoretical link relates to the centrality of shared knowledge. The definition of shared knowledge used in AiC relates to cognitive aspects. We find its counterpart in sociological terms in the phrase 'function-as-if-shared' used by the DCA approach. What is common between the two constructs is that each construct operationalizes when particular ideas or ways of reasoning are, from a researcher's viewpoint, "shared" or "accepted" by the participants.

These and other environmental and theoretical commonalities make AiC and DCA highly compatible. We made use of this compatibility and at the same time raised it to a higher level by articulating an argumentative grammar for our networking; that is, we were explicit about the rationale and theoretical foundation upon which the networking is based. A simplified description of our argumentative grammar is that we employ the relationships between the analytical constructs of AiC and of DCA to explicate the theoretical commonalities of the two frameworks in the empirical data we have from the classroom. The fact that the coordination of the two frameworks has succeeded in the three different studies described in the previous subsections validates this argumentative grammar. A detailed report on this research has been published elsewhere (Tabach et al., 2020).

4. Conclusion

In the research collaboration represented by the four studies reviewed in the previous section, we coordinated two theoretical frameworks, AiC and DCA, to achieve a comprehensive and at the same time detailed qualitative analysis of learning processes in intact inquiry-oriented classrooms. We explained why these two specific frameworks are compatible and allow coordination, and we described an argumentative grammar for such coordination. We found that knowledge agents play an important role in the interplay between different social settings in such classrooms, and we analyzed this role. We found that one path of mathematical progress in such classrooms is the construction of knowledge in small groups followed by a teacher-led whole class discussion in which the knowledge constructed is institutionalized. But we also found that this is but one path, and that mathematical progress in inquiry-oriented classrooms is complex and may be made in a variety of alternative ways.

We see what we achieved as just a beginning, with much research still needed in the future. For example, studies at different grade levels are in order: So far, we have neither a study at the senior high school level nor at the elementary school level. Studies in which all groups in a classroom are observed (video-recorded) would be desirable but of course present methodological challenges to deal with the large amounts of qualitative data collected. Studies with large classes, say of 30 or more students, are likely to present additional methodological demands.

In view of the promise of student-centered instruction at all levels of learning mathematics, we hope that other research teams will take up the challenge of research that aims at investigating learning in student-centered, especially inquiry-oriented classrooms in a comprehensive manner.

Acknowledgments

I would like to express deep thanks to Naneh Apkarian, Rina Hershkowitz, Chris Rasmussen and Michal Tabach for our thoroughly enjoyable and productive long-term collaboration.

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Lecture of Awardee 3

From Thinking in Action to Mathematical Models — A View from Developmental Psychology

Terezinha Nunes[1]

ABSTRACT Developmental psychologists agree that intelligent action precedes language in children's development and that language transforms children's thinking. In this lecture I explore the ways in which children's thinking in action is transformed by learning to use conventional mathematical signs to represent quantities and relations between quantities. Numbers have two types of meaning: a referential meaning, which connects numbers to quantities, and an analytical meaning, which is intrinsic to the conventional systems of signs. This dual nature of numbers means that, from the psychological perspective, numbers are models of the world. The referential meaning of numbers is based on children's use of schemas of action to establish relations between quantities; it is at the core of quantitative reasoning. The analytical meaning rests of the rules that define relations between numbers in a conventional system and provides the basis for arithmetic. This paper presents research that illustrates how teaching can build a bridge between thinking in action and mathematical models by promoting the coordination of quantitative reasoning with number knowledge.

Keywords: Quantitative reasoning; Multiplicative reasoning; Action schemas; Reasoning in action; Referential meaning of number; Analytical meaning of number; Cultural systems of signs.

This paper starts from what I take to be an uncontroversial idea: mathematical modelling is a form of intelligent action. Although this idea might seem trivial, it has profound implications for thinking about how mathematics is learned and used to model the world. By examining this idea from the perspective of developmental psychology, I consider here four fundamental questions:

1. What is the origin of intelligence?
2. How does the learning of numerical signs change children's thinking about quantities?
3. What are the basic types of relations between quantities that students need to master in primary school?
4. How can schools promote students' thinking about relations between quantities?

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These four questions are examined in the context of mathematics teaching in school, but it is important to stress that mathematical reasoning develops also outside school (Nunes et al., 1993). However, an analysis of what and how people develop mathematical reasoning outside school is beyond the scope of this paper.

1. The Origin of Intelligence

For the fathers of the study of human intelligence, there was no doubt that intelligence starts to develop before language; this means that language cannot be the origin of intelligence. Gesthalt psychologists, such Wertheimer and Köhler, as well as child psychologists, such as Binet and Piaget, studied intelligent action in subjects who did not have language; Gesthalt psychologists studied monkeys and developmental psychologists studies human babies. The paradigms that were used to investigate intelligent action in the absence of language had in common the fact that the subjects could not attain the aim of their action directly: they had to use inferences that connected different aspects of the world, and by doing so they were able to attain their goal. These classical paradigms can be illustrated by detour problems and the use of tools.

In detour problems, the subject seeks to obtain an object, but is separated from the object by a barrier, as in Figure 1. Because the direct movement towards the object is obstructed, the subject must conceive of two movements as a single path: the first movement, A, away from the object, is reversed by the second one, B, towards the object, which is located in position C. Such behavior exemplifies a practical inference, which puts two different pieces of information together and leads to the conclusion of an equivalence: the movement from A to C is equivalent to the sum of the movements from A to B plus B to C.

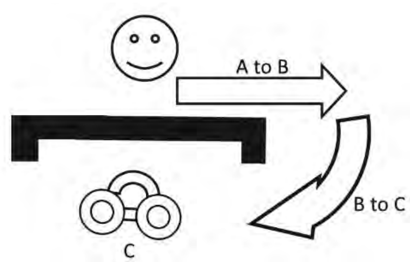


Fig. 1. The baby tries to reach the toy over a barrier but does not succeed. By thinking of a movement away from the toy as cancelled by another one, the baby can reach the toy.

A second classical paradigm in the study of intelligent action in the absence of language involves the use of tools: the subject tries to reach an object directly but the object is out of reach; by using a tool, such as a stick, the subject can move the object into reach. The use of tools also illustrates a practical inference: object A cannot be reached by the subject, but the subject can reach object B which in turn can reach object A, and B can be used to bring A within reach.

A large number of studies using the detour and tool use paradigms leaves no doubt that monkeys and babies can solve these problems and thus show intelligent action in the absence of language. Alternative theories were proposed early on in psychology to explain how detour and tool use problems were solved. The two initial hypotheses were proposed by Gesthalt psychologists and learning theorists.

Gesthalt psychologists suggested that the solution of detour and tool use problems involved a sudden re-organization of the perception of the situation, termed insight. Thus they argued that intelligence was rooted in perception, but there was no explanation for what led to this sudden perceptual re-organization. Insight provided a structure for further intelligent action: the subject became able to use previous insights to solve new problems that required the same sort of inference.

Learning theorists suggested that solution was accomplished by trial and error, also known as the law of effect, which involves a gradual approach to the solution based on the reinforcement of those actions followed by success. Such learning had no structuring role on further problem solving because learning was viewed as essentially guided by reinforcement, which is external to the learner.

Piaget criticized both theories. He argued that the Gesthaltists offered a description of structures for thinking, but no explanation for their origin, whereas the trial and error theory sought to explain the origin of the solution but provided no description of structures for thinking. Piaget proposed that the solution of problems in action was accomplished by learning to coordinate initially independent actions into structured sequences, that are repeatable and useful in different situations. Such sequences he termed schemas of action. According to Vergnaud (2009), action schemas contain theorems in action, i.e. involve a process of making inferences which are not explicitly recognized by the actor.

In summary, the existence of intelligent action in the absence of language is accepted wisdom in psychology: intelligence starts in action. Although the description of the processes by which problems are solved in action differs across theories, all theories about the development of human intelligence include a period during which babies are able to make practical inferences and to solve problems, even though they do not use language to communicate and cannot use language to represent the process of making inferences nor to represent the solution. It is also accepted wisdom in psychology that the acquisition of conventional systems of signs, such as language and number systems, changes the possibilities of human intelligent action. This idea is explored in the next section.

2. How Does the Learning of Numerical Signs Change Children's Thinking about Quantities?

All theories of cognitive development propose that the acquisition of signs transforms intelligence. Consider the case of numerical signs. Human babies can compare small quantities and distinguish two from three objects, for example; they can also compare larger quantities when the difference between the quantities is large, such as 6 versus 20 objects. But they do not succeed in distinguishing, for example, 19 from 20 objects visually, and neither do adults. Because babies do not know how to count, their ability to compare discrete quantities is restricted to what they can distinguish perceptually. Older children and adults, who are not restricted to visual comparisons and can use a counting system to compare discrete quantities, can easily tell the difference between a set with 19 and one with 20 items. They are no longer limited by their perceptual skills. Learning to count changes human's ability to compare sets.

Learning to count does not immediately transform children's ability to compare discrete quantities. Children learn how to count, but continue to try to compare quantities just by looking at them until about the age of 6 or 7 years. Many 4- and 5-year old children who can count objects up to ten do not count to compare sets with seven or eight items, for example, and do not succeed in making accurate comparisons when the perceptual arrangement of the items makes visual comparison difficult. This observation has been replicated in several countries with different counting and educational systems, such as Brazil, China, England, France, Switzerland and the US (for a review, see Nunes and Bryant, 2022 a).

The explanation for this phenomenon is a matter of controversy, which in my view boils down to the theory of number meanings upon which the different explanations are based. In this paper, I contrast two radically different explanations for how children learn the meanings of words, and in particular the meaning of number words. The general terms for these two basic explanations, associationism and representational theories, cover a variety of specific approaches that differ in their details, but still allows the theory to be classified as belonging to one of these two types. This discussion does not focus on the variations as its aim is to contrast the two general theoretical approaches.

The associationist view

According to the associationist view, which is considered the oldest theory of thought (for a review, see Mandelbaum, 2015), a word attains its meaning by being heard at the same time as its referent is perceived; i.e. by association based on contiguity in time and place, to use Hume's (1896) expression. In the same way, a number word attains its meaning by association to a perceived numerosity: the word "three", for example, becomes associated with the perceived numerosity of a group with three items.

Among the many criticisms of this view of how number words acquire meaning, I consider one the most crucial: in this theory, there is no principled connection between the meaning of the words "three" and "four", for example: each of these words acquires its meaning by association to a referent, which is a set with a specific numerosity. This problem of associationism as a theory of number meanings has been recognized by researchers, who tried to solve it by resorting to other, different processes which would be complementary to association. Carey (2004), for example, suggested that children learn the meaning of the number words "one" and "two" by association, but at the same time they distinguish each of the two items in the set with the numerosity of two. This process, termed parallel individuation, eventually allowed the child to realize that "two" means one more than one, and that "three" means one more than two. These realizations eventually would lead a child who is learning to count to generalize from the initial sequences "one, two, three" to all sequences in counting, and to infer that that each number word refers to a set that has one more item than the set represented by the number word that precedes it in the sequence. Thus, in Carey's theory, the meaning of numbers ceases to be the result of associations between words and referents and comes to rely on a new process.

Carey's theory ends up by breaking with associationism. To quote her conclusion: "We cannot just teach our children to count and expect that they will then know what 'two' or 'five' means. Learning such words, even without fully understanding them, creates a new structure, a structure that can then be filled in by mapping relations between these novel words and other, familiar concepts" (Carey, 2004, p. 68). Even though Carey does not clarify what she means by the other, familiar concepts, her explanation departs from associationism in so far as it depends on internal conceptual structures provided by the child rather than on association. It is worth noting that Carey explicitly relies on numerical comparisons as part of the internal structures when she describes the child's realization that two is one more than one. In order for this explanation to be in line with findings in mathematics education, numerical comparisons would have to be an early achievement, but research in mathematics education has shown that the mastery of numerical comparisons is a late achievement, rather than an early one (e.g. Carpenter et al., 1981; 1982; Hudson, 1983).

The representational view

The representational view offers an alternative to associationism and it is also a classical theory in psychology (von Helmholtz, 1921). I present here the synthesis to which we (Nunes and Bryant, 2015a; 2022a; 2022b) have arrived over the years. We consider numbers to be elements of a system of signs and to have two meanings.

<u>The first meaning of numbers is representational</u>, extrinsic to the number system, and connects numbers to quantities through measurement. According to Thompson (1993), a person constitutes a quantity when he/she thinks of a quality as susceptible of measurement. Measurement is the attribution of numbers to quantities according to rules (Stevens, 1946) that represent relations between quantities: thus, in the representational view, numbers are relational concepts.

When we measure extensive quantities, the measure is obtained by the addition of its units. If the quantity is discrete, the unit is often a natural unit: e.g. oranges, rabbits, pencils etc. When the child conceives of counting as the addition of each unit, as it is counted, to a set of already counted items (von Helmholtz, 1921), the child has a first insight into the representational meaning of natural numbers. The action schema of joining and its inverse, separating, are the sources for the representational meaning of natural numbers. If the quantity is continuous, a conventional unit is defined, and the total quantity is conceived as the sum of all the conventional units, such as centimeters or inches. The logic of additive relations between the units in a quantity supports counting systems in different languages, although the number words and the particular organization (e.g. base ten, mixed base ten and twenty, base 12) of counting systems differs. Understanding the connection between number and continuous quantities is a later achievement, but the logic of additive relations applies also to continuous quantities.

Measuring intensive quantities is yet a later achievement. Intensive quantities are measured by the ratio between two different extensive quantities (Tolman, 1917): e.g. speed is measured by km/hour and density is measured by mass/volume. However, a discussion of intensive quantities and rational numbers is beyond the scope of this paper, which focuses on natural numbers (for further discussion, see Nunes et al., 2022b).

In a representational theory, number meanings are not based on the association between words and referents, but rather on the ability to think logically about quantities and to make logical inferences on the basis of relations between quantities. Thus numbers are relational concepts that represent relations between the units within a quantity being measured; natural numbers can be used to determine the numerosity of groups of items, but their meaning does not result from the associations with numerosities.

The second meaning of numbers is formal, analytical, intrinsic to the system and defined by the rules of the system. Natural number systems use addition rules to define numbers: for example, 9 means $8 + 1$ because we add 1 to 8 to get to 9. In fact, any number has infinite analytical meanings: 9 means $8 + 1$, $7 + 2$, $6 + 3$, $10 - 1$, $11 - 2$ and so on. The rule of addition justifies the cardinal meaning of numbers: one can say that a set has, for example, 9 objects, because as each object is counted, it is added to the set of already counted items. Addition also justifies the ordinal meaning of numbers: 9 is more than 8 (and all its predecessors) because one gets to 9 by adding 1 to 8. In contrast, rational numbers are not based on addition, but on ratios, and thus lead to a rather different system of analytical meanings for numbers (for further discussion, see Nunes and Bryant, 2022 b).

Learning numerical signs and grasping their analytical meanings enables children to think about quantities in a new way. Whereas thinking in action depends on the quantities being present and on the child's ability to manipulate items, thinking that uses numerical representations is freed from this restriction. Steffe (1992) suggests that a simple example of this capacity to use numbers to think about quantities can be found when children are able to answer a problem such as: "There are seven marbles inside this cup; I will put four marble in; how many marbles will the in the cup?". The researcher puts the marbles in the cup as the child watches. If the child counts on from seven, saying "eight, nine, ten, eleven", the child has demonstrated that number words (in this case, seven) can now stand for a whole collection of objects; the seven marbles do not have to be seen to become part of a larger collection. Another example of this empowerment by the use of symbols is manifested in counting money: when a child can point to a single coin that has the value of 10 pence and count on from 10, in order to pay for example 13 pence using a combination of 10p and 1p coins, the child's thinking has been empowered by the use of symbols (Nunes et al., 2015 b). Neither of these tasks, in which a number word has to stand for a measure, is mastered by children immediately as they learn to count, but only later on, when they grasp the role of addition in counting.

One further example: If we tell a child that there are 12 objects in a box and 8 in another box, the child who understands the relational meaning of number words no longer needs to compare the quantities directly by looking at them and knows in which box there are more objects. Once again, this achievement cannot be taken for granted when a child has learned how to count: Davidson et al. (2012) showed that children who could count to, for example, 20 were not necessarily able to make comparisons between groups of objects inside boxes on the basis of their numerical labels. Thus, learning to count is necessary but not sufficient for grasping the analytical meaning of numbers: the analytical meaning depends on understanding the relational meaning of each number, defined as an addition of one to the previous number in a counting system.

Number meanings and mathematical skills

Each of these two types of number meaning is associated with a different mathematical skill: the representational meaning is related to quantitative reasoning and the analytical meaning is related to arithmetic (Nunes et al., 2016). Quantitative reasoning is the ability to make logical inferences on the basis of relations between quantities. One can reason about quantities without representing them with numbers: for example, if you know that I have red pencils and blue pencils, you can infer that the total number of pencils I have is greater than the number of either blue or red pencils, and that the number of red pencils is equal to the total number of pencils minus the number of blue pencils. You don't have to know the number of pencils to think about these relations, which are additive (i.e. based on addition and subtraction) and relate to part-whole reasoning. Thompson (1993) suggested that numbers are not relevant to quantitative reasoning; in fact, quantitative reasoning is the source for the representational meaning of numbers.

In contrast, arithmetic skill is the ability to analyze "the behavior of various numbers in four operations: addition, subtraction, multiplication, and division" (Guedj, 1998). Arithmetic skill is related to the analytical meaning of numbers. When students become able to coordinate quantitative reasoning and arithmetic, they are able to use mathematical models to understand the world; this is a radically new step beyond thinking in action, but originates from action schemas. The remainder of this paper does not focus on the analytical meaning of numbers and arithmetic, but on quantitative reasoning and on the relations between quantities that students need to master in primary school.

Quantitative reasoning and arithmetic are correlated, but distinct abilities, both theoretically and empirically, because each makes an independent contribution to the prediction of mathematics achievement, even after controlling for general cognitive ability (Nunes et al., 2012). Unfortunately, these two abilities have not been clearly distinguished in school nor in research: for example, word problems are designed to test students' skills in applying arithmetic to specific situations, but the focus is more often on the arithmetic operations that students have just learned to calculate than on the relations between quantities described in the word problems. With notable exceptions (e.g. Cheong et al., 2002; Kho et al., 2014; Ng et al., 2009; Nunes et al., 2015b; Thompson, 1993), teaching and research has been more concerned with arithmetic than with quantitative reasoning.

In summary, there are two different types of theory about how learning numerical signs changes intelligence in action. I suggest that the difference between the theories boils down to their explanations for how people learn number meanings. According to associationism, number signs acquire meaning by being associated with particular numerosities. In associationism, learning number signs doesn't actually change intelligent action because there is no principled connection between number words; this is a fundamental weakness of associationims. Attempts to deal with this weakness (e.g., Carey, 2004) led to a move away from associationism, because such attempts introduced the idea of change in the child's thinking structures, but the idea of thinking structures is incompatible with associationism. In our view (Nunes and Bryant, 2022a and 2022b), a representational theory of number meanings provides an alternative to associationism and a good description of how signs change intelligent action. When children realize that, as they count (i.e., use number signs) they are adding objects to the already counted group, they can explore the consequences of the action schemas of joining, separating and setting in one-to-one correspondence for relations between numbers of objects in collections.

Two distinct, even though correlated, mathematical abilities are related to each of the two meanings of number: quantitative reasoning and arithmetic. Because arithmetic has maintained such a prominent role in school mathematics, the teaching of quantitative reasoning has received comparatively little attention. In the two final sections of this paper, the focus is on quantitative reasoning and possibilities as well as obstacles to its teaching in school.

3. What are the Basic Types of Relations between Quantities that Students Need to Master in Primary School?

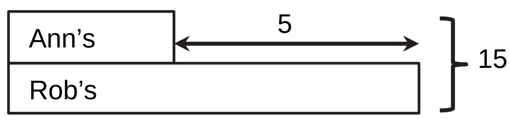
In order to think more about quantitative reasoning, it is useful to start from a classification of the types of relations between quantities that students need to master in primary school. Previous classifications by different researchers (e.g. Harel et al., 1994; Nesher, 1988; Vergnaud, 1983) have distinguished between additive and multiplicative reasoning. The distinction between these two types of relations between quantities becomes crystal clear when one considers the action schemas as well as the logical relations involved in understanding additive and multiplicative relations. Because these are the two types of relation between quantities that are crucial for learning mathematics in the first eight years in school, it is vital that research and teaching identify and promote each of these forms of reasoning. However, as Thompson et al. (2003) pointed out, there is still a school of thought that considers multiplication as repeated addition, an approach to teaching multiplication that seems to rest on the assumption that children's intuitions about multiplication is inevitably grounded in repeated addition. Teaching multiplicative reasoning as repeated addition obscures the difference between the two types of relation between quantities.

In our view (Nunes et al., 2015a), additive reasoning is grounded in the logic of part-whole relations between quantities, which in turn is anchored in the action schemas of joining, separating, and setting items in one-to-one correspondence. Multiplicative reasoning is grounded in the logic of a fixed ratio between quantities; the action schema that provides a source for understanding ratio is setting items in one-

Additive Reasoning

- Together Rob and Ann have 15 books (a sentence about a quantity)
- Ann has 5 books fewer than Rob (a sentence about a relation, which can be rephrased as Rob has 5 more books than Ann).
- How many books does each one have?

A diagram that shows part-whole relations between quantities



Multiplicative reasoning

- Together Rob and Ann have 15 books (a sentence about a quantity)
- Rob has twice as many books than Ann (a sentence about a relation, which can be rephrased as Ann has half the number of books that Rob has)
- Together they have 15 books.
- How many books does each one have?

A diagram that shows a fixed ratio relation between quantities

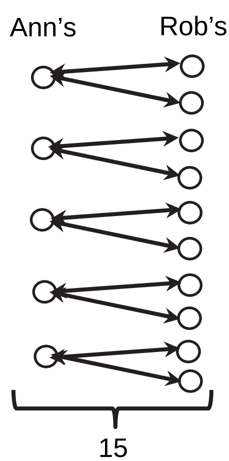


Fig. 2. The quantities in these problems are represented by the same numbers, but the relations between quantities are different. A part-whole representation of the multiplicative reasoning is possible, but conceals the fixed ratio between the two quantities.

to-many correspondence. Fig. 2 contrasts additive and multiplicative relations in problems that use the same numbers and that might seem rather similar to students, but can be solved by means of different action schemas. The diagrams used in the figure help to sharpen the contrast between the two types of relation between quantities and also illustrate how the use of a part-whole diagram is inappropriate to represent fixed ratios between quantities.

Much research has shown that 5 and 6-year old children can solve additive reasoning problems before school by using joining and separating schemas; comparison problems, which require setting items from two collections in one-to-one correspondence, are solved later (e.g. Carpenter et al, 1982; Ryley et al., 1983). Research also shows that from about the same age, children can solve multiplicative reasoning problems using the action schema of one-to-many correspondence before they have been taught about the operations of multiplication and division in school (e.g., Becker, 1993; Corea et al., 1998; Frydman et al., 1988; 1994; Kouba, 1989; Kornilaki et al., 2005; Mulligan, 1992; Mulligan et al, 1997; 2009; Nunes et al., 2008; 2010; Park et al., 2001).

Studies that analyzed children's competence in solving multiplicative reasoning problems have shown that it is critical that problem presentation fosters the use of the one-to-many correspondence schema by offering the children different types of material to represent each of the two quantities that have to be set in a fixed ratio (e.g. Ellis, 2015; Nunes et al., 2010; Piaget, 1952). In one study (Nunes et al., 2008b), children were asked to solve multiplicative reasoning problems of different types (multiplication, sharing and division in quotas) with different unit ratios (e.g., 1:2; 1:3; 1:6) using manipulatives. At the time the study was carried out, schools in England did not include the teaching of multiplicative reasoning in the curriculum until Grade 3; the children participated in this study before this teaching. The children solved three sets of problems under different problem solving conditions. For example, in one problem the children were told a boy had 2 fishbowls and that he could fit 12 tadpoles in each bowl; the task was to figure out how many tadpoles he could have altogether. In one problem solving condition, the children had different materials to represent each of the quantities mentioned in the problem: they had cut-out circles to represent the fishbowls and blocks to represent the tadpoles. In the second condition, they had materials to represent the quantity that was a unit in the ratio: in this example, they had circles to represent the fishbowls and had to imagine the tadpoles. In the third condition, they had materials to represent the second quantity in the ratio: in this example, they had blocks to represent the tadpoles and had to imagine the fishbowls.

The problems were systematically paired with each problem solving condition across children to control for problem difficulty; the order of problem solving condition varied systematically across children to control for practice effects. Fig. 3 presents the difference in mean accuracy as a function of problem solving condition.

The mean accuracy differed across the conditions: the children performed best when they had different materials to represent each of the quantities, even though they had not received instruction in solving multiplicative reasoning problems in school. This result was replicated subsequently by Ellis (2015) with younger children in their first year in school, after multiplicative reasoning problems with the support of materials became part of the English National Curriculum.

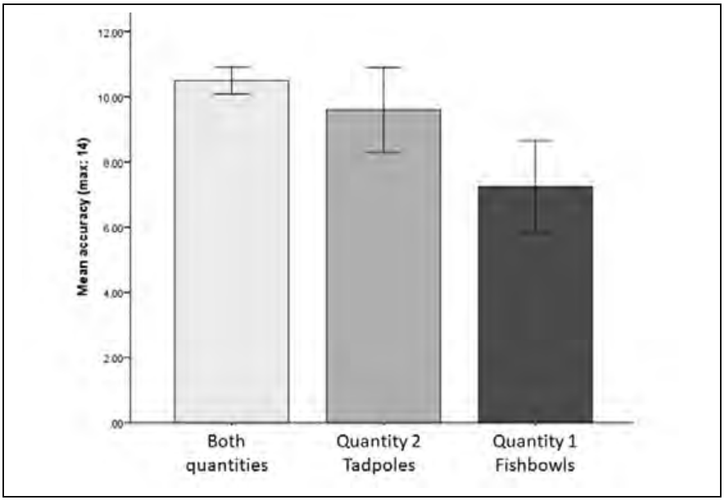


Fig. 3. Mean number of correct responses to multiplicative reasoning problems as a function of type of material that was offered to the participants to support their reasoning. Error bars represent 95% confidence interval.

In summary, even before learning about arithmetic operations in school, children can reason about relations between quantities in action. It seems common practice to encourage young children in pre-school and at the start of primary school to solve simple additive reasoning problems, but quantitative reasoning about multiplicative relations often receives little if any attention in the early years. It is possible that the belief that multiplicative reasoning stems from repeated addition is at the root of the negligence of such an important action schema in the education of young children, because it is expected that children require much practice with addition before they can start to think about multiplication. It is also possible that previous lack of awareness of the need to offer children different materials to represent the different quantities in a ratio led to the belief that young children cannot reason multiplicatively. Independently of the explanation for the neglect of teaching multiplicative reasoning in the early years, there is now plenty of research to show why it is important to include problems that involve part-whole and ratio relations in young children's mathematics education and how this can be done. I now turn to how schools can promote the use of action schemas to foster quantitative reasoning in primary school.

4. How can Schools Promote Children's Thinking about Relations between Quantities?

Even though research has shown that it is possible to cultivate students' quantitative reasoning from the time they start primary school (Nunes et al., 2007) and that this improves their learning of mathematics, quantitative reasoning has not been a traditional focus of teaching in schools (Thompson, 2011). It is now recognized that traditional mathematics curricula do not necessarily promote quantitative reasoning (Agustin, 2012; Gläser et al., 2015) and that there is an important place for quantitative reasoning in mathematics and science education (Elrod, 2014; Mayes, 2019; Panorkou et al., 2021; Smith et al., 2007). It is therefore crucial that teachers find methods to implement this new practice across the years in primary school. In this paper, some critical pointers to effective practice are presented briefly; Nunes et al. (2022a) reviewed research that documented the effectiveness of these pathways to promoting quantitative reasoning.

Encouraging the use of action schemas to solve problems: Young students aged 5-7 years can solve a variety of quantitative reasoning problems in action. Teachers can encourage students to use action schemas to solve problems by providing appropriate representations that can be manipulated (Fig. 3 for the analysis of manipulatives used in multiplicative reasoning problems). When students do not seem to know how to start, teachers can suggest a starting point: for example, they can suggest “pretend these are the … in the story” or ask “show me what happened in this story: how did it start?”

Schemas of action are characterized by being applicable in a variety of situations; thus teachers can rely on manipulatives (e.g. cut-out circles, rectangles, blocks, tokens) as representations for different objects (e.g. fishbowls, cars, children, balloons) mentioned in different problems. Students become more confident when they use the same action schema, such as one-to-many correspondence, in the context of different activities (e.g. arranging children in cars to go to the zoo; placing balloons in correspondence with children who are going to a party; paying the same amount of money for each chocolate bar; sharing pencils fairly among children; figuring out how many children can sit in a hall where there is a fixed number of chairs around a table). They have the opportunity to explore the action schema from different perspectives and to start to represent it in words, which leads to a greater awareness of the action schema’s organization and utility.

Reflecting about the consequences of actions in situations and using the action schemas forwards and backwards: Joining and separating are schemas of action; addition and subtraction are arithmetic operations. In order to take the step from thinking through action schemas to using thinking based on arithmetic operations, students need to understand addition as the inverse of subtraction and vice versa. The research literature has documented the difficulty of thinking of operations as the inverse of each other in problem solving (e.g., Brown, 1981; Booth, 1981) and it has also shown that there are effective approaches to support students to reflect about the inverse relations between operations (e.g., Fong et al., 2009; Nunes et al. 2008; 2009; Schneider et al., 2009; Squire et al., 2003).

Reflection about solutions obtained through action is facilitated by discussion, which requires the students to describe what they did in language and to articulate their thinking in words. Metacognition, i.e., thinking about one’s own thinking, plays an important role in promoting students’ thinking about relations between quantities (e.g.

Garofalo et al., 1985; Kramarski et al., 2002; Lamon, 1993; Lee et al., 2014; Schoenfeld, 1987).

From actions to drawings and diagrams: It is vital that schools promote thinking in action and it is just as essential that they help students to take the step from active to iconic (i.e. using drawings and diagrams) representation and then to symbolic representations. The step from action to drawing might seem very small to an adult, but it has an impact on students' problem solving performance. In one study carried out in London schools (Kornilaki, 1999), 5- and 6- year old children were shown rectangles that represented hutches and told that there 4 rabbits in each hutch; the students' task was to take the right number of food pellets (represented by tokens) from a box so that each rabbit could have one food pellet. Students' performance was quite good: 68% of the 5-year olds and all the 6-year olds took out the right number of tokens from the box. In another study carried out in the same schools (Watanabe et al., 2000), 6 and 7-year olds were presented with a similar problem, but this time there were no manipulatives: the students were presented the problem by means of drawings and asked to present their answers in drawing. Fig. 4 shows the item presentation and a students' response.

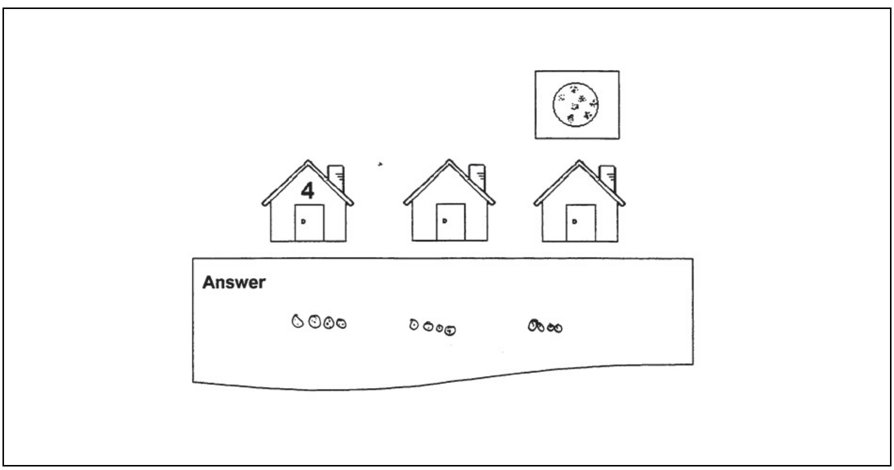


Fig. 4. The problem: In each house live 4 rabbits. They like carrot biscuits, like the one in the top box. Draw the right number of carrot biscuits so that each rabbit can have one biscuit.

In spite of the similarity in the problem, in the task presented by means of drawing the students' performance was much lower than in the task with manipulatives: 52% of the 6-year olds and 68% of the 7-year olds answer correctly. Thus it is important that schools support students in the transition from solving problems in action to solving problems using drawings and diagrams.

Drawings and diagrams are arguably a means towards abstractions of relations between quantities (Gravemeijer, 1997) and have been used widely to support quantitative reasoning in problem solving (e.g., Brooks, 2009; Csíkos et al., 2012;

Gravemeijer et al, 1990; Streefland, 1997). It is a central hypothesis in the Singapore Model Method Ang, 2001;2015; Kho et al., 2014; Ng, 2009; Yoong et al., 2009) and of the Tape Method used in Japan (Murata, 2008; She et al., 2022). There is much research about the use of drawings and diagrams, but relatively little emphasis on the distance between solving problems in action and using drawings and diagrams.

Drawings and diagrams can be used to extend students' reasoning to novel and rather problems. Cartesian or product of measures problems have been found to be significantly more difficult than the type of multiplicative reasoning problems presented earlier on in the paper (Brown, 1981; Vergnaud, 1982). However, they can be solved by the same action schema of one-to-many correspondence with some support. Figure 5 shows an example of a diagram used to encourage students aged 10–11 years to extend their multiplicative reasoning to Cartesian problems (left). Students who participated in this program initially had cut-out shapes to solve Cartesian problems and were later encouraged to use drawings and diagrams. At the end of the program, they used the same approach to describe the sample space in probability problems (Nunes et al., 2014).

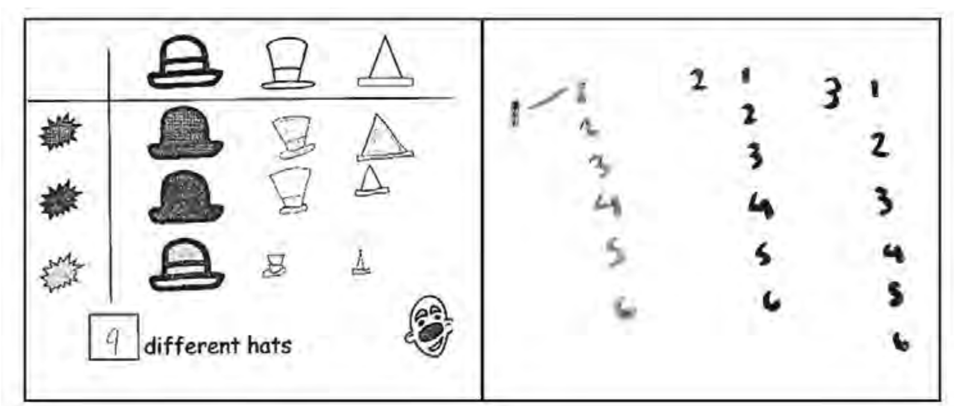


Fig. 5. Left - the problem and a student's production: A shop that sells hats lets the client choose from three different materials and three different shapes of hats. How many options does the clown have to choose from? Right – the problem and a student's production: When you throw two dice and add the numbers, is there one total that is more likely to come up than all the others?

5. Summary and Conclusions

This paper starts from the recognition that mathematical modeling is a form of intelligent action grounded in cultural mathematical practices. The aim of the paper is to explore what this assumption means for learning mathematical modeling from the perspective of developmental psychology. This aim was pursued by considering four fundamental questions. In this final section, I address these questions briefly, without contrasting the theoretical options adopted here with other possible views.

1. What is the origin of intelligence?

The origin of intelligence is in action. Psychological research about intelligent action has documented problem solving by babies before they have learned language. There is little controversy in psychology about the idea that intelligence precedes language. Piaget studied in detail how instincts and reflexes change as they are used and give rise to intelligent action. He adopted the concept of action schema to describe actions that are organized and applicable in a variety of situations and to a variety of objects, allowing the child to think in terms of relations between classes of objects and positions in space. Relational thinking is the essence of intelligent action.

2. How does the learning of numerical signs change children's thinking about quantities?

What happens to children's thinking when language is learned has proven to be hugely controversial amongst psychologists. The controversy starts with how children learn the meanings of words. In this paper, the theory adopted is the one held by eminent developmental psychologists, such as Piaget and Vygotsky, which proposes that word meanings are concepts rather than associations between two stimuli, words and objects. Therefore, children's conceptual development is intimately related to the meanings that they attribute to words. When children learn conventional systems of signs, such as natural language and number systems, they need to master two types of meaning: a representational meaning, that connects the signs to something external to the system of signs, and an analytical meaning, that is defined by the system. In the context of natural numbers, children must learn that, as they count, they are adding: the meaning of each number is based on a relation to the previous number $(+1)$ and also based on a relation to its successor (-1). This relational conception of number meanings implies that any number has an infinite number of analytical meanings (e.g. 8 means $7+1$, $6+2$, $5+3$, $9-1$, $10-2$ and so on).

Mastery of the relational meaning of numbers transforms counting. For example, children become able to count on from the number that represents a hidden group of objects and to count on from the value of a coin; learning the relational meaning of natural numbers means that children no longer need to see the items in order to account for them when trying to find the cardinal of a collection or the total amount of money that someone has. Additive relations between units are represented by natural numbers and ratios are represented by rational numbers. Learning conventional numerical systems empowers children's thinking.

3. What are the basic types of relations between quantities that students need to master in primary school?

In primary school, students need to master two types of relations between quantities: part-whole and ratio. Part-whole relations are additive and rely on students' action schemas of joining, separating and setting items in one-to-one correspondence. Ratio relations are multiplicative and rely on the action schema of one-to-many correspondence. Additive and multiplicative reasoning have distinct origins and students can be helped to recognize their difference from an early age by solving both additive and multiplicative reasoning problems in action.

4. How can schools promote students' thinking about relations between quantities?

Research has shown several ways in which schools can promote students' thinking about relations between quantities. The first step is to engage students in thinking in action: modeling a variety of story problems with manipulatives is an important pathway to develop their reasoning about relations between quantities. The next step is to support students' thinking about their problem solving in action. Talking about processes of solving problems enhances metacognition, which in turn promotes further understanding of the problem solving processes. The use of drawings and diagrams is also a pathway towards abstraction. Diagrams aim to capture the essential relations between elements rather than the particularities and support talking about ideas by pointing to aspects of the diagram.

To conclude, it is important to bear in mind that the development of intelligent action based on cultural tools, such as number systems and arithmetic operations, is not an instantaneous process: it takes time and nurturing. Schools are assigned the task and awarded the privilege of promoting students' development, from thinking in action to mathematical models.

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https://doi.org/10.1142/9789811287152_0019

Lecture of Awardee 4

Developing the Research Programme on History of Mathematics Teaching and Learning

Gert Schubring[1]

ABSTRACT Having been distinguished by ICMI with the Hans Freudenthal medal for having established the research programme on the history of mathematics education, and this area being relatively new within mathematics education, I thought it adapted to expose in my awardee lecture, in a somewhat autobiographical manner, the development of this research programme, to show its rationales, its methodological challenges and approaches and the relations to mathematics education at large.

Keywords: Mathematics history; Relation to teaching; Research methodology.

1. My Multiple Contacts with Hans Freudenthal

I should like to express my profound gratitude to ICMI for honouring me with its Hans Freudenthal medal. I might use this occasion to mention that I am probably – among the Hans Freudenthal medal awardees — the one who had the most differentiated types of contacts with him.

- My very first contact with him was on mathematics, by studying his results in the theory of Lie algebras, when I was working at the Mathematics Institute of Bonn University for a PhD thesis in mathematics.
- The next contact was a review he had written about the first paper published in 1974 by the group around Michael Otte at the just created *Institut für Didaktik der Mathematik* of Bielefeld University, severely criticising this contribution to the emerging of mathematics education as a scientific discipline. Freudenthal had qualified it as a product of an alleged Bielefeld *Zauberberg* – magic mountain (Freudenthal 1974, p. 124).
- Thereafter, I met him eventually personally at various mathematics education meetings, in particular those organised by Hans-Georg Steiner at Ohrbeck, a conference centre between Bielefeld and Osnabrück.
- At one of these meetings at Ohrbeck, in one of the usual getting together in the evening, I was instigated by him to my research about the history of negative numbers: he had claimed that the conception of negative numbers

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had already become settled since the 16th century – a remark which arose my doubts about traditional lore of mathematics history.

- And more recently, I was remarking the lasting influence of his conceptions for realistic mathematics education and its international impact via, in particular, the PISA testing machine (see Schubring 2021).

1.1. How to reach from mathematics to the history of its teaching?

As a first issue of my development I should like to comment upon how one can reach from studying mathematics to research on the history of mathematics education. Let me first report about my studies. I began studying mathematics and physics in 1963, at the University of Mainz (Federal Republic of Germany). In 1965, after having obtained the *Vordiplom* in mathematics and the *Vordiplom* in physics (a kind of intermediate exam), I moved to Bonn University, then a leading German research centre in mathematics. Finishing my studies there in 1969, I obtained the *Diplom* in mathematics — before the deconstruction of the German university structures by the Bologna process, from 1999, a highly valued degree in mathematics. I dare say that I never heard something about the history of mathematics or its teaching.

Shortly later, in 1973, I became a member of the just created *Institut für Didaktik der Mathematik* (IDM), a part of the also recently founded University of Bielefeld – aspiring to revive the research ethos of Wilhelm von Humboldt. There, I worked at first in the group led by Michael Otte — a group constituted of newcomers to mathematics education and thus not prejudiced by traditional conceptions of mathematics education, challenging traditional views and establishing new theoretical perspectives (see Schubring 2018). This group conceived of these perspectives from a strong interest in philosophy and epistemology of mathematics. Thus, there was always an interest present in the history of mathematics, though without any experiences in pertinent research.

The first publication of the group was the study *Zu einigen Hauptaspekten der Mathematik-Didaktik* (Otte et al., 1974). As a theoretical outline, it met an unprepared and rather shocked public. Freudenthal was one who voiced this strangeness (Freudenthal 1974).

My PhD thesis of 1977, on the genetic principle in mathematics education, had already searched historical roots of this prescriptive conception, with its great number of differing representatives, back until the 17th century (Schubring, 1978).

1.2. From ingenious approaches towards research standards

Focussing after finishing the PhD thesis upon historical research, I became aware that I did need professional formation in standards of such research. Happily enough, Bielefeld University itself provided an excellent context for such an extension of my formation. It constituted a highly propitious atmosphere for interdisciplinary research and for new approaches:

- there was the new branch of social history, established within historiography by Hans-Ulrich Wehler (1931–2014);
- there was a proper institute for science studies: the *Universitätsschwerpunkt Wissenschaftsforschung*: with the sociologists Peter Weingart and Wolfgang Krohn, and in particular Peter Lundgreen (1936–2015), a historian of science and technology, who would become an important partner of cooperation;
- their focus of research was the social function of the sciences, history of disciplines, and institutional history of science. All these approaches should become basic elements in my research;
- another sociological theory, which should become important for my understanding of the social functions of science and of education was the systems theory just developed in Bielefeld by the sociologist Niklas Luhmann (1927–1998) (see more below);
- cooperation with my friend Hartmut Titze, professor of history of education, specialised in research about quantitative data of the development of the German educational system.

Thanks to this broad net of connections, communications and cooperation, I had been introduced to methodological approaches for historical research understanding history of science not as a history of ideas, but as a complex system, of interacting sub-systems, which one can call social history of science, focussing in particular on:

- the conceptual development of science;
- processes of discipline-building and professionalisation;
- social supports for science: institutionalisation and career structures in related labour markets;
- priority of access to sources;
- use of quantitative and qualitative methods.

1.3. The first research project

Based on this autodidactic formation for interdisciplinary research, I chose independently, after the PhD, my first research project. It was on history of mathematics, but soon proved to evoke questions needing research into the history of mathematics teaching.

Background was the focus of some research into the history of mathematics in my group at the IDM: this focus was on the interrelation between the poles of research and education. According to my new methodological orientations, I understood the relations between development and application of science as missing and set out to search a subject for a pertinent case study.

In 1978, I chose as subject for such a case study: the projects for creating a Polytechnic Institute in Berlin, in Prussia (between 1817 and 1845), in the context of pure mathematics then there dominating — thus confronting two quite opposed patterns for the practice of mathematics.

Although there had been four phases to create the Institute, all of them had failed. There were several publications about these attempts, but none gave satisfactory reasons for the failures.

Upon searching for the reasons more effectively, it became clear that I should have to analyse the sources intensely and extensively. And for identifying the pertinent sources and for getting access, I am owing very much to the advice by Wolfgang Eccarius (born 1935), who had done important research on the history of mathematics — and he, already too, on the history of its teaching! Meanwhile he turned to studying orchids ...

His paper on the projects was the nearest one to the analysis of sources (Eccarius, 1977), and I learned from it that the major archival document was the volume with the deliberations within the Prussian Ministry of Education. And the archive of this Ministry was preserved, not in Berlin, but due to the last stages of World War II in Merseburg, in the *Deusches Zentralarchiv* of the GDR.

Using an archive in the other German state was rather adventurous at that time. I visited therefore Eccarius, living in Eisenach, in the GDR, in a not really legal manner during a trip to the Leipzig fair, and was generously received and perfectly informed about how to access and how to use files there. Therefore, I became first confronted and then familiar with the reality of working in archives exploring old documents.

The first challenge there was to decipher handwritten manuscripts. This turned out to be really challenging: German people used another script in the 19th century, no longer in use today — I had to learn to decipher texts written in this script unknown to me. The problem became worsened by understanding the practice that the officials wrote their drafts of documents in a type of shorthand script, known to the staff in the bureau who had to transform this into a fair copy which would be sent to the addressee while the draft remained in the ministerial files. See here an example, the last lines of a draft by Johannes Schulze, the powerful official of the Prussian Education Ministry, of 1828 (Fig. 1).

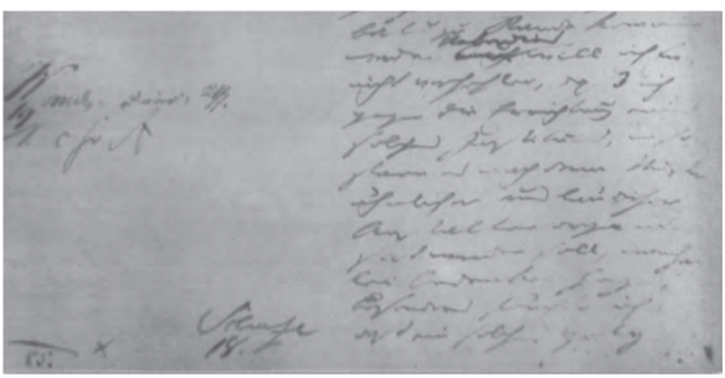

Fig. 1. Example of manuscript draft writing, by Johannes Schulze, in the Prussian Ministry's file on the Plans for the Polytechnic Institute, fol. 2v (Schubring, 1981)

My approach for analysing documents can be called “holistic”: not just searching a few documents in the files, supposedly being the decisive ones, but to assess the files in their entirety. Although it had been quite difficult at that time, I had asked the archive in Merseburg for complete copies of the entire volume regarding the Polytechnic Institute, and of the related volumes in the archives of the War Ministry, and the Commerce Ministry — for not missing any possibly relevant detail. Clearly there did not yet exist the technique to obtain digitalised copies; the standard form were microfilm copies. And paying invoices from the FRG to the GDR implied then quite intricate procedures.

Eventually, I obtained significant results. The major result was to succeed in revealing why all the four attempts for creating a polytechnic institute had failed and, at the same time, why the accounts of the projects given so far had failed, too, in understanding the reasons: August Leopold Crelle (1780–1855), the mathematics advisor of the Ministry, had planned the Institute to promote *pure* mathematics:

> So it is also important that pure mathematics should be explained in the first instance without regard to its applications and without being interrupted by them. It should develop purely from within itself and for itself. For only in this way can it be free to move and evolve in all directions. In teaching the applications of mathematics it is results in particular that people look for. They will be extremely easy for the person who is trained in the science itself and who has adopted its spirit (quoted from Schubring, 1989a, pp. 180f.).

Crelle explained his radical transformation of the nature of a polytechnic institution by a peculiar re-interpretation of the Paris model, claiming it to be: “an institution having as its essential task the training of mathematics teachers” (ibid.).

Thus, the projects eventually were not realised because one became aware that one had already an institution for training mathematics teachers: the universities, reformed in Prussia according to neohumanism!

This principal result led me to a new challenge: Researching the history of mathematics in Prussia, but also in Germany, afforded to investigate the history of the training of mathematics teachers! This led me to then to my second research focus, to the history of mathematics teaching and learning.

1.4. Entering the scientific community

Before addressing this second focus, let me first tell about publishing research results and entering the scientific communities. In fact, there occurred happy sequels of my research upon the never realised Polytechnic Institute: thanks to a circulating first version of my research results, I was invited to the first congress on social history of mathematics, in Berlin, July 1979, organised by three key researchers in mathematics history: Henk Bos, Herbert Mehrtens and Ivo Schneider. There, I had the chance to meet and get in contact with leading researchers in the area, thus establishing communication with the community.

As a result, already a second paper was published, in the Proceedings of this congress, on the conception of pure mathematics, then dominating in Prussia (Schubring 1981a). Moreover, there occurred another happy instance, regarding the publication of my first research. In fact, I was faced with the problem: how to publish as a youngster, without any international publication experience, a paper in good idiomatic English and in a scholarly style? I enjoyed an incredibly strong assistance by the two historians of science Roy Steven Turner and Lewis Pyenson, for the important international journal: *Historical Studies in the Physical Sciences.* There my paper was published in 1981 (Schubring, 1981b).

In continuing now research, I consolidated my research approaches. It proved as essential to have as a clear priority to get access as much as possible to primary sources; and especially for history of science the focus was on using the *Nachlass* of scientists. Actually, it proved that searching the *Nachlässe* had not been so far a major concern of historiography. I dedicated therefore much energy to search and localise *Nachlässe*, either already organised, kept in a library, archive or an Academy, or kept without an inventory, or even being preserved in the hands of some descendant.

1.5. Two major research strands

Based on the methodologies developed and meanwhile improved, and based on the research experiences made, I focused since now on two major research strands. The fist became:

- Analysis of the development of mathematical concepts.

A major issue for this research strand became, instigated by Freudenthal's remark, the history of negative numbers. And this research led me to a new pattern of conceptual research, to "social history of ideas", since the results made me understand that national communities, based on the specific structures of the social systems in their country, proved to practice proper epistemologies.

This conceptual research kept me occupied for over 20 years and resulted eventually in a voluminous book, revealing conceptual development as a contextualised history: *Conflicts between Generalization, Rigor and Intuition. Number Concepts Underlying the Development of Analysis in 17th-19th Century France and Germany* (Schubring, 2005).

2. The Second Research Strand

This second strand became the History of the Teaching and Learning Mathematics. It was based on methodological approaches and on various key results of the first research strand. I should name in particular:

Understanding the history of a school discipline as quite more complex than a scientific discipline, requiring even more an interdisciplinary approach:

- regarding methodology, modern society is structured, according to Luhmann's system theory, as a net of interacting functional sub-system;

- the education system, the labour market with its professional sub-systems, the system of the sciences with its institutions.

Origins: due to my results of research on pure mathematics in Prussia, the basis of pure mathematics practice had proved to be the profession of mathematics teachers in the *Gymnasien* (college) of Prussia.

This origin provoked the question: how did this profession come about? This profession of mathematics teacher turned out to constitute a historical novelty. I led a two-year postdoc project for these researches, 1981 to 1983. Of these two years, summing the days of research in the various institutions, I had spent six months in archives and libraries. The results of this quantitative and qualitative research resulted in the book: *Die Entstehung des Mathematiklehrerberufs im 19. Jahrhundert. Studien und Materialien zum Prozeß der Professionalisierung in Preußen (1810–1870)* (Schubring, 1983, second edition 1991).

Besides studies on the practices of mathematics teaching at Prussian Gymnasia during this period and searching for all kind of documents about mathematics teachers, a major dimension of the project was the evaluation of the teacher examination introduced in Prussia in 1810: to assess, firstly, whether the intention was to have encyclopaedic-qualified and multi-disciplinary practicing teachers or to provide disciplinary-specialised teachers. In the archive of the Prussian Education Ministry, it was preserved the collection of the files of the at first three examination boards (wissenschafliche Deputation), as well as from 1817 the six, and later seven, boards (wissenschaftliche Prüfungskommission) in the university towns. The second assessment was for identifying those who were declared qualified to teach mathematics. The revealing results were that the intention was to provide scientifically trained teachers, and not to continue with the earlier practice of encyclopaedic formation and practice. And it became documented that the — at first abstract — conception of a mathematics teacher was accepted by young people who began to study mathematics in a specialised manner and who qualified as a teacher in an impressively growing manner, thus constituting mathematics teaching as a profession (Schubring, 1991, pp. 126 ff.).

This quantitative analysis was complemented by a qualitative analysis, using the method of prosopography: having identified to "new" mathematics teachers, graduates of specialised studies and the teacher examinations from 1810, and having found also those who had taught mathematics before the reform period, thanks to histories of the older Gymnasia, I undertook it to systematise the profiles of their teaching activities. These profiles allowed it to group the teachers into three subsequent generations:

- those who taught already before the reforms of 1810: in general, without academic studies or with not specialised ones (encyclopaedic), and recruited without an exam;
- recruited since the beginning of the reforms, frequently without an exam, often just with autodidact studies — and frequently with weak teaching success;

- generation (since about 1820) with specialised academic studies and qualified with an exam (ibid., pp. 156).

A last stage of quantitative-qualitative analysis was devoted to the question about school teaching reality, namely whether the teaching of those who taught in the school reality mathematics corresponded to the qualifications attested to them in their exam. I undertook it to assess this question for the Gymnasia in three of the seven Prussian states, in distances of five years for these schools. The results showed a remarkable degree of correspondence between the attested teaching competence and the attributed classroom disciplines (ibid., pp. 147 ff. and especially p. 157).

2.1. Extending the research programme internationally

While my research into history of mathematics and of mathematics teaching at first had been confined to one of the great German states, to Prussia, I extended its reach rapidly, from Germany to an international scope.

A first extension had been to the history of teaching the sciences, in Prussia, too, and then to the other German states. This was due to an invitation to participate at the great project on the German history of education: the *Handbuch der deutschen Bildungsgeschichte,* editors Karl-Ernst Jeismann and Peter Lundgreen. For its volume III, for the period 1800 to 1870, I had been invited to write the chapter on the history of teaching mathematics and the sciences in Germany (Schubring, 1987a).

The first instance to extend the research programme internationally was the invitation by Roland Stowasser, colleague at the IDM, to update an earlier entry in *The International Encyclopedia of Education*, edited by Torsten Husen and T. Neville Postlethwaite, on the history of mathematics teaching. This entry was written together with Christine Keitel and Roland Stowasser (Keitel et al., 1985). The extension to an international vision entailed new methodological problems and approaches.

My aim became increasingly to investigate structural patterns determining general characteristics in order to differentiate them from patterns characteristic for one (or more than one) country.

Thus, I embarked on studying such patterns from the Antiquity, and focussed at first upon comparing France and Germany where the developments in the 19th century had been very different (Schubring, 1984). Thereafter, Italy presented for me a highly revealing case for confronting global patterns with local ones. I had become aware at first of Italy presenting a challenge when I learned that Legendre's geometry textbook had been refused there, since 1860, together with Euclid's *Elements* having been prescribed as a textbook in 1867. The refusal of Legendre's book, having been translated into Italian already various times and praised in all other countries for its rigour, was legitimised by its condemnation due to alleged lack of rigour (Schubring, 1994).

The surprise about Legendre's refusal and the adoption of Euclid was reinforced when I remarked as a problem for conceptualising history of mathematics teaching: all publications by Italians until then on their proper history of mathematics teaching had shared this denunciation of Legendre's textbook (see Vita, 1986, p. 7).

This peculiarity of the Italian case afforded to abandon to be kept within the educational system of one's own country, being no more bound to accept all characteristics of this country as evident and natural, and to be able to rather questioning all what constitutes matters of course of this system and to thus detect them as historical *variables*.

The Italian case made thus even more imperative to reveal social determinants of mathematics teaching: secondary schools in Italy were still functioning according to classicist values. This dominant classical education induced mathematicians to adapting the mathematics taught to the values of classicism.

It proved to be likewise imperative to consider epistemological determinants of the views of mathematics as a school subject. Mathematics was understood in Italy as also rooted in classicist values: geometry should be taught according to the values attributed to the Greeks: geometry as strictly separated from algebra, while Legendre's approach was: mutual support between geometry and algebra (Schubring, 2004).

The cases of the various German states, of France and of Italy led to conceive of research into the history of mathematics teaching and learning, on the one hand, as an interdisciplinary programme and, on the other hand, as an internationally and transversal comparative programme — to unravel, over periods and epochs, structural patterns for the functions of mathematics teaching. As one such rather universal pattern, the two-polar-relation between general education and professional functions had proved to provide an effectively structuring characteristic. My paper delivered at the so-called Fifth day of ICME 6 at Budapest in 1988, which had introduced to consider broader contextual research agendas in mathematics education within the issues of ICME, presented a first conceptualisation of this research programme (Schubring, 1988).

2.2. The traditional research focus analysing textbooks

The traditional practices of studying the history of mathematics teaching was to understand it as collecting facts, easily accessible, conceiving of the history of mathematics instruction as a series of administrative decisions, which supposedly were transformed into classroom practice. According to this perspective, the history basically is a history of the curriculum, of the syllabus, managed by centralist authorities — and transformed into schoolbooks.

The analysis of mathematics schoolbooks used to be restricted to just one book, assuming this should already reveal its meaning within the history of school mathematics. There, I developed a methodology for connecting textual analysis with *con*textual analysis; a first elaboration, with great overall reception is the paper (Schubring, 1987b).

This approach, which thus means a hermeneutical one, was established at first for one author, Sylvestre-François Lacroix (1765–1843), in his time the dominant French textbook entrepreneur, where I proposed to investigate the production along three dimensions, for contextualising the respective schoolbook — already understood not as an isolated document, but as an element in a series of editions, which indicate already changes requiring understanding:

- the first dimension consists in analysing the changes within the various editions of one textbook chosen as a starting-point, say an algebra textbook or an arithmetic one;
- the next dimension consists in finding corresponding changes in other textbooks belonging to the same *œuvre,* by studying those parts dealing with related conceptual fields, say geometrical algebra, trigonometry, etc.,
- the third dimension relates the changes in the textbooks to changes in the context: changes in the syllabus, ministerial decrees, didactical debates, evolution of mathematics, changes in epistemology, etc. (Schubring, 1987b, p. 45).

I developed this methodology, further — as of hermeneutical textual analysis, being contextualised in the respective educational system and the development of the mathematical knowledge, related to its respective elementarisations.[2] This proved to become a general approach, practiced in my endeavour of a first historical analysis of mathematics textbooks, from Antiquity, over periods and cultures (Schubring, 2003).

2.3. International establishment of the research programme

The systematic development of this research programme for international, comparative analyses, well disseminated and received by books, chapters and papers in journals, turned into a broad international area of research.

A decisive step for this new degree was enhanced at ICME-10, in Copenhagen, in 2004. Its president, Mogens Niss, proposed a new Topic Study Group, on the history of mathematics teaching and learning. I was called as one of the co-chairs. The TSG worked very successfully, and became a permanent TSG, realised at each ICME since then. As a thematic issue of the journal *Paedagogica Historica*, the main contributions of the TSG at ICME 10 were published, in 2006.

This first international event meant the take-off for the research programme. Already in 2006, the first international journal dedicated to it was launched: the *International Journal for the History of Mathematics Education*, IJHME (Fig. 2).

[2] This can be called the objective hermeneutics following its founder in philology, Friedrich August Wolf, and in philosophy Friedrich Daniel Schleiermacher; it should not be confounded with what one call subjective hermeneutics — from Wilhelm Dilthey to Hans-Georg Gadamer, aspiring only a personal sense-making and empathy (see Schubring, 2005, pp. 4).

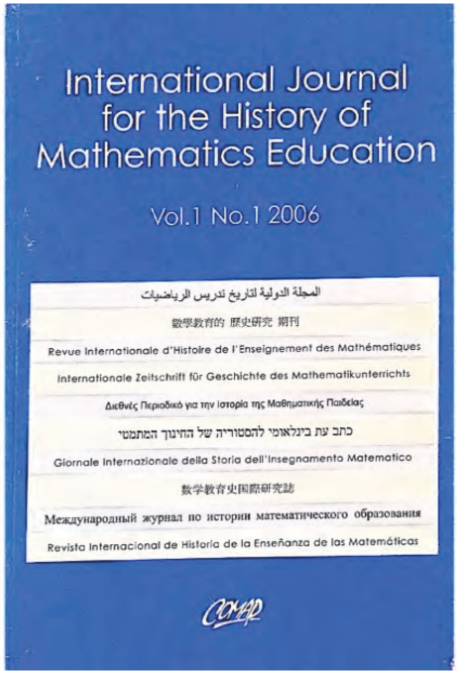

Fig. 2. Cover of the first issue of IJHME

And in 2009, Fulvia Furinghetti and Kristín Bjarnadottir launched the first *International Conference on the History of Mathematics Education*, ICHME. It took place in June 2009 in Reykjavik, Iceland. ICHME became a series, realised bi-annually. So far, there have been six ICHMEs:

- ICHME 2: Lisbon, Portugal 2011,
- ICHME 3: Uppsala, Sweden 2013,
- ICHME 4: Torino, Italy 2015,
- ICHME 5: Utrecht, The Netherlands 2017,
- ICHME 6: Luminy, France 2019

ICHME 7 is planned, due to the pandemics, for September 2022, in Mainz, Germany. Proceedings were published for each of the ICHMEs, with the recurring title *Dig where you stand*.

The most significant sign for the consolidation of the research programme was the publication of the *Handbook on the History of Mathematics Education*, edited by Alexander Karp and Gert Schubring. It presents the history from Antiquity till the end of the 20th century, according to cultures and periods, in 35 chapters, resp. sub-chapters.

The latest development in this international research area is the establishment of the series, in 2018: *International Studies in the History of Mathematics and its teaching*. So far, the following volumes were published in the series:

- Gert Schubring (ed.), Interfaces between Mathematical Practices and Mathematical Education. Cham: Springer, 2019.
- Alexander Karp (ed.). National Subcommissions of ICMI and their Role in the Reform of Mathematics Education (Cham: Springer, 2019).
- Évelyne Barbin, Marta Menghini and Klaus Volkert (eds.), Descriptive Geometry, The Spread of a Polytechnic Art (Cham: Springer, 2019).
- Alexander Karp (ed.), Eastern European Mathematics Education in the Decades of Change (Cham: Springer, 2020).

2.4. Recent dimension: relations between coloniality and decoloniality

This new aspect is instigated by a major break in the history of mathematics teaching: the ex-colonies reflecting which mathematics teaching did their colonial powers import and which are they needing? In this decolonialising movement, there has been a conviction, as a complement to the usual assumption of a universality of mathematics, a likewise universality of Western school mathematics was assumed, despite relating that each colony had to follow its respective metropolis, while not being aware of the significant differences between the educational systems of the metropoles and thus also of their conceptions of mathematics teaching:

> During the colonial period, the school system was patterned exactly after that of the colonising country. The norms of fit between school and society were quite precise: the school system was to come as close as possible to that of the mother country. It should produce graduates that would fit into the civil service and who would do well in universities in the mother country. With independence the above norms of fit between school and society were seen with mixed feelings. Leaders became conscious that a school system developed according to such norms would, among other things, simply contribute to the brain drain. They also became conscious that the school system had to respond to different cultures and classes in the country: a westernized elite, a growing lower middle class, urban workers, a traditional rural sector (Bienvenido Nebre, 1988).

Despite not being aware of the metropoles' decisive differences regarding social strata and hence regarding mathematics curricula, this was a first document calling for the necessity of decolonialising the teaching conceptions in the former colonies of the imperialist powers. It should be mentioned that there is also a widespread conviction of a global uniformity of the mathematics curriculum. The Finnish mathematics educator George Malaty is maintaining this:

> Till the end of the school year 1957/1958 school mathematics was quite the same everywhere. Primary school mathematics mostly consisted of arithmetic. Secondary school mathematics was mostly algebra and plane Euclidean geometry, and in the upper grades algebra, analytic geometry, solid Euclidean geometry and

trigonometry. In the 1950s calculus teaching spread in upper secondary school (Malaty, 1999, p. 231).

According to Malaty, diversity began as a consequence of the Sputnik shock and the ensuing modern mathematics movement launched by the Royaumont Seminar in 1959. There is likewise a widespread opinion that modern mathematics affected all countries in the same way.

There are examples of decoloniality practices in former colonies. For a certain time, after independence in 1975, schoolbooks in Mozambique evidence these approaches (Fig. 3). Yet, these achievements need not to persist. Pressures by the World Bank and International Monetary Fund are exerted on developing countries for "liberating" their national schoolbook market to international publishers, in order to obtain grant development aid funds (Schubring, 2017).

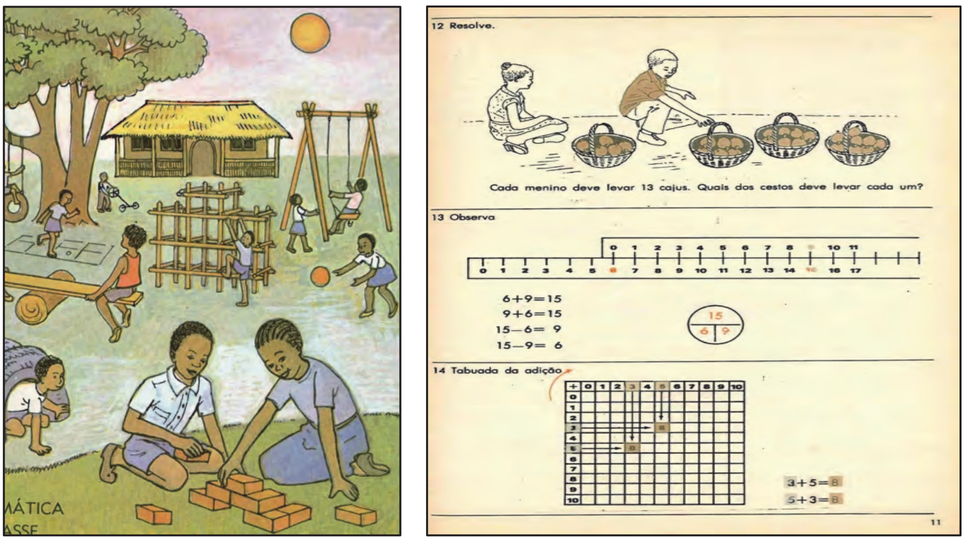

Fig. 3. Left: Cover of Draisma et al., *Eu gusto de Matemática*. Classe 2. 1984; right: ibid., p. 11.

And there are even rather global new coloniality pressures. they are exerted by the Organisation for Economic Cooperation and Development, OECD. This organisation, created by the USA after World War II to reconstruct the "West" as a fortress against the "East", and thus practicing politico-economic agendas also for the international education sector, is acting decisively for homogenising the mathematics curriculum globally, via pressures to the member governments to achieve good results in the international ranking established by the PISA test achievements (see Schubring, 2021).

I should like to mention here an excellent research project upon coloniality and decoloniality: a research into the history of mathematics teaching in French colonial times and in Khmer post-colonial times in Cambodia, by Sethikar SamAn (2018).

Fig. 4. Cover of Ramus's 1569 book

3. Closing Remarks

I should tell you that I have a dream: to edit the first book ever published on teaching of mathematics. This was the book published by the French Petrus Ramus in 1569: *Scholarum Mathematicarum Libri Unus et Triginta*, distinguished by the methodical critique of Euclid's *Elements*, as not suited as a textbook for teaching (Fig. 4). It inaugurated the special French way of conceiving of teaching geometry, differing from other European countries. The voluminous book, with 320 pages, written in Latin, but with many Greek terms inserted, is difficult to read due to the many abbreviations used then by printers to economise casting letters.

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Lecture of Awardee 5

Advocating for High Quality Mathematical Access for Each and Every Child: Our Collective Work, Our Passion, and Our Future

Trena Wilkerson[1]

ABSTRACT It is imperative that we advocate for the highest quality mathematics for each and every student across the world. All students must have access to mathematical learning experiences that will prepare them for success not only in the classroom but also to lead our world in the future. While we have seen much progress in mathematics education over the last 30 years, we continue to face significant challenges related to access. Disparities in learning opportunities based on race, ethnicities, class, language, gender, and perceived mathematics ability are far too prevalent in school mathematics—this has been made more evident during the COVID-19 pandemic. We must address these disparities to ensure equitable mathematical opportunities for each and every student. The National Council of Teachers of Mathematics identifies four key recommendations that serve as a catalyst for change to launch each and every student on a successful journey with mathematics. These are discussed in this paper.

Keywords: Advocacy; Teaching practices; Mathematical understanding

1. Introduction

The National Council of Teachers of Mathematics (NCTM) is honored to receive the International Commission on Mathematical Instruction (ICMI) Emma Castelnuovo Award for Excellence in the Practice of Mathematics Education. It is an honor to receive such a prestigious award that was named after Emma Castelnuovo, an Italian mathematics educator, to honor her pioneering work in mathematics education. Her work aimed at a way of teaching that actively engaged students, marked a key point in history for teaching and learning mathematics that fostered a discovery learning environment for all students from elementary through university. NCTM is honored to continue to build on this legacy so that each and every student has an engaging, high-quality experience in learning mathematics.

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As President of the National Council of Teachers of Mathematics, I would like to thank The United States Commission on Mathematics Instruction, chaired by John W. Staley for submitting the nomination of NCTM for this award. The U.S. Commission includes Solomon Friedberg, James Roznowski, James Barta, Marta Civil, Robert Gould, Maria Hernandez, Ilana S. Horn, Chris Rasmussen, Padmanabban (Padu) Seshiyer and April Strom. In their nomination they highlighted four continuing priorities in NCTM's work:

- NCTM has served the mathematics education community (nationally and internationally) for 100 years by providing leadership, publications and resources, professional development, and networking opportunities.
- NCTM has served its membership by supporting and growing educators and involving them in many of the organization's initiatives and projects, and providing various opportunities to develop members' leadership skills.
- NCTM continues to advocate for high-quality mathematics teaching and learning for each and every student. This advocacy extends to the work that helps educators who choose to advocate with their elected officials and policymakers. And
- NCTM continues to build and value collaborative relationships with educators throughout the world.

The U.S. Commission noted that the National Council of Teachers of Mathematics is the public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for each and every student through vision, leadership, professional development, and research. We also are grateful for the multiple letters of support that were included in the nomination.

Our thanks also goes to the International Commission on Mathematical Instruction for awarding the 2020 Emma Castelnuovo Award to the National Council of Teachers of Mathematics. I would like to thank Dr. Jill Adler, President of the International Commission on Mathematical Instruction. It was an honor for NCTM to receive notification of the 2020 Emma Castelnuovo Award from Dr. Adler in October of 2019. I also thank Professor Konrad Krainer, Chair of the Emma Castelnuovo Awards Committee and the entire committee for their work in reviewing nominations. We are honored.

NCTM's work in mathematics education is consistent with International Commission on Mathematical Instruction's principles:

- The development of mathematical education at all levels; and
- The promotion of reflection, collaboration, exchange, and dissemination of ideas on the teaching and learning of mathematics from the primary to the university level.

NCTM's mission is to advocate for high-quality mathematics teaching and learning for each and every student from early childhood through secondary school and beyond. NCTM includes mathematics educators from preschool, elementary, middle grades, high school, universities and colleges across the United States and

Canada and 171 other countries across the world with over 30,000 members and more than 230 Affiliates. NCTM also established the International Corresponding Societies (currently 19 organizations with representatives from South and Central America, Europe, Asia, Africa and Australia) to build ties with professional associations of mathematics education in other countries. In addition, NCTM has supported several initiatives with educators in Latin, Central, and South America.

I would like to take this opportunity to recognize Ken Krehbiel as Executive Director of NCTM. He has provided professional leadership for the organization for over 20 years. He has guided NCTM in multiple efforts to further the mission and vision of the organization. I also want to acknowledge the work and leadership of Dr. Robert Q. Berry, III who was NCTM President at the time of the nomination and continues as Past President. Dr. Berry is an accomplished writer, researcher, presenter, and leader in mathematics education. It was in his role as president-elect that NCTM's Catalyzing Change initiative for high school was launched under the leadership of then NCTM President Matt Larson. Dr. Berry has expanded the initiative to address early childhood, elementary and middle school levels. I was fortunate to be in the position of president-elect at that time to work with him and the other outstanding writers on the Catalyzing Change position and series of publications. It is this initiative, *Catalyzing Change in Mathematics* to which I would like to devote the remainder of this paper.

2. Advocating for High-Quality Mathematics Teaching and Learning

Let us consider our role and responsibility in advocacy for high quality mathematics teaching and learning. It is imperative for our profession that we advocate for the highest quality mathematics for each and every student across the world. All students must have access to mathematical learning experiences that will prepare them for success not only in the classroom but also prepares them to lead our world in the future. While we have seen much progress in mathematics education over the last 30 years, we continue to face significant challenges related to access. Disparities in learning opportunities based on race, ethnicities, class, language, gender, and perceived mathematics ability are far too prevalent in school mathematics — this has been made more evident during the COVID-19 pandemic. We must address these disparities to ensure equitable mathematical opportunities for each and every student. How can we do that? To that end, NCTM gathered writing groups to examine how we can address the issues we face today and move forward in the teaching and learning of mathematics for all. *Catalyzing Change* moves this conversation forward by focusing on structural policies, engaging in conversations on the purposes of school mathematics, and sustaining a focus on sense-making and reasoning.

2.1. Catalyzing change in early childhood and elementary mathematics

Why do we need Cata*lyzing Change* in early childhood (NCTM, 2020a)? We know that children's growth in mathematical knowledge in kindergarten and first grade is a strong predictor of later mathematics success. Mathematics instruction in early childhood and elementary school often places too much emphasis on memorizing basic number facts and following procedures at the expense of developing deep conceptual understanding. And mathematically powerful instruction in early childhood and elementary school is reaching too few children, particularly those most marginalized in our society. This leads to differential and unjust mathematics learning environments and outcomes (Adair 2015).

2.2. Catalyzing change in middle school mathematics

Why do we need *Catalyzing Change* at the middle grades level? At the middle grades level structures and traditions in mathematics education are deeply rooted. We must reconsider legacy practices and structures impacting students' mathematical identities and sense of mathematical agency (NCTM, 2020b). Instructional practices must be examined in order to systemically support, enhance, and adopt practices that are equitable and provide high-quality learning opportunities to motivate and engage students in learning. The evidence is compelling that students who are identified as marginalized learners based on certain ethnicities, Indigenous populations, language learners, poor, or those with disabilities do not have the same access to a high-quality mathematics program as their peers.

2.3. Catalyzing change in high school mathematics

Why do we need *Catalyzing Change* at the high school level? We are finding that in the United States the percentage of high school students enrolling in upper-level mathematics courses over the last three decades has increased (Dossey, McCrone, and Halvorsen, 2016). This is good, but there are major gaps on who has access to those courses and who does not. There is a concern over opportunity gaps and actions that are needed to support each and every high school student. It is an issue of access. There are times that the sheer amount of content expected to be addressed in high school is seen as a deterrent to addressing the desired level of rigor that is needed. Further, it is imperative that we have a high school mathematics experience for our students that prepares them for future college and career opportunities, particularly related to STEM fields, that is the fields of Science, Technology Engineering, and Mathematics. Thus, it was important to consider what changes were needed and how these might be addressed (NCTM, 2018).

3. Recommendations Serving as a Catalyst for Change in Mathematics

It is important to initiate these critical conversations and consider next steps and actions that are needed. As noted previously, NCTM began in 2018 with *Catalyzing Change in High School Mathematics: Initiating Critical Conversations* and followed in 2020 with the publication of *Catalyzing Change for Early Childhood and Elementary Mathematics* and then *Catalyzing Change for Middle School Mathematics.* Together these outline four key recommendations that serve as a catalyst for change to launch each and every student on a successful journey with mathematics. They are

(1) Broaden the Purposes of Learning Mathematics;
(2) Create Equitable Structures in Mathematics;
(3) Implement Equitable Mathematics instruction; and
(4) Develop Deep Mathematical Understanding.

While the four recommendations span the grade bands, they vary slightly in focus depending on the grade band. Let's unpack these four recommendations and consider what critical conversations we need to have that will lead to actions to address these recommendations. Consider who we should engage in those conversations. Who are our partners and stakeholders in mathematics education?

3.1. Recommendation #1: broaden the purposes of learning mathematics

For all grade levels there are three major areas of focus: develop a deep understanding of mathematics, understand and critique the world through mathematics, and experience the wonder, joy and beauty of mathematics with developing as confident and capable learners and contributing to a positive mathematics identity as central.

The early childhood and elementary authors note that "The power of the multiple purposes occurs when the purposes converge in ways that foster positive relationships between children and mathematics. The goal is for children to see themselves in the world of mathematics, not looking in from the perimeter or looking for the nearest exit door" (NCTM, 2020a, p. 23). We should also note that the mathematics students learn during middle school includes many of the most useful mathematics concepts that students will use as adults. "Middle school mathematics programs must challenge students to reason, and, most important, they must be respectful of students' distinctive cultural and developmental needs and interests" (Gutstein, 2003; Liptstitz and West, 2006; Lounsbury, 2015 as cited in NCTM, 2020b, p. 7). The purposes have to do with empowerment. Preparing learners for their future education and employment-opening and expanding opportunities. As students transition to high school, we need to remember that "A multipurpose high school mathematics curriculum plays a critical role in the cultivation of students who become fully engaged members of society, who contribute to society in positive ways and who become human beings capable of achieving their full potential, personally and professional, through the intellectual experiences of their mathematics education" (NCTM, 2018, p. 12–13).

Each and every learner should develop deep mathematical understanding, be able to use mathematics to understand and critique the world, and experience the wonder, joy, and beauty of mathematics. I challenge us to consider what short- and long-term work needs to be done toward achieving this vision of broadening the purposes of learning mathematics. That is, what can we do now and what can be done over time? This is an important, critical conversation to have.

3.2. Recommendation #2: create equitable structures in mathematics

Across grade levels we need to dismantle and attend to inequitable structures such as ability grouping and tracking of students and teachers, and challenge spaces of marginality and privilege that exclude many students from high-quality learning opportunities in mathematics. These may vary or look different across our countries, but we all need to carefully examine our structures that support or may be deterring the teaching and learning of mathematics. We need to position students as competent, confident, and capable learners and doers of mathematics affirming their strengths every day in ways that cultivate positive mathematical identities and a sense of agency.

As students move from early grades to middle school, they continue to build their mathematics identity. We need to continue to support their positive mathematics identity as they develop in their mathematical thinking and explore their world through mathematics. One's mathematical identity continues to develop from adolescence to adulthood. Our continual affirmation of students' positive mathematical identities through their learning experiences, building on their strengths will support them in developing strong and resilient positive identities (NCTM, 2018, 2020a, and 2020b).

We need to examine our own deficit-based beliefs about students or their families and communities and ensure that we truly believe that all students can and should do mathematics. Are there ways that we are grouping students that limit their access to high quality mathematics instruction and opportunities for deep understanding of mathematics? Are there historical, cultural or social beliefs about our students' mathematical abilities contributing to inequitable practices and opportunities in mathematics. Are structures in place that inhibit student learning in mathematics?

I encourage us to ask ourselves: What supports are needed in schools, districts, states/provinces, or our countries to discontinue inequitable practices such as ability grouping, tracking, and dead-end course pathways where students' opportunities for learning mathematics are limited and not always of high quality and inclusive? Have we considered our curriculum, instructional resources, assessment practices and professional development and support for our teachers? What work is needed to make sure all students — and specifically those often traditionally marginalized — have equitable structures in place to support their mathematics learning? Actions might include identifying, analyzing, and evaluating policies and practices to assess the impact of tracking; providing each and every student access to grade-appropriate intellectually challenging curriculum; providing ongoing professional development and support for our teachers; and providing time and space for educators to collaborate (Berry, 2018).

3.3. Recommendation #3: implement equitable mathematics instruction

"Teachers and their instructional practices have strong influences, often far greater than one realizes, on children as they learn mathematics" (NCTM, 2020a, p. 45). Equitable mathematics instruction requires that teachers take direct action stemming from intentional planning and reflection informed by data from their students (NCTM, 202b). We should approach through asset, strengths-based perspectives and not deficit views of students and their learning. Our instructional practices should be dedicated to implementing equitable instruction that engages all students in learning mathematics. Students bring multiple strengths to the mathematics learning experience. They begin formal school eager and ready to learn, and with multiple strengths from their daily learning experiences. These strengths continue to grow if nurtured in both formal and informal learning environments. We need to identify, foster and value these strengths, support students as thinkers and doers of mathematics, and leverage students' experiences, cultural perspectives, backgrounds, languages, and interests. This strengths-based approach will facilitate deepening mathematical understanding, helping students make sense of their world through mathematics.

In 2014 NCTM published *Principles to Actions: Ensuring Mathematical Success for All,* which identified eight effective mathematics teaching practices.

(1) Establish mathematical goals to focus learning
(2) Implement tasks that promote reasoning and problem solving
(3) Use and connect mathematical representations
(4) Facilitate meaningful mathematical discourse
(5) Pose purposeful questions
(6) Support productive struggle
(7) Build procedural fluency from conceptual understanding
(8) Elicit and use evidence of student thinking

These eight, taken together, provide research-informed mathematics teaching practices to create a classroom learning environment that supports ambitious, effective, and equitable mathematics instruction and nurtures children's positive mathematical identities and strong sense of agency with shared authority in learning mathematics.

Focusing on implementing equitable, effective mathematics instruction helps us to consider the quality of mathematics learning experiences rather than quantity of problems and provides a space for student voice, student interest, and student concerns. This focus supports mathematics as a collaborative endeavor and one in which problems may be solved in multiple ways, and it encourages students to share their mathematical thinking and not just solutions. All learners are viewed as thinkers and doers of mathematics across all grade levels (NCTM 2014, 2018, 2020a, and 202b). I challenge us to consider the question: Do our student see themselves as mathematically capable? Do all teachers see all students as *doers* of mathematics? Do we see ourselves as a *doer of mathematics*? Implementing equitable instructional practices positions all students for success. "effective teaching is the nonnegotiable core that ensures that all students learn mathematics at high levels and that such teaching requires a range of actions ….[at all]….levels" (NCTM , 2014, p. 4).

Let's consider this question: In what ways can and should we engage in discussions with multiple stakeholders to create shared experiences and a collective understanding of equitable mathematics instruction? Who are our partners and stakeholders? Are we inclusive of voices and supportive?

3.4. Recommendation #4: develop deep mathematical understanding

Considering broadening the purposes of mathematics, implementing equitable structures and equitable, effective mathematics teaching practices leads us to the last recommendation — develop deep mathematical understanding. “When mathematics instruction goes deep, children are empowered to explore the richness of the mathematical landscape” (NCTM, 2020a, p. 77). *Catalyzing Change* underscores the importance of engaging children — beginning in early childhood and elementary grades—as active doers of mathematics who author and generate strategies and share their mathematical insights. Doing mathematics involves engaging in the norms, routines, and habits that are central to the work of mathematicians (NCTM, 2018, 2020a, 2020b). These include representing and connecting mathematical ideas and concepts, explaining and justifying mathematical thinking, and noticing, using and understanding mathematical structures.

Moving to middle grades mathematics students need to engage “as active participants in their and their peers’ mathematics learning” (NCTM, 2020b, p. 67). They need to engage in mathematics that is relevant — often about sensitive or controversial topics — and requires careful attention and thoughtful implementation, but it needs to be a part of students’ middle school mathematical learning experiences. Across all grade levels to support students in developing a deep mathematical understanding there are multiple mathematical proficiencies to address as noted by the National Research Council (2001). They include conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Further, we need to include mathematical experiences and course or curricular pathways that support students’ development of key mathematical practices such ss those identified in the Common Core State Standards in the United States (National Governors Association Center for Best Practices [NGA Center)] and Council of Chief State School Officers [CCSSO], 2010). These include making sense of problems, persevering in solving problems, reasoning both abstractly and qualitatively, being able to construct viable arguments, model with mathematics, use appropriate tools strategically, attend to precision, along with recognize and use mathematical structures. We need to consider mathematical and statistical modeling and thinking threaded throughout all grades. *Catalyzing Change* calls for disrupting the cycle of rote learning of mathematics. Each and every learner deserves mathematically powerful learning spaces that emphasize reasoning and sense making on a daily basis (NCTM 202a).

Often in the United States in the high school grades we see a segmented approach to the curriculum that includes a course sequence of an Algebra I, Geometry, and Algebra II pathway with a rush to calculus. This is often to the detriment of deep understanding of major key mathematical concepts and connections, and it avoids addressing other essential mathematical topics such as statistics, quantitative literacy, and modeling which are essential in today's world and to our future. This is not necessarily the case in other parts of the world but there may be other challenges that exist when thinking about mathematical experiences across all grade levels in developing a deep understanding of mathematics. Important questions for us to consider are: What is the essential mathematics needed? What are varied possible pathways for students to pursue rich mathematical experiences? And do all students have access to those essential robust pathways?

4. Conclusion

In considering these four recommendations it is essential that we engage in critical conversations to move to actions that will provide and support powerful mathematical learning spaces to support access and equity for all. Currently there are many marginalized students who are not receiving equitable learning experiences and thus their education is limiting their future opportunities. We have an opportunity to change this by working together in mathematics education. To be effective and impactful we must advocate both individually and collectively across local, national, and international levels. The gathering at the 14th International Congress of Mathematical Education was a unique opportunity to engage in reflection, discussions and collaboration to address advocacy efforts in mathematics education.

In my April 2021 NCTM President's message, *Advocacy as a Mathematics Education Community-The Time is Now*, I shared that we have much we can engage in as an advocate in mathematics education. We need to be called to action to advocate for high-quality mathematics teaching and learning for all students (Wilkerson, 2021).

Advocating for high-quality mathematics teaching and learning for each and every student must be more than words. To be effective advocacy must include thoughtful actions both individually and collectively across local, national, and international levels. Advocacy should raise awareness and influence decision makers and the public on issues to expand high-quality mathematics teaching and learning and provide access to every student, school, and community. Why should we advocate? What are effective ways to advocate in mathematics education? These are just some of the questions to consider as we examine our role in advocacy in mathematics education.

Every voice matters in mathematics education. Just think what we can do working together on this journey advocating for high-quality mathematics teaching and learning for each and every student and supporting each and every teacher! We must challenge existing inequities in structures and practices related to teaching and learning mathematics. Together we can do this.

On behalf of the National Council of Teachers of Mathematics, I want to again thank the International Commission on Mathematical Instruction for honoring NCTM with the Emma Castelnuovo Award for Excellence in the Practice of Mathematics Education. It's been an honor to accept the award and to have this opportunity to offer this address. I look forward to our continued dialogue.

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Part V
Survey Teams

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https://doi.org/10.1142/9789811287152_0021

Survey Team 1

Research in University Mathematics Education

Marianna Bosch[1], Reinhard Hochmuth[2], Oh Nam Kwon[3], Birgit Loch[4], Chris Rasmussen[5], Mike O. J. Thomas[6], and Maria Trigueros[7]

ABSTRACT In this report we highlight significant advances in university mathematics education research as well as areas that are in need for additional research insights. We add here to the rich set of literature reviews within the last several years. A novel aspect of this literature review is the fact that the areas of accomplishment and areas for growth were identified based on thematic analysis of survey responses from 119 experts in the field. The review provides a useful overview for both seasoned scholars and those new to research in university mathematics education.

Keywords: University mathematics; Advances; Gaps.

1. Introduction

It is an exciting time for research in university mathematics education (RUME). There are now several major conferences every year across the globe, as well as the fairly new *International Journal of Research in Undergraduate Mathematics Education*, now in its seventh year. With the significant growth in the number of researchers focused on university mathematics education has come the development of research groups and the consolidation of a diverse academic community; RUME is coming to age as a field of research that is beginning to coalesce and develop an identity.

To explore this identity we first surveyed 218 RUME scholars across the world, both well-established scholars and rising stars. We invited these scholars to respond to the following prompt:

What do you see as the most significant advances, changes, and/or gaps in the field of research in university mathematics education? These advances, changes, or gaps might relate to theory, methodology, classroom practices, curricular changes, digital environments, purposes and roles of universities, social policies, preparation of

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university teachers, etc. Please elaborate on just **one or two advances, changes, or gaps** most relevant to your experience and expertise. If possible, please include a few key references.

We received 119 responses. Our next step was to conduct a thematic analysis, which led to the identification of five areas in which there has been considerable progress (advances) and seven areas that are less well-researched (gaps)[8]. Our next step was to conduct a literature review, guided by the identified themes. Perhaps not surprisingly, these two areas are not entirely disjoint.

2. Advances

2.1. Theoretical perspectives

One of the field's major advances considered by several respondents is that we now have a plethora of theoretical perspectives and hence tensions among them can sharpen their constructs and methodologies, and also open the possibility to find commonalities among some of them previously considered to be incompatible. This diversification has contributed to the development of new methods, research topics, and the development and research on theory-based teaching experiences. In particular, the growth of the networking of theories is a recent advance that has added power and depth for analyzing complex learning and teaching phenomenon (Bikner-Ahsbahs and Prediger, 2014; Prediger, et al., 2008)

Related to the networking of theories, recent years have seen the emergence of an interdisciplinary group of scholars interested in using a variety of approaches (logical, cognitive, historical, philosophical, etc.) to address questions which have always been of interest to RUME. This also has increased connections with other disciplines in mathematics and science education and in funding agencies supporting interdisciplinary projects.

Another theoretical advance that is of growing interest is the use of theories that enable insights into issues of equity and social justice. Adiredja and Andrews-Larson (2017) lay out a research agenda for this emerging domain that speaks to the interrelatedness of knowledge, identity, power, and social discourses. While there is still much research that is needed here, we see this new direction as an important advance for the field of university mathematics education research.

2.2. Instructional practices

The research of instructional practices at university level is a rapidly developing area of research. Much of the research on this topic relates to active or inquiry based mathematics education (Artigue, M. and Blomhøj, 2013; Laursen and Rasmussen, 2019). Given the myriad calls for instructional reform in university mathematics

[8] Special thanks to Antonio Martinez and Talia LaTona-Tequida, graduate students at San Diego State University, for their help in this analysis.

classrooms, researchers and educators have challenged traditional lecture-based instruction by conducting studies that have provided evidence for the positive effects of innovative student-centered instructions on students' cognitive and affective development.

Active learning, broadly defined as classroom practices that engage students in activities, such as reading, writing, discussion, or problem solving, that promote higher-order thinking, has repeatedly shown to improve student success and to reduce the equity gap for women and underrepresented students (Freeman et al., 2014; Laursen et al., 2014; Theobald et al., 2020). For example, Freeman et al. conducted a meta-analysis of 225 studies that compared student success in traditional lecture versus active learning in postsecondary science, technology, engineering, and mathematics (STEM) courses and found that average examination scores improved by about 6% in active learning sections, and that students in classes with traditional lecturing were 1.5 times more likely to fail than were students in classes with active learning and the effectiveness of active learning was found across all class sizes. On the other hand, RUME has only begun to deeply explore the culture, experiences, and gendered/racialized interactions in these classes — and how those social factors may be mitigating the students' opportunities to learn (Johnson et al., 2020).

2.3. Professional development of university teachers

Just over a decade ago, Speer et al. (2010) described tertiary level professional development as virtually non-existent and an unexamined practice. Since then there has been considerable progress in this area. In their review of recent research, Winsløw et al., (2021) characterise the literature as comprising primarily small-scale studies of lecturer preparation for teaching. Examples include the effect of backgrounds on teaching (Hernandes-Gomes and González-Martín, 2016; Mathieu-Soucy et al., 2018) and the knowledge used in teaching (Musgrave and Carlson, 2017). An inquiry-based experience can be found in (Florensa et al., 2017).

Another advance is the growing collaborative research that builds links between mathematics educators and university mathematicians, links that can increase the pedagogical awareness of the latter group through reflection on teaching practice and provide pedagogical tools (Bardini et al., 2021; Nardi, 2016). However, in order for mathematics educators and mathematicians to collaborate on professional development initiatives, the two groups need to build mutual understanding and trust. Some positive and productive examples of such collaborations are detailed in Jaworksi's (2020) overview of the professional development of university mathematics teachers.

Another very successful collaboration between a group of mathematics educators and mathematicians is detailed in Barton et al. (2014). This collaboration sought to open a two-way channel of communication between the two groups, with the aim to close the ideological perspective gap (Thomas, in press) by understanding the other group's thinking. In this manner, the theoretical and pedagogical knowledge of mathematics education could be conveyed to mathematicians in a manner that focused on their orientations and goals (Schoenfeld, 2010). In turn the mathematician's focus on the crucial elements of mathematics and its learning, along with the role of rigour, was conveyed to the mathematics educators.

2.4. Digital technology

As the review by Clark-Wilson et al. (2020, p. 1223) notes, "in the last two decades the range of digital technology (DT) available has expanded considerably and their facilities and power have also greatly increased. In the light of these changes, the research focus of many has moved from how computers can help with learning to how teachers can make practical use of different types of digital technology to provide students with activities that will enhance their mathematical learning." In addition to new digital technologies, new theoretical perspectives may be applied to the study of technology use at the tertiary level. There have been advances in this area that have emerged. For example, the notion of *Instrumental Orchestration* that arose to consider the collective knowledge building of teachers and students when a technology is appropriated for some mathematical pedagogical purpose (Trouche and Drijvers, 2010). This has recently been developed into the *Documentational Approach to Didactics* (Trouche et al., 2020). The latter combines theoretical elements related to the (instrumental) use of technology, resources, curriculum design, and teachers' professional learning and development to document meaning resource use in a given context with a pedagogical intention.

There have been many advances in terms of DT tools in recent years, such as an ever-expanding internet, clickers, pen-enabled tablets, powerful mobile technology, and interactive retinal screens on smartphones. Several recent studies (e.g., Loch et al., 2014; Maclaren et al., 2018) have demonstrated a very positive feedback from the students on the use of pen-enabled tablets in teaching mathematics, in particular showing a strong preference for this delivery mode compared to other delivery modes.

2.5. Service-courses in university mathematics education

Service-Courses in mathematics are courses provided by mathematicians for students who study engineering, natural sciences, economics, social sciences, psychology, medicine or life sciences, etc. The importance of mathematics for university education is reflected in institutionalized discipline specific mathematics working groups. For example, the European Society for Engineering Education has developed a competence orientated framework for mathematics curricula in engineering education. Research on mathematical service courses also plays an increasing prominent role at national and international conferences in university mathematics education.

Workplace studies figuring out the specific relevance of mathematics for vocational demands have so far mainly considered engineering. For example, considering structural engineers, Gainsburg (2007) has shown that reflections on mathematical concepts become mainly important and more explicit in situations where usual routine procedures do not lead to sufficient results. For this kind of situation Kent and Noss (2003) coined the notion of "breakdown situations".

There are many endeavors making service courses more helpful and relevant to students by implementing or strengthen the discipline related perspective, which intend both to improve the motivation for learning mathematics and the ability to transfer mathematics to discipline (mostly engineering) contexts (Czocher, 2017; Schmidt and Winsløw, 2021). To realize a better connection among service mathematics and other disciplines, a collaboration of mathematics lecturers and lecturers from the other discipline seems to be crucial. Jaworski and Matthews (2011) demonstrate how such a collaboration could be achieved where the lecturers involved conjointly design materials and plan teaching activities. Also related to an inquiry based perspective is the recently started design of so called study-and-research-paths (see for example Barquero et al., 2020) intending to support students in the integration and validation of mathematical practices from different institutional settings.

3. Gaps

3.1. Theories and methods

The development of novel research questions can contribute to the exploration of cogent theories of teaching and learning and the development of sound and innovative methodologies to answer them. As for existing coherent theories there may be a need to develop and extend them further to capture the complexities of the studied phenomena. Studies aiming at developing theory or to test and revise theories using empirical data, and their corresponding methodologies can help in making existing theories more robust and move the field forward. The lack of a shared discourse on meta-level learning is also reflected in the abundance of conceptual and theoretical frameworks suggested by the research literature, which are not necessarily compatible or even commensurable. Developing a shared and explicit discourse for the informal/meta-level content of university mathematics education is a crucial component in any effort to improve pedagogy at this level. While the networking of theories has made some progress on this front, more is needed.

There is in particular a need for frameworks for conducting evaluation research in realistic university settings for testing innovations integrating insights from broader educational research and from university mathematics teachers. Moreover, small-scale qualitative studies are still predominant (Artigue, 2021) or typically only one cycle of research is reported whereas reliability is often associated with multiple rounds of principled research. Also, knowledge of the process of scaling research-based innovations, including effective ways to navigate the political obstacles to shifting undergraduate mathematics courses to be more meaningful, coherent and mathematically engaging for students, is needed.

3.2. Linking research and practice

Some research results have been introduced into university mathematics curriculum, particularly through widely available textbooks for introductory courses. This is, however, not enough. Research results need to inform the teaching of all the mathematics courses through their introduction in the curriculum to significantly improve teaching practices. An important gap that impedes reaching this goal is the lack of research results in advanced university mathematics courses and in the possibility to introduce advanced mathematical ideas in introductory courses. This kind of research can illuminate new ways to motivate students through the teaching of both the usefulness and beauty of mathematics using interesting examples related to students' areas of interest, using what has been found in research about modelling and applications or by introducing new theory-based didactical approaches such as "training using challenges" and paradigms such as "questioning the world" (Chevallard, 2015).

Mathematics teachers and mathematicians need, on the one hand, to have access to different means of communication that go further than research published papers. Mathematicians often find it difficult to read and make sense of the theoretical points of view and the vocabulary used by researchers in mathematics education. Research results need to be made accessible to these other communities, and evidence based instructional practices in ways that are convincing to them. To make this possible not only through professional development programs but other types of media must be used and it is important to offer instructors and mathematicians incentives that can be financial or cultural together with longitudinal support through coaching and mentoring. These incentives and initiatives should clearly be to transform the university culture, which means researchers, teachers and students' transformation, and to support improvements in education. This change is considered fundamental because it is about the culture of mathematics departments, but we know very little about how to affect change (see Reinholz et al., (2019) for a research agenda centered on institutional change).

3.3. Professional development of university teachers

As reviewed in the previous section, there has been some progress in the professional development of university teachers, but much more is still needed. Research could consider the professional needs that university mathematics teachers have, in addition to completing a PhD and possibly a short course in general pedagogy (Winsløw, et al., 2021). How can university mathematics research be developed to contribute to filling those needs, and what measures are needed to engage university mathematics teachers in doing or learning from such research?

The development of programs to achieve strong teaching competence, along with study of their implementation, analysis and improvement comprise an important area for research in university mathematics education. While there have been a number of small studies in this area, the question of how to develop these further and extend them to scale is a topic for research. As Winsløw et al. note, there is a lack of large-scale international studies on teaching practice and its development in university mathematics teacher education.

One aspect to consider in the design of professional development programs is whether there is a role for technology in its design and implementation. For example, could use be made of recent online courses, such as MOOCs designed for in-service mathematics teachers where the aim has been to increase teachers' professional competencies and improve their practices? One advantage of this is the use of online mathematical communication in an open forum. Exchanges from such online mathematical forums have found fruitful use in similar research (Kontorovich, 2018).

There is still room for investigation of the orientations (Schoenfeld, 2010) of mathematics lecturers with respect to their pedagogy. For example, in what manner is mathematics taught by teachers and what do they believe about students who do not do as well as expected in their courses? Do they develop a "deficiency discourse" about their students (e.g., they can't do mathematics because of a lack of ability, are unprepared, unmotivated)? Is their practice influenced by their beliefs about students and is this discourse in any way related to their positive professional identity?

Another issue related to institutional factors that needs attention is to the culture of mathematics teaching and learning at tertiary institutions and how and why this culture continues to result in under-representative participation levels of women and historically marginalized groups (such as those from indigenous cultures and persons of color), particularly at higher levels, where retention rates are extremely low.

3.4. Digital technology

In many schools around the world there has been increased use of digital technology in mathematics learning. Discussing the transition from school to university, Gueudet (2008, p. 252) noted "the question of the effective and possible uses of technology in the secondary–tertiary transition has not been researched yet, as far as I know. Are the abilities with technology built at secondary school exploited at university?" (p. 252). Thus the possible disjoint between school and tertiary use of technology means that issues related to school-university transition may be in need of further research. One interesting question could be, how does a shift from a technological to a non-technological environment influence students' perceptions/interest/attitudes about/for/towards mathematics?

There is now a wealth of sources of mathematical information available to students, who have almost instant access to them, both at home and at their institution. There has been some research related to how they use these sources, such as how engineering students make use of mobile devices and the Internet (Puga and Aguilar, 2015), however, serious questions remain about how and what tertiary mathematics students access and the factors that influence and shape their help-seeking behaviors in the digital era. Examples include research that would systematically analyze how university students use the Internet and mobile devices as a source of mathematical help: What sites do they consult? Why do they consult them? Do they use online real-time support? What makes students trust or prefer one source of information over another one? And how are their mathematical reasoning processes affected by immersing themselves in the use of these digital resources?

Finally, due to the pandemic, courses have increasingly been converting to an online or hybrid format, and when this is pedagogically desirable or not is a topic for research. The research on DT as it relates to the pandemic is only just beginning, detailing both what the field has learned and what open questions remain. We trust that the next ICME survey on RUME will provide substantive insights on this topic.

3.5. Curriculum

Curriculum is an entity present but rarely taken as a unit of analysis in research in university mathematics education. Curricular questions are obviously at the center of all study and examination regulations of mathematics degree programs. However, their treatment rarely relies on specific research, except some empirical studies about students' difficulties or perceptions that are rarely published. At the same time, and at least in the case of mathematics undergraduate degrees, topics, contents, and the structure of study, programs appear surprisingly stable. In the case of mathematics subjects in other types of degrees (engineering, natural sciences, economics and business, etc.), the situation seems more evolving thanks to the introduction of new technologies (especially in the case of statistics subjects), but it is still stable in calculus and linear algebra subjects.

A comparative description of curricula across universities and countries, as well as the processes of external didactic transposition, i.e. the process of selecting and transforming scholarly knowledge into knowledge to be taught, have surprisingly seen little systematic investigation. This is possibly also due to the fact that in mathematics degree programs the knowledge to be taught is rarely questioned. According to (Bosch et al., 2021) for the case of mathematics degrees, there are some differences, for example, between Canada and the USA on the one hand, and Europe on the other, or between types of higher education institutions, such as classical universities and universities of applied sciences. Nevertheless, within each type, the knowledge to be taught has not strongly evolved. This also applies to the external framework conditions, as well as the various dynamics of decision-making processes for changing the curricula or processes of maintaining curricular orders.

3.6. Higher years

Didactic research on learning and teaching in advanced mathematics studies is significantly under-represented. The focus has so far been clearly on the transition from school to university and in the first year of study. This reflects the relevance of the transition and study entry problem, e.g. with regard to dropout rates.

Historically, Felix Klein should of course be mentioned here. Core parts of his "Elementary Mathematics from a Higher Standpoint" actually refer to mathematics that many of today's student teachers do not even get to know in their academic studies. This applies, for example, to knowledge of Fourier analysis that goes beyond the basics, but especially also to knowledge of function theory, e.g. Riemann surfaces and value assignment theorems. Even when students hear about function theory, for example, they usually do not get as far as understanding what Felix Klein considered appropriate knowledge for prospective teachers more than a century ago. Klein considered this knowledge appropriate because it explains why, for example, certain elementary operations have to be restricted in certain ways for mathematical reasons (and not just for didactic reasons of reduction!), and related curricular decisions.

3.7. Interdisciplinarity

Survey responses addressed gaps on many different levels. The most general level concerns research itself. On the one hand, this involves cooperation with mathematicians, engineers, economists, psychologists, etc. For many years, there have been many different kinds of cooperation, for example, agreements between faculties with regard to teaching. What does not seem to exist so far is, among other things, systematic research on these cooperations. What are the benefits of these? How do they take shape? How do they function? Possibilities, limits, etc.? On the other hand, it is about cooperation with researchers from other disciplines in our own research in the narrower sense, i.e. besides mathematicians, with psychologists, university teachers, pedagogues, sociologists, political scientists, historians, anthropologists. In the context of empirical research into learning processes, cooperation with psychologists and educationalists has now been established in many places and, with a view to professionalizing teaching, also with university didacticians. And of course there are also isolated cooperations with other academics. What seems to be missing, however, is a more systematic description and conceptualization of the links. This could be formulated as goals.

The relationship of mathematics to other sciences or the use of mathematics in other sciences also is an area that needs to be addressed. There are several places, such as philosophy or the history of science, in which such connections are examined and the question of what distinguishes mathematics itself and its respective role in other sciences is explored. Research on this is dependent on the respective ideological assumptions and accordingly there are no unambiguous and generally accepted answers here in depth. From the point of view of didactics, however, clarifications in this regard could certainly be regarded as desirable, since they would be of great help in answering the question with which goals, which and how mathematicians, but especially engineers, economists, psychologists, etc., are to be taught.

Last but not least, although in a slightly different way, this also concerns mathematics in itself. It, too, changes its inherent orientation and, to some extent, its character over time. New fields are emerging, such as Big Data and Data Science. Correspondingly, there are new fields of application in other sciences, such as discrete mathematics in electrical engineering, numerical methods also in psychology, etc. This leads directly to questions of teaching: the question of what should be taught in service courses and how is manifold but certainly not sufficient.

4. Conclusion

Grounded in the responses from our RUME scholar survey, we identified five areas in which the field has made significant progress (Theoretical Perspectives, Instructional Practices, Professional Development of University Teachers, Digital Technology, and Service-Courses in University Mathematics Education) and seven areas are in need or further development (Theories and Methods, Linking Research and Practice, Professional Development of University Teachers, Digital Technology, Curriculum, Higher Years, and Interdisciplinarity). These gap areas represent exciting opportunities for the mathematics education research community to conduct scholarly work and help advance the field at large. So while there is now much research-based wisdom, there are also exciting opportunities for new research.

Acknowledgements

The authors thank all those who took the time to respond to the survey.

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https://doi.org/10.1142/9789811287152_0022

Survey Team 2

A Survey of Recent Research on Early Childhood Mathematics Education

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ABSTRACT This is a summary report of the ICME-14 survey on Early Childhood Mathematics Education, which was conducted by the authors of this paper, with the aim to establish a review of the state-of-the-art of the most important developments, and of current tendencies, new perspectives and emerging challenges in the particular field. The survey was based on an analysis of the research literature published between 2012 and 2020.

Keywords: Literature review; Early mathematics learning, Early mathematics teaching; Technology; Early childhood teachers.

1. Introduction

In the past few years there has been an internationally growing interest in early childhood mathematics education (ECME). The interest in this field is induced firstly by the strong emphasis given on early childhood education in many countries. This is evident by the increase of their expenses and investments on early childhood education and of their access to pre-primary education (Kagan and Roth, 2017). The well documented, positive relation between children's early mathematical knowledge and their later success in mathematics learning is another factor for the growing research in this field (Dunkan et al., 2007).

This survey has been designed to establish an in-depth and comprehensive review of the state-of-the-art of the most important developments and contributions between 2012 and 2020, and of new perspectives and emerging challenges in ECME.

In the survey we identified six major research themes in recent literature on ECME. Three of these themes are content-oriented: number sense and whole number development, geometry education and children's competences in other content domains. Another theme that is systematically reviewed deals with the role of technologies in early mathematics teaching and learning. A cognition-oriented theme

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focuses on cognitive skills associated with mathematics learning and special education and comprises two parts: Abilities predictive of or associated with mathematical performance and special education. Finally, the sixth theme focuses on developments and trends in teacher-related issues, particularly on early childhood teachers' knowledge, education and affective issues in mathematics.

To identify relevant research for the survey, we drew from a broad range of sources, focusing on publications since 2012. For each identified theme we searched for relevant peer-reviewed papers in journals in two fields, that is, Mathematics Education and Early Childhood Education, for relevant chapters in prominent research books on Mathematics Education (such as PME Handbook, POEM, ICME-13 monographs), international peer-reviewed conference proceedings, including ICME, PME and CERME. Moreover, we used publications found in search engines using strings with relevant keywords for each theme.

After eliminating double records, we produced annotated bibliography with summaries of the papers that have been identified as relevant for each theme, leading to a comprehensive analysis of the issues raised by this research literature and to a qualitative synthesis of the pertinent findings. This led to the production of a written overview of the research in each theme. A summary of the key findings for each theme of the review and a final section with our concluding remarks are presented below.

2. Number Sense and Whole Number Development

Number sense development is globally recognized as the fundamental foundational knowledge for children's mathematical growth. Hence, literature argues for children's stimulation of numerosity as early as their toddler stage for future benefits. The complexities brought forward by diverse backgrounds of children as well as diverse provisions of stimulation enriches strategies and seek more conceptualization. Children's diverse numerical abilities reflect children's varied experiences from home and their immediate environment (Ramani and Siegler, 2011). These abilities are foundational blocks for children's development of numerical fluency, on the other hand low performance is proven to be associated with limited numerical experiences prior to kindergarten and inability to catch up with peers (Aunio et al., 2015).

Innate abilities of young children have been identified to be observable as early as six months. New-borns of 7 to 94 hours demonstrated that they could map space, number and time including brightness, and loudness (de Hevis et al., 2014). Robertson, Shi and Melancon (2012) discovered that 24 months babies were able to match objects with the defining number. Also, Norwegian toddlers demonstrated competencies using number words, however reciting lower competencies than previous literature (Reikerås, Løge, and Knivsberg, 2012).

Sella et al. (2017) explored numerosity and spatial mapping to three groups, preschool children, 4-year-olds, 1st Grade and 3rd Grade, to discover that spatial mapping favored high numerical abilities in all groups studied. Benz (2014) revealed that 4- to 6-year-olds were able to explain structures in quantities and why they used to compose or decompose, while a more recent study with the use of eye tracking by Schöner and Benz (2018) showed that children built structures in the collection of objects but could not explain their approaches and resort to counting as a strategy.

Strategies that are revealed to enhance numerical abilities are linear board games (Ramani and Siegler, 2011), numerical acuity and inhibitory control, tablet-based non-symbolic approximate arithmetic game, conceptual subitising (Sayers et al., 2016), fine motor skills (Asakawa et al., 2017), story problems (Jordan et al., 2012), catch up numeracy (Holmes and Dowker, 2013) differentiated approach in using games (Bay-Williams and Kling, 2014). Children who get exposed to numeracy stimulation before school demonstrate a positive gain in early schooling (Clerking and Gilligan, 2018; Segers et al., 2015). Particularly, benefits are observed in play-based approach for numeracy development at home, allowing children to learn through play and give opportunity to adults to pose challenging questions and listen to children illustrating their action and adding meaning to them (Magnusson and Pramling, 2018). Furthermore, mediation of patterns and structure influences children numerical fluency positively especially those who are low performers (Lüken and Kampmann, 2018). Problem solving was found to boost four basic operations with an effect size of 0.60 (Bicknell et al., 2016).

Dynamic assessment is one of the strong predictors for problem solving of word problems development (Seethaler et al., 2012). Polotskaia and Savard (2018) used Relational Paradigm in facilitating problem solving and this led to improved problem-solving skills and enabled students to solve problems demanding rational thinking. A longitudinal study favored students who begin to use derived fact strategies during mid-year than those using counting strategies (Gaidoschik, 2012). White and Szucs (2012) promoted modelling methods to increase understanding and developing mental representation through estimation tactics.

Although children's mathematical development through technology is systematically analyzed in a distinct theme of this review, it is worthwhile here to present a number of technology-based interventions that were found to support the numeracy abilities of children with diverse backgrounds. A Math Shelf intervention using a tablet was designed for at risk 4-year-old pre-schoolers with results that show significant improvement performance (Schacter et al., 2016). Also, an adaptive computer game "Number Race" yielded similar findings as Math Shelf intervention for disadvantaged children (Sella et al., 2016). Lady Bug Count and Fingu apps allow children to develop own mathematical concepts such as subitizing, estimation and finger motor-skills (Ladeland Kortenkamp, 2014).

Generally, this literature addresses vital concepts within number sense that need attention and recognition. Our findings indicate a need to understand or unpack children's language as some studies assert that children were able to articulate their strategies and reasons behind their selection, while others report that children were limited to counting in describing their strategies. Furthermore, there is too little literature on transitioning from informal numerosity to formal numerosity and how mediation should be structured to achieve the transitioning.

3. Geometry Education

The literature review about geometry education in early childhood focused on the following major threads: Spatial skills and their relation to mathematics and geometry learning, shape knowledge and understanding, embodied, dynamic and semiotic approaches in geometrical thinking and learning and enhancing and assessing geometry learning.

Regarding the first thread of the review, strong relations have been found between spatial skills and quantitative competences, including, number-line estimation, counting and arithmetic (addition and subtraction) in children already from age 3 up to 7 years (e.g., Verdine et al., 2017). The potential significance of young children's spatial training or learning in their mathematics performance is also highlighted (e.g., Hawes et al., 2015). The relationships between spatial skills and geometry learning are scarcely investigated and research on this issue suggests that the associations between geometry learning and teaching and spatial reasoning are quite complex (e.g., Dindyal, 2015). However, spatial knowledge is found to be the basis for building geometrical knowledge and understandings in problem solving (Soury-Lavergne and Maschietto, 2015).

Our findings on shape knowledge and understanding (second thread) suggest that a great deal of studies focused on plane shapes, taking a rather static perspective, e.g., studying children's shape recognition and sorting abilities, definitions/descriptions of shapes (e.g., Olkun et al., 2017). A major finding, revealed also by previous literature, is that children develop visual prototypes and they use these prototypes to compare shapes they are asked to identify or to draw (e.g., Dağlı et al., 2014).

A few studies adopted a more spatial perspective in investigating children's competences in geometry. Research on children's competences with 3D shapes has shown that children encounter difficulties when reasoning about plane and solid shapes across various kinds of geometric representations (Hallowell et al., 2015) and also in reconstructing a 3D figure with building blocks and in using the regularities of this geometrical object when needed (Maj-Tatsis and Swoboda, 2017).

A large part of research on embodied, dynamic and semiotic approaches in geometrical thinking and learning (third thread) investigated the role of body and gestures in children's learning of geometry. A major finding is that children's body and gestures reflect implicit knowledge and have a crucial and fundamental role in the development of geometrical reasoning, in solving geometrical problems and argumentation and in communicating geometrical/spatial relationships (e.g., Calero et al., 2019; Elia, 2018). The multimodality of children's learning, and specifically the synergy between talk, gesture, and material environment is regarded as a critical characteristic of children's development in geometry by many research studies (e.g., Thom, 2018). A number of studies on dynamic learning environments (DLE) in early geometry provided evidence for the potential of DLE to support children's developing understanding, and reasoning about, different geometry concepts, including the properties of specific shapes, angles and reflective symmetry (e.g., Ng and Sinclair, 2015).

Research on enhancing early geometry learning (fourth thread) suggested play as a significant learning approach. Using children's play as starting point to teach mathematical content supported children's explorations of shapes (Bäckman, 2016), while guided play also enhanced children's shape knowledge (Fisher et al., 2013). In addition, using picture book reading with or without the inclusion of additional mathematical activities was found to be a promising avenue to contribute to the development of children's understanding of shapes and spatial relationships (McGuire et al., 2021). Regarding the assessment of children's geometrical thinking and learning, a study by Thom and McGarvey (2015) showed that drawing serves as a means to access, assess, and attend to children's understanding. Tirosh et al. (2013) highlighted the need to use a combination of tasks to assess strengths and weaknesses of children's geometric knowledge, as not all children take advantage of the opportunities afforded by a given task.

The findings of the review on early geometry education indicate the need for further research on the body's role in children's geometrical learning (regarding both plane and 3D shapes), so as to deepen and enrich current knowledge about how children think and build geometrical understandings, on teaching strategies in geometry and spatial reasoning (e.g., picture books, DLE) to support children move into more abstract ways of thinking and, finally, on how early geometrical knowledge and understanding affect children's future mathematical performance.

4. Children's Competencies in Other Content Domains

Compared to the domains of number and geometry, other content domains have received less attention within ECME research. The major content domains within ECME that are studied in the reviewed literature are patterns and structures, measurement, statistical reasoning and early algebraic reasoning. A common focus of the reviewed studies across the different content domains is twofold: Firstly, offering insights into young children's competences and development and secondly, proposing and investigating the effectiveness of programs or interventions on children's learning. Research on patterns had increased rapidly since 2013, when Mulligan and Mitchelmore (2013) proposed Early Awareness of Mathematical Pattern and Structure showing that it generalizes across early mathematical concepts. In this study, not only can students' abilities of structural development be reliably categorized through particular levels, but students who show a superior level of development on one task also operate at a similar level on other tasks.

Besides this, research has investigated children's recognition of the unit of repeat and the structure of the repeating patterns and the relations with other mathematical contents, such as number and arithmetic, algebra, calculation, or geometrical thinking. It is shown that effective patterning instruction included instruction on symmetrical patterns, patterns with increasing numbers of elements, and patterns involving the rotation of an object through 6 or 8 positions. Moreover, the abilities to ascertain the structure of patterns and understand mathematical language were found as strong predictors of mathematical success, with the latter making a more significant contribution.

Concerning measurement, a significant number of studies in this theme have focused on general measurement, length, area, mass, and time. However, few research papers are about bulk, or volume. Regarding length measurement, which has received greater attention than measurement of other properties, research has provided evidence for the effects of different pedagogical approaches on kindergarten children's learning and evaluated and elaborated the developmental progression or levels of thinking on length measurement (e.g., Szilágyi, etc., 2013). A few pieces of literature are about area measurement. Clements et al. (2018) investigated the effects of instructional interventions through different levels of a learning trajectory designed to support young children's understanding of area measurement as a structuring process.

The literature on data modeling includes research on statistical reasoning, statistical learning, quantitative modeling, and probabilistic thinking. English and Crevensten (2013) explored data modeling with a specific focus on structuring and representing data, including the use of conceptual and meta-representation competence, informal inference, and the role of context through the longitudinal study of data modeling in grades one to three.

Kieran, Pang, Schifter and Ng (2016) investigated the nature of the research carried out in early algebra and how it has shaped the field's growth for the younger students, aged from about six years to 12 years. This study found that mathematical relations, patterns, and arithmetical structures lie at the heart of early algebraic activity, with noticing, conjecturing, generalizing, representing, justifying, and communicating central to students' engagement.

Based on the strengths that have been identified in this current research body about mathematics learning in other content domains in early childhood contexts, we consider important for future research pathways to investigate the connections of the development of early knowledge and skills in content domains beyond number, e.g., pattern and structure, data modeling and early algebra with future mathematical performance of students. Further research could be carried out about the nature of classroom culture and the role of the teacher, the curriculum, instructional strategies and new technologies on the learning in the above mathematical content domains, as well as on the relationship between children's competences in these content domains and their cognitive skills and affective characteristics.

5. The Role of Technology in Mathematics Teaching and Learning

A large body of research on this theme focused on design features of different technological tools used in ECME, such as spreadsheets, Interactive Whiteboards

(IWBs), dynamic geometry software and programmable toy robots, but the vast majority of the technological tools described are interactive applets for tablets.

An issue raised in many papers is the need for designing new apps for learning mathematics in order to create constructive opportunities to represent mathematical objects so that they can be manipulated. A second issue raised is the importance of well-founded guidelines at the basis of this design.

A number of studies focused on multi-touch devices, in which environments are designed supporting embodied and multi-player interactions. Among the most referenced and studied is the app TouchCounts, which takes advantage of the multi-touch and gestures functionalities of the device and provides multimodal visual and auditory feedback for every touch or gesture (e.g., De Freitas and Sinclair, 2017). Other studies have focused on activities with programmable toy robots designed to ameliorate preschool children's visuospatial reasoning, engaging them in a playful and tangible way (e.g., Di Lieto et al., 2017).

Frameworks used in this domain or developed from design research include the Theory of Semiotic Mediation (e.g., Bartolini Bussi and Baccaglini-Frank, 2015), the discursive approach, and a framework that makes a distinction between instructive, manipulable, and constructive multimedia (Goodwin and Highfield, 2013), through which a review of educational apps was conducted. An important conclusion was the value of open-ended tasks, also expressed in other studies.

An important area of research is that of computer assisted interventions. Various studies focused on educational software centered around specific topics, such as number sense. This research aimed at highlighting the educational potential (in terms of students' improvements in mathematical achievement) offered by certain apps that exploit affordances of multi-touch devices for fostering preschoolers' development of number-sense (e.g., Baccaglini et al., 2020).

Studies suggested that well-planned integration of apps in the classroom, with clear learning objectives and appropriate feedback could motivate children, enhance concentration, and support independent learning and communication (e.g., De Freitas and Sinclair, 2017; Kaur and Sinclair, 2014). Some studies showed that children make sense of the digital tools and are able to apply the tools purposefully as long as they also interact with an adult, within their zone of proximal development. Findings also suggest that when technology integration is accomplished successfully in early childhood education settings, children tend to interact more with one another and exchange information related to computer tasks as well as to the overall classroom on-going curriculum themes. Research on using dynamic geometry environments with young children suggested that gestures and motion play an important role in children's developing the mathematical conceptions at stake (e.g., Kaur and Sinclair, 2014).

Many of the studies in this theme explored the roles that preschool teachers give to technologies in mathematics education and the ways in which they structure their mathematics learning activities when using technological artefacts. As far as the teachers are concerned, having a negative attitude towards technological artefacts like an IWB led to a decrease in the likelihood to enrich the learning environment and lead to pedagogical change. The IWB does not seem to pose pedagogical challenges to teachers as its stable location offers the opportunity of using it in traditional teaching ways. Instead, tablets seem to pose a problem for some teachers because of their mobility and the need to reconfigure the organization and, to some extent, the roles of teacher and students.

Many semiotic resources were used with different mediating roles in different teaching-learning processes. However, an emerging challenge is that teachers frequently do not seem to be confident about their ability to teach mathematics using computers. These studies suggested providing effective professional learning and development programs so that teachers can employ a wider range of pedagogical strategies to support the children's use of ICT (e.g., Dong, 2018).

Overall, the findings on preschoolers' interactions with touch-screen-based virtual manipulative mathematics apps confirm the hypothesis that multi-touch technology has the potential to foster important aspects of children's development of number-sense, however such research is still in its infancy. A few studies have pointed to relationships between children's strategies used when interacting with certain apps that afford multi-touch inputs and their development of number sense abilities (e.g., Holgersson et al., 2016).

6. Cognitive Skills and Special Education of Young Children

6.1. Abilities predictive of or associated with mathematical performance

In this area of research, a distinction is made between domain general abilities (e.g., executive functions (EF), working memory (WM), long term memory, visuo-spatial abilities, inhibition), domain specific abilities (neurocognitive trend) and abilities related to the socio-cultural dimension and language.

Studies on domain general abilities highlighted the correlations between children's executive functions and their numerical abilities. Also, non-verbal number sense and working memory are central for early mathematical achievement in preschool. Both visuo-spatial working memory and the phonological loop have been associated to mathematical achievement as well as to the children's development of language, and linguistic competence, is also correlated with mathematical achievement (e.g., Fuchs et al., 2019)

Low visuo-spatial abilities do not seem to change the nature of the mental number line, but they can lead to a decrease in its accuracy. On the other hand, visualizing spatial arrangements is a key ability in the development of number sense.

Regarding domain specific abilities, studies have found that the nature and time needed for the development of children's mapping of the new symbolic representations of numbers they learn onto pre-existing non-symbolic representations are not yet clear. Moreover, there seems to be a tendency within neurocognitive research to support the development of cardinality before ordinality. However, some studies have suggested the importance of early focus also on ordinality (e.g., Coles and Sinclair, 2017).

The approximate number system (ANS) plays a key role in the development of basic numerical abilities, and it is thought to help children achieve cardinality. However, an important role is also played by children's spontaneous focus on numerosity (SFON) (e.g., Verschaffel et al., 2016). Research has widely recognized the importance of using fingers, suggesting that explicit teaching of finger counting in early primary school practices should be promoted to help weaker children overcome their difficulties in arithmetic.

The literature on the abilities related to the socio-cultural dimension and language indicated that the role of language is quite controversial. All of the studies analyzed suggested that there are links between early mathematical achievement and phonological competence. Moreover, the use of narration enhances mathematical learning.

Difficulties with mathematical language can also become obstacles in the development of mathematical meanings. This line of research highlighted the key role played by the teacher. Research converged on the main objective of developing multimodal approaches to mathematics education, within which language is a resource that involves various elements (signs, gestures, etc.) combining mathematical symbols (including formal ones) and images to promote the development of mathematical meanings.

Different ways of pronouncing numbers in different languages can support or hinder the development of number sense itself; for example, using languages with more transparent ways of denominating numbers has a long-lasting benefit on children and it places them at an advantage (e.g., Dunbar et al., 2017).

A number of studies emphasized how a deprived socio-economical-cultural environment has a significant negative influence on the development of number sense.

6.2. Special education

The reviewed research in special education within ECME involves two major directions: The first direction is focused on mathematical capabilities and their development in young students with special needs and the second one refers on ways to support and improve mathematics learning of young students with special needs. For both directions, the findings from the literature review reveal a greater emphasis on researching low-attaining children than high-achieving children. Considering the first direction, a large part of research in low performing students focused on students with mathematical learning disabilities (MLD) mainly with respect to numeracy. Several studies have investigated the relations between MLD and cognitive skills (e.g., working memory), language skills, other learning difficulties (e.g., dyslexia, non-verbal learning disabilities) or pathological disabilities. Regarding the second direction, a significant number of studies developed remedial numeracy programs or examined the effectiveness of interventions for young children at risk for mathematics difficulties or low-attaining children in mathematics (e.g., van Garderen et al., 2020). However, the remediation of MLD in educational contexts is a field that needs further improvement. There is also growing research that takes place on the use and effects of evidence-based instructional tools, ICT, teacher professional development, family-centered practices and parent involvement/ training on the mathematical development of children at risk for mathematics difficulties which could contribute to this endeavor.

7. Early Childhood Teachers' Knowledge, Education and Affective Issues in Mathematics

For preschool teachers, subject matter knowledge (SMK) often refers to the concepts and skills learned before first grade, such as number and operation, measurement, geometry, data representations, and patterns. Several studies presented models and frameworks for investigating teachers' knowledge and competencies for teaching early childhood mathematics (Gasteiger and Benz, 2018; Tsamir, et al., 2014). In addition to content, studies included elements of pedagogical content knowledge (PCK) such as knowledge of early mathematics development, ability to observe mathematical situations and to take appropriate pedagogical actions. Lindmeier et al. (2016) added reflective competence and action-related competence. Most studies of preschool teachers' SMK reported on general scores, (e.g., Opperman et al., 2016), with a few focusing on specific domains, such as patterns (e.g.,Tirosh et al., 2017) and geometry (e.g., Tsamir et al., 2015; Ulusoy, 2020). Investigation of teachers' content knowledge in areas such as measurement and data representation is scarce.

Studies that concern teachers' PCK are more frequent than studies of teachers' content knowledge, focusing on teachers' knowledge of students as mathematics learners (Tanase and Wang, 2013) and teachers' knowledge of task orchestration (Hundeland et al., 2017). Several studies focused on teachers' abilities to recognize mathematical situations that occur during children's natural play (e.g., Benz, 2016) or during play-based scenarios (Lee, 2017). In general, teachers note concepts related to classification, number sense, and measurement.

Teachers' attitudes towards teaching mathematics improved through professional development (e.g., Sumpter, 2020). Some teachers claim that mathematics in kindergarten is important because children need to be prepared for first grade (Schuler et al., 2013). Teachers mostly consider counting, rather than geometry, as relevant and important (Schuler et al., 2013).

Concerning pedagogical mathematical beliefs, teachers stress children's need to use their bodies as tools for learning mathematics, for example by climbing up and down to feel differences in height (Franzén, 2014). Most teachers believe that it is essential for children to be active learners (Li et al., 2019) and that the teachers' roles are to ask questions (Cross Francis, 2015), instill curiosity (Schuler et al., 2013), and encourage children to think and draw conclusions by themselves (Li et al., 2019).

Self-beliefs, such as anxiety, confidence, and self-efficacy were found to be linked to various contextual factors (e.g., Thiel and Jenssen, 2018). Gasteiger and Benz (2018) pointed out that teachers' attitudes and motivation influence how they use their knowledge and skills. Higher knowledge is associated with higher levels of student-centered beliefs (Ren and Smith, 2018). Tirosh et al. (2017) investigated the links between pattern knowledge and corresponding self-efficacy.

Few studies investigated the preparation of prospective teachers to teach mathematics in preschool. Instead, programs aimed at promoting practicing teachers' knowledge of their young students' mathematical thinking, and helping teachers develop and use tools to better understand children's engagement with mathematics. Several programs used video as a tool for reflection (e.g., Cross Francis, 2015). Other programs encouraged teachers to use clinical interviews (e.g., Polly et al., 2018).

In conclusion, although this section is about teachers, the common thread running through most studies is an emphasis on children. More studies focused on teachers' PCK than their SMK, and within PCK, research has focused on coming to know young students as mathematics learners. Studies of teachers' beliefs investigated the relevance of mathematics for young children, and what is appropriate mathematics for children. Finally, many professional development programs focused on enhancing teachers' knowledge of children's mathematical reasoning. Further research might focus on teachers' knowledge and beliefs related to the use of technology in preschool mathematics and interventions specifically for prospective preschool teachers.

8. Concluding Remarks

Our work on this survey has shown that there is a plethora of research work of a broad scope on ECME and that there will be continued growth and important progress in this field in the years to come. In the past few years we gained considerable amount of relevant knowledge about what mathematics children know and (can) learn before or at the beginning of formal education, how they learn mathematics and develop their mathematics skills, how early mathematics learning can be stimulated and enhanced and also on teachers' knowledge, acts and beliefs related to early years mathematics. We expect to see further development in the aforementioned areas with greater research attention on the following aspects: assessing and developing children's competences in content strands beyond number sense and also children's cognitive abilities that are strongly associated with mathematics performance (e.g., SFON, ANS), toddlers' mathematical development and learning opportunities, the use of digital tools to support children's learning and opportunities offered to children to engage in embodied ways of mathematical thinking and learning. The development of early childhood teachers' knowledge and skills on the above aspects would be another major challenge in the field.

Acknowledgments

We wish to thank Giulia Lisarelli and Alessandro Ramploud for their help in reviewing a part of the literature for the survey.

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https://doi.org/10.1142/9789811287152_0023

Survey Team 3

Teachers' Collective Work as Regular School Practice for Teacher Professional Development and Learning

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ABSTRACT In this report we present the results from an extensive search of the literature regarding mathematics teachers' collective work in schools, in the Eastern and Western literature. In particular, we try to answer the research questions related to the following themes: (1) the nature of mathematics teachers' collective work as regular school practice; (2) the participants of such school-based collective work and their roles; and (3) the professional development and learning that can be observed in school-based teacher collective work. In terms of theoretical frames, results show that different variations of Lesson Study have been the main frame for teachers' collective work at school level, in particular of course in Japan and China, but also increasingly in Western countries such as the UK, the Netherlands, Portugal and Spain. The choice of this frame also impacted on the nature of the collective work: working in cycles of lesson (and learning progression) planning, enactment, and evaluation, leading to the re-design of lessons. Whilst in Western countries participants comprised a mix of teachers and researchers, in most Eastern countries, teachers would also work on their own (as a group of teachers) or with so-called expert teachers in their collective groups. Teacher learning resulting from collective work was reported in terms of: (a) lesson planning and preparation; (b) pedagogical content knowledge(c) classroom practices; (d) general pedagogy; (e) social and personal issues in the mathematics classroom. The findings have implications for the conceptualization of school-based teacher collective work, for the support and facilitation of such work, and for research, in particular in terms of teacher agency.

Keywords: Teacher collaborative work; Regular school practice; Teacher professional development; Teacher learning; Lesson Study.

1. Introduction

During the past decades, teacher collaboration has received increasing attention from both the research and the practice fields. It has been claimed that teacher collaboration can positively influences the whole school community. DuFour et al. (2005) contend

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that collaborative learning communities "hold out immense, unprecedented hope for schools and the improvement of teaching" (p. 128). Amongst others, teacher self-efficacy has been found to have improved (e.g., Puchner and Taylor, 2006), increased teaching effectiveness (e.g., Graham, 2007), and improvement of instructional quality (e.g., Jackson and Bruegmann, 2009; Hochweber et al., 2012). These positive effects will improve their quality as professionals and as Hattie (2003) suggests, teacher quality alone accounts for 30% of the variance in student performance. Hattie (2015) also claims that teacher collaborative working communities will enhance teacher effectiveness and expertise. Moreover, selected research has shown that the positive influence of teacher collaboration transcends the teacher community, and it has been suggested that professional collaborative activities might have a positive effect on student achievement (e.g., Dumay et al. 2013; Goddard et al. 2010).

Whilst in many (Western) countries previously teacher professional development activities were mainly conducted at and by universities and teacher education institutions, nowadays they are often run by local or regional school boards and agencies at school level. This trend goes hand-in-hand with proposals that teachers become partners in the design of their curriculum, rather than "simply" implementing the curriculum, supported by (government) approved textbooks. Moreover, due to the availability of an enormous amount of free educational resources on the web, teachers ask for guidance and support for choosing and appropriating those resources for their classroom, and the closest support lies at school level, with their colleagues (in their or neighbouring schools). However, this trend also asks for teacher agency, their professional agency: e.g., their decisions to participate in or withdraw from the teacher collective; which resources to ask for and use; which foci to choose in the collective; how to collaborate with colleagues/peers; which role to take in the collective.

At the same time teachers' collective work as regular school practice has a long history in many (mainly Eastern) countries: Lesson Study in Japan and Teaching Research Groups in China are well known examples. However, varying forms of such practice exist in many countries and in varying educational contexts. Over time, and particularly in more recent years, these practices have been shared and researched leading to the evolution of a wide, yet dispersed, knowledge base.

In this paper, we present the results from our international survey of the literature regarding mathematics teachers' collective work in schools. In particular, we try to answer the following research question:

> ***What can be learnt from an examination of common features of mathematics teachers' collective work as regular school practice as well as from variations in practices and their rationales in different national contexts?***

We ask the following sub-questions:

1) *What is the nature of mathematics teachers' collective work as regular school practice, and how does this relate to situation, culture and context?*
2) *Who is engaged in such school-based collective work, what are the roles*

of those people involved, and how do they relate to each other in the different communities?

3) *What kinds of learning can be observed in school-based teacher collective work? (How does teacher collective learning happen in teacher collectives at school, what is the evidence for their learning?)*

In the following (second) section, we present the background to the present study. This will be followed, in the third section, by the methods we used to conduct the literature review. In the fourth section, we explain the theoretical frames. In the fifth section we present the results relating to the research sub-questions, and in last section (section six) we refer to the main research question with our conclusions.

2. Background

In the previous section we have provided a rationale for conducting the survey we did, as we contend that teacher collaborative work is an important area for study. In earlier and recent ICME- related studies this has been acknowledged by: e.g. Borko and Potari 2020 (ICMI-25 Study); Robutti et al. 2016 (ICME Survey 13); Jaworski et al. 2016; Adler et al. 2005 (ICME 2004 survey). We build on this body of work, and attempt to establish in which ways teachers' collaborative work at school level has developed and how it varies across contexts.

3. Methods

To identify the relevant literature to answer our research questions, we conducted a systematic literature review. This review mainly relied on the procedures of a thematic synthesis (Xiao and Watson, 2019) with the overarching aim to build on the current body of literature, to summarize what is known about teachers' collective work at school level. Since research on teachers' collective work has enormously increased over recent years, we reduced our literature review to publications that were published since 2015. We went through all titles and abstracts from the following list of journals, conference proceedings and books in order to identify the papers comprising recent research on mathematics teachers' collaborative work at school level:

- Journals:
  - ✓ Educational Studies in Mathematics
  - ✓ International Journal of Science and Mathematics Education
  - ✓ Journal of Mathematics Teacher Education, Research in Mathematics Education
  - ✓ ZDM — Mathematics Education; Mathematics Teacher Education and Development;
  - ✓ Professional Development in Education
  - ✓ Mathematics Education Research Journal
  - ✓ Journal of Science and Mathematics Education in South-East Asia
  - ✓ Mathematics Teacher Education and Development; Mathematics Teacher
  - ✓ The Montana Mathematics Enthusiast

  - ✓ The Mathematics Educator
  - ✓ International Journal of Lesson and Learning Study
  - ✓ Teaching and Teacher Education
  - ✓ South African Journal of Education
  - ✓ Journal of Research in Mathematics Education
  - ✓ Teacher Education Quarterly
  - ✓ Professional Development in Education
- Conference Proceedings:
  - ✓ Congress of the European Society for Research in Mathematics Education (CERME),
  - ✓ Conference of the International Group for the Psychology of Mathematics Education (PME)
  - ✓ Mathematics Education Research Group of Australasia (MERGA)
- Books:
  - ✓ The International Handbook of Mathematics Teacher Education (Vol. 3, Participants in mathematics teacher education- individuals, teams, communities and networks; Krainer and Wood (eds.) 2008)
  - ✓ The Resource Approach to Mathematics Education (Trouche et al., 2019)
  - ✓ Mathematics lesson study around the world (Quaresma et al. 2018)
  - ✓ Lesson study research and practice in mathematics education (Hart et al. 2011)
- National/regional publications (in the respective languages) in the following countries: Japan, China, Netherlands, Lebanon, Australia
- Reports by researchers from particular countries (based on selected questions): Italy, Denmark, Singapore, Israel.

Altogether we searched through 21 journals and 4 different conference proceedings between 2015 and 2021, to collect about 200 articles (including national publications in the native languages), in addition to reports from four countries. After identifying a corpus of the relevant literature, we went through the literature identifying results on mathematics teacher collective work at school level that related to the three research questions.

4. Theoretical Frames

The following theoretical frames are explained below:

(1) teacher collective work (at school level);

(2) Lesson Study;

(3) teacher agency.

4.1. Teacher collective work at school level

Within mathematics teachers' collective work, we can basically distinguish between two types: (1) Lesson study (please see section 4.2) with its three distinct features: planning a research lesson collaboratively; conducting and observing the planned lesson; jointly reflecting on the lesson based on observations of student activity (Murata, 2011); and (2) teachers' collective work on a proposed (or agreed, e.g., by a project) theme: e.g., reasoning and proof in commonly used textbooks. This would not necessarily include lesson preparation or indeed the enactment of a planned lesson.

Another distinction of teachers' collective work relates to **the place of the collective work**: e.g., in school, at a distance (online), or at university. Clearly, we refer here only to work that is school-based. However, school-based teacher collective work can also be online, that is at a distance. The latter have enormously increased, possibly due to COVID-19 pandemic related measures.

It can be said that most of the collective work at school level includes planning and enacting lessons, as this is the main part of teachers' daily work. Hence, we can say that most of mathematics teachers' collective work at school level relates to Lesson Study, one way or another, and this is the reason why we have emphasized this way of working in our theoretical frames.

In order to conceptualize lesson study adaptations (also in their home Danish context), Skott and Moeller (2020) have used the notion of *figured worlds* (Holland et al., 1998), asking 'What characterizes the dominant figured worlds when groups of teachers engage in lesson study in a Danish context?'. Their results and insights stress

> the importance of working on adaptations of approaches such as lesson study in order to transform issues of culture and power in the teachers' local setting. This applies in particular to those related to the three characteristics of a Danish teaching culture identified earlier …: teacher methodological autonomy (as interpreted from a teaming perspective), teacher collaboration characterized by functionality of teaching and a family culture, and the tendency to shake off macro-level demands. (p. 8/9)

This importance was supported by parts of their data showing that some teachers occupying senior-teacher positions would alternate between "old-hand" and "development-oriented" positions. Hence, they conclude that in their (Danish context) "it is necessary to address these broader issues of culture and power in order to adapt lesson study in a Danish context" (p.9). From this study (and others), we conclude that it is indeed necessary to distinguish between (research on) **lesson study adaptations within and outside the East Asian region.**

In terms of Lesson Study adaptations, Ding and Jones (2020) compared three such adaptations/models, each designed for supporting (and studying) in-service teacher collaboration and learning: (1) The Action-Education Model (AE) (Gu and Gu, 2016), a combination of Keli study (exemplary lesson development) practiced by researchers and teachers in schools in China and action research; (2) Learning Study (LS) (Lo and Marton, 2012), a combination of Lesson Study and design study originally conducted in Hong Kong SAR, China; (3) The Community-Centered (CC) model for teacher learning (Borko et al., 2005), a university-based summer institute program for supporting mathematics teacher collaboration (and learning) in the United States. The authors note that both Lesson Study and Learning Study (LS) address simultaneously lesson plan design and implementation as a whole teacher learning process, and (referring to Huang and Shimizu, 2016) how theory can be used to guide teaching and how teaching experiments can further refine theory (p.115). Interestingly, the western design studies (e.g., Cobb et al. 2017) share this view: whilst practically supporting teachers in improving specific aspects of their instructional practice, theoretically, they aim at designing and evaluating (and possibly re-designing) learning progressions (in association with instructional practices) and the teacher learning that goes with it.

4.2. Lesson study

Lesson Study is a complex professional learning approach. Several researchers have used the metaphor of an iceberg to capture the unseen features of lesson study with respect to the task for exposing student thinking and impacting student learning. Their metaphor is useful, in as much as the iceberg has much beneath the surface, many of the features (or essentials) of lesson study are not immediately obvious, and exposing them is said to assure fidelity of implementation of those essential features (Hart et al. 2011).

Historically, Lesson study is a collaboration-based teacher professional development approach that originated in Japan (e.g., Fernandez and Yoshida 2004) and also in China. Over the past decade it has attracted the attention of an international audience: e.g., in 2002 it was one of the foci for the Ninth Conference of the International Congress on Mathematics Education (ICME).

Lesson study incorporates many characteristics of effective professional development programs identified in prior research: e.g., it is site-based, practice-oriented, focused on student learning, collaboration-based, and research-oriented (e.g., Borko 2004; Cochran-Smith and Lytle 2001; Darling-Hammond 1994). Lesson study places teachers at the center of the professional activity with their interests and a desire to better understand student learning based on their own teaching experiences. The idea is straight forward and authentic: teachers share a question/goal regarding their students' learning and they come together based on that question; they plan a lesson to make student learning visible, and examine and discuss what they observe. Through multiple iterations of the lesson design, refinement, enactment and collection of data on student learning, reflection on lesson, and re-design process, teachers have many opportunities to discuss student learning and how their teaching affects it. Lesson study typically has a research lesson (live lesson observation) as the centerpiece of the study process (e.g., Fernandez and Yoshida 2004; Wang-Iverson and Yoshida 2005). The main purpose of this step is not to plan a perfect lesson but to test a teaching approach (or investigate a question about teaching) in a live context to study how students learn. During lesson planning, teachers also have an opportunity to study curricular materials, which may help teachers' content knowledge development. During the lesson, teachers attend to student thinking and take notes on different student approaches. In the discussion after the lesson, teachers discuss student learning based on the data they have collected during the observation (Murata 2011).

There are other professional development programs that incorporate many of the characteristics of lesson study (e.g., action research). For example, in China, the concept of Lesson Design Study has been known to work well in the Teaching Research Groups in China (Ding et al. 2019). And there are also many adaptations to Lesson Study, in particular in the United States (e.g., Amador and Carter, 2018) and in Europe (e.g. Manolino, 2020) However, what is typically different in Lesson Study is the *live* research lesson. This is said to create a unique learning opportunity for teachers. Shared classroom experiences, such as teacher noticing of certain aspects of teaching and learning, might not otherwise be shared.

4.3. Teacher agency

From the work on agency, agency is known to be related to social systems or individual characteristics: e.g., making choices among alternatives, taking initiative or being able to influence oneself and others; and is both constrained and afforded by social relations and structures, particularly power relations (e.g., Mercer 2011). Mercer (2011, p. 428) argues:

> humans as agents [are] able to influence their contexts, rather than just react to them, in a relationship of ongoing reciprocal causality in which the emphasis is on the complex, dynamic interaction between the two elements.

At the same time, Etelapelto and her colleagues (2013) argue for a subject-centered, sociocultural view of professional agency, which takes the individual and social contexts of agency to be analytically separate but mutually constitutive (2013, p. 45). In understanding agency from this perspective, they say, we need to investigate:

> how agency is practiced and how it is resourced, constrained and bounded by contextual factors, including power relations and discourses, and further by the material conditions and cultures of social interaction (2013, p. 61).

The same group of researchers also argue that agency has a temporal aspect, in that people's life histories and prior experiences influence their agency in relation to their contexts (Etelapelto et al. 2013). Biesta et al. (2015) argue that agency is an emergent phenomenon of actor–situation relations and is something that people do, rather than have, i.e. agency is enacted in context and denotes the 'quality of engagement of actors with temporal-relational contexts-for-action' rather than a property, capacity or competence of the person (2015, p. 626). This means that agents act 'by means of their environment rather than simply in their environment' (2015, p. 626).

5. Results

In this section we answer the research sub-questions with data and examples from the literature review.

5.1. What is the nature of mathematics teachers' collective work as regular school practice, and how does this relate to situation, culture and context?

We now summarize insights from studies and reports of mathematics teacher collective work at school-level (1) within and (2) outside the Austral-Asian region, and (3) at school and (4) school-based but at a distance (online).

5.1.1 Austral-Asia

Starting with **Japan**, one of the major characteristics, or the nature, of Japanese mathematics teachers is their **voluntary in-service training**. Baba et al. (2018) discusses the background of "mathematics education Lesson Study in Japan" from four perspectives; Historical, Community, Institutional, and Development Assistance. In the Community perspective, the following facts are highlighted. Some schoolteachers, who usually are excellent teachers, have had a chance to get long-term in-service training under the supervision of university researcher. After their training, these teachers returned to their school and became "Leader Teachers" in the school. They also play an important role in their district teachers' communities. School teachers often have voluntary workshop in their district communities, in which teachers discuss about mathematics materials (Kyozaikenkyu, or material research), preparations for their Lesson Study in their school, writing papers about the results of their in-service trainings, and so on. Such community or workshop is called **"Kenkyukai", or math teacher circles**.

At the same time, schools sometimes have opportunities to get funding for their in-school teacher trainings. Such projects are assigned by district education office, by prefectural education office, or sometimes by ministry of education, and usually done by the strong leadership of "Leader Teachers", who are not only the teachers who have had long-term in-service training, but also the teachers who actively attend to "Kenkyukai", or math teacher circle activities. In this sense, Japanese Teachers' Collective Work as a Regular School Practice is done with strong implicit support of "Kenkyukai", or math teacher circles.

Another major characteristic in Japanese education is the existence of "Fuzoku schools", which are attached schools to university. Especially, Fuzoku schools attached to faculty of education (or university of education) have had special role in Japanese education. It is said that the major role of Fuzoku school are: 1) education to students just as regular school, 2) preservice teacher training, 3) practical study. Regarding (1), Fuzoku teachers are also schoolteachers who do the same work as other schoolteachers. Concerning (2), Fuzoku schools are the place for prospective teachers to do their student teaching. Regarding (3), each Fuzoku school usually has its own "research or study topics", and play an important role to serve practical information about education. Mathematics teachers who are working for Fuzoku schools are usually the "Leader Teachers" in their math education communities. Obviously, they have more opportunities to write reports about their practical work in mathematics education. The number of Fuzoku schools attached to faculty of education is very small. There are about 70 Fuzoku within 20,300 elementary schools, 71 Fuzoku within 11,000 Jr. high schools, and 15 Fuzoku within 5000 senior high schools.

In a paper by Isoda (2020), the author reports on the historical development of Japanese Lesson Study. The author briefly sketches the Japanese theories for designing and reproducing better lessons to share and transfer the challenges and experiments of lesson study. Whilst Lesson study was initiated in 1873, it developed over more than a century, whilst "the major theories of mathematics education for designing and reproducing sciences were developed on the elaboration of theories proposed by various lesson study groups." (p.15) At the time of the author's writing, these can be summarized as the theories for: "clarify the objectives; distinguish teaching concept; establish the task sequence; and teaching approaches which includes assessments" (p. 15).

One of the differences of Lesson Study in Japan as compared to Western practices is that the importance of lesson preparation is largely underestimated in the West, and the collaborative work among teachers that goes into creating that lesson plan is largely under-appreciated by non-Japanese adopters of Lesson Study. This might be due to the effort involved being largely invisible to outsiders, with attention going to its most visible part, the live research lesson. The paper by Fujii (2016) makes visible "the process of lesson planning and the role and function of the lesson plan in Lesson Study" (p. 411). The paper identifies key features of the planning process in Lesson Study, including its focus on task design and the flow of the research lesson, and offers suggestions for educators seeking to improve Lesson Study outside Japan.

In **China**, mathematics teachers' collective work as regular school practice has been guaranteed by the teaching research system, because each mathematics teacher is "naturally" (by default, as part of the job as a teacher)) a member of the mathematics TRG (Teaching Research Group) and LPGs (Lesson Preparation Group) in each school in Chinese mainland. In *Secondary School Teaching Research Group Rule-book* issued by MOE in 1957, the duty of TRG was emphasized:

A Teaching Research Group is an organization to study teaching. It is not an administrative department. Its task is to organize teachers to do teaching research in order to improve the quality of education, but not to deal with administrative affairs (MOE, 1957).

Chinese Lesson Study (CLS) is just one of the forms of collective learning based on school-level TRG activities.

Not only mathematics teachers, every subject teacher belongs to a specific subject TRG for the reason of the teaching research system as the fundamental context in Chinese mainland. Because most of the Chinese teachers who teach just one subject two or three times a day, the same subject teachers are easily organized into subject-specific TRGs. This multi-tiered teaching research system is a network where province-level TRO oversee city-level TRO (see figure below), and city-level TRO oversee county-level TRO which oversee school-level TRGs (Yang, 2009; Yang and Ricks, 2013). The TRG is the basic unit in this network; its main responsibility is conducting research on teaching to solve the practical problems from teachers. So, mathematics teachers' collective work rooted deeply in the school-level TRG activities, which linked the lessons and the studies in their daily work (Fig. 1).

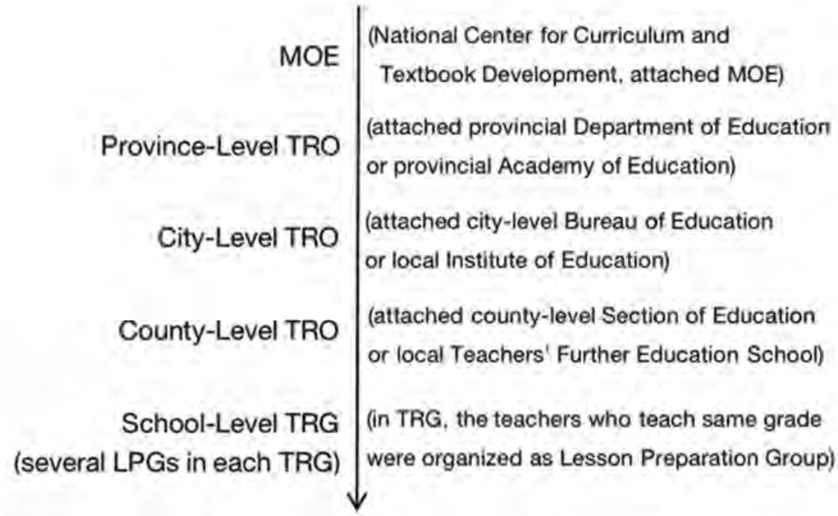

Fig. 1. The top to down guiding structure in Teaching Research System

In terms of content in such CLSs, a study by Huang et al. (2016) reports on student learning being studied by teachers, to improve teaching that promotes students' understanding. Interestingly, this CLS included didacticians (practice-based teaching research specialist and University-based mathematics educators) and mathematics teachers in China, who explored and documented how teacher participants "shifted their attention to students' learning by incorporating two notions of teaching: learning trajectory (LT) and variation pedagogy (VP)" (p.425). The former describes conjectured routes of children's thinking and learning with pertinent tasks to move towards the learning goals along the route, while the latter suggests strategies for using systematic tasks progressively. The concepts of LT and VP were used to guide planning, teaching, and debriefing throughout the LS process. Results revealed that "by building on the learning trajectory and by strategically using variation tasks, the lesson has been improved in terms of students' understanding, proficiency, and mathematical reasoning" (p.425). It was claimed that (and how) "theory-driven Lesson Study could help teachers improve their teaching and develop the linkage between theory and practice." (p.425)

In **Australia**, there has also been a particular interest in Japanese Lesson Study, as a vehicle to improve mathematics teaching practice. In their paper Groves et al. (2016) report on a small-scale research project, implementing structured problem-solving mathematics lessons through lesson study. The two major aims of the project were to investigate critical factors in the adaptation and effective implementation of (1) structured problem-solving mathematics lessons, and (2) Japanese Lesson Study as a model for teacher professional learning in the Australian context. Critical factors of Lesson Study were identified by the teachers as contributing to the success of the project. These included “the opportunities for in-depth lesson planning, the presence of large numbers of observers at the research lessons and the post-lesson discussions, and the insight provided by the knowledgeable other” (p. 501). Major constraints included the difficulty in finding suitable problem-solving tasks to match the Australian curriculum, and the teaching culture that emphasizes small-group rather than whole-class teaching.

Reporting on Lesson Study in **Korea**, Pang et al. (2016) describe how a lesson study using five practices for mathematics discussion was implemented in the Korean context. They contend that Lesson Study has had an effect on improving the quality of mathematics instruction and supporting teachers’ professional development, in the sense that “the lessons were changed to specify learning goals for students, to devise mathematical tasks in a rigorous and meaningful way, and to design the lesson structure to maximize students’ engagement” (p. 471).

5.1.2 Europe and Middle-East

During the past two decades, in Europe and Western countries (including North America) professional learning communities (PLCs) have been established, as they are seen as levers for teacher professional development. PLCs are generally defined as groups of teachers who come together to engage in regular, systematic and sustained cycles of inquiry-based learning, with the intention to develop their individual and collective capacity for teaching to improve student outcomes (Brodie 2021). PLCs are said to create spaces for ongoing, sustained professional development, in particular at school level, different from the often-fragmented professional development programs that many teachers are exposed to (Borko 2004, Cobb et al. 2018). PLCs can be seen as a special case of communities of practice (Wenger 1998), where members engage in professional learning (see section 5.3). One of the main intentions for PLCs is to deliberately position teachers as professional agents in their own professional development, through their making professional decisions as to what they need to do to enhance their teaching, in particular based on their understandings of their learners’ needs. While much of the work on PLCs argues for teacher agency as a key driver of PLCs, it is not yet known what it means to develop teachers as agents and what teacher agency actually entails (Brodie 2021; Horn et al. 2018). However, the literature on PLCs converges on five key characteristics of successful PLCs (e.g., Brodie 2021): focus, long-term inquiry, collaboration, leadership support and trust. How these characteristics play out in PLCs is central to their sustainability as spaces for professional development.

In Europe, these PLCs meet, for example, at school or at university, or in other commonly agreed spaces. In many European countries (e.g., UK) it is common to meet in school, on a voluntary basis. However, there is typically no institutionalized system of PLCs, as we see in China and Japan. Often, the PLCs are initiated by (European) projects and conducted by university academics in regional schools. In selected countries (e.g., NL), the PLCs (sometimes called DOTs- design teaching teams; see articles by Verhoef 2013, 2015) are initiated by national institutions to implement curriculum changes at school level. In recent years, Lesson Study in various forms (see earlier discussion; Skott and Moeller 2020; Ding and Jones 2021) has been promulgated as a suitable vehicle for professional development.

In Israel, Karsenty et al.'s (2019) team explored how secondary mathematics teachers, participating in a school-based video club (Sherin, et al. 2009) communicated with each other and with the facilitator along the different sessions of the club. Whilst there are different forms of video clubs, in this context a group of teachers met on a regular basis, usually under the guidance of a facilitator, to watch and discuss classroom video selected according to a certain aim. Analyzing their evaluative comments (with respect to the non- judgmental norms that this club aimed to nurture), three types of evaluative comments were identified, "reflecting varying degrees of teachers' capability to interpret and discuss observed teaching moves while attributing possible rationalizations to the filmed teacher's decisions." (p.3400) They found that as the club proceeded the communication became more productive.

In the UK, we found an example of Lesson Study in the context of the introduction of a New National Curriculum for Mathematics in England; this was not supported by a mathematics teacher educator (Warwick et al. 2016). They claim that Lesson Study is "rapidly becoming one of the most adopted models of teacher professional development worldwide" (p.555). They examined the teachers' discussions that were an integral part of the Lesson Study research cycle. In particular, they investigated "the 'dialogic mechanisms' that enable teachers' pedagogical intentions to be developed within the context of discussions that stem from observations of students as they address mathematical problems" (p.555). Findings suggested that a focus on student outcomes enabled teachers to collaborate effectively on developing pedagogical intentions to directly address student need.

Leaning on teacher collaboration for lesson planning, the paper by Pepin et al. (2017) reports on mathematics teachers re-designing their lessons due to curriculum changes in selected countries in Europe. Whilst the goal of this paper was to develop enhanced understandings of mathematics teacher design and design capacity when interacting with digital curriculum resources, it also offered new understandings of teacher collaboration in different context: e.g., France and Norway; small group collaboration (France) vs large group of teachers (Norway). Drawing on two different collective environments and two individual teacher cases working within these environments, the authors investigated and illustrated teachers' design processes (and design capacity building) across a range of contexts and curriculum formations, with the focus on how digital resources can help to develop teacher design capacity.

In terms of teacher collaboration at the distance, we found several papers, all using different ways of communicating at a distance. One of these ways were MOOCs (e.g. Taranto et al. 2020). In this project the authors used two theoretical lenses (Meta-Didactical Transposition, Connectivism) to investigate teachers' learning processes (see also section 5.3). Results showed two different teachers' learning processes: one that evolved dramatically because of the interventions — they called it an 'explosion'; the other less proactively — they called it 'linear'. In the Danish context, as another example, Tamborg (2021) investigated how a national platform brought teachers together for professional development and how it affected teachers' work. The platform was the main tool to implement at scale "an evidence-based, objective-oriented approach to teaching". He concluded that the design of platforms conflicted with the needs of mathematics teachers.

5.1.3 Americas

Over the past decade, Lesson Study has become very popular, in particular in North America. Stigler and Hiebert (2016) reported that lesson study is gradually spreading around the globe, and that the Western community has "much to learn from how it is implemented in a variety of cultural contexts". (p. 581) They reflect on the goals of lesson study, the organizational supports required to sustain the practice in various contexts, and "the benefits that may be derived from making more explicit the connections between lesson study and the wider field of improvement science" (p.581). They claim that both research and practice can benefit from learning about and from such different practices.

To provide an example of such 'borrowing', Lewis (2016) presents a theoretical model of lesson study's impact on instruction by impacting on teachers' beliefs and their learning community, amongst others. She also describes four different types of lesson study in Japan: (1) incorporation of high-quality tasks and materials; (2) attention to processes that illuminate student thinking; (3) attention to system features; and (4) models for scale-up. (p.581) She points out their "synergies in producing a system where local teachers "demand" knowledge for their lesson study work and lesson study provides a collaborative, practice-based venue to try out recent innovations in curriculum and instruction" (p. 581).

In several Western countries (e.g., USA, France) we found teacher collaboration being established around how to make sense of new academic standards and how teachers may shape the implementation of those standards. In the US context, Johnson et al. (2016) reported on a study where professional development was organized around the analysis of mathematical tasks, to support teachers to prepare for standards implementation by helping them develop common understandings of standards and how to help students meet ambitious new learning goals. However, in reality designers and teachers brought different goals to the professional development context, which became evident when teachers engaged in task analysis. Using a particular 'design tensions framework', they analyzed tensions within a research–practice partnership comprised of university researchers, district curriculum leaders, mathematics teachers, and Web engineers. Results showed the need for designers of professional development focusing on standards implementation, to be 'adaptive and willing to evolve activities to satisfy multiple stakeholders' goals for participation'.

In terms of distance learning in collectives, we found many ways of collaborating 'at school level'. For example, Larsen and Liljedahl (2017) used Twitter posts to analyze stimulating sustainable mathematics teacher collaboration in a 'distant professional development context'. To their surprise, an unprompted, unfunded, unmandated, and largely unstudied mathematics teacher community emerged where the mathematics teachers use social media to communicate about the teaching and learning of mathematics. Results indicated that enough redundancy and diversity among members is necessary to make conversations productive.

In summary, it can be said that the contexts and cultural education traditions influence the professional learning communities: their set-up, their practices, the tools used, and the expected outcomes. In the Western countries, many communities are driven by a desire to innovate or renew the curriculum and the pedagogy or to come to a meaningful integration of technology. These communities tend to be part of projects with a limited time frame. In the Eastern countries professional learning activities are more connected to the everyday teaching activities and focus on values and perceptions of 'good mathematics teaching' and 'good lesson planning'. In particular, countries like Japan and China have established a 'tradition' of teacher professional learning communities at school level.

5.2. Who is engaged in such school-based collective work, what are the roles of those people involved, and how do they relate to each other in the different communities?

Depending on the context (e.g., research project in Europe, Lesson Study in Japan, Learning Study in China), there are often different people involved in the collective work of teachers. As explained earlier, in the European context, teachers often work with teacher educators on Lesson Study or similar project that is most of the time financed by outside (school) funding bodies (e.g., EU financing), whilst in the Japanese and Chinese lesson study collaborations, classroom teachers (e.g., of the same grade) work together, sometimes with the support of expert teachers (Pepin et al. 2017). Despite these differences, a common thread running throughout the surveyed articles is the need for learning to be situated in collaboration with others. However, the collaboration can take on very different structures in supporting teachers' professional learning due to the different purposes and roles of the teachers, expert teachers or teacher educators in the studies (see 5.3).

Professional learning communities with mathematics teachers and teacher educators and/or expert teachers working and learning in collaborative groups show a huge diversity of roles, identities and interactions. This makes it difficult to get an insightful overview of this diversity, to compare initiatives or to grasp the specificity of individual initiatives. In their article Krainer and Spreitzer (2020) selected seven recent articles (covering all continents) and analyzed them along the following dimensions: relevant actors, relevant targets, and relevant environments of the collaboration (RATE). In terms of 'relevant actors', they claim that (using the RATE scheme) apart from mathematics teachers, the seven articles show “a variety of actors”, including teacher educators (6 initiatives), mathematicians (4), and policy makers (2). As social entities they found (video) “clubs”, different “communities”, (lesson study) “groups”, (project) “partners” and (design and project) “teams” (p. 34).

Regarding Lesson/Learning Study in China, Gu et al. (2016) reported on the roles of experts and other participants. The team investigated how “mathematics teaching research specialists” mentor practicing teachers during post-lesson debriefs of a lesson study in China. Results of fine-grained analysis of post-lesson study debriefing revealed that the “Chinese teaching research specialists (expert teachers, see Pepin et al. 2017) pay a great deal of attention to practical knowledge, which consists of setting students’ learning goals, designing instructional tasks, formative assessment of students’ learning and improving instructional behaviors” and that “less attention is paid to mathematics content knowledge and general pedagogical knowledge” (p. 441). The teaching research specialists apparently also pay less attention to address issues raised by the teachers or to engage in dynamic dialogue with them. Using a purposefully-developed framework for analyzing mentoring activities emerges, the strengths and weaknesses of the teaching research specialists’ mentoring strategies were identified.

It has been noted that in the USA, it is rare that teachers work with university colleagues in their school settings even though this collaboration often improves classroom instruction (Herrenkohl et al. 2010). Overall, university partnerships with teachers for professional development has been considered beneficial because of the potential of collaborative work in the teacher’s own classroom to be relevant to practice. From this perspective, both teachers and researchers can draw on their own expertise and work as authentic partners. In a study by Jung and Brady (2016) in the USA, they investigated how a teacher and a researcher performed their roles when collaboratively implementing mathematical modeling tasks within a context of in situ professional development. The researcher–teacher partnership shown in this study demonstrated how such collaboration can be supported by sharing knowledge and resources (Lau and Stille 2014). Through this in situ professional development focusing on mathematical modeling tasks, “several teacher and researcher roles were highlighted: (1) the researcher’s ways of opening the discussions and addressing the teacher’s concerns, (2) the researcher’s approaches to acknowledging the teacher’s expertise, (3) the teacher’s strategies for overcoming difficulties, and (4) the teacher’s process of reflecting on the factors that helped student development” (p.291). While the teacher learned about the new mathematical modeling tasks and related research, she helped the researcher recognize classroom realities and implement modeling tasks in these realistic settings. They also shifted roles at different stages of instructional practice (e.g., the researcher led classroom instruction or the teacher analyzed student work), which ensured that both teacher and researcher took ‘‘the role of expert’’ depending on the classroom situation (Lau and Stille 2014). The study supports the value and viability of this form of in situ professional development, indicating that significant changes in teachers' thinking (in this case about their students' mathematical model development) can occur in relatively short periods of time.

To summarize this section, it can be said that the different forms of professional learning communities include different actors. From the Chinese and Japanese cases of Lesson/Learning study, we learnt that these either include teachers (e.g., teaching the same grade) working on their own or with an expert, in a collaborative community. In these set-ups, the expert teachers (who can also be university teacher educators) are greatly appreciated, due to their seniority, their experience and expertise (e.g., Pepin et al., 2017). Teachers are expected to learn from the expert, perhaps even by 'imitating' the expert. In the European settings, university teacher educators often work with classroom teachers, not because of their seniority or teaching experience, but due to their knowledge about mathematics didactical theory — this is expected to 're-source' the teachers. However, in these settings teachers are expected to become involved in curriculum design (to take agency), including planning lessons and learning progressions, often according to newly implemented curriculum guidelines.

5.3. What kinds of learning can be observed in school-based teacher collective work?

In the studies we reviewed, teacher professionalization often has taken place in several dimensions. For example, teachers have gained content related insights, and have also changed their teaching practice based on the new insights and collectively designed lessons. We have categorized the kinds of learning reported in the studies, while we are aware that the categories can be distinguished, but in teacher learning processes they can often not be separated. We distinguish the following categories of teacher learning: (a) Lesson planning and preparation (including design capacity); (b) Pedagogical Content Knowledge (PCK); (c) Classroom practices; (d) General pedagogy; (e) Social issues and teacher identity in the mathematics classroom (e.g. teaching for equity, identity development). In the following paragraphs we describe each category in more detail.

(a) Lesson planning and preparation (including design capacity)

The focus on lesson planning and preparation appears typical for Lesson study approaches. Participation in Lesson Studies has helped teachers in a Korean and Chinese context to realize the importance of creating detailed lesson plans to accomplish mathematical learning goals. Teachers reported that, in several rounds of lesson study, they learned to anticipate student reasoning and to design tasks that evoked this reasoning (Huang, Gong, and Han, 2016; Pang, 2016). Also in an Australian study on structured problem-solving primary-school mathematics teachers reported that they had learned to appreciate the value of creating a detailed lesson plan in Japanese lesson study (Groves, Doig, Vale, and Widjaja, 2016). It had made the teachers realize "just how much there is to the teaching and learning when you step back from the actual lesson or class itself" (ibid, pp508). Enacting the lessons and receiving feedback from observers in post-lessons discussions were driving forces for their learning, which had led to changes in their classroom pedagogy. However, the teachers also noted that the Japanese approach was difficult to implement in an Australian classroom due to the different classroom cultures: in the Australian context, small-group rather than whole-class teaching was emphasized.

(b) PCK

Several studies reported PCK-related learning gains, which typically depended on the mathematical topic of the professional development project described in the study. This learning was even relevant for teachers at pre-school level. Thouless and Gifford (2019) studied the learning of UK teachers from 6 schools who participated in a two-year professional development project. The teachers formed a community of practice, in which also researchers were involved. The teachers developed their knowledge of patterns and changed their teaching of this topic by jointly developing pedagogical approaches. Other examples of PCK-related learning gains in teacher collective work at different school levels include: proportional reasoning in the primary school curriculum (Hilton and Hilton, 2019), exploring the functions between two variables by students at middle school level (Wilkie, 2016), implementation of mathematical modeling tasks in middle school (Jung and Brady, 2016), meaningfully integrating the concepts of functions and graphs in combined science / mathematics tasks in upper secondary school (Potari et al., 2016).

Involvement in the design of educational technology can contribute to the development of teachers' technological pedagogical content knowledge. This was demonstrated in a study by Hansen, Mavrikis, and Geraniou (2016) with a group of primary school mathematics specialists in the UK, who co-designed virtual manipulative on fractions and used it in their classrooms. A promising to develop teachers' technological pedagogical content knowledge was described by Misfeldt and Zacho (2016): in their project teachers developed digital learning environments, using GeoGebra and Google sites to create open-ended projects for students, based on the concepts of educational scenarios and games. More research was needed to overcome among others the steep technological learning curve for some participants.

(c) Classroom practices

Some studies on PD projects including teacher collective work aimed at a change of classroom practices, generally with the purpose to move away from mathematics focused on procedures, towards student conceptual understanding and mathematical reasoning. In lesson study projects, classroom practices change as a result of enacting the collectively developed lesson. For example, a Chilean lesson study project focused on primary teachers developing classroom practices to maintain high cognitive demand (as opposed to procedural or routine efforts) in the implementation of statistic lessons (Estrella, Zakaryan, Olfos, and Espinoza, 2020). In types of PD programs other than lesson studies, teacher collective work was included as an effective way for teachers to learn (e.g. see Wiliam, Lee, Harrison, and Black, 2004). Veldhuis and van den Heuvel-Panhuizen (2020) took this approach when developing PD workshops for primary school teachers in the Netherlands to help them develop classroom assessment techniques, methods that allow the teacher to get a quick overview of students' skills and knowledge of relevant mathematical content, so as to provide meaningful formative feedback. In the workshops, teachers and researchers collaboratively developed classroom assessment techniques, based on mathematical and pedagogical analysis of the mathematical content. Significant increases in student achievement scores on standardized mathematics tests were found. A change of classroom practices was also the purpose of a design research project in New Zealand in which teachers and researchers collaboratively aimed to improve statistics lessons for Pasifika students whose home language was not English (Sharma, 2019).

In Sweden, a large scale professional development program took place in which more than 33,000 mathematics teachers participated (Bergqvist, Liljekvist, van Bommel, and Österholm, 2017). Purpose of the program was, among others, to develop the teaching culture at the schools towards teaching for the development of mathematical competencies in line with a new national curriculum (e.g. problem solving, conceptual understanding, mathematical reasoning, and modelling). The program consisted to a large extent of supervised teacher collaboration and discussions, including the use of web-based support modules. An evaluation study in 35 schools, based on observations and interviews before, during and after the program showed that significant and sustained changes took place in teachers' classroom practices towards the development of mathematical competences. The researchers argue that the program was successful, because "the teachers were given organized possibilities to develop their knowledge and abilities to teach in line with the new curriculum documents" (pp160).

(d) General pedagogy

In several studies, teacher learning was reported that took place in the context of teaching mathematics, but was not typical for mathematics teaching. This learning typically consisted of an increased ability to notice, to reflect on teaching and learning, and to take the perspective of the student. For example, Tan and Lim (2017) studied in Malaysia how the primary teachers' reflections on lessons developed by participating in several rounds of lesson study. They found that with increasing experience in the reflection process, teachers reflected in more detail on student learning, shifted their perspective from the teacher to the students, changed perspective during their reflections and were able to anticipate student responses when refining lesson plans and student tasks. Such a shift took also place in a one year PD program studied by Haßler et al. (2015) with primary school teachers in Zambia, in schools serving disadvantaged communities. This study aimed to promote interactive forms of subject teaching in conjunction with Open Educational Resources (OER) and technology. An increasing ability to reflect and the development of a reflective language that supports deep discussions about core issues was found in a study on PD project in which Israeli mathematics teachers watched and discussed videotaped lessons of unknown teachers (Karsenty and Arvaci, 2017). Another study on the use of video, a video club for rural mathematics teachers in the USA, reported an increasing ability to notice student thinking, and to use it for instructional decisions (Wallin and Amador, 2019).

Collective work of teachers in PD is not a guarantee for effective teacher learning. Dalby (2021) studied a design research project in which groups of secondary school mathematics teachers in the UK designed lessons to explore the use of iPads for formative assessment. Findings show that teachers made progress towards this aim, both technically and pedagogically. However, comparing two groups, she found that they did not develop as equally effective professional learning communities. Group leadership, how often communication between members took place, and the extent to which group members felt ownership of the aims had an impact on the effectiveness. For individual members, also their prior technical knowledge influenced their learning.

(e) Social issues and teacher identity in the mathematics classroom

Several studies reported teacher learning in terms of in terms of doing justice to students of mathematics and developing their own professional identities as mathematics teachers. In New Zealand, professional development focusing on the collective redesign and enactment of classroom practices in schools serving disadvantaged communities helped mathematics teachers to see more and different mathematical capacities in their students (Hunter et al., 2020). Such developments of supporting student engagement may result in a shift of mathematics teachers' professional identity from knowledge providers to "'facilitators', 'learners' and 'co-creators' of knowledge" (Bobis et al.). Others (e.g. Nicol et al., 2017) used discussion and reflection as to explore possibilities and challenges of teaching mathematics for justice. The process was described as complex, with dialogue, contradictions and discomfort playing a role.

6. Conclusions

In this section we analyze the findings from the previous three sections and link them to our earlier conceptualizations of the theoretical frames (see section 4).

First, we ask 'what means school-based teacher collective work', because we have seen that teacher collective work can happen in person, or at a distance (whether teachers sit at home or in school). Different initiatives for distance collaboration can be activated through MOOCs (e.g., Italy), through platforms (e.g., Denmark, France, Netherlands), or through websites (e.g., Israel). These initiatives ask for a more nuanced description and conceptualization of school-based teacher collaborative work.

Second, this re-conceptualization of teacher learning communities also needs to include the ways these communities are supported: e.g., are teachers given time, as a matter of course, to collaborate at school (or at a distance), or do they have to ask the head teacher to carve out time for such activities? Do teachers have opportunities to meet and discuss their lesson planning? If teachers are expected to participate in such communities (that provide opportunities for discussions with colleague professionals), a culture of collaboration is needed. This is particularly needed, so the literature argues (e.g., Lamb and Visnovska 2012), in rural communities and small schools, where there is a smaller number of mathematics teachers who can support greater collegial collaboration. Leaning on Millet and Bibby's (2004) 'zone of enactment', school-based teacher collective work needs to be supported, for example with low-cost digital resources that allow for video conferencing from different sites. They can also be supported by teachers' working condition: e.g., where the professional learning in school -based communities is counted as a 'normal' daily task of a teacher (as it is in China and Japan). Or they can be supported by individual projects (e.g., as in Europe by EU projects); however, the sustainability of such initiatives is not ensured, and often the learning community 'disconnects' when the project finishes.

Third, we have seen that Lesson Study (albeit in different forms) exists all across the world. However, and according to local or regional or national practices, they are differently 'lived' in different cultures.

Fourth, we have seen that different forms and participation of professional learning communities provide different forms of agency for teachers. It seems that teachers always enact agency, even when they seemingly accept practices from others. These enactments have to be seen in relation to particular social and material conditions in their environment (and often relations of power). More research is needed to understand how teachers enact agency in school, and more particularly in school based collaborative communities.

Fifth, we have seen that teacher learning in collective work takes place in many domains. Teachers may gain competence in preparing their lessons, in mathematical classroom practices, in general pedagogy and in helping their students learn mathematics in more equitable ways. They gain PCK and develop their professional identities as a result of their activities. A common element of many studies is that teachers learn from enacting new practices in the classroom, collectively reflecting on these practices, and developing them further. In many cases knowledgeable others, experts or researchers facilitate the learning processes (and may be the initiators of the professional development programs in which teachers participate). The Lesson Studies in Japan and China are, indeed, part of teachers' regular school practice. However, such initiatives do not exist in all countries and other studies we examined were set up as projects with a limited duration. It can be expected that many mathematics teachers do not take part in these projects, but do participate in collective activity as part of their regular practice. Hence, there is a need to do further research on how these teachers learn and develop professionally.

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https://doi.org/10.1142/9789811287152_0024

Survey Team 4

Interdisciplinary Aspects of the Teaching and Learning of Mathematical Modelling in Mathematics Education Including Relations to the Real World and STEM[1]

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ABSTRACT This report presents preliminary results from a survey commissioned for ICME-14 on *teaching and learning of mathematical modelling and interdisciplinary mathematics educations.* The systematic literature review focusses on how a well-understood relation between mathematics and the real world underpins interdisciplinary work, interdisciplinarity in research and teaching teams, issues and challenges in the relationships among mathematical modelling, mathematics, the real world and interdisciplinarity, and mathematical modelling and a well-understood relation to the real world ensuring mathematical depth in STEM integration.

Keywords: Interdisciplinary; Mathematics; Mathematical modelling; Relations to the real world; STEM.

1. Background

1.1. The survey team's terms of reference

This paper reports preliminary findings from Survey Team 4, commissioned for ICME-14, focusing on the topic, *The teaching and learning of mathematical modelling and interdisciplinary mathematics educations,* expanding on aspects presented by the authors during the congress in July 2021. The terms of reference of Survey Team 4 were the teaching and learning of mathematical modelling and interdisciplinary mathematics education with a focus on relations to the real world and connections and implications for Science, Technology, Engineering and Mathematics (STEM) education.

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The importance of mathematical modelling and mathematical applications to real world context has been growing in mathematics education over recent decades supported by regular activities on applications and modelling at the ICME's and through the series of International Conferences on the Teaching of Mathematical Modelling and Applications (ICTMAs), held biennially since 1983 (except for the recent interruption due to the COVID-19 pandemic). ICME Proceedings and Survey Lectures indicate the most recent developments at the relevant time and contain many studies, conceptual contributions, and resources addressing the relation between the real world and mathematics, as do the ICTMA Series books (e.g., Leung et al., 2021). Additionally, ICMI study 14 on *Modelling and Applications in Mathematics Education* (Blum et al., 2007) addressed a variety of topics related to modelling bringing together international perspectives. An increasing number of empirical research projects which focus on special aspects of mathematical modelling and applications, as well as national and international comparative studies, generate further interest.

In 1989, D'Ambrosio proposed that mathematical modelling is the thread by which individual disciplines in the curriculum can be interconnected to promote curricular integration. English has long argued (e.g., English 2007, 2008; English et al., 2016) for mathematical modelling becoming an enabler of interdisciplinary practices in schools, as the means by which "creative and flexible use of mathematical ideas within an interdisciplinary context where students solve substantive, authentic problems that address multiple core learnings" is promoted (2007, p. 275). The survey therefore reviews the current state-of-the-art on the teaching and learning of mathematical modelling under the specific consideration of interdisciplinary aspects. For this reason, a well understood relation between mathematics and the real world is an important focus. This is particularly relevant in the context of STEM, which has recently come to political prominence (Kelley and Knowles, 2016; Moore et al., 2020) in several educational jurisdictions around the world.

1.2. Interdisciplinary approaches to education and research

The construct, *interdisciplinarity*, entered the literature in 1972 (Miller, 2020) in an Organization for Economic Cooperation and Development report (Apostel, 1972), although the idea was well known in education long before this. Like most constructs, interdisciplinarity has morphed to have several meanings. In our survey, we were guided by the USA National Academies definition for *interdisciplinary research*:

> a mode of research by teams or individuals that integrates information, data, techniques, tools, perspectives, concepts, and/or theories from two or more disciplines or bodies of specialized knowledge to advance fundamental understanding or to solve problems whose solutions are beyond the scope of a single discipline or area of research practice. (National Academy of Sciences, 2005, p. 39)

In actual teaching and learning situations in classrooms, different approaches to integration can be seen as *isolated* into separate disciplines, or *connected* deliberately relating separate disciplines to show connections, or *nested/fused* where content from one or more other disciplines is taught in a discipline to enrich it, or *multidisciplinary* incorporating two or more disciplines around the same theme or topic but the disciplines keep their identity, or *interdisciplinary* when two or more disciplines interact to become something new such as mathematical ecology, or *transdisciplinary* where there is a transcendence of the disciplines and the focus becomes the field of knowledge as exemplified in the real world (Gresnigt et al., 2014, p. 52). Gresnigt et al. use the metaphor of a staircase of increasing complexity where more elements of teaching are shared between disciplines rather than assuming the higher up the ladder the better the integration. Instead, they characterise these steps as different ways to integrate. Others such as Williams and Roth (2014) include *meta-disciplinarity* where there is awareness of the nature of the disciplines involved in their relation and the differences within an inquiry or problem solving. As Roehrig et al. (2021) point out, “It is the multidisciplinary nature of real-world problems, as opposed to the disciplinary structure within formal schooling, that grounds arguments for curricular integration” (p. 2).

1.3. Mathematical modelling and interdisciplinarity

Mathematical modelling can be understood as real-world problem solving, although we acknowledge it does not have to be. Mathematical modelling in this interpretation is then the process of applying mathematics to a real-world problem or situation with the goal of understanding it (Niss et al., 2007). It is more than applying mathematics in a closed situation where nothing needs to be assumed or estimated — just a known mathematical technique applied. Thus, multiple interpretations of the situation being modelled are possible. “The modelling enterprise involves identifying and addressing open-ended questions, creating, refining and validating models, and arguing the case for implementation of model informed outcomes” (Niss et al., 2007, p. 17).

In addition, from an epistemological perspective, “essential characterisations of modelling… involve posing and solving problems located in the real-world, which for our purposes includes other discipline areas such as engineering or medicine, and general contexts of living as they impact on individuals, groups, and communities” (Niss et al. 2007, p. 17). Researchers have therefore developed or proposed theoretical views on the interplay between mathematical modelling and interdisciplinarity in mathematics education (e.g., Borromeo Ferri and Mousoulides, 2017; English, 2013; Michelsen, 2006, 2015).

English (2013), for example, argues that there is a need to build a stronger foundation in the mathematical sciences through future-oriented learning experiences to equip students for the challenges of the 21st century. She lists core competencies that are key elements of productive and innovative workplace practices to ensure such a foundation. To achieve this aim, she recommends an increased focus on interdisciplinary problem solving that engages students in complex modelling with challenging, life-based scenarios. In such learning experiences, knowledge and skills from at least two disciplines are applied to real-world scenarios with the aim of shaping the total learning experience. English sees such mathematical modelling having applicability in primary and middle schooling where the focus is on engaging students in the kinds of mathematical and scientific thinking needed for challenges beyond the classroom. She supports her argument from two design-based studies, a data modelling study in Year 1 and engineering-based modelling experiences in Year 7.

1.4. Mathematical modelling and a well understood relation to the real-world

Relations between mathematics and the real world have existed since the very beginnings of mathematics (see, e.g., Joseph, 2011). If mathematical modelling is conceived as real-world problem solving, a well understood relation between mathematics and the real world is essential. When such a relationship is in play, the real-world situation encourages a deeper understanding and processing of mathematics; but simultaneously use of mathematics encourages deeper understanding and processing of the real world. Each enriches the other.

The example of water falling from two gates on a dam wall into the spillway below illustrates our point and is a context to design modelling materials for a secondary school classroom. From a mathematical point of view, the maintaining of a well-understood relation of any in-class modelling to the real world is an issue for teaching mathematics in school. The situation to be modelled is derived from the observation that the water from the gate that is lower in the dam wall appears to have stronger momentum than that from the higher gate; however, the horizontal distances from the dam wall to the impact point on the spillway seem to be similar. We pose the real-world problem: What is the relation between the location of a gate and the horizontal distance of where the water lands? This situation is concerned with mathematics and physics. (See Stillman et al., submitted, for a full solution to this problem.)

From a mathematics perspective, students realise the importance of generating and selecting variables, setting up a simplified situation and validating the solution derived from the model by experiment. During their modelling, students have opportunity to appreciate the utility of mathematics to understand (represent, explain, predict) parts of the world. From a real-world perspective, knowledge of physics is enriched. Students learn that the velocity of the spilling water is proportional to the square root of the distance of the gate from the top and that the horizontal distance of the landing site of the spilling water becomes a maximum when the gate is located at the mid-point of the height (depth) of the water in the dam at the wall where the gates are located.

Two questions are crucial in both solving an authentic real-world task like this and in planning its implementation and management in class. Firstly, what type of mathematics can be applied and secondly, how can the real-world situation be conceptualised. Interaction between these two leads to a well-understood relation between mathematics and the real world *when they enrich each other*.

2. Focuses and Research Questions

It is timely for the international mathematics education community to survey, synthesise, and propose new directions for research that is focused on interdisciplinary aspects of research and teaching in mathematical modelling. The survey team has been addressing the specific scope and foci of relevant work that has been developing in these areas over the last decade in different educational systems around the world. The interdisciplinarity aspects that have been focused on are (a) interdisciplinarity in research teams as well as research focussed on interdisciplinarity and (b) interdisciplinarity in teaching and teaching design teams through all levels of schooling and into tertiary education.

Broad research questions, guiding the analysis the survey team has conducted for this paper are:

- How does a well-understood relation between mathematics and the real world underpin interdisciplinary work in mathematics education?
- How have interdisciplinary teams contributed to knowledge about mathematical modelling and the relation of mathematics to the real world?
- What issues and challenges are there in the relationships among mathematical modelling, mathematics, the real world and inter-disciplinarity in both teaching and research?
- How could contributions from research and teaching on mathematical modelling and relations of mathematics to the real world contribute to ensuring mathematical depth in STEM integration?

3. Methodology

The team started collection and collation of potential sources for the survey by calling for contributions on several online list serves which resulted in contributions by individuals of lists of publications, including projected future publications, and chapters about research and teaching projects with interdisciplinary connections of mathematics as well as mathematical modelling. An initial surveying of literature (including English, German, Japanese, Portuguese and Swedish) from selected geographical regions was conducted by different team members. This established a basis for what we might expect to locate in the time period and the likely fruitfulness of potential sources, so a systematic survey of sources began in preparation for a systematic analytical review of literature (Newman and Gough, 2020). Synopses of all selected sources were collated in one database where they were coded by reading the original source and the synopsis, re-reading the source and adding to the synopsis when necessary. Sources that did not appear to be within the remit of the terms of reference were starred for potential culling following cross checking. This was carried out by two team members independently. Our initial focus was on the period 2012–2020 but then we conducted a more in-depth search of 2016–2021, once the date of the conference and reporting was extended. Our sources included refereed journals, edited books (especially in relevant book series in mathematics education and other fields such as STEM), conference proceedings, and theses (see Tab. 1). Our database extends beyond this because of the initial cross geographical regions search and individual contributions. As well as major mathematics education journals, we have read and collated articles from other journals to augment understanding of the breadth and depth of research and teaching in other fields such as science education and to gain insight into the STEM field, especially in relation to mathematical modelling outside of what is usually cited in mathematics education. We, thus, examined articles from the *International Journal of STEM Education*, amongst other journals.

Tab. 1. Literature sources systematically surveyed

Sources
Journals
Educational Studies in Mathematics 2016–2021
Eurasia Journal of Mathematics Science and Technology Education 2018–2020
European Journal of Science and Mathematics Education 2016–2021
Journal for Research in Mathematics Education 2016–2021
Journal of Mathematical Behavior 2016–2021
Journal of Science Education in Japan
International Journal of Science and Mathematics Education 2016–2021
International Journal of STEM Education 2016–2021
Mathematical Thinking and Learning 2016–2021
Mathematics Education Research Journal 2016–2021
Nordic Studies in Mathematics Education
ZDM — Mathematics Education 2016–2021
Book Series in Mathematics Education
NCTM Monographs APME 1 volume (Hirsh and Roth McDuffie, 2016)
Realitätsbezüge im Mathematikunterricht book series (REIMA) 11 volumes (e.g., Frank et al. in Greefrath and Siller, 2018)
Springer Series: *Advances in Mathematics Education* — 1 volume (Chamberlin and Sriraman, 2019)
Springer Series: *Early Mathematics Learning and Development* — 1 volume (Suh, Wickstrom and English, 2021)
Springer Series: *ICME-13 Monographs* — 2 volumes
(Doig, Williams, Swanson, Borromeo Ferri and Drake, 2019; Stillman and Brown, 2019)
Springer Series: *International Perspectives on the Teaching and Learning of Mathematical Modelling* — 4 volumes (Stillman, Kaiser, Blum and Brown, 2013; Stillman, Blum and Biembengut, 2015; Stillman, Blum and Kaiser, 2017; Stillman, Kaiser and Lampen, 2020; Leung, Stillman, Kaiser, and Wong, 2021)
Books from other fields — examples
Comparison of mathematics and physics education I: Theoretical foundations for interdisciplinary collaboration (Kraus and Krause, 2020)
Asia-Pacific STEM teaching practices (Hsu and Yeh, 2019)
Handbook of research on STEM education (Johnson, Mohr-Schroeder, Moore and English, 2020)
Conference and Symposium Proceedings
CERME 8, CERME 9, CERME 10, CERME 11
CNMEM 8th, 9th, and 10th editions of the National Conference on Modelling in Mathematics Education (Conferência Nacional sobre Modelagem na Educação Matemática) — Brazil 28 works analysed
MACAS 2017; MERGA 2016–2019; NORMA17; PME 43, 42, 41, 40; PME-NA 41
Research projects
Funded research 2011–2021 projects in mathematics education in the Nordic countries also involving interdisciplinarity and STEM
Theses and dissertations
Craig (2017); Gibbs (2019)

In the grey literature we examined outputs in recent years in several conference series in different parts of the world and added those where there was no similar literature in other scholarly sources. Research projects, both by established researchers and teams and by early career researchers in theses and dissertations, were also surveyed. For example, when Prof. Arleback joined our team at the end of 2020, he began by surveying funded research projects in mathematics education in the Nordic countries which also involved interdisciplinarity and STEM to gain an understanding of what was happening in the time period of the survey. It was expected this would lead to published literature on the earlier projects but most likely not on the recent ones, given COVID-19 had restricted conference travel and possibly the research proceeding.

A coding scheme was developed, and the codes for this paper were structured around our research questions according to four overarching categories: (a) relations between mathematics and the real world (RQ 1), (b) interdisciplinary team contributions (RQ2), (c) issues and challenges in relationships (RQ3), and (d) mathematical depth in STEM integration (RQ4). Tab. 2 provides an example of our coding scheme for a selection of the reviewed literature for the interdisciplinary team contributions category. Similar coding schemes were developed for the other three categories. Initial and final coding was carried out by the first author. Another member of the team checked the first coding and sent all queries to the initial coder who then carried out a second coding. Subsequently a third re-coding round was conducted 6 months later to check coding reliability. A configurative synthesis (Newman and Gough, 2020) of the different literature sources was then conducted to answer our research questions, focusing on the research questions and problems or topics the selected literature addressed, noting confirmatory and contradictory findings and rival explanations of such findings.

Tab. 2. Example coding scheme for interdisciplinary team contributions category

Code	Brief Description	Examples of Literature
CoA	Co-authorship from other domain(s) other than mathematics education	Sala et al. (2017); Sawatzki et al. (2019); Viirman and Nardi (2021)
CoA MN	Co-authorship multi-national team	Chang et al. (2020); Frejd and Geiger (2017); Guerrero-Ortiz et al. (2016)
Contribution – CC	Cross cultural validation	Durandt et al. (2022)
Contribution – IC	International comparison	Chang et al. (2020)
Contribution – KT	Knowledge transfer between countries	Krawitz et al. (2022)
IDT	Interdisciplinary design team	Viirman and Nardi (2021)
IRT	Interdisciplinary research team	Durandt et al. (2022)
ITT	Interdisciplinary teaching team	Gardner and Tillotson (2019)
MM	Contribution to mathematical modelling research/ teaching/ design/curriculum	Gardner and Tillotson (2019); Viirman and Nardi (2021)
R to RW	Consideration of/new contribution to research on/ relations to real world	Guerrero-Ortiz et al. (2016); Viirman and Nardi (2021)

4. Findings

We present our findings in four threads to address our four research questions where overall trends, issues and challenges will be illustrated and exemplified.

4.1. Relation between mathematics and the real world underpinning interdisciplinary work in mathematics education

As seen in subsection 1.4, to try to understand the relation between mathematics and the real world, we think about two worlds, the real world and mathematics. On the one hand, the real world encourages deeper understanding and processing of mathematics. On the other hand, mathematics encourages deeper understanding and processing of the real-world situation. If both are satisfied, enriching each other, we say the relation between mathematics and the real world is well-understood and this is what we see as ideal to underpin interdisciplinary work in mathematics education.

Mathematical modelling is sometimes described as a transformation of a real-world problem to a mathematical problem and back again, so the real-world and mathematical world appear distinct. Czocher (2018) raises and considers a critical issue in the modelling process related to a deep understanding of the *relation of mathematics to the real world*: "How do modellers determine if the transformation from the real world to mathematics was conducted well?"

From a study of the modelling activity of four engineering students, Czocher (2018) presents an empirically derived typology of five validating activities to explain how validating functions to ensure a mathematical model will yield a reasonably accurate prediction. To capture the complexity of modelling due to validating, she offers an empirically grounded schematic adapted from the modelling cycle diagram of Blum and Leiß (2007). Two circular regions show the real world and the mathematical world, respectively; however, a honeycomb-patterned annulus surrounds the mathematical world representing reasoning that is mathematically structured but constrained by real-world conditions. Triangles and circles are used to indicate the model construction stages (e.g., situation model) with solid arrowed arcs showing the transitions between these. Dotted arrowed arcs show validating actions which lead to, or from, the mathematically structured real world. The schematic shows that most of the observable reasoning occurs in the annulus as a blend of the real-world and the mathematical world, thus validating ensures the real world and mathematical world stay intertwined. Czocher (2018) proffers this model because she found no evidence of a separation into purely mathematical thinking and purely real-world thinking and switching between the two in her study.

The four validating activities that perform this function are comparing the mathematical expression, its constituent components or relationships, to the interpretation of the problem setting; comparing the mathematical expression, its constituent components or relationships, to the idealized version of the problem setting; comparing the real results to empirical, or based-on-empirical, expectations (predicted by theory); and comparing real results against physical principles accounted-for in the real model. The fifth validating action she identified occurred purely in the mathematical world. "The nuances of validating suggest that creating and maintaining relationships between reality and mathematics [are] more complex than a transformation" (p. 137). As a consequence, Czocher (2018) suggests a more prominent role should be afforded to validation in the modelling process. Czocher et al. (2018) argue that in order to foster learners who are confident and capable in STEM fields, it is necessary to revisit how verifying and validating activities are conceptualised and developed across the years of schooling and in different subject areas. (See also Jensen, 2018.)

Another issue in teaching modelling is how to facilitate students' conceptualisation of the real-world situation. Bearing this in mind, Wernet (2017) focused on the interaction between teachers and students about contextual features in written tasks. During whole class discussions, teachers and students discussed context of written problems in multiple ways, and these interactions often led to higher authenticity (Palm 2008) being enacted in discussions, than was written in the task descriptions. In a similar vein, Chang et al. (2020) focused on making assumptions to conceptualise a real-world situation. Such conceptualisation necessarily involves both realistic considerations and non-mathematical knowledge. Inadequate assumption making leads to an inadequate situation model and to an inadequate mathematical model for the problem situation.

4.2. Contributions from interdisciplinary teams to knowledge about teaching and learning mathematical modelling and relations to real-world

With respect to contributions from interdisciplinary teams, there were some limitations with respect to determining disciplines of researchers and team composition from published work and how, and to what extent, a researcher contributed to a study. At times this information was available in the biographies of authors or in acknowledgements but at other times information had to be sought from project and university staff personal webpages. In many cases, many experts who contributed to a project are not visible (e.g., curriculum designers, methodology experts). Despite these limitations, there was evidence of contributions of several interdisciplinary research teams and interdisciplinary teaching and curriculum design teams and combinations of these to knowledge about teaching and learning of mathematical modelling and relations to the real-world. Some of the interdisciplinary teams were multi-national, whereas others addressed topics in which interdisciplinary teams are essential. The latter included international comparative studies where both culture and language had to be taken into account, knowledge transfer from one country to others, and validation of results of a study across countries and cultures.

The study by Chang et al. (2020), for example, involved a team of four mathematics educators, two from Chinese Taiwan and two from Germany. The aim was to compare Year 8 Chinese Taiwan students and Year 9 German secondary school students' knowledge use in solving structured modelling problems set in familiar contexts. In this study, the researchers intentionally designed two types of assumptions in two modelling tasks, namely, one requiring only non-numerical assumptions and another requiring both non-numerical and numerical assumptions. As few current studies comparing modelling performance between Western and non-Western students have considered the differences in students' knowledge, students' relative performance was investigated when students' mathematical knowledge in solving modelling problems was matched. The results showed that the Chinese Taiwan students had significantly higher mathematical knowledge than did the German students, whether conceptual or procedural. However, when students had the same level of mathematical knowledge, the German students showed higher modelling performance on the same modelling problems, no matter what type of assumptions were necessary. Chang et al. suggest that their findings imply that Western mathematics education may be more effective in improving students' ability to solve holistic modelling problems. It should be noted, though, that the German school mathematics curriculum provided opportunities for students to learn modelling, whereas the Chinese Taiwan students did not, so the findings could also be reflecting opportunity to learn.

Viirman and Nardi (2021) report on the work of an interdisciplinary research team comprised of three tertiary mathematics education researchers and one research mathematician with extensive experience of mathematical modelling and teaching applied mathematics. Mathematical modelling was being used as a vehicle to integrate mathematics and biology to improve tertiary biology students' engagement with mathematics and their competencies in both mathematics and biology. Viirman and Nardi traced the students' meta-level learning about mathematical modelling, particularly as they fluctuated between deploying graphs for mere illustration of data and as sense-making tools. Some students, however, used graphing only for illustration of work done and not to make meaning from data in their modelling. Previous mathematical experiences with graphing were relied on and the necessity to maintain a good relation, between mathematics and the real or extra-mathematical worlds in the situations they were modelling, was not noticed. Hankeln (2020) noted similar results from a comparative study of upper secondary school students' modelling processes in Germany and France. Although French students were more unfamiliar than German students with real-world context being of more importance than as a mere motivation to engage in mathematics, German students often just exposed the mathematical content as they had accepted socio-mathematical norms that a deep understanding of the real-world context was not the focus of mathematical tasks and could even be a hinderance. These findings in both studies highlight the importance of teachers at the secondary and tertiary levels making students aware of the depth of engagement with real-world aspects of tasks that are needed to be productive in adjusting modelling methods and models used previously to fit a new situation when constructing, interpreting, and validating models or the modelling used.

4.3. Issues and challenges in the relationships among mathematical modelling, mathematics, the real world and interdisciplinarity

Several issues and challenges in the relationships among mathematical modelling, mathematics, the real world and interdisciplinarity in teaching and research were raised in the literature reviewed. We have space to raise only a few.

As Carreira and Baioa (2015) point out, it is rare in mathematics classes for secondary students to engage in examining material objects and artefacts of the real world to generate mathematical ways of viewing reality. However, such experiences enable the making of connections between students' everyday life, mathematics, and the real world. Carreira and Baioa designed and implemented a mathematical modelling learning activity where students collected data about staircases in their neighbourhood. This allowed students to come to the realisation that linearization was the key to designing the best staircase for a house despite finding slight variations from the ideal in reality. A linear model came from mathematical notions such as average step to compensate for variability in tread and riser measurements in producing a staircase of constant slope. As Diego-Mantecón, Haro et al. (2021) note, in teaching activities such as this, there is a need for contextualised knowledge that is not taught in school but, as Carreira and Baioa (2015) have demonstrated, there are ways to gain some of this such as going out into the locality where students live.

In other work, Diego-Mantecón, Prodromou et al. (2021) found that in-field mathematics teachers avoided using transdisciplinary projects where school mathematics content was difficult to address, whereas out-of-field mathematics teachers tended to overlook the mathematics in interdisciplinary projects, oversimplified it, or used basic mathematics below curriculum standards. In-field mathematics teachers promoted high cognitive demand and productive dispositions towards mathematics in projects. Some students, however, did not want to invest time into such projects, being more examination oriented. Students also contributed to projects where their strengths were (e.g., practical skills) rather than use the opportunity to improve areas where they were less confident (e.g., mathematical analysis).

4.4. Ensuring mathematical depth in STEM integration

STEM integration can foreground mathematics teaching, learning and use in two ways: *intra-mathematical* providing opportunities to engage with mathematics in the development and application of mathematical ideas, concepts and skills in the context of meeting curriculum obligations in mathematics or *extra-mathematical* as a unique mathematical viewpoint and its practice contributing to understanding and development of ideas, concepts and skills in other STEM disciplines. According to several authors, the optimisation of these ways of foregrounding mathematics so there is a noticeable effect on mathematical achievement and learning is yet to be realised as mathematics is seen as benefiting least from STEM integration (English, 2016; Fitzallen, 2015; Maass et al., 2019). A study by Li et al. (2020) of publicly funded STEM projects in the USA from 2003-2019 perplexingly showed that the majority of projects focused on single disciplines, especially mathematics. However, these projects showed a strong emphasis on mathematics, particularly before 2012.

To address the rather shallow treatment of mathematics in STEM education, several sources (e.g., Ärlebäck and Albarracin, 2019; English et al., 2016; Gonçalves and Pires, 2014; Leung, 2018; Turner et al., 2019) expressed the expectation that mathematical modelling would be an enabler of interdisciplinary practices and integrating professional disciplines with secondary school subjects and also in primary/elementary school (e.g., Baker and Galanti, 2015). However, from a literature survey of STEM integration practices 2013-2016, Bajuri et al. (2018) identified mathematical modelling as the least focused on integrative practice. They proposed using metacognition and social interaction development to promote these abilities in a mathematical modelling approach to STEM. Such a proposal would seem eminently sensible given the nature of mathematical modelling as conducted by professional modellers and in school classrooms and the fact modelling is included in each of the STEM disciplines and is "an opportunity to express and to develop disciplinary knowledge and ways of thinking" (Hjalmarson et al., 2020, p. 229).

5. Closing

As our preliminary findings from our systematic analysis of relevant literature show, all four threads we have explored have not been exhausted. These areas represent opportunities for the mathematics education research community to conduct further scholarly work and help advance the field at large. Our final report will also contribute to this.

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Part VI

Topic Study Groups

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https://doi.org/10.1142/9789811287152_0025

Topic Study Group 1

Mathematics Education at Preschool Level

Marja van den Heuvel-Panhuizen[1], Angelika Kullberg[2], Ineta Helmane[3], and Xin Zhou[4]

1. Aims of the TSG

TSG-1 is about the foundations of learning mathematics and the contexts in which the first steps are taken towards achieving mathematical understanding. The aim is to share and discuss contemporary research on early childhood mathematics learning and teaching and their theoretical and methodological frameworks. TSG-1 involves research on children's mathematical development from birth until entering formal schooling in first grade (children up to 6). The nurturing of this development can take place in care centers, preschool, and kindergarten, and at home.

Although it is currently widely accepted that the development of mathematical skills in the early years is essential for later mathematics learning, it is not so obvious what mathematics should be fostered in young children. Mathematics as a subject has traditionally been considered above the preschool and kindergarten levels. Moreover, researching young children's mathematical understanding has for a long time been a privilege of psychology and pedagogy. These sciences have provided much knowledge about conditions and variables that influence children's mathematical development but do often not consider very deeply the mathematics that is, or has to be, developed by young children and generally do not cogitate about why certain mathematical competences are important or what activities are crucial to stimulate the development of these competences. To gain a better insight in this what aspect of mathematics education at preschool level, we invited contributions from the didactics of mathematics, but also for example from (neuro-) cognitive, developmental, socio-cultural and other approaches to the learning and fostering of young children's mathematical understanding. TSG-1 intends, from multiple perspectives, to contribute to the improvement of knowledge and understanding of issues that early childhood mathematics education encounters in different contexts and come eventually with

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proposals for advancing research, development and practice in mathematics education at preschool level.

To achieve this TSG-1 we invited submissions of substantial research-based theoretical or empirical contributions within the following four subthemes:

1. Unpacking early childhood mathematics. Opening up the thinking about the mathematics content (broadly interpreted as knowledge, skills, conceptual understanding, mathematical reasoning and attitude) to be fostered in young children. What mathematics is worth to be developed? What mathematics anticipates future learning and opens the road to future learning?
2. Pedagogical and didactical approaches in early childhood mathematics education. What are meaningful learning environments for young children in a school setting or home environment? What tools, including manipulatives and technology, supports early mathematics learning? How can play and story reading be used? In what way can learning environments for young children be improved by embodiment theories on learning?
3. Assessing mathematical understanding in early childhood. How to get a better understanding of young children's mathematical development?
4. Preparing early childhood educators to foster children's mathematical development. How can professional development provide appropriate support and flexibility to allow teachers, care-givers and parents to develop new knowledge and understanding about mathematics education for young children?

1.1. Submissions

The 2021 round of submissions ended up in 17 accepted papers from 14 countries (North America: 3; Asia: 5; Europe: 9; Australia: 2).

1.2. Sessions

The TSG met during three days. In the first session, the TSG chair Marja van den Heuvel-Panhuizen, and the rest of the organizing team, described the organization of the sessions. Before the conference all papers and one power point slide for each paper were sent to the participants. The sessions were organized around two themes. The first session was on the theme Investigations of children's learning, whereas the two last session was on the theme Investigations of children's learning environment. Each paper was presented and discussed during 10 minutes. At the end of each session there was a joint discussion for 25 minutes on topics related to the papers as a whole and developments in the field. The list of papers and order of presentation are shown in Tab. 1.

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Application of number line estimation strategy for 5–6 years old children: Effect of reference point marking. ***Xiaoting Zhao*** *and Xiaohui Xu* (China).
[2]	Unraveling the quantitative competence of kindergartners. ***Marja van den Heuvel-Panhuizen*** (Norway) *and Iliada Elia* (Cyprus).
[3]	Insights about constructing symmetry with 5-year-old children in an artistic context, ***Yuly Vanegas***. *Carla Rosell and Joaquin Giménez* (Spain).
[4]	Kindergartners' use of symmetry and mathematical structure in representing SELF-portraits. ***Joanne Mulligan*** *and Gabrielle Oslington* (Australia).
[5]	Investigating evidence of girls' and boys' early symmetry knowledge through multiple modes of assessment. ***Nicole Fletcher***, *Diego Luna Bazaldúa, and Herbert P. Ginsburg* (USA).
[6]	4-Year-olds children's understanding of repeating patterns: A report from China. ***Fang Tian*** *and Jin Huang* (China).
[7]	Investigating how kindergartners represent data with early numeracy and literacy skills through a performance task. ***Insook Chung*** (USA).
[8]	Counting activities for young children: Adults' perspectives. *Dina Tirosh, Pessia Tsamir,* ***Ruthi Barkai****, and Esther S. Levenson* (Israel).
[9]	Asking early childhood teachers about their use of finger patterns. ***Miriam M. Lüken*** *and Anna Lehmann* (Germany).
[10]	Performance expectations in the area of "Shapes and Spaces" of early childhood educators in an international comparison. ***Catherine Walter-Laager*** (Austria), ***Manfred R. Pfiffner*** (Switzerland), *Xin Zhou* (Chian), *Douglas H. Clements* (USA), *Julie Sarama* (USA), *Linh Nguyen Ngoc* (Vietnam), *Lars Eichen* (Austria), *and Karoline Rettenbacher* (Austria).
[11]	Mathematics in play. ***Ronald Keijzer***, *Marjolijn Peltenburg, Martine van Schaik, Annerieke Boland, and Eefje van der Zalm* (Netherlands).
[12]	Does preservice teacher training change prospective preschool teachers' emotions about mathematics? ***Oliver Thiel*** (Norway).
[13]	Bishop's (1988, 1991) mathematical activities reframed for pre-verbal young children's actions. ***Audrey Cooke*** *and Jenny Jay* (Australia).
[14]	When math meets games — The active construction of children's core mathematics experience in games. ***Jianqing Wen*** (China).
[15]	Analysing a Danish kindergarten class teacher's instructional support in mathematics with the tool Class. ***Birgitte Henriksen*** (Denmark).
[16]	Mathematical learning environments in Norwegian ECEC child groups. ***Øyvind Jacobsen Bjørkås***, *Dag Oskar Madsen, Anne Grethe Baustad, and Elisabeth Bjørnestad* (Norway).
[17]	"More Gooder": children evaluate early numeracy apps. ***Ann LeSage*** *and Robyn Ruttenberg-Rozen* (Canada).

2. Conference Themes

The studies discussed in TSG-1 involve mainly research on children's mathematical development in the years before they enter in formal schooling in first grade. The nurturing of this development can take place in various environments: care centers, preschool, kindergarten, and at home. The 17 submitted papers to TSG-1 was divided in the categories: "Investigations of children's learning" (papers [1]–[7]) and "Investigations of children's learning environment" (papers [8]–[17]).

The papers in the first category are all based on data collected from children. For several mathematical content areas and competences it is investigated what children are capable of. The papers of the second category are based on data collected through observing classrooms and interviewing early childhood teachers and educators, prospective preschool teachers, and other adults. Interestingly, in one study the learning environment was also investigated by interviewing children themselves. In this second category are also two papers which have a more theoretical stance. One is proposing a revision of a framework for mathematical activities and the other is recommending the use of mathematical games in kindergarten.

2.1. Investigations of children's learning

The collection of papers in this section addresses mathematical competences in the domain of early number, symmetry, patterns, and representation of data.

With respect to early number, one of the Zhao and Xu[1] investigated children's competence in making estimations on the number line. This is a topic that is not everyday dealt with in kindergarten classes. By eye-tracking technology the study showed that the ability of kindergarten children (aged five to six) of making estimations on a 0–10 number line can be effectively improved by using a midpoint marker instead of a marker at every quartile.

Heuvel-Panhuizen and Elia[2] aimed to unravel the composition of the quantitative competence of kindergartners. By analysing data from a collection of paper-and-pencil items it was revealed that in addition to counting, subitizing, and additive reasoning, also multiplicative reasoning belongs to this early number ability. Furthermore, an implicative analysis at item level showed that in general, multiplicative reasoning and conceptual subitizing items were found at the top of the implicative chain, counting and perceptual subitizing items at the end, and additional reasoning items in the middle.

Three studies investigated the development of the notion of symmetry in kindergartners. In the study[3] of Vanegas et al. a sequence of 16 symmetry-related activities was developed in the context of art work. In these activities kindergartners had to work with various axes of symmetry. The authors found that the designed sequence can constitute a hypothetical path by which children in early childhood education can progress in their learning of symmetry.

Mulligan and Oslington[4] contributed the second study on kindergartners' competence in symmetry, which looked for an alternative for the often used "butterfly" pictures. To make the context more meaningful for the children they had to work with drawings of their portraits which they had to analyse for features of line symmetry and mathematical structure. The authors found that over thirty percent of children represented explicit structural features such as equal spacing, congruence, partitioning and alignment of facial features. The third symmetry study[5] by Fletcher et al. explored the assessment of early symmetry knowledge. In the study an intervention with symmetry software took place in which first and second grade children were taught reflection, translation and rotation. After the intervention the children were assessed by a paper-and-pencil test and by interviewing them. The authors found that children who reached higher scores on reflection and translation tasks, in the interviews also provided explanations indicating conceptual understanding of the symmetric transformations. The similar relationship was reported for girls and boys.

Recognizing and being able to work with patterns is considered a vital element of young children's mathematical development. To know more about children's understanding of patterns, Tian and Huang[6] investigated in a sample of 134 four-year-olds preschool children how able they are in solving tasks on repeating patterns. The results showed that the children could fill and expand repeating patterns, but also difficulties came to the fore in the abstraction of the pattern, especially in identifying the unit of a repeating pattern.

The last content domain that is reported in this section is the representation of data. Chung[7] describes a study in which it was investigated whether kindergartners (aged five-six) can sort and group objects, identify the quantity of each group of objects, and then can draw pictures or write names and numbers to organize and present the data. One of the results of the study is that half of the 35 children involved failed to represent the quantities using numerals and pictures.

2.2. Investigations of children's learning environment

The papers in this section lift in different ways the veil of the conditions and circumstances in which the early learning of mathematics can come about. To gain knowledge about this, in most studies data were collected by interviewing early childhood teachers. In the study[8] by Tirosh et al. a broader response group was surveyed and adults (not being preschool teachers but including grade school and high school teachers, psychologists, occupational therapists, engineers, municipal workers, and accountants) were asked what types of activities they perceive as the ones that can promote numerical skills. Many participants suggested counting objects. Sub-skills such as counting forward from some number other than one or focusing on one-to-one correspondence, were less mentioned.

In Lüken and Lehmann's study[9], when 23 early childhood teachers were asked about their use of finger patterns in their daily interaction with children it was found that they all use finger patterns in a variety of everyday (such as age/birthday, finger games, board games) and mathematical contexts (verbal counting, object counting, referring to quantities or number signs, and when calculating). The frequency and type of used finger patterns varied among the teachers. Only four teachers used finger patterns doing calculations. Two of them used the fingers in a dynamic way and two in a static way. No more then ten teachers used finger patterns as a visualization to help children develop an understanding of numbers.

Because what early childhood educators think about the mathematical abilities of their children may influence the learning environment they offer to them, an international study[10] by Walter-Laager et al. was set to investigate the performance expectations of early childhood educators in five countries. The focus was on shapes and space. The data of 1343 early childhood educators revealed that the expectations for this content area were more accurate in Austria and Switzerland than in China, Vietnam and the USA. Also, the estimations for 3–6-year-old children were more appropriate than those for the 1–3 year olds.

In addition to learning through focused activities, young children's learning of mathematics also takes place to a large extent through free play. In Keijzer et al.'s report[11], to figure out what interactions between preschool/kindergarten teachers and preschoolers (2 to 6 years) can be considered as useful for stimulating young children's language and mathematical development a professional learning community (PLC) was set up consisting of preschool and kindergarten professionals and researchers. Based on discussions held within PLC-meetings and the analysis of the mentioned interaction characteristics three guidelines for interactions were identified that can stimulate children's mathematical development during children's spontaneous play: observing (understand the child's interest and feelings), connecting (confirm what the child is playing) and enriching (cooperatively construct mathematical meaning).

Preschool teachers' positive feeling about mathematics is a determining factor of the quality of the early childhood learning environment. Therefore, in Thiel's longitudinal study[12] with an experimental pre-test post-test control group design, it was investigated whether and how a preservice teacher training can change prospective early childhood teachers' emotions about mathematics. The study was carried out with full-time and part-time teacher students. Only the part-time students showed after the training an increase in mathematics enjoyment and a reduction of mathematics anxiety. For almost all the part-time students the lessons at the university were the most important reason of this change. For only half of the full-time students this was the case, while 35% indicated that it was the five-weeks practical period they spent in an early childhood institution.

In the two following papers, instead of an empirical approach, the learning environment is considered from a theoretical point of view. Cooke and Jay[13] discussed with what mathematics young, pre-verbal children might be engaged. The authors used for this Bishop's framework of the six mathematical activities which are fundamentally mathematical: counting, locating, measuring, designing, playing, and explaining. By reframing each of these activities by putting the focus rather on actions than on language, the framework is made appropriate for pre-verbal children and may provide assistance in identifying the mathematical thinking that is evident in pre-verbal children's actions. Wen[14] focuses on games as the basic form of activity for preschool children and describes the mathematics that children can meet in games and through which they can achieve the ability to think mathematically. Questions to be answered are how the gameplay and the core mathematics experience are related and how the fun of games can be combined with the effectiveness. The paper continues by giving examples of teachers playing games with children and children playing alone or cooperatively.

A tool to measure the quality of the early childhood learning environment in a standardized way is the Classroom Assessment Scoring System (CLASS). With this tool, among other things, the given instructural support can be investigated with respect to three dimensions: the development of concepts, the quality of the feedback and the language modelling. Henriksen[15] used this tool in a kindergarten class and analyzing the classroom interaction in an observed lesson. It was revealed that there was a low score on Instructional Support: the teacher did not prompt children to explain their strategy, did only focus the feedback on the correctness of the answers, and asked mostly close-ended questions. By proving this information, the tool can give indications in what way the teacher may develop.

In a large national study[16] by Bjørkås et al., the quality of the learning environments in the child groups of Early Childhood Education and Care centers were investigated by means of data based on observations with the Infant/Toddler Environment Rating Scale —Revised and the Early Childhood Environment Rating Scale — Revised. The focus in these observations was on the learning area “Number, Spaces and Shapes”. In addition, questionnaires were used to collect from directors of ECEC centers. A comparison of the results with a study done some seven years ago showed that the centers worked more systematically on this learning area. However, the quality of the learning environments as measured through the observations varies greatly and are to a large extent qualified as inadequate. For example, most of the centers only provided one kind of blocks on a daily basis, giving little opportunity for children to investigate different kinds of properties of space and shape.

In the final paper[17] by LeSage and Ruttenberg-Rozenan an alternative research perspective was chosen. In this study children themselves was given a voice when investigating the quality of the early childhood learning environment. The focus was on the quality of educational software. In particular five early numeracy apps were investigated, which were uploaded onto the classroom iPads. Data from 12 children (4 to 6-year-olds) were collected through multiple sources, including observations, interviews and videotaped child-led ‘tours’ of their favorite apps. As criteria for good apps were identified the quality of the game experience (frequent positive verbal reinforcement and earning rewards) and the autonomy in making choices.

https://doi.org/10.1142/9789811287152_0026

Topic Study Group 2

Mathematics Education at Tertiary Level

Ghislaine Gueudet[1] and Irene Biza[2]

ABSTRACT In this report we summarize the activities and the studies presented at the TSG-2: Mathematics education at tertiary level of the 14th International Congress on Mathematics Education (ICME-14) that took place online and in Shanghai, China on July 11 to 18, 2021. The activities of the group spanned across four themes: mathematics teaching; students' practices and experiences in mathematics; transitions to, across and from studies of mathematics at tertiary level; and, mathematics for other disciplines. New themes and emerging theoretical directions are amongst the suggestions of the group.

Keywords: Mathematics for other disciplines; Secondary-tertiary transition; Students' practices and experiences; Teaching at tertiary level.

1. Introducing TSG-2 at ICME-14

1.1. Scientific scope of TSG-2

Topic Study Group 2 (TSG-2) at ICME-14 aimed to share and discuss the recent results of research and practice on learning and teaching mathematics at tertiary level, and to identify perspectives for future research. The works in the group drew on findings discussed at ICME-13 related to themes such as: mathematical practices; teaching, professional and curriculum development; connections to engineering; transition to university; preservice teachers; student thinking; and, research related to specific courses such as calculus, differential equations, and linear algebra — see Topical Survey on Research on Teaching and Learning Mathematics at the Tertiary Level (Biza et al., 2016).

The scope of TSG-2 was described in the call for papers as follows:

The questions studied can concern "traditional" courses, such as "chalk and talk" lectures in relations to teachers' practices and students' difficulties or achievements. They can also relate to "innovative approaches", such as design, implementation and evaluation of experimental courses. The contributions can address particular mathematical domains or mathematical practices. They can also concern teaching and learning practices such as assessment, use of technologies or resources; or university teachers' professional knowledge and professional development. Also, we expect submitted proposals to address the variety of tertiary programs that include

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mathematics, such as pure mathematics, engineering, teacher education, etc. Contributions can engage explicitly with theory (e.g., cognitive, socio-cultural, institutional, discursive, etc.) and certain methodological approach or can share a systematic reflection on teaching and learning practices."

More precisely, TSG-2 initially proposed the following five themes:

- Mathematics Teaching at the Tertiary Level;
- Students' Practices and Experiences in Mathematics at the Tertiary Level;
- Mathematical Topics Teaching and Learning at Tertiary level;
- Transitions to, across and from Studies of Mathematics at Tertiary Level;
- Mathematics for Other Disciplines at the Tertiary Level.

All these themes were covered by the submissions we received. Nevertheless, the "mathematical topics" (e.g., Linear Algebra, Calculus, Arithmetics etc.) theme was discussed in studies which could be also categorized in one of the other themes. As a result, the Sections 2 to 5 below address the four other themes with attention to the specific mathematical topics under consideration. Finally, Section 6 addresses emerging perspectives for further research that were discussed in the group.

1.2. TSG-2 at ICME-14: Organization and participants

The work in TSG-2 at ICME-14 was prepared by a team including the authors of this report, as chair and co-chair of the group; Rongrong Cao (Qingdao University, China); Victor Giraldo (Universidade Federal do Rio de Janeiro, Brazil); and, Azimeh Khakbaz (Bu-Ali Sina University, Iran). The IPC Liaison person was Frode Rønning (Norwegian University of Science and Technology, Norway). During the hybrid conference, TSG-2 welcomed 1 invited talk (30 minutes); 8 long presentations (15 minutes); 16 short presentations (10 minutes) and 3 posters (brief 'teaser' presentation before the poster sessions). The authors came from more than 20 countries, representing all the different parts of the world (Brazil, Canada, Chile, China, Colombia, Finland, France, Germany, India, Iran, Israel, Netherland, Philippines, Russia, Spain, South Africa, Sweden, Thailand, Uganda, UK, USA, etc.). Tab. 1 (on the next page) lists the titles and the authors of the papers and posters presented.

2. Mathematics Teaching at the Tertiary Level

The communications in TSG-2 highlighted the ever-growing interest in mathematics teaching at the tertiary level. Different complementary aspects were considered by the participants of TSG-2. Watkins et al.[22] studied university teachers' mathematical knowledge for teaching algebra. Analyzing videos of lessons, they evidenced that for instructors with high MKT for teaching algebra, much more segments without errors or imprecisions can be observed. Chen and Niu[26] studied the impact on class size, in the context of an evolution in Chinese universities to reduce this size. While it did not directly impact students' achievement, the class size influenced teaching practices.

Tab. 1. List of papers and posters presented at TSG-2

	Paper and author(s)
[1]	Errors of engineering students on the vector subspace concept. ***Andrea Cárcamo*** *and Claudio Fuentealba* (Chile).
[2]	Transition between paradigms in the university: The role played by the theoretical framework. ***Ignasi Florensa*** *and* ***Marianna Bosch*** (Spain).
[3]	Gendered patterns in university students' use of learning strategies for mathematics. ***Lara Gildehaus*** *and Michael Liebendörfer* (Germany).
[4]	First year university students' goals and strategies. ***Robin Göller*** (Germany).
[5]	Comparing two self-assessment models in a mathematics course — an exploratory study. ***Jokke Häsä****, Johanna Rämö, and Juulia Lahdenperä* (Finland).
[6]	Geometry for student teachers — capstone course in mathematics with a multitude of links to school mathematics. *Max Hoffmann and* ***Rolf Biehle****r* (Germany).
[7]	Engineering students' approach to studying mathematics and its influence on their achievement. ***Helena Johansson****, Magnus Oskarsson and Hugo von Zeipel* (Sweden).
[8]	The quality of mathematics teacher education at tertiary level in Uganda: is it relevant for 21st Century Mathematics Teachers? ***Marjorie Sarah Kabuye Batiibwe*** (Uganda).
[9]	How university students perceive the importance of resources to study calculus and linear algebra. ***Zeger-Jan Kock****, Birgit Pepin* (The Netherlands)*, and Domenico Brunetto* (Italy).
[10]	Success of mathematics training and talent search programme in India. ***Ajit Kumar*** *and S. Kumaresan* (India).
[11]	Conceptualizing agency and autonomy in tertiary mathematics. ***Mariana Levin****, John P. Smith III, Shiv S. Karunakaran, Valentin A.B. Küchle, and Sarah Castle* (USA).
[12]	The relational nature of supports for high priority mathematics students. ***Behailu Mammo*** *and Signe E. Kastberg* (USA).
[13]	BullsEyes and circles: Alternative scoring practices in collegiate mathematics courses. M*ichelle Morgan and* ***Jeffrey J. King*** (USA).
[14]	From student scribbles to institutional script: Towards a commognitive research and reform programme for university mathematics education. ***Elena Nardi*** (UK)*, Irene Biza* (UK)*, Bruna Moustapha-Corrêa* (Brazil)*, Evi Papadaki* (UK)*, and Athina Thoma* (UK).
[15]	The Secondary-Tertiary transition: An international perspective on where we are and how to move forward. ***Alon Pinto****, Hadas Levi Gamlieli, and Boris Koichu* (Israel).
[16]	An innovative hands-on activity to facilitate the learning of group of symmetries in abstract algebra. ***Tika Ram Pokhrel*** *and Parames Laosinchai* (Thailand).
[17]	The double discontinuity in teacher education — How to face it? ***Cydara Cavedon Ripoll*** *and Luisa Rodríguez Doering* (Brazil).
[18]	Instructors, Mentors, and Students: A Cross-comparison of perceptions of student-centered instruction. ***Kimberly Cervello Rogers****, Sean P. Yee, Jessica Deshler, and Robert Petrulis* (USA).
[19]	From a "strict and scary" class to the "active and favorite" subject: A long-lasting change in the teaching of mathematics at a first-year military school in Chile. ***Antonio Salinas Layana****, Sergio Celis, and Farzaneh Saadati* (Chile).
[20]	An approach to transition of mathematics of secondary to tertiary level mathematics. ***Gloria Inés Neira Sanabria*** (Colombia).
[21]	Mentoring of mid-career and early-career faculty. ***James Sandefur****, Michael Raney, Erblin Mehmetaj, and David Ebenbach* (USA).
[22]	Investigating mathematical knowledge for teaching and quality of instruction in US community colleges. ***Laura Watkins****, Irene Duranczyk, Vilma Mesa, and April Ström* (USA).
[23]	Student reasoning about eigenequations in mathematics and quantum mechanics. ***Megan Wawro****, John Thompson, and Kevin Watson* (USA).
[24]	Characteristics of collective mathematical activity associated with states of student engagement. ***Derek A. Williams****, Jonathan López Torres, and Emmanuel Barton Odro* (USA)
[25]	Flipping a general education mathematics course. ***Fei Xue*** *and Robert Nanna* (USA).
[26]	Study of the influence of class size on the teaching effect of college mathematics. ***Chaodong Chen*** *and Dunbiao Niu* (China). (Poster)
[27]	The relationship between conceptual and procedural knowledge. ***Janine Hechter*** (South Africa). (Poster)
[28]	Meaning of good mathematics teaching from the university students' point of view. ***Seyed Hadi Afzali Borujeni*** *and Azimehsadat Khakbaz* (Iran). (Poster)

Nevertheless, some teachers did not adapt their practices for small classes (50–60 students).

Other studies focused on the development of teacher practices, and the support needed, in particular towards more student-centered approaches. Mammo and Kastberg[12] observed a teacher improving his practice especially for underperforming students, in a setting where peer-tutors were helping these students. The teacher developed their awareness of factors that favor or hinder the efficiency of peer-tutoring. Salinas Layana et al.[19] studied the development of teaching practices on long term, towards more student-centered practices. Through interviews with two teachers, they evidenced that the support of the institution and the agency of teachers were both crucial for this development. Indeed, promoting student-centered practices requires specific teacher support and education. Sandefur et al.[21] investigated the impact of an intensive mentoring experience. Two teachers were involved in a course using a flipped approach and more active learning strategies, new to both of them. Teachers were supported by experienced mentors, and used videos and other material prepared by their mentors. The authors observed evolutions of the two teachers' practices, and of their awareness of students' learning processes. Rogers et al.[18] analyzed the teaching practices of novice collegiate mathematics instructors, also supported by mentors for implementing student-centered techniques. The analyses evidenced discrepancies between the declarations of the novice instructors (who consider that they actually used students-centered techniques), and those of the students and of the mentors. Even with the support of mentors, implementing students-centered techniques remains challenging.

Some studies also presented successful interventions. Kumar and Kumaresan[10] presented a program called: "Mathematics Training and Talent Search (MTTS)", and emphasized some of its aspects, such as personal attention and collective work. This four-week summer school for university students in India has convinced many students to go further with their mathematical studies. Pokhrel and Laosinchai[16] presented an innovative hands-on activity for the learning of group of symmetries in undergraduate level. Working on this activity, students developed an inquiry stance and explore groups of symmetries. Xue and Nanna[25] presented a flipped course about modeling with elementary functions. They evidenced that students learned better with this course than with a traditional lecture-based course, and that the classes were more active and dynamic. Some of the interventions concerned the assessment practices. Häsä et al.[5] study the impact of students' self-assessment on their learning practices. Comparing two models of self-assessment, they show that the assessment of their own skills seems to promote deep learning, more than the assessment of coursework. Morgan and King[13] studied alternative scoring practices and how they impact the students' learning experience. They evidenced the importance of feedback, allowing the students to improve their scores; and the positive effect of non-numerical scores.

3. Students' Practices and Experiences in Mathematics at the Tertiary Level

Students' practices and perceptions at tertiary level were also an important theme in TSG-2. Several theoretical evolutions have been proposed, and new themes emerged.

Williams et al.[24] investigated students' engagement during collective mathematical activity. They proposed a specific theoretical construct about students' engagement in the context of collective activity, linked with the interpretive framework introduced by Cobb and Yackel (1996). They observed that participating in argumentation is not sufficient for high engagement, while participation leading to collective progress and further understanding corresponded to higher engagement. The theoretical construct they proposed evidences that sociomathematical norms influence associations between engagement and mathematical activity. Levin et al.[11] proposed a conceptualization of students' agency and autonomous actions. Through interviews with students, they observed that agency and autonomy should not be described as qualities that participants either had or did not have. Agency and autonomy depend on the context, and exist on a continuum. Another theoretical and methodological construct was proposed by Göller[4] to investigate students' goals and strategies. Drawing on self-regulated learning theory, Göller introduced three overarching categories of strategies and associated goals. Learning strategies aimed to understand and remember new content, problem-solving strategies aimed to solve mathematical problems; nevertheless, students also used coping strategies to deal with institutional requirements. An empirical study confirmed the relevance of this theoretical construct. Gildehaus and Liebendörfer[3] also investigated students' strategies, searching for gendered patterns in the choice of strategies. They observed that, across different courses, female students reported a higher use of organization, time investment and peer learning strategies. This confirmed the view of female students being diligent and social, and can explain gender differences in the learning of university mathematics.

Other studies considered students' perceptions. Khakbaz and Afzali Borujeni[28] investigated what "good mathematics teaching" means for students. They observed different meaning, linked with the students' specializations. For example, for engineering students a good teaching of mathematics emphasizes on applications. Kock et al.[9] study concerned students' perception of the usefulness of different kinds of resources to study mathematics. Analyzing students' answers to a survey, they observed three different groups of students, those who see more importance (a) to lecturer explanations; (b) to the textbook; and, (c) to other curriculum resources (e.g., worked examples, materials prepared by the teacher). Nevertheless, these groups were depended on the courses as the resources proposed by the teachers and the institution vary across courses, and can influence students' practices.

4. School and Tertiary Mathematics Education, Transitions

The secondary-tertiary transition was, not surprisingly, one of the issues addressed in TSG-2. Pinto et al.[15] conducted an international survey about this transition, collecting the views of 310 university mathematics teachers in 30 countries. Exploring the discourses of the respondents, they noted that the concerns about the secondary-tertiary transition were aggravated during the last decades; many universities have organized concrete measures to face the difficulties, but communication between university teachers, secondary school teachers and mathematics education researchers still seems to be lacking. Several other papers addressed issues pertaining to teacher education at university, in different countries: Uganda (Kabuye Batiibwe[8]), Colombia (Sanabria[20]), Brazil (Ripoll and Doering[17]) and Germany (Hoffmann and Biehler[6]). Future teachers experience indeed "double discontinuity" as identified by Klein (1908): from secondary school mathematics to university mathematics, and then back to secondary school mathematics. The content of the teacher education programs has to acknowledge this double discontinuity, for example, by connecting university and school mathematics (see Hoffmann and Biehler[6], for an example in geometry).

5. Mathematics for Other Disciplines at the Tertiary Level

Research about mathematics and non-mathematics disciplines has developed internationally in the last five years; communications within TSG-2 at ICME-14 reflect this development. Wawro et al.[23] investigate students' meanings for mathematics (namely eigentheory) when solving quantum physics tasks. Using the theoretical framework of Knowledge-in-Pieces (diSessa 1993), they analyzed students' discourses and evidenced in some cases synergies between the mathematical and physical meaning of the same concept, but also incompatible interpretations. Other studies related to this theme concern mathematics in engineering education. In this context, the combination of theoretical and procedural aspects is a complex issue. Hechter[27] evidenced that conceptual understanding and procedural fluency are intertwined and should not be separated in the mathematics courses for future engineers. Cárcamo and Fuentealba[1] studied the difficulties encountered by engineering students working on linear algebra tasks, and observed difficulties related to proof and definitions. Johansson et al.[7] studied the learning practices of engineering students in a differential calculus course and their consequences in terms of students' achievement. While regular personal work across the semester had positive effects on exam results, some students worked with mathematics in the preparation for the exams only.

6. New Themes and Emerging Theoretical Directions

Studies in TSG-2 evidenced new themes and emerging directions for research. We note that Inquiry-Oriented practices in mathematics at tertiary level seem to be now well developed in the USA. Which kind of teacher support (including teacher education programs) can promote these new practices? This question is not only considered in the USA, for example, *Study and Research Paths* (SRPs, Bosch, 2018) are Inquiry-Based courses that have been increasingly developing in Europe and South America.

SRPs are also linked to emerging theoretical and methodological development we identified at TSG-2's works. The studies about SRPs are grounded in the

Anthropological Theory of the Didactics (ATD, Chevallard, 2015), which is increasingly used in studies at tertiary level. Moreover, these studies introduce new methodological approaches: Question-Answer maps, essential for the epistemological analysis needed before designing an SRP, and also allowing a collective work of researchers in mathematics educations and university teachers (Florensa and Bosch[2]). Related to emerging theoretical approaches, we also would like to mention the increasing number of studies using discursive approaches. Such studies consider the teaching and learning phenomena as discursive phenomena, and use methods for analyzing discourses. In TSG-2, Nardi[14] raised the potential of the commognitive approach (Sfard, 2008) to observe discursive shifts in university mathematics, beyond the micro-level of a student working on a precise task. She claimed that this approach can inform a reform agenda. Besides the aforementioned frameworks, some studies proposed innovative conceptualization to address particular aspects of students' practices at tertiary level. Williams et al.[24] proposed a conceptualization of students' engagement in collective mathematical activity. Göller[4] proposed categories for the analysis of students' goals and strategies. Levin et al.[11] introduced a conceptualization of students' agency and autonomy in the context of mathematics teaching and learning at tertiary level. This conceptualization leads to consider that students' agency and autonomy can change in response to context and over time. We really look forward to seeing further advances to research in tertiary mathematics in the next ICME conference and in other events such as INDRUM, CERME, RUME or DELTA conferences.

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https://doi.org/10.1142/9789811287152_0027

Topic Study Group 3

Mathematics Education for Gifted Students

Florence Mihaela Singer[1], Joseph Li[2], and Viktor Freiman[3]

1. Aims of the TSG

The goal of TSG-3 was to promote research and practice in the field of mathematical ability, mathematical potential, and giftedness in different cultures and contexts. The topic study group involved educational researchers, research mathematicians, mathematics teachers, teacher educators, curriculum designers, doctoral students, and others in a forum for exchanging insights related to the research and practice in mathematics education. The main purpose was to contribute to the development of our understanding of the nature and nurture of high mathematical ability in individuals.

1.1. Submissions

We received 28 submissions from 16 countries: Austria, Canada, China, Germany, Israel, Japan, Peru, Romania, Russia, Singapore, Slovenia, South Africa, USA, Sweden, Thailand, and The Netherlands, thus reaching our goal of diverse cultural representation. Of the submissions, five were accepted as long-paper presentations, seventeen were accepted as short-oral presentations, and six as posters. Among these, 21 papers have been presented.

1.2. Sessions

Throughout the three days of the TSG sessions, participants' dialogue and networking were focused on identifying emerging research themes, potential interdisciplinary approaches, and future research opportunities. To create a diversity of approaches and interactions, chairing the sessions was distributed among the TSG-3 organizers. Thus, first day: Florence Singer and Joseph Li, second day: Viktor Freiman, Florence Singer, Joseph Li, and third day: Joseph Li, Viktor Freiman, Florence Singer alternatively chaired the respective sessions. During the third day, we succeeded to have an interesting ad-hoc interaction between the online community and the participants in situ, in Shanghai, thanks to the excellent translation provided by Joseph Li. This online — offline interaction has shown the large interest of our TSG topic and the desire to extend the discussions concerning important issues related to high achievement and giftedness in mathematics.

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1.3. Paper topics

As mentioned above, of the 28 accepted papers, 21 papers have been presented during the online conference. A list of these papers and authors, organized in Tab. 1 in the order of presentation, is next included.

Tab. 1. List of the presented papers

Papers and author(s)
[1] Student perceptions of support provided by a summer math camp. *Michael Hicks,* ***Hiroko Kawaguchi Warshauer****, and Max Warshauer* (USA).
[2] Derivation of regression equations predicting Japan mathematical olympiad preliminary qualifiers from within arbitrary groups. ***Atsushi Tamura*** (Japan).
[3] How do math students use informal representations? A comparison between gifted and not gifted. ***Florence Mihaela Singer****, Cristian Voica* (Romania).
[4] School stages of educating the mathematician-investigator. ***Aleksandr Vasilevich Yastrebov*** (Russia).
[5] Problem solving and creativity among talented students from a multi-age perspective. ***Odelya Uziel****, Miriam Amit* (Israel).
[6] Educating prospective teachers in the field of mathematical giftedness - comparing experiences. ***Matthias Simon Brandl****, Attila Szabo, Elisabeth Mellroth, and Ralf Benölken* (Germany).
[7] Questions about the identification of mathematically gifted students. ***Marianne Nolte*** (Germany).
[8] What do prospective teachers express as to mathematical giftedness? An exploratory study. *Daniela Assmus and* ***Ralf Benoelken*** (Germany)
[9] Mathematical thematic content and didactic skills for the teaching of mathematics of students of the primary education career of the Catholic University Sedes Sapientiae, Peru. *Norma Fuentes Supanta De Fukunaga and* ***Patricia Edith Guillen Aparicio*** (Peru).
[10] Role of peer and teacher recognition for students' talents in STEM projects. ***Viktor Freiman*** *and Jacques Kamba* (Canada).
[11] Egalitarianism in inclusivity: thwarting the intellectual growth of mathematically gifted students in South African schools. ***Michael Kainose Mhlolo*** (South Africa).
[12] Activities for the mathematically gifted and their evaluation in Slovenia. ***Bostjan Kuzman****, Mojca Juriševič, and Urška Žerak* (Slovenia).
[13] Using interdisciplinary problem posing to promote gifted students in the regular classroom. ***Sara Hinterplattner****, Zsolt Lavicza, and Marca Wolfensberger* (Austria).
[14] Mathematically gifted students: challenges and opportunities in the primary years. ***Ban Har Yeap*** (Singapore).
[15] Discovering and educating the gifted students with excellent problems. ***Xiangrui Chan*** (China).
[16] Mathematical culture and teaching of equation. ***Yanchun Liu****, Lili Gao, and Peng Zhao* (China).
[17] Study of construction by quadratic curve addition method. ***Hideyo Makishita*** (Japan).
[18] Intuitive sense constructions of children with mathematical giftedness. ***Alena Witte****, Franziska Strübbe* (Germany).
[19] LEMAS — a joint initiative of Germany's Federal Government and Germany's Federal States to foster high-achieving and potentially gifted pupils. *Friedhelm Käpnick, Philipp Guillaume Girard, Julia Kaiser, Yannick Ohmann,* ***Lea Martina Schreiber****, and Wiebke Auhagen* (Germany).
[20] University students self-evaluation: digital solutions for identifying highly motivated students. ***Mirela Vinerean Bernhoff****, Yvonne Liljekvist, and Elisabet Mellroth* (Sweden).
[21] Experimental study on intellectual development in elementary school students. ***Yuwen Li*** (China).

2. Conference Themes

The congress papers can be summarized across five main themes:

- analysis of high-achieving students' skills and perceptions,
- identification and prediction of mathematical giftedness,
- informal training programs for the gifted and talented students in mathematics,
- adequate training for teachers and prospective teachers dealing with mathematically gifted students, and
- educational policies related to mathematical giftedness in different countries.

Concerning the *analysis of high-achieving students' skills and perceptions*, Uziel and Amit[5] explored the problem-solving capacities of 118 talented students in grades 5–12 who took part in the enrichment program known as "Kidumatica", examining the students' solutions from a multi-age perspective. Data was gathered from students' products and teacher observations during a series of workshops devoted to 10 non-routine problems with multiple solution paths. Their findings revealed a troubling phenomenon: as the age of students rises, they are less prone to looking for creative and holistic solutions when solving problems, and more likely to be "held hostage" by their habitual use of algebra. Somehow strengthening this conclusion, Witte and Strübbe[18] noticed that very young mathematically gifted children show a strong fascination for mathematical questions, and they develop intuitive conceptual constructions regarding various mathematical relationships. For older students, in a qualitative case study, Hicks, Warshauer and Warshauer[1] examined the perceived support as described by three female African American students enrolled in a summer camp for high achievers with an interest in STEM. The students' perceptions on the given support for developing their mathematical competence and sense of belonging to a community of mathematics learners highlighted that the key solution for progress in learning is adequately addressing individual needs at the right time. From another perspective, Singer and Voica[3] investigated students' representations of abstract mathematical concepts beyond reproducing definitions and theorems. They exposed 51 undergraduate university students to tasks that required the association of mathematical concepts as limits or convergence to as many as possible images that potentially generate suggestive mental representations in school students. The researchers found that high-performing university students have focused on mathematical properties of concepts, for which they found meaningful images that revealed deep mathematical meanings, compared to low performers who only pointed some surface characteristics of the mathematical concepts involved.

Identification and prediction of mathematical giftedness was a topic of high interest in the TSG-3 community, as the discussions have shown. Thus, in her review article, Nolte[7] gave an overview of the questions on diagnostics and procedures of high mathematical talent. Various methods such as intelligence tests, school achievement tests and checklists were presented and discussed. The conclusions favored multidimensional approaches with a focus on specially designed mathematical tests. Identification of the gifted students is important when we talk about Mathematical Olympiads. Thus, Tamura[2] developed a model based on derivation of regression equations that could predict Japan Mathematical Olympiad preliminary qualifiers from within arbitrary groups. Within the presentation, he explained how his mathematical talent checklist was used to identify preliminary qualifiers among a group of average high school students and Mathematical Olympiad preliminary qualifiers at a true discriminant ratio of over .. % and how this tool was refined through a logistic based on regression analysis to improve the result. As he pointed out, an analysis of the characteristics found in the sample of students with advanced mathematical abilities are indispensable in the development of "math for excellence" educational materials. Beyond complex mathematical tools for the identification of talented students, Bernhoff, Liljekvist, and Mellroth[20] found that it is possible to identify highly motivated individuals among engineering students by exposing them to the use of Learning Management System (LMS) to self-evaluate their work on recommended tasks, which then provided the lecturer with some statistical data. The research is just at the beginning, we wait for the next steps of it.

Some of the TSG-3 papers were focused on the presentation of *informal training programs for the gifted and talented students* in mathematics, giving the audience some concrete hints and procedures. Thus, Li[21] described a training organized for students who participated on a voluntary basis from first to fifth grade at Changhe Elementary School in Dezhou, China, and compared their mathematics achievement with students who did not participate in the training. Through comparison studies on all participated students and prospective longitudinal studies on randomly selected students from both the experimental group and control group, he found that students who participated in the training had significantly better mathematical achievement and abilities in geometry, logic, and innovation than their comparable peers. Some relevant tasks for talented students could come also from history and traditions. Makishita[17] refered to a traditional Japanese mathematics from the Edo period described in Wasan books, showing that people learned mathematics for fun to solve quizzes, puzzles, and other entertainment problems, as well as for monetary exchange, and other everyday work activities. The author has used Wasan contents and applications in modern mathematics education, recording success with various types of mathematics classes. Diverse and substantially rich examples of working with mathematically gifted students exist in many countries. Kuzman, Jurišević, and Žerak[12] from Slovenia presented activities such as mathcamps, research projects, competitions provided by different parties (schoolteachers, educational institutions, expert groups, learning societies), concluding that often the success of these activities largely depended on the enthusiasm and competencies of the involved individuals. These activities were evaluated within the PROGA project (2017–2020), and a support system for high school students with special talents in mathematics was established. While in many countries, alternative programs are searched to improve students' mathematical abilities at higher levels, Yastrebov[4] made a three-folded plea: 1) Shaping of skills and habits of the future mathematician can be started at the early stages of his/her education and upbringing within school; 2) Educating the mathematician-investigator at the early stages of education contributes to the implementation of general goals of school education, regardless of the student's future profession; and 3) Experience of the pedagogical society is sufficient for shaping the uniform, integral system of educating the mathematician-investigator at the school level.

The issue of *adequate training for teachers and prospective teachers dealing with mathematically gifted students* was also one of high interest for the TSG-3 community. Brandl, Szabo, Mellroth and Benölken[6] focused on how prospective teachers can be trained in the field of mathematical giftedness. By comparing three independently developed concepts to deduce cornerstones of appropriate seminar concepts, they concluded that a combination of theoretical and practical parts is particularly important for a sustainable education in the context of giftedness. As Freiman and Kamba[10] have stressed, over the past decades, novel integrated STEM learning spaces have emerged in K-12 schools providing students with new interdisciplinary enrichment opportunities to express their gifts and talents. Based on the study of provincial makerspaces and maker projects they have conducted since 2016, they identified a clear trend related to an increasing role of an expertise of some students who became recognized by their peers and was essential for the collective success of the projects they developed. Therefore, enhancing the monitoring role of high achieving students among their peers can promote better learning in the whole class. Within a study investigating prospective teachers' perceptions regarding mathematical giftedness, Assmus and Benölken[8] found that future teachers' ideas and knowledge are rather shadowy compared to the state of research on the modeling of mathematical giftedness. Yeap[14] has proposed a model of professional development of teachers comprising three phases. In the first phase, teachers learn by experiencing the doing of challenging mathematics. In the second phase, they observe students engaging in mathematically challenging tasks. In the third phase, they teach lessons that include such tasks. Following this strategy, three categories of opportunities emerged: for the mathematically students the teachers teach, for the other students who are also taught by these teachers, and the final category includes opportunities for the teachers, their colleagues, and the school culture. De Fukunaga and Aparicio[9] analyzed the mathematical thematic content and the didactic skills of prospective primary education teachers from Universidad Católica Sedes Sapientia, Lima-Peru. Their results show that student-teachers require training in mathematical content and personalized pedagogical help, as well as the necessary didactic material for their children to achieve meaningful learning and develop their mathematical potential applied to daily life and society. Still, mathematical culture is the treasure of human culture, and its contents, ideas, methods, and language are important components of modern civilization; following this assumption, Liu et al.[16] insisted on the fact that teachers in the new era should think deeply about how to integrate mathematics culture into the teaching practice, so that students can be influenced by the mathematics culture in the process of learning. They provided an example of introducing mathematical culture into equation teaching for students in different grades. They explored ways to maximize the charm of cultural acquisitions to stimulate students' interest in learning, and their passion for mathematics and mathematical culture.

The topic of *educational policies related to mathematical giftedness* in different countries inevitably goes into the eternal debate between egalitarianism and the promotion of giftedness — how this two can be approached and complemented. As Mhlolo[11] has stressed, since 1994 the focus moved from separate and specialized education for gifted learners to inclusive education with all learners being educated in regular classrooms. Although inclusive education policy initiatives in theory aimed at ensuring quality education for all, current empirical evidence shows that in many African countries including South Africa, excellence and egalitarianism have become out of balance as gifted students from previously disadvantaged communities do not reach their full potential in regular classrooms. Similarly, Hinterplattner et al.[13] noticed that meeting the needs of all students in an inclusive classroom is a rather challenging task. Gifted students often need more or different tasks and activities than teachers in regular classes can or are willing to offer. In such situations, it may happen that gifted students face boredom, which may lead to various behaviors driven by unsatisfaction. To prevent boredom and misbehavior, an experiment with 10 gifted secondary school students was carried out based on findings from neuro-didactics, to ensure a good learning experience. The students were taken out from one of their three regular mathematics classes per week for 9 weeks and challenged with an interdisciplinary problem that was based on STEAM ideas. To solve this problem, a combination of problem-posing abilities, the capacity of gathering knowledge from various fields, and their applied interdisciplinary ideas were necessary. After this experiment, students were asked about their experiences concerning the project itself, and its impact on their regular classes. Results show that the project was described as quite challenging and motivating. Also, its impacts on their regular classes were described as highly positive. Students reported that they used the time they were usually waiting in the regular classrooms for solving the problems they found in their project. Educational policies addressing strategies to foster high-achieving and potentially gifted students might not be only developed from a research perspective, but also from a systemic one. Käpnick et al.[19] presented a joint initiative of Germany's federal government and Germany's federal states for an interdisciplinary network of scientists from 16 universities, together with 300 schools, under the name of "LEMAS" to develop guiding principles and adaptive concepts to support gifted and talented students. This long-term program has just started, and the entire community is expecting relevant results.

3. Future Directions for Research and Practice

The participants agreed that they will continue the international exchange of ideas related to research on the identification of mathematical talent, didactics of teaching highly able students, as well as the promotion of mathematical challenge and enrichment for all. The focal topics will continue to include empirical, theoretical, and methodological issues related to students' excellency in mathematics. Discussions and research within community will aim at better understanding of: useful tools for identifying and assessing mathematically gifted students; the educational approaches and organizational settings more effective for training gifted individual students or groups of students of various ages; the nature of mathematical tasks and activities that are challenging, free of routine, inquiry-based, and rich in authentic mathematical problem solving and posing; the relationship between exceptional mathematical abilities, motivation and mathematical creativity; the relationship between mathematics education for the gifted and equity of education for all students; teacher education aimed at mathematics teaching that encourages and promotes mathematical talents, and the development of interdisciplinary programs (STEAM included), for gifted students. New areas of research will be opened towards the relationship between mathematics education for gifted students and the talent development in the areas that are important in the future, such as artificial intelligence, genetic technology, computational thinking, big data, cryptocurrencies, etc. We hope that the network of professionals in the field will continue to increase, for the benefit of the students around the world.

https://doi.org/10.1142/9789811287152_0028

Topic Study Group 4

Mathematics Education for Students with Special Needs

Michelle Stephan,[1] Yan Ping Xin,[2] Anette Bagger,[3] and Juuso Nieminen[4]

1. Aims of the TSG

In TSG-4, we focused on a variety of theoretical and practical topics related to supporting the mathematical development of students with special needs as well as teachers' support of students. Throughout the presentations we actively searched for connections both in regards of methodology, theory and the possible impact of the presented research. We aimed to explore five related themes: 1) how do we define "special needs", 2) what are the benefits of differing instructional contexts (e.g., 1-1 teaching as opposed to inclusive settings), 3) how do we reconcile the variety of frameworks that originate from two different fields, special education and mathematics education, 4) what are the characteristics of effective professional development programs that aim to support teachers of students with special needs, and 5) what are the pros and cons of different research methodologies within our context (e.g., single case design and classroom teaching experiments)? We encouraged submissions that offered theoretical and/or empirical contributions and sought to include research from a variety of cultural contexts to enhance our discussions.

1.1. Submissions

We received 35 submissions from 21 countries (South America: 3; North America: 8; Asia: 3; Europe: 17; Africa: 2; Australia: 1; Eurasia: 1), thus reaching our goal of diverse cultural representation. Of those 35 submissions, twenty-seven were accepted as paper presentations, seven as posters, and 1 was rejected.

1.2. Sessions

There were so many high-quality submissions, the ICMI organizing committee granted our TSG one more time slot for presentations. In our first 90-minute session, the TSG Chair, Michelle Stephan, introduced the rest of the Team and described the format of the sessions. Generally, all four-time sessions led with a 25 minutes long oral presentation and discussion and two to three short oral presentations with a 10-minute collective discussion. Throughout the four days, we attempted to facilitate participant dialogue in order to collective identify emerging research themes, potential

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interdisciplinary approaches and future research opportunities. Discussion time was critical for fostering participant networking as well as countering "virtual conference" fatigue. At the end of our last session, we built in 50 minutes for whole group reflection, discussion and suggestions for needed research trajectories.

1.3. Paper Topics

Of the 27 accepted papers, only 18 papers were able to be presented during the online conference. A list of these papers and authors are included in order of presentation and are organized in Tab. 1.

Tab. 1. List of papers presented

Paper and author(s)
[1] Mathematics learning difficulties? The impact of a constructivist oriented approach to intervention for young learners who struggle the most. ***Ann Gervasoni*** *and Anne Roche* (Australia).
[2] Conceptual model-based problem-solving computer tutor for elementary students struggling in mathematics. ***Yan Ping Xin****, Soo Jung Kim, Bingyu Liu, Qingli Lei, Shuang Wei, Wudong Wang, Sue Richardson, Signe Kastberg, and Yingjie Chen* (USA).
[3] Interventions in micro-spaces for learners with mathematics difficulties. ***Robyn Ruttenberg-Rozen*** *and Ann LeSage* (Canada).
[4] An inclusive child's enactment to a task in dynamic geometry environment. ***Shajahan Haja-Becker*** (Germany).
[5] The effect of schema-based instruction on the resolution of addition problems by a student with autism spectrum disorder. ***Irene Polo-Blanco****, Steven Van Vaerenbergh, Maria González, and Alicia Bruno* (Spain).
[6] Emergent technological practices of middle year students with mathematical learning disabilities. ***Alayne Armstrong*** (Canada).
[7] Introduction to probability in an inclusive setting — insights by a student with learning difficulties. ***Nadine da Costa Silva*** (Germany).
[8] Preparing teachers for mathematics and special education consultations. A collaboration across four continents. ***Sarah Van Ingen*** (USA), *Samuel Eskelson, David Allsopp, Steffen Siegemund, Anna-Sophia Bock, Vera Lúcia Messias, Fialho Capellini, Ana Paula Pacheco Moraes Maturana, and Di Liu* (China).
[9] Criteria used by teachers to identify students with difficulties in learning mathematics. ***Shemunyenge Hamukwaya*** (Namibia).
[10] Becoming a mathematician: The role of learning environments in the identity narratives of mathematics students with learning disabilities. ***Juuso Nieminen*** (Finland).
[11] Tactile drawings and 3-D objects: Two keys to geometry for a blind student in an inclusion university course for preservice K-8 teachers. ***Patricia Baggett*** (USA).
[12] Arithmetical achievements of children with Trisomy 21 supported on geometrical basis. *José Ignacio Cogolludo-Agustín,* ***Elena Gil-Clemente****, and Ana Millán Gasca* (Spain).
[13] Beyond ability rankings: Educational assessment as relational rigor and accountability. ***Anette Bagger****, Alexis Padilla, and Paulo Tan* (Sweden).
[14] Mathematics and blind students: The problem of representations. *Elisabete Marcon Mello* (Brazil).
[15] Mathematics difficulty of students with special needs from the perspective of memory theories. ***Chi To Lui*** *and Ida Ah Chee Mok* (Hong Kong Sar, China).
[16] A teacher's attitude and approaches to high and low achieving students. *Julie Vangsøe Færch, Signe Gottschau Malm, and* ***Steffen Overgaard*** (Demark).
[17] Intervention based on mathematical thinking improves student outcomes: Math disabilities and difficulties. ***Jessica Hunt*** *and Kristi Martin* (USA).
[18] The variety of mathematical braille notations and their underlying principles. ***Annemiek van Leendert****, Michel Doorman, Johan Pel, and Johannes van der Steen* (The Netherlands).

2. Themes

Although there was a large variety in research topics presented during the sessions, the majority of the work can be summarized across four themes. First, a few research teams focused on supporting students who are visually impaired as they learn to symbolize in geometry, probability and other mathematical areas. For example, van Leendert's group[18] analyzed a variety of braille readers to see how they read and express mathematical notations and images. The results show that most of their ways of representing mathematics are very close to the graphical notation used by people who are not visually impaired or to the notation used in Excel or LaTeX. Baggett[11] shared a variety of manipulatives that were used to support a university student who is visually impaired as she learned geometry. And Marcon Mello[14] found that having students who are visually impaired draw their mental image of a mathematical object can reveal much more about their understanding than simply having them describe an already-drawn object.

A second theme that arose across the sessions involved advocating for a strengths-based, critical approach to both teaching and research with students with special needs. In particular, Bagger's group[13] argued that local and national assessments have been used to marginalize students with disabilities and make a compelling case for disrupting that deficit narrative. Hamukwaya[9] explored the criterion that teachers use to determine if a student has mathematics difficulties, while Færch's team[16] showed how a teacher's knowledge that a student was either high or low achieving impacted the quality of instruction given to them. Finally, Nieminen[10] interviewed university mathematics students to explore their social and cultural identities as mathematics majors and what experiences led to those identity formations.

In a related theme on identity and asset-based approaches to teaching mathematics to students with special needs, many participants argued that traditional research and teaching that focuses solely on developing students' procedural knowledge is inequitable to students with disabilities. Rather, students with disabilities should be engaged in mathematical sense making, like their regular education peers. Hunt and Martin developed an instructional unit for fractions that builds on constructivist learning progressions as did Gervasoni and Roche[1] for elementary students. Ruttenberg-Rozen and LeSage[3] introduced the term *microspace* and argued that students with disabilities should engage in these small, micro instances of sense making that, over time, build conceptual understanding. Cogolludo-Agustín's group[12] and Polo-Blanco's group[5] each showed how instructional units focusing on students' sense making in elementary concepts can improve students' with special needs mathematical understanding.

Finally, several research teams explored the use of manipulatives and technologies with students with disabilities. Xin and colleagues[2] created a computer tutor program to help students with learning disabilities/difficulties make sense of additive word problem solving after priming their early number concepts while Haja-Becker[4] found that students only partially used the provided technology when given the choice.

Similarly, Armstrong interviewed middle school students with disabilities to learn what types of mathematics technologies they regularly use besides calculators and found that students do not utilize assistive technologies much outside of the classroom.

3. Areas for Future Research

On the final day of the conference, the participants discussed three potential future research topics. First, we wondered what kinds of assistive or instructional technology are widely available to *simultaneously* support a particular need a student might have and would also engage them in mathematical sense making. Many technology programs facilitate procedural fluency but are not as strong in developing deep conceptual understanding (an exception is the computer tutor developed by Xin and colleagues[2]). How can we attend to both when designing technology for students with special needs and for inclusive purposes (i.e., technologies that are accessible to all students)? What kind of design principles are needed to strengthen such technologies and what research needs to be conducted in order to better understand their impact?

Participants also noted that there continue to be a relatively small number of research studies conducted in classroom settings. With the exception of da Costa Silvas' findings[7] regarding a student's learning of probability in an inclusion classroom, there were no other studies that explored students' learning in the context of a classroom with a teacher with multiple students learning together, such as in an inclusion setting.

Another research strand that was under-represented at the conference concerned supporting teachers in their work as both regular and special educators. Van Ingen and colleagues[8] presented a framework to describe the complex knowledge necessary for teachers to support their students with mathematics difficulties. Their singular work suggests that much more research is needed to determine the types of professional development and university preparation that is needed to support the work that teachers do with students who are struggling in mathematics.

https://doi.org/10.1142/9789811287152_0029

Topic Study Group 5

Teaching and Learning of Number and Arithmetic

Andrea Peter-Koop[1], Arthur Powell[2], and Rui Ding[3]

1. Aims of the TSG

The purpose of this TSG was to gather congress participants who were interested in research and development in the teaching and the learning of number systems and arithmetic through activities in and out of school. The mathematical domains include whole numbers, integers, ratio and proportion, and rational numbers as well as representations and problem-solving using numbers related to each of these domains:

- *research-based specifications of domain-specific goals,*
- *analysis of learning processes and learning outcomes in domain-specific learning environments*
- *and classroom cultures,*
- *new approaches to the design of meaningful and rich learning environments and assessments.*

We encouraged submissions that offered theoretical or empirical contributions and sought to include research from a variety of cultural contexts to enhance our discussions.

1.1. Submissions

We received 37 submissions from 18 countries (US: 5; Canada: 2; Brazil: 6; China: 1; Japan: 1; UK: 2; Germany: 3; Mexico: 3; South Africa: 4; Spain: 1; France: 1; Australia: 1; Sweden: 2; Isrel:1; Italy: 1; Nigeria: 1; Bruni: 1; Chile: 1), thus reaching our goal of diverse cultural representation. Of those 37 submissions, twenty-nine were accepted as paper presentations, six as posters, and two was rejected.

1.2. Sessions

There were so many high-quality submissions the ICMI organizing committee granted our TSG one more time slot for presentations. In our first 120-minute session, the TSG Chair, Arthur Powell, introduced the rest of the Team and described the format of the sessions. Generally, there are four 20 minutes long oral presentations and discussions and 13 short oral presentations, and after every 2 or 3 presentations, there were 10 minutes of collective discussions. Throughout the four days, we attempted to facilitate participant dialogue in order to collectively identify emerging research themes,

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potential interdisciplinary approaches, and future research opportunities. Discussion time was critical for fostering participant networking as well as countering “virtual conference” fatigue. At the end of our last session, we scheduled 50 minutes for whole group reflection, discussions, and suggestions for needed research trajectories.

1.3. Paper topics

Of the 29 accepted papers, only 18 were able to be presented during the online conference. In Tab. 1 below is a list of these papers and authors in order of presentation.

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Representational flexibility linked to higher attainment in early number learning. ***Samantha Morrison*** (South Africa).
[2]	Conceptual and procedural understanding on addition of fractions among Year 5 primary children. ***Nor’Arifahwati Abbas****, Masitah Shahrill, Mohd Khairul Amilin Tengah, Nor Azura Abdullah* (Brunei).
[3]	South African learner’s patterns of performance on additive word problems. ***Herman M. Tshesane*** (South Africa).
[4]	The case against coherence in mathematics instruction. ***Ola Helenius*** *and Linda Marie Ahl* (Sweden).
[5]	Identifying South African primary learners doubling and halving reasoning through a written assessment. ***Sameera Hansa*** *and Hamsa Venkat* (South Africa).
[6]	The use of arrays in solving multiplication word problems in Grade 4. ***Mayamiko Malola*** (Australia).
[7]	Toward a universal cognitive core: A cross-cultures (USA, China) progression in multiplicative reasoning. ***Ron Tzur*** (invited speaker, USA) *and Rui Ding* (China).
[8]	Flexible mental calculation: A study with 2nd and 4th Grade Brazilian students. ***Luciana Vellinho Corso****, Sula Cristina Teixeira Nunes, and Évelin Fulginiti de Assis* (Brazil).
[9]	The flexibility in mental calculation: Characterizing the profiles of a group of Brazilian Elementary Students. ***Évelin Fulginiti de Assis****, Sula Cristina Teixeira Nunes, and Luciana Vellinho Corso* (Brazil).
[10]	Precursors of problem-solving in two Brazilian cities: the role of social and economic differences. ***Beatriz Vargas Dorneles****, Camila Peres Nogues, and Elielson Magalhães Lima* (Brazil).
[11]	The performance in domain-specific cognitive abilities among low and typical mathematical achievers. ***Camila Peres Nogues****, Elielson Magalhães Lima, and Beatriz Vargas Dorneles* (Brazil).
[12]	Improving student knowledge of fraction magnitude in the early grades. ***Arthur Belford Powell*** *and Candell V. Ali* (Israel).
[13]	Elementary teacher professional learning to explore and extend nuanced meaning of number. ***Krista Francis****,* ***Sharon Friesen****, Miwa Takeuchi, Armando Paulino Preciado Babb, and Barb Brown* (Canada).
[14]	Difficulties of learning the decimal positional numeration (DPN) system: The principle of exchange. ***Daniela Fernandes*** *and Jeanne Koudogbo* (Canada).
[15]	Decimal number system in Quebec Mathematics Program and in textbooks: What Knowledge and for which mathematical education. ***Jeanne Koudogbo*** *and Daniela Fernandes* (Canada).
[16]	Errors in ratio and proportion: A framework for analysis. ***Özdemir Tiflis*** *and Gwen Ireson* (UK).
[17]	School-readiness in mathematics: Development of a screening test for children starting school. ***Andrea Peter-Koop*** (Germany).
[18]	Students performance when solving word problems involving fractions. ***Maria T. Sanz****, Carlos Valenzuleza, Olimpia Figueras, and Bernardo Gómez* (Spain).

2. Conference Themes

Although there was a large variety of research topics presented during the sessions, the majority of the work can be summarized across three themes. First, a few research teams focused on whole number learning. For example, Samantha Morrison's study[1] focused on "representational flexibility linked to higher attainment in early number learning," and Ola Helenius' group[4] presented a case against coherence in mathematics instruction. Second, Tshesane[3] talked about the South African learner's patterns of performance on additive word problems.

A second theme was multiplicative thinking and reasoning. In particular, Tzur and Ding[7] argued that there was a universal cognitive core in multiplicative reasoning cross-culture (USA and China). Furthermore, they used quantitative data to support that Same-Unit Coordination is the screener in students' multiplicative reasoning and place value concept in base ten. Finally, Corso and her group presented two related papers[8,9], one reported on the quantitative results of 2nd and 4th grade Brazilian students' flexible mental calculation, and the other one described the characteristics of the flexibility in mental calculation of a group of Brazilian elementary students.

Finally, several research teams explored the teaching and learning of fractions. Powell discussed how to improve students' knowledge of fraction magnitude in the early grades. Fernandes and Koudogbo[14] studied the difficulties of learning the decimal positional numeration system and talked about the principle of exchange. They also discussed the decimal number system in the Quebec mathematics program and textbooks.

3. Areas for Future Research

On the final day of the conference, the participants discussed potential future research topics and recommended some papers to publish in related journals.

https://doi.org/10.1142/9789811287152_0030

Topic Study Group 6

Teaching and Learning of Algebra at Primary Level

Jodie Hunter[1], Doris Jeannotte[2], Eric Knuth[3], Ann Gervasoni[4], and Xiaoyan Zhao[5]

1. Aims of the TSG

TSG-6 had a key aim of bringing together a variety of mathematics educators with research interests in early algebra (working with students up to 12 years old). The key overall focus of the group was examining the characteristics and nature of algebraic teaching and learning. This included but was not restricted to the study of numbers, operations, and properties in the context of early algebra, reasoning about functional relationships, the study of structure, and processes related to early algebra such as making conjectures, justifying, generalizing, and developing age-appropriate forms of proof. We called for submissions that offered empirical and/or theoretical contributions and also examined the teaching and learning of early algebra across a wide range of contexts including teacher education, different cultural settings, and with different groups of students.

During the presentations for TSG, we worked collectively to explore themes as well as similarities and differences in relation to the methodologies, theories, and results of both the empirical and theoretical contributions. Overall, we attended to four inter-related areas which included 1) Characteristics and nature of algebraic thinking and reasoning including across different mathematical strands and contexts; 2) Classroom culture and the role of the teacher (including pedagogical practices) in fostering early algebraic thinking for all students; 3) Nature of teacher education and professional development that supports teachers' capacity to foster early algebraic reasoning in classrooms and; 4) analysis of forms of curricular activity that support early algebraic reasoning.

1.1. Submissions

We received 27 submissions from 14 countries (South America: 1; North America: 9; Asia: 6; Middle East: 1; Europe: 6; Australasia: 4) which represented a diverse cultural spread from different international settings. We accepted twenty-three as paper presentations (including both long and short paper presentations) and two as poster presentations, and one submission was rejected. This included three invited paper presentations.

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1.2. Sessions

TSG-6 hosted three sessions for presentations which were led and moderated by different members of the TSG organisers given the time difference in the respective locations. Each session followed a similar structure beginning with an introduction from the TSG convening group chair and short oral presentations (10 minutes) with questions, answers, and discussion and then long presentations (15 minutes) or an invited presentation (25 minutes) with questions, answers, and discussion. Over the three sessions, we engaged all participants in a short collective discussion at the end of the session to foster networking opportunities along with identifying themes across the research field and to reflect on potential new ideas or existing gaps in the research field.

1.3. Paper topics

Of the 23 accepted papers, only 15 papers were able to be presented during the online conference. A list of these papers and authors are included in order of presentation and are organized in Tab. 1.

Tab. 1. List of papers presented

Papers and author(s)
[1] Mathematical learning disabilities in algebra. ***Francesca Gregorio*** (Switzerland).
[2] The pedagogical journey from arithmetic to algebraic reasoning in a professional development project through the theme of fractions. ***Yuriko Yamamoto Baldin*** *and Aparecida Francisco de Silva* (Brazil).
[3] Generalizing about odd and even numbers. ***Susanne Strachota***, ***Karisma Morton***, *Ranza Veltri Torres, Ana Stephens, Yewon Sung, Angela Murphy Gardiner, Maria Blanton, Rena Stroud, and Eric Knuth* (USA).
[4] Toward a common view of algebraic thinking through design of resources by primary and secondary teachers. ***Jana Trgalová***, *Mohammad Dames Alturkmani, and Sophie Roubin* (France).
[5] Cognitive routes of algebraic thinking in pre-school and elementary school: Literature review. ***Passaro Valeriane***, ***Elena Polotskaia***, *and Azadeh Javaherpour* (Canada).
[6] Highlighting the potential for developing early algebraic thinking: A praxeological framework of reference. ***Doris Jeannotte***, *Hassane Squalli and Virginie Robert* (Canada).
[7] Development and implementation of the unit of pattern and correspondence to foster functional thinking. ***Jeongsuk Pang*** (invited speaker) *and Sunwoo Jin* (South Korea).
[8] The relation between the evolution of generalization and the development of relational thinking and functional thinking: a study with grade 4 students. ***Celia Maria Mestre*** (Portugal).
[9] Enhancing elementary teachers' functional thinking. ***Ahmad Reza Haghighi*** *and Nasim Asghary* (Iran).
[10] Arithmetic problems with natural numbers in a multi-grade primary school. ***Lorena Trejo Guerrero*** (Mexico).
[11] Investigating early algebraic thinking in primary school: An empirical study from China. ***Siyu Sun*** (China).
[12] Multiplication and division problems as a context for developing young children's algebraic thinking. ***Ann Gervasoni*** *and* ***Anne Roche*** (Australia).
[13] Young students noticing and generalising growing pattern tasks. ***Jodie Louise Miller*** *and Jodie Hunter* (Australia).
[14] Designing an evidence-based learning progression for algebraic reasoning. ***Lorraine Day***, *Max Stephens, Marj Horne, and Derek Hurrell* (Canada).
[15] Fraction tasks which identify algebraic reasoning. ***Catherine Anne Pearn***, *Max Stephens, and Robyn Pierce* (Australia).

In addition, we had two posters presented during the online conference. A list of the posters and authors are included in Tab. 2.

Tab. 2. List of posters presented

Poster and author
[16] Reasoning with patterning tasks. ***Adam Ross Scharfenberger*** (USA).
[17] Research to improve education guidelines for promoting children's understanding of mathematical functions. ***Yoshiki Nisawa*** (Japan).

2. Conference Themes

A review of the conference papers and presentations highlights a range of key themes which were evidence. The first theme is encompassed by the characteristics and nature of algebraic thinking and reasoning including across different mathematical strands and contexts. Passaro and colleagues[5] reported on an analytical literature review to distinguish the approaches identified in research studies to facilitate algebraic reasoning in elementary classrooms. They identified three key approaches across the literature. First, an emphasis on grounding the teaching of algebra in the arithmetic knowledge that students develop. Second, developing tasks to introduce algebraic topics not covered by arithmetic. Third, developing student understanding of quantitative relationships and general laws.

A growing area of interest is the opportunities for early algebraic reasoning across different strands of mathematics. This was represented by a paper which focused on the teaching and learning of algebra through fractions. Pearn and colleagues[15] investigated students' responses to three reverse fraction tasks and from this developed a classification scheme for students' written responses. They argue that students' generalizations related to the strategies that students used to solve the task. Similarly, Guerrero[10] examined how arithmetic problems support students to engage in argumentation and develop understandings of natural numbers. Another paper by Sun[11] highlighted the capability of students from different grade levels in China in solving algebraic tasks. All students demonstrated strengths in in generalized arithmetic tasks, however, older students were more likely to use symbolic representation.

The second key theme identified across the conference papers was the relationship between task design and differing types of algebraic reasoning. Within this theme, there continues to be ongoing interest in the types of reasoning and generalization that students use when asked to solve patterning tasks. Two different presentations, one a poster by Scharfenberger[16] and the second a paper by Mestre[8] illustrate the importance of considering task design and characteristics when supporting students to engage in early algebraic reasoning. Scharfenberger developed a clinical interview using TIMMS items to analyse student responses to patterning tasks. He highlighted that dependent on the context or representation of the pattern, students were provided with differing opportunities to engage in_recursive, covariation, and correspondence thinking. Mestre also focused on generalization and used data from a teaching experiment to analyze the relation between the development of generalization and the development of relational thinking and functional thinking. This study found that task characteristics had the potential for both acting as an enabling factor or barrier to the development of algebraic reasoning. Specifically, pattern exploration tasks and those that focused on relational thinking contributed to functional thinking development.

Also drawing on the theme of task design, a number of papers focused on both the task characteristics and the use of materials to develop algebraic reasoning. Baldin and de Silva[2] focused on using fractions and concrete material to develop algebraic reasoning. These researchers highlight the value of manipulatives in supporting student understanding of operations (multiplication and division) with fractions. This has some overlap, with the work by Gervasoni and Roche[12] who examined multiplication and division problems as a context for developing young students' early algebra understandings. They highlight the important role of models to support students to shift to seeing structure when solving problems. Additionally, Strachota and colleagues[3] examine both task design and identify the affordances of specific tools (concrete material) to support students to make generalizations about odd and even numbers.

Building further on the theme of task design was the facilitation of professional development to support teachers to design tasks and resources. Two papers focused on different aspects of professional development. Firstly, Haghighi and Asghary[9] reported on an intervention that supported teacher capacity to develop and implement resources to support functional thinking in the classroom. Secondly, Trgalová and colleagues[4] worked collaboratively with primary and secondary teachers to design, implement and re-design tasks. Both studies highlight how professional development can support teachers to develop a shared view of algebraic reasoning and the types of classroom activity that support this.

Another key theme was focused on frameworks of learning progressions and the associated teaching approaches to develop algebraic reasoning. Jeannotte et al.[6] highlights the approach of some countries which do not explicitly introduce the development of algebraic thinking in primary school. Their research instead introduces a framework to be used to analyse the potential for developing algebraic thinking in curricula such as textbooks. In contrast, Day and her colleagues[14] report on the explicit development of an evidence-based learning progression across different aspects of early algebra. This included teaching advice designed to support teachers to use a targeted teaching approach to move students along the progression. Three of the papers focused on functional thinking as a route to foster early algebraic thinking. For example, Pang and Jin[7] highlighted the development of a pattern and correspondence unit in a Korean elementary mathematics textbook and the resulting student thinking. Miller and Hunter[13] examined how young students notice structure in growing patterns and the teaching actions that supported this and Nisawa[17] developed a learning framework to support students' understanding of mathematical functions both with the use of numerical values and in other activities without numerical values.

Finally, a key aspect of curricular activity that supports early algebraic reasoning is providing access to all students to engage in early algebra. Gregorio[1] highlights the lack of equity for specific groups of students including those with mathematical learning difficulties. This research focused on classification of different types of algebra tasks and the development of an assessment tool to identify students with difficulties in accessing algebraic concepts.

3. Areas for Future Research

Across the conference paper and presentations, there continues to be potential for further research to address existing gaps in the field. Firstly, future research could include a focus on diverse and marginalized learners including students from different cultural backgrounds or students with specific learning needs including learning difficulties. Also of interest is a focus on specific teacher actions beyond those related to task design and enactment. This could include questioning and prompts, specific classroom practice, and formative assessment methods that facilitate or support student development of early algebra.

We note that at the conference, there were a number of areas that were under-represented. This includes the use of digital tools such as virtual manipulatives in developing early algebraic thinking. Additionally, while a number of papers focused on task design, learning trajectories or curriculum development, there is a lack of work that focuses on cross-national comparative analyses. Finally, an important addition to the research field would be longitudinal studies which analyse the impact of early algebraic thinking on students' later study of algebra or mathematics in general.

https://doi.org/10.1142/9789811287152_0031

Topic Study Group 7

Teaching and Learning of Algebra at Secondary Level

Boon Liang Chua[1]

1. Scope and Focus of TSG-7

The teaching and learning of algebra at the secondary level is a well-researched field. TSG-7 aims to bring together international researchers, teacher educators and teachers who investigate students' ways of doing, thinking and talking about algebra, and investigate teachers' ways of designing and implementing the teaching of algebra at the secondary level and the knowledge needed to support effective algebra student learning. The group envisages integrating young researchers and established scholars in the field with the intention of sharing new findings and current research trends in the teaching and learning of algebra at the secondary level. In addition, we aim to foster discussion of theoretical and methodological issues challenging the field. The topic study group will engage a group of interested participants in rigorous discussions emphasising the following themes:

- Algebraic thinking: defining and characterising algebraic thinking in students; issues of representation, symbolisation, and manipulation, and how algebraic thinking is identified and assessed; relationships between conceptual and procedural knowledge of algebra; and, how students progress from arithmetical to algebraic thinking
- Proving and justifying: Their role in the learning of algebra; ways of characterising and understanding their features and processes (e.g., in expressing generality); and, socio-mathematical norms and didactical contracts associated with generalising, proving and justifying
- Mathematical tasks: Principles of task design aiming at developing algebraic thinking, and analysis of algebraic tasks used as instruments in classroom research
- Relationships between teacher knowledge, teaching practice, and student learning: Mathematical knowledge for teaching algebra; classroom practices that support algebra learning and their connection to teacher knowledge; links between teacher practice and changes in student learning of algebra

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- Teaching experiments and design research studies: Conditions that enable or hinder the teaching and learning of algebraic thinking; how new teaching and learning opportunities (e.g., the role of technology, the design principles used) are created and studied in terms of their impact on teachers, students and other actors; the classroom discourse during the teaching experiment; and, how the transition of research into practice is studied

2. Submission to TSG-7

Tab. 1 below shows the number of papers submitted and accepted for presentation.

Tab. 1. Number of papers submitted and accepted for presentation

Number of papers	Received	Accepted
Total	30	17
Long oral		5
Short oral		10
Poster		2

The 30 papers came from 17 countries. Tab. 2 shows the breakdown of papers by countries.

Tab. 2. Breakdown of papers by countries

Country	Number of papers	Country	Number of papers
Australia	1	Luxembourg	1
Belgium	1	Malaysia	1
China	4	Nepal	2
Indonesia	1	Norway	1
Iran	2	Panama	1
Israel	2	Russia	1
Japan	1	Rwanda	1
South Korea	1	South Africa	2
		USA	7

3. Program Overview

During the congress in July 2021, three sessions were organised for TSG-7. Of the 17 accepted papers, there were eventually 4 long oral presentations, 8 short oral presentations and 1 invited talk (Tab. 3 on the next page).

4. Future Directions and Suggestions

Many studies in the literature on teaching and learning of algebra at the secondary level had investigated students' learning difficulties and misconceptions. There is already much for researchers in different countries to learn from these existing studies. Instead

Tab. 3. The list of papers presented

Paper and author(s)
Session 1
[1] Knowledge for teaching algebra: variation in the use of knowledge in the light of classroom constraints. ***Demonty Isabelle*** (Belgium) *and Vlassis Joëlle* (Luxembourg).
[2] Constructing the link between graphical visualization and algebraic computation by means of analogy: the case of a system of equations. ***Klila Copperman*** *and natoli Kouropatov* (Israel).
[3] Using an online card game-based activity to build algebra foundation. ***Jiqing Sun*** (Australia).
[4] Investigating students' algebraic proficiency from a symbol sense perspective. ***Al Jupri*** (Indonesia).
[5] Diagnosis and treatment of students' algebraic misconceptions and errors. ***Mukunda Prakash Kshetree*** (Nepal).
[6] Examining the quality of classroom interactions in the teaching of algebra for upper secondary schools. ***Aline Dorimana****, Alphonse Uworwabayeho, and Gabriel Nizeyimana* (Rwanda).
Session 2
[7] Generalization as a marker for robust mathematical meanings among in-service algebra teachers. ***Lori Burch*** *and Erik Tillema* (USA).
[8] Student knowledge of exponential functions. ***Robert Powers****, Alees Lee, Melissa Troudt, and Jodie Novakl* (USA).
[9] The importance of teacher–student interactions in mathematical learning: the example of generalization. ***Vlassis Joëlle*** (Luxembourg) *and Demonty Isabelle* (Belgium).
[10] Learners' number patterns generalizations in a south African evaluative assessment. ***Zwelithini Dhlamini*** (South Africa).
Session 3
[11] Thinking about algebra from the anthropological theory of the didactic: reference models for the analysis and the design. ***Noemí Ruiz-Munzón****, Marianna Bosch, and Josep Gascón* (Spain).
[12] Students' unconventional graphical representations of covariational reasoning. ***Laurie Cavey****, Tatia Totorica, and Patrick Lowenthal* (USA).
[13] The impact of an online learning platform in algebra. ***Zachary Stepp*** (USA).

of duplicating some of these studies on students in different countries, perhaps, in order to add new knowledge to this field, future studies could examine students' learning (a) using different theoretical lenses, for example, through a constructivist lens, (b) using different concrete manipulatives or technological tools or approaches that might improve and deepen students' understanding of algebraic concepts and procedures, and (c) in a different learning environment, for example, a virtual setting which becomes vital during the COVID-19 pandemic. For the teaching of algebra, future studies could consider examining the "invisible" professional thinking of teachers behind their teaching practices. One such potential aspect for research is to capture teachers' pedagogical reasoning and actions.

https://doi.org/10.1142/9789811287152_0032

Topic Study Group 8

Teaching and Learning of Geometry at Primary Level

Nathalie Sinclair[1], Michael Battista[2], Eszter Herendiné-Kónya[3], Haiyue Jin[4], and Jesús Victoria Flores Salazar[5]

1. Aims of the TSG

This group provided a forum for discussion of the learning and teaching of geometry, with a focus on the elementary grades, K-6 (or preK-8). We had short presentations on, and discussions of, important new trends and developments in research or practice, in geometry teaching and learning, and expositions of outstanding recent contributions to it.

The focus of the group was on theoretical, empirical or developmental issues related to the themes below. The issues raised were considered from the historical, epistemological and ontological, cognitive and semiotic, and educational points of view. The following subthemes are proposed:

- Studies of the use of new/alternate geometry curricula or curriculum components (including topological ideas, ethnomathematical approaches, etc.);
- The use of tools/resources such as physical manipulatives (e.g., pattern blocks, cubes, paper folding, mirrors) and digital technologies;
- Problem solving in geometric contexts;
- Task design for the teaching and learning of elementary geometry;
- Explanation, argumentation, and proof in elementary geometry education;
- Spatial and geometric reasoning about two- and three-dimensional shapes;
- Psychological roots of spatial, visual and geometrical thinking;
- The role of geometrical transformations in learning and teaching geometry;
- Teacher knowledge and preparation in geometry education.

1.1. Submissions

We received 16 submissions from 12 countries (Canada 1, China 2, Japan 2, Great Britain 1, France 1, Austria 1, Denmark 1, Switzerland 1, Hungary 1, Turkey 2, Australia 1, New Zealand 2), thus reaching our goal of diverse cultural representation. Of those 16 submissions, 8 were accepted as long orals and 8 as short orals.

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1.2. Sessions

In our first 90-minute session, the TSG co-Chair, Michael Battista, introduced the rest of the Team and described the format of the sessions. The second 90-minute session was introduced by co-Chair Nathalie Sinclair. The third 120-minute session was introduced by TSG-8 members Eszter Kónya and Haiyue Jin. Generally, all three sessions led with two or three long oral presentation and discussion and two short oral presentations with a short collective discussion. Throughout the three days, we encouraged participants to engage in discussion around each presentation, as well as across the presented papers.

1.3. Paper Topics

Of the 16 accepted papers, only 13 papers were able to be presented during the online conference. There were seven long oral presentations and six short oral presentations. A list of these papers and authors are included in order of presentation and are organized in Tab. 1.

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Mathematical knots as a teaching material to improve pupils spatial abilities in elementary school. ***Tomoko Yanagimoto****, Akiyo Higasio, Madoka Koyama, Hisashi KinoShita, and Moe Miyazaki* (Japan).
[2]	The transition from informal to formal area measurement. ***Eszter Herendiné-Kónya*** (Hungary).
[3]	Tilings and symmetry using the labosaique. *Paolo Bellingeri,* ***Emmanuelle Feaux De Lacroix****, Eric Reyssat, and Andre Sesboue* (France).
[4]	The knowledge to be taught: a novice mathematics teacher plans to teach quadrilaterals in 5th grade. ***Nazlı Akar*** *and Mine Işıksal Bostan* (Turkey).
[5]	Supporting the development of young children's spatial reasoning: insights from the math for young children (m4yc) project. ***Catherine Diane Bruce,*** *Zachary Hawes, and Tara C Flynn* (Canada).
[6]	The basic routines of spatial thinking and acting. ***Marion Zoggeler*** (Austria).
[7]	Developing spatial abilities and geometrical knowledge with use of a virtual city. *Jean-Luc Dorier and* ***Sylvia Coutat*** (Switzerland).
[8]	Understanding path descriptions in a Manhattan-like map — a comparison of German 2nd and 3rd graders. ***Elisabeth Mantel*** (Germany).
[9]	Impact of teacher professional learning on students geometric reasoning relating to prisms. ***Ann Patricia Downton*** (Australia).
[10]	Unpacking language in geometry lesson on shapes in a New Zealand multilingual primary class. ***Shweta Sharma*** (New Zealand).
[11]	Spatial visualisation reasoning about 2d representations of 3d geometrical shapes: the case of G4–6. ***Taro Fujita****, Yutaka Kondo, Hiroyuki Kumakura, Susumu Kunimune, and Keith Jones* (UK).
[12]	Exploring second graders performances on reading comprehension of mathematics picture book with words and no-word. ***Yan-Hong Chen*** (China).
[13]	Implementing the project-based approach in teaching the area of circle: An explorative study. ***Jinyu Yu*** (China).

2. Conference Themes

The highest proportion of papers focused on the subtheme of spatial reasoning, with two long orals and two short orals presented on the second day. While two of these papers related specifically to geometry, two others were broader in terms of their mathematics curriculum connection. It seems clear that spatial reasoning has become a significant area of attention, and that it can offer an interesting opportunity for connecting the arithmetic/algebra and geometry components of curriculum. The researchers all emphasized the strong correlations between spatial reasoning and success in school mathematics, while also pointing to the relative paucity of focus on spatial reasoning activities in most primary school mathematics classroom. The tasks studied across the four research presentations covered a variety of contexts (using physical materials as well as digital technology; occurring in both indoor and outdoor situations). The third day's presentations also included presentations on spatial reasoning skills. In addition, the role of everyday language in concept formation was discussed, with a special focus on multilingual classrooms. 2D-3D visualization, interpreting representations, and switching between representations were also discussed. The presentations also touched on related areas of teacher training beyond the teaching of geometry in primary schools.

We list below the main TSG-8 Meeting Themes/Research Questions

- How can instruction support the learning of geometric measurement and formulas relationally (conceptually) in a way that is integrated with procedural fluency?
- What ways of visualizing and representing (including verbally describing) enable students to understand the structure of 3D shapes?
- How is spatial structuring — in say, tiling and using isometries — related to knowledge of geometric properties?
- How does the depth of teacher knowledge of geometric content affect how they teach that content?
- How can mathematics education researchers productively investigate ideas in cognitive science spatial ability research in ways that reveal students' actual sense making in mathematics contexts? That is, how can mathematics education researchers (perhaps in collaboration with cognitive scientists) use qualitative research to elaborate and deepen the knowledge produced in spatial reasoning quantitative research?

3. Areas for Future Research

With respect to spatial reasoning, there has emerged a variety of ways of describing, characterizing and identifying instances of it. The conference offered an excellent opportunity for researchers to find some convergence and overlaps, and continued work in this direction would facilitate future research collaboration. While spatial reasoning is recognized as an important aspect of mathematical thinking, more research on how the spatial reasoning activities are used by teachers in ways that support specific conceptual development is warranted. Finally, greater attention to the theoretical framing of spatial reasoning would help researchers better understand how engaging visual and kinaesthetic ways of thinking can support learning — in particular, theories of embodied cognition could offer much to future research in this area.

As preschool and primary school children are already part of the alpha generation, it would be important to extend research on how digital manipulations can contribute to a better understanding of geometric concepts. What differences in cognitive development are caused when physical manipulative activities are replaced by digital applications in early childhood?

4. TSG-8 Future Research Ideas

Research investigating interrelationships between spatial and geometric reasoning is essential — unfortunately, it has been neglected in recent mathematics education research. Future research should include:

- detailed research that investigates, from a cognitive perspective, exactly how spatial reasoning supports geometric reasoning, and vice versa.
- an understanding of how visualization and geometric property knowledge become integrated to form abstract geometric reasoning that is grounded in students' real-world experiences, and how instruction can support this integration.
- investigating the role that mental models play in mathematical reasoning, with spatial reasoning playing a critical role in constructing and operating on mental models.
- going beyond general theories of relationships between spatial and geometric reasoning to detailed elaborations of spatial-geometric interrelationships for specific topics in geometry (which may lead back to general theories).
- investigating, with student interviews and teaching experiments, the nature of, and causes for, the correlation between spatial and mathematical reasoning found by cognitive psychologists, moving towards understanding underlying reasons for how spatial reasoning is related to, and supports, mathematics reasoning for various topics.
- investigating area measurement through decomposition and composition, that is, to consider the possibility of finding the measure of the area of a polygon from its reconfigurations. Also, the existence of other variables that favor the decomposition and composition operation, such as the use of the grid mesh, can be incorporated in other studies.
- carrying out more research about the learning of 3D geometry, because spatial geometry, deserves a different reflection and analysis, since drawing representations of three-dimensional figures, on the board or in paper, does not allow the student to fully observe the characteristics and properties.

https://doi.org/10.1142/9789811287152_0033

Topic Study Group 9

Teaching and Learning of Geometry at Secondary Level

Keith Jones[1], Matthias Ludwig[2], Liping Ding[3], Joris Mithalal[4], and Yiling Yao[5]

ABSTRACT At ICME-14, the Topic Study Group (TSG) on the teaching and learning of geometry at the secondary school level, TSG-9, enabled participants from around the world to share research results, research projects, new developments, and updates on ongoing projects concerning geometry education at the secondary school level. The TSG embraced the four themes of connections between geometry education and mathematical practices and processes, teacher preparation and teacher knowledge for geometry, developments in geometry teaching, and curricular issues in school geometry. The discussion during the TSG sessions at the congress benefitted from the good range of quality presentations on each of the themes.

Keywords: Teaching; Learning; Geometry; Secondary school.

1. TSG-9 Themes and Description

Geometry holds a major place within the secondary school mathematics curriculum. In preparing for the Topic Study Group (TSG) on the teaching and learning of geometry at the secondary school level, TSG-9, at ICME-14, the TSG-9 Team built on existing research and development on geometry education, including the equivalent TSG at ICME-13 (Herbst, Cheah, Richard and Jones, 2018) and reviews such as Jones and Tzekaki (2016), to identify a set of themes to guide and encourage contributions and discussions. The themes were as follows:

- Connections between secondary school geometry education and mathematical practices and processes such as argumentation and proof, visualization, figuration, and instrumentation;
- Teacher preparation and teacher knowledge for geometry at the secondary school level;

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- Developments in secondary school geometry teaching, including geometrical modeling and out-of-school problem solving;
- Curricular issues in secondary school geometry, including reform initiatives in school geometry, and new forms and applications of geometry.

In advance of ICME-14, TSG-9 received 43 submissions from all around the world, thereby providing diverse cultural representation that addressed the full range of the identified themes. Of the submitting participants, a number were impacted by the COVID-19 pandemic prevalent at the time and could not take part in the TSG at ICME-14. During the congress, 23 papers and 4 posters were presented in the final programme for TSG-9.

2. TSG-9 Programme

During the TSG-9 sessions, one part of the programme of presentations and discussion was devoted to each of the themes. Within three of the four themes, the TSG-9 Team identified one paper for extended presentation to enable deeper discussion. Each themed session was chaired by a member of the TSG-9 Team.

2.1. Geometry education and mathematical practices and processes

The theme of connections between secondary school geometry education and mathematical practices and processes (such as argumentation and proof, visualization, figuration, and instrumentation) comprised the four papers set out in Tab. 1.

Tab. 1. Geometry education and mathematical practices and processes

Paper and author(s)	
[1]	A teacher's use of dynamic digital technology to address students' misconceptions concerning the use of additive strategies within geometric similarity. ***Ali Simsek***, *Celia Hoyles, and Alison Clark-Wilson* (UK).
[2]	Students spatial ability and solving-strategies for spatial geometrical, mathematical, and physical task. ***Marion Zoeggeler*** *and Guenter J. Maresch* (Austria).
[3]	Introduction of an auxiliary element as a shift of attention. ***Alik Palatnik*** *and Avi Sigler* (Israel).
[4]	Construction program as a link between drawing and language to prepare proof process. ***Joris Mithalal*** (France).

Within this theme chaired by Joris Mithalal, the paper for extended presentation was by Alik Palatnik.

2.2. Teacher preparation and teacher knowledge for geometry

The theme of teacher preparation and teacher knowledge for geometry at the secondary school level comprised the four papers set out in Tab. 2.

Tab. 2. Teacher preparation and teacher knowledge for geometry

	Paper and author(s)
[5]	Understanding student teachers' mathematical knowledge for teaching geometry in a history of mathematics course. ***Svein Arne Sikko***, *Iveta Kohanová, Magdalini Lada, and Liping Ding* (Norway).
[6]	Teacher knowledge related to secondary school level geometry: Evidence from teacher development in South Africa. ***Jogymol Alex*** (South Africa).
[7]	A pre-service teacher mental structure development for understanding the geometric reflection in terms of motion and mapping view: Alexis case. ***Murat Akarsu*** (Turkey)
[8]	Distinguishing content knowledge and pedagogical content knowledge for geometry teaching. ***Liping Ding*** (Norway) *and Keith Jones* (UK).

Within this theme chaired by Liping Ding, each presentation was allotted the same time.

2.3. Developments in geometry teaching

The theme of developments in secondary school geometry teaching (including geometrical modeling and out-of-school problem solving) comprised the eight papers set out in Tab. 3.

Tab. 3. Developments in geometry teaching

	Paper and author(s)
[9]	Possibility of the pirates' treasure problem for teaching elementary geometry. Satoshi Takahashi, ***Ryoto Hakamata*** *and Koji Otaki* (Japan).
[10]	Inquiry-based learning using the centroids of the circumscribed equilateral triangles. ***Yuki Osawa*** (Japan)
[11]	Study of angles and trigonometric ratio for 7th grade. ***Tsuyoshi Sonoda*** (Japan).
[12]	Decomposing proof in secondary classrooms: A promising intervention for school geometry. ***Michelle Cirillo*** (USA).
[13]	Distance under the magnifying glass: Developing series of problems around fundamental concepts in geometry. ***Eszter Varga*** (Hungary).
[14]	The grasp of the Pythagorean Theorem and its proof by Chinese pre-service mathematics teachers. ***Hai Li*** (China).
[15]	Implicative relationships among spatial perception, mental rotation and spatial visualisation: Implications for teaching geometry. ***Melih Turgut*** *and Iveta Kohanová* (Norway).
[16]	Geometry modelling outdoors with MATHCITYMAP. ***Matthias Ludwig***, *Iwan Gurjanow, Simone Jablonski, and Moritz Baumann-Wehner* (Germany).

Within this theme chaired by Matthias Ludwig, the paper for extended presentation was by Michelle Cirillo.

2.4. Curricular issues in secondary school geometry

The theme of curricular issues in secondary school geometry (including reform initiatives in school geometry, and new forms and applications of geometry) comprised the seven papers set out in Tab. 4.

Tab. 4. Curricular issues in secondary school geometry

Paper and author(s)	
[17]	Online formative assessment in geometry proving. ***Yael Luz*** *and Michal Yerushalmy* (Israel).
[18]	Geometric reasoning and mechanics experiment: A case study of interdisciplinary integration teaching with graphic center of gravity as an example. ***Feishi Gu****, Zhenzhen He, and Liya Ban* (China).
[19]	A study on the performance of seventh-grade students in mathematical spatial reasoning. ***Zhikun Zhang*** *and Jian Liu* (China).
[20]	Didactic suitability characterization of three levels of achievement on geometric drawing of secondary school students. ***Javier Díez-Palomar*** *and Elvira García-Mora* (Spain).
[21]	Let's make a circle by three persons. ***Ken-ichi Iwase*** (Japan).
[22]	Reconfiguration of polygons to determine the measurement of their area. ***Melissa Denisse Castillo Medrano*** *and Jesus Victoria Flores Salazar* (Peru).
[23]	High school learners' preconceptions on the classification of quadrilaterals. ***Judah Paul Makonye*** (South Africa).

Within this theme chaired by Keith Jones, the paper for extended presentation was by Yael Luz.

3. Summary Discussion and Future Directions

Topic Study Group 9 (TSG-9) at ICME-14 brought together participants from around the world to share research results, research projects, new developments, and updates on ongoing projects concerning geometry education at the secondary school level. The discussion at TSG-9 benefitted from the good range of quality presentations on each of the themes.

The discussion during the theme of connections between secondary school geometry education and mathematical practices and processes ranged from considering the role of digital technologies in supporting this connection to the place of spatial reasoning and geometrical construction. Future directions are likely to continue to be on argumentation and proof in school geometry, and on visualization, figuration, and instrumentation processes.

During the theme of teacher preparation and teacher knowledge for geometry at the secondary school level, discussions focused primarily on teacher knowledge for secondary school geometry teaching. Future directions are likely to continue to be on such teacher knowledge and on the design of geometry education teacher development both pre-service and in-service.

The theme of developments in secondary school geometry teaching was wide ranging and addressed geometry teaching concerns from spatial perception and visualisation to geometric proof. The presentation of teaching ideas was a strong and beneficial element. Future directions are likely to continue to be on the teaching of spatial and geometrical reasoning, along with geometrical modeling and out-of-school problem solving.

The theme of curricular issues in secondary school geometry was equally wide-ranging, with discussions enriched with ideas being used in schools alongside research into the secondary school geometry curriculum and its assessment. Future directions are likely to continue to be on the scope of the secondary school geometry curriculum, including reform initiatives and possible ways of incorporating new forms and applications of geometry that occur in mathematics.

Acknowledgments

The TSG-9 Team began as Keith Jones (Chair), Matthias Ludwig (Co-chair), together with Liping Ding, Flordeliza Francisco, and Joris Mithalal. During the preparation work for the TSG that took place at the time of the COVID-19 pandemic, Flordeliza Francisco had to withdraw. The TSG-9 Team thanks her for her efforts and thanks Yiling Yao for stepping in to join the Team and for playing such a helpful role during the presentations at the congress. The TSG-9 Team is especially grateful for all the submissions to the TSG, and most particularly to participants who presented so well and contributed so much to the success of the TSG. The TSG-9 Team thanks the ICME-14 International Programme Committee liaison person, Maria Alessandra Mariotti, and all the members of the ICME-14 organising committees for all their help in ensuring that TSG-9, and the entire congress, was such a success.

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https://doi.org/10.1142/9789811287152_0034

Topic Study Group 10

Teaching and Learning of Measurement

Christine Chambris[1], Richard Lehrer[2], Florent Gbaguidi[3], and Yuquan Wang[4]

ABSTRACT This chapter presents the aims, the work done during the online conference, and perspectives of TSG-10 — teaching and learning of measurement.

Keywords: Teaching and learning of measurement; Variability of phenomena; School arithmetic; Multiplicative reasoning; Interplay between space and measure; Interdisciplinary practices.

1. Aims of the TSG

Measurement topics in TSG-10 include typical domains such as length, area, angle, volume, and mass but also those less studied, such as time, and those commonly visited in science and engineering education. Overall, internationally, there seems to be a lack of attention to measurement instruction in mathematics education, especially at the primary levels. This is in spite of measure's links to everyday contexts and to STEM disciplines. Although the historic role of measurement has declined in some areas of mathematics, substantive mathematical ideas, such as number and quantity, originated in practices of measure, and these origins continue to be important for student learning about these ideas.

The main objective of the TSG is to better understand the conditions and constraints on teaching and learning measurement in international contexts (from primary to university levels) and to consider new approaches to learning both of measurement and of related forms of mathematics. A diversity of perspectives was expected, e.g., theoretical, methodological, historical, epistemological or empirical, and from various points of view, including teachers' practices, students' learning, as a mathematical subject, teacher education, curriculum, and so on.

1.1. Submissions

There were 18 submissions in the TSG. These submissions were from a diversity of locales (North America: 3; Asia: 5; Europe: 4; Africa: 3; Australia: 2; Eurasia: 1). Thirteen oral presentations and one poster were planned to be presented in Shanghai during the online conference (but eventually only twelve orals presented).

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1.2. Sessions

The TSG work was organized in three sessions: 90 minutes, 90 minutes, and 120 minutes. A brief presentation by the chair (Christine Chambris) introduced the work of the three sessions. She presented the different themes. Then, each presentation was followed by time for discussion. Last, Rich Lehrer drew perspectives for the TSG, at the end.

1.3. Paper list

Tab. 1. List of papers presented

	Paper and author(s)
[1]	Rethinking measure. *Petronilla Bonissoni, Marina Cazzola, Gianstefano Riva,* ***Ernesto Rottoli****, and Sonia Sorgato* (Italy).
[2]	Can length measurement estimation activities contribute to learners' improvement on number line estimation tasks? ***Pamela Vale*** (South Africa).
[3]	Measurement units and numeration units: What reveals the introduction of a "mixed" table in decimals teaching. *Christine Chambris,* ***Lalina Coulange****, and Grégory Train* (France).
[4]	Role of "error" in teaching-learning measurement. ***Ishan Santra*** *and Jeenath Rahaman* (India).
[5]	An investigation of teachers' explanatory talk when introducing standard units of measuring length to standard 4 learners in Malawi. *Liveness Mwale* (Malawi). (Due to connection issues this paper was unfortunately not presented)
[6]	Insight into pupils' errors in solving problems involving calendar dates through analysis of knowledge states. ***Phei Ling Tan*** *and Liew Kee Kor* (Malaysia).
[7]	Measuring the teacher's arm span: Interpreting a data modeling sequence through an aesthetic lens. ***Russell Tytler****, Peta White, and Joseph Ferguson* (Australia).
[8]	The use of geometric construction problems to solve measurement problems at middle school. ***Gbaguidi Ahonankpon Florent*** (Benin).
[9]	Conceiving volume as a multiplication of three quantities: the cases of Stan and Sloane. ***Samet Okumus*** (Turkey).
[10]	Articulations between mathematics and physics education: the concept(s) of unit of measurement, from geometry to formulas. ***Charlotte de Varent*** (France).
[11]	Dynamic measurement for area and volume. ***Nicole Panorkou*** (USA).
[12]	Teaching with clocks: Instrumental dynamics' effects on time learning. ***Chaereen Han*** *and Oh Nam Kwon* (South Korea).
[13]	Young students learning the mathematics of measurement through an interdisciplinary approach. ***Peta White****, Russell Tytler, Joanne Mulligan, and Melinda Kirk* (Australia).
[14]	An explorative study of using Picture books to support students' learning of measurement in primary education. *Lanjie Sun* (China). (Poster. Not presented during TSG sessions)

2. Conference Themes

Among the diversity of the topics in the papers to be presented, four main themes emerged that structured the sessions: relationships between numbers and units, negotiating measure and its meanings, interplay between conceptions of space and measure, and fostering development of measure through interdisciplinary practice.

In the first session, papers focused on relationships between numbers and units. For decades now, it is acknowledged in mathematics education that contrary to academic number sets constructions, teaching fractions and decimals should be based on measurement. That said, the specific roles that measurement units might play in teaching and learning mathematical concepts such as these were considered.

Rottoli and colleagues[1] discussed a perspective that views natural number as generated by counting and fraction as generated by comparison. In this perspective, the unit is a common unit between the two compared quantities obtained through subtractions. Vale[2] suggested that through preparing sticks of relative sizes to a referent (i.e., subunits), students develop knowledge in estimation on the number line. Chambris et al.[3] highlighted that poor understanding of units of measurement and of numeration impedes students' abilities to relate place value to metric units. They suggested that a multiplicative understanding of units is required for making this bridge.

The second session consisted of empirical studies of students and teachers that were negotiating measure and its meanings. Observing students and teacher's prompts, Santra and Rahaman[4] demonstrated missed and successful opportunities for "errors" to be a resource in teaching and learning of measurement, and specifically in stressing the variability of phenomena. Tytler and colleagues[7] collected a series of data involving grade-4 students measuring their teacher's arm-span and then constructing representations of the collection of measures. Analyses indicated how students' aesthetics of data display guided how they constructed their representations, and also how their teacher elicited these aesthetic judgments to help students understand some of the common conventions employed to visualize data. Tan and Kor[6] focused on students' errors in calendar learning, and this led to a discussion of relevant tasks to assess student knowledge about the measurement of time, including students' understandings of the meaning of time.

The third session focused on interplay between conceptions of space and measure. Florent[8] presented three geometrical problems that aimed to involve students in the construction of a measured quantity. Such problems seem to be promising for fostering reasoning in geometry to expand the meaning of a geometrical quantity. Okumus[9] investigated the students' understandings of the meaning of multiplication in the volume formula of a rectangular prism. Despite forming stacks of triangles, students were not able to relate the area of a triangle and the height of the stack to the measure of the volume of the prism. Panorkou[11] studied students' development in measurement reasoning mediated with dynamic geometry. She captured instances of meaning students generated by manipulating the software. Students seemed to perceive how dragging a surface area through a length generates volume. They also were able to identify units of area formed through dragging a certain distance a line of a given length, similarly for units of volume formed through dragging a surface of a given area through a particular distance. Students linked the latter to a volume formula (base × height) for a prism. Last, students seemed to reason in terms of continuous change: to make the volume n times bigger, they need to make the area of the base n times bigger. This also demonstrates co-variational reasoning in students. Opportunities for students to reason dynamically about area and volume could be very productive but are not yet incorporated into school curriculum. For instance, in a study of textbooks commonly used in France to teach concepts of measure, de Varent[10] found that length units and area units were related in only one of 13 textbooks in France, in the setting of the formula of the area rectangle. This questions not only coherence between geometry and arithmetic, but also how co-variation between the different quantities is taught. Regarding relationships between time and space, Han and Kwon[12] demonstrated how specific use of two kinds of educative devices — clocks with linked and independent hands — depending on teachers' competences in Korea, engaged students in co-variational reasoning or not.

Finally, the last theme of the TSG was about fostering development of measure through interdisciplinary practice. White and colleagues[13] gave an inspiring example of how interdisciplinary practice supported grade-1-and-2 students to learn robust mathematics of measurement. Based on ecological inquiry of their surroundings, students developed means of measuring, recording, and constructing a display of a class collection of measures within a pedagogical context that notably encouraged exploration of ideas and constructing consensus.

3. Summary, Future Directions and Suggestions

The fundamental problematic of measurement can be seen as inventing symbolic means for describing and quantifying attributes of objects and of quantitative relations among them. The relationship between measure of an attribute and understandings of the nature of the attribute are co-constituted. Thus, communities render measures and phenomena reproducible, so that individual acts of measure are guided and enlarged by a community that values reproduction and clarity about acts of measure. Expressions of and work with measures require development of means, notably more elaborate systems of units and of numbers.

Beyond these epistemological considerations, major teaching and learning issues in measurement appear. Reproducibility of phenomena raises questions about variability: Why are our measures varying? Variability expands the reach of measure, notably in terms of characterizing its sources, of reasoning about co-variation, and of important ideas like distribution. Numbers emerge from the need to record relations through comparison processes that involve units (i.e., a given attribute that can at time be chosen arbitrarily) that enable records of phenomena. Grounding measure in space both via construction, and symbolic means reflect interaction between structures of phenomena and measures. Yet, though promising experiences, analyses of actual learning or teaching process and resources reveal the need of continuing efforts in the domain.

This leads to future directions for research. Interdisciplinary practices appear to be a powerful means to develop meaningful practice of measure, both for students and teachers. Further research is needed that clarifies conditions of teaching and learning of practices and concepts of measure that foster deeper understanding of arithmetic, including multiplicative reasoning, and co-variational reasoning. There is also a critical need to develop longitudinal studies in order to better understand changes in students, and teachers. An emphasis on students' changing conceptions of measure suggests the need to develop means and tools for broader assessment, both for concepts to be learnt and teaching practices.

Generally speaking, TSG-10 had regular attendants who were ready to engage in rich discussion throughout the three sessions in a receptive atmosphere. The relatively small number of papers confirms that there is a lack of attention to this domain. Despite this, new issues in teaching and learning were raised, and the necessity for further international studies in the domain of measurement was reaffirmed. We hope that the topic study group dealing with measurement continues as a well-recognized group of the congress.

https://doi.org/10.1142/9789811287152_0035

Topic Study Group 11

Teaching and Learning of Probability

Emesta Sánchez[1], Sibel Kazak[2], and Egan J. Chernoff[3]

ABSTRACT An overview of the papers that were submitted and accepted to TSG-11 is presented. First, we present the document "Purposes and subthemes" with which we call for papers. Second, we make a very brief description of some features of the papers, organized by school level (primary, secondary and tertiary) and each of these divided according to whether they refer to students or teachers. Next, some relationships between the subthemes of the initial document and the topics that are actually present in the papers are pointed out. Finally, some recommendations are formulated.

Keywords: Probability; Probability education; Probabilistic thinking.

1. Purpose and Subthemes

The general aim of the Topic Study Group on Teaching and Learning of Probability (TSG-11) was to continue the relatively recent, albeit ever-growing trend of providing a dedicated venue to promote the discussion of a variety of perspectives related to probabilistic thinking and the learning and teaching of probability. TSG-11 at the 14th International Congress on Mathematical Education (ICME-14) attempted to provide an overview of the international discussion on probability education, as broadly as possible, by building upon the more recent literature from the field. Further, TSG-11 made every effort to display the progress of the discussion in the intervening years since ICME-13 and ICME-12. Lastly, we would be remiss not to mention that we, to the best of our ability, allowed for insight into less well-known strands of the discussion from researchers around the world, especially those from underrepresented countries. To meet these general and specific objectives, we identified five subthemes for TSG-11.

- *Conceptual frameworks to develop probabilistic thinking.* To continue the emerging creation of frameworks to describe or model the development and growth of probabilistic thinking of students especially at intermediate and tertiary levels. We recognized the importance of discussing models of students' process of integrating the different philosophical interpretations (e.g., classical, frequentist, subjective, and others) of probability.

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- *Connecting probability with statistics.* The development of probabilistic notions through experiments, data explorations and simulations can help students to build basic connections between statistics and probability, but it is required to understand how the process of concept formation emerges in students under such conditions.
- *The role of technology in teaching and learning probability.* The availability of increasingly powerful technology and software for statistical and probabilistic education requires, in addition to the inherent innovation, a parallel development of the theoretical reflection and conceptualization of empirical experiences.
- *Task design and learning trajectories.* One way to ensure that the knowledge accumulated by research in education in probability develops into educational practices is through the design of tasks and learning trajectories to promote the thinking and reasoning of integrated probabilistic concepts, including modeling processes.
- *Probabilistic knowledge for teaching.* Understanding and deepening the knowledge that teachers need to teach probability can help solve potential problems with their learning so that they provide a comprehensive education that includes probability. The availability of models that describe and conceptualize the probabilistic knowledge of teachers and their relationship with their teaching practice is important.

We, of course, welcomed submissions that fell outside the presented topics but within the teaching and learning of probability.

2. General Organization of Papers Presented

TSG-11 at ICME-14 had 21 presentations: three invited lectures of 20 minutes each, seven "long" presentations of 15 minutes and eleven "short presentations" of 10 minutes (Tab. 1 on the next page). Invited lecturers were: Amy Renelle, Stephanie Budgett and Rhys Jones who presented "A consideration of alternative sample spaces used in coin-toss problems"[10]; Vincent Martin, Mathieu Thibault and Marianne Homier presented "Self-reported practices of probability teaching: the use of the frequentist approach, manipulatives and technological tools"[21]; and, Gale Russell who presented "From towers of linking cubes to the binomial expansion theorem: what can be learned about combinatorics?"[18]. In what follows, we summarize the distribution of all 21 papers across three variables: Students/Teacher, School level (Primary, Secondary, Tertiary), and the use (or not) of Digital technology.

In addition to the first fifteen papers[1–15] on studies with students across different ages, the other six papers[16–21] involved pre- or in-service teachers. Parsing a bit further, that is, considering the school level at which the study is focused, the first 15 are distributed as follows: Two papers[1,2] refer to primary school students, nine[3–9; 13,14] to secondary school (middle and high school) and four[10–12;15] at the tertiary level. Of the remaining six, two papers[16,20] do refer to primary school teachers, three[17–19] to secondary school teachers, and one[21] to both primary and secondary school teachers. Five papers included technological resources in their research, two[13,14] were with high school students, one[15] with university students. Of the remaining two, one paper[20] involved pre-service teachers and the other[21] with primary and secondary in-service teachers. We hope that this context, presented in this manner, will help all interested readers better navigate our list of conference papers.

Tab. 1 Papers presented in TSG-11

Paper and author(s)	
[1]	The emerging interplay between subjective and objective notions of probability in young children. ***Sibel Kazak*** (Turkey) *and Aisling Leavy* (Ireland)
[2]	Children's apatial cognitive strategies and their development from the perspective of microgenesis. *Zikun Gong and* ***Du Zhang*** (China).
[3]	Developing a learning progression for probability based on the GDINA model in China. ***Shengnan Bai****, Jiwei Han, Kaijun Zhang, and Xueming Gao* (China).
[4]	How can probability reasoning protect adolescents from problem gambling? ***Catterine Primi*** *and Maria Anna Donati* (Italy).
[5]	Confidence and competence of Indonesian secondary school students in completing probability tasks: findings from a pilot study. ***Bustang Bustang*** (UK).
[6]	Problem sequences for developing two basic notions: probability and expected value in Hungarian secondary schools. ***Öedoen Vancsó*** *and Eszter Varga* (Hungary).
[7]	The frequentist approach of probability, from random experiment to sampling fluctuation. ***Jannick Trunkenwald*** (France)*, Fernand Malonga-Moungabio* (Congo)*, and Dominique Laval* (France).
[8]	Secondary school students' strategies in solving permutation problems. ***Luca Lamanna*** (Italy)*, Magdalena M. Gea-Serrano* (Spain)*, and Carmen Batanero* (Spain).
[9]	Establishing connections between language and probabilistic notions through a wodb task. ***Maria Ricart****, Pablo Beltrán-Pellicer, and Assumpta Estrada* (Spain).
[10]	A consideration of alternative sample spaces used in coin-toss problems. ***Amy Renelle****, Stephanie Budgett, and Rhys Jones* (New Zealand).
[11]	Is it in the cards? Revealing consequential probability. ***Egan J. Chernoff****, Nat Banting, and Ryan Banow* (Canada).
[12]	Use of the empirical rule in the course of probability: an application proposed by students. ***Beatriz A. Rodríguez González****, Omar Alejandro Guirette Barbosa, Gabriela Noemi Figueroa Ibarra, Hector Antonio Durán Muñoz* (Mexico)*, and Difariney González Gómez* (Colombia).
[13]	High-school students' probabilistic reasoning when working with random intervals. *Sandra A. Martínez Pérez and* ***Ernesto Sánchez*** (Mexico).
[14]	The computer simulation as a resource to teach normal distribution. ***Jesús Salinas*** *and Julio César Valdez* (Mexico).
[15]	Modeling eliciting activities for the teaching of the probability in a computational environment. ***Sandiago Inzunza*** (Mexico).
[16]	Alice in randomland: differences in attitudes of future primary school teachers towards probability and its teaching. *Claudia Vásquez,* ***Flavio Guiñez****, Camila Brito, and Salomé Martínez* (Chile).
[17]	Teachers' epistemological assumptions that tend to govern their pedagogy while teaching probability. ***Haneet Gandhi*** (India).
[18]	Concretely developing the binomial expansion theorem: where did the permutations go? ***Gale Russell*** (Canada).
[19]	The mathematical work of secondary teachers in the domain of probability in Chile. ***Katherine Machuca Pérez*** (Chile).
[20]	Understanding elements of a randomization test. ***Susanne Podworny*** (Germany).
[21]	Self-reported practices of probability teaching: the use of manipulatives and technological tools. ***Vincent Martin****,* ***Mathieu Thibault****, and* ***Marianne Homier*** (Canada)

3. Brief Indications on the Topics of the Papers

Regarding the research with primary students, Kazak and Leavy[1] focused on the children's estimations of the likelihood of outcomes from chance experiments observing the interplay between subjective and objective notions of probability, and Gong and Zhang[2] addressed emerging cognitive strategies when children face sample space tasks, as well as how and how quickly they develop. The studies with pre- or in-service primary teachers focused on different topics: Vásquez et al.[16] on attitudes towards probability and its teaching; Martin et al.[21] on the self-reported practices of how teachers use the frequentist approach to probability, manipulatives, and technology in their teaching of probability; and Podworny[20] on the understanding and difficulties about the elements of randomization test. We would also note that these last two do include technology, one asking teachers how they use it and the other using computer simulations.

Considering the secondary level, Bai et al.[3] focused on developing a learning progression of probability for 7–11th grade students by using a diagnosis test of 26 items administered to 1490 Chinese students. Primi and Donati[4] reported on developing and evaluation of a school-based preventive intervention aimed to modify gambling-related distortions on at-risk adolescents, focusing the training activity on the concept of probability. Bustang[5] investigated the confidence and competence of Indonesian high school students and wonders if the biases and misconceptions that affect Western students are also present in other cultures or if the culture of the students affects their probabilistic reasoning. Vancsó and Varga[6] proposed a sequence of problems in the context of betting to develop the secondary students' notions of probability and expected value. Trunkenwald[7] pondered on students' understanding of the frequentist approach to probability, particularly, the relationship of the empirical observation of frequencies fluctuation with the idea of measuring a probability. Lamanna et al.[8] explored the effect of instruction in combinatorial reasoning of secondary school students in Italy by analyzing the students' strategies in solving two permutation problems with and without instruction. Ricart et al.[9] used the technique "Which One Doesn't Belong? (WODB)" to explore, through the mathematical vocabulary used, the ideas that students at different educational levels have about probabilistic notions. Martínez and Sanchez[13] reported on a design experiment to introduce the concept of random intervals from a frequentist approach with the aid of a technology tool and observe, in this context, the students' reasoning for making sense to frequentist approach of probability. Their paper concerned the potential of the use of technology to explore the high school students' reasoning when using the software to understand the normal distribution.

Considering teachers at high school, Gandhi[17] raised questions about the epistemological assumptions with which high school teachers approach the curricular material in their probability classes and the way in which epistemological approaches to probability become part of their pedagogy. The paper explains the evolution of an activity, carried out with pre-service teachers, consisting of starting from towers of linked cubes to arrive at the binomial expansion theorem and what can be learned with it about combinatorics. Machuca[19] presented the design of research project for secondary teachers in which mathematic activity in front of probability modelling tasks will be developed and she wonders about the features of teachers' work when they solve the tasks.

Regarding tertiary level students, Renelle et al.[10] told us about alternative sample spaces used by participants in coin-toss problems and the paper is a reflection on whether one sequence could be more likely depending on the interpretation of the question. Chernoff at al.[11] recounted how different ways of analyzing a problem by students, leads to explore and quarrel the probabilities stemming from a simple standard deck of cards sitting on a table. Rodríguez et al.[12] proposed a perspective on problem posing in probability and statistics that involves the topics of experimental probability, the empirical rule and hypothesis testing with the aim of developing creativity and learning skills of students. And Salinas and Valdez[14] developed modelling eliciting activities with their students and asks: What challenges university students face with a model and computer simulation approach? How is their reasoning when they interact and build statistical models?

Having presented brief indications on the topics of the papers, we now switch our focus. Relationships that emerged with initial subthemes are now presented.

4. Some Relationships with Initial Subthemes

Conceptual frameworks to develop probabilistic thinking. All the papers are based on some theoretical considerations, but only a few[13–15,19] include a section of theoretical or conceptual framework. Gong and Zhang[2] built a four-level hierarchy of children's cognitive strategies, whereas others mention some conceptual framework in the introduction section or in the method. Two papers[19,14] stand out, in that the former mentions the Mathematical Working Space (MWS), and the latter addresses a documentational approach to didactics.

Connecting probability with statistics. Four papers address topics related to statistics: Podworny[20] carried out activities to understand the randomization test technique; Rodríguez et al.[12] reported on the use of the empirical rule in a probability course; in the presentation of Martínez and Sanchez[13] the students solved a problem of random intervals that later can be related to confidence intervals; and Salinas and Valdez[14] showed how to approximate the normal distribution with the help of technology.

The role of technology in teaching and learning probability. Six papers included, in the investigations they report, some use of technology, but only in the case of paper[14] a question is asked about the role it plays in learning; in the other cases it has an ancillary function. Three papers[1,15,20] utilize *TinkerPlots* and two papers[13,14] use *Fathom*. Of note, a paper[21] is included, albeit indirectly because the paper asks teachers about how they use technology in teaching.

Task design and learning trajectories. Vancsó and E. Varga[6] proposed a series of betting problems to promote the understanding of probability and expected value. Podworny[20] mentioned a learning trajectory for inferential reasoning with randomization tests. Others use problems in their investigations. For example, Kazak and Leavy[1] proposed a task to subjectively evaluate the probabilities, Ricart et al.[9] explored students' probabilistic language using the WODB technique, and Chernoff et al.[11] promoted a quarrel about the probabilities stemming from a deck of cards problem.

Probabilistic knowledge for teaching. In six papers the object of study was the probabilistic knowledge of teachers. Martin et al.[21] studied the self-reported practices of teaching by primary and secondary teachers. Gandhi[17] explored the assumptions that teachers adopt in their classes on the curricular material and the role of the three epistemological approaches of probability in their pedagogy. Vásquez et al.[16] examined the attitudes of primary teachers towards probability and its teaching. The other three focus more on teachers' probabilistic content knowledge: combinatorics and binomial expansion[18], randomization test[20], and modelling[19].

Other related notions. Several topics other than those stated in the purposes and subthemes were addressed in the papers, of which we can highlight attitudes and values[2,4,16], problem solving[6,10,11] and combinatorics[8,18].

5. Looking Ahead

Recognizing the dissonance associated with the leaders of a Topic Study Group entitled Teaching and Learning of "Probability" (which is colloquially, albeit widely known as how likely something is to happen) attempting to peer into the future, we nevertheless wish to end this proceedings report with what, we see, comment on a few possible research directions stemming from the papers presented in TSG-11 at ICME-14.

First, we contend that the use of technology in supporting learning and teaching of probability appears to shift the attention for advanced probabilistic concepts, relationships, and procedures. More research on this aspect of technology is needed to develop effective ways of promoting students' conceptual understanding. Second, while most papers addressed student learning at different educational levels, studies involving pre-service and in-service teachers were limited. With the availability of educational technology tools to support student learning, we note that there is need for further research on teacher knowledge and practices on the use of technology in their teaching and learning of probability. Third, a modeling approach to probability, especially with the use of computer simulations, is an emerging area of research. More interest in researching the modeling approach in teaching and learning of probability with teachers and students are welcome. Lastly, we wish to underscore the continual untethering of probability education from that of statistics education. Perhaps particularly present at ICME-14 due to dedicated working groups for Teaching and Learning of Statistics (TSG-12) *and* Teaching and Learning of Probability (TSG-11), we hope to see, again, a Topic Study Group dedicated specifically to probability education at the 15th International Congress on Mathematics Education in 2024 in Sydney, Australia.

https://doi.org/10.1142/9789811287152_0036

Topic Study Group 12

Teaching and Learning of Statistics

TSG-12 Working Team[1]

1. Theme and Description

In today's world rapid technological advances have facilitated the production and management of large data sets in diverse forms. Being able to create value with data by converting them to meaningful information, to critically evaluate and effectively utilize the information for decision-making, and to understand social and natural phenomena are important 21st century skills for all citizens. Thus, the importance of teaching and learning statistics as the "science of data" is increasingly gaining recognition at all educational levels. The study of statistics provides students with tools, skills, ideas and dispositions to use in order to react intelligently to information in the world around them. Reflecting this need to improve students' ability to think statistically, statistical literacy and reasoning are becoming part of the mainstream school and university curriculum in many countries. Emerging issues in statistics education relate to dealing with "big data" and dealing as a "data scientist" and to the use of statistics in thinking about social changes and policy decisions and the impact of these on both the school and university curricula. In light of these trends, statistics education is a growing and exciting field of research and development that will enable us to build from the knowledge we have accumulated in the past about teaching and learning statistics to move forward in productive ways.

At TSG-12, academic work on major issues in statistics education research were presented and discussed along any one of the four themes:

1. Recent research on teaching and learning statistics in school and at the tertiary level including global trends;
2. Development and assessment of statistical literacy, including the connection of statistics education to social and political issues;

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Chair: Enriqueta Reston, University of San Carlos, Philippines
Co-chair: Andreas Eichler, University of Kassel, Germany
Members: Leandro de Oliveira Souza, Universidade Federal de Uberlandia, Brazil
Gail Burrill, Michigan State University, USA
Qian Chen, Sichuan Normal University, China

3. Preparation and professional development of statistics teachers and of statistics teacher educators; and
4. The impact of “big data” and technology-rich learning environments in statistics education, and the connection between learning statistics and learning data science.

2. Program Overview

At ICME-14, TSG-12 provided the venue for statistics educators, teachers and researchers for presentation of research and discussion of issues on these themes. The discussion included time to reflect on the status of research in statistics education related to the various themes and highlighted areas of high priority for the statistics education research community. There were four group sessions at TSG-12 which included invited and contributed papers, as well as posters, primarily delivered through online conference platforms.

2.1. Online Paper Presentations

These online presentations in four sessions start with invited papers and move to contributed papers. Except for one on-site presentation by a Chinese national, all these papers were presented virtually using Zoom conference platform with around 20–30 participants per session. The papers presented in these four sessions are outlined in Tab. 1 on the next page.

2.2. Poster Presentations

The posters presented in these four sessions are outlined in Tab. 2 following Tab.1.

3. Future Directions and Suggestions

After TSG-12 at ICME-14, the post-conference discussion focused on submission of selected papers form TSG-12 into a monograph to be published by Springer. The Call for Papers among the TSG-12 presenters was initiated by Gail Burril, one of the TSG-12 team members. This volume of papers from TSG-12 of ICME-14 will not be proceedings but rather collections of papers, each of which shares some common ground with the original paper presented during TSG-12. With the exception of the opening chapter with short country reviews, this volume consists of 14 chapters each representing a paper presented at ICME-14 for TSG-12 and the discussion that ensued. These papers will constitute a book publication entitled Reasoning with Data and Statistical Thinking: An International Perspective to be published by Springer with five sections, namely: Statistics Education Across the World, Data and Young Learners, Data and Simulation to Support Understanding, Data and Society, Statistical Learning, Reasoning and Attitudes. The editors Gail Burrill, Enriqueta Reston and Leandro Souza are members of TSG-12 technical working group.

Tab. 1. List of papers presented

Paper and author(s)
Session 1
[1] Designing embodied tasks in statistics education for grade 10–12. ***Lonneke Boels*** (The Netherlands).
[2] Teaching statistics and sustainable learning. ***Hanan Innabi*** (Sweden).
[3] Reading and interpreting distributions of numerical data in primary school. ***Daniel Frischemeier*** (Germany).
[4] Statistical literacy as central competence to critically understand big data. ***Karen François*** and *Carlos Monteiro* (Brazil).
[5] Students beliefs about statistics and their influence on students´ attitudes toward statistics in introductory courses. ***Florian Berens*** (Germany).
Session 2
[6] Interdisciplinary data workshops. ***Danny Parsons,*** *David Stern, Balázs Szendröl, and Elizabeth Dávid-Barrett* (UK).
[7] Distinctive aspects of reasoning in statistics and mathematics: implications for classroom arguments. ***Anna Marie Conner*** *and Susan A. Peters* (USA).
[8] A school experiment for introductory inferential statistics in Hungarian secondary schools. ***Péter Fejes Tóth*** *and Ödön Vancsó* (Hungary).
[9] An informal statistical inferential reasoning experience with seventh graders: a lesson study. ***Soledad Estrella,*** *Maritza Méndez-Reina, Tamara Rojas, and Rodrigo Salinas* (Chile).
[10] Research on teaching strategies of the mean from the perspective of statistical literacy. ***Jiaqi Wu*** (China).
Session 3
[11] Margin of error: connecting chance to plausible. ***Gail Burrill*** (USA).
[12] Critical citizenship in statistics teacher education. ***Lucía Zapata-Cardona,*** *Cindy Alejandra Martínez-Castro, Lucía Zapata-Cardona, and Gloria Lynn Jones* (Colombia).
[13] Mathematics ability and other factors associated with success in introductory statistics. ***Adam Molnar*** *and Shiteng Yang* (USA).
[14] Elementary students' responses to quantitative data. ***Karoline Smucker*** *and Azita Manouchehri* (USA).
Session 4
[15] Implementation of a course on Tidyverse in Pakistan under the ASA Educational Ambassador Program. ***Saleha Naghmi Habibullah*** (Pakistan).
[16] Young learners' reasoning with informal statistical models and modeling. ***Michal Dvir*** *and Dani Ben-Zvi* (Israel).
[17] The binomial model: coin tosses or clay pots? ***Von Bing Yap*** (Singapore).
[18] Variability modeling and data-driven decision-making using socially open-ended problems: a comparative study of high school students in Thailand, Brunei and Zambia. ***Orlando González*** (Thailand).
[19] Algebraization levels of statistical tables in secondary textbooks. ***Mara Magdalena Gea*** *and Jocelyn D. Pallauta, Pedro Arteaga, and Carmen Batanero* (Spain).
[20] Data modelling with young learners as experiences of allgemeingbildung. ***Stine Gerster Johansen*** (Denmark).
[21] Investigating mathematics teacher educators' conceptions for informal line of best fit. ***Jale Gunbak Hatil*** *and Gulseren Karagoz Akar* (Turkey).

Tab. 2. List of posters presented

Poster and author(s)	
[22]	Does climate change really exist? high school students discover statistical methods by solving a modeling problem. ***Maren Hattebuhr*** *and Martin Frank* (Germany).
[23]	Do students in grade 10 generate ideas of statistical hypothesis testing spontaneously? ***Hiroto Fukuda****, Naoya Miwa, and Yoshiki Hashimoto* (Japan).
[24]	Model proposal to promote the construction of the strong meaning of volatility. ***Miguel Andres Diaz Osorio*** (Columbia).
[25]	Improving statistical pedagogy among K to 12 mathematics teachers in the Philippines. ***Enriqueta Deguit Reston*** (Philippines).
[26]	Developing mathematical knowledge for teaching mean and median of prospective mathematics teachers through the lesson study. ***Thi Ha Phuong Nguyen*** (Vietnam).
[27]	Analysis of the most frequent errors in practical works on tables and graphs in biostatistics. ***Teresita Evelina Teran*** (Argentina).
[28]	Comparing the statistical content of elementary school mathematics textbooks from Japan, India and China. ***Yuqi Li****, Xue Li, and Zhemin Zhu* (China).
[29]	Comparing the statistical content of elementary school mathematics textbooks from Japan, India, the United States, Singapore and China. ***Zhemin Zhu****, Yuqi Li, Yilin Li, Lulu Li, and Xue Li.* (China).
[30]	Aspects of critical thinking in statistical education-research survey on sixth-grade elementary school. ***Naoki Ohta*** *and Ken Teraguchi* (Japan).

https://doi.org/10.1142/9789811287152_0037

Topic Study Group 13

Teaching and Learning of Calculus

David Bressoud[1], Kristina Juter[2], Elizabeth Montoya[3], Armando Cuevas[4], and Xuefen Gao[5]

1. Aims of the TSG

This Topic Study Group sought contributions on research and development in the teaching and learning of Calculus, both at the upper secondary and tertiary levels. Contributions accounted for advances, new trends, and important work done in recent years on the teaching and learning processes of Calculus. These included:

- Introducing and building basic concepts of Calculus in upper secondary education,
- Meeting the challenges of teaching and learning Calculus and Analysis at universities and through online courses,
- Teaching and learning of Calculus for special audiences (*e.g.* professional training, engineering, life sciences),
- Use of technology in the teaching and learning of Calculus, including online courses
- The role of visualisation in the teaching and learning of Calculus,
- Analysis of textbooks concerning the presentation of the concepts of Calculus and Analysis,
- Easing the transition between secondary and tertiary education in the teaching and learning of Calculus, and between Calculus and Analysis at the tertiary level,
- Theoretical approaches to study the phenomena related to the teaching and learning of Calculus.

Contributions also described theoretical or pragmatic research into effective practices for the teaching and learning of key concepts of Calculus such as co-variation

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of functions, limits, continuity, differentiation, integration, or the Fundamental Theorem of Calculus, among others.

1.1. Submissions

We received 36 submissions from 19 countries (South America: 6; North America: 10; East Asia: 7; Southeast Asia: 1; South Asia: 2, Europe: 5; Middle East and North Africa: 5), thus reaching our goal of diverse cultural representation. Of those 36 submissions, eleven were accepted for long paper presentations, nineteen for short paper presentations, and eight as posters. None were rejected. One presenter selected for a long paper and one presenter selected for a short paper withdrew after the congress was rescheduled for 2021.

1.2. Sessions

Because almost all presenters were joining remotely and representing many different time zones, the presentations were grouped geographically, first those from East and South Asia, then those from Europe, the Middle East, and North Africa, and finally those from the Americas. Each session began with three or four 15-minute presentations, followed by five or six 5-minute presentations. There was little time for discussion.

1.3. Paper Topics

A list of the papers and authors are included in order of presentation and are organized in Tab. 1 (on the next page).

2. Conference Themes

There were three main themes for the papers presented in this topic study group. The first dealt with student understandings and misunderstandings of basic concepts of calculus. These included rate of change, limits, continuity, derivatives, differentials, and definite integrals. At a more basic level, there was also discussion of how to improve the covariational reasoning of students and general student difficulties with the language of mathematics and how mathematics uses language.

A second theme turned to the use of technological tools to help students build understanding of certain fundamental ideas.

Quite a few of the presentations focused on the third theme, presenting a variety of techniques for improving instruction in the calculus classroom. Several described the use of inquiry-based learning. There was discussion of other approaches to creating an active learning environment as well as the use of flipped instruction, an emphasis on modeling, and the use of writing assignments.

Other topics included a comparison of textbooks, a comparison of how physics and mathematics differ in their approach to ordinary differential equations, the introduction of tangents and asymptotes without reference to limits, and a discussion of how to deal with student overgeneralization of the concept of linearity.

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Mathematical knowledge for teaching of calculus: an exploratory study of secondary school teachers mathematical thinking related to concepts in calculus. ***Jonaki B Ghosh*** (India).
[2]	Modeling concepts of derivative and differential with educational software. ***Vladimir Nodelman*** (Israel).
[3]	Constructing knowledge using digital tools: the case of the inflection point. ***Regina Ovoenko*** *and Anatoli Kouropatov* (Israel).
[4]	Students' interpretations of the definite integral. ***Inen Akrouti*** (Tunisia).
[5]	Comparison of mathematics textbooks in IB school and Chinese public high school: take core concept — calculus as an example. ***Yun Lu*** (China).
[6]	Research in calculating areas between curves. ***Gordana Stankov*** *and Djurdjica Takaci* (Serbia).
[7]	The concept of continuity through different types of representations of the function. ***Matthias Antonopoulos*** *and Leonora Antonopoulou* (Greece).
[8]	Actions in the learning environment: analyzing physics and mathematics lessons in the case of ODE. ***Kristina Elisabeth Juter****, Örjan Hansson, and Andreas Redfors* (Sweden).
[9]	From upper secondary school to university calculus: language difficulties versus conceptual difficulties. ***Arne Hole****,* ***Inger Christin Borge****, and* ***Liv Sissel Grønmo*** (Norway).
[10]	The discrete-dense-continuous phenomenon and its implication in continuous. ***Elizabeth Montoya Delgadillo*** (Chile).
[11]	A limit free calculus for introducing the concepts of tangent and asymptote. an educational proposal inspired by the past. ***Maria Astrid Cuida Gomez*** (Spain).
[12]	An approach to reduce the number of failure students in a large calculus class. ***Jianhui Pan*** (China).
[13]	The exponential function from the viewpoint of mathematical modelling: a Chilean lesson study. ***Carlos Andres Ledezma Araya*** *and Elizabeth Montoya Delgadillo* (Chile).
[14]	Using open education resources to promote the active learning of calculus in urban districts. ***Kenneth Horwitz*** (USA).
[15]	Mathematics anxiety levels among students in an inquiry-based calculus class. ***Harman Prasad Aryal*** *and Otto Joshua Shaw* (Nepal).
[16]	Learning difficulties in calculus: an investigation through students' written solutions. ***Raquel Carneiro Dorr*** (Brazil).
[17]	The design and use of low instructional overhead tasks in undergraduate calculus: making student reasoning more accessible to calculus instructors. ***David C. Webb*** (USA).
[18]	The observed impact implementing inquiry-based learning at a calculus classroom. ***Su Liang*** (China).
[19]	Teaching calculus based on complexity theory of teaching and learning. ***Mehmet Turegun*** (USA).
[20]	Notions of continuity of the pre-service teachers: reflections for a problematization. ***Antonio Bonilla*** *and Ricardo Cantoral* (Mexico).
[21]	Resignification of the derivative in a school situation with a perspective of an exclusion-inclusion dialectic: from emulation of the concept to autonomy of uses. ***Jose Luis Morales Reyes*** *and Francisco Cordero Osorio* (Mexico).
[22]	Covariational reasoning: an axis in the construction process of the definite integral concept. ***Mihaly Andre Martinez Miraval*** *and Martha Leticia Garcia Rodríguez* (Peru).
[23]	The "overgeneralization of linearity": difficulty, conflict or obstacle? ***Nicolas Lopez*** *and Gloria Ines Neira Sanabria* (Colombia).
[24]	Rate of change: meanings students have in accordance with context. ***Dafna Elias****, Tommy Dreyfus, Anatoli Kouropatov, and Noah Sella* (Israel).

3. Areas for Future Research

None of the presentations specifically addressed the problems of preparation for calculus. This is a huge issue, especially in places such as the United States where student preparation for university is so varied in quality and so highly correlated with socio-economic status. Good work is being done in trying to address these disparities. They require attention from the research community to understand what works in what situations and why.

Student understanding and misunderstanding of the concepts of calculus has been a rich source for research in the teaching and learning of calculus. Thirty years ago, the focus was on how students misconceive so many of these fundamental ideas. Within the past decade, this has shifted to a more productive line of exploring natural student understandings that can be encouraged and developed to improve student grasp of and ability to use fundamental aspects of calculus. Good examples of this are the development of covariational reasoning and the development of an understanding of limits expressed in terms of narrowing bounds on the distance from the target value, placing the emphasis on what happens to the dependent variable rather than the independent variable. There is still work to be done in identifying productive approaches to basic ideas of calculus and understanding how they can be effectively encouraged.

Technology in its many forms is a constant presence. While most of the work has focused on exhibiting the effectiveness of a clever new tool, much more work needs to be done on how to balance the use of what has become basic and ubiquitous technology such as computer algebra systems. How have they changed what students need to learn and be able to carry with them beyond the calculus class? What procedures for differentiation or integration are still essential and why?

Finally, as the emphasis on improved approaches to teaching and learning in this topic study group has revealed, there is a great deal of work being done on the implementation of a variety of active learning approaches. There is no question that when undertaken by a dedicated and enthusiastic instructor, these can greatly improve student outcomes. The questions that require exploration revolve around how these efforts can be scaled up. How does one convince reluctant colleagues to attempt active approaches to their teaching? What kinds of supports are most helpful? How can departments deal with the fact that later adopters are often discouraged by the difficulties they encounter? What are the ingredients of active learning that are easiest to implement on a broad scale and most effective? These are very broad questions with applications to any aspect of the teaching and learning of mathematics, but there is good research into a variety of approaches to active learning that apply specifically to the context of calculus instruction. Narrowing the focus in this way promises to generate good ideas and significantly improve the teaching and learning of calculus.

https://doi.org/10.1142/9789811287152_0038

Topic Study Group 14

Teaching and Learning of Programming and Algorithms

Chantal Buteau[1], Maryna Rafalska[2], Xuemei Chen[3], and Bakhyt Matkarimov[4]

1. Teaching and Learning of Programming and Algorithms: A New TSG at ICME!

We see the introduction of this new TSG-14 as a response to the increased integration of programming and algorithmics in our school curricula around the world, sometimes within or in relation to mathematics. At the onset of this TSG, is undoubtedly the work and vision by pioneer Seymour Papert. We opened our TSG sessions by recalling the following quote from Papert's influential 1980 Mindstorm book, reminding us that although the broader integration of programming and algorithmics in our schools (compulsory programs) is rather recent, the vision had long been laid out:

> In many schools today, the phrase "computer-aided instruction" means making the computer teach the child. One might say the computer is being used to program the child. In my vision, the child programs the computer and, in doing so, both acquires a sense of mastery over a piece of the most modern and powerful technology and establishes an intimate contact with some of the deepest ideas from science, from mathematics, and from the art of intellectual model building. (Papert, 1980, p. 5)

2. Aim of the TSG

The aim of this new ICME Topic Study Group was to explore questions that raise from such a rapid and widespread interest of integrating programming and computational thinking in education and to exchange information about evolving trends and perspectives within various educational contexts from around the world. Questions and themes at centre of our interests were for example:

- What are the current realities of teaching and learning of algorithmics and programming in relation to school and university mathematics classrooms?
- To what extent and how is research informing teacher education and practices to support the development of computational thinking?

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- What theoretical perspectives and methodologies are relevant for studying the teaching and learning of programming and algorithms in relation specifically to learning mathematics and what theoretical or practical results have emerged? For example, what is the relation between mathematics (teaching and learning) and programming and algorithmics (teaching and learning)?
- What obstacles to integration have occurred and how have they been overcome?
- What affordances have been observed and how were they exploited?

To this end, we received and discussed research- and practice-based contributions concerning different levels of education (from elementary to university) and various topics related to the teaching and learning of programming and algorithmics, either in support of or as supported by the teaching and learning of mathematics. In the next section, we provide more details about the contributions that were presented and discussed.

3. Program Overview

3.1. Submissions

We received 17 submissions from 12 countries (South America: 1; North America: 2; Asia: 3; Europe: 5; Australia: 1), thus reaching our goal of diverse cultural representation. Of those 17 submissions, one was an invited long paper, five were accepted as long paper presentations for 2020 (3 were presented in 2021), seven as short paper presentations for 2020 (6 were presented in 2021), and three as posters for 2020 (1 was presented), and one was rejected.

3.2. Paper and poster presentation sessions

Each of our three sessions had a similar format: first an introduction by one of the TSG co-chairs of the themes and schedule for the session. It was followed by the oral presentations of (40-minute) *invited* talk (IT) or (25-minute) *long* papers (LO), (15 minute) *short* papers (SO), and the (5-minute) *poster* for the last session. We kept a 20 minutes time window at the end of each session for a collective discussion at which some guiding questions were provided as a way to prompt the conversation. We saw this discussion as critical to promote networking and to engage in deepening our understanding of different issues presented. We ended the last session by summarizing some key points that were raised during our three sessions, as well as highlighting some under-represented topics that should be part in future conversation.

Among the 16 accepted submissions, 11 of them were presented at the 2021 hybrid conference. We list these contributions and authors below in Tab. 1 in order of presentation:

Tab. 1. List of Contributions Presented

Paper and author(s)	
[1]	Algorithmic thinking: emerging implications for school mathematics education. ***Max Stephens*** (Australia) *and Djordje M. Kadijevich* (Serbia). (LO)
[2]	Mathematics education and computational thinking. ***Takuma Takayama*** (Japan). (SO)
[3]	Teachers' perceptions of computational thinking as part of the teaching of mathematics: a hermeneutic literature review. ***Camilla Finsterbach Kaup*** (Denmark). (SO)
[4]	Three important aspects of research on computational/algorithmic thinking. ***Djordje M. Kadijevich*** (Serbia) *and Max Stephens* (Australia). (LO)
[5]	On enumeration in mathematics and computer science: some didactical issues. ***Simon Modeste*** (France). (LO)
[6]	A framework for analyzing the integration of algorithms and programming into mathematics textbooks. ***Tran Kiem Minh****, Nguyen Thuy Viet Anh, and Tran Trong Ha* (Vietnam). (SO)
[7]	Working mathematically and thinking computationally: capitalising on commonalities for integrated teaching. ***Elena Prieto*** *and* ***Kathryn Holmes*** (Australia). (IT)
[8]	Modelling and 3D priniting a circular staircase for a doll's house: teaching computational thinking using a range of different tools. ***Gregor Milicic*** *and Matthias Ludwig* (Germany). (SO)
[9]	Researching the teaching and learning of programming for university mathematical investigation projects. ***Chantal Buteau*** (Canada)*, Eric Muller* (Canada)*, Ghislaine Gueudet* (France)*, Joyce Mgombelo* (Canada)*, Ana I. Sacristán* (Mexico). (SO)
[10]	"Math & CS Labs": a bi-disciplinary course for second-year undergraduates in mathematics or computer science. ***Antoine Meyer*** *and David Doyen* (France). (SO)
[11]	Matlab as a tool for experimental mathematics. ***Yevgeny A. Gayev*** (Ukraine). (Poster)

3.3. Conference themes

Each session focused on different themes. In the following, we list the themes of each session, together with selected guiding questions that were proposed in order to facilitate the discussion session. We see those questions pointing to particular interests from scholars and needs from practitioners in the area of teaching and learning of programming and algorithmics in mathematics education.

In the first session, we focused on the joint development in curriculum of mathematics and algorithmic/computational thinking; pedagogical approaches; and, attitudes and knowledge of (prospective) teachers. We noted that there is a historical and epistemological proximity between mathematics and computer sciences and questioned for example: *In what ways do or 'should' school mathematics curricula exploit this proximity? And is an understanding of this proximity necessary for teachers in order for them to meaningfully integrate algorithmic/computational thinking in their teaching of mathematics?* Furthermore, we wondered, due to the emerging integration in many curricula: *To what extent and how should research inform mathematics teacher education and practices to support the integration and assessment of algorithmic/computational thinking? What are the barriers and challenges experienced by teachers who are integrating algorithms and programming as part of their mathematics teaching? And which pedagogical approaches support students' learning of mathematics in a context with algorithms and programming?*

In this second session, we focused on two main themes, namely the conceptualization of algorithmic and computational thinking, and the interactions between computer science and mathematics and their potentialities in the teaching and learning of mathematics. Early on in our discussion, emerged the need to articulate a conceptualization of algorithmic thinking and computational thinking, and we wondered about its link(s) with: *problem solving, different types of mathematical thinking (in particular, algebraic thinking and statistical thinking), mathematical reasoning (logical thinking, argument, justification, generalisation), and design thinking*. We wondered about the *theoretical frameworks that can be relevant to study algorithmic/computational thinking for mathematics teaching and learning.* And we also asked questions, such as: *How does/should the computer science curriculum impact problem solving in mathematics?*

Finally, in this third and last session, we focused on the incorporation of algorithmic/computational thinking into mathematics curriculum at secondary school and university levels, as well as teaching practices and resources at these levels. Different aspects were discussed, such as the *kind of articulations between programing and mathematics activities that support the students' learning of mathematics and computer science concepts in the most efficient way.* We also questioned, for example, *which domains/subject areas that are the most fruitful basis for the integration of algorithmic/computational thinking in mathematics curriculum*; *and whether there are differences between integration of algorithmic/computational thinking in mathematics curriculum at high school and university levels.*

4. Future Directions (Areas for Future Research)

Different approaches that bring together the learning of programming and algorithmics with mathematics learning were discussed during the sessions, including multiple examples of activities, different models of integration in the curriculum, resources for teachers, and the identification of different areas of mathematics that are particularly fruitful. As we ended our TSG, we identified three under-represented topics in our conversation that we deemed as key to incorporate in future conversation, namely:

- Initial and professional teacher education
- Classroom realities of teacher and student practices
- Theoretical and methodological frameworks to analyse the above two

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https://doi.org/10.1142/9789811287152_0039

Topic Study Group 15

Teaching and Learning of Discrete Mathematics

Elise Lockwood[1], Cecile Ouvrier-Buffet[2], Ambat Vijayakumar[3], Mariana Durcheva[4], and Han Ren[5]

1. Aims of the TSG

Discrete mathematics is the study of discrete (as opposed to continuous) structures. It has many applications in a variety of fields, and it often exists at the interface of several disciplines, making it increasingly relevant in our digital world. Discrete mathematics offers many accessible points of entry for students to engage in rich mathematical thinking, as students can interact with ideas and reason about problems without requiring considerable prior knowledge of mathematical content. Further, its accessible nature makes it an excellent context in which students can engage in important mathematical practices such as conjecturing, generalizing, justifying, and proving. For these reasons, we view discrete mathematics as an indispensable part of mathematics education that deserves attention at all levels of education.

The main goal of TSG-15 was for researchers and educators to share current developments in the teaching and learning of discrete mathematics at all levels, ranging from elementary through postsecondary school. We sought to extend previous work on the teaching and learning of discrete mathematics by sharing new research and pedagogical innovations about a variety of topics related to discrete mathematics. We were particularly interested in identifying and exploring the variety of ways in which discrete mathematics is studied and taught across the world. We acknowledge that the teaching and learning of discrete mathematics may involve investigations into both mathematical content (particular mathematical topics within discrete mathematics and other disciplines) and mathematical practices (more general mathematical approaches or habits of mind), and that it may be a setting in which to explore other relevant issues in mathematics education. In terms of content, in this TSG we characterized discrete mathematics broadly as consisting of a variety of topics. This includes topics traditionally associated with discrete mathematics (such as algorithms, coding theory, combinatorics, cryptography, graph theory, languages and automata, logic, number theory, recursion, and set theory), as well as topics that might be considered relatively new (such as complexity theory, existence and constructability, and computational

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number theory, algebra, and group theory). In addition, there are a number of mathematical practices that could be related to discrete mathematics, including problem solving, conjecturing, justifying, generalizing, proving, and more. We also acknowledge that there may also be additional topics in mathematics education that might particularly be emphasized — these might include, for example, issues of affect and beliefs, equity and inclusion, classroom discourse, pre-service teacher preparation, or in-service teacher training.

We envisioned that this TSG would include presentations of papers on any of the wide range of topics discussed above, focusing on any level of school. We welcomed papers that were related to the teaching and learning of discrete mathematics, which may include, but were not limited to:

- research on student thinking about relevant concepts in discrete mathematics;
- research demonstrating effective instructional strategies in teaching discrete mathematics;
- research-based ideas for innovative activities and pedagogical interventions in classrooms at a variety of age levels;
- research-based ideas of incorporating technology into the discrete mathematics classroom;
- explorations of discrete mathematics as a setting in which to investigate mathematical practices;
- explorations of discrete mathematics as a setting in which to investigate other relevant issues in mathematics education;
- ways of thinking (or habits of mind) that may be productive in discrete mathematics, such as combinatorial reasoning, algorithmic or computational approaches, or recursive thinking;
- curriculum and educational policy issues related to discrete mathematics.

1.1. Submissions

We received 22 submissions from 12 countries (South America: 2; North America: 4; Asia: 3; Europe: 13). Of those 22 submissions, 18 were accepted as paper presentations, 2 as posters, and 2 were rejected. Four accepted paper presenters were not able to present in 2021 in the virtual format, so we had a total of 14 papers presented during the online conference (of these, one did not show up to the presentation, so there were 13 total presentations).

1.2. Sessions

In general, most presenters were given 20 total minutes, which included time for questions and the transition to the next speaker. Two exceptions were one 30-minute invited presentation (by Erik Tillema and Lori Burch[6]), and one shorter 10-minute presentation (who was a last-minute addition that had not indicated they intended to present). Our first session had four 20-minute presentations and one 10-minute presentation; our second session had one 30-minute presentation and three 20-minute presentations; and our third session had four 20-minute presentations. We also had some time at the end of the third session to have some overall discussion, although this did not seem like enough time to be able to have the kinds of in-depth conversations and discussions we would have liked to have. Still, there was a sense of community during the sessions, and even though we did not have much time for additional discussion, it felt like a productive and valuable shared experience.

1.3. Papers presented

Of the 18 accepted papers, four authors had to withdraw from the 2021 virtual participation. Thus, only 13 papers were able to be presented during the online conference. A list of these papers and authors are included in order of presentation and are organized in Tab. 1.

Tab. 1. List of papers presented

	Paper and author(s) in order of presentation
[1]	Suggestions for an integration of cryptology into a math curriculum. ***Tomas Borys*** (Germany)
[2]	Enriching pre-service teachers' conceptions about proof with discrete mathematics. ***Cécile Ouvrier-Buffet*** (France)
[3]	Graph theory in primary school mathematical education — a quantitative study on the impact of graph theory concepts on psychological characteristics of fourth grade students. ***Melissa Windler*** (Germany)
[4]	The role of discrete mathematics in secondary mathematics for non-STEM paths. ***Jaime Carvalho e Silva*** (Portugal)
[5]	Discrete mathematics in the Hungarian mathematics curriculum. ***Katalin Gosztonyi****, Csaba Csapodi, and Eötvös Loránd* (Hungary)
[6]	Leveraging combinatorial and quantitative reasoning to support the generalization of advanced algebraic identities. ***Erik Tillema*** *and Lori Burch* (U.S.A.)
[7]	Combinatorial counting problems in elementary school: a comparative analysis of German textbooks. ***Karina Höveler*** *and Janet Winzen* (Germany)
[8]	Preliminary levels of sophistication for enumerating permutations. ***Joseph Antonides*** *and Michael T. Battista* (U.S.A.)
[9]	Guiding students' reinvention of combinatorial operations. ***Belmiar Mota*** *and Rosa Antónia Tomás Ferreira* (Portugal)
[10]	Preservice teachers' development of mathematical knowledge for teaching via combinatorial tasks in a computational setting. ***Elise Lockwood*** *and Adaline De Chenne* (U.S.A.)
[11]	Relation between algorithmic and combinatorial thinking of undergraduate students of applied informatics. ***Janka Medová*** *and Sona Čeretková* (Slovakia)
[12]	Some approaches for incorporation of CAS in a discrete mathematics course. ***Mariana Durcheva*** (Bulgaria)
[13]	How can poly-universe sets develop creativity during the solution of combinatorial exercises? ***Eleonóra Stettner*** *and Szabina Tóth* (Hungary)

2. Conference Themes

We had presentations on a variety of topics, and we note some big ideas and themes that emerged during the sessions. These themes mostly align with areas of focus of our papers, and we describe categories of papers that highlight overall areas of emphasis that were covered in our topic study group.

One theme is that there are a number of ways in which discrete mathematics is and can be integrated in school mathematics. We saw examples of papers that demonstrated discrete mathematics for non-STEM majors (Carvalho e Silva[4]), for and for K-12 students in general (Gosztonyi et al.[5]). We also saw ideas for innovative ways to incorporate topics into the discrete mathematics curriculum (including Borys' focus on crytology[1]). This suggests that there are a variety of ways around the world in which discrete mathematics is being integrated into curricula, and this demonstrates that we have many opportunities to explore effective ways to teach discrete topics for students at a variety of age and grade levels.

A second theme is that the field is currently conducting (and would benefit from continuing to conduct) research about specific topics within discrete mathematics. The most common topic that is being regularly investigated is combinatorics, and eight of our papers focused on combinatorics (including the invited presentation by Tillema and Burch[6]). In addition, though, we saw promise for focusing on other topics, including graph theory (Windler[3]) and cryptology (Borys[1]). This suggests that there are opportunities for additional topics to be studied in more depth, and perhaps a next direction for the field is to jointly study other topics as thoroughly as combinatorics is being studied.

A third theme is that it may be productive to explore teacher preparation related to discrete mathematics. A couple of studies in our topic study group focused on pre-service teacher preparation (Ouvrier-Buffet[2]; Lockwood and De Chenne[10]), and this may continue to be a fertile area of research, where we may focus on the preparation of teachers who will teach topics in discrete mathematics. Particularly given the role of discrete mathematics in the curriculum in many different countries (as indicated in theme 1), it may be valuable to investigate in more depth how teachers are prepared to teach discrete topics.

Finally, our last theme is that discrete mathematics interfaces meaningfully with computing and technology. Several of our papers (including those by Medová and Čeretková[11]; Durcheva[12]) examined the role of technology and computing in discrete mathematics, suggesting that there may be valuable connections with researchers who study informatics or computer science. We see opportunities for interdisciplinarity in the future, particularly in this context of computing.

3. Areas for Future Research

There are several opportunities that we as a community identify as areas for future research, and our ideas for future research are related to the themes we described above. In particular, the distribution of topics about which we had presentations highlights different opportunities for more research in certain areas. While combinatorics has been increasingly well-researched in the last decades (which is underscored by our eight combinatorics-focused topics), other particular topics are as yet relatively under-researched. These include topics like cryptology and graph theory, as well as other areas of discrete mathematics like relations, sets, logic, and recursion. In terms of particular topics in discrete mathematics, then, there are many opportunities for researchers to investigate multiple aspects of these understudied topics, including students' reasoning about them and effective instructional interventions. We also hope that the field will continue to expand to investigate the teaching and learning of discrete mathematics as related to teacher preparation, exploring effective ways to prepare teachers to teach discrete topics in particular, and studying pre-service and in-service teachers' understandings of and experiences with discrete mathematics. Finally, there is a clear connection between discrete mathematics and computing, and we see valuable opportunities to continue to explore and examine the intersection of these ideas. We hope researchers will investigate effective ways to leverage computing in the teaching and learning of discrete mathematics, as well as ways to teach discrete mathematics to other populations such as informatics and computer science students.

https://doi.org/10.1142/9789811287152_0040

Topic Study Group 16

Reasoning, Argumentation, and Proof in Mathematics Education

Viviane Durand-Guerrier[1], Samuele Antonini[2], Kotaro Komatsu[3], Nadia Azrou[4], and Chao Zhou[5]

1. Aims of the TSG

There is international recognition (see Stylianides and Harel, 2018) of the importance of reasoning and proof in students' learning of mathematics at all levels of education (elementary, secondary, university) and in all tracks (general, vocational). Indeed, reasoning, argumentation, and proof are at the very heart of mathematical activity, playing a crucial role in learning processes. There is also international research-based evidence showing that many students face difficulties with reasoning about mathematical ideas and constructing or understanding mathematical arguments. This is particularly the case when these arguments meet the standard of proof; in addition, teachers often lack adequate resources for helping their students to develop skills in reasoning, argumentation, and proof. Although the existing body of research offers important insights into this area, there are still many open questions for which theoretical and empirically based responses are needed.

We have welcomed submissions of theoretical or empirical research reports on any topic related to reasoning, argumentation, and proof in mathematics education, including interaction between mathematics and other disciplines (e.g., Computer Sciences, Physics, Economy etc.). The reports could cover any level of education: elementary, secondary, and university (including pre-service teacher education, or in-service teacher professional development).

2. Submissions

We received 45 submissions (38 papers and 7 posters) from 17 countries: Algeria: 1; Brazil: 3; Cameroun: 1; Canada: 1; Chile: 1; China: 7; Colombia: 1; Germany: 4; Italy: 2; Japan: 4; Norway: 2; Peru: 1; South Korea: 2; Tunisia: 1; Turkey: 2; United

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Kingdom: 1; United States of America: 11. Among the 38 submitted papers, 12 were accepted as long oral presentations (Algeria: 1; Cameroon: 1; Canada: 1; China: 1; Germany: 1; Italy: 1; Japan: 1; Norway: 1; United Kingdom: 1; United States: 2; Tunisia: 1) and 26 were accepted as short oral presentations, and the seven submitted posters were accepted. During the conference, in July 2021, there were only 10 long oral presentations[1–10], and 18 short oral presentations[11–28]; during the poster sessions, 2 posters[29,30] have been presented (see Tab. 1 on the next page).

3. Sessions

Considering the high number of submissions, the ICMI organizing committee granted our TSG one more time slot for presentations. Each long oral presentation lasted 10 mins and was followed by 5 min discussion; the short oral presentations were 5 minutes each followed by a collective discussion. Due to the pandemic, the sessions were in hybrid form with a small number of participants and presenters in Shanghai, the majority attending online.

In the first session on July 13th, 2021, after the introduction of the team and of the agenda of the TSG, there were two long oral presentations by Azrou[1] and Chellougui[2], and five short oral presentations by Bae[11], Na and Knuth[12], Meyer et al.[13], Solar et al.[14], and Lin[15]. A 20-min discussion on the short oral presentations followed. The session was chaired by Kotaro Komatsu (online).

In the second session on July 14th, 2021, there were 2 long oral presentations by Jablonski and Ludwig[3] and Soldano[4] and 6 short oral presentations by Shibata and Misono[16], Kempen[17], Lee[18], Dallas[19], Damrau[20], and Murata[21], followed by a 20-min discussion on the short oral presentations together with a 10-min discussion on general issues from the two first sessions in order to prepare the final collective discussion. This session was chaired by Nadia Azrou (online).

In the third session on July 16th, 2021, there were three long oral presentations by Yan and Hanna[5], Zhuang and Conner[6], Buchbinder and Crone[7] (US), and four short oral presentations by Hao and Lin[22], Huitzilopochtli er al.[23], Wong[24], and Mazzi[25], followed by a 20-min collective discussion on the short oral presentations. This session was chaired by Samuele Antonini (online).

In the fourth session on July 17th, 2021, there were three long oral presentations by Makino[8], Gustavsen et al.[9], Stylianides and Stylianides[10], and three short oral presentations by Dong and Liu[26], Zheng and Cheng[27], and Zhang and Wu[28], followed by a 20-min collective discussion on the short oral presentations, and a 45-min collective discussion on future research agenda and possible collaborations. This session was chaired by Viviane Durand-Guerrier (online) and Chao Zhou (from Shanghai).

Obayashi[29] and Barut[30] presented their posters during the related session.

Tab. 1. List of papers and posters presented

	Paper and author(s)
[1]	Writing a proof text at the university level: the role of knowing what a proof is. ***Nadia Azrou*** (Algeria).
[2]	Formalisation of proof: a tool for researcher. ***Faïza Chellougui*** (Tunisia).
[3]	Changes in the argumentation characteristics of mathematically gifted students — a longitudinal study. ***Simone Jablonski*** *and Matthias Ludwig* (Germany).
[4]	An inquiring-game for discovering and proving a geometric theorem. ***Carlotta Soldano*** (Italy).
[5]	Computer-assisted proving in the classroom. ***Xiaoheng (Kitty) Yan*** *and Gila Hanna* (Canada).
[6]	An application of habermas' theory of validity claims for classroom-based argumentation. ***Yuling Zhuang*** *and Anne-Marie Conner* (USA).
[7]	Characterizing mathematics teachers' proof-specific knowledge, dispositions and classroom practices. ***Orly Buchbinder*** *and Sharon Mc Crone* (USA).
[8]	Cognitive characteristics generating incomplete proof: analyzing the solving process of a geometrical problem by Japanese ninth graders. ***Tomohiko Makino*** (Japan).
[9]	Caught in-between tensions in teaching proof and proving. *Sikunder Ali,* ***Trond Stoelen Gustavsen****, Sigurd Johannes Hals, Andrea Hofmann, and Silje Trai* (Norway).
[10]	Posing new researchable questions as a dynamic process: the case of research on students's justification schermes. ***Andreas Stylianides*** *and Gabriel Stylianides* (UK).
[11]	Student interpretation of diagram in hyperbolic geometry: changes in the ontology of Geometric models. ***Younggon Bae*** (South Korea).
[12]	A comparative study of example uses in the proving-related activities of Korean and American students. ***GwiSoo Na*** (South Korea) *and Eric Knuth* (USA).
[13]	When is an argument an argument? Area-specific aspects of argum*ents reception.* ***Michael Meyer****, Christoph Koerner, and Julia Rey* (Germany).
[14]	Articulation of argumentation and mathematical modelling in the math classroom. ***Horacio Christian Solar*** (Chile)*, Manuel Goizueta* (Italy)*, Maria Aravena-Diaz* (Chile)*, and Andres Ivan Ortiz Jimenez* (Chile).
[15]	Fostering third graders fraction conceptions through argumentation and technology. ***Ho-Chieh Lin*** (USA).
[16]	Is there any difference in students' descriptions due to direction differences in a deductive reasoning task? ***Yoshiki Shibata*** *and* ***Tadashi Misono*** (Japan).
[17]	Investigating the differences between generic proofs and purely empirical verifications. ***Leander Kempen*** (Germany).
[18]	Proof and reasoning in high-stakes testing systems: the senior secondary mathematics curricula in Hong Kong and international baccalaureate diploma programme. ***Chun-Yeung Lee*** (UK).
[19]	Mathematics classroom argumentation: an international perspective. ***Markos Dallas*** (Norway).
[20]	Understanding the generality of mathematical statements and the role proofs play. ***Milena Damrau*** (Germany).
[21]	The function of definition in Japanese textbooks. ***Shogo Murata*** (Japan).
[22]	A comparative study of geometric proof opportunities in Chinese Taiwan and Chinese mainland middle school textbooks. ***Lei Hao*** *and P-Jen Lin* (Chinese Taiwan).
[23]	Using writing and discussions to support mathematical arguments in early algebra. ***Salvador Huitzilopochtli****, Daniel Lopez-Adame and Judit Moschkovich* (USA).
[24]	Justifications in exposition in algebra in school mathematics textbooks in Hong Kong. ***Kwong Cheong Wong*** (Hong Kong SAR, China).
[25]	Different types of reasoning in geometry in Brazilian high school mathematical textbooks. ***Lucas Carato Mazzi*** (Brazil).
[26]	Analysis of analogical reasoning exercises in primary school mathematics textbook: taking geometry field as an example. ***Yaoyao Dong*** *and Jian Liu* (China).
[27]	Regional and gender differences in Chinese 8th grade students' mathematical reasoning competency. ***Xin Zheng*** *and Jing Cheng* (China).
[28]	A study of the teaching process of mathematical concept argumentation based on tap - taking function concept teaching between expert and novice teacher in China as a case. ***Yi Zhang*** *and Xiaopeng Wu* (China).
[29]	The transient stages of inductive and deductive reasoning. ***Masanori Obayashi*** (Japan).
[30]	The Last Decade of Proportional Reasoning: A Systematic Review. ***Betül Barut*** (Turkey).

4. Collective Discussions and Future Research Agenda

We have tried to keep, as possible as we could, the collective discussion along the sessions despite the tiny time due to the high number of presentations. In the last session, a 45-min slot was dedicated to discussion on future research agenda. The main issues that emerged from the presentation and discussion in perspective of future research agenda were the followings:

- The need for going on exploring epistemological and philosophical references in mathematics education, considering the increasing role of digital technologies in mathematical activity, including the role of computer assisted provers.
- The role of logic in proof and proving has long been a controversial issue in mathematics education; the body of research has been developing during the last decade and need to be still developed, considering both the use of logic in the teaching and learning of proof, and the role of logic for analyzing students' proving activities.
- The value of symbolism and of formalization in proof and proving and their interplay with heuristic aspects.
- The scope and the role of generic proof in mathematics education is still under research: in which respect such generic proofs could be recognized as *genuine mathematical proofs* by teachers is one among the open questions.
- Examining argumentation and proof in textbooks is not an easy task, because the importance of students' activities and exchanges is crucial in argumentation and proving. Nevertheless, such analysis could inform on what is likely to happen or not in classroom.
- Another important issue concerns the teacher's knowledge related to argumentation and proof, this being related with teachers' own practices, and then with their training in proof and proving in their studies; this last point being related to the double Klein transition, from secondary school to university and then back from university to secondary school.

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https://doi.org/10.1142/9789811287152_0041

Topic Study Group 17

Problem Posing and Solving in Mathematics Education

Tin-Lam Toh[1], Manuel Santos-Trigo[2], Puay Huat Chua[3], Nor Azura Abdullah[4], and Dan Zhang[5]

ABSTRACT This report presents a summary of the content of the various presentation by the participants in ICME-14 under the Topical Study Group 17: Problem Posing and Solving in Mathematics Education. Some trends in the research on problem solving and problem posing are identified through this study. Other areas which were less explored were also highlighted

Keywords: Problem solving; Problem posing; Processes; Teaching and learning; Teacher education.

1. TSG-17 — A Brief Introduction

1.1. TSG-17 on Problem Solving/Problem Posing

The TSG-17 on Problem Solving (PS) and Problem Posing (PP) attracted large numbers of paper submissions. After the review process and confirmed registration of the participants, there were 35 paper presentations out of which 6 were long paper presentations and 29 were classified as short orals. There were a good mix of presenters from all over the world from the five continents. Hence, the views and trends in this study is a balanced views from researchers from all parts of the globe.

1.2. Questions on PS/PP to be addressed by the TSG

The focus of the TSG-17 is on four aspects of PS/PP: (1) the teaching and learning of mathematics in relation to PS/PP; (2) the enactment of PS/PP in the mathematics classroom; (3) teacher education and (4) the use of digital technologies in PS/PP activities. Mathematical PS has been the focus of a long line of inquiry in mathematics education for more than half a century, which gets back to as far as the publication of the seminal work How to Solve It of George Pólya (1945). With this attention on PS, the mathematics curricula around the world have placed PS as the heart of the national mathematics curriculum in many countries. The four-phase Pólya's problem solving

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model became well-known and is an icon for PS. Schoenfeld (1985) introduced his framework to further enhance Pólya's model. Although PS has been in attention for many years, its emphasis has not diminished. For example, in the Singapore context, PS has been the heart of the school mathematics curriculum. Despite the regular curricular revision, the PS framework remains unchanged, with the new initiatives of mathematics education introduced by the Singapore Ministry of Education serving to unpack and elucidate the various aspects of the PS framework. In China, PS and PP are highly emphasized in the national curriculum standard and are the main objectives of the national curriculum. PS/PP are widely used in the mathematics classrooms, although it is recognized that there is still the need to strengthen teachers' ability of encouraging students engage in PS and PP. PS has continued to attract researchers' attention in areas such as the enactment of PS in the mathematics classroom (Toh et al., 2008a, 2008b). For example, in Brunei schools, teachers are studying effective strategies to enact PS in the mathematics classroom. Not only has PS continued to attract attention of researchers even recently, new trends on PS have continued to emerge (Liljedahl and Santos-Trigo, 2019). In particular, in this technological era, studies on how PS can ride on the affordance of technology to enhance students' PS ability emerges (e.g., Santos-Trigo, 2019). Recently in Mexico, curriculum reforms at elementary and high school levels recognize that both PS and PP activities are central for students' development of mathematical thinking. Thus, in the post-pandemic instructional approaches, students are encouraged to always look for different ways to represent, explore, and solve problems and to constantly pose and pursue new questions or problems.

By comparison, PP is a much younger field of inquiry in mathematics education. A quick search of the education literature shows clearly that attention to this topic has grown rapidly in recent decades. Researchers and educators in many countries have incorporated PP as a research and instructional focus respectively. The juxtaposition of these two topics PP and PS in this TSG thus merges a mature field of inquiry with a more nascent one.

2. Content of the Paper Presentation

Although there are 35 paper presentation, we were able to identify three main trends of PS and PP throughout all the paper presentations. In giving a brief description of the papers, the session and the order of the speakers are presented in Tab. 1 (on the next page). There were four sessions: Session 1 (13 July, 14:30 to 16:30 GMT+8), Session 2 (14 July 19:30 to 21:30 GMT+8), Session 3A (17 July, 14:30 to 16:30 GMT+8) and Session 3 (17 July, 21:30 to 23:30 GMT+8). As this is a synchronous online presentation, the presentation slots were arranged mainly based on the time zone of the speakers, instead of the content. In re-classification of the content of the paper, four trends were visible: (1) teaching and learning in relation to PS/PP; (2) processes involved in PS/PPS; (3) teacher education in relation to PS/PP; and (4) textbook analysis and comparative studies on PS/PP. The details are discussed in the following subsections.

Tab. 1. List of papers presented

Paper and author(s)
Session 1
[1] Analysis on creating problem situation in middle school mathematic teaching. ***Peijun Zheng*** (China).
[2] Historical comparison and analysis of problems and problem-sovling in middle school mathmatics textbooks. *Rong Wang and* ***Cuiqiao Wang*** (China).
[3] Problem posing among Pre-service and in-service mathematics teachers. ***Ma Nympha Beltran-Joaquin*** (Philippines).
[4] Regulation of cognition during problem posing — a case study. ***Puay Huat Chua*** (Singapore).
[5] Characterizing the problem-solving processes used by Pupils in classroom: propositioin of a descriptive model. ***Stephane Favier*** (France).
[6] A framework on examining mathematical commuication in problem posing. ***Ling Zhang****, Jinfa Cai, and Naiqing Song* (China).
[7] Using problem posing to disgnose and understand perservice teachers conceptual understanding. ***Yiling Yao*** *and Jinfa Cai* (China).
[8] Elementary mathematics teachers learning to teach through problem posing:initial findiings of a longitudinal study. ***Dan Zhang****, Yiling Yao, and Jinfa Cai* (China).
[9] Primary school teachers' behaviors, beliefs, and their interplayt in teaching for problem solving. ***Benjamin Rott*** (Germany).
Session 2
[10] Teaching students how to pose mathematical questions. ***Peter Juhasz****, Reka Szasz, Lajos Posa, and Ryota Matsuura* (Hungary).
[11] How elementary and middle school teachers formulate multiplication and division word problems. ***Sintria Lautert****, Alina Galvao Spinillo, Rute Elizabete Borba, Juliana Silva, and Ernani Martins dos Santos* (Brazil).
[12] Gifted students strategy felxibility in non-routine problem solving. ***Yeliz Yazgan*** (Turkey).
[13] Types of reasoning promoted in mathematics classes in the context of problem-solving instruction in Geneva. ***Maud Chanudet*** (France).
[14] Investigating elementary school students' STEM problem posing: the walkstem after-school club. ***Min Wang*** *and Candace Ann Walkington* (USA).
[15] Designing professional development programs that support teachers' incorporation of problem solving in their mathematics instruction —the DCP mode. ***Jillian White****, Patrick Johnson, and Merrilyn Enid Goos* (Ireland).
[16] Mathematics problem multicontextual exploration, solving and posing in the classroom and teacher education: a perspective in critical education. ***Silvanio de Andrade*** (Brazil).
Session 3
[17] How do Chinese textbooks incorporate mathematical problem posing in different stages? ***Jiajie Yan****, Yufeng Guo, and Wenjia Zhou* (China).
[18] Appreciation of the aesthetic qualities of mathematical objects: an analysis of students problem sovling. ***Hayato Hanazono*** (Japan).
[19] Towards LITE, a local instructional theory for mathematicall explorations. ***Jayasree Subramanian****, K. Subramaniam, and R. Ramanujam* (India).
[20] Graphic organizers for problem-solving in primary mathematics: teachers' reflections. ***Nor Azura Abdullah*** (Brunei).
[21] The effect of problem-posing strategies on primary Pre-service teachers conceputual knowledge of fractions. ***Eda Vula*** (Albania).
[22] Investigating mathematics teachers' knowledge for teaching problem-solving. ***Brantina Chirinda*** *and Patrick Barmby* (South Africa).
[23] Elements of mathematical activity that emerge when future teachers of secondary school mathematics use digital technologies to solve problems. ***Matias Camacho-Machin****, Alexander Hernandez, and Josefa Perdomo-Diaz* (Spain).
[24] A study on evaluating prospective teachers' problem posing activity. ***Zoltan Kovacs*** (Hungary).
[25] Use of video clips to engagestudents in mathematical problem soling. ***Tin Lam Toh*** *and Eng* Guan *Tay* (Singapore).
[26] Problem solving and generalization with an advanced computing environment. ***Marina Marchisio****, Alice Barana, Alberto Conte, Cecilia Fissore, and Fabio Roman* (Italy).
[27] A study on improving flexibility in problem solving: unit teaching based on big-idea in mathematics. *Hongyun Li, Jian Dun, and* **Qilei Feng** (China).

Session 4

[28] Supporting students to compress mathematical knowledge while problem solving. ***Rogier Bos** and Rona Lemmink* (Netherlands).
[29] A strategy for enhancing mathematical problem solving. ***Miguel Cruz Ramirez*** (Cuba).
[30] A study on primary school mathematical problem-posing abilities in China. ***Na Yan*** (China).
[31] The process of posing problems: development of a descriptive process model for problem posing. ***Lukas Baumanns** and Benjamin Rott* (Germany).
[32] Automation of math discovery support: reinforcement of problems with criteria for evaluating partial solutions. ***Sergei Nickolaevitch Pozdniakov*** (Russia).
[33] Divison problem posing of fifth graders: a cross-national study in China and the United States. ***Fenqjen Luo**, Yali Yu, Monte Meyerink, and Ciara Burgal* (USA).
[34] Students engagment in problem posing while solve a fermi problem. ***Nelia Amado**, Susana Carreira, and Monica Alexandra Robelo Valadao* (Portugal).
[35] Develop your own problem! — problem posing in given real-world situations. ***Luisa-Marie Hartmann**, Stanislaw Schukajlow, and Janina Krawitz* (Germany).

2.1. *Teaching and learning in relation to PS/PP*

This category contains both empirical and theoretical discussion papers that were classified into several categories.

2.1.1. Creating Problems (or PP) in classroom situations

Both Zheng[1] and Harmann et al.[35] discussed the creation of problems by students. Zhang focuses more on the theories and characterization of problems created by middle school students while Harmann et al., to find out the type of problems students can pose and solve, which are problems that are related to the real world. Peter Juhasz et al.[10] propose how best to facilitate problem posing for mainstream students based on a method (which they term Posa method) that was originally developed for gifted students.

2.1.2. Processes involved in PS/PP

Lukas Baumanns and Rott[31] presented a descriptive model of the processes involved in students' problem posing through his empirical studies. Chua[4], through a case study, presented the regulatory cognitive phases during PP of students. Favier[5] characterizes the processes used by students when they solve problems in the mathematics classrooms. Zhang et al.[8] presented a framework for examining mathematical communication in problem posing, which refers to the process of conveying and expressing information during activities of problem posing.

Yazgan[12] examined students' flexibility in solving non-routine problems. Chanudet[13] studied the types of mathematical reasonings that students exhibited in problem solving instruction. Wang[14] presented a paper on investigating students problem posing ability. A study by Yan et al.[17] compared the problem posing ability between Han Chinese student and the students from the minority groups in China. Hanazono[18] discussed how students are able to appreciate the aesthetic qualities of mathematical objects through appropriate teacher intervention discussed in his paper.

Studies on intervention to improve students' processes in PS/PP are presented in this paper. Regarding the flexibility, Li et al.[27] discussed a teaching model that has been shown in their studies to improve students' flexibility in problem solving. Bos and Lemmink[28] presented strategies to compress mathematical objects, procedures and statements, which could play a major role in achieve success in mathematical problem solving. Ramirez[29] presented a strategy, heuristic strategy, for enhancing mathematical problem posing. With the use of a type of modelling problems, the Fermi problems, Amado et al.[34] showed that students' assumptions made of a problem are very closely connected to the types of problems that they pose.

In a study conducted by Luo et al.[33], the similarities and differences of the problems posed on division posed by students from China and the United States were analyzed.

2.1.3. Technology and PS

Three papers on the use of technology to enhance PS/PP were among the paper presentation. Pozdniakov[32] proposed the use of modern computer technologies to support independent PS, based on Polya's model of problem solving as the framework. Marchisio et al.[26] proposed the use of technologies, an ACE (Advanced Computing Environment), to support PS through representing and exploring mathematical tasks. Toh and Tay[25] discussed the creation and adaptation of video clips for the teaching of PS.

2.1.4. Enactment of PS/PP in mathematics classrooms

Subramanian et al.[19] presented a Local Instructional Theory for Exploration for classroom enactment of mathematical exploration. The study was based on the Realistic Mathematics Education framework. Abdullah[20] proposed the use of graphic organizers in developing PS ability amongst students in the Brunei schools.

2.2. Teacher education in relation to PS/PP

Beltran-Joaquin[3] presented a study on pre-service and in-service teachers' PP, she highlighted the need to strengthen PP skills among mathematics teachers. Lautert et al.[11] offered a zoom-in view in studying how elementary and middle school teachers formulate multiplication and division word problems, showing that the teachers could have difficulty posing complex and challenging word problems. Kovacs[24] presented a study on evaluating pre-service teachers' PP activity, by examining how the mathematical background of the original problem changes during PP.

Yao and Cai[7] presented a study and asserted that PP contributes to pre-service teachers' conceptual understanding of division of fractions, but also to diagnose and appreciate their mathematical understanding. Another study by Vula[21] showed that PP resulted in positively impact pre-service teachers' conceptual understanding of fractions. Further, Zhang et al.[8], in their large-scale longitudinal study, claimed PP could be an approach to build teachers' pedagogy in mathematics.

Chirinda and Barmby[22] investigated South African in-service teachers' knowledge for teaching PS using Chapman's MPSKT framework. White et al.[15], through an extensive literature review, summarized into the DCP model into the features of an effective professional development that support infusing of PS into mathematics instruction. The finding of Rott[9] suggests the importance of teachers' beliefs in the success of the teaching of PS. Matias Camacho-Machin et al.[23] discussed the impact on a group of pre-service teachers in using digital technology (GeoGebra) in solving mathematics problems. There was evidence of mathematical creation and reasoning and much activities among the pre-service teachers.

2.3. Textbook analysis in relation to PS/PP

There are two papers on this sub-topic. Wang and Wang[2] presented a historical comparison of the problems and PP tasks of three series of middle school mathematics textbooks in China and found an increasing trend in PP tasks in the textbooks over the years. Yan et al.[17] examines how PP tasks were introduced into the Chinese textbooks in different stages. The nature of the PP tasks was different in the primary and junior high levels.

2.4. Social context in relation to PS/PP

There is one paper on this sub-topic. Andrade[16] in his presentation calls for a PS and PP interconnected approach vis problem exploration in a critical education perspective. PS/PP should be seen, in addition to the pedagogical consideration, from the wider level of socio-political-cultural context, as are the other dimensions of education.

3. What Next?

The TSG-17 had very rich discussion over the four sessions described in the above section. All the papers were compiled into pdf format and made available to all the participants of the TSG. The work of the TSG does not end with the conclusion of the congress. As an afterwork of the congress, selected presenters from the above presentation were invited to contribute their work to (1) a special issue in Hiroshima Journal of Mathematics Education which has been published in October 2022; and (2) a book on PS/PP to be published by Springer which targeted before the next ICME. The authors were invited to refine their papers, which will be peer-reviewed for the publication process.

Acknowledgment

The TSG-17 team would like to thank the organizing committee for this opportunity to helm the TSG on PS/PP. In particular, the chair would like to acknowledge Professor Ed Silver who was the original chair of TSG-17, and started the initial work with the chair in the paper review process prior to the COVID-19 pandemic. Professor Silver

withdrew from this participation due to health reasons. The TSG-17 team wish he could continue to enjoy a good health.

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https://doi.org/10.1142/9789811287152_0042

Topic Study Group 18

Students' Identity, Motivation, and Attitudes towards Mathematics and Its Study

Maike Vollstedt[1], Masitah Shahrill[2], Karin Brodie[3], Donglin Chen[4], and Bozena Maj-Tatsis[5]

ABSTRACT This report provides a short summary of the work of TSG-18 on "Students' Identity, Motivation, and Attitudes towards Mathematics and Its Study". We begin with a characterization of the field of students' affect before we provide information on the review process, the participants, sessions, proceedings, and the mode of publication of papers after the conference.

Keywords: Students' affect.

1. TSG Description

In TSG-18, we focused on students' affect with a special focus on students' identity, motivation, and attitudes towards mathematics and its study. There was a parallel Topic Study Group on teachers' affect (TSG-34: Affect, beliefs, and identity of mathematics teachers).

Affective variables can be seen as either hidden or explicit factors that influence learning processes and outcomes. The different research perspectives used in the study of students' affect include psychological, sociological, philosophical, and linguistic, and all these as well as other perspectives were welcome. In addition to the general domain "affect", the title of this Topic Study Group highlighted three concepts that have been popular in the field of mathematics education: identity, motivation, and attitudes. This was not seen as restrictive. On the contrary, we invited discussion on all areas of affect, encompassing anxiety, attitude, beliefs, emotion, flow, goals, identity, interest, meaning, motivation, needs, norms, self-concept, values etc. All of them play a crucial role in students' learning of mathematics and there are also subtle differences among them. In addition, we welcomed the analysis of the mutual relationship between

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affective constructs and their connection to cognition and other constructs studied in mathematics education as well as the description of programs for promoting aspects of affect.

According to the call for papers, the activities of the working group were aimed at

- clarification of the dimensions of affective constructs and their relationships;
- development of measurement instruments (questionnaires, rubrics for qualitative analysis etc.) and other methodological tools for research on affect;
- role of the different affective concepts (see the list given above) in learning of mathematics, problem solving, proof, etc.;
- developmental aspects of affect, e. g. development of interest, anxiety etc.;
- intervention or comparative studies aimed at changes in affective variables;
- relationships between students' and teachers' affect, role of affect in communication among students or between students and teachers;
- affect as sociocultural phenomenon and lifelong learning;
- development of learning communities that foster positive affective climate;
- relationships between affect and gender/ethnicity/mathematical activity etc.

2. Participants

The participation in the Topic Study Group highlights the growing interest in research on affective issues. At ICME-13, there was only one affect related TSG in which 86 researchers were involved and 22 papers were presented on both students' and teachers' affect (Hannula et al., 2017). For TSG-18 in 2021, that focused on students' affect only, we had 2.5 times as many papers. In total, 57 contributions were considered in the review process. Based on the reviews as well as gender and regional balance, the TSG organizing committee decided to welcome 38 short oral presentations, and eleven authors were invited to hand in an extended version of their paper. In addition, eight posters were accepted for presentation.

The COVID-19 pandemic caused the shift from an on-site conference in 2020 to an online-conference format in 2021. Therefore, some authors were not able to attend the conference, and we finally had 35 presentations in our topic study group in which 80 researchers from all around the world were involved. Most of the researchers participated online, but in addition there was always a significant number of participants in the room in Shanghai.

3. Sessions

As TSG was the biggest TSG at ICME-14, we were allowed an additional time slot for the organization of our program. Thus, we had two sessions of 120 minutes and two sessions of 90 minutes time for the work in our TSG. Still, we had a very strict time limitation due to the very high number of presentations. There were 20 minutes time for presentation and discussion of a long paper, 10 minutes for a short paper, and five minutes for a poster (Tab. 1).

Presentation times were primarily organized according to the time zones of the presenters to enable every presenter to give their presentation at their daytime. In addition, we also tried to group presentations on similar topics wherever possible.

The first session started with an opening and welcome message by the TSG organization committee. Each session was then chaired by a different member of the committee. They paid special attention to making sure that contributions came both from the online participants and from the room in Shanghai. The discussions prove to be fruitful for the presenters.

Tab. 1. List of papers presented

Paper, poster and author(s) in order of presentation
Session 1
[1] Mathematics-anxiety students reasons and feelings when choosing to solve particular problems. ***Kai Kow Joseph Yeo*** (Singapore).
[2] Mathematical problem-solving beliefs of Filipino seventh graders. ***Katrina Grace Q. Sumagit*** *and Nympha B. Joaquin* (Philippines).
[3] Understanding the intentions of shadow education in Brunei Darussalam. ***Masitah Shahrill*** *and Ai Len Gan* (Brunei).
[4] Developing and validating a scale for measuring students' critical thinking disposition in mathematics educatioin. *Changgen Pei and* ***Jiancheng Fan*** (China).
[5] Exploration of math mindset changes over time in an urban sample of elementray and secondary school students in the United States. ***Beijia Tan****, Jenee Love, Leigh M. Harrell-Williams, and Christian E. Mueller* (USA).
[6] Classroom goal structures, Chinese students' goal orientations and mathematics achievement. ***Meng Guo*** *and Xiang Hu* (Hong Kong SAR, China).
[7] Applying the Theory of Planned Behaviour to 2012 Australian PISA data. ***Mun Yee Lai*** *and* ***Pauline Wong Wing Man Kohlhoff*** (Australia).
[8] The non-intellectual level of efficient mathematics learning of junior high school students and their influence pathways on mathematics learning performance. ***Rui Yang****, Guangming Wang, and Shuang Li* (China).
[9] Different contributions of parental expectations and teacher's behaviors to students' mathematics-related beliefs. ***Sheng Zhang*** *and Guangming Wang* (China).
[10] Does parents' attitude towards math matter to young adolescents' math achievement in China? Meditating effects of math anxiety. ***Mingxuan Pang*** *and Xiaorui Huang* (China).
[11] Mathematical identities of a high school mathematics learner in landscapes of mathematical practice. ***Wellington Munetsi Hokonya*** *and Pamela Vale Mellony Graven* (South Africa).
Session 2
[12] Make a tutorial! The impact of a classroom video project on emotions, motivations and achievement. ***Daniel Barton*** (Germany).
[13] Perceived difficulty in answering mathematical task: reflections on metacognitive factors. ***Marta Saccoletto*** *and* ***Camilla Spagnolo*** (Italy).
[14] Affective issues in the learning of abstract algebra. ***Marios Ioannou*** (Canada).
[15] A framework of learners' mathematical identities. *Aarifah Gardee and* ***Karin Brodie*** (South Africa).
[16] A conceptual framework relating mathematics clubs and mathematical identities. ***Lovejoy Comfort Gweshe*** *and* ***Karin Brodi*** (Zimbabwe).
[17] Influence of collaborative learning on student attitudes toward mathematical problem solving. ***Farzaneh Saadati*** (Chile).

Session 3

[18] A quantitative analysis of six aspects of student identity and creativity-fostering instruction. ***Paul Regier****, Miloš Savić, and Houssein El Turkey* (USA).

[19] Does types of problem influence on interest? A replication of a German study in the Spanish context. ***Clara García-Cerdá*** *and Irene Ferrado* (Spain).

[20] Attitudes, beliefs and emotions towards graph theory. ***Claudia Vargas-Díaz*** *and Victoria Núñez-Henríquez* (Italy).

[21] Predicting college major choice in STEM with students data at grades 9 and 11. ***Jihyun Hwang*** *and Kyong Mi Choi* (USA).

[22] The role of interpersonal discourse in small-group collaboration in developing mathematical arguments and student identity. *Shande King* (USA)*, Lynn Hodge* (USA)*, and Qintong Hu* (China).

[23] Exploring pre-service teachers persistence through multiple strategies tasks. ***Amanda Meiners****, Kyong Mi Choi, and Dae Hong* (USA).

[24] Meaningful reasons for learning mathematics. ***Maike Vollstedt*** (Germany).

[25] Positive emotions in early algebra learning. ***Yewon Sung****, Ana Stephens, Ranza Veltri Torres, Susanne Strachota, Karisma Morton, Maria Blanton, Angela Murphy Gardiner, Eric Knuth, and Rena Stroud* (USA).

Session 4

[26] Stereotype on female's success boosts female's math learning. ***Xiaorui Huang*** *and Bo Dong* (China).

[27] Peer pressure effect on student teachers' affective relationship with problem posing. ***Bozena Maj-Tatis*** (Poland)*,* ***Konstantinos Tatsis*** (Greece)*, and Andreas Moutsios-Rentzos* (Greece).

[28] Questionnaire of attitudes toward statistics for junior high school students in Japan. *Yoshinori Fujii and Koji Watanabe* (Japan).

[29] "Dear Kingos, it's all right to be noisy!" why is it so hard to get them talking? ***Natanael Karjanto*** (South Korea).

[30] The character of students mathematical values in learning mathematics. ***Miho Yamazaki*** *and Wee Tiong Seah* (Japan).

[31] A case study of mathematical research presentation in a public junior high school: focus on the relationship of assumption of others and the quality of learning. ***Tomoaki Shinobu*** (Japan).

[32] Mathematics anxiety: a Portuguese study in higher education. ***Vanda Santos****, Anabela Pereira, Teresa Neto, and Margarida M. Pinheiro* (Portugal).

[33] The transition from school to university mathematics: which roles do students interest and beliefs play? ***Sebastian Geisler*** (Germany).

[34] Exploring 11$^{\text{th}}$ grade students' attitudes towards mathematics. ***Jiraporn Wongkanya****, Naruon Changsri, Kiat Sangaroon, and Maitree Inprasitha* (Thailand).

[35] High school students images, anxieties and attitudes toward mathematics. ***Shashidhar Belbase*** (United Arab Emirates).

The closing at the end of our last session provided some information with respect to the proceedings of our TSG and a very heartful summary of our work: The word cloud in Fig. 1 was created from all papers that were presented in TSG-18.

4. Publication

All papers were collected in the proceedings of the TSG (Vollstedt, 2021), which were also accessible during the conference. In addition, the TSG participants were given the opportunity to publish extended versions of their papers in *Didactica Mathematicae*, an international journal of mathematics education. Four papers were published in volume 43 (2021), while five papers are currently under review for publication in volume 44 (2022).

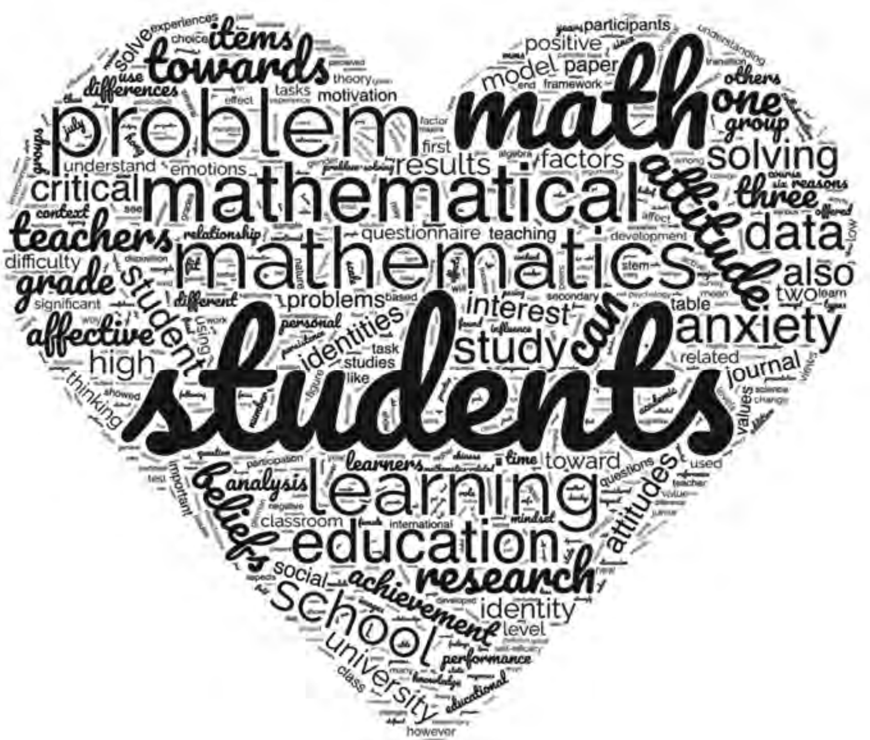

Fig. 1. Heart-shaped word cloud created from all papers of TSG-18

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https://doi.org/10.1142/9789811287152_0043

Topic Study Group 19

Mathematical Literacy, Numeracy and Competency in Mathematics Education

Sarah Bansilal[1], Ratu Ilma Indira Putri[2], Vince Geiger[3], Bo Zhang[4], and Kathy O'Sullivan[5]

1. Aims of the TSG

The Mathematical Literacy Topic Study Group 19 at ICME-14 was organized around four key themes that drew from emerging findings in the literature related to discussions about the Mathematical Literacy field:

The "place" of mathematical literacy: What are the specific focuses and topics that can characterize the notion of mathematical literacy? How does the notion of mathematical competency relate to mathematical literacy? How should mathematical literacy be taught directly, as a by product of regular mathematics or integrated across subjects?

Theories of mathematical literacy: What are some theories and methodologies that can help us understand the issues central to the teaching and learning of mathematical literacy?

Research issues: What can research tell us about the teaching and learning of mathematical literacy? What do empirical results from large- and small-scale studies indicate that can inform our thinking about the conceptualization, teaching, learning, or assessment of mathematical literacy? What are some understandings of mathematical literacy, and how do these permeate the curricula, teachers' identities, beliefs, attitudes and practices, teacher education, learning materials, and assessments, etc.?

Views about the future of mathematical literacy: If we are committed to developing mathematical literacy at the school level, what barriers should we overcome? What new types of initiatives, policies, or collaborations (across subject areas, outside schools) are needed? What are potential gains or losses with possible initiatives?

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1.1. Submissions

Papers and discussions related to these themes were intended to stimulate discussion about key directions for future research related to mathematical literacy. Overall, at the conference our TSG had a total of 17 oral presentations and one poster presentation.

1.2. Sessions

There were three sessions in total, the first one was 2 hours and the second and third ones were 90 minutes each. There were two invited speakers who presented in the first two sessions.

1.3. Paper topics

The list of papers that were presented in the sessions appears in Tab. 1.

Tab. 1. List of papers and authors

Paper and author(s)	
[1]	Common European numeracy framework — a multifaceted perspective on numeracy. ***Kees Hoogland*** (The Netherlands)*, Javier Diez-Palomar* (Spain).*, and Niamh O'meara* (Ireland).
[2]	Mathematical literacy: what, why and how. ***Ross Turner*** (Australia).
[3]	Elements and definitions of the core literacy of mathematics in primary school from an international perspective: based on NVivo 12.0 coding analysis. ***Xuan He*** *and Yunpeng Ma* (China).
[4]	Top-level design and systematic thinking for the cultivation of math competencies-case study and inspirations. ***Feng Ma*** (China).
[5]	It is time pre-service teachers develop their numerate abilities to support their students numeracy learning. ***Kathy O'Sullivan*** (Ireland).
[6]	Aspects of fair-minded critical thinking in mathematics education: based on the perspective of critical mathematics education. ***Yuichiro Hattori*** *and Hiroto Fakuda* (Japan).
[7]	How teachers generate ideas for classroom numeracy tasks. ***Vince Geiger*** (Australia).
[8]	Pre-service teachers' experiences with the Australian national numeracy test. ***Jennifer Hall*** *and Anna Podorova* (Australia).
[9]	Mathematical Literacy in pre-service teacher-designed mathematics picture books. ***Zetra Hainul Putra****, Gustimal Witri, and Syahrilfuddin Syahrilfuddin* (Indonesia).
[10]	Identifying 9th grade students' errors in solving a mathematical literacy problem. ***Maryam Mohsenpour****, Mahbobeh Rohanipur, and Zahra Gooya* (Iran).
[11]	A new model design to improve mathematical literacy: A dual focus teaching model. ***Cigdem Arslan****, Murat Altum, Tugce Kozakli-Ulger, Isil Bozkurt, Recai Akkaya, Furkan Demir, Zeynep Ozaydin, and Burcu Karaduman* (Turkey).
[12]	Unpacking some challenges of learning mathematical literacy in South Africa. ***Sarah Bansilal*** (South Africa).
[13]	Designing PISA-like mathematics task using Asian games context. ***Ratu Ilma Indra Putri*** *and Zulkardi Zulkardi* (Indonesia).
[14]	Assessing PISA-like tasks considering levels of context use for mathematics problems. ***Ahmad Wachidul Kohar****, Tatag Yuri Eko Siswono, and Dayat Hidayat* (Indonesia).
[15]	Financial numeracy practices in secondary school: A study with mathematics teachers from Quebec. ***Alexandre Cavalcante*** *and Annie Savard* (Canada).
[16]	A semantic network analysis of information literacy in school mathematics in Korea. ***Eun Hyun Kim*** *and Rae Young Kim* (South Korea).
[17]	Mathematical literacy in Norway. ***Oda Heidi Bolstad*** (Norway).
[18]	A survey on primary school mathematics teachers conception of mathematics core literacy in the context of Chinese curriculum reform. ***Qiuchan Li*** (China). (Poster)

2. Themes

There were a large variety in research topics presented during the sessions, however there seemed to be three overarching themes that emerged across the presentations. The first related to interpretations of mathematical literacy and its broader purposes in different countries. The second theme concerned issues related to developing mathematical literacy skills or practices. The third theme related to issues about task design for mathematical literacy. The discussion around these themes are presented below.

2.1. Understanding mathematical literacy and exploring interpretations of constructs related to mathematical literacy

In his invited talk[2] Ross Turner raised the point that although there is no agreed definition of the term "mathematical literacy", there are components which need to be considered in any definition. These are: practical aspects including number sense, arithmetic and spatial skills; procedural knowledge which includes knowledge about procedures, theorem and definitions; and, a more pervasive way of thinking about mathematics including thinking mathematically, reasoning and communication. Hoogland in his invited talk[1] presented findings raising from the project that aimed to develop a Common European Numeracy Framework (CENF) for adult learning. The framework recognizes numeracy as a social practice and takes into account meta-cognitive aspects, psychological and sociological facets, and power-related factors, which influence the quality of numerate behavior among adults.

Many presentations focused on the curriculum interpretation or on teachers' interpretations of constructs related to mathematical literacy in various countries. He and Ma[3] looked at the elements considered as the core literacies in primary school mathematics curricula in the US, UK, Australia, Japan and Singapore while Ma[4] focused on the competencies covered in the mathematics curriculum of the 2-year IB programme designed for high school learners. Bolstad[17] analysed curriculum documents to identify how mathematical literacy was treated in Norway and suggested that teachers need more advice and support about how to implement the ideas in the classroom. Kim and Kim[16] conducted an analysis of 30 mathematics teacher guidebooks from Korea to better understand how information literacy was interpreted and recommended for use by teachers in their classrooms. Li[18], in a poster session, reported on a survey on primary school mathematics teachers' conceptions of mathematics core literacy from a rural district in the context of Chinese curriculum reform. Arslan and her colleagues[11] outlined a dual focus model that was introduced to middle school mathematics teachers in Turkey which focuses on acquisition of concepts as well as on applications in order to develop the mathematical literacy skills of their students. Cavalcante and Savard[15] conducted a study with Canadian teachers to understand how they incorporated financial numeracy in their classrooms and found that the teachers delved into issues mainly related to personal finances and not those of citizenship or social justice.

2.2. Research about developing skills related to mathematical literacy

Hattori and Fakuda[6] looked at the notion of fairminded critical thinking and how this could be actualised amongst their students. They shared an example of how this was targeted in lesson practices. The lessons focused on developing critical citizenship through the implementation of statistics education in the context of the environment.

O'Sullivan[5] researched the numeracy skills of 204 preservice-teachers in Ireland and found that most were not able to complete all of the numeracy tasks. Hall and Podorova[8] explored the experiences of 458 PST's with the high stakes, mandatory test, Literacy and Numeracy Test for Initial Teacher Education (LANTITE). They found that there were connections between the students' preparation and their perceptions of the test as well as differences by demographic groups.

Mohsenpur and her colleagues[10] conducted interviews with nine Grade 10 students in Iran, to better understand the errors they made when solving a mathematical literacy problem. Bansilal[12] presented a contextual attributes framework for identifying and describing some of the challenges experienced by learners in working with contexts within mathematical literacy tasks.

2.3. Research related to the design of Mathematical literacy tasks

Putra[9] and her colleagues from Indonesia looked at tasks designed by 13 groups of preservice teachers and found that the most popular contexts were those based on personal issues. Kohar[14] and his colleagues from Indonesia analysed 130 mathematical literacy task to identify the levels of context use and found that 29%, 65% and 6% displayed a zero, first and second order use of contexts respectively. Putri and Zurkardi[13] from Indonesia set out to assess how well tasks set within the Asian Games context, could work to support learning. Geiger[7] looked at the different ways in which teachers generate ideas for the design of numeracy tasks, including taking advantage of incidental events; bringing together elements of curriculum from different learning areas; and archiving ideas.

3. Areas for Future Research

The closing discussion touched on a number of areas such as the fact that there is no established definition of mathematical literacy for which there is wide agreement. It is clear that there are varying ideas about the need for, and value of, mathematical literacy and how it fits into the curriculum. One presenter commented that if one thinks about mathematical literacy as the ability to implement mathematics, then it reproduces the view that mathematical literacy is a discipline instead of it being recognised as a practice. This discussion highlights the need for more interrogation of what mathematical literacy is and how mathematical literacy practices could be enhanced. Most participants agreed that we need increased attention to teaching practices, as well as teacher preparation and professional development programmes which can help teachers to improve the numerate behaviour of their students. The presentations also articulated the need for more attention in future research to assessment design as well as curriculum policies.

https://doi.org/10.1142/9789811287152_0044

Topic Study Group 20

Learning and Cognition in Mathematics (Including the Learning Sciences)

Gaye Williams[1], Pablo Dartnell[2], and Wenjuan Li[3]

ABSTRACT This paper includes the themes and descriptions for TSG-20 Learning and Cognition, the session topics developed in response to the themes, a report of the review process and results of this, the TSG-20 program including the research focus for each invited speaker, profiles of the invited researchers, the content of their submitted papers, and the authors and titles of other papers. It also includes participant reflections about sessions that provide indicators of future intended research directions.

Keywords: Learning; Cognition; Teacher practices; Student activity.

1. The Theme, Subthemes, and Descriptions

The scope of research on learning and cognition in mathematics education is extensive and diverse in relation to questions posed, theoretical frameworks selected, and methodologies employed. Theoretical perspectives include (but are not limited to) forms of cognitive constructivism, and social constructivism, and more recently, interconnections between these (including integration, partial integration, and networking of such theories). Affective, and embodied elements, and personal characteristics of learners and teachers are amongst the many other constructs that form part of various theoretical frameworks. Learning and Cognition in Mathematics, TSG-20, 2020 specifically included 'the Learning Sciences' which interrogates interplays between cognitive, social, psychological and cultural elements of learning processes in diverse contexts, for the purpose of 'improving' learning environments. Research into learning mathematics through STEM (Science Technology Engineering and Mathematics) Education although increasing, is not yet reflected in the proportion of STEM related papers submitted to TSG-20 in ICME14. Although this description contains illustrations of research foci within TSG-20, there are opportunities for intending contributors to focus within these, or to justify other foci associated with learning and cognition in mathematics education.

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1.1. Subtheme 1: teacher change processes and influences upon them

Processes of teacher learning, that can enhance student learning of mathematics, have been researched with various theoretical frameworks employed (including but not limited to cognitive, social, belief based, and dispositional frameworks, and interconnections between various of these frameworks). Areas of research into influences on teacher change processes include but are not limited to professional learning models employed, types of pedagogical approaches under focus: teacher-controlled or teacher-guided learning, and approaches enabling various degrees of student autonomy in the learning of mathematics. Depending on the personal characteristics of the teacher, some approaches may be easier to employ than others. Such personal characteristics include but are not limited to types of knowledge possessed, prior experiences, and whether resilience and/or self-efficacy are possessed. There are many other factors associated with teaching that could become the focus of a submitted paper, as long as that focus can be justified as belonging to TSG 20 Learning and Cognition.

1.2. Subtheme 2: student learning processes and influences upon these

Theoretical frameworks employed to study processes of student learning of mathematics include, but are not limited to, those associated with cognitive constructivism, social constructivism, and embodied, cultural, and material conceptions of mathematics cognition. Various combinations of these theoretical framings have also been developed. Influences on the nature of mathematical understandings developed include, but are not limited to, the degree of student autonomy in the learning situation, affective elements of the process, the nature of the learning environment, and personal characteristics of the student. Study of processes associated with the construction of mathematical insights are crucial to this theme because they can build deep mathematical understandings and positive student personal characteristics. Study of learning processes in situations in which students have little to no autonomy is also important as many mathematics teachers employ such pedagogical approaches. Studies of how to increase students' feelings of safety in such controlled learning situations or decrease the boredom of other students are important areas of research, as is study of learning in particular mathematical situations.

1.3. Subtheme 3: the learning sciences

The Learning Sciences is dedicated to furthering the scientific understanding of learning processes for the purpose of designing and implementing learning innovations to increase learning opportunities. This research field highlights the social nature of learning and the many different settings in which learning may occur. Studies interrogate various interplays between cognitive, social, psychological and cultural factors in learning processes in diverse contexts. To enable study of learning as it occurs in messy naturalistic settings, creative research designs have been, and continue to be, developed.

Studies in this multidisciplinary field include but are not limited to foci such as: the situated nature of knowledge and ways of knowing and learning; individual, and group learning processes; and mathematics learning in out-of-school settings, such as, museums, and homes; mathematics learning difficulties and disabilities. Student learning and teacher learning are two of the many areas of researcher attention within this field. Further study of mathematics learning in out of school settings has the potential to inform mathematics learning more generally.

2. Organizing TSG-20

2.1. Panel

The Study Topic Group, TSG-20 Learning and Cognition included the following panel members, all of whom contributed in various ways to the development of the TSG:

Chair: Gaye Williams, University of Melbourne;
Co-chair: Pablo Dartnell, University of Chile;
Members:
Wenjuan Li, New York University,
Zain Davis, University of Cape Town, and
Chunli Zhang, Beijing Normal University.

This team was drawn from universities in various countries across the world. All five team members took part in the review process. Gaye Williams, Pablo Dartnell, and Wenjuan Li hosted the three sessions of TSG-20 at ICME14.

2.2. Invited speakers

Selection of invited speakers was guided by the themes of focus. These speakers were Alison Castro Superfine (USA), Keiko Hino (Japan), Lieven Verschaffel (Belgium) and Alejandro Maiche (Uruguay). However, Alejandro was unable to participate due to circumstances at the time.

2.3. Review process

25 submissions were received for TSG-20. The 4 from invited speakers were accepted without review. One was a poster, and one paper was referred to another TSG for which it was more appropriate. The other 19 were research papers. The 5 TSG-20 panel members undertook the reviews. Each paper was reviewed by 2 panel members (one chair/cochair and another panel member). Where there was disparity between these two reviews, the other chair/co-chair also reviewed the paper before it was discussed by the chair and cochair. Where a paper was close to being judged a long paper, authors were provided with advices and invited to resubmit their papers before a final judgement was made. 14 papers had two reviewers, and 5 papers had 3 reviewers. The review process classified the 19 contributions as 5 long papers and 14 short papers, and authors from 12 of these 19 papers accepted and presented in TSG-20. The 12 presentations, together with the invited talks, were set into 3 sessions and listed in Tab.

1, in which IT stands for invited talks, LO for long oral presentations, and the others are short oral presentations.

Tab. 1. List of presentations presented in TSG-20

Paper and author(s)	
Session 1	
[1]	Exploring new models for teacher professional learning: Working *with* teachers rather than on. ***Alison Superfine Castro*** (USA). (IT)
[2]	Introduction of STEM education through collaborative action research practices. ***Fatlume Berisha*** *and Eda Vula* (Kosovo). (LO)
[3]	Theorizing teachers' learning of students' mathematical thinking in the context of student-teacher interaction. ***Biyao Liang*** *and Kevin C. Moore* (USA). (LO)
[4]	Students' ways of thinking in a computer-based mathematics investigation project. ***Joyce Mgombelo****, Wendy Ann Forbes, Chantal Buteau, Eric Muller* (Canada)*, and Ana I. Sacristán* (Mexico).
[5]	Reciprocity between teachers' and students' problem-solving actions enables teacher change. ***Gaye Williams*** (Australia).
Session 2	
[6]	Interactive patterns that lead to children's discursive changes in lessons comparing fractions. ***Keiko Hino*** *and Yuka Funahashi* (Japan). (IT)
[7]	Assessing mental abstraction activities using eye-tracking techniques. ***Eivind Kaspersen*** *and Trygve Solstad* (Norway). (LO)
[8]	Mathematics itself: reflections about an often neglected, but pivotal dimension. ***Michael Neubrand*** *and Carl von Ossietzky* (Germany). (LO)
[9]	On the epistemological significance of contextualizing in mathematical cognition. ***Marcia M. F. Pinto*** *and Thorsten Scheiner* (Australia).
[10]	Learning strategies used by high achieving and low achieving students in mathematics. ***Bishnu Khanal*** (Nepal).
Session 3	
[11]	The amazingly frequent, efficient, and flexible use of the subtraction-by-addition strategy in elementary school children's mental multi-digit arithmetic: A challenge for cognitive psychology and mathematics education. ***Lieven Verschaffel****, Joke Torbeyns, Gwen Verguts, and Bert De Smedt* (Belgium)*,* (IT)
[12]	Numerical processing profiles in children with varying degrees of arithmetical achievement. ***Nancy Estévez*** (Cuba)*, Danilka Castro* (Chile)*, Eduardo Martínez* (Cuba)*, and Vivian Reigosa* (Uruguay). (LO)
[13]	How proper use of mathematics can help students to build quantum physics thinking to learn the subject: simple harmonic oscillator. ***Jose Vieira Do Nascimento Junior*** (Brazil).
[14]	A cognitive model of learning applied to data analysis of mathematics learning. ***Jairo Alfredo Navarrete*** (Chile).
[15]	Exploring basic numerical capacities in children with varying degrees of arithmetical achievement. *Danilka Castro Cañizares,* **Pablo Dartnell** (Chile)*, and Nancy Estévez Pérez* (Cuba).

2.4. TSG-20 Program

Challenges associated with program organization arose from the online nature of the program (caused by a) the global pandemic, and time constraints associated with panel decision that all participants should be involved in all sessions. Safety nets were constructed to cater for technological problems that might arise. They included setting up multiple ways that a presentation could be uploaded, and construction of a TSG-20 Website where presentations could be made available, session information provided, and additional questions and discussions uploaded. Time constraints were reduced through email and website introductions to sessions and to invited researchers. The sessions and researchers are displayed below with capitalization of paper presenters. Discussion times were factored into each presentation and a short discussion time occurred at the end of each session. Twenty to thirty participants attended each of the three sessions.

2.4.1. Session 1, July 13th. Host: Wenjuan Li

Processes of teacher learning, that can enhance student learning of mathematics, have been researched with various theoretical frameworks employed (including but not limited to cognitive, social, belief-based, and dispositional frameworks, and interconnections between various of these frameworks). Areas of research into influences on teacher change processes include but are not limited to professional learning models employed, types of pedagogical approaches under focus: teacher controlled or teacher-guided learning, and approaches enabling various degrees of student autonomy in the learning of mathematics. Presenters shared their recent work on new models or approaches to support teacher learning, and their theory or framework to analyze teacher change.

The invited talk[1] in Session 1 was given by Alison Superfine Castro, who is a Professor of Mathematics Education and Learning Sciences at the University of Illinois at Chicago. Her research interests focus primarily on studying and supporting mathematics teacher learning. Alison has developed different analytic approaches to study mathematics teacher educators and the knowledge needed to teach teachers. She has received various grants to design and study learning environments for mathematics teacher preparation courses and published extensively in the areas of mathematical knowledge for teaching, professional noticing, mathematics teacher's learning trajectory-based formative assessment practices. Alison is an active member of the international mathematics community. She is currently serving as an associate editor for the Journal of Mathematics Teacher Education and the EURASIA Journal of Mathematics, Science & Technology. In this ICME-14 TSG-20 invited talk, she presented new models and the design principles to support teacher professional learning.

The author gave the abstract as follows:

> New models for supporting teacher professional learning generate new conceptualizations of teacher learning, afford new designs for studying teacher learning over time, and situate teacher learning in problems of practice relevant to their own circumstances. In this paper, I describe two examples in which we engaged teachers in new models to support their professional learning, including examples of the various forms of inquiry we developed, as well as ways in which teachers engaged in the activities as part of these efforts. I then discuss a set of design principles underlying both examples. Finally, I discuss tensions that emerged from these efforts.

2.4.2. Session 2. July 16th. Host: Gaye Williams

This session focused around the learner — student learning processes, influences upon these learning processes, learning strategies employed, the degree of student autonomy in the learning process, the nature of the mathematical objects developed, and theoretical frameworks employed.

The invited talk[6] in Session 2 was developed by Keiko Hino with her colleague Yuka Funahashi. Hino is a Professor of Mathematics Education at Utsunomiya University in Japan. Her scholarly interests include the development of students' mathematical thinking through classroom teaching, international comparative study of the teaching and learning of mathematics, and mathematics teachers' professional development. Hino has produced many scholarly publications and undertaken positions to improve mathematics education, including as an editor of Japanese Primary and Lower Secondary School Mathematics Textbooks, as an external expert for Lesson Study in Mathematics, and as a member of the Editorial Board of MTED (Mathematics Teacher Education and Development Journal). Keiko Hino and Yuka Funahashi (Nara University of Education, Japan) have collaborated on research that informs the professional learning of teachers for more than ten years now, undertaking detailed analyses of problem-solving activity during mathematics lessons.

The abstract of the talk is as follows:

> The analysis presented in this paper examined the changes in the way children explained equivalent fractions and explored the teacher's key interactions that enabled such changes. Data from nine consecutive fifth-grade lessons in Japan taught by an experienced teacher were examined using a guided focusing pattern framework, from which it was found that the changes in the explanations were mostly in the focusing phase. The teacher's key interactive actions were classified into three categories: proposing focus, modifying focus, and narrowing focus. In particular, it was found that the teacher consistently attempted to change the children's focus from procedure to quantity and quantitative relationships using intervening language and by evoking discursive rules.

2.4.3. Session 3. July 17th. Host: Pablo Dartnell

This session is built mostly around TSG-20 ICME-14's 3rd subtheme: The Science of Learning, although it necessarily has some components from the other two subthemes. The science of learning is dedicated to furthering the scientific understanding of learning processes for the purpose designing and implementing learning innovations to increase learning opportunities. Among the presentations scheduled for this session, our invited speaker shares findings about surprisingly frequent and efficient use among Belgian elementary students of a mental subtraction strategy. In addition, results are shared about basic cognitive numerical capacities and their relationship with arithmetic difficulties, conducted in two different Latin American countries; relationships between the use of mathematics and the building of knowledge in quantum physics; and a proposed cognitive model of learning with implications for the analysis of data regarding the learning of mathematics.

The invited talk[11] in this session was given by Lieven Verschaffel, who is a full professor in Educational Sciences, and director of the Center for Instructional

Psychology and Technology (CIP & T) at the Katholieke Universiteit Leuven, Belgium. His research work in Mathematics Education covers a wide variety of topics, many of them related to TSG-20, such as problem solving, strategy choice and change, conceptual change, metacognitive and affective aspects of learning, and early and elementary mathematical education. The quality of his research work has led to him. receiving many awards, invitations as plenary lecturer, and member of plenary panels of many international conferences, including some previous versions of ICME. In addition, he. has become a member of many editorial boards. On this occasion he presented the result of two studies (conducted in collaboration with other researchers from KU Leuven) dealing with a mental subtraction strategy used by elementary school students.

The abstract of the talk is as follows:

> In two related studies — a first study with 6th grade elementary school children and a second study with children from 4th until 6th grade of elementary school, we investigated the use of the subtraction-by-addition strategy in children with different levels of mathematics achievement. In doing so, we relied on Siegler's cognitive psychological model of strategy change, which defines strategy competencies in terms of four parameters — strategy repertoire, distribution, efficiency, and selection — and the choice/no-choice method, which is essentially characterized by offering items in two types of conditions — choice and no-choice conditions. In both studies, children of different mathematics achievement levels solved multi-digit subtraction problems in the number domain up to 1,000 in one choice condition (wherein they could choose between direct subtraction or subtraction by addition on each item) and two no-choice conditions (wherein they had to use either direct subtraction or subtraction by addition on all items). Distinction was made between two types of subtraction problems: problems with a small versus large difference between minuend and subtrahend. Although mathematics instruction only focused on applying direct subtraction, most children reported using subtraction-by-addition in the choice condition. Subtraction-by-addition was also applied surprisingly frequently and efficiently, particularly on small-difference problems, and children flexibly fitted their strategy choices to both numerical item characteristics and individual strategy speed characteristics. Interestingly, these results were obtained for children of all grades and all mathematical achievement levels. These remarkable findings — both from a cognitive psychological and a mathematics educational perspective — add to our theoretical understanding of children's strategy acquisition and challenge current mathematics instruction practices that pay exclusive attention to direct subtraction.

3. Reflections and Future Directions

Time for discussion of future directions was limited by participants' interest in continuing discussion of research from the third session. Post-session reflections from participants (see italics below) indicated intended future directions though. Slight changes to quotes were made to increase clarity. Names of invited speakers or presenters are used to refer to studies in this section.

In Session 1, Alison Superfine Castro[1] stimulated new thinking about models to support teacher professional learning: *I enjoyed the presentation of Dr Castro very much, it was an eye-opener for future research paths*. Participants connected other research from this session to her work: *Alison Castro Superfine's contribution was quite informative to me, as was Williams'*[5] *and other presentations (e.g., Berisha's*[2]*). Collaboration seems to be a key point in teacher education.*

In Session 2, Keiko Hino's presentation[6] was appreciated by participants for the detailed way it examined interactions between the teacher and the students, and diagrammatic representations communicating this: *The way the research and the teaching in the classroom were analysed was very good*, and *the diagram helped in seeing the large amount of time the classroom teaching dedicated to interactions between teachers and students*. This research was referred to in subsequent presentations and reflections. Comparisons were made between Pinto's *proposal*[9] *for working with fractions, developed from the perspective of contextualizing* and Keiko Hino's presentation *typical of Japanese teaching undertaken without the usual context (real world etc) but still in the reflection mode.* Pinto's team intend to use Keiko Hino's work to extend their *thinking about other possible approaches to equivalence of fractions.*

The interconnected nature of most presentations in Session 2 was also recognised: *the research presented is interwoven in many senses. Four presentations focused on strategies of learning or modes of learning (or a specific mode of learning), to inform teaching or educational policies. Neubrand*[8] *and Kaspersen*[7] *focused on abstraction but differed in their conceptions of the interpretative model built. Further discussion of these two approaches and methodologies should be productive. Pinto*[9] *and Neubrand*[8] *proposed different categories for organizing understanding of the same phenomena — the leaning of maths*. *Both also attempted to avoid dichotomies*.

In Session 3, Lieven Verschaffel's presentation[11], participants were *surprised by the prevalence of use of the Subtraction-by-Addition method and the accuracy and speed of responses found in Belgium*. This presentation raised questions for future research including a*) could these results be replicated in other places* and b) *might there be alternative strategies that could be employed for other mathematical procedures, that could produce similarly strong results?* Questions for future research were also raised by Dartnell's presentation[15]: *Given that very low achievement in mathematics can have a variety of causes—not always related to a disability—what might be found if students with Mathematics Learning Disabilities were the focus of such research?*

In Summary, sessions for TSG-20 were vibrant and extended the thinking of various participants in different ways. Our thanks to all participants.

https://doi.org/10.1142/9789811287152_0045

Topic Study Group 21

Neuroscience and Mathematics Education/Cognitive Science

Inge Schwank[1] and Marie-Line Gardes[2]

1. State of the Art

Without a doubt, doing mathematics depends on creative, problem-solving thinking. Mathematics didactics provides mathematics with some cloak: What is to be understood is the teaching and learning of mathematics and how the acquired knowledge can be used for teaching processes. In a fundamentally oriented manner, borrowings from cognitive science are taken up and mental processes are examined using neuroscientific methods. Mathematics is a vast field of knowledge full of concepts and tools, mathematical thinking is highly complex. Fig. 1 displays the complex of areas and approaches relevant to TSG-21.

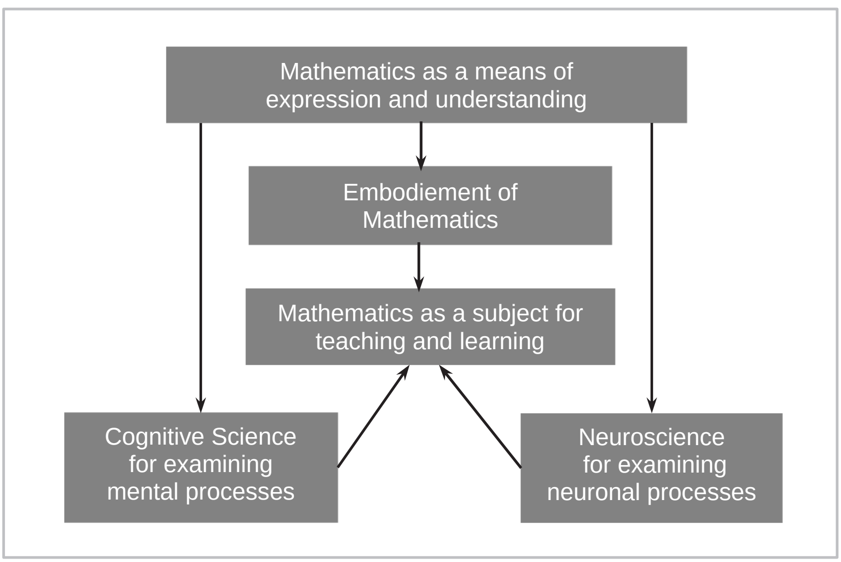

Fig. 1. Relevant complex of areas and approaches

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2. Contributions

The contributions by scientists from six countries (Tab. 1).

Tab. 1. List of papers presented at TSG-21

Paper and author(s)	
[1]	General Spatial Ability Other than Special Mathematical Ability Correlates with Ill-Structured Problems in Junior Students. *Xinlin Zhou, Chunxia Qi, Li Wang, and* ***Chen Cao*** (China).
[2]	Behavioral Processing of Fractions in Adults with and Without Mathematics Learning Difficulties. *Parnika Bhatia, Jessica Leone, Jerome Prado, and* ***Marie-Line Gardes*** (France).
[3]	Consideration of Characteristics of Eye Movement and Brain Activity During Mental Rotation Tasks. ***Tatsuki Kondo****, Naoko Okomato, and Yasufumi Kuroda* (Japan).
[4]	Learning Representations of Mathematical Objects in Computational Models of Mathematical Cognition. ***Trygve Solstad****, Silvester Sabathiel, and Celestino Creatore* (Norway).
[5]	Electrophysiological Characteristics of First-Grade Children at Different Levels of Number Sense. ***Yuqing Zhao****, Feidan Yu, and Zikun Gong* (China).
[6]	Declarative Knowledge and Procedural Knowledge: Learning Processes in the Case of Pound Arithemtic. *Roland Grabner* (Austria), *Stefan Halverscheid* (Germany), *Jochen A. Mosbacher* (Austria), *and* ***Kolja Pustelnik*** (Germany).
[7]	Even Young Children Are Able to Grasp and Apply Logical Rules in Mathematically Structured Environments — the Puzzle of Cognition. ***Inge Schwank*** *and Elisabeth Schwank* (Germany).

Contribution by Zhou et al. (Cao's presentation)[1] falls into the area of cognition-oriented mathematics didactics. By the use of relevant tests the connections between general cognitive and mathematical abilities were examined. Apparently, general spatial ability has a greater impact on ill-structured problem solving than special mathematical ability.

Contribution by Bhatia et al. (Gardes' presentation)[2] also comes from the field of cognition-oriented mathematics didactics and examines, how fraction knowledge and competencies are processed in adults with mathematical learning difficulties. Results indicate that different pathways are utilized for accessing the magnitude of non-symbolic line ratios and symbolic fractions. Mathematical learning difficulties when dealing with fractions had a greater impact on failures in calculation and estimation and less on representing symbolic fractions in verbal form and vice versa.

Konto et al.[3] used cognitive science methods. Teaching spatial geometry is challenging. This contribution's objective is to better understand the basics of cognitive processes in view of spatial geometry. For this purpose, the present study examines the characteristics of eye movement and brain activity during mental rotation tasks. The result shows: The mental rotation tasks that required lesser time involved a high frequency of looking at the same parts of left and right solids and reached max activation time quickly.

Solstad et al.[4] applied to the mathematical modeling of cognitive processes when doing mathematics. The initial question is: Can our mathematical abilities be explained in neuroscientific or computational terms? Some of the properties of representations generated by idealized neural network models for numbers are examined and described. Furthermore, the question of how computational tools can contribute to a greater understanding of the relation between mathematics education and neuroscience is addressed.

Zhao et al.[5] applied a neuroscientific method and examined the electrophysiological characteristics in the first-grade children at different levels of number sense in a number comparison task. In fact, there are differences in brain activity in children with a high level of number sense compared to children with a middle or low level of number sense. The results suggest that children with different levels of number sense show different electrophysiological characteristics during number sense processing.

A joint contribution[6] by German and Austrian researchers comes from the field of cognition-oriented mathematics didactics with the use of mathematical modeling. Learning processes when dealing with a special arithmetic are examined. A distinction is made between "know-how"; i.e.: learning how to solve arithmetic problems and "know-that"; i.e.: learning arithmetic facts. The power-law function is used for the description with a view to its coefficients and any correlations between these coefficients. This worked satisfactorily only with regard to fact learning. Correlations between the two learning processes turn out to be weak.

Finally, Schwank and Schwank[7] came from the field of cognitive mathematics. Based on the theory of functional-logical thinking versus predictive-logical thinking, different scenarios are examined. Access to (first) arithmetic is easier for children dominantly using the approach of functional-logical thinking, mathematically gifted children show special talents here. The development of mathematical thinking can be promoted through the use of special mathematical play worlds as the results from an early support study with preschoolers indicate. Particularly children with difficulties in the area of early mathematics seem to benefit from the support provided by mathematical play worlds.

3. Outlook onto Future Research Activities

There is still a large discrepancy between the possibilities of gaining knowledge based on theoretical concepts as well as experimental methods offered by Cognitive Science and Neuroscience and the needs for knowledge acquisition, which arise due to the complexity mathematical thinking, mathematical knowledge, mathematical problem solving as well as mathematical creativity. Nevertheless, it can be expected — especially if interdisciplinary research is intensified — that the mental processes when dealing with Mathematics will be increasingly understood more thoroughly and that these findings for mathematical teaching and learning processes can be used specifically and successfully. An important issue will be gaining professional mathematicians' cooperation as well as further people, especially children with so-called special needs. The digital transformation of society will play a big role in this advancement. Accessibility issues have never been easier to adapt to the addressee (e.g. in terms of font size, contrast, images, interactions, problem variations). The variety that is thereby made possible in a considerably easier way, sets a whole new potential for research questions and scientific studies. Cognitive Mathematics will be the state of the art (Fig. 2).

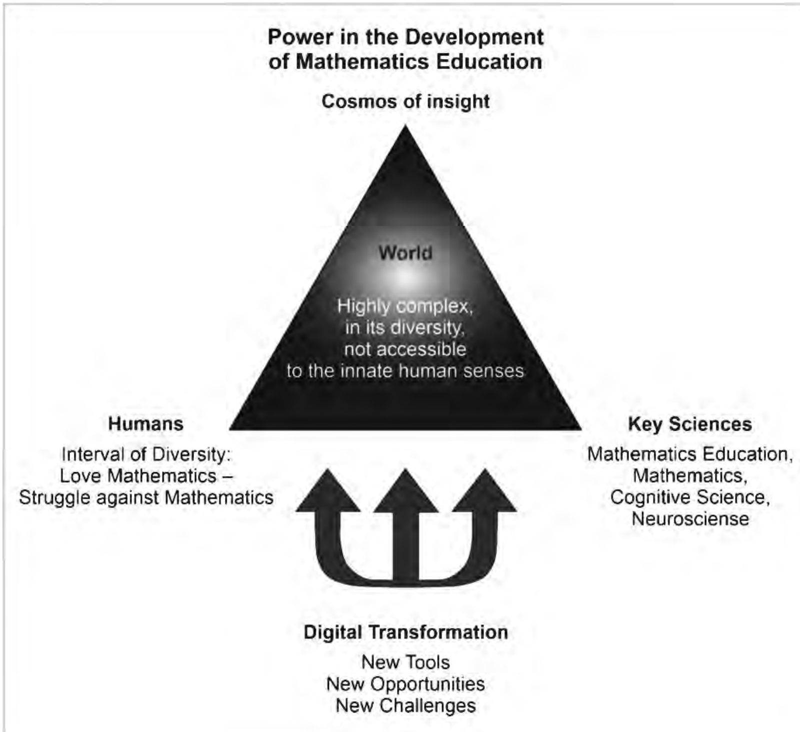

Fig. 2. Cognitive Mathematics as a combination of important approaches to the world of understanding, learning, teaching and strengthen mathematics

https://doi.org/10.1142/9789811287152_0046

Topic Study Group 22

Mathematical Applications and Modelling in Mathematics Education

Gilbert Greefrath[1] and Susana Carreira[2]

ABSTRACT In the field of applications and mathematical modelling there is intensive research. Within the TSG, we have thematically addressed the teaching of mathematical modelling, teacher education, and modelling processes and competencies of school and university students. Future research directions are expected to consider theory building, empirical studies, including developing standardized research instruments, and the use of technology.

Keywords: Mathematical modelling; Teaching modelling; Teacher education; Modelling competencies and processes.

1. Theme and Description

The teaching and learning of mathematical applications and modelling is a world-renowned field of research in mathematics education and it has been an important theme for teachers and researchers especially during the last 50 years; and the importance has been growing worldwide during the last decade. This is evident, for example, in the International Congress on Mathematical Education (ICME) regular topic study groups and lectures on applications and modelling, and the series of conferences of the International Community on the Teaching of Mathematical Modelling and Applications (ICTMA) since 1983. This increasing interest is a consequence of several factors; on the one hand there is the public demand for the relevance of mathematics outside the discipline, and on the other hand there is an increasing number of research projects and empirical studies which focus on specific aspects of applications and modelling in mathematics teaching and learning. Many recent qualitative and quantitative research studies on mathematical modelling in school and higher education have focused on students and their modelling processes; however, teachers clearly play an important role in implementing mathematical modelling into mathematics lessons and in fostering students modelling competencies. Furthermore, classroom settings also play an important role. Enriching the focus on teacher practice in proposing and implementing interventional activities, there has been

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a research approach to the design of single modelling lessons as well as to the whole modelling learning environments at different school levels.

This topic study group (TSG-22) considers the importance of exploring relations between mathematics and the real world that occur in educational environments. It also recognizes the value of examining the discussions in research and development on the applications and modelling issues at the primary, secondary and tertiary school levels, including the mathematics teacher education. The TSG also recognizes the interplay between research and development of modelling learning environments (Greefrath et al. 2023).

2. Program Overview

2.1. Team and participants

The TSG-22 team was composed of Xiaoli Lu (China), George Ekol (South Africa), Susana Carreira (Portugal, Co-Chair) and Gilbert Greefrath (Germany, Chair). From the large number of submissions, 10 long oral, 26 short oral and 8 posters were presented during the conference. The first authors of these papers and posters came from 20 different countries. The countries most strongly represented in TSG-22 were Chile, China, Germany and Japan.

2.2. Structure of the sessions

For the TSG, four sessions of 90 to 120 minutes each were available. Each session was chaired by one of the team members. The sessions were structured thematically. During the session, long and short oral contributions alternated. For some presentations, a joint discussion took place together, as far as the high number of contributions allowed. In the first session, in addition to the thematic part, there was an introduction for the group and a presentation as a thematic overview of the current state of research.

2.3. Theme 1: introduction and teaching mathematical modelling

Following a welcome and overview of sessions by the chair, Gabriele Kaiser made a long oral presentation on *The Teaching and Learning of Mathematical Modelling. A Description of the Current State-of-the-Art*. In particular, she addressed theoretical perspectives (Kaiser and Sriraman, 2006) and modelling competencies (Niss and Blum, 2020). This was a very relevant starting point for the work of the TSG. Subsequently, the *teaching of mathematical modelling* was examined from different angles. For example, sociocultural and geographical aspects were discussed, but also specifics of statistical modelling were highlighted (see Tab. 1).

Tab. 1. Presentations on the themes "introduction and teaching mathematical modelling"

	Paper and author(s)
[1]	The teaching and learning of mathematical modelling. a description of the current state-of-the-art. ***Gabriele Kaiser*** (Germany).
[2]	Sociocultural influences on mathematical modelling: an ethnomathematical perspective. ***Milton Rosa*** *and Daniel Clark Orey* (Brazil).
[3]	Teaching methods for modelling problems. ***Stanislaw Schukajlow*** *and* ***Werner Blum*** (Germany).
[4]	Examining the geographical features of the nasu area. analysing the origin of the nasu area using mathematics. ***Masahiro Takizawa*** (Japan).
[5]	A mathematical modelling technique as tool for teaching mathematics. ***Eloisa Benitez-Mariño*** (Mexico).
[6]	Theorizing tensions between mathematical modelling processes and conventional mathematics instruction. ***Wenmin Zhao*** *and Samuel Otten* (China).
[7]	The rationales of statistical modelling in education research from a mathematical modelling perspective. ***Takashi Kawakami*** *and Jonas Bergman Arleback* (Japan).
[8]	Modelling in a teacher education programme. ***Dragana Martinovic*** (Canada).

2.4. *Theme 2: teacher education*

The second topic of the TSG was teacher education. On the one hand, there were contributions on specific topics such as global warming or subject areas like STEM. On the other hand, there were contributions to promote certain aspects of professional competencies such as noticing skills or self-efficacy. Different types of school levels were taken into account (see Tab. 2).

Tab. 2. Presentations on the theme "teacher education"

	Paper and author(s)
[9]	Mathematical modelling in STEM contexts. Characterization of STEM skills and gender gaps in initial formation of mathematics teachers. ***María Aravena Diaz****, Marcelo Alejandro Rodriguez, Susan Valeria Sanhueza Henriquez, Maria Jose Seckel, and Angelica Urrutia Seplveda* (Chile).
[10]	Using assessment for learning to support students modelling activities. ***George Ekol*** (South Africa).
[11]	Epistemic states of university mathematics teachers in mathematical modelling education. ***George Gotoh****, Mitsuru Kawazoe and Hirofumi Ochiai* (Japan).
[12]	Using staged videos to foster pre-service teachers noticing skills. ***Alina Alwast*** *and Katrin Vorhölter* (Germany).
[13]	Prospective teachers self-efficacy for teaching mathematical modelling. ***Hans-Stefan Siller****,* ***Gilbert Greefrath****, Raphael Wess, and Heiner Klock* (Germany).
[14]	Pedagogy that supports mathematical modelling. One elementary school teachers story. ***Rejoice Akapame*** *and Robin Angotti* (USA).
[15]	Pre-Service mathematics teachers project-based mathematical modelling instruction: conception, task design, and enactment. ***JooYoung Park*** (USA).
[16]	The development of a modelling teacher education program starting from the transformation of a mathematised task into modelling tasks. ***Akihiko Saeki****, Masafumi Kaneko, Takashi Kawakami, and Toshikazu Ikeda* (Japan).
[17]	Prospective teachers of mathematics suspend common sense in solving word problem. ***Abolfazl Rafiepour*** *and Zohreh Khazaei* (Iran).

2.5. Theme 3: students modelling processes

The topic with the most contributions was on modelling processes and modelling competencies of students. Here the particular situations in different countries were considered and also different age groups were considered. Also, different ways of measuring modelling competencies were discussed. Various models for describing modelling processes were also discussed here, and the use of technology in modelling was explored (see Tab. 3).

Tab. 3. Presentations on the theme "students modelling processes"

Paper and author(s)	
[18]	The mathematical modelling landscape: a literature review on perspectives, methodology, content, unit of analysis, and geography. ***Armando Paulino Preciado Babb****, Fredy Peña Acuña, Andrea Ortiz Rocha, and Armando Solares Rojas* (Canada).
[19]	Distinguishing the distinctions: observing the solving of a mathematical modelling task. ***Paola Andrea Ramirez Gonzalez*** (Chile).
[20]	Mathematical modelling skills of secondary students. ***Kwan Eu Leong*** (Malaysia).
[21]	Mathematical modelling in the new curriculum: are chinese students ready? ***Jian Huang*** *and Binyan Xu* (China).
[22]	Student presentations of mathematical modelling as a site for fostering reflective discourse. ***Hyunyi Jung****, Corey Edison Brady, Jeffrey Allen McLean* (USA)*, Angeles Dominguez* (Mexico)*, and Aran Glancy* (USA).
[23]	How do undergraduate students hold the individual assumptions in collaborative modelling? ***Kazuhiko Imai*** *and Akio Matsazaki* (Japan).
[24]	Investigating students data moves in a citizen science based data-rich model-eliciting activity. ***Jeffrey Allen McLean****, Corey Edison Brady, Hyunyi Jung, Aran Glancy* (USA)*, and Angeles Dominguez* (Mexico).
[25]	Differences in students conceptions about mathematics when participating in a mathematical modelling contest. ***Flavio Guiñez*** (Chile).
[26]	Measurement mathematical modelling competency and its relationship to mathematical interests of seventh grade. ***Zhiyong Xie****, Yaling Li, Tian Wang, and Jian Liu* (China).
[27]	Assessment of four-grade students mathematical modelling competency: take one city of china as an example. ***Tian Wang****, Zhiyong Xie, and Jian Liu* (China).
[28]	Study of a problem solving using the extended mathematical working space framework. ***Laurent Moutet*** (France).
[29]	Introducing a composite model for investigation in real world problem. ***Kazem Abdollahpour Chenary*** *and* ***Abolfazl Rafiepour*** (Iran).
[30]	A computer-based learning environment on mathematical modelling: research design and pilot studies. ***Lena Frenken*** (Germany).

2.6. Theme 4: university students modelling processes

The fourth and last topic of the TSG was dealing with modelling processes and modelling competencies of students at the university. Various models for describing modelling processes were discussed and the use of technology in modelling at the university was considered. Also, different instruments for the assessment of modelling competences at the university were presented. It was also described how one can learn certain mathematical contents through mathematical modelling (see Tab. 4).

Tab. 4. Presentations on the theme "university students modelling processes"

Paper and author(s)
[31] Undergraduate students' modelling routes mediated by technology in the learning of linear transformations. ***Susana Carreira***, *Guillermo Enrique Ramirez Montes, and Ana Claudia Henriques* (Portugal).
[32] Is quality teaching favourable for the development of modelling competency? an empirical study with engineering students over two years. ***Rina Durandt***, *Werner Blum, and Alfred Lindl* (South Africa).
[33] Validating a modelling competencies assessment. ***Jennifer A. Czocher***, *Sindura Kandasamy, and Elizabeth Roan* (USA).
[34] Mathematical modelling with biology undergraduates: using activity theory to understand tensions. ***Yuriy Rogovchenko*** (Norway).
[35] Calculus learning competency through mathematical modelling. ***Lorenza Illanes*** *and* ***Roberto Retes*** (Chile).
[36] Research on evaluation of college students' mathematical modelling ability based on AHP and BP neural network. ***Yixin Dong***, *Huanhuan Zhang, Meng Ci, and Ziyi Wang* (China).

3. Future Directions and Suggestions

Even though mathematical modelling in mathematics education is currently being very intensively discussed and researched, there are still some open questions for the future. These relate to both theoretical areas and empirical areas. For example, the further development of central constructs such as modelling competencies or professional competencies for teaching modelling is an interesting field of consistent research development. For further empirical studies, appropriate standardized test instruments are needed, which should be developed and shared in the community. This could also contribute to a better interlinking of studies on the methodological and thematic side. Furthermore, the changes caused by a heterogeneous school population and the use of technology in all school levels up to university should be responded to accordingly.

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https://doi.org/10.1142/9789811287152_0047

Topic Study Group 23

Visualization in the Teaching and Learning of Mathematics

Cristina Sabena (chair)[1], Marc Schäfer (co-chair)[2], Marei Fetzer[3], Hui-Yu Hsu[4], and Zhiqiang Yuan[5]

1. Introduction and Aims of the TSG

In mathematics education research, visualization is generally referred to as the product and the process of creating, using, interpreting, and reflecting on visual information. It plays an important role in mathematical thinking and in most branches of mathematics: there is general consensus in the mathematics education community that visualization is a vital component of conceptual understanding, reasoning, problem solving and proving.

The aim of the TSG-23 was to interrogate the significance for research in understanding the role of visualization processes in the teaching and learning of mathematics at all school levels. Specifically, it was the aim of TSG-23 to not only show-case this research in a global context, but also to start thinking about and considering possible visualization research trajectories and frameworks that could support scholars with articulating their own visualization research agendas.

In the call for papers we proposed some subthemes, which highlighted the close connection of visualization with different aspects involved in mathematics learning and teaching, such as:

- *Visualization as a cognitive process*, including visualization and reasoning, justification, argumentation, imagination, and difficulties with visualization.
- *Visualization as a mathematical construct*, including visualization and mathematizing, visualization and generalizing, visualization as a mathematical proof.
- *Visualization and new technologies*, including technologies such as interactive dynamic software, 3-D printing, augmented reality, virtual reality and other digital media.
- *Visualization and neurological functioning*, including research into neurological activities in the brain associated with visualization processes and

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their implications to mathematical thinking and teaching-learning processes

- *Visualization and language*, including interrogating the relationships between visualization, signs and language(s), including embodied aspects such as gestures and bodily actions.
- *Visualization in school practice and in teacher education*, including research into the explicit inclusion of visualization in school curriculum, practice and assessment, and interrogating the development of visualization skills in teacher education programs.
- *Visualization as a social process*, including research about the negotiations on visualization in classrooms as well as diversity aspects across various cultural contexts.
- *Visualization and research methodology*, including asking questions about visualization research design and methodological approaches that foster visualization.
- *Visualization and theory*, including research into possible overarching theoretical frameworks that could frame and orient visualization research such as embodied cognition theories, learning theories, socio-cultural theories.

This list was not meant to be exhaustive, and we encouraged contributions based on empirical research, including ongoing studies, as well as theoretical elaborations and reflections on a theme.

2. Submissions

We received 29 submissions of papers and 13 submissions of posters from 16 countries (South America and North America, Asia, Europe and Africa), which shows a very encouraging cultural diversity. Papers underwent a peer review process that involved two submitting authors and one team member for each paper submission, and two team members for each poster submission. The review process was guided by criteria that included innovation, theory and methodology, coherence, interest for an international audience and clarity. Paper proposals could be accepted as either long or short papers. After the review process was completed, and the conference finally took place, 13 long papers and 9 short papers were presented and discussed during the conference sessions.

3. Sessions

During the conference, we met in a blended modality (in presence and at distance) in four sessions, for a total of six hours. Onsite facilitators in Shanghai helped us to manage a reliable connection between onsite and online participants.

As TSG-23 we wanted to provide sufficient time for scientific exchange and discussion. As we had not enough time to discuss and interact with the presenters during our official sessions, we met in additional interactive sessions during dinner times (Shanghai times) in order to share moments of scientific discussion and exchange pertaining to the specific presentations. These meetings were realized only at distance and made use of the parallel rooms facilities. Throughout the four days we attempted to facilitate dialogue between participants in order to give constructive feedback to the presented studies, identify possible research themes and opportunities that emerged from the presentations. We found that the discussion time proved very fruitful in facilitating networking opportunities and sustaining the interest momentum that was generated during the presentations. At the end of the last session, we hosted a 50-minute whole group reflection session to consider some research implications that arose.

4. Paper Presentation and Emerging Themes

The long papers (LO) were allocated 15 minutes for presentation, and short papers (SO) were allocated 10 minutes. Papers were briefly discussed in the official sessions, and then discussed in more depth in the parallel rooms during the additional sessions.

Our programme was organized around eight themes or clusters which characterized the submissions. Below we present each theme as they appeared in the programme and list the related papers in Tab. 1 (on the next page).

- *Theme 1: Visualization and problem-solving.* This theme related specifically to research that looked at how visualisation process and mathematical problem-solving articulated with each other.
- *Theme 2: Classroom interaction.* This theme interrogated how issues of visualization played out in particular classroom interaction contexts.
- *Theme 3: Visualization and teaching*. The focus of this theme was how selected teachers used visualization tools and media (including dynamic geometry software) to teach mathematics.
- *Theme 4: Different kinds of representations, different technologies.* In this theme the presenters looked at how different representations and diagrams could be meaningfully utilized in the mathematics classroom.
- *Theme 5: Diagrams and mathematics visualization.* Here the specific focus was on how mathematical visualization related to mathematical representations.
- *Theme 6: Math, visualization and other disciplines.* In this theme researchers engaged with mathematical visualization processes using means from other disciplines.
- *Theme 7: Visualization and (latest) technologies.* Here researchers considered how different digital resources were used either as a research tool or means to engage with mathematics.
- *Theme 8: Educational materials.* In this theme the presenters considered how specific educational materials, in the context of visualization processes, could be used in the mathematics classroom.

Tab. 1. List of papers presented

Paper and author(s)
Session 1
Theme 1: Visualization and problem-solving
[1] Imaging and visualizing in geometry: Explorations by mathematics university students. ***Ferdinando Azarello**, Cristina Sabena, and Carlotta Soldano* (Italy). (LO)
[2] Visualization as an embodied problem-solving process. ***Beata Dongwi** and Marc Schäfer* (South Africa). (LO)
[3] Characterizing visualization and spatial analytic reasoning for solving isometry problems. ***Leah Michelle Frazee** and Michael Battista* (USA). (LO)
[4] The role of visualization towards student's mathematical abstraction and representation: The case of probability. ***Dennis Lee Jarvis Baring Ybanez** and Catherine Vistro-Yu* (Philippines). (LO)
Theme 2: Classroom interaction
[5] The use of gestures and language as co-existing visualization teaching tools in multilingual classes. ***Clemence Chikiwa** and Marc Schäfer* (South Africa). (LO)
[6] On objects and visualizations — An interactionistic perspective. ***Marei Fetzer*** (Germany). (LO)
Session 2
Theme 3: Visualization and teaching
[7] How teachers scaffold students in visualizing diagram for understanding geometric problem solving. ***Hui-Yu Hsu*** (Chinese Taiwan). (LO)
[8] Preservice and Inservice teachers' mathematics visualization skills. ***Vimolan Mudaly*** (South Africa). (LO)
[9] Dynamic visual instructions by GeoGebra for introducing Takada's theorem on pentagons. ***Hirotshi Furutsu**, Yukiko Ishii, Hisashi Kato, Yusuke Washio, and Noriko Hirata-Kohno* (Japan). (SO)
[10] High school mathematics inquiry teaching based on GeoGebra visualization environment. ***Wei Wang** and Xue Huang* (China). (SO)
Theme 4: Different kinds of representations, different technologies
[11] The development of 3D representations using physical manipulatives, technology-aided design and 2D drawings. ***Jill A. Cochran*** (USA). (LO)
[12] The social construction of knowledge in a new pedagogical setting: The same activity presented as three different interactive diagrams. ***Elena Navtaliev*** (Israel). (LO)
Session 3
Theme 5: Diagrams and mathematics visualization
[13] Visualization as vision, imagination and intuition: reflections on graduate students struggling with a visual conjecturing problem. ***Francesco Beccuti*** (Italy). (LO)
[14] Mapping diagrams: Function visualization of real and complex analysis and matrix algebra. ***Martin Flashman*** (USA). (SO)
[15] Interactive visualizations of topics in engineering mathematics. ***Antti Rasila*** (China). (SO)
Theme 6: Math, visualization and other disciplines
[16] Drawing (on) diagrams: Typicality of geometric shapes in concept image elicitation for secondary students. ***Santanu Dutta**, Charudatta Sharad Navare, and Harita Raval* (India). (SO)
[17] Research on visualization in mathematics learning based on mathematical drama performance or by video. ***Yan Li**, Pan Liu, and Xinyu Liu* (China). (SO)
Session 4
Theme 7: Visualization and (latest) technologie
[18] Some like it social: Looking into the interplay between math and internet memes. ***Giulia Bini** and Ornella Robutti* (Italy). (LO)
[19] Children's ambiguous interpretation of visualizations — eye tracking as a diagnostic tool for division concepts. ***Daniela Götze*** (Germany). (LO)
[20] A review of the application cases of augmented reality (ar) in mathematics education. ***Luona Wang*** (China). (SO)
Theme 8: Educational materials
[21] Using geometric intuition in the domain of number and algebra: From textbook designers' perspective. ***Jiling Gu** and Fei Zhang* (China). (SO)
[22] Methodology visual experience based mathematics education 2019. ***Janos Szasz Saxon** and Zsuzsa Dardai* (Hungary). (SO)

5. Areas for Future Research

In our last session we consolidated our deliberations by identifying possible visualisation avenues for further research that emerged. These were:

- Visualisation as a cognitive process. This includes visualisation and reasoning, visualisation and imagination, difficulties with visualisation.
- Visualisation and mathematising.
- Visualisation and new technologies, such as interactive dynamic software, augmented reality and other digital media.
- Visualisation and language, specifically the relationship between visualisation, signs, language(s), including embodied aspects such as gestures and bodily actions.
- Visualisation as a social process, including negotiations on visualisation in classrooms as well as diversity aspects across various cultural contexts.
- Visualisation and theory, which includes researching possible overarching theoretical frameworks that could frame and orient visualisation research such as embodied cognition theories, learning theories and socio-cultural theories. Do we have a visualisation theory?

https://doi.org/10.1142/9789811287152_0048

Topic Study Group 24

The Role and the Use of Technology in the Teaching and Learning of Mathematics at Primary Level

Sitti Maesuri Patahuddin[1] and George Gadanidis[2]

1. Aims and Themes

The aim of TSG-24 at ICME-14 was to share, discuss and advance knowledge and understanding of key aspects of research and practices related to the role and use of technology in the teaching and learning of mathematics at the primary school level.

For these aims, we invited contributions within four sub-themes.

1.1. Sub-theme 1: student interaction

In the ICME-13 Monograph Uses of Technology in Primary and Secondary Mathematics Education, several contributions concern students' learning. Some research suggests that digital technologies have the potential to support the learning of mathematics, giving a unique experience. At the same time, contributions highlight that few works focus on student interaction with digital media, and why and how these technologies have an impact on learning. Therefore, in this TSG, we are interested in continuing to deal with this theme:

- How does the use of apps enhance students' learning?
- What kinds of activities/tasks are proposed for students?
- What are the differences, if any, in the use of touchscreen apps (mobile or not) from the perspective of learning processes of specific mathematical content?
- What is the potentiality of coding activities for exploring mathematical concepts?

1.2. Sub-theme 2: digital and analog tools

Digital technologies are not considered alone. In many primary schools, depending on the culture of each country, there are physical manipulatives also in use. This is taken into account in some projects, focusing on digital and analog tools, from different perspectives (e.g., artefacts, coding to model mathematical relations, DGS simulations). We solicited contributions on this aspect:

- How do analog and digital technologies support students'learning?

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- In which specific situations do digital and physical tools display advantages for overcoming students' difficulties?

1.3. Subtheme 3: new technologies

From the discussion on technology, it emerges that many kinds of technology (e.g., tablets, IWB, and personal computers) are available for primary education. Nevertheless, their diffusion and how they are used at school are different from one country to another one (one-to-one tablet, BYOD, IWB). At the same time, new technologies are available and experimented in the classroom (e.g. VR, AR). In this sense, we would discuss:

- Which types of technology use are emerging to enrich and foster mathematics learning in kindergarten and primary school?
- Which digital technology for education do enable primary children to inquire, problem solve and think mathematically and share their learning?
- Which are the most spread technologies at kindergarten and primary school? Which mathematical contents do they concern about?

1.4. Subtheme 1: teacher's role

We were also interested in questions about teacher evaluation of apps. We aim to investigate the different aspects of this question. In particular, which criteria could be suggested to teachers for choosing apps in their teaching.

The spread of digital technologies at school depends on teachers' engagement (training and practice). A pedagogical approach taken by the teacher is complementary to the potential of the affordance of the apps to influence students' learning. The tasks given to students and the classroom culture the teacher develops are key elements of the learning. So, we aim to deepen the discussion about the teacher's role:

- How do schools and teachers use technology to enrich mathematics learning at the primary level?
- How do teachers choose the technology they use in their classrooms?
- A specific question is addressed to each participant, in order to know the state of the art: What are the typical digital technologies in used in kindergarten and primary school classrooms in your country?

2. The Sessions

Due to the global pandemic caused by coronavirus disease (COVID-19 pandemic), only 9 out of 21 contributors participated in this conference. Therefore, the work of TSG-24 was organized into three main TSG Sessions, and these were supplemented by oral communication and poster presentation (Tab. 1). The three sessions were chaired by Sitti Patahuddin and supported by George Gadanidis.

The contributors of this TSG were from seven countries: China, Malaysia, Japan, Australia, USA, India, and Canada. The nine contributions are as follow:

Tab. 1. List of papers presented

Paper and author(s)
Session 1
[1] Impact of computer-mediated sharing on classroom activities. ***Shaikh Sashid Rafikh****, Harita Raval, Harshit Agrawal, and Nagarjuna Gadiraju* (India).
[2] ELPSA framework uses in designing lessons with web-based resources: A case of equivalent fractions. ***Sitti Patahuddin*** *and Jonathan Adam* (Australia).
Session 2
[3] Using mathematically-focused text messages to connect families with their childs learning. ***Mollie Helen Appelgate****, Christa DeAnn Jackson, and Kari Nicole Jurgenson* (USA).
[4] Exploring the use of digital platforms in supporting dialogue in primary mathematics classrooms. ***Qian Liu*** (China). (Poster)
[5] Computational modelling in Grades 1-3 mathematics. ***George Gadanidis****, Janette Hughes, Immaculate Namukasa, and Ricardo Scucuglia* (Canada).
[6] Building computational thinking (CT) readiness: a self-assessment framework and tools for integrating CT in primary math classrooms. ***Heater Sherwood*** (USA). (Poster)
Session 3
[7] Proposal on how to use digital textbooks at primary level and research directions. ***Manabu Goto*** (Japan).
[8] Coding in elementary mathematics lessons. ***K. M. Leung*** *and P. Y. Tang* (Hong Kong SAR, China).
[9] Effectiveness of digital game-based learning (DGBL) in enhancing fraction skills among primary four pupils. ***Jia Yi Boo*** *and Kwan Eu Leong* (Malasya).

The TSG-24 started with a short introduction from each participant followed up by an overview of the TSG-24 programs by Sitti and George. The TSG sessions progressed well and we did not face any technological issues. All contributors did a live presentation and followed up by discussions. They shared their screen without a need to send their pre-recorded video or slide presentation.

TSG-24 also had an additional discussion forum led by George Gadanidis. George shared practices related to “Coding in the Ontario Mathematics Curriculum for Grade 1-8”. This session stimulated great discussion among the participants. George invited the participants to share in what ways coding was a part of the school curriculum in their countries. George emphasised that despite the fact that coding is not a new thing, being an explicit part of the curriculum offered opportunities to show the relevance and the value of mathematical thinking in this digital world.

In conclusion, TSG-24 was a learning space where various topics were shared and discussed, including digital game-based learning, theoretical frameworks in designing mathematics lessons with technology, digital textbooks, computational thinking and coding. The challenge of this TSG was mainly a lack of connections among the participants due to the online setting and the time zones as some participants had to join the session in the middle of the night and very early in the morning.

https://doi.org/10.1142/9789811287152_0049

Topic Study Group 25

The Role and the Use of Technology in the Teaching and Learning of Mathematics at Lower Secondary Level

Morten Misfeldt[1], Hans-Stefan Siller[2], Mariam Haspekian[3], Arthur Lee[4], and Mailizar Mailizar[5]

ABSTRACT Topic Study Group 25 (TSG-25) on the role and the use of technology in the teaching and learning of mathematics at lower secondary level discussed this topic over four sessions. These sessions focused on diverse topics such as immersive learning environments in the second session topics as self-efficacy, use of digital technologies outside the classroom and pedagogical aspects of using digital technologies were discussed. In the third session we discussed cultural aspects of technologies in the classroom, and in the last session we discussed communication and the mediating role of digital technologies. The group had contributions from all continents (apart from Antarctica), with a slight overweight of the contributions from European countries. The contributions consisted of five long papers, 21 short papers and eight posters

Keywords: Digital tools; Technology; Immersive learning; Computational thinking.

1. Themes and Description

Topic Study Group 25 (TSG-25) on the role and the use of technology in the teaching and learning of mathematics at lower secondary level was focused on three interrelated themes: (1) Technology in lower secondary education as a scientific endeavor. (2) The role of technologies in the teaching and learning of mathematics, and (3) Teacher in- and pre-service training with technologies or as a reply to new demands of technologies.

The work departed in an acknowledgement that technology and mathematics has a huge and increasing influence on many aspects of society, and hence that the educational attendance to the combination of mathematics and technology is of paramount importance.

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We had contributions from all continents (apart from Antarctica), with a slight overweight of the contributions from European countries. The contributions consisted of 5 long papers, 21 short papers and 8 posters.

1.1. Technology in lower secondary education as a scientific endeavor

The focus on technology in lower secondary teaching as a scientific endeavor was initiated in order to frame a theoretical and methodological discussion. The concern being that several theoretical constructs and methodological approaches have been applied and developed in order to investigate the role of technology in mathematics learning, as well as in teachers' difficulty to integrate them. But these are somehow compartmentalized in local theoretical traditions. Networking of theories and the development of shared knowledge and methodologies are important, and in the group, we were trying to cross internal barriers and build a paradigmatic organization.

1.2. The role of technologies in the teaching and learning of mathematics

The range of technologies that suggest themselves to the mathematical classroom is wide and expanding. Some technologies come in relatively stable form (such as calculators spreadsheets and physical manipulatives), while others are in rapid flux (intelligent CAS tools as Wolfram Alpha, Applications within Virtual Reality and programming languages are examples of this). Some are specifically designed for school mathematics teaching (such as GeoGebra), some have been imported from business area (such as spreadsheets). These technologies influence the teaching and learning process in mathematics, in ways that have been studied by research for the last three decades, and continue to generate results.

1.3. Teacher in- and pre-service training

The discussion of what mathematical knowledge, skills and competence that technological development requires, or favors reaches decades back to the introduction of handheld calculators in schools, but more recently the rapid digitalization of the social and economic spheres is increasing the demand for students to develop algorithmic and computational competences. Technologies and digital tools are of growing importance to all educational systems and call for teacher training. For this reason, there is an ongoing research interest in the field of professional development of teachers using technology. Even if evidence-based methodologies can tell us whether these tools are succeeding and should be introduced in pre-service teacher education, it remains still the question of the transferability of successful experiments and, more generally, that of how to train teachers to integrate these technologies in their practices. In the Topic Study Group we had the possibility of taking a genuine international audience and perspective on this important issue.

2. Overview of the Program and the Sessions

The four sessions where thematically organized and ran in sprints with 2–5 presentations followed by a shared discussion.

2.1. First session

In the first session we discussed technologies that somehow changed the learning environment through immersive artifacts such as augmented and virtual reality, game-based learning and simulations, but also by a combination of historical sources and digital tools as well as other ways of bringing CAS and DGS into students mathematical reasoning.

Session one contained the opening, followed by two long oral presentations (LO, each 10 minutes) and a shared discussion of the two papers (also over 10 minutes). After a break the session continued with three short oral presentations (SO, each of eight minutes), and a shared discussion of these papers. Following one more break five posters where presented and discussed (Tab. 1).

Session one had many and rather diverse contributions. In the discussions it became clear that even though trends as game-based learning, virtual reality are both new and important to the teaching of mathematics it also makes sense to consider the pedagogical situations that these tools can lead to in continuity with other aspects of mathematics education, such as organization and implementation, and the use of history and cultural artifacts. In this session we also aimed at establishing a collaborative atmosphere.

Tab. 1. List of papers presented in Session 1

Paper/Poster and author(s)	
[1]	An immersive learning experience for teaching equations equation lab. ***Morten Elkjaer*** *and Lui Albaek Thomsen* (Denmark). (LO)
[2]	Student's autonomy and digital technologies: collective documentation work in preservice teacher education. ***Ghislaine Gueudet*** *and Sophie Joffredo-Le Brun* (France). (LO)
[3]	Using augmented reality technology for instructional media in mathematics education. ***Shiwei Tan*** (China). (SO)
[4]	Mediations and rules when working with the interplay between original sources and GeoGebra. ***Marianne Thomsen*** (Denmark) *and Uffe Thomas Jankvist* (Afghanistan). (SO)
[5]	Developing spatial skills in a virtual reality environment for carpentry apprentices. ***Sylvia Van Borkulo*** *and Paul Drijvers* (The Netherlands). (SO)
[6]	Application of GeoGebra in the function study: the use of ICT in teaching mathematics. ***Wesley Matheus Moura Balbino****, Medeiros de Oliveira, and Francismar Holanda* (Brazil). (Poster)
[7]	EVA: an educational tool to simulate evacuations of buildings. ***André Greubel*** *and Hans-Stefan Siller* (Germany). (Poster)
[8]	Perspectives on the use of ICT in the high school mathematics classrooms. ***Erin Herz and George Ekol*** (South Africa). (Poster)
[9]	Role of ICT to enhance mathematics teaching. ***Santosh Paudel and Binaya Bhandari*** (Nepal). (Poster)
[10]	The mathema kids research seed: a GeoGebra youth club that tells stories. ***Carlos Eduardo Leon and Jefer Camilo Sachica-Castillo*** (Colombia). (Poster)

2.2. Second session

Session 2 addressed key elements relevant to the use of digital technologies from a scientific perspective — subject knowledge, self-efficacy, use of digital technologies outside the classroom and pedagogical aspects of using digital technologies.

All these issues are highly relevant and can have a lasting positive or negative impact on the use of digital technologies. Therefore, effectiveness research, subject knowledge testing and pedagogical measures to understand the content are enormously important. In this session we discussed these topics through two long oral presentations (LO, 10 minutes each) and 4 short oral presentations (SO, 8 minutes each) and discussed in detail future research possibilities (Tab. 2).

Tab. 2. List of papers presented in Session 2

Paper and author(s)	
[11]	Instrumental orchestration with dynamic geometry: A Chinese case study. ***Fangchun Zhu*** (China). (LO)
[12]	Gray-boxing as a means for mathematical communication. ***Cecilie Carlsen Bach*** (Denmark). (LO)
[13]	Desmos App in the mathematics classroom: limitations and potentialities. ***Jair Dias de Abreu*** *and Silviano de Andrade* (Denmark). (SO)
[14]	Augmented reality for outdoor modeling tasks: bridging real problems with mathematical concepts. ***Adi Nur Cahyono**, Yulius Leonardus Sukestiyarno, Mohammad Asikin and Matthias Dieter Ludwig* (Indonesia). (SO)
[15]	Micro-teaching of landmark jobs fostering self-efficacy for teaching mathematics with technology. *Daniel Thurm and* **Baerbel Maria Barzel** (Germany). (SO)
[16]	Digital competency found by prospective secondary teachers according ontosemiotic approach. ***Joaquin Gimmenez**, Silvia Carvajal, and Vicenç Fon* (Spain). (SO)

2.3. Third session

Session 3 had focus on different cultural influences on different cultures and countries approaches to teaching mathematics with technologies. Case studies from China, France, India, Mexico and Denmark, and the ways that different school cultures and local policies was in interplay with the pedagogical possibilities and difficulties of various tools and technologies. We also discussed the way that cultural artifacts such as historical sources, buildings and museums can be used to enhance teaching and learning of mathematics and how digital tools can be instrumental in that respect. The session included presentation and discussion of two long papers (LO, 10 minutes for each presentation and 10 minutes for a shared discussion), three short papers (SO, presented for eight minutes) and a short presentation of a poster. These four contributions were discussed jointly (Tab. 3).

Tab. 3. List of papers and poster presented in Session 3

Paper and author(s)	
[17]	The development of technological craft knowledge within a community of inquiry. ***Ahlam Anabousy*** *and Michal Tabach* (Israel). (LO)
[18]	Mobile learning of mathematics with apps for math trails. ***Ana Donevska-Todorova*** (Germany). (LO)
[19]	Media, cognition and assemblage perspectives on ict in education: a three-part study in an indian school. ***Prateek Shah****, Harshit Agrawal and Sanjay Chandrasekharan* (India). (SO)
[20]	Evolution of teaching practices with ICT: a case study with scratch in the French new mathematics curricula. ***Mariam Haspekian*** (France). (SO)
[21]	Connecting conjectures and proof using dynamic geometry environments and a toolbox puzzle approach. ***Ingi Heinesen Hojsted*** (Denmark). (SO)
[22]	Technology in classroom: a report of 3 researches. ***Alejandro Miguel Rosas Mendoza*** (Mexico). (Poster)

2.4. Fourth session

In session 4 we departed in the fact that effective use of digital technology inevitably leads to the mediating role of it. This is not only about how digital technologies are used in the classroom, but also about how digital tools enable or change interaction between teacher and learner. This consideration is not new in the discussion of digital technologies, but it is precisely through the use of new technologies and through the linkage with other process-related activities that this perspective continues to gain attention and contains sufficient research potential.

In the fourth session, which just put the scientific perspective on the mediating role of technology, 7 short-papers (SO) for 8 minutes each and 1 poster in the duration of 3 minutes were presented. An extensive discussion in the group rounded off this session, which then also closed TSG-25 (Tab. 4).

Tab. 4. List of papers and poster presented in Session 4

Paper and author(s)	
[23]	Impact of online automated learning path on student learning: the mindmath project in elementary algebra. ***Brigitte Grugeon-Allys****, Elann Lesnes-Cuisiniez and Fabrice Vandebrouck* (France). (SO)
[24]	Engagement and moderation of mathematical modelling tasks in virtual environments. ***Joseph Simon Madrinan*** *and Catherine Vistro-Yu* (Philippines). (SO)
[25]	Computer-dependent mathematics teaching in schools. *Rabindra Kumar Bhattacharyya* (India). (SO)
[26]	Type of mathematics tasks with dynamic geometry software. ***Liping Yao*** (China). (SO)
[27]	Strategic use of content-specific and content-neutral technologies to cater learning diversity in mathematics. ***Thomas K. F. Chiu*** (Hong Kong SAR, China). (SO)
[28]	Digital tools and mediation in informal justification. ***Rikke Maagaard Gregersen*** (Denmark). (SO)
[29]	Digital technology in relation to the mathematical thinking competency. ***Mathilde Kjaer Pedersen*** (Denmark)*, Uffe Thomas Jankvist* (Afghanistan)*, and Morten Misfeldt* (Denmark). (SO)
[30]	Students mathematics experience of the technology self-directed learning (TSDL) pedagogy. ***Hoi Kei Melody Wong*** *and I. A. C. Mok* (Hong Kong SAR, China). (Poster)

3. Future Directions and Suggestions

The work in the TSG gives a good outset for continuation. Firstly, it became clear that we have a solid base of research on how technology is used in lower secondary mathematics. The work is most elaborated in relation to tools such as CAS and DGS where a shared language and some sort of paradigmatic organization. The language coming from the theory instrumental genesis has facilitated this positive development.

Apart from consolidating the concerns about CAS and DGS the TSG also showed the development of two new areas for mathematics education and technology in lower secondary teaching. The first of these areas can be described as embodiment, immersion and virtual/augmented reality. This area stood out as a promising future direction for mathematics education in the sense that augmented and virtual reality tools provide platforms for new mathematical experiences, and for the development of teaching materials. Lastly it was clear that computational thinking and programming is becoming very prominent in the curriculum and educational practices of mathematics teaching in various countries but is still under researched in mathematics education.

After the conference we had a survey-based evaluation of the work in TSG-25. Rather few participants answered the survey, but there was agreement that TSG was a good academic experience despite the online and hybrid format. There was also an interest in continuing collaboration and discussion on the topic. One participant asked for better possibilities for post conference publishing of the conference papers.

https://doi.org/10.1142/9789811287152_0050

Topic Study Group 26

The Role and the Use of Technology in the Teaching and Learning of Mathematics at Upper Secondary Level

TSG-26 Working Team[1]

1. Introduction

The TSG-26 counts 17 papers presented, in a double modality: as a long paper (three in the 1st Session), and as a short paper, (fourteen, in the 2nd and 3rd Sessions). The participants to the Sessions have been 24, divided in these countries: Australia, Botswana, Brazil, Canada, Chile, China, Colombia, France, Italy, Mexico, the Netherlands, Philippines, South Africa, United Kingdom, and USA.

The call for papers of TSG-26 has been divided into 5 themes:

- Theoretical and methodological aspects: current/new frameworks for developing and analyzing new technology's integration in mathematics teaching and learning from didactical, cognitive and epistemological perspectives.
- Role of emerging devices and technologies, such as tablets, smartphones, virtual learning environments, augmented reality environments, and haptic technologies.
- Interrelations between technology and the mathematics taught at this age level.
- Students' education and the relationships between teaching and learning.
- Teachers' professional development.

Each theme has been articulated in different questions, ideas, suggestions for discussions, research results, and methodologies that guided the authors to the submission of papers and posters.

2. Sessions

The works of TSG-26 took place in three sessions, two of 90 minutes and one of 120 minutes. In the first session, a panel of three long papers was organized with the three presentations followed by a discussion. The other two sessions have been divided into two sub-sessions each, around common topics, with presentations of short papers and

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discussions in the end. Each communication was videotaped and made available for all participants before the sessions. During sessions, presentations were limited to 10 minutes (long papers) and 5 minutes (short papers) in order to allow the maximum time for the discussion within the TSG.

This report is structured according to the sessions and sub-sessions of the TSG.

2.1. Session 1

The first session of the TSG was dedicated to three long papers on the role of the students in a "digital class" at different levels (Tab. 1). The presentations of the long papers have been organized as a panel, followed by a discussion. The panel dealt with the different roles that students may have in educational processes: as protagonists of learning processes, as active designers of resources for education purposes, and as subjects evaluated by a system of formative assessment.

Tab. 1. Papers presented in Session 1

Paper and author(s)	
[1]	Students as designers of digital curriculum resources. ***Annalisa Cusi*** *and Agnese Ilaria Telloni* (Italy).
[2]	Formative assessment and technology: an attempt of framework. ***Gilles Aldon*** (France)*and Monica Panero* (Switzerland).
[3]	Straightening the bend: sequencing embodied experiences with high and low-tech designs for the notion of Radian. ***Rosa Annalucia Alberto***, *Anna Shvarts, Arthur Bakker, and Paul Srijvers* (The Netherlands).

The first paper[1] presented by Cusi shows an educational programme aimed at involving upper secondary students in the design of digital curriculum resources (DCR) using the GeoGebra software. The study characterizes the praxeologies (made of practices and theoretical reflections on them — Chevallard, 1991), developed by the students in relation to the task of DCR-design, through the analysis of the reflections they proposed during semi-structured interviews at the end of the educational programme. This characterization of students' praxeologies highlighted their awareness both on the characteristics of the DCR that supports students' learning and on the role of the design process in fostering the designers' learning itself.

The second paper[2], by Aldon and Panero, presents a research study on the place and the role of technology in the assessment process, specifically on formative assessment (FA) processes. It addresses the professional development of teachers integrating technology into their practices in order to enhance formative assessment and to give students the awareness and the ownership of their learning. The main claim of this paper is that technology does modify classroom assessment processes, but at the cost of a reorganization of the act of teaching by promoting student ownership of their learning. Technology is not only a facilitator in the implementation of FA-strategies, but more profoundly a modifier of the didactic contract, making teacher and students together responsible for teaching and learning.

Alberto, Shvarts et al.[3] aimed to investigate how to support students in moving from calculation to reasoning strategies in trigonometry by the design of technology-enhanced learning activities inspired by embodied mathematical cognition and embodied design. They conjectured that novel mathematical relations need to be enacted physically to provide opportunities to actively conceptualize these relations. They followed up on this by exploring embodied interactions that foster students' understanding of the input of trigonometric functions in the unit circle and the sine graph. Students appeared to lack physical experience with measuring circular arc lengths and using the radius as a unit of measurement. Low-tech paper materials and practices — like folding and bending — seem to afford sensible enactments better than would digital materials. Embodied approaches and interactions are essential for the emergence of mathematical conceptualization.

In the three presentations, students learn through an awareness of a reversal of responsibilities: this awareness is provoked by the teacher and the situations he/she sets up in the classroom. Technologies and actors' relationships with artifacts are responsible for the awareness of an evolving role for students in their learning processes. The use of digital technologies seems to favor a change of roles in students and teachers and also a change in the dynamics of the classroom: co-responsibility and cross-responsibility foster an active participation, modifying students' praxeologies used in learning processes (design, assessment, conceptualization). Technologies, in these cases, are no longer studied for themselves but rather for the role they play in the new organization of learning.

2.2. Session 2

Session 2 was split into 2 parts, totally 7 papers were presented (Tab. 2).

Tab. 2. Papers presented in Session 2

Paper and author(s)
Part 1
[4] Questions of design research: a technology mathematics lesson framed by the didactical triangle. ***Marie Joubert****, Geoff Wake, and Marc North* (UK).
[5] Merlo item as boundary object in teachers professional development. *Ornella Robutti* (Italy)*,* ***Theodosia Prodromou*** (Australia)*, and Gilles Aldon* (France).
[6] Acceptability of the proposed multimedia instructional module in selected pre-calculus topics among STEM students of Muntinlupa National High School. ***Maxima Joyosa Acelajado*** *and Arlene B. Miyas* (Philippines).
[7] Twitter, emotion and mathematics. ***Mario Sanchez Aguilar*** (Mexico).
Part 2
[8] Integrating GeoGebra in classroom teaching of 3D geometry: contrasting a French and a Chinese case. ***Mingyu Shao*** (China/France).
[9] Mathematics prospective teacher display of technological content knowledge in a GeoGebra-based environment. ***Kim Agatha Ramatlapana*** (Botswana).
[10] An implementation of technological pedagogical content knowledge framework for analysing the design of tasks in a digital environment. ***Carolina Guerrero-Ortiz*** (Chile).

2.2.1. Session 2 part 1

This sub-session was devoted to the role technology can play in the interactions between teachers and students, but also between teachers and researchers or teacher trainers.

Joubert et al.[4] analyzed these interactions by means of knowledge, technology playing a role of support of interactions between students and knowledge. Joubert's communication concerns the process of design research, within the context of teaching a 'technology' mathematics lesson in Further Education colleges in England. It explains the context, then uses the "didactic triangle" to frame an analysis of the design requirements for a lesson on factors and multiples. Having determined the design requirements, an example of a lesson was given, with an explanation of how it meets the requirements. It concluded that, although the design requirements had been met, the end product perhaps lacked a sufficiently coherent narrative. It ended by speculating on whether a fourth vertex, to represent technology, should have been added to the didactic triangle or whether there was a way of capturing a coherent narrative within the design requirements.

Prodromou et al.[5] considered the technology as a component of a boundary object providing a place to interact between teachers and researchers. The presentation was focused on the possibility to consider Meaning Equivalence Reusable Learning Objects (MERLO) itemed as a boundary object in crossing the boundary between two communities: researchers and teachers. The boundary crossing was seen as a process of transformation that can influence a modification (more or less stable) in the meta-didactical praxeologies (namely practices and theoretical reflections on them — Arzarello et al., 2014) of the teachers. Primary pre-service teachers were engaged in this experiment, during their professional development. Results were on the possible existing intertwining of their praxeologies and the MERLO items they produce, seen as boundary objects in their evolution over time.

Acelajado et al.[6] presented a tool, the Multimedia Instructional Module, aiming at helping learners to overcome difficulties in understanding mathematics. The authors developed and analyzed a new technological tool, the Multimedia Instructional Module, whose goal was to help students with mathematics learning difficulties, specifically on basic concepts of calculus. Statistically significant differences in the post-test meant scores between the two groups reveal that the experimental group performed better and that the participants of this group perceived the MIM to be "Highly Acceptable".

Aguilar[7] started by the use of a communication tool considered as a medium that provides us with an insight into students' emotional experiences related to the teaching and learning of mathematics. Aguilar took advantage of the familiarity of Twitter in students' everyday life to engage mathematics students in and out of the classroom. In his article, he argued that this social network could serve as a medium that provides us with an insight into students' emotional experiences related to the teaching and learning of mathematics. To illustrate this, a selection and categorization of tweets about mathematics was presented.

The discussion following the paper presentation focused on the theoretical background that supports the study of these interactions. More precisely, what can be the role of digital technologies in the development of educational actors' interactions? Examples given in all the presentations show that digital technology provides a powerful media helping for communication between actors, but also with knowledge at stake. It could also be the place of interactions encouraging actors to look at knowledge from a different perspective. The discussion that followed the presentations returned on the concept of boundary object (Prodromou et al.[5]), showing at a micro level how two communities (researchers and teachers) can approach each other by crossing the boundary. In this case of crossing, technology played the role of a shared place where different points of view from two communities could crystallize on a common one then adopted by both communities. At a meso level, interactions between two communities (teachers and students) could profit from social media (Aguilar[7]) and emotional experiences that students face during their mathematics studies were revealed. More investigation should be done for recognising and studying in different educational contexts objects at the boundary between communities, and their evolution over time, along with the causes of this evolution process, including not only cognitive reasons, but also meta-cognitive aspects that may have propulsor roles in these processes.

2.2.2. Session 2 part 2

The second part of the session was dedicated to different uses of Digital Geometry Software in the teaching of Geometry.

Shao[8] reported on orchestration of lessons using 3D geometry software in contrasted cases in France and China. Drawing on the instrumental orchestration (Trouche, 2004) and the instrumental genesis frameworks, Shao's paper contrasted the case of a Chinese mathematics teacher with a French one, investigating how they have managed to integrate GeoGebra in their class on 3D geometry. The results opened some perspectives for further investigation, including the teacher's documentation work before the class, and the factors that could influence their choices of instrumental orchestration.

Ramatlapana[9] explored geometry technological content knowledge of mathematics prospective teachers within a GeoGebra environment. The presenter explored geometry technological content knowledge displayed by mathematics prospective teachers when working on a high school circle geometry task within a GeoGebra-based environment. The investigation analyses six prospective mathematics teachers' thinking as displayed in their solutions to the technological content knowledge-based task, particularly on what the GeoGebra constructions revealed about teachers' competence with geometry diagrams within a GeoGebra environment. The narratives and constructions were expected to reflect the teachers' ability to transform the statements from a static environment to a dynamic construction, employing GeoGebra as a construction tool. The affordances and constraints of GeoGebra when making connections between the construction and geometric principles emerged.

Guerrero-Ortiz[10] dealt with task design, highlighting the possibilities offered by a technological environment, according to the TPACK (Koehler and Mishra, 2009) framework and following a qualitative perspective. The paper presents different tasks designed by pre-service mathematics teachers, analyzing their peculiar elements related to the domains of TPACK. In the study aspects related to modeling, simulation, visualization and the use of the tools of a Dynamic Geometrical System (DGS) are highlighted. This work allows us to know how in the tasks design the pre-service teachers' knowledge can be evidenced, the results also show how they conceptualize modeling in a technological environment.

The discussion started with the three following questions:

- How is it possible to go beyond the identification of teachers' knowledge in the use of DGS?
- Is it possible to think through a networking of theories both the use of DGS and the design of resources?
- How can we contrast and compare different contributions of these theories?

The first question refers to works such as the one by Guerrero-Ortiz[10] and Ramatlapana[9], who paid attention to the specialized knowledge manifested by prospective teachers when facing the resolution or design of mathematical tasks based on the use of dynamic geometry systems. These studies confirmed how the analysis of the use of digital tools by mathematics teachers could serve as a window to teachers' specialized knowledge. However, we ask ourselves how we can use this identified specialized knowledge to promote an adequate and rational implementation of technological tools in the teaching of mathematics.

The second and third questions are theory-oriented investigations whose purpose was to promote the discussion of the potential that theory networking could offer for the study of situations involving the use of DGS as a teaching tool, but also as a means to design and implement mathematical tasks. Likewise, the particular affordances that different theories could offer for the study of the role and the use of technology in the teaching and learning of mathematics at upper secondary level were put at the center of the discussion. These presentations, as well as the plenary discussion, lead us to consider DGS through different theoretical filters which each of them brings a particular highlighting on the relationships between the artefact and its use.

2.3. Session 3

Session 3 was also split into 2 parts with 7 papers presented (Tab. 3).

Tab. 3. Papers presented in Session 3

Paper and author(s)
Part 1
[11] Mathematics VR teaching design mode and its practice at upper secondary level: based on VR All-in-one Computer. ***Jijian Lu****, Xiaoyuan Shen, and Yi Lv* (China).
[12] Mobilizing mathematics: how technology enhances embodied learning. ***Stefan Rothschuh*** (Canada).
[13] The reading and the comprehension of mathematics text: an eye-tracking study with primary pre-service teachers. *Roberto Capone, Federica Ferretti, Alessandro Gambini, and* ***Camilla Spagnolo*** (Italy).
[14] Computational thinking for mathematical learning. ***Yahya Tabesh*** (USA).
Part 2
[15] Students' understanding of the notion of collinear vectors in dynamic geometry environment. ***Jose Orozco-Santiago*** (Mexico).
[16] Enhancing metacognition by using flipping classroom with GeoGebra. ***Chak-Him Fung****, Kin-Keung Poon, and Michael Besser* (Hong Kong SAR, China).
[17] Students coping with a post-16 mathematics course: flipped learning, self-regulation and technology. ***Sofya Lyakhova****, Marie Joubert, and Dominic Richard Oakes* (UK).

2.3.1. Session 3 part 1

The first part of the third session was dedicated to discuss the research on interactive technologies, particularly Augmented and Virtual Reality, embodiment and reasoning.

Lu et al.[11] addressed the creation and use of VR all-in-one virtual simulation software and hardware platform as an alternative to overcome the limitations of traditional online technologies. They used VR head display and VR glasses in students' collaborative interaction and integration in mathematics teaching at upper secondary level, building a mathematics VR teaching design mode, including: resource selection, interaction design, development and innovation.

Their team constructed the high school mathematics teaching design mode assisted by VR all-in-one computer, and would address more studies towards the extended practice of mathematical maker education, teachers and students could carry out interdisciplinary inquiry learning practice.

Rothschuh[12] discussed the theoretical considerations and the practical implementation of a research on technologically enhanced embodied mathematics learning. It sought to study and improve the practice of learning and teaching mathematical functions at the secondary level by incorporating embodied learning designs. It drew on established theories of how individuals learn mathematics, recent developments that aimed to incorporate embodiment and technology in mathematics learning processes, and the desire to study learning where it naturally occurs, as the long paper by Alberto et al.[3]. Using a design-based research approach, the researcher and partnering teachers developed and implemented a set of technologically enhanced embodied lesson designs. Over the course of three iterations, the lesson designs were continuously revised and improved, to promote embodied learning of the function concept, and harmonize technology-integration in these learning environments. It showed how calculus became meaningful for modeling everyday experiences in technologically augmented classroom inquiry, rather than being a domain that is merely focused on number and calculation.

Capone et al.[13] dealt with the type of text — by using eye track tools — to investigate students' attention during reading tasks. It pointed out that, being recognized in literature, the central role of argumentation in the teaching-learning process and the type of text affecting students' reading, the use of eye-tracking might render it possible to understand students' reading of a mathematical text. In particular, we may understand if selective readings with a focus on some textual elements considered essential may lead students to a lack of understanding of the problematic situation. The research, carried out with the innovative tool of eye track, shows a first exploratory study conducted with primary pre-service teachers while dealing with mathematics texts.

Tabesh[14] presented an intuitive digital learning model, focused on problem-solving through computational thinking and is targeted to empower teenagers. The proposed model is a hands-on interactive web platform for mathematical problem-solving that enables creative engagement, develops mathematical skills, and supports a growth mathematical mindset. It illustrated the benefits arising from engaging youth with progressively more complex tasks and giving them increased ownership of their learning. As a theoretical foundation, the development teams consider the use-modify-create framework to offer a helpful progression for developing mathematical thinking. It presented a computational thinking playground and a functional programming paradigm in a platform for creative problem-solving. In the platform one can use models and simulations to represent phenomena which, by playing with a mathematical framework, will be learned through creative and innovative thinking. The gained knowledge and skills of this cognitive learning both empower learners and enhance creativity.

After all the presentations, the discussion started by the following three questions:

- What advances can the use of interactive technologies provide to research in Mathematics Education?
- How can interactive technology be used in mathematics education? How to use or combine embodiment, virtual reality, augmented reality, particularly when solving mathematical problems?
- What is the role of these environments in the development of computational thinking itself?

Motivated by the third presentation — that highlighted innovative tools for research methodology advances — the first discussion question supported reflections on different lines of innovation in the studies. It used eye-tracking contact with the text as a tool to collect data for the study about argumentation. It led us to question the processes shown by tracks. The first presentation focused on the advances regarding cognitive development by using VR devices. The second and fourth presentations focused on developing tools. The second regarded a model of lesson design with embodiment technology. The fourth was a playing platform to support students' problem-solving. The second discussion question was motivated by the different uses and the variety of interactive technologies such as Virtual Reality all in one computer and computing thinking, creativity on embodied cognition with the integration of technology and graphs generation through movements linked to the classification of gestures. However, the uses are more isolated for a while. The fourth presentation led to our third discussion question. The TSG-26 discussion to reach the importance of computational thinking incentive through a platform to solve problems; using interactive games. Nonetheless, we need more research on tasks and students' work associated with these innovative tools to improve mathematical learning.

2.3.2. Session 3 part 2

The second part of the session was focused on the ways (i.e. flipped classroom) of using dynamic geometry software to promote students' understanding and teaching effect. Orozco-Santiago[15] proposed a contribution to the current/new frameworks for developing and analyzing new technologies integration in mathematics teaching and learning from didactical, cognitive and epistemological perspectives. They designed the tasks considering the potential of the dragging tool in a dynamic geometry environment. The tasks were assigned in a linear algebra course in engineering, and the work of one student in France was analyzed, as a case study, to examine the actions instrumented under the framework of instrumental genesis. They showed that the suggested design supported the student to explore what remains invariant under dragging, and to conjecture about the meaning of collinearity.

Fung et al.[16] followed a quasi-experiment design to study the Flipping Classroom (FC) assisted by GeoGebra to increase Chinese college students' mathematical metacognition in comparison to FC assisted by video and/ or direct instruction. The result revealed that the main effect of the metacognition was significant, while the interaction between the metacognition and the teaching methods was not significant. It suggested that significant improvement could be observed among students but no significant difference could be observed among the teaching methods in terms of the metacognition. In other words, the FC supported by GeoGebra is an effective teaching method in terms of students' metacognition development.

Lyakhova et al.[17] interviewed sixteen students from two research projects in the UK which employed technology to compensate for the lack of in-school resources (advanced post-16 mathematics course). The study showed the evidence that technology could create new learning situations as well as new learning materials that students perceive as beneficial, although perhaps an effort is required from students (i.e., self-regulation skill) to successfully adapt to these.

Inspired by the three reports mentioned above, our panel discussion focused on the following two areas:

- What is the theory behind dynamic geometry software to promote students' understanding of geometric concepts?
- How to effectively improve the effectiveness of flipped classroom teaching?

The discussion of the first question revealed the challenges that iterative updates in technology pose on how to study the effectiveness of classroom instruction that incorporates information technology, and that research on this issue needs to be supported by further theoretical exploration. The discussion of this problem focused on how to develop theoretically relevant assessment frameworks to track students' development in mathematical thinking at a higher level. The second discussion topic focused on the popular flipped classroom model, in which participants were very concerned about how to improve students' self-discipline and initiative in the flipped classroom? The effective use of flipped classrooms requires a change in students' attitudes toward learning. On the one hand, the content of the flipped classroom needs to be carefully designed, and on the other hand, the improvement of students' self-learning ability, an issue that involves further exploration of meta-cognition such as self-monitoring and self-evaluation.

3. Poster Presentations

In a special session, ten posters also composed TSG-26 studies with researchers from Belgium, Cambodia, China, Morocco, Nepal, and Peru (Tab. 4).

Tab. 4. Posters with TSG-26

Poster and author(s)
[18] The study of mathematic classroom teaching integrated information technology and mathematic multirepresentations. ***Hua Wu*** (China).
[19] A blended approach to support aspiring engineering students. ***Paul Georges Igodt*** (The Netherlands).
[20] The study of mathematical multi-representations and teaching scaffolding in the smart-classroom environment. ***Na Han*** (China).
[21] Integration of ICT in modeling and experimentation of interdisciplinary problems. ***My-Lhassan Riouch*** (Morocco).
[22] Instrumentation of the symbolic artifact quadratic function. *Daysi Julissa Garcia Cuellar*.
[23] Application of GeoGebra based on AR/VR technology in high school solid geometry teaching. ***Xue Huang*** (China).
[24] Students self-regulated learning strategies, perceptions and mathematics performance in a mobile technology-integrated mathematics classroom. ***Gerald Cristobal Apostol*** (Nepal).
[25] The effective strategies of teaching trigonometry function using ICT applications: GeoGebra and Wolfram. ***Leangsim Im*** (Cambodia).
[26] Practical research on the application of information technology in function review lectures. ***Xiayan Shao*** (China).
[27] A case study on TPACK performance of Chinese middle school mathematics teachers. ***Huishui Ye*** (China).

These posters also discussed Interactive technologies with a study applying GeoGebra based on AR/VR technology in high school solid geometry teaching by Huang[23]. Multirepresentation was the focus of two posters. Wu[18] discussed IT and its integration into the classroom. Han's study[20] focused on approaching it in a smart-classroom environment with the lens of teaching scaffolding. As for different teaching methodologies, Igodt[19] presented a blended approach specially built to support students aspiring to study engineering, and Riouch[21] discussed interdisciplinarity as modelling and experimentation of interdisciplinary problems. Regarding the instrumental approach, Cuellar[22] presented an analysis of the instrumentation of the symbolic artifact quadratic function. Mathematics functions were also highlighted by Im[25], while discussing the effectiveness of a teaching trigonometry function approach using GeoGebra and Wolfram. As for the literature on ICT integration, Shao[26] showed a review of practical research on the application of IT in Function. Apostol's paper[24] about self-regulated learning strategies, perceptions, and mathematics performance in a mobile technology-integrated mathematics classroom represented students' learning study. Finally, on the side of teachers' knowledge, Ye[27] presented a case study on the TPACK performance of Chinese middle school mathematics teachers.

4. From the Past to the Future: Challenging Themes of Discussion

The TSG-26 has been the occasion for contrasting, discussing, and comparing different themes with arguments in relation to theories, to teachers' education and engagement, and to students, in relation to the use of technologies in educational settings.

What emerged in relation to theories is the use of TPACK to investigate teachers' knowledge, when they are engaged in design cycles, or the new emerging approach of boundary crossing between communities, when speaking of a methodological tool to favor deep understanding: MERLO item. Commognition theory seems to have a fundamental role in analyzing the learning present in virtual reality context. Likewise, the notion of praxeology — including the particular case of meta-didactical praxeologies — was the key element to analyse the involvement of students in designing digital curriculum resources, and to outline the evolution of teachers' interactions when designing learning objects. More than in the past studies, recently the importance of explicitly defining the role of the teacher in multiple didactical tasks, and their engagement in design/use/orchestrate mathematical activity emerges in the papers and posters presented. And, moreover, the students' role has been addressed with a higher level of engagement and sharing of responsibility: not only in solving tasks, but also both in co-designing tasks and in being engaged in the assessment process with more awareness. Here the notion of instrumental orchestration (Trouche, 2004) becomes relevant to establish and organize the conditions that favor the involvement of students and the sharing of responsibility for their learning. This gaining in responsibility of teachers and students is possible also thanks to the technology used, which mediates the participation in common activities, and gives immediate feedback of the actions done. Technologies such as dynamic software, or Web 2.0 environments give the educational support to learning and teaching mathematics in many different kinds of activities: inquiry tasks from simple variation to co-variation situations. Also technologies for embodied approaches that detouch motions/eye movements/virtual exploration are providing us with empirical data that support new ways of looking at the construction of knowledge.

These tools and the data that they produce have the potential to offer a fresh look at the study of the complexities of mathematics teaching and learning. Future research in this area could address how these technologies can be integrated into the rapidly changing instructional landscape.

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https://doi.org/10.1142/9789811287152_0051

Topic Study Group 27

The Role of History of Mathematics in Education

Ysette Weiss[1] and Desiree Agterberg[2]

1. Rationale and Aim of the TSG

Mathematics is a human intellectual enterprise with a long history and a vivid present. Thus, mathematical knowledge is determined not only by the circumstances in which it becomes a deductively structured theory, but also by the procedure that originally led or may lead to it, and which is indispensable for understanding processes of change in mathematics. Therefore, learning mathematics includes not only the "polished products" of mathematical activity but also the understanding of (implicit) motivations, the sense-making actions, and the reflective processes of mathematicians, which aim to the construction of meaning. Hence, teaching mathematics should include giving the opportunity to students to "experience mathematics in the making." That is, although the "polished products" of mathematics form that part of mathematical knowledge that is communicated, criticized (in order to be accepted or rejected), and serving as the basis for new work, the process of producing mathematical knowledge is equally important, especially from a didactical point of view. This perception of mathematics should be central in the teaching of mathematics, and the image of mathematics communicated to the outside world. In this perspective, putting emphasis on integrating historical and epistemological issues in mathematics teaching and learning constitutes a possible natural way for exposing mathematics in the making that may lead to a better understanding of specific parts of mathematics and to a deeper awareness of what mathematics as a discipline is.

TSG-27 aims to provide a forum for participants to share their research interests and results, as well as their teaching ideas and classroom experience in connection with the integration of the history of mathematics in mathematics education. Special care is taken to present and promote ideas and research results of an as broad as possible international interest, while still focusing due attention to the national aspects of research and teaching experience in this area. Every effort will be made to allow researchers to present their work, get fruitful feedback from the discussion, and stimulate the interest of newcomers by giving them the opportunity to get a broad overview on the state-of-the-art in this area. This TSG refers to all levels of education

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— from primary school to tertiary education, including in-service teachers' training — preferably on work and conclusions based on actual classroom experiments and/or produced teaching and learning materials.

1.1 Submissions

We received 41 submissions from 16 countries (South America: 2; North America: 5; Asia: 23; Europe: 10; Africa: 1). Of those 41 submissions, 6 were rejected, 2 were redirected to another TSG, 6 were accepted as poster and 27 were accepted as paper presentations (long or short). The main part of the review process was organized and carried out by former TSG Chair Kathleen M. Clark, former Co-Chair Constantinos Tzanakis and former team member Uffe T. Jankvist.

Three papers were withdrawn by the authors after the postponement of the conference.

Of the remaining 24 accepted papers and 6 posters, only 13 papers and 2 posters were able to be presented during the conference. A list of the papers and authors are included in order of presentation and are organized in Tab. 1:

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Methodological proposal for the analysis of historical sources of mathematics. ***Erika Zubillaga-Guerrero****, Flor Monserrat Rodríguez-Vásquez* (Mexico)*, and María Teresa González-Astudillo* (Spain).
[2]	Towards qualitative and participative research on history of mathematics in mathematics education: some arguments and possible paths. ***David Guillemette*** (Canada).
[3]	The application of HPM micro-video in the teaching of the binomial theorem. ***Jiaye Han*** (China).
[4]	The design and cases of primary school HPM micro-video. ***Zhuochen Li*** *and Jiaye Han* (China).
[5]	Combining cognitive demand with history of mathematics in mathematics (teacher) education. ***Desiree Agterberg*** (The Netherlands).
[6]	The Gradual Linearization of German Geometry Teaching. ***Ysette Weiss*** (Germany).
[7]	An empirical study on the impact of students' cognition through the concept of function teaching from the perspective of HPM in senior high school. ***Silu Liu*** *and Zhongyu Shen* (China).
[8]	Organum Mathematicum — a mathematical shrine as source for modern math education. ***Silvia Schöneburg-Lennert*** *and Thomas Krohn* (Germany).
[9]	The binary tree and its avatars: From Xiantian to the eternal symmetree. ***Jorge Soto-Andrade*** (Chile)*, Dandan Sun* (China)*, Daniela Diaz-Rojas* (UK)*, and Alexandra Yáñez-Aburto* (Chile).
[10]	An empirical research on the intension of mathematical culture based on the history of mathematics. ***Qing-chun Yu*** (China).
[11]	The development of teachers' MKT: a case study of HPM learning community. ***Zhongshu Shen*** *and Jiachen Zou* (China).
[12]	Enhancing mathematics teaching self-efficacy in pre-service teachers: effects of an HPM learning community in Shanghai. ***Haozhe Jiang*** (China).
[13]	A comparative study of the history of mathematics in high school mathematics textbooks in Chinese mainland and Chinese Taiwan. ***Peiyao Lei*** (China).

1.2 Sessions

A new team was formed to organize the hybrid sessions in which the papers were presented. The TSG Chair Ysette Weiss, Co-Chair Desiree Agterberg and team member Silvia Schöneburg-Lennert led the sessions. The sessions were attended by at least 21 participants. This number does not account for some of the live audience in the Shanghai conference room that were not on the list of authors nor paid participants for TSG-27.

The ICMI organizing committee granted our TSG four timeslots for presentations. Generally, all four sessions started with one or two 15 minutes *long* oral presentations and a discussion afterwards and one or two 10 minutes *short* oral presentations with collective discussion afterwards. At the end of our last session, we did a short group reflection.

2. Conference Themes

The thematic call for proposals for TSG-27 was broad and reflected main research areas in the history of mathematics in mathematics education:

- Theoretical and/or conceptual frameworks — in particular from general mathematics education research — for integrating history in mathematics education;
- History and epistemology implemented in mathematics education: Classroom experiments and teaching materials, considered from various perspectives; e.g., cognitive, didactical, pedagogical, affective, etc.;
- Surveys on the history of mathematics as it appears in curriculum and/or textbooks;
- Original sources in the classroom, and their educational effects;
- The role of history of mathematics in relation to the use of digital technologies in the teaching and learning of mathematics;
- History and epistemology as a tool for an interdisciplinary approach in the teaching and learning of mathematics and the sciences by unfolding their productive interrelations;
- Cultures and mathematics fruitfully interwoven.

Almost all presentations contributed not only to one of these subject areas, but affected several topics. The introduction to a conceptual frame for the inclusion of the history of mathematics and mathematics teaching in the teaching of mathematics was often accompanied by the discussion of its implementation, the display of empirical results and the consideration of possible educational effects of the use of historical sources as a tool.

Guillemette[2] reflected in his contribution on existing theoretical frameworks and their potentials and limits. Conceptual frameworks were also in the focus of the contribution by Zubillaga-Guerrero et al.[1]. They presented a methodology for the qualitative analysis of historical sources and demonstrated their tool in the analysis of the concept of isomorphic groups in Arthur Cayley's work. Agterberg[5] introduced a cognitive demand framework that she developed for the analysis of tasks and classroom activities involving historical facts and sources and illustrated it with various examples. As the discussion showed, in several European and Asian countries there is an increasing involvement of history of mathematics in mathematics lessons and school curricula. Associated with this, one can find a growing number of contributions in mathematics textbooks that take the history of mathematics into account in varying ways.

Three contributions were directly devoted to curricular developments in textbooks, but in very different ways: in Weiss' study[6] of representations of conic sections in mathematics textbooks and curricula during the last 150 years, the focus was on the connections between institutional and curricular reforms. Lei's comparative textbook analysis[13] of high school mathematics textbooks in Chinese mainland and Chinese Taiwan concentrated on the integration of mathematics history. It was shown that the differences in the two editions were mostly related to applications. Yu[10] analysed 20 lessons from 2012–2018, regarding the implementation of the intention of mathematical culture in the reformed senior high school mathematics curriculum. All three contributions also examined the development of conceptual frameworks for the selection of curricular content.

Three papers with different emphases addressed the use of the history of mathematics to promote the understanding of mathematical concepts. Liu and Shen[7] studied the development of students' conceptual understanding of the concept of a function, using milestones of the historical development of this concept. Shen and Zou[11] presented a study on the development of teachers' mathematical knowledge over a semester at the HPM Studio in Shanghai.

The contributions by Li and Han[3,4] on the use of history-based micro videos (HPM micro-videos) in the teaching and learning of mathematics dealt with history of mathematics in relation to the use of digital technologies. The study focused on strengthening the motivation to learn mathematics and the selection of suitable topics for the promotion of general educational aspects. Jiang[12] presented a study, which also aimed on strengthening motivation, improvement of self-efficacy and changing beliefs, but this time in the context of an HPM teacher professional development program. The empirical study, he carried out, served him as a starting point for further conceptual development of this program.

In the contribution by Schöneburg-Lennert and Krohn[8], as well as in the contribution by Soto-Andrade et al.[9], the use of historical sources in mathematics lessons was foregrounded. The first contribution on the Organum Mathematicum presented teaching material from the 17th century and its educational potential. The second presentation of intercultural history of the concept development of the binary tree, beginning in ancient China in the 11th century up to the present, showed a variety of inspiring, enlightening and educational possibilities to approach mathematics and related subjects.

The great diversity of the presented historical sources made for an interesting and inspiring discussion. The wide range of implementations, which included primary and secondary education as well as university education and in-service teachers' training, also enabled a holistic view of mathematical-historical education.

3. Areas for Future Research

The topics discussed in the TSG-27 "The Role of the History of Mathematics in Mathematics Education" reflect actual research areas and significant recent scientific development in mathematics education.

The great importance of action and developmental research in our group raised the question of new formats with an international reach.

Another new accent was set by the question of possible cooperation between historians of mathematics, mathematicians, mathematics educators and mathematics teachers.

Since many topics that are relevant in other TSGs are also relevant to our TSG, the question of cooperation with other TSGs arose. A lot of TSGs come to mind, such as the teaching and learning of algebra/geometry/calculus, teacher education, mathematics and interdisciplinary education, the use of technology and so on. For example, digital media offer new opportunities to have historical sources more easily accessible in the classroom. This opens up new possibilities for accessing historical materials, for instance virtual museum tours or film material. Tools such as GeoGeobra can be used to rediscover Greek geometry or transcendental curves.

https://doi.org/10.1142/9789811287152_0052

Topic Study Group 28

Preservice Mathematical Teacher Education at Primary Level

Salvador Llinares[1], Craig Willey[2], Hui Jiang[3], Rukiye Didem Taylan[4], and Ban Heng Choy[5]

ABSTRACT The rationale of TSG-28 was to engage participants in reflection on, and discussion of, the theoretical, empirical and practical issues. Twenty-five papers were reported, from twelve countries showing a great diverse cultural. The papers were grouped in four themes: Noticing, Preservice Teachers' Learning, Preservice Teachers' Knowledge, and finally one group with various issues (Other: University-school partnership, beliefs, and textbooks).

Keywords: Preservice primary teachers; Noticing; Knowledge; Learning; Beliefs; University-school partnership.

1. Background and Rationale of TSG-28

It is a multi-faceted task to prepare preservice teachers of mathematics (PSTs) to conduct high-quality teaching. Many agree that teaching competence is not based merely on PSTs' academic qualifications, and consequently the concern for problems of practice has grown in programmes for teacher education. This is evidenced, for instance, in attempts to reconsider the knowledge needed in instruction; to work with representations of practice in college-based parts of programmes; to have prospective teachers rehearse and plan for instructional activity while at college; and to capitalize on prospective teachers' field experiences. This links research and development in mathematics teacher education (MTE) to scholarship on teaching and raises issues of (1) what high-quality instruction entails in different contexts and (2) how prospective teachers may develop their capacity to conduct teaching accordingly under different circumstances. In spite of the common interest in preparing PSTs for practice, it is still contentious how best to do so, and the answer to this question may vary across contexts. The rationale of TSG-28 was to engage participants in reflection on, and discussion of, the theoretical, empirical and practical issues.

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Twenty-five papers were reported in this TSG (add two posters), from twelve countries showing a great diverse cultural. The papers were grouped in four themes: Noticing, Preservice Teachers' Learning, Preservice Teachers' Knowledge, and finally one group with various issues (University-school partnership, beliefs, and textbooks). TSG-28 had assigned two time slots of 120' and two time slots of 90'. Authors of papers presented key points from their work and for each theme one member of the team synthesized the relevant aspects and underlined differences and similarities (Tab. 1), and all the contributions to TSG-28 are listed in Tab. 2 (on the next page).

Tab. 1. Themes and countries

	Noticing	PSTs' Learning	PSTs' Knowledge	Other	TOTAL
Germany	1				1
USA	3	1	1	2	7
Spain	1		3	1	5
Turkey	2			1	3
UK		1			1
Canada		2			2
South Africa		1			1
China				1	1
Netherlands			1		1
Malaysia			1		1
Chile			1		1
Malawi			1		1
TOTAL	**7**	**5**	**8**	**5**	**25**

2. Noticing

Mathematics teacher noticing is a set of core practices through which teachers interact with the different aspects of a teaching situation. Conceptualizing it and understanding how it develops are the focus of some papers discussed. The studies about noticing focused on three aspects: factors influencing the preservice teachers' reasoning process involved in noticing; on the relationships between different skills such as attending to, interpreting and decision making; and on the design principles of tasks to enhance noticing.

Firstly, the papers discussed how PSTs attend to the students' answers to mathematics problems and which factor might or might not influence their interpretations. This set of papers makes explicit the relationship between the PSTs' reasoning process and their decision making. So, Laschke et al.[4] examines the explicit criteria that PSTs refer to when confronted with an unexpected student solution of a probability problem and to which extent these criteria are influenced by students' social status background. The findings showed that although PSTs mostly used content-specific criteria, also implicit criteria were relevant as evaluations of student work were biased by students' social background.

Tab. 2. List of papers presented

Paper and author(s)	
Noticing	
[1]	A study on written feedback on preservice teachers' teaching practices and its impact on noticing. *Müjgan Baki and Zeynep Medine Özmen.*
[2]	Designing tasks for support preservice primary teachers' noticing of geometrical thinking. ***Melania Bernabeu***, *Mar Moreno, and Salvador Llinares* (Spain).
[3]	Relationships between preservice teachers' knowledge and their responses to students errors: Making word problems for the concept of division. ***Qintong Hu*** (China), *Lynn Hodge, and Shande King* (USA).
[4]	How preservice teacher judge and unexpected student solution — explicit and implicit criteria. ***Christin Laschke***, *Bettina Rösken-Winter, and Sven Schüler* (Germany).
[5]	An analysis of preservice teachers' noticing of student pattern generalization strategies. ***Ji-Eun Lee*** *and Mi Yeon Lee* (USA).
[6]	Prospective teachers' noticing of student's algebraic thinking: Pattern generalization. ***Zeynep Özel***, *Mine Işiksal-Bostan, and Reyhan Tekin-Sitrava* (Turkey).
[7]	Developing preservice teachers' noticing of productive struggle with video analysis. *Hiroko Warshaue*, **Christina Starkey**, *Christine Herrera, and Shawnda Smith* (USA).
Preservice teachers' learning	
[8]	Preservice Mathematics Teacher Education for the Montessori Teachers. ***Kinful Lartebea Aryee***, *Immaculate Kizito Namukasa, and Marja Bertrand* (USA).
[9]	Tracing threads of awareness in initial teacher education: peer-collaboration. ***Gwen Ineson, Julie Alderton***, *Chronoula Voutsina, Kirsty Wilson, Gina Donaldson, and Tim Rowland* (UK).
[10]	Preservice teachers designing meaningful digital learning environments using makerspaces for math. ***Anjali Khirwadkar*** *and Candace Figg* (Canada).
[11]	Exploring pre-service teachers' mathematics learning experiences and self-efficacy in teaching primary level mathematics. ***Sangyeon Park*** (USA).
[12]	Where the journey to reflective practice begins: A case of preservice teachers. ***Chikiwa Samukeliso*** *and Graven Mellony* (South Africa).
Preservice teachers' knowledge	
[13]	Exploring how prospective teachers pose problems: The case of $\mathbf{8 \times (-2)}$. ***Miguel Angel Montes***, *Juan Pedro Martin, Maria Isabel Pascual, Nuria Climent, and Jose Carrillo* (Spain)
[14]	Integrating EDTPA Preparation in a methods of teaching elementary mathematics course. ***Norma J. Boakes*** (USA).
[15]	Contribution of a didactic course on the development of primary preservice teachers' knowledge of Measurement and Geometry. ***Israel García-Alonso***, *Josefa Perdomo-Díaz*, ***Diana de las Nieves Sosa-Martín*** (Spain).
[16]	Using Technology for virtual representations of Teaching for developing math talk during problem solving. ***Melva R. Grant*** *and Signe Kastburg* (USA).
[17]	Torpedo: A Digital learning environment for developing mathematical problem-solving ability in primary teacher education. ***Marjolein Kool*** *and Ronald Keijzer* (The Netherlands).
[18]	Design of a learning unit for preservice elementary school teachers: definition of the boundary of a 2D Shape. ***Alejandro López***, *Salomé Martínez, Aldo Ramírez, and Ricardo Salinas* (Chile).
[19]	Explanatory talk in the teaching of number concepts and operations to preservice teachers: A case of one mathematics teacher educator. ***Justina Longwe-Mandala*** (Malawi).
[20]	Preservice teachers' conceptual understanding of fraction: Implications for improving curriculum Standards and classroom practices. ***Suhaidah Tahir***, *Masami Isoda, Munirah Ghazali, and Dominador Dizon Mangao* (Malaysia).
University-school partnership, reliefs, and textbooks	
[21]	In what ways does a mathematics curriculum based on the theory of multiple intelligences affect the attitudes and beliefs of preservice elementary school teaches toward mathematics? ***Mark Arvidson*** (USA).
[22]	Mathematics workshops: Changing the perceptions of both inservice and prospective teachers with regard to mathematics. V*alentina Celi* (France), *José Ignacio Cogolludo, Raquel García Catalán, Elena Gil Clemente,* ***Inmaculada Lizasoain*** (Spain), *Ana María Millán Gasca, and Luigi Regoliosi* (Italy).
[23]	School University Partnership in Mathematics Teacher education: How Prospective Mathematics Teachers view their experiences. ***Rukiye Didem Taylan***, *Zelha Tunç-Pekkan, and Mustafa Özcan* (Turkey).
[24]	Building a university-school partnership: from early missteps to emerging success. ***Ryan G. Zonnefeld*** *and Valorie L. Zonnefeld* (USA).
[25]	Features of exemplary lessons over different decades: A comparative analysis of eleven elementary mathematics lessons in China. ***Dongchen Zhao*** *and Yunpeng Ma* (China).

Other three papers focused on how PSTs decide what to do next in teaching based on their interpretations of students' mathematical thinking. Lee and Lee[5] examine the characteristics of elementary PSTs' attention to students' work in order to suggest how to follow up in supporting the students' progress. They conclude the lack of sophistication in the analysis of student thinking influence how PSTs decide about the teaching. Hu et al.[3] revealed the difficulties of PSTs to pose word problems for the measurement model of the concept of division and respond to students' errors. Also, Özel et al.[6] indicated that PSTs had difficulties in interpreting students' algebraic thinking, but they could support the student's algebraic thinking when the solution was incorrect by asking follow-up questions. These studies underline two aspects. First, the complex relationships between the skills of attending to students' mathematical thinking and how to support the student's progress. Secondly, the relationship between PSTs' knowledge and their reasoning processes

The findings of these studies suggest the important role of professional tasks in PST noticing skills in mathematics education. A third set of papers focused on aspects to enhanced noticing. Bernabeu et al.[2] provide designing principles for the tasks in primary teacher education that are designed to enhance the core practices which constitute professional noticing. The authors consider three dimensions: a sociocultural perspective of PSTs' learning, the necessary introduction of information about the development of students' mathematical thinking (e.g. learning trajectories of mathematical topics), and practical registers from mathematics teaching (e.g., a set of students' answers indicating different features of students' development). Baki and Özmen's paper[1] underlines the impact on noticing of the written feedback that preservice teachers received. Finally, Starkey et al.[7] underlined the role of providing PSTs with a framework about noticing. Their findings suggest that the use of a productive struggle framework helped PSTs develop a language for discussing.

3. Primary Level Preservice Teachers' Learning

Five papers focused on PSTs' learning using different theoretical frameworks and with specific foci. One common aspect of these papers is to identify specific intervention (or contexts) that influence PSTs' learning.

Inesson and colleagues' study[9] draws on aspects of enactivism and the notion of reflective spection to trace threads between PSTs' retrospection of their own learning and pro-spection of their approach to teaching. The findings suggest the importance of collaboration in 'seeing' what others 'see' and its influence on PSTs' own teaching. Aryee and colleagues' study[8] underlines the role played by the Montessori mathematics-for-teaching training on how teachers organize and prepare classroom learning environments. Samukeliso and Mellony[12] focused on the PSTs' reflective practice to identify its influence on learning. Park's study[11] underlines the role played by the mathematics content courses in reforming the PSTs' self-perceived competence in the mathematical concepts and in teaching mathematics to students. Finally, Khirwadkar and Figg's study[10] focuses on how to prepare PSTs for teaching in digital classrooms to create meaningful digital learning experiences. This study introduces makerspaces to PSTs and analyses how PSTs connect mathematics teaching and makerspaces experiences.

Globally these studies underline different types of activities influencing the PSTs' learning and their approaches to teaching. Some of these activities are the collaboration in seeing what others see when analyzing mathematical tasks as tools for teaching; the nature of their own experience as mathematics learners influencing on their self-perceived competence; participating in a training with a specific approach (e.g. Montessori teacher education system); and participating in a reflective practice in the context of video-based lesson analysis.

4. Primary Level Preservice Teachers' Knowledge

PSTs' knowledge and features of learning environments aimed to improve the PSTs' knowledge was the focus of eight papers. What PSTs should learn and in which environments are presented from different perspectives and relationships. For example, PSTs' knowledge about posing problems or about the knowledge necessary to teach mathematics (and how we can measure them). Other papers focus on the features of learning environments for enhancing the reflective activities and its relation with problem solving ability, or to learn to facilitate productive math talk during problem solving.

Montes et al.'s study[13] focuses on how prospective teachers pose problems involving a specific operation with negative numbers. The findings showed inconsistencies in the problems posed by the prospective teachers (regarding the relationship between negative numbers and the contexts). Also, Tahir et al.[20] analyzed PSTs' conceptual understanding and pedagogical content knowledge in teaching fractions. Boakes' study[14] presents how to prepare the PSTs for the mathematical aspects of the performance-based assessments that measure PSTs' pedagogical competencies. Garcia-Alonso et al.[15] reported the results of a study aimed to measure the primary PSTs' geometric and measurement knowledge. Globally, these studies support the hypothetical relationship between teacher's knowledge and the development of teaching skills (such as planning the mathematical lessons).

Features of different approaches to improve PSTs' knowledge are the foci of other group of papers. Kool and Keijzer[17] presented a digital learning environment (TORPEDO) for developing PSTs' mathematical problem-solving abilities. The environment enhanced the PSTs' reflections after solving non-routine mathematics problems. The relationship between how PSTs perceived that the reflection contributed to their problem-solving ability was analyzed. Also, Grant and Kastburg[16] analyzed how the participation in a virtual teaching (classroom simulation environment) help PSTs to facilitate productive math talk during problem-solving. The teach-reteach approach to microteaching within a simulated representation of practice using avatars as students seem to support the PSTs' knowledge of how facilitate productive math talk during problem-solving. Lopez et al.[18] presented the features of a learning environment (learning unit) aimed at developing the specific mathematical knowledge involved in the construction and use of a definition of the boundary of 2-D shapes.

Finally, Logwe-Mandala[19] studied the talk of a mathematics teacher educator teaching number concepts and operations to PSTs as a key feature of the learning environments in teacher education programs. Globally, these studies present features of learning environments and the activities generated in them to improve the different dimensions of PST knowledge.

5. Others: University-School Partnership, PST Beliefs, and Textbooks

The foci of five contributions were on contextual and institutional factors such as partnership between university and school; cognitive factors such as beliefs and attitudes, and finally characteristics of the lessons from a textbook. The issue of how to change the PSTs' beliefs and attitudes toward mathematics and its teaching reflects the recognition that PSTs sometimes do not hold productive beliefs or do not have attitudes compatible to support students in primary education. So, some studies have the goal to change PSTs' beliefs and attitudes. Lizasoain and colleagues[22] presented a mathematical workshop to bring the PST mathematical training closer to the school classroom reality. The workshop has as a goal to influence the PSTs' confidence and beliefs about the nature of mathematics and their attitudes towards the mathematics teaching. Arvidson[21] reported a course based on Multiple Intelligences aimed to change the attitudes toward mathematics showing the difficulties of this change.

The school-university partnership in Primary Mathematics Teacher Education is a key issue to the success of the primary teacher training. School-University partnership is the focus of two papers. Taylan and colleagues[23] reported a model of university-school partnership addressed to integrate theoretical knowledge of teaching with school–based practical knowledge. Different forms of school and university partnership experiences and PSTs' views of these experiences are described together. Zonnefeld and Zonnefeld[24] reported the trajectory of how to build the university-school partnership (from early missteps to emerging success).

Finally, Zhao and Ma[25] reported a comparative study on exemplary lessons of primary mathematics over different decades in Chinese mainland using five dimensions in order to identify the influence of the mathematics education reforms. This study reported how mathematics teaching can be reflected in mathematics lessons and how this can influence the practices of primary level teachers.

6. Final Remarks and Future Implications

In TSG-28, we discuss research and development work on MTE, including the underlying assumptions about classroom practice and PSTs' and school students' learning. We also discuss the potentials of, and challenges for, the research endeavour itself, that is, questions concerned with the use of different theoretical frameworks and methodologies.

https://doi.org/10.1142/9789811287152_0053

Topic Study Group 29

Preservice Mathematical Teacher Education at Secondary Level

Olive Chapman[1], Jing Cheng[2], Tracy Helliwell[3], Benita Nel[4], and Immaculate Kizito Namukasa[5]

ABSTRACT Topic Study Group 29 (TSG-29) addressed preservice secondary mathematics teacher education. Its aim was to engage participants in sharing and discussing significant new trends and development in research, theory, and practice related to the various aspects of the initial education of secondary mathematics teachers. This report includes the themes of the TSG, overview of the TSG program, themes of the presentations, and future directions for research.

Keywords: Secondary preservice teachers; Mathematics teacher education; Topic Study Group.

1. TSG-29 Aims and Themes

1.1. Aims

The aim of Topic Study Group 29 (TSG-29) was to engage participants in sharing and discussing of significant new trends and development in research, theory, and practice related to the various aspects of the initial education of secondary mathematics teacher. The intent was to offer a program consisting of an overview of the current state-of-the-art, invited contributions from experts in the field, presentations of high-quality research reports from an international perspective, and discussion of directions for future research. Through this program, participants were expected to learn about and discuss research studies from different countries as well as have opportunity to learn about practices used around the world in relation to the education of preservice secondary mathematics teachers such as similarities and differences in the formal mathematics education of teachers, types and routes of teacher education and pathways

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to certification, curricula of mathematics teacher education, and factors that can influence similarities or differences.

1.2. Themes

Five themes were selected to frame the program on Preservice Secondary Mathematics Teacher (PSMT) education and satisfy the aim of the TSG. The call for proposals invited submissions of papers on research that addressed the suggested topics or other related topics for each theme. Tab. 1 consists of the themes and related topics.

Tab. 1. TSG-29 themes and topics

PSMT knowledge

- Nature of PSMTs' content and pedagogical content knowledge
- Theoretical and methodological frameworks for studying PSMTs' knowledge
- Development of PSMTs' knowledge during teacher education
- Relationship between PSMTs' knowledge and their practices

PSMT professional beliefs and identities

- Theoretical and methodological frameworks for studying PSMTs' professional identities
- Experiences contributing to the development of PSMTs' identities
- Experiences contributing to changes in PSMTs' professional beliefs during preservice education
- Relationship between PSMTs' practicum teaching and professional identity
- Nature of PSMTs' productive disposition

PSMT field experience

- Effective teaching practices
- PSMTs' experiences in mathematics classrooms and issues related to their school placements
- Mechanisms that foster bidirectional relationships between partner schools and higher education institutions to support PSMTs in their field experiences
- Experiences that PSMTs should have prior to student teaching
- Activities to help PSMTs to become reflective practitioners during student teaching
- Different types of field experiences required for PSMTs' certification
- Best practices for preparing mentor teachers to work with PSMTs

Technologies tools, and resources

- Characteristics of technology, pedagogy and content knowledge (TPACK), development and assessment of PSMTs' TPACK
- Video cases and online interactive environments that support PSMTs' learning
- Mathematics tasks, textbooks, and other curriculum materials to support PSMTs' learning
- Assessment tools used in PSMTs' mathematics education programs
- Tasks to assess PSMTs' mathematics knowledge for teaching

Teacher educator knowledge

- The nature of mathematics teacher educators' knowledge for teaching PSMTs

2. TSG Program Overview

The TSG-29 program was based on the proposals submitted and the presenters who were able to participate in the online sessions. The following overview of the program outlines the TSG's submissions, format, presentations, and presentation themes.

2.1. Submissions

The call for submissions to the TSG resulted in 51 proposals: four from invited presenters, 40 for oral presentations, and seven for poster presentations. The paper and poster proposals were reviewed by two reviewers — one from the organizing team and one from the list of authors. Based on the reviews, 17 papers were accepted as long oral presentations, 18 as short oral presentations, and 12 as posters. The invited paper proposals were reviewed by the organizing team to provide feedback to the authors. The final breakdown of the 51 proposals accepted consisted of: 4 invited presentations; 17 long oral presentations; 18 short oral presentations; and 12 poster presentations.

2.2. Format

The initial plan was to structure the TSG program based on the TSG themes (Tab. 1). However, this plan was revised when several participants indicated that they were not able to attend the online conference. The remaining papers were not well distributed among the TSG themes, which made organizing them into meaningful discussion groups across the three sessions impractical. The final format of the program consisted of individual presentations ranging from eight to 20 minutes depending on whether long or short oral or invited. There was limited interaction and discussion of questions due to the tight timeframes for the presentations and the online setting. Each session was opened by the chair of the organizing team, Olive Chapman, and chaired by one of the team members. For the first session (120 minutes), Tracy Helliwell chaired the first hour with five presentations and Jing Cheng chaired the second hour with five presentations. Benita Nel chaired the second session (90 minutes) with six presentations and Immaculate Namukasa chaired the third session (90 minutes) with six presentations.

2.3. Oral presentations

Of the 39 accepted oral presentation submissions to the TSG, only 22 were presented during the online conference because some authors were unable to attend. Tab. 2 consists of the oral presentations given at the conference. In the table, IT stands for Invited Talk, LO for Long Oral presentations, and SO for Short Oral presentations. It is organized alphabetically within the different categories of presentations. Each session consisted of at least one invited presentation, one long oral presentation, and two short oral presentations.

Tab. 2. TSG-29 Oral presentations

Paper and author(s)	
[1]	Using multiple scripting tasks to probe preservice secondary mathematics teachers' understanding of visual representations of function transformations. ***James Mendoza Álvarez***, *Theresa Jorgensen, and Janessa Beach* (USA). (IT)
[2]	Measuring prospective secondary mathematics teachers' knowledge. ***Kim Beswick*** (Australia). (IT)
[3]	A case study on the development of pedagogical design capacity of mathematics prospective. ***Meiyue Jin*** (China). (IT)
[4]	Mentor teachers as inductors of preservice mathematics teachers at secondary schools. ***Kakoma Luneta*** (South Africa). (IT)
[5]	Developing an identity as a mathematics teacher: connecting with the community of teacher graduates. ***Judy Anderson*** *and Debbie Tully* (Australia). (LO)
[6]	Teacher educators' use of technology to represent instruction. ***Daniel Chazan*** *and Patricio Herbst* (USA). (LO)
[7]	Tertiary and secondary mathematical knowledge for prospective teachers: a comparison of teacher employ-ment tests for secondary math in Korea and China. ***Xiaoying Chen*** *and Bomi Shin* (South Korea). (LO)
[8]	Developing preservice teachers' ability to enact formative assessment for mathematical practices. ***Jacqueline Coomes*** (USA). (LO)
[9]	A situated approach to assess prospective mathematics teachers' professional competencies. ***Le Thi Bach Lien*** *and Tran Kiem Minh* (Vietnam). (LO)
[10]	Transforming secondary mathematics teacher preparation: a multi-dimensional problem. ***Gary Martin*** *and Marilyn E. Strutchens* (USA). (LO)
[11]	Instrumental genesis and the growth of preservice secondary mathematics teachers' technological content knowledge. ***Xiangquan Yao*** (USA). (LO)
[12]	Prospective secondary mathematics teachers' learning of problem solving and modelling for teaching. ***Olive Chapman*** (Canada). (SO)
[13]	Direct and indirect effect sizes on secondary mathematics teacher candidates' content knowledge & pedagogical content knowledge as measured by national examinations. ***Jeremy Zelkowski*** *and Tye Campbell* (USA). (SO)
[14]	Emotional awareness and support for preservice teachers during micro-teaching. ***Réka Szász*** (Hungary). (SO)
[15]	Teacher candidates' and mentor teachers' perspectives of using co-planning and co-teaching during clinical experiences in secondary mathematics. *Ruthmae Sears,* **Cynthia Castro-Minnehan**, *Laurie Riggs, Pier Junor Clarke, Jamalee Stone, Charity Cayton, Maureen Grady, Jennifer Oloff-Lewis, Patricia Brosnan, and Marilyn Strutchens* (USA). (SO)
[16]	Should school and university mentors agree in their feedback to pre-service mathematics teachers? ***Viren Ramdhany*** (South Africa). (SO)
[17]	Physical representations and understanding of multivariate functions. ***M. Kathleen Heid*** *and Matthew Black* (USA). (SO)
[18]	Developing prospective teachers' knowledge to promote students' mathematical reasoning: design of a teacher education experiment. ***Ana Claudia Henriques***, *Hélia Oliveira, Leonor Santos, and Henrique Guimarães* (Portugal). (SO)
[19]	A case study on applied lesson study for Korean secondary pre-service teachers. ***Na Young Kwon*** (South Korea). (SO)
[20]	Investigating the professional learning of pre-service mathematics education students using reflection and collective feedback to enhance teaching. ***Benita Portia Nel*** (South Africa). (SO)
[21]	Concept cartoon design in preservice teacher training: an opportunity to learn from the practice. ***Cristina Ochoviet*** (Uruguay). (SO)
[22]	Integrating computational making tools in mathematics thinking activities. ***Immaculate Namukasa***, *George Gadanidis, and Derek Tangredi* (Canada). (SO)

2.4. Poster presentations

Of the 12 posters accepted, only six were presented at the conference. The other authors were unable to attend the online conference. The poster presentations occurred during a separate poster session and not as part of the program of this TSG. Tab. 3 consists of the posters that were presented.

Tab. 3. TSG-29 Poster presentations

Paper and author(s)	
[23]	Online live teaching of mathematics methodology course with tencent classroom. ***Peijie Jiang*** *and Bin Xiong* (China).
[24]	Common construction of pre-service mathematics teachers practical capacity. ***Xiaofeng Lan****, Ying Zhou* (China)*, and Tommy Tanu Wijaya* (Indonesia).
[25]	Pre-service teachers problem solving in trigonometry. ***Kristi Renea Martin*** (USA).
[26]	Encouraging student success: exploring the use of technology based pedagogic strategies within mathematics higher education milieus. ***Jayaluxmi Naidoo*** (South Africa).
[27]	Development of TPACK of preservice secondary mathematics teachers. ***Mária Slavíčková*** (Slovakia).
[28]	A didactic model to favor the positive use of error in the initial teacher training. ***Osvaldo Jesus Rojas Velazquez*** *and Carlos Berrío Pérez* (Colombia).

2.5. Themes of oral presentations

The areas of research covered in the TSG oral presentations addressed seven themes regarding PSMT education. Each theme was addressed in different ways by the studies.

Theme 1 studies focused on the development of the knowledge and ability of PSMTs, which received the most attention by the presentations. The studies included investigations of the following topics: (i) Development of the PSMTs' ability to understand and enact formative assessment as a way to leverage mathematical practices for student learning. (ii) Development of the PSMTs' mathematical and didactical knowledge to promote students' mathematical reasoning. (iii) Use of applied lesson study to help PSMTs to learn about teaching. (iv) A technique involving emotion cards used to give emotional support to PSMTs during micro-teaching. (v) Multi-dimensional issues faced by PSMT preparation programs. (vi) PSMTs' learning through designing and using open-ended concept cartoons. (vii) Supporting PSMTs' learning of problem solving and modelling for teaching.

Theme 2 studies addressed measuring PSMT knowledge. They investigated ways of measuring/assessing the PSMTs' knowledge for teaching mathematics; professional competencies for teaching mathematics from a situated perspective; content and pedagogical content knowledge based on national examinations; and mathematical knowledge based on teacher employment tests for secondary mathematics.

Theme 3 studies focused on PSMT field experience. They investigated: (i) PSMTs' learning using reflection and collective feedback of practice teaching to enhance teaching; (ii) practicum experiences of PSMTs supervised by school and university mentors; and (iii) PSMTs' and mentor teachers' perspectives of using co-planning and co-teaching during practicum.

Theme 4 studies focused on PSMT content knowledge. They explored: (i) the PSMTs' understanding of visual representations of function transformations and capacity to connect multiple representations of functions and (ii) the PSMTs' physical representations and understanding of multivariate functions. One study used a method involving multiple scripting tasks to conduct the exploration.

Theme 5 studies focused on PSMT teacher educators. They investigated: (i) a training program around mathematics teacher induction and mentorship of PSMTs and (ii) PSMT teacher educators' use of technology to represent instruction and facilitate collaboration on both teaching and research.

Theme 6 studies focused on PSMTs' use of technology. They explored: (i) PSMTs' integration of "computational making tools" in mathematics thinking activities in teaching mathematics and (ii) types of technological content knowledge emerging in the process of instrumental genesis when PSMTs engaged in problem-solving with the Geometer's Sketchpad.

Theme 7 focused on identity. It consisted of only one study that investigated a strategy to build community of practice to support PSMTs' development of identity as a mathematics teacher.

3. Future Directions for Research

Research on mathematics teacher education continues to be of importance to support international efforts to reform mathematics education for a digital and changing world. As reflected in the number of submissions to this TSG, there is high level of interest in engaging in research on PSMT. The presentations and the themes of the TSG suggest future directions for ongoing research on PSMT. Tab. 4 presents summaries of the TSG themes and the themes emerging from the oral presentations to highlight how they are related. While all of the topics for the TSG themes (Tab. 1) are important to guide future research of PSMT education, the following discussion focuses on aspects of them that overlap with the presentation themes as outcomes of the TSG program with implications for future directions of research.

Tab. 4. Summary of themes of TSG-29 and oral presentations

TSG-29 themes	Themes of oral presentations
PSMT knowledge	Exploring PSMT content knowledge Development of PSMT pedagogical knowledge
PSMT professional beliefs and identities	Development of PSMT identity
PSMT field experience	Field experience
Technologies, tools, and resources	Measuring PSMT knowledge Technology
Teacher educators' knowledge	Teacher educators

PSMTs' knowledge for teaching mathematics (the first TSG theme) is central to teacher education programs. Thus, ongoing research is necessary to offer further insights of the nature of this knowledge and instructional approaches to effectively support the PSMTs' development of it. The TSG presentations included studies that explored PSMTs' knowledge of functions and the use of unique ways of conducting the exploration. The presentations also included studies that investigated PSMTs' development of different aspects of their pedagogical ability (e.g., use of formative assessment; promoting mathematical reasoning) and instructional approaches to support PSMTs' learning of pedagogical knowledge for mathematics (e.g., use of lesson study; emotion cards; concept cartoons). Based on these studies, one implication for future research is consideration of different mathematics content topics associated with secondary school curricula and of innovative approaches to explore these topics to offer alternative ways of understanding PSMT content knowledge for teaching. Another implication is the need for further research to deepen our understanding of: (i) the development of different aspects of PSMTs' pedagogical ability and (ii) innovative, effective approaches to support PSMTs' development of both pedagogical ability and pedagogical content knowledge for mathematical practices.

The TSG theme of PSMT professional beliefs and identities was the least represented by the TSG presentations. One study addressed the development of PSMTs' identity through community of practice. However, while math-related beliefs have received lots of attention in research, the same has not been the case for identity, which is also an important component in defining the mathematics teacher. Thus, identity remains an area that should receive more attention in future research on PSMT.

The TSG theme of technologies, tools, and resources highlights specific ways (e.g., use of digital tools, mathematical tasks, textbooks) of engaging PSMTs in their learning and ways of measuring the level of what they learned or know. The TSG presentations included studies that addressed technology in the context of exploring PSMTs' use of "computational making tools", PSMTs' technological content knowledge, and teacher educators' use of technology in instruction. The presentations also included studies that addressed ways/tools of measuring or assessing content and pedagogical knowledge (e.g., performance measures; a situated perspective framework; national examinations; teacher employment tests). The implications from this group of studies is that more attention is needed on this TSG theme in future research to explore it with more breadth and depth. For example, future research could further investigate innovative and effective ways for integrating technology in teacher education and tools for determining the quality of PSMT mathematics knowledge for teaching to inform teacher education. The current pandemic also opens up the importance of considering and researching its impact on PSMTs' education regarding what technological knowledge they should hold (e.g., regarding remote learning) and how to support their learning of it.

The TSG theme of PSMT field experience highlights another central area of PSMTs' education that requires ongoing attention in research. The TSG presentations included studies that investigated PSMTs' learning using reflection and collective feedback of practicum teaching to enhance it, PSMTs and mentor teachers working together, and joint supervision of PSMTs by school and university mentors. Thus, implications for future research include the need to further investigate innovative, effective ways for PSMTs to learn from their field experiences and the types of practices under university and school advisors that are best for practicum experience.

The final TSG theme is teacher educator knowledge, which recently has been receiving growing attention in research. The TSG presentations included studies that investigated an education program for PSMTs' mentor teachers and how teacher educators represented classroom interaction through use of digital tools. Based on this theme, implications for future research on teacher educators at school or postsecondary levels include exploration of innovative, effective approaches or programs to support their development of knowledge for teaching PSMT; the nature of the content and pedagogical content knowledge needed to prepare PSMT in the context of a digital age and changing world; and alternative approaches to researching this knowledge. For example, self-based methodologies (Chapman et al., 2020) such as narrative inquiry and self-studies need more attention as promising ways of exploring and understanding mathematics teacher educators.

Other important areas of PSMT education that should be addressed in future research but not explicitly addressed in the TSG themes include the following: (1) The cooperation between mathematicians and educational researchers is of value in the field of mathematics education, in general, and PSMT education, in particular. Thus, research needs to explore how mathematicians could play a more important role in PSMT education in addition to teaching advanced mathematics courses. (2) Different countries and regions have rich theories and practices in PSMT education. Research should explore new paths of international cooperation in addressing these theories and practices for the benefit of the international field of PSMT education. (3) Preservice teacher education is closely related to previous primary/elementary and secondary school education and subsequent in-service teacher professional development. Thus, research should attend to the correlations and connections among the three in the context of PSMT education. (4) The development of artificial intelligence is constantly changing people's way of life and even the way students learn inside and outside schools. Research should attend to what should or would happen to the content and methods of PSMT education in this changing context.

To conclude, TSG-29 was successful in achieving the goal of sharing meaningful, insightful research on PSMTs from around the world. Through the themes of the TSG and the themes of the paper presentations, the outcomes of the TSG offer implications for future research to advance the field of PSMT education. In general, research needs to address PSMT education in the context of a digital age and rapidly changing world.

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https://doi.org/10.1142/9789811287152_0054

Topic Study Group 30

In-Service Mathematical Teacher Education and Mathematical Teacher Professional Development at Primary Level

TSG-30 Working Team[1]

1. Introduction

The entire organizing team worked together before the congress in planning and organizing TSG-30 since 2018. There are 36 proposals submitted, and finally 4 accepted and presented as long-oral reports and 15 accepted and presented as short-oral reports. The TSG-30 was well attended in two 90-minute sessions and one 120-minute session in July 2021, which indicates strong interest in this topic by congress delegates. This report provides an overview of the aim and focus of TSG-30 and a summary of the presentations and discussions that occurred throughout the sessions.

2. Aims, Focus, and Themes

As set by the organization team, the general aim of TSG-30 was, in the international mathematics education community, to provide a venue for congress participants to share research, policy, design or practice that focuses on in-service mathematical teacher education and mathematical teacher professional development at primary school level.

The focus of TSG-30 was a discussion of research related to in-service mathematical teacher education and mathematical teacher professional development. In-service mathematics teacher education and professional development are integral parts of teachers' life-long learning process, and take many different formal or informal formats with various focuses and requirements within and across education systems. The situation becomes especially complex at primary school level, where teachers can be generalists in many education systems but content specialists in some other systems. Understanding and researching in-service mathematical teacher education and

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mathematical teacher professional development at primary level, therefore, call for special attention to policy, design, and practice situated in special system and sociocultural contexts. For example, it is not difficult to notice contrasting practices in which mathematics teachers work and learn in different ways through various forms of collaborations in the East versus in the West. Efforts to understand what in-service mathematics teachers may do in and for improving their teaching and expertise have led to ever-increased interest in exploring and examining different programs, activities, and the nature of various collaborations and processes through which primary school teachers are engaged to learn. Consistently, new theoretical perspectives have also been developed and proposed about in-service teachers' professional development (e.g., practice-based professional education of teachers, locating teacher learning in communities, lesson study in Japan and China, communities of teachers working in contact with communities of researchers and evolving in their professional practices). It is important to understand through research the nature of different programs and activities, the focus and process of various teachers' knowledge learning and professional development, the extent of teachers' learning effects, the roles of policy and administrative support, and specific system and sociocultural factors associated with different teacher education programs and activities.

With this focus, TSG-30 was intended to provide an international gathering place for all interested parties (e.g., mathematics educators, teacher educators, school teachers, educational researchers, etc.) to share and disseminate findings from their research on in-service mathematical teacher education and mathematical teacher professional development at primary level, with the use of various theoretical perspectives and methodologies, and to exchange ideas in research, development, and evaluation of in-service mathematical teacher education and mathematical teacher professional development at primary level.

In TSG's call for contributions, the following five themes were outlined:

1. Theoretical perspectives and methodological advances in research on in-service mathematical teacher education and mathematical teacher professional development at primary level;
2. Research on the design and/or implementation process of specific programs, approaches, and practices, such as the use of video clips and IT, for in-service mathematical teacher education and mathematical teacher professional development at primary level;
3. Research on documenting the effectiveness of specific programs, approaches, and practices for in-service mathematical teacher education and mathematical teacher professional development at primary level;
4. Research on comparing and documenting system and sociocultural factors contributing to in-service mathematical teacher education and mathematical teacher professional development at primary level;

5. Issues concerning possible (dis)connections between research and practice in in-service mathematical teacher education and mathematical teacher professional development at primary level.

The original plan was then to organize accepted proposals for presentations in theme-based sessions. Unfortunately, due to the unexpected pandemic that started in early 2020, the organizers had to not only postpone the Congress from July 2020 to July 2021, but also to offer all the sessions in a hybrid format due to the international travel restrictions. At the same time, however, the hybrid format itself won't be able to solve the issue of time zone differences for international contributors as they may not be able to join a specific session. In fact, many international contributors decided not to join the Congress to present, even though their proposals were accepted. At the end, there were 19 accepted proposals remaining with the contributors agreed to participate and present during the Congress. Thus, the session organization was no longer to follow specific themes, but mainly to accommodate presenters' availability and preference to present at certain times.

Given the three sessions (two 90-minute sessions and one 120-minute session) that were allocated for TSG30, the first 90-minute session on July 13th was devoted to four long-oral presentations (LO), and the other 90-minute session on July 16th and the 120-minute session on July 17th were organized for the 15 short-oral presentations (SO) (see Tab. 1 on the next page). In the following sections, we briefly summarize the paper presentations and discussions during these sessions.

2.1. Session 1

The first long oral presentation (Huang and Zhang)[1] shared the scope of a program that involved mathematics teachers from Shanghai and Britain that, in a collaborative environment, taught two weeks in the other country. Taking the Anthropological Theory of the Didactic as a theoretical framework, this strategy provided the professional development of the teachers participating in the project. The study focused on two primary teachers from Shanghai that had their teaching experience in Britain. The results revealed that they encountered difficulties in the implementation of the lesson (the approach they used in their country was not suitable in this new context of practice), which led them to reflect on their practice and to adjust their didactic tasks and techniques in time for teaching improvement.

The long presentation by Livy et al.[2] was to understand how the lesson structure of teaching a challenging task might impact on a Year 2 teacher's pedagogical approaches for teaching mathematics. An inquire-based approach of teaching students with a challenging task was explored in three phases: Launch the task (without telling); Explore the task (students' attempt on the task by themselves); and Summarise (the teacher's selection of particular students to share their work during the lesson to support student learning). The results revealed that the work sample chosen by the teacher in the summarise phase helped students to learn from each other, and permitted

Tab. 1. List of papers presented

Paper and author(s):	
Session 1	
[1]	Chinese teachers' learning as transformation of didactic praxeologies in a cross-cultural teacher exchange programme. ***Xingfeng Huang*** *and Yunji Zhang* (China). (LO)
[2]	Developing teachers' classroom actions and pedagogical knowledge through the facilitation of teaching a challenging task. ***Sharyn Livy****, Janette Bobis, Ann Downton, Sally Hughes, Maggie Feng, Melody McCormick, James Russo, and Peter Sullivan* (Australia). (LO)
[3]	Changes in mathematical knowledge for teaching and belief on practices through professional development based on reasoning-modeling approach. ***Kyong Mi Choi****, Jihyun Hwang, Jessica Jensen, Dae Hong, and Wesley Cox* (USA). (LO)
[4]	Are elementary in-service teachers confident and well prepared in mathematics they teach? — the case of fraction division. ***Yeping Li*** (USA)*, Huirong Zhang, and Naiqing Song* (China). (LO)
Session 2	
[5]	Mathematical reasoning and teacher education. ***Leonor Santos****, Ana Henriques, Joana Mata-Pereira, and Lurdes Serrazina* (Portugal). (SO)
[6]	In-service teacher education for promoting mathematics reasoning in primary school. ***Lurdes Serrazina*** *and Joana Brocardo* (Portugal). (SO)
[7]	Growing through inquiry: a story of three primary teachers investigating their practice. ***Derek J. Sturgill*** (USA). (SO)
[8]	Math teachers competence assessment to develop personalized professional learning. ***Ilze France****, Dace Namsone, Liga Cakane, and Ilze Saleniece* (Latvia). (SO)
[9]	Assessing the efficacy of the math for all professional development program for primary teachers and their students. ***Babette Moeller****, Matt McLeod, Teresa Duncan, Jason Schoeneberger, John Hitchcock, and Marvin Cohen* (USA). (SO)
[10]	Drawing on the didactical suitability criteria to analyse a lesson study enhancing teachers competence of didactical reflection. ***Viviane Hummes****, Adriana Breda, Elvira García-Mora, Vicenç Font, Javier Díez-Polomar* (Spain)*, and Maria José Seckel* (Chile). (SO)
[11]	Insights on Shanghai in-service primary mathematics teachers' acquisition of pedagogical content knowledge through teaching research group activities: a case study. ***Hong Yuan*** (USA). (SO)
Session 3	
[12]	Kyozaikenkyu as well-formed story making for developing quality mathematics lessons. ***Masakazu Okazaki****, Keiko Kimura, and Keiko Watanabe* (Japan). (SO)
[13]	Teaching as professional learning: small steps towards sustainable mathematics teacher professional development. ***Ban Heng Choy*** *and Jaguthsing Dindyal* (Singapore). (SO)
[14]	Improvement of a preschool teacher's reflection on pedagogical content knowledge during a professional development programme in Japan. ***Nagisa Nakawa*** *and Nanae Matsuo* (Japan). (SO)
[15]	Teachers views of the effects of the fostering inquiry in mathematics project. ***Jill Cheeseman*** (Australia). (SO)
[16]	Developing teachers' knowledge of fractions: a case from Karachi, Pakistan. ***Munira Amirali*** (Pakistan). (SO)
[17]	Contingencies as moments of collaboration: a report on investigating and supporting mathematics teachers' knowledge. ***Shikha Takker*** *and K. Subramaniam* (India). (SO)
[18]	Re-conceptualizing primary mathematics in-service teacher professional development in nigerian context. ***Lawan Abdulhamid*** (South Africa) *and Balarabe Yushau* (Nigeria). (SO)
[19]	Development of critical lenses among teachers in lesson study. ***Tan Saw Fen*** (Malaysia). (SO)

the teacher to experience success with her own teaching by building knowledge of pedagogy from practice.

The long presentation by Choi et al.[3] shared a programme with its main objective to develop teachers' mathematical knowledge for teaching (MKT) and beliefs (B) on instructional practices. Using the Reasoning and Modelling approach, the study aimed to understand if this programme has the effects as expected and in which way the 22 participating teachers grew in these two domains. The results suggested that there were significant changes in mathematical knowledge for teaching and beliefs on instructional practices.

The last long presentation, by Li et al.[4], focused on both in-service Chinese teachers' (ITs) confidence about their knowledge and the extent of their conceptual knowledge for teaching (MCKT) on the topic of fraction division. The results revealed how these ITs' confidence may or may not be supported by their knowledge for teaching fraction division, an important topic they need to teach as part of the curriculum standards in China. The results also illustrated the importance of specifying knowledge components in mathematics instruction in order to help build and support ITs' confidence for classroom instruction.

2.2. Session 2

Santos et al.[5] shared the project REASON (Mathematical Reasoning and Teacher Education) that investigated ways to support primary and secondary prospective and practicing teachers' development of mathematical and didactical knowledge to promote students' mathematical reasoning. The presentation highlighted that, when systematic intervention was carried out with prospective and in-service mathematics teachers, their capacity to engage students in mathematical reasoning was evident. The initial result showed that the tasks developed were helping teachers to build their knowledge, skills and capacity to teach students to promote their reasoning skills.

Serrazina and Brocardo[6] reported part of the research developed by the project REASON (Mathematical Reasoning and Teacher Education). They highlighted the key findings of the project with primary school teachers (grades 1–6) following a Design-Based Research approach. The presentation included examples from the training material used in developing teachers' understanding of the nature of tasks that promote mathematical reasoning among students. Also, the students' work indicated that teachers were able to implement their learning acquired in the project to promote mathematical reasoning in their class.

Sturgill[7] presented the research findings from working with three Grades 4–6 teachers of mathematics who taught in the Midwestern United States in the classroom inquiry projects. The study highlighted that classroom inquiry, a structured form of teacher research, is a powerful tool for improving teachers' knowledge for teaching and their practice. The study findings included improvement in teachers' knowledge of teaching and student learning, knowledge of classroom inquiry and action research, and students' engagement with the support and time to enact their respective inquiry projects.

France et al.[8] presented a study that aimed to examine 25 mathematics teachers' competencies in the context of Latvia's curriculum reform in general education, with a particular focus on teachers' preparedness to develop student cognitive skills and ways to stimulate more appropriate and individualized teacher professional development. Based on the study findings, four groups of teachers were identified, each requiring individualized professional development to ensure implementation of reform-relevant ideas into the school practice.

Moeller et al.[9] shared Math for All (MFA), an intensive professional development (PD) program for in-service teachers. They reported on a randomized controlled trial (RCT) of MFA involving 32 schools, 98 4th and 5th-grade general and special education teachers, and approximately 1,500 4th and 5th-grade students. The findings indicated that MFA had statistically significant, positive effects on teachers' self-reports of their preparedness and comfort with teaching. A school-level analysis found a moderate MFA effect on student achievement. Quasi-experimental analyses of a subgroup of teachers being observed showed initial evidence of MFA impacts on their classroom practices.

Hummes et al.[10] discussed the combination of two major instruments for professional mathematics teachers' development: the lesson study (LS) and the didactical suitability criteria (DSC). Drawing on a literature review, presenters argued that combining LS and DSC offers teachers the opportunity to draw on a consensual structured approach covering the main educational dimensions embedded within their classroom practice.

Yuan[11] focused on one of the job-embedded and expert-assisted professional development programs. The study examined Shanghai in-service primary mathematics teachers' acquisition of pedagogical content knowledge (PCK) through participating in Teaching Research Group (TRG) activities. The study findings showed that teachers developed their PCK by creating supplementary teaching materials, studying students' thinking, and teaching mathematical thinking by working closely with teaching research coordinators during TRG activities; and writing reflection reports afterwards. The study has implications for teachers' community of practice which, in turn, improves students' learning of mathematics.

2.3. Session 3

Okazaki et al.[12] shared how teachers' in-depth study of instructional material can be examined in the context of lesson study. In particular, the four levels of teachers' instructional material study were reported, which were identified as a result of the studies that compared three types of teachers who were different in their experiences.

Choy and Dindyal[13] discussed teachers' professional development in the in-service training program in Singapore. They conceived the aspects of the professional development in terms of Desimone's framework, and focused on the first phase of needs analysis. As a result of analysis, they clarified the teachers' insufficient understanding of the connections among mathematics, students' learning difficulties, and teaching approach. Thus, they suggested the importance of positioning every teaching action as opportunities for professional development and of exploring teaching approach in terms of mathematics and students' learning.

Nakawa and Matsuo[14] reported a pre-school teacher in Japan who improved her mathematical abilities in teaching through experiencing PD program based on the ALACT model under the guidance of the researchers. The results showed that the teacher could develop knowledge of content and teaching and knowledge of content and curriculum in the framework of mathematical knowledge for teaching through the program.

Cheeseman[15] reported the change in the teachers' views through participating in the professional development project FliM that focused on problem solving and inquiry. The results showed that the teachers believed they had improved professional skills, knowledge, pedagogies, enthusiasm, and confidence in their teaching of mathematics with young children through the FliM project.

Amirali[16] shared the urgent situation in Pakistan, where primary school teachers are struggling to teach basic mathematical concepts as they have been recruited as the "primary school teachers" rather than "subject specialist teachers" ... Teachers lose the opportunities of choosing grades they want to teach as the administrators would assign mathematics teaching to those whom they think are able to teach mathematics, especially in grades IV and V or even higher grades. Moreover, the researcher also pointed out the basic situation in Pakistan that after the schooling, teachers who teach mathematics received very limited training in mathematics by themselves.

Takker and Subramaniam[17] pointed out that contingencies arising in the context of teaching practice are important moments in the teacher-teacher educator collaboration. These moments would require teacher educators to revisit their goals and use these moments as learning opportunities for all participants. Researchers discussed two episodes (connection between a method and the algorithm & teaching the missing ideas in the textbooks) and found out when teachers sought supports, that dynamic contingent situations emerged. If researchers wanted to respond well to those situations, that would mean they needed more flexibility in their roles and had the knowledge and awareness to deal with. Moreover, these contingent situations created possibility for teachers to try alternative practices and for the researchers to take a more active role in the practice of teaching, and also challenge teachers' existing knowledge of the content, students and specific topics.

Abdulhamid and Yushau[18] presented a re-conceptualization of mathematics teacher professional development that highlights the policy implications for addressing the gaps among Nigerian primary mathematics teachers in their fundamental understandings of basic mathematics. Despite the fact that huge rollout of PDs in Nigeria that focused on teachers' content knowledge and content-specific pedagogy, the researchers found that PDs typically were not based on the research and teachers' specific needs. They called for the need to disaggregate the levels at which in-service mathematics teacher professional development interventions could usefully start across lower (grade 1–3), middle (grade 4–6) and upper (grade 7–9) basic teachers, and the need for a longer period of PD, with interim assessments.

Fen[19] presented a study that explored the development of teachers' critical lenses when they were conducting lesson study within two lesson study groups. The two lesson study groups, comprised of 6 teachers and 3 teachers respectively, conducted five lesson study cycles. Teachers from both lesson study groups developed student and curriculum developer lenses. But those teachers only developed curriculum developer and student lenses within this study, rather than the researcher lens as the development of their critical lenses might be affected by the knowledgeable others, anticipation of students' responses and difference in experience and seniority among the participating teachers. Researchers pointed out that when planning to set up lesson study group in a specific school, administrators might want to focus on the experience and seniority of teachers which showed great importance to teachers' professional development.

3. Closing Remarks

Among the main points discussed and summarized across these three sessions we highlight the following ones:

- Efforts to explore and document effective programs and approaches to improve mathematical teachers' knowledge for teaching and related beliefs in diverse system and cultural contexts;
- The search for possible approaches and solutions to address the weak mathematical training that many primary school teachers often have in diverse system and cultural contexts, and possible policy implications;
- The changes in teacher professional development needs in the context of educational reforms, including curriculum and in-service learning, and efforts to address such changes;
- The development of new conceptions (or re-conceptions) of what teachers need to know and be able to do in teaching, and teacher professional development;
- The development and use of various assessment and analytical tools to document possible changes in teachers' knowledge, beliefs, and/or practices through participating in professional development programs.

Naturally, in such a broad topic as mathematical teacher education and mathematical teacher professional development at primary level in diverse system and cultural contexts, many questions remain to be addressed. The diversity of approaches and foci presented suggests many different perspectives that contributors took to develop, search, and document effective programs or approaches, which aspects of professional development were to be focused, what may need to be developed and implemented to address the critical needs in teacher professional development in various contexts, and how professional development may be positioned to facilitate educational reforms. The participants shared a strong interest in various topics covered in the TSG's presentations through online discussions across these sessions.

https://doi.org/10.1142/9789811287152_0055

Topic Study Group 31

In-Service Mathematical Teacher Education and Mathematical Teacher Professional Development at Secondary Level

Konrad Krainer[1], Betina Duarte[2], Talli Nachlieli[3], Craig Pournara[4] and Youchu Huang[5]

ABSTRACT The focus of TSG-31 for ICME14 was the study of in-service and/or professional development initiatives aimed at improving secondary mathematics teaching on a *large scale*. We adopted the definition of *scaling up* as reaching many classrooms, and potentially whole schools, districts, cities, or even a whole state or nation. We also encouraged submissions dealing with the adaptation and implementation of an initiative from another country. TSG-31 covered a wide range of secondary in-service courses and professional development programs, as well as school development projects, and collaborative networks of practitioners and researchers.

Keywords: Scaling up; Professional development; Large scale programs.

1. Topic under Study: General Description

At ICME-14, the Topic Study Group "In-Service Mathematical Teacher Education and Mathematical Teacher Professional Development at Secondary Level" (TSG-31) focused on the study of in-service and/or professional development initiatives aimed at improving secondary mathematics teaching on a large scale. *Scaling up* means to reach many classrooms ($n \geq 10$), and potentially whole schools, districts, cities, or even a whole state or nation. The TSG team expressed openeness for a broad range of initiatives, including in-service courses, professional development programs, school development projects, and collaborative networks of practitioners and researchers. Each initiative was asked to be research-based and to provide new insights into the challenge of improving mathematics teaching on a large scale. Some of the research questions considered were: What does it take to up-scale a professional-development

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program? What are the factors that need to be considered when adapting a certain program to new cultural settings? Which aspects of the intervention could be scaled up and which couldn't? How could the impact of large-scale approaches be evaluated? What types of diagnostics about students' mathematics learning can be applied and why? To what extent can a steady collaboration between research and practice be achieved on a systemic level? What are the key factors in sustaining such collaboration? What challenges arise during such collaborations?

In order to facilitate the organization of activities, the TSG-31 team carried out a voluntary online-meeting one day before the first session. Among others, we explained our chat & notes system which helped us to record questions, answers and comments by onsite and online participants as well as by us during the week.

2. Contributions

Overall, 40 papers and posters had been submitted to TSG-31. Due to several reasons (review process, pandemic etc.), we finally had 20 paper presentations and one invited talk (IT) (18 online and 3 onsite), representing countries on six continents (Tab. 1 on the next page). In addition, five posters were presented in an extra session outside the topic study groups.

2.1. Core ideas about scaling up

Paul Cobb[1], the invited speaker, pointed out the significant progress reached by research on the teaching and learning of mathematics in recent years. However, these findings have limited impact on classroom instruction in many countries, including the USA.

On a recent investigation, Cobb collaborated with mathematics teachers, school leaders, and the leaders in several large urban school systems for eight years to investigate what it takes to support improvements in the quality of instruction and thus students' learning on a large scale. Their findings from this work take the form of an empirically-grounded theory of action (ToA) for instructional improvement at scale that spans from the classroom to system instructional leadership and encompasses: curriculum materials and assessments; pull-out teacher professional development; school-based teacher collaborative meetings; coaches' practices in providing job-embedded support for teachers' learning; school leaders' practices as instructional leaders in mathematics; and system leaders' practices in supporting the development of school-level capacity for instructional improvement.

Scaling up is a special case of implementation which can be understood as "a change-oriented process of endorsing an action plan" (Koichu et al. 2021). Implementation, that aims at scaling up (e.g., to thousands of mathematics classes or schools) has much more complex issues to deal with than a smaller project (e.g., working with some mathematics classes or schools). When reaching a larger regional or national level, a sound interaction between research, practice and policy is needed. Regarding the balance between research, practice and policy, two contrasting approaches — Technical rationality and Reflective rationality (Schön, 1983; Altrichter et al. 2008) — have been described so far. In order to combine the strengths of both approaches and to avoid weaknesses, a third approach — Societal rationality — is introduced (Krainer, 2021) and discussed.

Tab. 1. List of presentations and invited talk (IT) in TSG-31

Paper and author(s)	
[1]	Investigating what it takes to improve the quality of mathematics teaching and learning on a large scale. ***Paul Cobb*** (USA). (IT)
[2]	How Chinese mathematics teachers prepare for teaching competition in community? ***Chenfei Zhu*** *and Hongbing Wang* (China).
[3]	Linking theories and practices: understanding teachers' learning in Chinese lesson study through activity theory perspecitve. ***Wenjun Zhao****, Rui Ning, Xiaoxia Zhang, Chuan Zeng, Xianjia He, and Jun Wen* (China).
[4]	Shifting cultural contexts: a professional development program towards congnitively demanding instruction*.* ***Talli Nachlieli*** *and Einat Heyd-Metzuyanim* (Israel).
[5]	Scaling up a mathematics professional development course in South Africa and its impact on students. ***Craig Pournara*** (South Africa).
[6]	Action learning: a tool to help teachers promote self-regulation (SR) in students. ***Tamsyn Margaret Terry*** (Australia)
[7]	Collaboration between mathematics and special education teachers to promote argumentation as an inclusive practice. *Pilar Peña,* **Horacio Solar***, Constanza San Martín, and Florencia Gómez* (Chile)
[8]	Developing and supporting exemplary mathematics educators in high need schools. *Lillie R. Albert, Chi-Keung Cheung, and* **Solomon Friedberg** (USA).
[9]	An investigation on mathematics teachers' professional development in rural China. ***Limin Chen*** (China)*, Caroline Williams-Pierce* (USA)*, and Min Jing and Lieven Verschaffe*l (Belgium).
[10]	Sustainablity and scaling up of school-based teacher professional development programme. ***Zhen Feng Eric Koh****, Leng Low, and Ngan Hoe Lee* (Singapore).
[11]	Effective design of massive open online courses to support mathematics teachers' professional learning. ***Karen Hollebrands*** *and Hollylynne S. Lee* (USA).
[12]	Using videos to foster facilitators' noticing in the field of language-responsive mathematics teaching. ***Christoph Look****, Christin Laschke, Bettina Roesken-Winter, and Rebekka Stahnke* (Germany).
[13]	Investigation on the identification and group differences of professional development approaches of mathematics teachers. ***Luyishou Ma*** (China)
[14]	Developing an e-mentoring professional development program in supporting pedagogical content knowledge of novice mathematics teachers: A design-based study. ***Derya Çelik****, Mustafa Güler, Rukiye Didem Taylan, Müjgan Baki, Esra Bukova Güzel, Fatma Aslan Tutak, Damla Kutlu, and Aytuğ Özaltun Çelik* (Turkey)
[15]	Enhancing students' mathematical reasoning through a professional development experiment. *Joana Mata-Pereira and* ***João Pedro da Ponte*** (Portugal).
[16]	Exploring online learning environments in professional development for scaling-up educational innovations. *Robert Weinhandl and* **Stefanie Schallert** (Austria).
[17]	Professional development facilitators and their learning goals towards a PD course on teaching probability and inferential statistics. ***Ralf Nieszporek****, Birgit Griese, and Rolf Biehler* (Germany).
[18]	Out-of-field teachers' acquisition of school-related content knowledge during a professional development course. ***Steffen Lünne*** *and Rolf Biehler* (Germany).
[19]	Windows on the backstage of the classroom: Using video to support mathematics teachers conceptual change about instruction. ***Ilana Horn*** (USA).
[20]	Survey and analysis of confusion of the implementation of the new curriculum for high school mathematics teachers in Henan Province, China. ***Deming Yan*** *and Hongwei Wang* (China).
[21]	In-service mathematical teacher education in Morocco: impediments and challenges. ***Nouzha EI Yacoubi*** (Morocco).

2.2. Professional development issues at large scale

China has a level-by-level teaching competition system. Every selected contestant teacher from a lower-level competition, together with their community, prepares for a higher-level competition through iterative and incremental preparing sessions. These preparing sessions all start with a mock-teaching lesson with randomly selected students and are followed by discussions within the community. As a part of an ongoing research, Zhu and Wang[2] reported findings from these preparing sessions through an analysis of 28 interviews. These sessions share similar features in objectives, procedures, participants, resources, and effects; as they mature, there are changes and trends; all participants improve their knowledge, abilities and beliefs related to mathematics teaching. The process of the preparing sessions is constructive, resourceful, inspirational, and problem-solving-like. Moreover, the influence of this type of work is progressive, productive, transferrable and pointed to all participants' long-term professional development.

Chinese lesson study is powerful in linking theories and practices in the reform context. However, the mechanism behind such effectiveness is still under-researched. Zhao et al.[3] contributed to this issue exploring how teachers learn to use theories to guide their teaching in a Chinese lesson study. Taking activity theory as the theoretical lens, their research identified contradictions between activity systems of research and teaching, and how they were dealt with through the lesson study activities. This can shed light on how teachers learn in lesson study and necessary conditions that support their learning.

Nachlieli and Heyd-Metzuyanim[4] provided another contribution to the understanding of the processes of teachers change through professional development programme. The study explored processes of change in teachers' practice as a result of participating in a professional development (PD) for cognitively demanding, discourse-rich, instruction, titled TEAMS (Teaching Exploratively for All Mathematics Students). The TEAMS PD was "imported" from the USA to Israel, relying on two programmes: "The 5 Practices for Orchestrating Productive Discussions" and "Accountable Talk". Participants included 50 middle- and elementary-school teachers who participated in PD meetings during one school year and videotaped themselves teaching cognitively demanding tasks. Lessons were coded using a lesson-observation protocol. Individual Growth-Curve Model analysis indicated statistically significant growth in several parameters. Their findings contributed to shed light to the challenges of transferring a PD programme between two cultural contexts. As an example, it is pointed out that the "travel" of observational measures for studying the cognitively demanding instruction across cultures, is limited.

Schallertn and Schallert[16] agreed in her presentation with Cobb on the difficulty of getting educational innovations beyond the pilot phases. To scale up educational innovations, they combined professional mathematics teacher development and online learning environments (OLE). By applying a grounded theory approach and design-based research, they have investigated how OLE should be designed to support scaling-up. Analyzing written and oral research data indicated that (a) teachers make their own decisions concerning online learning, (b) OLE highlights benefits and practical relevance of an approach/technologies, (c) OLE does not lead to additional work, and (d) security and privacy of OLE could be crucial for teachers.

Related to online environments, Hollebrands and Lee[11] investigated the use of MOOCs for professional development. The design of three MOOCs for mathematics and statistics teachers based on principles of effective online professional development were guided by design principles and describe characteristics and engagement of 5,767 registrants with these designed features. Through this experience, the researchers claim that design principles provided opportunities for educators to develop their pedagogical skills.

Coming back to sustainability issues, the paper [10] of Koh and colleagues discussed the factors that influence the sustainability and scale-up of a school-based professional development programme for mathematics teachers in the Singapore context. In particular, this study considered factors such as shared vision and mutual accountability, influencing the scale-up and sustainability of a school-based professional development programme's impact on mathematics teachers' knowledge or practice.

Special characteristics of professional development are to be considered in the case of novice teacher as it was pointed out by Çelik et al.[14]. Although many studies have determined the professional needs of novice teachers, there are limited articles that have introduced interventions to address these needs. As part of a large-scale project, this paper presented two cycles of a design-based study aimed to develop the pedagogical content knowledge (PCK) of novice mathematics teachers. The study was conducted with the participation of twelve novice mathematics teachers in total and the change between two cycles is described. The analyses of the data show that teachers in the cycle which was enriched with additional video content presenting student thinking made better progress in knowledge of students, particularly related to misconceptions and learning difficulties compared to other teachers. Overall, both cycles appeared to support novice teachers' PCK. Finally, some suggestions are made for further research and practices for teacher learning.

The challenge of improving students' mathematical reasoning is addressed by a study aimed at improving the professional development of secondary teachers in charge. Mata-Pereira and Ponte [15] identified the main characteristics of a professional development experiment (PDE) centered on developing secondary teachers' mathematical and didactical knowledge to enhance students' mathematical reasoning. This design-based research is one of the strands of project REASON and concerns a PDE with secondary teachers that begins in October 2019. One of the main features of this PDE is the close link between research-based knowledge about enhancing students' mathematical reasoning and participant teachers' practice, thus providing innovative PD strategies and materials to enact such link.

Linked to the intention to consider the PD's specific characteristics are needed to improve school performance of students from lower secondary schools in the South African context, Pournara[5] showed some evidence to promote mathematics teachers' interventions. The Transition Maths 1 course was designed for teachers many of whom were under-prepared for this task. Following the initial pilots, a quasi-experimental study revealed conditions to impact student attainment. The intervention was scaled up and has now been completed by more than 150 teachers from approximately 80 schools. A recent quasi-experimental study provides further evidence of impact at student-level although this impact is not necessarily evident in the year immediately following course completion, suggesting a delayed impact of PD on students.

In a different context, the shortage of mathematics teachers in secondary schools of many German Federal States have promoted professional development courses in mathematics for teachers who already teach or want to teach mathematics out-of-field, which means teaching mathematics without official qualification. Since 2014 the German Centre for Mathematics Teacher Education and the regional government in Detmold (North Rhine-Westphalia) have conducted three of these professional development courses for out-of-field teachers in mathematics (secondary schools). Lünne and Biehler[18] presented an experience with the research aim to improve the knowledge about the group of the participating teachers and about aspects of success in the design of the courses as well as with the goal to develop curriculum material for such courses, which can be put to broader use all over Germany. During the second and the third course they investigated participants' development of school-level content knowledge in elementary algebra. The results show that participants with little prior knowledge can build up content knowledge in the PDC, while participants with a high level of prior knowledge are probably under-challenged by the test.

2.3. Collaborations between researchers and teachers

Some presentations showed the complexity of working with in-service teachers and their great potential. Terry[6] explored how action learning processes contribute to teachers' understanding and practice of self-regulation (SR) in secondary mathematics classrooms. Three constructs of SR, cognitive, metacognitive and motivational informed this study. Ten secondary teachers who participated in this study during 2019 reported that action learning was key in building teacher understanding of SR and improved pedagogical approaches that enhance student SR. Observations of their classroom behavior confirmed the targeted impact on the classes.

Peña et al.[7] presented preliminary results of a qualitative study with the purpose to analyze the collaboration processes between teachers' dyads of special education and mathematics that favor the development of argumentation in the mathematics classroom. Using case studies methodology, they conducted interviews and classroom video recordings. From a sample including 24 pairs of 7th-grade math and special education teachers, three pairs of cases were selected because they showed high levels of argumentation and had worked collaboratively. Through analyses of interviews, they identified facilitators and obstacles of collaboration to promote argumentation. They supported that, in order to promote argumentation, teacher dyads need to have common goals and shared workspaces for planning and decision making, but time devoted to shared work is still insufficient.

Albert et al.[8] addressed the collaboration with educators in high need schools. As students in high need school districts of United States generally do not do as well in mathematics as students in other districts, they are ultimately less likely to become part of the STEM workforce. Addressing this gap requires both the development and the retention of high-quality math teachers in high need districts. They reported on a project, now in its seventh year, to do so. The project features university level math educators and mathematicians working together, allowing for foci on content knowledge, pedagogical content knowledge and expertise in pedagogy as well as the development of a professional community concerned with supporting secondary math teachers. The project has been broadly successful, and the experience provides lessons that may be taken for other programmes with similar concerns.

In a similar way, students showing low language proficiency struggle with learning mathematics, especially in comprehending conceptually. Thus, teachers challenged with the language-responsive mathematics teaching needed, engage in Professional Development (PD). Look et al.[12] presented a design research approach to prepare PD facilitators who provide support teachers' learning. To prepare facilitators for their challenging role, and to foster their noticing of teacher learning they opted for using videos, taken from PD. Their approach was twofold: by an expert rating, involving five experienced facilitators, they confirmed that the videos address PD-PCK aspects, and yielded noticing prompts to be added to the videos. In the implementing study they showed that 60% of the discussion was related to PD-PCK, the remaining 40% on general PK. When facilitators reflected on content-specific aspects, 31% of the time was dedicated to describing, 44% on interpreting the situation, and 25% on making suggestions for alternative actions. On a re-design they will concentrate to further push discussions towards PD-PCK.

The need of qualified facilitators is found in different scenarios and Nieszporek et al.[17] presented the case of professional development for the teaching of probability and statistics. Although facilitators and their competencies play an important role for the success of PD courses, there is only little research on their orientation towards central learning goals. The need of providing facilitators a strong development including material for their work justifies this study as a scaling-up context. This case study casts a light on facilitators' decision-making, using an expertise model for the PD level. Preliminary results on the choice of learning goals by facilitator Mike (the study case) and his justifications enable a better understanding of his practices and thinking.

2.4. Constructing professional development maps

Chen et al.[9] presented a research base on a questionnaire administered to 61 rural middle school mathematics teachers from China to investigate their professional development and their views of its influencing factors on their professional development. The questionnaire was designed in four dimensions: teachers' personal information; teachers' professional development (i.e., identity as rural teachers, self-development consciousness); impact of other influencing factors (i.e., rural students and their parents, school atmosphere); and their training requirements. Firstly, results revealed that rural teachers held slightly positive beliefs in their professional development, as well as the impact of influencing factors on their professional development. Secondly, results revealed a significant positive relationship between teachers' points of view in their identities, self-development consciousness, the impact of rural students and parents, and the impact of school atmosphere.

Ma[13] presented a qualitative survey designed to document on 110 primary and secondary mathematics teachers' independent development, teaching and research activities, routine training practices, induction culture and exceptional professional development approaches. The study seeks to determine the basis for the selection of the methods of mathematics teacher education.

2.5. The challenge of a new curriculum and education general reforms

The development of a new curriculum is a starting point for a professional development process insofar this kind of changes arouses confusion among teachers. Yan and Wang[20] used an open questionnaire and statistical methods to investigate the confusion for high school mathematics teachers in Henan Province. The researchers found that the core literacy of high school mathematics is the most conspicuous, the new curriculum structure and content related issues need to be solved, teaching quality evaluation needs to be operated, and in consequence new curriculum training needs to be expanded and upgraded. Based on these findings, suggestions from four aspects are established: strengthening the implementation of new mathematics curriculum standards of high school mathematics, giving play to the leading and radiating role of the experimental area, giving full play to the leading role of the college entrance examination, and training the new high school mathematics curriculum well.

Several reports reveal that the Moroccan Educational System, despite some registered progress, is still facing some dysfunctions, in particular the In-service Mathematical Teacher Education and Mathematical Teacher Professional Development has not yet been placed in a strategic position to respond to teachers' real needs. El Yacoubi[21] presented the priorities of the system reform, one of them aiming to enable teachers to complete and perfect their training.

3. Summary

In the following, we sketch some important assumptions and questions that emerged from our discussions in TSG-31:

- The starting point for many scaling up initiatives is the wish to take an instructional innovation that has proved effective in supporting students' learning in a small number of classrooms and reproducing that success in a large number of classrooms (Cobb and Smith, 2008). Clearly we could replace classrooms with schools, districts or regions etc.
- Scaling up initiatives must take into consideration a wide range of contextual constraints. This is particularly important when importing from or exporting

to vastly different cultural settings. For example, different countries have different general conditions for their education systems (centralized versus decentralized, public or private sector, relatively well resourced versus relatively poorly resourced, urban areas versus rural areas, top-down versus bottom-up steering, voluntary versus compulsory professional development etc.).

- Also, it makes a difference whether an initiative focuses on a specific mathematical content or theme (e.g., algebra or proving), on a reform that aims at improving mathematics teaching, or a larger reform focusing on STEM teaching (where mathematics is only one subject).
- Scaling up an initative might change the character of the initiative: For example, will it lose its focus on the micro-level (e.g., students' learning, student-teacher interaction etc.)? Does it get more policy-driven? What does this mean for research and teaching practice?
- Scaling up of initiatives poses new challenges for researching the initiative. For example, what kind of data become more important? What kind of knowledge is aimed at: for the scientific community (publications, presentations etc.), for practice (suggestions, materials etc.), and for policy (steering information, policy advice etc.)? Which knowledge regarding teachers gets specific focus (content knowledge, pedagogical content knowledge, pedagogical knowledge etc.)? Are larger projects more under pressure than the initial interventions to produce "success stories"?
- Studies of scaling up do not necessarily involve only quantitative methods. A good mixture of quantitative and qualitative methods helps to generate "numbers and stories".
- Scaling up involves working with stakeholders who may be less relevant in smaller initiatives (e.g., from mathematics teachers to district lead teachers, from single mathematics education researchers to research institutions, from superintendents to a country's policy makers).
- What can we expect from school-based interventions on a large scale? How can we respect/consider the particularities of a school in a large-scale education intervention project? (e.g., the existence of teachers with specific qualifications in mathematics education)? Which financial implications need to be taken into consideration?
- Collaboration among mathematics teachers and collaboration between teachers and researchers in mathematics education are both important means to make initiatives successful (Borko and Potari, 2020). How can the benefits of such collaborations be drawn into the scaling up of initiatives where such collaborations may not easily occur?
- What can we learn from lesson and learning studies as collaborative approaches in mathematics education? And how might these learnings inform the scaling up of lesson and learning studies?

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Topic Study Group 32

Knowledge in/for Teaching Mathematics at Primary Level

Stéphane Clivaz[1], Kam Ling Lao[2], Janne Fauskanger[3], and Verónica Martín-Molina[4]

1. Themes and Description

Following Shulman's (1986) suggestion that teaching requires knowledge that is distinctive of the teaching profession, teachers' knowledge in/for teaching mathematics has attracted researchers worldwide. Different approaches have emerged on how this knowledge can be studied, developed, and strengthened and Topic Study Group 32 (TSG-32) has extended this conversation, with a focus on several emerging issues. TSG-32 at ICME 14 invited paper submissions on significant (new) trends and developments in research, theory, and practice about all different aspects that relate to the knowledge in/for teaching mathematics at primary level (learners' ages 5–13).

The following (often partly overlapping) themes have been considered.

(1) Focus on children's mathematics
- How attending to children's mathematical thinking can influence teachers' knowledge
- How elementary teachers use the leverage of core mathematical knowledge to nurture children's mathematical minds

(2) Focus on teacher learning
- Learning in collaborative communities (e.g., professional learning communities, lesson study, etc.)
- Acquisition of mathematical knowledge in teacher training
- Learning through teaching

(3) Focus on various aspects and uses of knowledge in/for teaching mathematics
- Cultural aspects such as cultural responsiveness, equitable teaching, teaching of mathematics in social and political contexts, etc.
- Aspects of teaching practice, such as high-levering/ambitious practices, attention to diverse learners, etc.

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- Use of mathematical knowledge in the different phases of teaching (lesson planning, observation of students, task design, situations of contingency, etc.)

(4) Focus on methods for studying mathematical knowledge in/for teaching

2. Program Overview

TSG-32 had 3 sessions with presentations of papers and discussions of them. 26 papers and posters were submitted to TSG-32 and 3 papers were invited, resulting in the acceptance of 2 invited talks (IT), 8 long oral presentations (LO), and 8 short oral presentations (SO). However, several authors did not attend the TSG-32 sessions, probably due to the COVID-19 pandemic and postponement of ICME 14 and/or because of the online mode of the conference. There was also a joint poster session, and 4 posters were accepted. Tab. 1 includes the 10 papers and 2 posters that were actually presented during the conference.

Tab. 1. Papers and posters presented at TSG-32

Paper and author(s)	
[1]	Seeing mathematics through the lens of children's mathematical thinking: a perspective on the enhancement of mathematical knowledge for teaching. ***Randolph A. Philipp***, *John Siegfried, and Eva Thanheinser* (USA). (IT)
[2]	Towards a dialogic analysis of mathematical problem-solving knowledge for teaching in a lesson study group. ***Stéphane Clivaz***, *Valérie Batteau, Audrey Daina, Luc-Olivier Bunzli, and Sara Presutti* (Switzerland). (LO)
[3]	Exploring preservice teachers' noticing of resources that support productive struggle and promote equity. ***Christine Alyssa Herrera***, *Shawnda Rae Smith, Christina Starkey, and Hiroko Kawaguchi Warshauer* (USA). (LO)
[4]	A comparative study on the professional knowledge of elementary mathematics teachers in Shanghai and Hong Kong — from two scenarios in data handling and geometry. ***Kam Ling Lao*** (Hong Kong SAR, China). (LO)
[5]	Primary teachers' recognition of students' mathematical reasoning and beliefs about teaching and learning. ***Carolyn A. Maher***, *James A. Maher, and Louise Cherry Wilkinson* (USA). (LO)
[6]	Pre-service primary teachers' knowledge and the mathematical practice of defining. ***Verónica Martín-Molina*** (Spain). (LO)
[7]	Teacher time out as site for studying mathematical knowledge for teaching. ***Reidar Mosvold***, ***Janne Fauskanger***, *Kjersti Wæge, and Raymond Bjuland* (Norway). (LO)
[8]	Addition and multiplication teaching in the multi-grade primary school. ***Yolanda Chávez Ruiz*** *and Lorena Trejo Guerrero* (Mexico). (SO)
[9]	Elementary preservice teachers' expected challenges in teaching pattern generalization. ***Mi Yeon Lee*** *and Ji-Eun Lee* (USA). (SO)
[10]	Why does 1/4:1/5 equal 5/4? A case of a post-graduate student's understanding of common fractions division. ***Barbara Beata Pieronkiewicz*** (Poland). (SO)
[11]	Unpacking performance indicators in the TPACK (Technological Pedagogical Content Knowledge) levels rubric to examine differences in the TPACK levels for teaching mathematics in primary schools. ***Aleksandra Kaplon-Schilis*** *and Irina Lyublinskaya* (USA). (Poster)
[12]	Developing analytical models of pedagogical content knowledge: a case study of mathematics teachers in Macao. ***Huey Lei*** (Macao SAR, China). (Poster)

Despite the difficulty of attending some sessions for many participants, probably due to the time difference, the discussions were of high quality.

3. Future Directions and Suggestions

Some of the papers presented in TSG-32 offered suggestions for future directions. Firstly, more studies are needed to determine how teachers develop knowledge in/for teaching mathematics in collaborative settings, which is very complex. Secondly, it would be interesting to study teachers' beliefs about the importance of their students' conceptual understanding of the mathematics that they should learn in school, or how that conceptual understanding could be promoted. While there is a need to investigate the diverse mathematical knowledge base among in-service teachers, there is also much to be discovered concerning how to help pre-service primary teachers to acquire an appropriate level of mathematical content knowledge. Focusing on mathematics through the lens of mathematical thinking could be used to help teachers (since their view of their students would become "richer and more nuanced"), but how to implement this in teacher training programs remains to be seen.

In their review of studies of mathematical knowledge for teaching, Hoover et al. (2016) identified 190 studies published between 2006 and 2013. Based on this review, they suggested that "a central problem for progress in the field is a lack of clearly understood and practicable methodology for the study and development of mathematical knowledge for teaching" (Hoover et al., 2016, p. 20). They further argue that the use of measures and interviews might draw attention away from the actual work of teaching (Ball, 2017). In the presentations in TSG-32, different methods for studying knowledge in and for teaching mathematics — more or less close to the work of teaching — were explored. Based on this exploration, a suggestion for future research might be similar to what Hoover et al. (2016) conclude based on their review, namely to "use sites where professional deliberation about teaching are taking place as sites where we might productively research the work of teaching and its mathematical demands" (p. 23). A lot of work remains to develop methodologies for studying mathematical knowledge in/for teaching through the work of teaching mathematics. Future papers, posters and discussions in TWG-32 at future ICME conferences will be an important site for discussing and developing such methodologies.

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Topic Study Group 33

Knowledge in/for Teaching Mathematics at Secondary Level

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ABSTRACT The chapter summarizes the results of TSG-33: "Knowledge in/for teaching mathematics at secondary level". We provide all titles of all the scientific contributions in the program overview and address suggestions for future directions of research in this field in the report about the discussions.

Keywords: Mathematics teacher knowledge; Knowledge for teaching mathematics; Secondary education.

1. Themes and Description

The TSG-33 assembled international mathematics educational researchers on the topic of "Knowledge in/for teaching mathematics at secondary level". Since ICME-13 in Hamburg 2016 (Even et al., 2017), research, theory, and practice in this research topic have evolved and, in particular, questions about the relationship between teachers' knowledge and the practice of mathematics teaching at secondary level have been taken up by many researchers around the world. The goal of TSG-33 was to focus on a number of critical issues in research on knowledge in/for teaching mathematics at secondary level and to foster international discussion about the findings and challenges researchers, mathematicians, teacher educators, teachers, and policy makers face in addressing issues in this area of research. In particular, the role of teachers' knowledge in practice and practical implications for teacher training and professional development also played a stronger role. The discussions in TSG-33 focused on the following issues and respective key questions driving current research in this research field:

(1) Conceptualization of knowledge in/for teaching mathematics at secondary level

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- What kind of knowledge in/for teaching mathematics should be considered to become a proficient/effective mathematics teacher at secondary level?
- What core characteristics, basic abilities, attitudes, and beliefs are in play? (Are there some normative orientations?)
- What aspects are considered in the various existing theoretical frameworks?

(2) Measurement of knowledge in/for teaching mathematics at secondary level
- What aspects are measured in the study of knowledge in/for teaching mathematics at secondary level?
- How is knowledge in/for teaching mathematics at secondary level measured?
- Are these measurements/instruments appropriate for different contexts?

(3) Relationships between knowledge in/for teaching mathematics at secondary level and teaching practice in mathematics, including instructional quality in mathematics teaching, and student achievement
- What distinguishes (theoretical) knowledge in/for teaching mathematics at the secondary level from teaching practice (enacted knowledge)?
- What are the relationships between teachers' affect and their knowledge in/for teaching mathematics at the secondary level?
- What kind of situational knowledge and skills are needed or observable in practice and how does teachers' knowledge influence the quality of mathematical instruction and relate to students' mathematical achievement?

(4) Practical implications for teacher education and professional development and validation of research findings on knowledge in/for teaching mathematics at secondary level
- What are appropriate measures in teacher education and professional development to develop knowledge in/for teaching mathematics at secondary level?
- What kind of knowledge is relevant for teaching practice and where and how do teachers learn this knowledge?
- How can studies on teachers' knowledge be used to improve the quality of teacher education and professional development?

2. Program Overview

As one of the larger Topic Study Groups at ICME-14, TSG-33 was able to benefit from the possibility of a fourth program session. Due to the ongoing COVID-19 pandemic, the program was offered in a hybrid format (i.e., presentations were given both remotely and on-site). The program provided an opportunity for participants to discuss the questions stated in Section 1 in depth. However, as many researchers participated remotely, the time difference presented challenges for the scientific discussion with researchers working in different parts of the world. The names of the presenters in TSG-33 are listed in Tab. 1, along with the country of their affiliated institution and the title of their presentation.

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Critical remarks on the notion of unpacking mathematics in discourses of teacher knowledge. ***Thorsten Scheiner*** (Australia).
[2]	What subject matter knowledge do Chinese in-service junior middle school teachers lack? ***Dandan Sun*** (China).
[3]	Assessing the relationship between teachers' knowledge and classroom practices in the use of ICT in the secondary mathematics classroom. ***Mailizar Mailizar*** (Indonesia).
[4]	Number sense of teachers in different school levels. ***Rahmah Johar****, Munirah Ghazali, Mailizar, and Suci Maulina* (Malaysia).
[5]	Arts integrated pedagogy for meaningful mathematics teaching and learning. ***Binod Prasad Pant****, Bal ChandraLuitel, and Indra Mani Shrestha* (Nepal).
[6]	Uncovering mathematics teaching knowledge of out-of-field mathematics teachers. ***Achmad Nizar***[6]*, Merrilyn Goos, Miamh O'Meara, and Ciara Lane* (Ireland).
[7]	A study of Sri Lanka's pre-service mathematics teachers' pedagogical content knowledge. ***G.M. Wadanambi*** (Sri Lanka) *and Frederick K. S. Leung* (Hong Kong SAR, China).
[8]	Interweaving mathematics-news-snapshots in class: implications for teachers' horizon content knowledge. ***Ruti Segal****, Atara Shriki, Boaz Silverman, and Nitsa Movshovitz-Hadar Oranim* (Israel).
[9]	Comparing German and Slovak teachers' knowledge of content and students related to functions. ***Veronika Hubeňáková****, Ute Sproesser, and Ingrid Semanišinová* (Slovakia).
[10]	A focus on the specificities of teachers' knowledge for improving teacher education: the case of the MTSK conceptualization. ***Miguel Ribeiro****, Marlova Caldatto, and Milena Policastro* (Brazil).
[11]	The influence of teaching experience on mathematical teacher content knowledge at middle school level. ***María D. Cruz Quiñones****, Mourat Tchoshanov, Héctor Jesús Portillo Lara, Carlos Paez, and Rocio Gallardo* (Mexico).
[12]	Implementation of eight teaching practices for teaching problem solving. ***Sarah Sparks****, Alees Lee, Katie Morrison, and Gulden Karakok* (USA).
[13]	A preservice secondary mathematics teacher's specialized knowledge: the case of limit. ***Rüya Savuran*** *and Mine Işıksal-Bostan* (Turkey).
[14]	What do teachers learn about what mathematics is in academic mathematics courses? ***Anna Hoffman*** *and Ruhama Even* (Israel).
[15]	Mathematical quality of geometry instruction of a novice high school teacher in terms of richness of mathematics. ***Fetma Aslan-Tutak*** *and Buket Semercioglu Kapcak* (Turkey)
[16]	Investigation of preservice mathematics teachers' translations among multiple representations. ***Zeynep Pehlivan*** *and Fetma Aslan-Tutak Achmad* (Turkey).
[17]	Preservice secondary school teacher's errors when translating between representations. ***Florence Thomo Mamba*** (Malawi).
[18]	Connecting knowledge for teaching geometry at the secondary level with instructional quality in mathematics teaching. ***Agida Manizade*** (USA) *and Dragana Martinovic* (Canada).
[19]	Upgrading learning for teachers in real analysis (ULTRA): an instructional model for secondary teacher education. ***Nicholas H. Wasserman****, Keith Weber, Juan Publo, Mejia-Romos, Timothy Fukawa-Connelly* (USA).
[20]	Applications of teaching secondary mathematics in undergraduate mathematics courses. ***Elizabeth G. Arnold****, Elizabeth A. Burroughs, Elizabeth W. Folton, James A. Mendoza* (USA).
[21]	Mathematics teachers' perceptions of teaching competencies: a study of grades 5 through 8. ***Heather Bleecker*** *and Polly Dupuis* (USA)
[22]	Identifying mathematical learning opportunities in a task as a missing, essential skill of teaching. ***Michelle King****, Jodie D. Novak, Robert A. Powers, Alees T. Lee, Adam Ruff, and Shweta Naik* (USA).
[23]	The validation of an assessment instrument for measuring mathematical knowledge for teaching (MKT). ***Mihyun Jeon*** (USA). (Poster)
[24]	The specialized knowledge of a new generation of mathematics teachers under STEM training. ***Jenny Patricia Acevedo-Rincon*** (Colombia). (Poster)
[25]	A case study on MPCK of junior middle school mathematics teachers with different characteristics. ***Ruifang Zhao*** (China). (Poster)

[6]Achmad Nizar unfortunately passed away a few days before the conference, but we would like to highlight his contribution to TSG-33 nevertheless.

The presentations were accompanied by introductory remarks and discussion prompts from the team members.

3. Results of the Discussion, Future Directions and Suggestions

The discussions of the presentations led to various observations that will be documented here as a result of the conference. Of course, such summaries can only reflect a subjective impression of the organizing team. Compared to the presentations of the corresponding Topic Study Groups at ICME-12 in Seoul and ICME-13 in Hamburg, an increase of studies on the differences between mathematical knowledge at the university level and secondary school level could be observed. Considerations of the nature and kind of subject matter knowledge for teaching have evolved in various directions, including directions that diverge from Shulman's original notion of pedagogical content knowledge. Moreover, with the increasing recognition of the situated nature of teacher knowledge (Even et al., 2017), it is not surprising that the role of teachers' school mathematics knowledge is again becoming the focus of several research contributions.

The greater importance of professional practice for the development of teacher knowledge in teacher education and professional development is also accompanied by a stronger focus in research. The need to conduct studies of the development of teachers' knowledge over the course of a teacher's career (especially the need for longitudinal studies of high scientific standard) was again evident.

There have also been developments in research at the level of conceptualizing theoretical frameworks for teacher knowledge, partly due to the influence of the significance of the unfolding of teacher knowledge in situated (and culturally shaped) teaching practice[7]. On the basis of the still prevalent work of Shulman, various dimensions of teacher knowledge were distinguished and delineated, depending on certain content-related aspects or aspects of teaching practice. New conceptualizations are partly more fine-grained or take into account specific cultural conditions, such as the teaching knowledge of indigenous populations or national traditions. Overall, a more critical use and enrichment of common conceptual frameworks was observed. The frameworks discussed were not mainly analytical, but reflected both normative and descriptive approaches. However, for research on situated teacher knowledge, the more context-oriented knowledge is analyzed in teacher practice, the more difficult it becomes to empirically distinguish the knowledge from other factors like teachers' personality or affect. As the field seeks to better account for the ways in which teacher knowledge and its frameworks are culturally constituted, the question of how findings of studying teacher knowledge can be generalized to other cultural contexts becomes even more significant. In this context, there is a great need for further development of research methods (both quantitative and qualitative) to study situated and contextual teacher knowledge that meet high scientific quality criteria.

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[7] For further research in this field, it is worth looking at the developments in research on teacher knowledge in other disciplines (i.e., Science education), where consensual models of practice-relevant teacher knowledge have emerged in recent years (Hume et al., 2020).

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https://doi.org/10.1142/9789811287152_0058

Topic Study Group 34

Affect, Beliefs, and Identity of Mathematics Teachers

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ABSTRACT We present the organization of Topic Study Group 34 and summarize the main themes that were discussed during the session. Finally, we outline direction for further research on the topic, as emerged during the TSG work and discussions.

Keywords: Affect; Beliefs; Identity; Teachers.

1. Aims of the TSG

In TSG-34 we addressed the themes of affect, beliefs and identity in mathematics education, with a special focus on mathematics pre-service and in-service teachers.

Following Hannula (2012), we conceptualize theories related to affect into three dimensions. The first dimension describes three different types of affect: cognitive (e.g. beliefs), motivational (e.g. value, motivation), and emotional (e.g., emotion, engagement). The second dimension concerns stable aspects of affect (i.e. traits) versus dynamically changing aspects of affect (i.e. states). The third dimension concerns the different research traditions for theorizing affect: physiological theories, psychological theories, and socio-cultural theories.

Research on affect-related theme is indeed traditionally rich and diverse and such a diversity and richness was evident in the contributions to the TSG-34.

The starting point for the work of the Topic Study Group was the previous work in ICME-13 (Hannula et al., 2019) that led us to outline a list of relevant themes:

- Theoretical and methodological issues concerning research on teacher's affect and identity.
- The analysis of the mutual relationship between affective constructs.

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- The connection of affective constructs to cognition and other constructs studied in mathematics education.
- The design and implementation of teacher education programs for promoting aspects of affect and identity.
- The domain (mathematics) specific research on affect and identity (i.e. in which ways research and results take into account the specificity of mathematics teaching and learning?).

1.1. Submission

We initially received 32 submissions, that underwent a first round of review and were accepted as papers (13), short papers (16), and posters (3).

Because of the pandemic there were some withdrawals and some new submissions, so that finally we had 24 submissions, from 16 countries. Of those 24 submissions, 8 were long oral presentations (LO), 15 short oral presentations (SO), and 1 poster.

1.2. TSG sessions

The accepted papers were presented during the three TSG online sessions. In order to improve the quality of communication during the online sessions, presenters were asked to prepare a video presentation in advance (5 minutes presentation for short papers, 8 minutes communication for long papers). Such presentations were shown during the session. There were discussions after each long paper presentation and after the presentation of a group of short papers.

Forty minutes of the last session were devoted to a general discussion on the emerging themes and directions for further research.

Moreover, participants could write their comments and questions to the presenters in a shared virtual board (Padlet), so that the discussion could be carried out also in an asynchronous way.

1.3. Presented papers

A list of the accepted papers (in order of appearance) and authors is presented in Tab. 1 (on the next page).

2. TSG Themes

As evidenced by the titles in Tab. 1, the presentations during the sessions addressed a wide range of themes and issues, with a variety of research methods, including surveys, interviews, observations and focus groups**.** Analytical methods varied from statistical inferences, to thematic analysis, narrative inquiry and more. The focus varied from studies on teachers' affect and identity to studies on professional development interventions to make teachers reflect on their affect and identity and possibly promote

Tab. 1. List of presented papers

Paper and author(s): long presentation (LO) and short presentation (SO)	
[1]	Examining teachers' emotional experiences through the process of mathematics instructional change.***Dionne Cross Francis*** (USA)*, Ji Hong* (USA)*, Jinqing Liu* (USA)*, Ayfer Eker* (Turkey)*, Pavneet Kaur Bharaj, and MiHyun Jeon* (USA). (LO)
[2]	Investigating changes in attitudes toward calculus of pre-service mathematics teachers enrolled in a pedagogy course. ***Wilfred W.F. Lau*** (Hong Kong SAR, China). (LO)
[3]	Comparing espoused values in mathematics teaching between novice and experience primary teachers: a case study in Chinese mainland. ***Hui Min Chia****, Xuanzhu Jin, and Qiaoping Zhang* (Hong Kong SAR, China). (LO)
[4]	Mathematics student Teachers' Self-Efficacy Beliefs on teaching. ***Kanita Pamuta****, Narumon Changsri, and Maitree Inprasitha* (Thailand). (SO)
[5]	PTeacher's and students' beliefs concerning higher order thinking in mathematics: are they on the same page? ***Elizar Elizar*** *and Cut Khairunnisak* (Indonesia). (SO)
[6]	What kind of students should deserve challenging, laboratory and inquiry-based mathematical activities?***Gabriella Pocalana*** (Italy). (SO)
[7]	Understanding open exploration in a classroom. ***Harita Raval*** *and Aaloka Kanhere* (India). (SO)
[8]	A study on conceptions of trainers of mathematics teachers in pedagogical superior educational institutes of Peru in relation to mathematics and their teaching. ***Candy Clara Ordoñez Montañez*** *and Gina Patricia Paz Huamán* (Peru). (SO)
[9]	'There are so many ways to fail': pre-service elementary school teachers define failure in mathematics. ***Sonja Lutovac*** *and Raimo Kaasila* (Finland). (LO)
[10]	Teacher's identity negotiation while presenting themselves on video in a professional development setting. ***Einat Heyd-Metzuyanim*** *andTalli Nachlieli* (Israel). (LO)
[11]	The changing professional identities of mathematics teachers within further education in England. ***Diane Dalby*** *and Andrew Noyes* (UK). (LO)
[12]	Identity construction of female mathematics teachers in professional life: a narrative inquiry. ***Tara Paudel*** (Nepal). (SO)
[13]	Learning and developing as a mathematics teacher educator. ***Forster D. Ntow*** (Ghana) *and Jill Adler* (South Africa). (SO)
[14]	Two-year college: teacher self-efficacy and knowledge levels for effective mathematics instruction. *David Tannor* (USA). (SO)
[15]	Shame: a significant emotion influencing pre-service primary school teachers' mathematics education. ***Lars Jenßen****, Regina Möller, and Bettina Roesken-Winter* (Germany). (LO)
[16]	Prospective teachers' attitude towards mathematics and its teaching: stories of development. *Annalisa Cusi and* ***Francesca Morselli*** (Italy). (LO)
[17]	Using A quantitative approach to explore teachers' identity in mathematics. ***Wanda Masondo*** (South Africa). (LO)
[18]	Mathematics teacher emotions during classroom practice: A case study in Chinese mainland. ***Zheng Jiang****, Ida Ah Chee Mok* (Hong Kong SAR, China) *and Jinbo Tang* (China). (SO)
[19]	Touching the untouchables: promoting non/linear mathematics pedagogy. ***Indra Mani Shrestha****, Bal Chandra Luitel, and Binod Prasad Pant* (Nepal). (SO)

through that professional development. Both pre-service and in-service teachers were taken into account.

Besides the main theoretical constructs, such as emotions, beliefs, attitudes, identity, the authors often introduced other affect-related constructs such as curiosity, failure, shame, challenge. The issue of teacher change was often referred to.

We may note that for the first time in ICME, Topic study Group on affect and identity was divided into two strands: student dimension and teacher dimension. This was due to the growing amount of research on the field. Such an organization was efficient in order to address more in detail teachers' affect and identity, even if all the participants recognized that teachers' affect and identity are strictly linked to students' ones.

Participants also noted that, even if there is mutual relationship, focusing on teachers entails focusing on their professional learning and not (only) on their mathematical learning. However, the division between the two TSGs (one on students' the other on teachers') may have left those researchers whose research focuses on relationships between students and teachers without a "home".

3. Directions for Further Research

During the last session the participants were engaged in a discussion on directions for further research in the field.

A general reflection concerns the emerging role of the context (cultural, institutional), that should be considered in an even more explicit way in studies on affect and identity.

The contributions showed a transition in research from focusing on "describing" affect to "intervening" with affect. Further research is needed in the design of "interventions", and it is important to reflect on a reliable methodology to study such interventions.

The issue of teacher change was deeply addressed in the contributions. It would be interesting to go on in this direction, carrying out long-term studies on affect and identity development as part of professional development.

From a methodological point of view, it was noted that small-scale studies are prevailing. Further research should take the challenge of large-scale studies.

Despite the organization of the TSGs in two separate strands, there is the need for studies that take into account both students and teachers' dimensions, and the relations between them. For instance, it is important to realize classroom experiments in order to study how teachers' teaching practice may impact on students' affect.

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https://doi.org/10.1142/9789811287152_0059

Topic Study Group 35

Knowledge and Practice of Mathematics Teacher Educators

TSG-35 Working Team[1]

1. Introduction of the Background

The last decades of research in mathematics education are best summarized by Anna Sfard's survey team at ICME-10 (2004) as "the era of the teacher" due to researchers' uncontested focus on teachers. Such attention is also represented in the launching in 1998 of an international journal dedicated to mathematics teachers' education, the Journal of Mathematics Teacher Education. Questions about what teachers need to know and be able to do, as well as how they develop their knowledge, skills, and beliefs have become central to the mathematics education research literature.

More recently, there is also growing attention on mathematics teacher educators (MTEs), that is, those who educate mathematics teachers, who design and implement opportunities for mathematics teacher education and development (MaTED). The goal of the TSG-35 at ICME-14 is two-fold: to collect information about mathematics teacher educators working in a variety of MaTED programs around the world and understand their contexts and cultures; and to discuss growing research about mathematics teacher educators, their knowledge, practice and beliefs.

The wording "mathematics teacher educator" (MTE) in some sense suggests a focus on academics only. This may be true for those countries/regions where MaTED is mainly at universities. But there are countries/regions where MaTED takes place within the instruction system or in teacher education institutes that are independent of universities. The recently launched ICMI Study 25 (co-chaired by Potari and Borko) focuses on the idea of mathematics teachers learning through collaboration in schools or larger communities, drawing on an ICME-13 survey team by Robutti et al. Collaborative groups may be teams, communities, schools and other educational institutions, professional development courses, local or national networks. This means that mathematics teacher educators can be working in formal or informal groupings, in

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either face-to-face or distance settings. They can be facilitators such as trainers, coaches, or mentors.

2. Paper Invitation and Submission for TSG-35

Given the variety of ways in which mathematics teacher educators can work, and the different settings in which they can operate, we invite papers that address the growing need to further understand these professionals.

Some questions to be answered might be:

1. Who are mathematics teacher educators?
2. What do we know about their knowledge, practice, and beliefs?
3. What is their work and under what conditions do they operate?
4. What framework should we adopt to illustrate different aspects of their knowledge?
5. What and how do the different avenues/contexts contribute to the growth in their practice?
6. What counts as experience and what difference does it make in their practice?
7. In what ways do mathematics teacher educator and teacher's knowledge and beliefs come into play in teacher education contexts?

To answer similar questions several teacher educators from different parts of the world met virtually on the occasion of the ICME-14 in shanghai. The list of papers presented can be found in Tab. 1 (on the next page).

3. A Brief Description of the Contributions

To address the first question, Goos and Marshman[1] took a sociocultural perspective on MTEs learning as identity formation. Brief case studies of MTEs who participated in two Australian research studies were illustrated.

MTEs' competence including knowledge is concerned in many ways under different contexts. Alacaci et al.[20] compared two frameworks of competencies for MTEs in mathematics and in technology, aiming to contribute to the discourse on the nature of teacher educator competency frameworks, identify areas of variances, and suggest possible reasons as well as inherent tensions. Huang[21] examined pedagogical practices of MTEs who provide online professional education programs for prospective and practicing mathematics teachers to unpack MTEs' knowledge. Three MTEs' knowledge structure was analyzed using teacher educator knowledge tetrahedron. Cross-case analysis revealed knowledge that was specific for MTEs' decisions about online mathematics teacher educating.

Focused on the practice, Kumar[2] made an analysis of math teacher educator's practice using vignettes illustrate the way different knowledge needs to be integrated in practice of teacher educator in designing and facilitating the tasks and discussion using them. The analysis revealed that though the dialogic approach in workshops

Tab. 1. The list of papers presented

Paper and author(s):
Session 1
[1] Boundary crossing and mathematics teacher educators' hybrid identities. ***Merrilyn Goos*** (Ireland) *and Margaret Marshman* (Australia).
[2] Analyzing challenges in the practice of a math teacher educator for developing community of math educators. ***Ruchi S. Kumar*** (India).
[3] Mathematics and science teacher educators learning induced by common research on professional vision. ***Nada Vondrova*** (Czech).
Session 2
[4] Teacher educators' preparation model: example from a successful professional development. ***Paola Sztajn,*** *Kristen Malzahn, and Reema Alnizami* (USA).
[5] Using a community of practice perspective to analyze mathematics teacher educator learning during lesson study. ***Melissa Soto***, *Lara Dick, Mollie Appelgate, and Dittika Guptal* (USA).
[6] Characterizing mathematics teaching research specialists' mentoring in the context of Chinese lesson study. ***Zhenzhen He***, *Feishi Gu, and Lingyuan Gu* (China).
[7] Didactical suitability criteria used by Italian teachers in lesson studies. ***Carola Manolino*** (Italy), *Viviane Hummes, Adriana Breda, Alicia Sánchez, and Vicenç Font* (Spain).
[8] The lesson study cultural transposition: from Chinese lesson study to Italian lesson study. ***Alessandro Ramploud***, *Maria Mellone, Silvia Funghi, and Simone Esposito* (Italy).
[9] Using a nested structure of lesson study approach: a self-study as a mathematics teacher educator. ***Yinkang Wu*** (China).
[10] A collaborative work of four mathematics teacher educators. a study in Uruguay. ***Daniela Pages*** (Uruguay).
Session 3
[11] A collaborative self-study of two mathematics teacher educators learning and growing as culturally responsive pedagogues. ***Lindsay Keazer*** *and Kathleen Nolan* (Canada).
[12] Exploring power and oppression: a study of mathematics teacher educators' professional growth. ***Craig Joseph Willey***, *Michael Richard Lolkus, Jill Newton, and Troy Bell* (USA).
[13] Differing contexts and tensions mathematics teacher educators experience in content courses for elementary preservice teachers. ***Hwa Young Lee***, *Emily Miller, Travis Weilan, Tuyin An. And Daniel Clark* (USA).
[14] Developing mathematics education leaders in schools in Guatemala and implications for work in other countries. ***Chadd McGlone*** (USA).
[15] Transitioning between different identities: how the different positions assumed by the mathematics teacher educator impact their practice. ***Natalia Ruiz***, *Nicole Fuenzalida, and Luz Valoyes-Chávez* (Chile).
[16] Mathematics teacher educators as role model: intentions and strategies. ***Helena Montenegro***, *Salomé Martínez, and Francisco Rojas* (Chile).
[17] Experience of learning to teach mathematics: what do prospective teachers learn from their mathematics teacher educators? ***Francisco Rojas***, *Helena Montenegro, and Flavio Guiñez* (Chile).
[18] Narratives of maths teachers: students & teacher ratio in mathematics classes in private schools. ***Sagar Dahal*** (Nepal).
Session 4
[19] Integrated mathematics teacher educators' professional development program. ***Haw-Yaw Shy***, *Ting-Ying Wang, Yen-Ting Chen, Chi-Tai Chu, Chen-Ju Pai, and Mei-Hsien Chen* (Chinese Taiwan).
[20] Talking across professional communities: teacher educator competencies in mathematics and in technology. ***Cengiz Alacaci*** (Norway)., *Bulent Cetinkaya, and Ayhan Kursat Erbas* (Norway).
[21] Mathematics teacher educators' knowledge for designing online professional development. ***Dinglei Huang*** (USA).
[22] Mathematics teachers' professional noticing in teaching of inverse functions and graphs in grade 12? ***Annie Mamoretsi Kgosi*** (South Africa).
[23] Examining teacher educator noticing during rehearsals of teaching: a focus on attending. ***Marta Kobiele*** (Canada).
[24] Mathematics teacher educator care and questioning in mathematics methods early field debriefing discussions. *Signe Kastberg, Lizhen Chen,* ***Sue Ellen Richardson***, *and Mahtob Aqazade* (USA).
[25] Un/intelligent way to professional development of mathematics teachers: a case from Nepal. ***Amrit Bahadur Thapa*** (USA).

allows teachers to articulate their thoughts and ideas, further research is needed to identify the knowledge and practice of math teacher educators for developing the sense of the community. According to Ruiz et al.[15], the processes through which a novice facilitator navigates between teachers' and facilitators' identities (triggered by contextual aspects) is explored. The results indicate that to faithfully and flexibly replicate professional development programs in new and unfamiliar school contexts, it is critical to understand the processes of identification experienced by novice facilitators as part of their process of learning.

There are also several researches regarding collaborative work. Vondrova[3] reported the experience of a biology teacher educator and a mathematics teacher educator which started as collaborative research on professional vision but gradually became a learning experience for them. Pages' talk[10] was about a collaborative work of four MTEs in Uruguay. The researchers proposed them to plan a lesson in the first Calculus course of the mathematics teacher education program, to implement that lesson, and analyze it in a collective way. A theoretical model was found by using Classic Grounded Theory. The process called looking for agreements is resolved by the activation and eventual mobilization of the personal theories built in practice of each MTE, which constitutes the core category that emerged during the study. Keazer and Nolan[11] presented a collaborative self-study of two MTEs developing their own culturally responsive pedagogies (CRP) when teaching mathematics education courses. To answer the question "What do MTEs learn from attempts to grow and reflect on their own CRP?", they developed an MTE framework for growing CRP, which applied to their practice for data collection and for further iterations of examining their CRP.

MTEs play an important role in working with teachers. Paper by Sztajn et al.[4] described a model used to prepare MTEs to facilitate a professional development program that has demonstrated it can be implemented with integrity and has also shown positive impact on elementary teachers' knowledge and practice. In paper by Lee et al.[13], five MTEs who teach mathematics content courses for elementary preservice teachers (ePTs) at institutions across the USA present the differing contexts in which they teach such courses. The sequencing and integration of content and pedagogy, content coverage and mathematical rigor, and interactions with ePTs views and prior experiences in learning mathematics were explored from the perspective of MTEs. Montenegro et al.[16] reported a phenomenographic research which aimed to explore the approaches to modeling held by MTEs. Data were collected through semi-structured interviews conducted face-to-face with fifteen MTEs working in three Chilean primary initial teacher education programs. The analysis identified four approaches to modeling, ranging from performing pedagogical activities and interactions to developing teaching practices linked to the school classroom. Rojas et al.[17] reported on part a Self-Study aimed to investigate the challenges of two Chilean MTEs when teaching how to teach mathematics. The prospective teachers' perceptions of the teaching practices enacted by their MTEs was explored. Data were collected through focus groups and analyzed using thematic analysis. Conclusion showed prospective teachers look at the mathematics teacher educators as a role model and would replicate some of their teaching practices when they become schoolteachers. The teacher educator facilitating the rehearsal plays a key role in supporting pre-service teachers' learning opportunities. Kobiele[23] presented an analysis of four teacher educators' noticing when facilitating rehearsals of one instructional activity — quick images. Using video-based interviews with each of the teacher educators, eight aspects within rehearsals that they attended to were identified. Kastberg et al.[24] examined how caring-relations influenced MTE questioning practice in the context of debriefing discussions with prospective teachers (PTs) during early-field experience linked to a mathematics methods course. Findings revealed that the MTE's ability to maintain focus on the PTs' objects of interest was informed by the MTE's feelings of reciprocal care.

Some contributions concerned the LESSON STUDY, a model of teacher education spread all over the world after the origin in the far east. Soto et al.[5] shared research on the use of lesson study for MTEs' professional development. Using Wenger's (1998) Social Theory of Learning framework based on communities of practice, they demonstrated how MTEs' learning changed across the process of the lesson study. He et al.[6] examined how mathematics teaching research specialists mentor practicing teachers during post-lesson debriefs of a lesson study in China. On the basis of the data analysis, a framework for analyzing mentoring activities emerged. The strengths and weaknesses of the teaching research specialists' mentoring strategies are identified through the framework, and suggestions to improve the teaching research specialists' mentoring strategies are discussed. Manolino et al.[7] aimed to identify the Didactical Suitability Criteria used by a group of Italian teachers participating in Lesson Studies. The written reflections and report documents — such as the Lesson Plans — are qualitatively analysed. The results suggest that all the Didactical Suitability Criteria are considered: the Epistemic, Cognitive, Interactional and Ecological criteria are particularly prominent; the Emotional and Mediational criteria sporadically appear. Ramploud et al.[8] aimed to show the process of deconstruction, functional to start a Cultural Transposition. Starting from the awareness of the "disorientation" generated by different cultural approaches to mathematics teaching, this process aims to produce versions of a didactic practice that are compatible with other cultural context and are suitable to support changes in teacher' beliefs. Wu[9] reported a nested structure of lesson study approach adopted in teaching pre-service mathematics teachers the course entitled Design of Mathematics Teaching. The underlying consideration and activities of each phase of lesson study at levels of MTE and preservice secondary teachers are both presented. Some preliminary findings regarding the effect of this approach on both the enrolled preservice secondary teachers and the MTE herself were provided.

Part of the researches concern the professional development of MTEs. Paper by Willey et al.[12] showcased the process and findings from an examination of MTEs professional growth as a result of engaging in a collaborative interrogation of critical texts outside of mathematics education. Findings suggest that this series of structured reading and dialogue led MTEs to develop a deeper understanding of the historical movements and events that created todays local and global status quos. Shy et al.[19]

reported a four-year (two stages) professional development program called "New Horizon of Mathematics (NHM)" in Chinese Taiwan. The findings indicate that all MTEs and in-service teachers participating in the study have a great transformation on the understanding of CK and PCK and some satisfactory results on mathematics teacher education are obtained.

Other interesting research such as McGlone[14], reported that a program "Teachers2Teachers Global", starting in Guatemala, has developed and implemented teacher training strategies that are rooted in the best practices in mathematics education. The research problem of Dahal[18] was to know the practices of class size with student teacher ratio in private school in the context of Kathmandu district with academic performances in mathematics. Two maths teachers' stories were collected for knowing the gain and pain of the classroom with larger number of student and smaller number of student. Kgosi[22] discussed Mathematics Teachers' professional noticing in the teaching of inverse functions and graphs in Grade 12 in South Africa. As a result, the paper looks deeper in how professional noticing can be used to provide assistance for teachers to notice learner(s) mathematical thinking and how to interpret their mathematical understanding while learning inverse functions and graphs. Under the attention to the strong beliefs teachers possess of mathematical intelligence, Thapa's Paper[25] showed a case in Nepal using multi-paradigmatic research space to inquire about the dis/empowering environment teachers create in math class. The researcher discussed and shared the exploration and practices towards 'un/intelligent educational approach' for a reform in content, pedagogy and assessment of mathematics teaching through teacher development.

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https://doi.org/10.1142/9789811287152_0060

Topic Study Group 36

Research on Classroom Practice at Primary Level

Shuhua An[1], Birgit Brandt[2], Benedetto Di Paola[3], and Jiushi Zhou[4]

ABSTRACT The aim of TSG-36 was to share the experiences of research on classroom practice at the primary level, address its research methods and theories, describe innovative classroom practice, and discuss the impact of research on classroom teaching and learning mathematics in different countries. A total of 32 submissions of research articles, project reports, and posters addressed related topics.

Keywords: Classroom practice; Primary level; Integrating technology; Action research; Teacher education.

1. Themes and Description of TSG-36

1.1. The aim of TSG-36

The aim of TSG-36 at ICME-14 was to share the experiences of research on classroom practice at the primary level, discuss its methods developed, and address its impact on classroom teaching and learning mathematics in different countries. The experiences of research on classroom practice can come from various levels of practitioners, educators, and researchers. The complexity of teaching practices in the current rapidly-developing technology era raises a variety of questions for research on classroom practice, such as teacher as researcher, how to use research-based teaching strategies and evidence-based teaching strategies to support effective classroom teaching, appropriate methods for classroom teaching research that informs teaching practice, development of multidisciplinary integration (STEM) projects in mathematics classrooms, effective methods for different teaching approaches, using new technologies in classroom teaching research, effective collaboration on classroom teaching research between classroom teachers and researchers, and effective training

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programs for research expertise in higher education and professional development. The principles of teaching have been addressed in many national standards, but there is no clear answer on the principles of effective instruction in elementary mathematics classrooms. These challenges call for action to reflect and discuss important needs in research on elementary classroom practice. The TSG-36 explored state of the art strategies and approaches to address the concerns and problems, and advance the research on classroom practice at the primary level from international perspectives with an ultimate goal of supporting elementary mathematics classroom teaching and learning.

1.2. Themes and description of TSG 36

TSG-36 included the following four themes from:

- **Theme 1.** Empirical studies that investigate using effective classroom practices to support teaching and learning elementary mathematics, integrating technology and/or STEAM education into elementary classrooms, assessing student mathematics learning outcome, and using adaptions to support diverse student mathematics learning.
- **Theme 2.** Effective programs and projects related to teachers as researchers who conduct action research in practices which support effective classroom teaching and learning.
- **Theme 3.** The challenges, diverse and emerging research methods, and tools of effective research on classroom practice at the primary level.
- **Theme 4.** Effective approaches in training and developing expertise in research on elementary classroom practice in teacher education programs and in professional development for classroom teachers.

2. Program Overview

2.1. Format of TSG-36

The format of TSG-36 was a hybrid format — onsite and online meetings synchronously.

2.2. Submissions of TSG-36

A total of 32 submissions were received from 16 countries (Canada: 1; China: 9; Denmark: 1; France: 1; Germany: 1; India: 1; Italy: 1; Japan: 3; Malaysia: 1; Mexcio:1; Sweden: 1; Switzerland: 1; The Philippines: 1; UK: 1; USA; 7; Uzbekistan: 1). These submissions cover a variety of important topics in four TSG-36 subthemes by authors from different cultural backgrounds and countries. Of the 32 submissions, 15 were accepted as paper presentations (seven long presentations and eight short presentations), seven as posters, and 10 could not be presented.

2.3. Sessions and presentations

TSG-36 had three Class B sessions: sessions 1 and 2 had 90 minutes and session3 had 120 minutes. Each session started with a brief introduction, followed by an invited talk, then long oral presentations, and short oral presentations, see Tab. 1. Each session also included at least 20 minutes' open discussion for participants to ask questions and reflect on their learning. In addition, each session included not only diverse topics on classroom practice at the primary level, but also included presenters from different countries and regions to provide an opportunity for participants to interact and exchange their research expertise among various scholars. Furthermore, session 3 provided a 35 minutes next step for the whole group discussion and reflection. The following tables show the papers presented in the three sessions:

Tab. 1. List of papers presented in session 1

Paper and author(s)
Session 1
[1] The benefits of using videos from research studies for teacher education: attending to students reasoning and argumentation. ***Carolyn A. Maher*** (USA).
[2] Examining U.S. elementary teachers' perceptions of and comfort with students' mathematical mistakes. ***Jinqing Liu*** (USA), *Dionne Cross Francis* (USA), *and Ayfer Eker* (Turkey).
[3] Problems with variation: an educational experience of cultural transposition with prospective Primary teachers. ***Benedetto Di Paola*** (Italy).
[4] Shanghai practice of primary mathematics classroom activities. ***Min Zhang*** (China).
Session 2
[5] Conjecturing teaching as competency-based instruction. ***Pi-Jen Lin*** (Chinese Taiwan).
[6] How does a Japanese primary school teacher manage the whole-class discussion named Neriage? ***Valérie Batteau*** (Switzerland).
[7] Teaching mathematics at Mexican elementary schools. ***Edith Arévalo Vázquez***, *Hilda Alicia Guzmán Elizondo, and Elvira Alicia Sánchez Díaz* (Mexico).
[8] Action-research group on Go game as classroom practice to learn mathematics at primary level. ***Antoine Fenech*** *and Richard Cabassut* (France).
[9] A grade 2 teacher's shift in the use of mediational means within and across two addition lessons. ***Fraser Gobede*** (Malawi).
Session 3
[10] Using math clinic to support classroom teaching practice and sharpen teachers' pedagogical content knowledge. ***Shuhua An*** (USA).
[11] Data use to inform mathematics instruction: an exploratory study. ***Jong Cherng Meei*** (Malaysia).
[12] Concept of collective milieu to understand the Japanese mathematics lesson. ***Takeshi Miyakawa*** (Japan), *Valérie Batteau* (Switzerland), *and Minbom Ryu* (Japan).
[13] Exploring the differences between expert and pre-service teachers noticing. ***Yiru Pei***, *Min Chen, and Qiaoping Zhang* (Hong Kong SAR, China).
[14] From loser to user, from special to general education, learning Inside mathematics through outside actions. ***Allan Tarp*** (Denmark).
[15] Storytelling as a resource for fostering 'love of challenge' for mathematics in primary grade students. ***Pooja Keshavan Singh*** *and Haneet Gandhi* (India).

2.4. Main outputs

The three TSG-36 sessions resulted many important outcomes. The main findings included the following:

2.4.1. *Effective approaches in elementary classroom practice*

A number of studies addressed effective classroom practices in elementary classrooms. For example, Lin[5] introduced a conjecturing teaching model with five stages as a competence-based instructional approach. The development of the model was characterized into six periods of investigation. Each period had a focus for eliciting the model. Some studies used a lesson study or classroom teaching as a focus on data collection and analysis. For example, Batteau[6] analyzed teacher practices in a Japanese context with a focus on a specific phase of structured problem solving lessons, a whole-class discussion named Neriage. The study by Gobede[9] also observed teaching practice on mediational moves made by a grade 2 teacher while teaching the addition of whole numbers for the first time at this level and suggested the need to move further to more efficient calculation strategies.

Other presentations introduced interdisciplinary in mathematics classroom. For example, Fenech and Cabassut[8] demonstrated a game activity — Go game — as classroom practice to learn mathematics at the primary level by an action-research group. Singh and Gandhi[15] examined the cognitive engagement of primary grade students with a mathematical content that was embedded in the story situation. It was observed that the attachment of the students with the story characters motivated them to go beyond the basic requirements of the task, seek challenges and expand their vistas for more complex tasks. The study recommends storytelling as a resource for fostering a ‘love of challenge’ for doing mathematics with primary grade students. Zhang[4] introduced using three types of classroom activities — perceived experience, exploration and discovery, and understanding and application to stimulate students’ desire for inquiry, enhance students’ participation, improve students’ learning styles, and realize Mathematics Subject’s multiple-quality function and advantages in fostering qualified personals.

2.4.2. *Teacher education programs for effective classroom practice*

Various presentations related to this theme. Maher[1] addressed the benefits of using videos on attending to students’ reasoning and argumentation from research studies for teacher education programs. Di Paola[3] studies effective approaches in training and developing expertise research on elementary classroom practice in teacher education programs and in professional development for classroom teachers. An’s study[10] examined the impact of Math Clinic on classroom teachers’ questioning strategies, understanding students’ thinking and misconceptions and their intervention strategies of correcting errors in a mathematics graduate program. Using pre-recorded exemplary lessons, Pei et al.[13] examined the noticing ability among expert teachers and explores the difference and similarities between expert teachers and pre-service teacher's noticing on exemplary lessons via a multiple case study. The findings showed that the pre-service teacher has a relatively low noticing ability and tend to focus on the pedagogy and classroom environment.

2.4.3. Important issues and challenges in elementary classroom practice

The presentations in TSG-36 investigated important issues and also address various challenges in classroom practice at the elementary level. For example, a study by Liu et al.[2] examined seven U.S. elementary teachers' perceptions of and comfort with students' mathematical mistakes via interviews. The results showed that teachers believed mistakes are important for proactively teaching, essential for supporting student learning as well as for lesson planning. However, the challenge was that teachers did not feel very comfortable in addressing student mistakes. A study by Vázquez et al.[7] examined teaching mathematics in the categories of teaching strategies, forms of class organization, classroom organization, use of teaching materials, assessment tools and textbook at Mexican elementary school in public elementary schools in Mexico by assessing the practices of 70 elementary school teachers. The results show that despite the implementation of the current curriculum, most educators continue to use teaching strategies that are far from the suggested didactic recommendations. Meei[11] shared an exploratory study on data use to inform mathematics instruction by investigating the state of data use to inform instruction among primary school mathematics teachers in Malaysia.

Results of the questionnaire and interviews indicate that the data which was most frequently used was classroom-based assessment data. Although teachers indicated that training needs and support for data use were adequate and they were confident in using data to inform mathematics instruction, they also like to have more professional development courses so that they can use data effectively and systematically to inform their practice. Miyakawa et al.[12] addressed the concept of collective milieu to understand the Japanese mathematics lesson by highlighting a collective construction of the inquiry or the problem solving process step by step, in terms of the collective milieu due to a lack of theoretical tools to analyze the Japanese mathematics lessons with their specificities: the approach by problem solving, the collective dimension of the teaching, and the focus on the development of mathematical thinking. Based on the observation of how children communicate about Many before school, Tarp[14] indicated that accepting numbers with units means that counting, recounting and solving equations come before adding on-top or next-to introduce integral and differential calculus as well as proportionality in early childhood education.

3. Future Directions and Suggestions Themes

3.1. Future directions

Throughout the presentations and discussions at three TSG-36 sessions, the participants explored and identified current and future trends, merging research themes, and areas of research interest for classroom practice at the primary level, as follows:

(1) Training and developing expertise in teaching and research on elementary classroom practice for classroom teachers;
(2) Training and developing expertise in teaching and research on elementary classroom practice in teacher education programs Role of expert teachers in teacher education programs;
(3) Using videos in teacher education programs;
(4) STEM Education in pre-service elementary teacher training program;
(5) Using interdisciplinary approaches into mathematics instruction;
(6) Data science in mathematics education;
(7) Theoretical tools for analyzing mathematics lessons;
(8) Innovative approach in early children education.

3.2. Suggestions

The participants enjoyed their learning from all presentations on diverse topics at TSG-36 sessions. Despite the promising results from the presentations, some questions remain unanswered at present. Future studies on the classroom practice at the primary level are therefore recommended:

(1) Future work is suggested to design the effective training and profession development on the 21st century teaching and research expertise for elementary classroom teachers and pre-service teachers in teacher education programs;
(2) Future work is suggested to establish the sound theoretical framework for research on using videos, classroom observations, and lesson studies;
(3) Further research should be undertaken to explore topics related to current trends, such as: STEM, data science, and other innovative approaches.

https://doi.org/10.1142/9789811287152_0061

Topic Study Group 37

Research on Classroom Practice at Secondary Level

Yoshinori Shimizu[1], Carmel Mesiti[2], Jarmila Robova[3], and Li Tong[4]

1. Aims of the TSG

The aim of this Topic Study Group was to improve understanding of the research practices, methodologies, results, and supporting theories related to classroom teaching and learning at the secondary level. We intended to promote exchanges and collaboration around the identification and examination of issues of interest to classroom researchers across different education systems with the goal to enhance the quality of research on teaching and learning in secondary mathematics classrooms.

The TSG focused on research related to mathematics teaching and classroom practice at the secondary level. Research on the activities that teachers and students do within the mathematics classroom can involve a variety of methodologies including videography, ethnography, self-reports by participants, scenario-based assessments, first-person research, stimulated commentary by practitioners, simulations, and others. Research on the classroom practice recorded with those approaches involve examination of the interactions among the mathematical content to be taught and learned, the instructional practices of the teacher, or the work and experiences of the students within educational settings. As the report below shows papers presented in the TSG relied on a variety of theories and contributed to the growth of knowledge of a variety of research foci: the mathematics transacted in classroom practice, the complexity of the work of teaching, the roles of teacher and students vis-à-vis the mathematical content at stake, the knowledge used in practice, and more.

This TSG served as an international forum for mathematics education researchers who wanted to disseminate findings and practices from their research on teaching and classroom practice and for practitioners who were interested in learning about how this research was done and on its possible implications for practice. The organizers made a balanced use of the time allocated, with presentations by two invited speakers and

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devoting the majority of the time to the discussion of contributed papers, with the goal to maximize exchanges among participants.

1.1. Submissions

We had 47 submissions from 17 countries (South America: 1; North America: 5; Asia: 23; Europe: 12; Africa: 4; Australia/New Zealand: 2) including two invited talks. Each proposal was reviewed by two members of organizing team. Of those 45 submissions, 30 proposals were accepted as paper presentations, 12 as posters, and 3 proposals were rejected. For paper presentations, 16 papers were assigned for a long oral presentation while 14 papers were for a short oral presentation.

1.2. Sessions

TSG-37 had sessions in four time-slots as follows.

- Session I: 14:30-16:30 on 13th of July
- Session II: 19:30-21:00 on 14th of July
- Session III: 14:30-16:30 on 17th of July
- Session IV: 21:30-23:00 on 17th of July

After a short overview of the aim and topics of the TSG, session I started with three *long* oral presentations (10 minutes) of paper followed by a 10 minutes collective discussion for two rounds. Then, three short oral presentations (8 minutes) were made with a short question and answer time. Generally, paper presentations were grouped based on the similarity of topics presented. Session II was exclusively allocated for two invited lectures. Sessions III and IV also included both long and short oral presentations. In each session, the organizers attempted to facilitate participants' discussion and dialogue in order to identify emerging research questions and themes, alternative approaches, and future research opportunities. A whole group reflection time was taken at the end of Session IV, for discussing some issues and suggestions for the next step of research on teaching and learning in secondary mathematics classrooms.

1.3. Invited talks

We were privileged to invite two prominent researchers in the area of research on teaching and learning in secondary mathematics classrooms. Tab. 1 shows two speakers with the title of the papers. These two invited talks provided TSG participants perspectives on the complexity of teaching and learning in secondary mathematics classrooms and pointed out the need of theoretical frameworks to explore the complexity with describing the quality of instruction in the classrooms embedded in sociocultural contexts for students' learning.

Tab. 1. Invited talks

	Paper and author
[1]	Studying instructional quality in mathematics: the need for content-specificity and other open challenges. ***Charalambos Charalambous*** (Cyprus).
[2]	An Approach of mathematics teaching and learning based on activity theory: principles and examples of results. ***Aurelie Chesnais*** (France).

Charalambous[1] discussed the issues related to capturing instructional quality of teaching and learning in mathematics classroom. He first underlined the importance of studying instructional quality through content-specific lenses, in addition to generic lenses, in order to avoid obtaining partial delineations of the quality. After substantiating the thesis with reference to four arguments, he raised three challenges related to studying instructional quality through classroom observations that need to be addressed to move the field forward.

Chesnais[2] presented a theoretical framework based on an activity theory designed to investigate the mathematics learning and teaching process in classrooms with a focus on the questions related to the logics of teachers' practices and the way they impact students' learning. After sharing the theoretical principles of the framework and its methodological consequences, she exemplified the use of the framework in her specific study. The study aimed at investigating how the relationships between the sociocultural background of students and their mathematics achievement were constructed within the mathematics classroom.

1.4. Paper topics

Of the 30 accepted papers, only 22 papers were able to be presented during the online conference. A list of these papers and authors are included in order of presentation and are organized in Tab. 2 (on the next page).

2. Themes to Topics

A variety of research related to mathematics teaching and classroom practice at the secondary level was presented in the TSG as shown in Tab. 2. The presented papers relied on a variety of theories and contributed to the growth of knowledge of a variety of research foci: the mathematics transacted in classroom practice, the complexity of the work of teaching, the roles of teacher and students vis-à-vis the mathematical content at stake, the knowledge used in practice. Given the variety of the presentations, the classical didactic triangle in which student, teacher, and content form the vertices of a triangle may be useful to conceptualize research topics and themes arose across the sessions, although such conceptualization needs to be extended (Goodchild and Sriraman, 2012).

Tab. 2. List of papers presented

Paper and author(s)	
[3]	A large-scale study of teachers' practices in algebra. ***Julie Horoks**, Julia Pilet, Brigitte Grugeon-Allys, Sylvie Coppé, and Marina De Simone* (France).
[4]	Teaching functions using RME approach to improve students' perceptions of mathematics learning and learning functions. *Ayse Kaya and F**atma Aslan-Tutak*** (Turkey).
[5]	Teachers promoting student interaction: what happens when teachers enter a mathematical discussion? ***Marie Aasa Viktoria Sjöblom**, Paola Valero, and Clas Olander* (Sweden).
[6]	The lexicon project: seeking a structure for the australian mathematics teachers' professional lexicon. ***Carmel Mesiti**, David Clarke, and Jan van Driel* (Australia).
[7]	The lexicon project: understanding the universality and applicability of the czech teachers professional lexicon. ***Jarmila Novotná**, Alena Hošpesová, Hana Moraová, and Iva Žlábková* (Czech).
[8]	Technical vocabulary of Japanese mathematics teachers: the Japanese lexicon in the tradition of lesson study. ***Yoshinori Shimizu**, Yuka Funahashi, and Hayato Hanazono* (Japan).
[9]	Inquiry-based learning in the mathematics classroom: insights from a case of two lessons. *Cheng Lu Pien, **Cynthia Seto**, Lee Ngan Hoe, Wong Zi Yang, and June Lee* (Singapore).
[10]	The practice of project-based mathematics extended curriculum at secondary level. ***Dan Shen*** (China).
[11]	The implementation of project-based learning (PBL) in middle school mathematics classroom in Malaysia and South Korea. ***Abdul Halim Abdullah*** (Malaysia) *and Bomi Shin* (South Korea).
[12]	A multi-stage attempt at narrowing the gap between contemporary mathematics and high school mathematics. ***Nitsa Movshovitz-Hadar**, Ruti Segal, Karni Shir, Atara Shriki, Boaz Silverman, and Varda Zigerson* (Israel).
[13]	Puzzle-based class format to foster students' mathematical oral production and exchange. ***Luca Agostino**, Bruno Durand, Laetitia Sonia-Doucet, Dimitri Zvonkine, and Varda Zigerson* (France).
[14]	Developing students' metacognitive practice: a systematic approach. *Low Leng, **Ang Yue Hua**, and Lee Ngan Hoe* (Singapore).
[15]	Learning situation analysis: problem, focus and method. ***Yu Hongyu*** (China).
[16]	A lesson design model to enhance students' activities with examples. ***Mayumi Kawamura**, Kazuya Kageyama, and Masataka Koyama* (Japan).
[17]	Re-visiting instructional explanations: how might the organisation of a lesson contribute to an explanation. ***Vasantha Moodley*** (South Africa).
[18]	Anthropological perspective on japanese mathematics teachers' professional knowledge of board writing. ***Yukiko Asami-Johansson*** (Sweden).
[19]	The implementation of a set of tasks for the development of spatial ability in secondary schools. ***Jarmila Robová** and Vlasta Moravcová*(Czech).
[20]	Productive struggle: a focus on sense making and connecting. ***Azita Manouchehri** and Reyhan Safak* (USA).
[21]	Promoting student questions in mathematics classrooms. ***Melissa Kemmerle*** (USA).
[22]	English language learners learning statistics in multilingual classrooms. ***Sashi Sharma*** (New Zealand).
[23]	A class for conceptualizing lagrange's four-square theorem. ***Tomohiko Shima** and Minoru Ito* (Japan).
[24]	Different Learning opportunities for students provided by teachers in high school mathematics classrooms: a classroom video analysis. ***Changjie Li** and Yun Lu* (China).

First theme that arose across the sessions related to teaching a particular content and topics of secondary school mathematics (Robová and Moravcová[19]; Shima and Ito[20], Sharma[22]). Kaya and Aslan-Tutak[4] for example, proposed the use of RME approach to improve students' perceptions of mathematics learning and learning functions. Movshovitz-Hadar et al.[12] presented their research on the instructional materials developed for narrowing the gap between contemporary mathematics and high school mathematics. These studies challenged the long-standing issue of teaching mathematics at secondary school so as to be meaningful and useful to the students. Also, there were reports of the analysis of classroom activities related to the new trends in teaching and learning in secondary mathematics classrooms, such as project-based mathematics extended curriculum (Shen[10]) and Project-based learning (PBL) (Abdullah and Shin[11]), inquiry-based learning (Pien et al.[9]) emerged a new area of exploratory studies.

A second theme that arose across the sessions was related classroom interaction between the teacher and students as well as communication among students (Manouchehri and Safak[20], Asami-Johansson[18], Sjöblom et al.[5]). Also, there were presentations of particular focus on teacher or students. Further, a particular method of developing students' metacognitive practice (Leng et al.[14]) and "puzzle-based class format" was proposed for fostering students' mathematical oral communications (Agostino et al.[13]). In this context, the role of example (Kawamura et al.[16]) and the importance of explanations (Moodley[17]), and promotion of student's questions (Kemmerle[21]) in mathematics was emphasized through the empirical studies.

A third theme that arose across the sessions involved the analysis of lexicon, a focus on teachers' use of technical vocabulary that describe activities in mathematics classrooms. Three papers from the same project provided the analysis of lexicon in the different cultural traditions (Mesiti et al.[6], Novotná et al.[7], Shimizu[8]).

Research on the classroom practice recorded with those approaches involve examination of the interactions among the mathematical content to be taught and learned, the instructional practices of the teacher, or the work and experiences of the students within educational settings. Research on the activities that teachers and students do within the mathematics classroom involved a variety of methodologies including both a large scaled study of teachers' practices (e.g. Horoks et al.[3]) and case studies of lessons and video analysis (e.g. Pien et al.[9], Li and Lu[24], Sharma[22]). Also, for the empirical studies, various methods were used in the study presented in this TSG. videography, ethnography, self-reports by participants, and so forth.

3. Areas for Future Research

At the end of Session IV, the participants had an opportunity of reflecting on the trends and issues in research on teaching and learning in secondary mathematics classrooms based on the presentations and discussion in this TSG. A few potential future research topics were discussed. First, given the complexities of teaching and learning in secondary mathematics classrooms, empirical studies need to capture the quality of instruction with a focus on content and educational values of mathematics taught in secondary mathematics classrooms. Second, classroom interaction between the teacher and students as well as communication among students need to be examined further in relation to the development of students' mathematical thinking and learning. Third, research on teaching and learning in secondary mathematics classrooms needs to be scrutinized from a "meta-level". That is, an overarching theoretical framework is needed to discuss and integrate findings of empirical studies.

Acknowledgement

The original Chair (Professor Patricio Herbst) and Co-chair (Professor Fabrice Vandebrouck) were unfortunately unable to participate in the TSG due the postponement of ICME-14 caused by the COVID-19 pandemic. They had a major contribution in planning and preparing for the TSG, including reviews of the submitted papers and inviting speakers. We greatly appreciate their initial organization and contributions.

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https://doi.org/10.1142/9789811287152_0062

Topic Study Group 38

Task Design and Analysis

Minoru Ohtani[1], Michiel Doorman[2], Berta Barquero[3], Heather Johnson[4], and Xuhua Sun

1. Themes and Descriptions the TSG

1.1. Themes of the TSG

In TSG-38, we focused on a variety of theoretical and practical topics related to task design and its analysis. Since there exists a complex, layered relationship between task designers, teachers, and students, which is well-illustrated by successful, theoretically-based long term design research projects, which have resulted in novel materials and approaches impacting teachers and students. Throughout the presentations we actively searched for connections both in regards to methodology, theory and the possible impact of the presented research. We aimed to explore seven related themes: ① frameworks and principles for task design; ② methodological advances for studying task design in mathematics education; ③ relationships between task design, anticipated pedagogies, and student learning; ④ the role of tools in task design; ⑤ task sequences for promoting conceptual understanding and/or higher order thinking skills; ⑥ task design in innovative learning environments; and ⑦ textbook task analysis. We encouraged submissions that offered theoretical and/or empirical contributions and sought to include research from a variety of cultural contexts to enhance our discussions.

1.2. Submissions

We invited two presentations and received a total of 38 submissions from 17 countries (South America 1, North America 2, Asia 7, and Europe: 7), with diverse cultural representation. Of those 38 submissions, thirty-one papers were accepted as paper presentations (fourteen as long oral and seventeen as short oral), five as posters, and two were rejected.

1.3. Paper topics

Of the 31 accepted papers, only 24 papers were able to be presented during the online conference.

A list of these papers and authors are included in order of presentation and are organized with related themes in Tab. 1.

[1] Kanazawa University, Japan. E-mail: mohtani@ed.kanazawa-u.ac.jp
[2] Utrecht University, Netherlands. E-mail: m.doorman@uu.nl
[3] University of Barcelona, Spain. E-mail: bbarquero@ub.edu
[4] University of Colorado Denver, USA. E-mail: Heather.Johnson@ucdenver.edu
[5] University of Macau, Macau-China. E-mail: sunxuhua@gmail.com

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Action, process or object? Can they all be perceived in a single task? ***Maria Trigueros****, Asuman Oktaç, Rita Xochitl Vázquez Padilla, and Avenilde Romo Vázquez* (Mexico). ①③⑤
[2]	The design of tasks for automatic formative assessment: Supporting teachers and students. ***Willy Viviani*** *and Kayla White* (USA). ①②③⑥
[3]	A joint embodied and simulation design for graphing: Coordinating distances that change together. ***Heather Lynn Johnson*** (USA), *Anna Shvarts* (The Nethelands), *and Amy Smith* (USA). ①②③⑥
[4]	Collective work on task design through study and research path for teacher education. ***Berta Barquero*** *and Sonia Esteve* (Spain). ①②③
[5]	Exploring mathematical task designed by pre-service teachers. ***Ruchi Mittal*** *and Alprata Ahuja* (India). ①⑤
[6]	The fundamental idea of task design in China for algebraic development. ***Xuhua Sun*** (Macao SAR, China). ①③⑤
[7]	Schooling experience as mediating variables in preservice teachers' beliefs and instructional practice when designing mathematical tasks. ***Eugenio Chandia Muñoz*** (Chile). ②③
[8]	Transforming mathematics tasks: an important mathematics teacher's role. ***Guillermina Ávila García****, Liliana Suárez Téllez, and Víctor Hugo Luna Acevedo* (Mexico). ①③⑤
[9]	Developing silent video tasks' instructional sequence. ***Bjarnheidur Kristinsdottir****, Freyja Hreinsdottir* (Iceland), *and Zsolt Lavicza* (Austria). ②④⑥
[10]	A possible pathway of mathematical inquiry: how to calculate the cube root of a given number by using a simple pocket calculator? ***Koji Otaki****, Hiroaki Hamanaka, and Takeshi Miyakawa* (Japan). ①⑤⑥
[11]	Research on designing and teaching of worked examples in reviewing of sequence based on the SOLO taxonomy. ***Junyi Li*** *and Chao Zhou* (China). ①③⑤
[12]	Fermi problems as a hub for task design in mathematics and stem education. ***Jonas Bergman Ärlebäck*** (Sweden)*and Lluís Albarracín* (Spain). ①②⑤⑥
[13]	Opportunities for inquiry-based learning provided by Chinese and Dutch lower-secondary school mathematics textbook tasks. ***Luhuan Huang****, Michiel Doorman and Wouter van Joolingen* (The Netherlands). ②⑤⑦
[14]	Developing digital mathematical tasks to promote students' higher order thinking skills. ***Meryansumayeka****, Zulkardi, Ratu Ilma Indra Putri, and Cecil Hiltrimartin* (Indonesia). ②⑤⑥
[15]	Potential, actual and practical variations for teaching functions: cases study in China and France. ***Luxizi Zhang*** (China), *Luc Trouche* (France), *and Jiansheng Bao* (China). ③⑤⑦
[16]	Students' opportunities to engage in mathematical problem solving. ***Jonas Jäder*** (Sweden). ①⑤⑦
[17]	Tasks and scenarios for promoting inquiry-based mathematics teaching. ***Michiel Doorman*** (The Netherlands), *Matija Bašić* (Croatia), *Zeljka Milin Sipus* (Croatia), *and Rogier Bos* (The Netherlands). ①③⑤
[18]	Towards differentiated instruction: Insights from constructivist learning design. ***Ng Kit Ee Dawn****, Lee Ngan Hoe, Cynthia Seto, Mei Liu, Lee June, and Zi Yang Wong* (Singapore). ①③⑤
[19]	Task for introducing the vector concept using technology. ***Sofia Paz Rodriguez****, Carlos Armando Cuevas Vallejo and Hosé Orozco-Santiago Cinvestav* (Mexico). ①④⑤
[20]	Design tasks in MLR environment: Constructing examples for proving logical statements. ***Galit Nagari-Haddif*** (Israel). ①⑤⑥
[21]	Didactic sequence planning for the study of the teaching and learning of isometries in future primary school teachers. ***Marta Martin Nieto*** *and Natalia Ruiz-Lopez* (Spain). ①③⑤
[22]	Analyzing primary two pupils' errors answering fractions' task using the Newman procedure. ***Rosmawati Mohamed*** *and Munirah Ghazali* (Malaysia). ①⑤
[23]	Effects of low floor high ceiling mathematical tasks on students' mathematical proficiency in seventh-grade geometry. ***Franklin Falculan*** *and Maria Alva Aberin* (Philippine). ①⑤
[24]	Collaborative design of unit that fosters reification of a mathematical object. ***Minoru Ohtani*** (Japan). ①⑤

2. Program Overview

2.1. Sessions

There were so many high-quality submissions, the ICMI organizing committee granted our TSG one more time slot for presentations. In our first 90-minute session, the TSG Chair, Minoru Ohtani, introduced the rest of the Team and described the format of the sessions. Generally, all four-time sessions led with 20 minutes *invited long* oral and presentation and discussion and *long* and *short* oral presentations with a 30- and 10-minutes collective discussion respectively. Throughout the four days, we attempted to facilitate participant dialogue to collectively identify emerging research themes, potential interdisciplinary approaches and future research opportunities. We used an online discussion platform, "Padlet," to facilitate communication among participants. We shared roles of online chair, note taker, and onsite organizer, to foster participant networking as well as to counter "virtual conference" fatigue. At the end of our last session, we built in 20 minutes for whole group reflection, discussion, and suggestions for needed research trajectories.

2.2. Session 1

In the session 1, we had one invited and three long oral presentations. The invited presentation by Trigueros[1] discussed an example of a task designed from the viewpoint of APOS theory to examine students' understanding of linear transformations. The analysis of students' responses from the lens of APOS theory, coupled with observations made by using the Anthropological Theory of Didactics (ATD) offer important elements that can be used in the redesign of the task. The design of task with APOS theory made it possible for students with different conceptions to start working on the task and obtain an answer that they considered satisfactory. At the same time APOS theory enables researchers to look closely into details involved in students' responses and discern different conceptions. The design of task with ATD evokes the issue of students' interpretation and performance of the task in reference to the social and institutional situations. Viviani and White[2] proposed task design for automatic formative assessment of student responses in a calculus class which provides students with opportunities to engage in exploration to change and develop their mathematical perspectives. Johnson et al.[3] elaborated a joint embodied and simulation design methods which are rooted in different theoretical traditions. The empirical evidence and implications of two phases of joint design method were discussed. Joining genres brings in a correspondent theory and highlights aspects of learning, together providing the rich affordances for covering a variety of possible gaps in mathematics understanding. This approach is particularly relevant for task sequences involving socially shared, and yet spatially articulated, mathematical notations, such as Cartesian graphs. Barquero and Esteve[4] reflected on the work of transposing research and methodological tools to teacher education for primary school teachers' practice of designing and analysing tasks. The case of the study and research paths had theoretical underpinning of the anthropological theory of the didactic and the research methodology proposed by didactic engineering.

2.3. Session 2

In the session 2, we had seven short oral presentations. Mittal and Ahuja[5] investigated the exploration of mathematical task design by pre-service teachers from three aspects: their source, type, and appropriateness, all of which can give holistic understanding to the researchers. Sun[6] delineated the fundamental idea of task design for algebra knowledge development with specific emphasis on Chinese cultural context. Muñoz[7] presented research results on the relationship between the schooling experience by prospective teachers, their instructional beliefs and practices, and the way they design mathematical tasks. The analysis showed a predominant pattern oriented towards constructivist teaching practices, which was influenced by the schooling of the prospective teacher. García et al.[8] presented results of a teacher's task design in combinatorics based on the "mode 5e". Transformation of task sequences with the 5e mode broadened teacher's vision and promoted high school students' perception. Kristinsdottir et al.[9] presented results of design research project on developing silent video tasks in collaboration with researcher, upper secondary teachers, and students. The research showed that the silent video tasks could be used as formative assessment and in classroom discussion. Otaki et al.[10] analyzed an authentic inquiry in didactic situations within the framework of the anthropological theory of the didactic (ATD), especially by the Herbartian schema. Inquiry process by prospective mathematics teachers who engaged with a task about calculation cubic root by pocket calculators was deliberately analyzed by the schema. Li and Zhou[11] reported on SOLO taxonomy-based action research on the design of multiple task examples for high school students. The examples had holistic, hierarchical, and self-explanatory nature and benefit most students at different levels.

2.4. Session 3

In the session 3, we had one invited and seven long oral presentations. The invited talk[12] by Ärlebäck discussed research on so called "Fermi problems" and the fundamental principles underlying this type of tasks and their use. Based on the model and modeling perspective on teaching and learning, the research developed the "FPAT-framework" for supporting the design and use of Fermi problems to facilitate not only students' learning mathematics concepts and higher order thinking skills but also interdisciplinary collaborations with other subjects, especially STEM subjects. Luhuan Huang et al.[13] presented a comparative textbook task analysis of lower-secondary schools in Beijing and the Netherlands, using an inquiry-based learning (IBL) framework. The analysis showed that tasks in both textbooks provide some opportunities for IBL in phases related to solution procedures and representations.

Meryansumayeka et al.[14] illustrated design research on digital task design which aims at developing higher order thinking skills (HOTS) for junior high school students. The digital tasks were developed based on the PISA problems and were validated and evaluated for the field test. Zhang et al.[15] posited that the variation in task design has a profound theoretical foundation and developed an analytic model of "teaching mathematics through variation" which distinguishes potential variation and practical variation. The model illustrated a teacher's documentation work from potential to practical variation in in China and France. Jäder[16] addressed how students' opportunities to engage in mathematical problem solving is limited by the prevalence of routine tasks in textbooks. An analytic framework was developed to better understand some of the important components of mathematical problem solving and possibly also be of support in the design of mathematical problems. Doorman et al.[17] discussed task design with scenarios in promoting inquiry-based mathematics teaching. The combination of the RME and the TDS afforded the development of open and context-rich tasks and to support teachers in balancing phases of student-led inquiry with phases for creating a whole class shared understanding of mathematical structures. Dawn et al.[18] proposed four interacting elements for consideration when developing a mathematical activity to support the construction of mathematical concepts. Analysis of students' work from the activity revealed different trajectories of how teachers can plan for differentiated instruction to promote students' robust construction of concepts.

2.5. Session 4

In the session 4, we had seven short oral presentations. Rodriguez et al.[19] prezented a sequence of tasks to help university students to go from an elementary conception of vector in physics to an element of a vector space. The task sequence involved a contextual problem in a digital environment with increasing abstraction. It began with movement of a robotic arm then as an arrow with magnitude and direction, and finally as an ordered pair of real numbers in a geometric environment. Nagari-Haddif[20] demonstrated design pattern of tasks using the Seeing the Entire Picture (STEP) online assessment platform in which students were to construct and submit examples for refuting or supporting a statement in an MLR environment with the activity "asymptotes and parametric functions". Nieto and Ruiz-Lopez[21] reported design reserch on the creation of a didactic sequence of problems and a technological tool to guide the resolution for teaching and learning isometries for prospective primary teachers. Mohamed and Ghazali[22] devised fraction tasks and test items which were validated by experts. Through these instruments and the Newman procedure types of errors concerning fractions were identified. Falculan and Aberin[23] investigated the effects of using Low Floor High Ceiling (LFHC) mathematical tasks on students' mathematical proficiency in seventh-grade geometry by closely examining their conceptual understanding and procedural fluency. Ohtani[24] presented an activity-theoretical approach to collaborative design of task and learning environment in which researchers of different expertise, and secondary teacher play different roles.

3. Recent Trends and Future Directions

We laid out seven themes in section 1.1. In view of the seven related themes, we point out recent trends and pose future directions for further research that arose from the TSG.

1) frameworks and principles for task design

We had many presentations that evidenced productive coordination and/or joint of different frameworks and principles. Among others, emergence of the embodiment perspective demonstrates promise for furthering task design. Chinese variation theory also drew much attention in the TSG. We need to investigate further how different design principles reflect or generate different perceptions of mathematical concepts.

2) methodological advances for studying task design in mathematics education

Many presentations adopted Design Research methodology, including the iterative processes of thought experiments and teaching experiments. The model and modelling perspective were also prevalent in the TSG presentations. In view of the wide interest in STEM movement, it is crucial to incorporate methodological advancements from other disciplines.

3) relationships between task design, anticipated pedagogies, and student learning

In this theme, recent research emphasizes inquiry-based learning, study and research pass, and cooperative learning. In this regard, it is necessary to reflect how different combinations of tasks and pedagogy influence learners' perceptions and mathematical activity.

4) the role of tools in task design

Some research developed digital tasks and video tasks. It is also relevant topic in task design and analysis to investigate how visual features of task presentation affect mathematical activity.

5) task sequences for promoting conceptual understanding and/or higher order thinking skills

In our knowledge-based society students need so called 21st century skills. A way to create opportunities for addressing these skills in classroom is through inquiry-based learning (IBL). In the TSG, we had promising presentations that suggested the design of appropriate tasks could be an important prerequisite for successful implementation of IBL.

6) task design in innovative learning environments

The development of technologically rich environment and assessment system enables us to understand the complex relationship between task design and individual learner differences.

7) textbook task analysis

This theme was not listed in the discussion document of the TSG. However, we had several presentations that compare textbook task in different countries, specifically comparison of Chinese and European countries.

In addition to the aforementioned seven themes, the following themes were prevalent in the TSG.

8) teachers' professional learning on task design and analysis

The professional learning of prospective and practicing teachers about task design, sequencing, and adaptation. This emerging theme, which goes beyond superficial aspects, would require socio-cultural perspectives in task design.

9) notworking among stakeholders

The communities involved in task design are naturally overlapping and diverse and they may act in several of these roles. Stakeholders can include designers, professional mathematicians, teacher educators, teachers, researchers, learners, and policymakers and so on. The effectiveness of forms of collaboration and communication between task designers, classroom teachers, educators, and policymakers.

10) political and ethical dimension of task design

This theme was less obvious in the TSG. The role of task design in promoting equity and other values are less attentive but relevant theme which should be considered in future research.

Looking to the future, TSG participants who are examining interdisciplinary efforts, instructional tools, online resources, and research methodologies could expand their vision by sharing empirically grounded contributions that underlie design principles, theoretical approaches, and carefully analyzed cases and examples of tasks designed for promoting mathematical development. Another area to explore are the multi-dimensional aspects of task design: tasks and sequences of tasks can shape possibilities for interactions between teachers and students. Teachers' pedagogies can include the selection, modification, design, sequencing, installation, observation and evaluation of tasks, through which they may learn more about their students' thinking and experiences. In turn, students' interactions with tasks can afford opportunities to learn mathematical concepts, ideas, strategies, and also to use and develop higher order thinking skills and critical literacy.

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https://doi.org/10.1142/9789811287152_0063

Topic Study Group 39

Language and Communication in Mathematics Classroom

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1. Core Topics and Research Interests for TSG-39

To begin with, we give a brief overview of the thematic foundations and challenges of TSG-39.

The variety of research shared by the presenters at the ICME conference makes the wide range of individual approaches to theory and methodology visible that research in language and communication in mathematics education is based on. Morgan (2006) refers to this focus to the role and nature of language and communication in relation to the learning of mathematics as a ‘turn to language’, although Pimm (2018) describes a long history of research connecting language within mathematics education. But a recognizable shift in the recognition of the complexity of the relationship between language and the learning of mathematics is taking place.

The research results presented in TSG-39 are based on an understanding of language and communication in a broad sense. The authors presented work focusing on the following fields of research: classroom interactions; interactions between children at play; multimodal analysis; as well as research that focuses on the multi-semiotic nature of mathematical activities, or even the role of silences (e.g., Boistrup and Samuelsson, 2018; Elliott and Ingram, 2016; O’Connor, Michaels, Chapin, and Harbaugh, 2017). Judith Moschkovich (2018) formulated four recommendations concerning the research of language and communication in mathematics education at ICME-13:

(1) using interdisciplinary approaches;
(2) building on existing methodologies;
(3) defining central constructs;
(4) recognizing central distinctions while avoiding dichotomies.

With respect to points (1) and (2) of Moschkovich, it can be stated that research in our field is influenced from a variety of theoretical and methodological ideas from

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mathematics education as well as from other fields such as sociology, psychology, anthropology, linguistics, semiotics and many more. Working at the intersection of theories of learning and teaching mathematics and theories of language, interaction and communication is fundamental to doing research in mathematics education with a focus on language, interaction and communication. The connection of existing theories and methodologies is equally important as the development from new theories to drive innovation. A particular challenge when combining theories from different disciplines is on (3) defining terms and central constructs such as language, register and discourse and also to use them in a way that is consistent with the approach to research being taken. Finally, we turn to the recommendation of (4) recognising central distinctions while avoiding dichotomies. Moschkovich emphasises the often-made dichotomy between quantitative and qualitative approaches in contrast to the many examples of work that combines both. Another aspect of the dichotomy that should be considered in more detail is the type of attention given to language in research. However, language and mathematical activities are often intertwined, making it difficult to distinguish whether language is the focus of research or the medium through which researchers can access mathematical thinking or learning (Andersson and Wagner, 2019). The recommendations of Moschkovich (2018) shed light on the topic of mathematics and language from very different perspectives and were addressed in the various papers and posters as well as invited lectures.

2. Presentations in TSG-39

ICME-14 brought together researchers from around the world and TSG-39 included a range of researchers from 13 different countries, drawing on a range of different theoretical perspectives, different methodological approaches and with different focuses. Due to the COVID-19 pandemic, the meeting was held in a hybrid format. Part of the presentations were given online via Zoom, and another part at the location of ICME-14, East China Normal University, Shanghai, China in a lecture hall. A total of 21 papers and 7 posters were accepted for presentation in TSG-39. The papers were presented in 3 sessions over 3 days. In addition, two sessions were dedicated for one invited talk each (Tab. 1).

2.1. Invited talks

The first invited talk[1] is given by Krummheuer from Kassel, Germany and Schütte from Leibniz University Hannover, Germany, and the second invited talk[2] is given by Herbal-Eisenmann from Michigan State University, USA and Ingram from Oxford University, UK. These invited talks enabled TSG participants to reflect on new broader perspectives regarding mathematics and language and to discuss the insights gained on this topic.

Tab. 1. The list of papers presented

Paper and author(s)
Invited talks
[1] Cooperation, argumentation and learning — Three basic concepts referring to everyday procedures in teaching-learning situations in mathematics classes. ***Götz Krummheuer*** *and* ***Marcus Schütte*** (Germany).
[2] A political look at math communication. ***Beth Herbal-Eisenmann*** (USA) *and* ***Jenni Ingram*** (UK).
Session 1
[3] Meeting the challenges of research language and communication in mathematics education. ***Jenni Ingram*** (UK), *Marcus Schütte* (Germany), *Fengjuan Hu* (China), *Máire Ni Riodáin* (Ireland), *and Tran Vui* (Vietnam).
[4] Lifeworld connections in mathematics education — unquestioned, indispensable, and undefined? ***Elisa Bitterlich*** (Germany).
[5] The threshold of multiple representations for students to discover possible solutions for communicating their new ideas in integrated closed-open approach. ***Tran Vui*** (Vietnam).
[6] The practice and examination of opportunities to translate representation through problem-solving. ***Kunihiko Shimizu*** (Japan).
[7] Tau Ke: a software solution for capturing multiple representations of pangarau (mathematics) language. ***Piata Allen*** (Australia).
[8] The effects of using a modified Frayer Model to teach mathematics vocabulary to junior-form English learners in a Chinese medium-of-instruction secondary school. ***Wing-kwan Li*** *and Simon S. Y. Cha* (Hong Kong SAR, China).
[9] It always equalled an odd number: observing mathematical fluency through students' oral responses. ***Katherin Cartwright*** (Australia).
[10] Achieving meaningful statistics classroom learning through bilingualism and multilingualism: a case of selected grade 10 students in Marikina city. ***Mary Jane A. Castilla*** *and Catherine P. Vistro-Yu* (Philippine).
Session 2
[11] Discourse as the place for the development of mathematical thinking through an interactionist perspective. ***Judith Jung***, *Marcus Schütte, and Götz Krummheuer* (Germany).
[12] Language-responsive support of meaning-making processes for understanding multiplicative decomposition strategies. ***Annica Baiker*** *and Daniela Götze* (Germany).
[13] A study on the evaluating of learning opportunities in mathematics classes of secondary schools based on discourse analysis techniques. ***Zhihui Chen*** *and Yuting Tong* (China).
[14] Mathematical expression in different languages: The need for systematic description. ***Cris Edmonds-Wathen*** (Australia).
[15] A comparative study on teaching language of algebra classroom between novice teachers and expert teachers taking linear equation in one unknown as an example. ***Si-kai Wang*** *and Li-jun Ye* (China).
[16] Interactional obligations for collective argumentation in pair and group work. ***Rachel-Ann Böckmann*** *and Marcus Schütte* (Germany).
[17] How pre-service primary teachers engage in language responsive mathematics teaching while working on a scriptwriting task. ***Victoria Shure*** *and Bettina Rösken-Winter* (Germany).
[18] Support Systems as intersubjective processes between Teachers and Students. ***Ann-Kristin Tewes*** (Germany).
Session 3
[19] Epistemic (In)justice in mathematical communication between teachers and students. ***Lauren Hickman McMahon*** (USA).
[20] Identifying language demands for understanding the meaning of similarity. ***Kirstin Erath*** (Germany).
[21] Exploring a teacher's enactment of explanatory communication in a mathematics lesson. ***Fatou Sey*** (South Africa).
[22] Dissent and consensus situation structures in mathematics and computer science learning environments. ***Peter Ludes-Adamy*** *and Marcus Schütte* (Germany).
[23] Quadrilateral woop-de-doos: Language use and geometric property development of two fifth graders in a dynamic geomtetry learning environment. ***Candace Joswick*** *and Michael T. Battista* (USA).

2.2. Session 1

In Session 1 Bitterlich[4] focusses on situations with a lifeworld connection within mathematics lessons. Via interactional analyses and the analysis of linguistical markers her study aims at reconstructing the consequences of lifeworld connections on language use and the negotiation of (mathematical) meaning. Bitterlich underlines that lifeworld connections are frequently posed by the teacher, but seemingly seldom reflected concerning the underlying mathematical content. Shimizu[6] reports research that children will be proactive in their use of diverse mathematical representations when they have questions and explorative tasks in learning problem-solving. Based on this classroom practice, the process of representation was translated by exploring student questions, while students' feelings concerning their approach to mathematics were also important. Further, as students' inquiries deepened, their representation gradually became more sophisticated, and in the process, along with trial and error, a process of returning to representing thought was seen. Allen'***s*** study[7] focused on the use of digital technology in Māori-medium schools as a way of supporting Māori language, Māori knowledge and the acquisition of school mathematics. Following a period of Indigenous language and culture loss, in Aotearoa New Zealand, there has been rapid development of a corpus of mathematics terms and language to enable the teaching of mathematics in the Indigenous language, Māori. Allen's paper highlights the questions and concerns that continue to be raised, about the role of Indigenous mathematical practices in modern schooling. Li and Cha[8] conducted a study about learning Mathematics vocabulary using a Modified Frayer Model in a local secondary school. The model, also a graphic organiser, included four components which were specific to Mathematics: Mathematics symbols, diagrams or pictures, related vocabulary and sample sentences. The results showed that the model not only expanded the participants' Mathematics vocabulary, but also helped them remember it.

Cartwright[9] reports from a deductive analysis of student narrative data (transcripts) and student group work samples (artefacts) aiming to discover what characteristics students displayed both orally and written as evidence of mathematical fluency. She presents the oral and written language features students employed to explain their method and justify their strategies when solving mathematical tasks. Regarding mathematical fluency, Cartwright discusses that students' oral explanations were either procedurally-driven or findings driven. Students that were findings-driven could be identified at a higher level of fluency based on their ability to shift from low to high modality language. Cartwright proposes the need to analyze the language features (as a representation mode) of students' responses–particularly oral responses–as they provide data that might usually be missed or not present within written numerical work samples. Finally, Castilla and Vistro-Yu[10] examined the linguistic interactions that took place in a Statistics classroom using the framework of the second generation Cultural Historical Activity Theory (CHAT) with bilingualism and multilingualism as a potential primary CHAT construct. They looked into how the varying roles of students' alternating use of the Filipino and English languages combined with their mathematical language contributed to the students' participation in the activity.

2.3. Session 2

In Session 2 Jung et al.[11] focus on early childhood mathematics learning. Based on interactionist perceptions of mathematical learning, the development of mathematical thinking is described as increasing participation in mathematical discourses. For a more detailed description of these discourses, the so far common focus of interactionist approaches to mathematics learning on the analysis of mathematical negotiation of meaning is expanded to include a description of emerging argumentative structuring of the mathematical negotiation processes. Baiker and Götze[12] present a study investigating the impact of a language-responsive intervention on students´ understanding of decomposition strategies. Three second grade primary school teachers introduced multiplication ($n = 66$) by using meaning-related phrases of unitizing (e.g., '6 times 4 means 6 fours'), whereas three other classes taught without this focus served as control group ($n = 58$). The analyses of a multiplication post-test and a follow-up test showed a deeper understanding of decomposition strategies of the intervention group children.

Chen and Tong[13] addresses the question of the development of literacy skills of senior high school students in China and how to promote students' skills based on the model of mathematical core competencies in the classroom. Discourse analysis techniques are used to analyze two video-based lessons with different teaching methods. The results show that the evaluation model based on the idea of learning opportunity is reliable in terms of how students can benefit from the mathematical tasks and interaction (initiated questioning and feedback) provided by the teacher in the teaching process. Edmonds-Wathen[14] described a need for more systematic description of the variation in mathematical expression in different languages and the observed or speculated effects of this variation on mathematics education in those different languages, proposing a functional typology approach where languages are classified according to structural similarities and differences. Wang and Ye[15] reported a comparative study about the teaching language of a novice teacher and an expert teacher in algebra instruction. They classified the teacher's teaching language from the perspective of pragmatics, on the basis of which they specifically discussed the similarities and differences between the two teachers' use of teaching language.

Böckmann and Shütte[16] describes interactional obligations for bringing forth warrants or backings within collective argumentations which occur in social interactions of students working collaboratively in multi-age groups. She presents three interactional obligations — contradictions, mistakes and certain types of questions — as well as discusses how students' interpretation of interactional obligations can change within an interaction. Shure and Rösken-Winter[17] report on the results of a scriptwriting task study aimed at examining how pre-service primary mathematics teachers enrolled in a Master's program address language difficulties to support students in gaining mathematical reasoning competencies. They present differences in practices between higher and lower performing pre-service teachers and discuss the study's relevance for teacher education. At the end Tewes[18] focusses on different support systems between students, primary school teachers and special needs education teachers. She therefore located support systems between the participants of the interaction and describes them as intersubjective processes which are designed together. The aim of the study is to reconstruct the potential effects of these support systems for the participation in inclusive mathematic lessons.

2.4. Session 3

The Session 3 begins with McMahon[19] who describes in her paper, two forms of epistemic injustice offered by Fricker (2007) — testimonial and hermeneutic. Using a real-world example, she considers how such injustice can manifest itself in teacher-student communication about mathematics and discusses features of mathematical knowledge and skill that are necessary for children to experience epistemic justice in their interactions with teachers. Erath[20] reports from a Design Research study aiming at developing a language-responsive teaching-learning arrangement for the geometrical topic of similarity with a particular focus on supporting students' interaction in phases of unmoderated group work. She presents and discusses identified discourse practices and language means alongside a intended sequence of larger steps in the process of knowledge construction. Ludes-Adamy and Schütte[22] report on their research on learning environments with core topics of mathematics and computer science are examined. In the focus of digitalization, this topic and its connection to mathematics will play an important role in future curricula, making it an interesting object of investigation. Ludes-Adamy and Schütte present an ongoing study that examines, how the topic of computer science connected to mathematics learning can be approached in primary schools and what and how meanings are negotiated. They focus on the question what roles consensus and dissent play in interactional processes of negotiation and how the learning of the fundamentally new occurs in collective argumentation between pupils. Joswick and Battista[23] use a longitudinal analysis to track the language and geometry concept development of two 5th grade students regarding dynamic geometry for the study of quadrilaterals. In their paper, the authors describe the students' initial use of the term "whoop-de-doo" in their reasoning about quadrilateral shapes and point to the importance of our findings for productive classroom discourse.

3. Summary and Prospect

The papers that have been presented during our TSG present a wide range of research perspectives. They are published in the HAL open archive and can be freely accessed. The challenges that researchers in language, interaction and communication in mathematics education encounter, can also be seen as opportunities to foster innovation and influence teaching and learning of mathematics in a variety of contexts. Working together as a Topic Study Group at ICME, discussing and connecting the differences and similarities in the research, enabled us all to take advantage of these opportunities and develop the field further. This collaboration will continue due to an ever-growing core of collaborating scientists in the field of mathematics, language and communication at the upcoming international meetings, such as those of CERME-2022, ETC-2022 and the following ICME meetings.

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https://doi.org/10.1142/9789811287152_0064

Topic Study Group 40

Research and Development on Mathematics Curriculum

Masataka Koyama[1] and Jeremy Hodgen[2]

ABSTRACT The theme of Topic Study Group 40 (TSG-40) at the 14th International Congress on Mathematical Education (ICME-14) (Shanghai, China) is Research and Development on Mathematics Curriculum. TSG-40 was held worldwide on-line style in three sessions of July 13, July 16, and July 17, 2021. This article reports a concise summary of TSG-40 including its organization, theme and description, the list of presentations and program overview, the summary of presentations in the theme of four topics at TSG-40, and future directions and suggestions in the area of research and development on mathematics curriculum.

Keywords: Mathematics Curriculum; Policy; Research; Development.

1. Organization, Theme, and Description of TSG-40

1.1. Organization and theme of TSG-40 for ICME-14

TSG-40 was organized by the organizing team[3].

The theme of TSG-40 at ICME-14 is Research and Development on Mathematics Curriculum. Its aim is to share and discuss the recent results of research and development on mathematics curriculum at all levels, and to identify perspectives for future research and development. Recent mathematics curriculum study has expanded to explore a range of important topics, including policy issues, curriculum development and analysis, and curricular impact on teachers' teaching and students' learning (Li and Lappan 2014, Vistro-Yu and Toh 2019).

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Co-chair: Jeremy Hodgen, University College London, UK
Members: Gulseren Karagoz Akar, Bogazici University, Turkey
Shelly Dole, University of the Sunshine Coast, Australia
Ruilin Wang, Capital Normal University, China

1.2. Description of TSG-40

We called for papers for TSG-40 as follows.

TSG-40 welcomes researchers, teacher educators, teachers, curriculum developers, test developers, and policy makers with research interests in research and development on mathematics curriculum. We invite both theoretical and empirical research contributions that address one or more of the following topics in the research and development on mathematics curriculum.

Topic 1: Mathematics Curriculum Policy

This topic includes policy issues related to mathematics curriculum in different education systems, and the process of curriculum decision-making, curriculum changes, curriculum policy, and education changes viewed from a historical perspective.

Topic 2: Mathematics Curriculum Development and Analysis

This topic includes curriculum design and development in different education systems, explicating and comparison of diverse ideas and practices in curriculum development, textbook design, and changes in curriculum development in different system contexts.

Topic 3: Mathematics Curriculum, Teacher, and Teaching

This topic includes perspectives on the process of improving mathematics education by reform of curriculum and teaching, and the challenges of developing, implementing, and evaluating change in the content objectives and teaching of mathematics.

Topic 4: Mathematics Curriculum and Student Learning

This topic includes curricular impact on students' learning and the challenges of reforming the curriculum to improve students' learning.

2. List of Presentations at TSG-40 and Program Overview of TSG-40

2.1. List of presentations at TSG-40

As a result of both peer-reviews and the payment of registration fee for participating in ICME-14, TSG-40 contributions included 1 long paper (LO), 10 short papers (SO) and 3 posters. Tab. 1 (on the next page) lists the title and author(s) of the papers and posters presented at TSG-40.

2.2. Program overview of TSG-40

TSG-40 had three sessions with 90-90-120 minutes' timeslots for papers. We gave a careful consideration to the worldwide on-line style of TSG-40 especially the time difference and made the program of presentations at TSG-40 including an opening session in Session 1 and a closing session in Session 3 as follows.

Tab. 1. List of papers and posters presented at TSG-40

Title and *author(s)*	
[1]	Identifying the quality of teacher created curriculum shared via the teachers' pay teachers online platform. *Lara K. Dick, **Amanda G. Sawyer**, and Margaret A. MacNeille* (USA). (LO)
[2]	Comparative study on statistical contents in Chinese and Japanese mathematics textbooks. ***Xinqi Zhang** and Masataka Koyama* (Japan). (SO)
[3]	The implementation of a reformed mathematics curriculum: Mathematical processes in practice. ***Anna Klothou** and Charalampos Sakonidis* (Greece). (SO)
[4]	The mathematical literacy in Korean mathematics curricula. ***Eun Young Cho** and Rae Young Kim* (South Korea). (SO)
[5]	Financial education in the Romanian mathematics curriculum: Policy and implementation in elementary textbooks. ***Daniela Căprioară*** (Romania)*, Annie Savard, and Alexandre Cavalcante* (Canada). (SO)
[6]	Formative evaluation of a tool for representing ideas in mathematics curriculum design: A Delphi study example. ***Ellen Jameson** and Lynne McClure* (UK). (SO)
[7]	Images of mathematics curriculum and pedagogical influences. ***Laxman Luitel** and Bal Chandra Luitel* (Nepal). (SO)
[8]	A participative approach to designing a new mathematics course for all college and university students in the Philippines. ***Catherine P. Vistro-Yu*** (Philippines). (SO)
[9]	A comparison of U.S. and Chinese geometry strands through the lens of van Hiele. ***Lili Zhou**, Jinqing Liu, and Jane-Jane Lo* (USA). (SO)
[10]	Curriculum proposal from El Salvador for improving math learning, description, structure, first results and effectiveness. ***Francisco Antonio Mejia Ramos*** (EI Salvador). (SO)
[11]	A course design for mathematical modeling in high school based on STEM education. ***Shengkui Su**, Lin Miao, and Qinghua Chen* (China). (SO)
Posters	
[12]	Investigating third level lecturers awareness of second level curriculum reform four years on. ***Fiona Faulkner**, Cormac Breen, Michael Carr, and Mark Prendergast* (Ireland).
[13]	Mathematical curriculums for five-year junior college programs in ChineseTaiwan. ***Yu Jr Tsai** and Shao Ying Li* (China).
[14]	The curricular statute of the discrete mathematics discipline in the Brazilian systems analysis and development public technological course. ***Jefferson Biajone** and Vinicio de Macedo Santos* (Brazil).

- Session 1 (Tuesday July 13) 19:30~21:00 Beijing time (90 minutes), only one long oral presentation[1];
- Session 2 (Friday July 16) 21:30~23:00 Beijing time (90 minutes), six short oral presentations[2—7];
- Session 3 (Saturday July 17) 14:30~16:30 Beijing time (120 minutes), four short oral presentations[8—11];
- Poster Session (Saturday July 17) 13:00~14:00 Beijing time (60 minutes), three posters[12—14]

The authors came from 13 countries — Brazil, Canada, China, El Salvador, Greece, Philippines, Ireland, Japan, Nepal, Romania, South Korea, UK, and USA — representing the different parts of the world. In the next section 3, we will classify and summarize the all 14 presentations into the theme of four topics at TSG-40.

3. Summary of Presentations in Theme of Four Topics at TSG-40

3.1. Topic 1: mathematics curriculum policy

We had four presentations related to the theme of mathematics curriculum policy.

Klothou and Sakonidis[3] examined six primary school teachers' practices concerning four mathematical processes adopted after their involvement with piloting a reformed mathematics curriculum in Greece. Analysis of the data revealed contradictions in teachers' teaching practices which can be attributed to local recontextualization procedures activated during the implementation of the reformed curriculum. Căprioară et al.[5] focused on the introduction of financial education in the school curriculum. As the introduction occurs in many ways depending on educational policies and systems, they showed to what extent the Romanian mathematical curriculum for the primary level corresponds to the concepts derived from the definitions of financial education. The recent mathematics curriculum requires teaching a lot of financial concepts. So, they insisted the elementary school teachers play an important role though they have not been trained to teach financial learning, even if they are currently being asked to do so.

Tsai and Li[13] showed the arrangement of mathematical curriculum in five-year junior college programs which combines with compulsory education and higher education in Chinese Taiwan. No matter what schools are, achievement gap caused by the system of mathematical curriculum in compulsory education can be clarified effectively after analyzing mathematical curriculum guidelines. They suggested this may help teachers to understand the contents students are learning in class, and moreover, teachers can find some proper strategies to assist interdisciplinary students to learn mathematics effectively. Biajone and Santos[14] presented research on the statute of the discrete mathematics (DM) course curriculum production in terms of objectives and contents for the system analysis and development (SAD) undergraduate course offered by 134 public technological colleges and universities in Brazil. Developed in 2018, they investigated the DM discipline constitution at the undergraduate level according to what contents and purposes are needed for the SAD course and its prescribed curriculum under the perspective of curriculum policy cycle and history of disciplines.

3.2. Topic 2: mathematics curriculum development and analysis

Mathematics curriculum development and analysis was an important theme at TSG-40. We had six presentations related to the theme.

There were two presentations of comparative study in the theme. Zhang and Koyama[2] compared the statistical contents in Chinese and Japanese mathematics textbooks as a part of intended mathematics curriculum. The similarity and difference were reflected on the structure of statistical contents and means of data analysis. There were deficiencies in problems, plans and conclusions of the statistical investigative cycle in China and Japan. Therefore, they suggested that we can use mathematical history materials of statistics to help students promote statistical thinking process. Zhou et al.[9] compared the geometry standards in U.S. Common Core State Standards of Mathematics (CCSSM) and Chinese Compulsory Education Mathematics Curriculum Standards (CMCS) through the lens of van Hiele levels. By examining the van Hiele level distributions of the learning expectations and major topics, they investigated how CCSSM and CMCS propose the development of the geometric thoughts of students. Implications of this study and suggestions for future revisions for both standards were discussed.

Cho and Kim[4] analyzed the nature of mathematics and the goals of mathematics education represented in 10 Korean mathematics curricula (from the 1st to the 2015 revised curriculum) to find out how the concept and meaning of mathematics have changed over time. They conducted semantic network analysis by keywords of each curriculum to identify the word change trend by extracting the frequency and degree centrality of each word, and matrix charts among words. On the other hand, Luitel and Luitel[7] assessed the beliefs about the mathematics curriculum and its pedagogical influences in classroom aiming to improve the teaching and learning environment. They followed metaphorical approach to represent the beliefs of mathematics curriculum. The knowledge constitutive interest, transformative learning theory and social constructivism were considered the major theoretical lenses. The moments and situations they experienced during their own teaching and learning activities has represented through multiple genres.

Ramos[10] showed the mathematics curriculum in El Salvador. Since 2016, El Salvador in cooperation with JICA has developed a new mathematics education policy. The proofreading strategy based on El Salvador students' needs, the rearrangement of contents in the courses of study, and the approach to specific classes were essentially explained. The implementation based on a 'student-centered approach' and a suitable 'teacher support' based on some specific formative assessment statements were briefly presented as well. Finally, some findings of the first years of implementation and apparent success were shown. Su et al.[11] focused on the strong correlation of mathematical modeling literacy among multiple disciplines in high school. They have built a progressive course system including mathematical modeling basic courses (M), innovation practice courses based on school-enterprise cooperation (I), research-based learning advanced courses (R) and STEM higher-order courses (S), jointly constituting the MIRS course. On this basis, they illustrated the implementation of the MIRS course through four course cases.

3.3. Topic 3: mathematics curriculum, teacher, and teaching

Mathematics curriculum, teacher, and teaching were also an important theme at TSG-40. We had three presentations related to the theme.

Dick et al.[1] studied on identifying the quality of teacher created mathematics curriculum. Teachers Pay Teachers claims to be "the world's most popular online marketplace for original educational resources." The Teachers Pay Teachers (TpT) website offers more than five million free and paid resources and has over seven million teacher users. Despite the growing popularity of websites such as TpT, the mathematics education community knows little about the quality of these curricular resources. In their presentation, they sought to address this lack of knowledge for elementary mathematics. They shared results from a research study that compares 500 free vs. 500 paid elementary mathematics activities each with the highest rating found on TpT.

Vistro-Yu[8] described and analyzed the design process that went into the development of the new mathematics course for the general education curriculum (GEC) required of all students at colleges and universities in the Philippines, beginning SY 2018-2019. The new GEC was conceptualized in 2013 to accompany the new K-12 mathematics curriculum. Entitled “Mathematics in the Modern World” (MMW), this new course was envisioned to help provide for the holistic development of the Filipino student in tandem with courses from other disciplines. Proposals for a more systematic curriculum development process were offered. Faulkner et al.[12] made a further analysis of the transition of a second level curriculum reform to higher education in Ireland. At ICME-13, an initial study was presented by the authors investigating third level mathematics lecturers’ awareness of the second level reform. The findings determined that although many lecturers were mindful of the concept of Project Math, they were not aware of the changes in full and how that affected their own course content, teaching, and assessment strategies. This study was a follow-up to the original, and comparisons were made with the 2015 data to see if the situation had changed.

3.4. Topic 4: mathematics curriculum and student learning

Although the relationship between mathematics curriculum and student learning was an important theme, there was one presentation related to the theme at TSG-40. Jameson and McClure[6] discussed some contributions of a Delphi study conducted for the formative evaluation of such a tool, the Cambridge Mathematics Framework. A panel of curriculum researchers responded to questions arising from the design, theoretical framework and methodology. Their presentation focused on the panel’s responses regarding the contributions of motivation to mathematical thinking and doing. The panel assigned motivation lower priority in total for consideration in the design work, but also expressed the highest levels of professional disagreement about it.

4. Future Directions and Suggestions

The above-mentioned paper and poster presentations are classified into four topics of TSG-40 at ICME-40 as follows. The presentations[3,5,13,14] are related to the theme of Topic 1 which includes policy issues related to mathematics curriculum in different education systems, and the process of curriculum decision-making, curriculum changes, curriculum policy, and education changes viewed from a historical perspective. The presentations[2,4,7,9,10,11] are related to the theme of Topic 2 which includes curriculum design and development in different education systems, explicating and comparison of diverse ideas and practices in curriculum development, textbook design, and changes in curriculum development in different system contexts. The presentations[1.8,12] are related to the theme of Topic 3 which includes perspectives on the process of improving mathematics education by reform of curriculum and teaching, and the challenges of developing, implementing, and evaluating change in the content objectives and teaching of mathematics. The presentation[6] elated to the theme of Topic 4 which includes curricular impact on students' learning and the challenges of reforming the curriculum to improve students' learning.

Many of the papers presented at TSG-40 are descriptive. As future directions and suggestions in the research and development on mathematics curriculum, it would be valuable to consider the design process, the process of implementation and assessing the effects on students, and to take a more critical perspective on the comparative work.

Acknowledgments

As the Chair and Co-chair of TSG-40, we want to thank and acknowledge our organizing team members Gulseren Karagoz Akar, Shelly Dole, and Ruilin Wang. Indeed, TSG-40 wouldn't be possible without their dedicated efforts in the process of planning and implementing TSG-40 held worldwide on-line style in three sessions of July 13, July 16, and July 17, 2021. Thanks also go to all our presenters who took the time to help peer review submitted papers and joined the on-line sessions from across the world.

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https://doi.org/10.1142/9789811287152_0065

Topic Study Group 41

Research and Development on Textbooks and Resources for Learning and Teaching Mathematics

Sebastian Rezat[1], Jana Visnovska[2], Guorui Yan[3], Moneoang Leshota[4], and Hussein Sabra[5]

ABSTRACT This chapter gives an overview of the themes that grounded the work of Topic Study Group 41 at ICME-14, and an account of the congress sessions, in which the contributions were reported and discussed. The chapter authors were the members of the TSG-41 organizing team, chaired by S. Rezat and co-chaired by J. Visnovska. We highlight the directions for the continuation of the work of this TSG in the areas of (a) production of detailed accounts and justifications of the principles used in resource design, (b) resource design and evaluation for the opportunities to teach, (c) exploration of the supports the students require to navigate the creation of their own learning trajectories, and (d) development of analytical tools for determining the specific ways in which teaching resources could contribute to teacher support needs within broader systemic improvement efforts.

Keywords: Mathematics textbooks; Curriculum resources; Digital resources; Teaching resources; Instructional design.

1. Themes and Description of TSG-41

The efforts of Topic Study Group 41 (from here on TSG-41) focused on explorations of the issues related to the contents, design, development, use, and implementation of print and digital resources for teaching and learning of mathematics. The resources included extend beyond the print and digital school textbooks, and include teacher manuals, professional development materials, student learning and assessment materials, and a variety of online resources.

Directly linking to, and building on, the earlier work of TSG-38 at ICME-13 (Fan et al., 2017; Fan et al., 2018) the aim of TSG-41 at ICME-14 was to bring to the foreground and examine various theoretical and methodological approaches used to design, analyze, and empirically study mathematics learning and teaching resources and their use in diverse geographic regions and contexts. The pre-congress call for contributions for TSG-41 highlighted three broad leading themes of (1) role and effects,

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(2) design and development, and (3) use and implementation of mathematics resources. The aim was to attend to resources across their creation, use, and effects, with attention paid to the changes introduced by the emergence of digital resources. Each of the three themes in the call included illustrative questions.

Theme 1: The role and effects of print and digital textbooks and other resources in mathematics classrooms.

- What teaching and learning resources are available in mathematics classrooms in different countries? What role do they play in mathematics teaching, learning, and assessment? What are the current effects of the use and implementation of these resources on student learning?
- How is the role that the resources play affected by the digitalization of information and communication and the growing availability of digital and online resources? What are the effects of modern ICT (particularly internet) on students' use of learning resources?
- How does the availability and use of digital resources affect student behavior, learning, and relationships to the subject of mathematics?

Theme 2: The design and development of print and digital mathematics textbooks and other resources.

- What are the theoretical foundations that guide the aspects of development and design of mathematics textbooks and other resources, such as the selection and progression of tasks, development of student competencies, considerations of and supports for envisioned teacher learning or change of practice, features of interactive elements and feedback in digital resources?
- What do we know about designing resources for supporting specific pedagogical intentions for mathematical learning (e.g., project-based, inquiry-based or problem-based learning) and for supporting mathematical learning in environments with blended agendas (e.g., integrated, STEM, multiliteracies)?
- What are the key differences in features and contents between print and digital resources that result from the affordances of digitalization? How do we conceptualize interactions between resource designers and users? Specifically, what is the role of teachers and students in developing textbooks and other teaching and learning materials?

Theme 3: The use and implementation of print and digital mathematics textbooks and other resources and related interactions among resources, teachers, and students.

- What are the influences on the use and implementation of textbooks and other resources? How are teachers supported in their interaction with and the implementation of textbooks and other resources?
- How do teachers adopt and adapt new resources in their professional work?
- How do teachers' individual resources interact with collective resources, and how could we model such relationships?
- What are the consequences of the use of particular resources for the teaching of mathematics, and for teacher knowledge and professional development?
- What resources do students use for learning mathematics and how do they use them?
- How do students, as well as their teachers, interact through resources?

2. Organization of TSG-41 Sessions

Being the group that deals with research on textbooks, TSG-41 received comparatively high number of submissions. After the review process, 40 contributions were accepted, of which six were presented as long oral presentations, 20 as short oral presentations, eight as posters, and six author teams withdrew their submissions due to the ICME-14 date changes and transition to a hybrid congress mode in 2021.

2.1. Oral presentations

In addition to submitted contributions, TSG-41 engaged three top scholars as invited speakers and discussants, tasked with providing direction for the work of the group.

The ICME-14 congress assigned each TSG three sessions, two 90 min long, and one 120 min long. We included one invited talk in each session, and gave all long oral presenters time for an individual presentation.

Aiming for fruitful experiences online as well as on site, the organizing team sought creative ways of fostering discussions and collaborations, in which authors of all accepted short and long oral presentations could actively take part. Therefore, in the lead up to the congress, thematic small groups of paper authors have met online to discuss their respective work and co-create a pre-recorded 5 min video update on the issues and questions that stood out in their combined contributions.

Subsequently, the congress TSG blocks were shared between three 20-minute discussant contributions, six 10-minute long-paper presentations, and seven 5-minute pre-recorded reports from small group pre-congress debates. Moreover, we arranged $10 + 15 + 20$ minutes of discussions (scheduled respectively at the end of each presentation block), as well as opening and closing remarks from the organizing team. The list of oral presentations is given in Tab. 1, while the list of all accepted paper contributions that reflects the pre-congress collaborative work in small groups is given in Tab. 2 (each long oral presentation appears in both tables with the same numbering).

Tab. 1. List of invited talks and long oral presentations

Paper and author(s)	
Invited discussant contributions	
[1]	Textbooks as teacher support for engaging students in active knowledge organization. ***Susanne Prediger*** (Germany).
[2]	Digital mathematics curriculum resources: Towards design principles of educative materials for students and teachers. ***Birgit Pepin*** (The Netherlands).
[3]	Instructional materials as tools for instructional improvement. ***Paul Cobb*** (USA).
Long oral presentations	
[4]	Identifying educative features in scripted mathematics lesson plans. ***Moneoang Leshota*** (South Africa).
[5]	Learning to design resources for teachers. ***Jana Visnovska*** (Australia), *José Luis Cortina* (Colombia), *and Pamela Val* (South Africa).
[6]	Elements of a theory of textbook design. ***Sebastian Rezat*** (Germany).
[7]	Didactic considerations regarding the iterative development design of dynamic digital tools. ***Anatoli Kouropatov***, *Regina Ovodenko, Michal Fraenkel, Maureen Hoch* (Israel).
[8]	Teaching and learning with dynamic textbooks: studying student uses at scale. ***Vilma Mesa*** *and Saba Gerami* (USA).
[9]	Investigating the use of mathematics textbooks by students in Shanghai and England: a comparative study. ***Yi Wang*** (China) *and Lianghuo Fan* (China/UK).

Tab. 2. List of papers presented in groups

Paper and author(s)	
Group 1	
[10]	An analysis of data and probability tasks in US and Chinese elementary mathematics textbooks. ***Xiang Gao*** (China).
[11]	Constructing a textbook analysis framework of statistics and probability areas in elementary math. ***Shiqi Lu*** *and Wenbin Xu* (China).
[12]	Learning to design resources for teachers. ***Jana Visnovska*** (Australia)*, José Luis Cortina* (Colombia)*, and Pamela Val* (South Africa).
[12]	The effect of the curricula on textbooks for the teaching of probability and statistics. ***Gergely Balazs Wintsche*** (Hungary).
[13]	Mathematics Education according to the textbook: opportunities to learn investigated. ***Marc van Zanten*** (The Netherlands) *and Marja van den Heuvel-Panhuizen* (Norway).
Group 2	
[7]	Didactic considerations regarding the iterative development design of dynamic digital tools. ***Anatoli Kouropatov****, Regina Ovodenko, Michal Fraenkel, Maureen Hoch* (Israel).
[4]	Identifying educative features in scripted mathematics lesson plans. ***Moneoang Leshota*** (South Africa).
[14]	A comparative study of bidirectional connections in U.S.A. and Chinese high school mathematics textbook problems. ***Shuhui Li*** (USA).
[15]	Translations of function representation in different textbooks. ***Yang Shen*** *and Bao Jiansheng* (China).
Group 3	
[16]	A comparative study of problem solving in Chinese and U.S.A. primary mathematics textbook. ***Suijun Jia*** (China).
[17]	A comparative analysis of tasks contexts in mathematics textbooks in China and Singapore. ***Yao Li*** *and Lianchun Dong* (China).
[18]	A comparative study of mathematical inquiry activities in textbooks in China and Singapore. ***Hongwei Ran*** *and Lianchun Dong* (China).
[6]	Elements of a theory of textbook design. ***Sebastian Rezat*** (Germany).
Group 4	
[8]	Teaching and learning with dynamic textbooks: studying student uses at scale. ***Vilma Mesa*** *and Saba Gerami* (USA).
[19]	The elements of textbooks that Indonesian mathematics teachers use as they adopt student-centered instructional approach. ***Dewi Rahimah*** *and Jana Visnovska* (Australia).
[9]	Investigating the use of mathematics textbooks by students in Shanghai and England: a comparative study. ***Yi Wang*** *and Lianghuo Fan* (China).
Group 5	
[20]	Sesamath resources and collective work from mathematical laboratory to classes in Arabic environment. ***Karima Sayah*** (Algeria).
[21]	Promoting the teaching and learning of mathematics through visualising connections in post-16 resources. ***Dominic R. Oakes*** *and Sofya Lyakhova* (UK).
[22]	Comparing naming systems used by Chinese and Ukrainian teachers: exploring teachers' resource system. ***Maryna Rafalska*** (France)*, Chongyang Wang* (China)*, and Luc Trouche* (France).
Group 6	
[23]	Toward systematic support for preservice teachers' learning of productive resource use. ***Ok-Kyeong Kim*** (USA).
[24]	Analysing teachers' individual and collective resources through the lens of their digital resources. ***Katiane de Moraes Rocha*** (Brazil).
[25]	Student understanding of textbook visual representations of natural and fractional numbers: a collaborative international research. ***Everaldo Silveira*** (Brazil) *and Arthur B. Powell* (USA).
[26]	How expert mathematics teacher design curriculum based on textbook use: a case study in Beijing. ***Guorui Yan*** (Hong Kong SAR, China).
Group 7	
[27]	Long-term use of a digital mathematics textbook with integrated digital tools: investigating the influence on students' achievement and self-efficacy. ***Maxim Brnic*** (Germany).
[28]	The relationship between mathematical examples in Malawian grade 1 primary school mathematics teachers' guide and the goals of outcome-based education. ***Lisnet Mwadzaangati*** (Malawi).
[29]	Career mathways: a teaching & learning intervention to show the relevance of mathematics in careers. ***Niamh O'Meara****, Olivia Fitzmaurice, and Patrick Johnson* (Ireland).

2.2. Contributed posters

Posters in this TSG are listed in Tab. 3.

Tab. 3. List of posters

Poster and author(s)
[30] The extent of creative reasoning opportunities and aspects of cognition demand in textbooks in Nepal: a case of high school mathematics textbook. *Deepak Basyal and* ***Mohan Thapa*** (Nepal).
[31] This is the way I use textbooks and other resources for design mathematics lessons: A case of teaching the area of circle. ***Ya Cheng*** (China).
[32] Educative curriculum materials: Teachers' continuous training in the step by step of the materials designing process. ***Pauli Diniz*** (Mozambique).
[33] About textbooks on mathematical logic and theory of algorithms for prospective mathematics teachers. ***Vladimir I. Igoshin*** (Russia).
[34] The presentation of core knowledge acquisition process in junior middle school mathematics textbooks. ***Tianzhuo Jiang*** *and Shuwen Li* (China).
[35] The presentation of linear function in Chinese school mathematics textbooks. ***Na Li*** (China).
[36] A comparative study on fractions in primary school's mathematics textbooks of China and the United States. ***Fulin Liu****, Yiming Cao, and Dengfeng Liang* (China).
[37] A comparative study of "figures and geometry" in junior middle school mathematics textbook by PEP edition and Kangxuan edition. ***Yihan Wang****, Meiyue Jin, and Jiadi Zhang* (China).

3. Contributions to the Themes

3.1. Theme 1

The three contributions to TSG-41 in group 7 were related to Theme 1 (the role and effects of print and digital textbooks and other resources in mathematics classrooms), and particularly related to the question of how the availability and use of digital resources affect student behavior, learning, and relationships to the subject of mathematics. Brnic[27] reported on the long-term effects of a digital textbook on secondary students' achievement and their self-efficacy, where, so far, no significant effects were found for either construct. O'Meara et al.[29] investigated how Irish textbooks include examples, which show the relevance of mathematics in careers. Finding the lack of such examples in the textbooks, the authors created other materials that show the relevance of mathematics and tested them in a pilot study. Mwadzaangati[28] analyzed Malawian primary level mathematics textbooks in order to identify how the examples in these textbooks contributed to the goals of outcome-oriented education. The starting point of all three papers was that textbooks still appear to be the dominant resource used in classrooms across the world. While the paper by Mwadzaangati[28] focused on the contribution of the textbooks to achieve the goals of the official curriculum, the two papers by Brnic[27] and O'Meara et al.[29] focused on alternatives to the traditional textbook and explored the affordances of other resources for enhancing the learning of mathematics.

3.2. Theme 2

Papers in Groups 1, 2, and 3 contributed to Theme 2 (the design and development of print and digital mathematics textbooks and other resources). Rezat[6] raised the need for the theoretical foundations capable of guiding the development and design of mathematics textbooks and other resources. He offered a preliminary attempt at systematizing the empirically evaluated design principles and features of mathematics textbooks, and at forming a structural theory of textbook and curriculum material design.

Some contributions within this theme could be regarded as being related to the design of particular opportunities to learn that mathematics textbooks and other resources come to embody (or capture, or represent). Design research provides a methodological framework that aligns theoretical considerations in the design process with empirical evaluations of the design aims. Two papers exemplified these design processes, while using different methodological approaches. Kouropatov et al.[7] described the iterative design process of three digital resources related to transformations of functions. The paper demonstrated how mathematical and didactical considerations must go hand in hand with empirical evaluations in the design process to develop resources that have the desired learning effects. Similarly, Lisnani and Sariyasa[6] described the design and testing of a digital resource, in which they leveraged comics for learning integers.

A number of contributions to TSG-41 did not bring accounts of design principles or the design process but analyzed and evaluated the opportunities to learn as they were provided in mathematics textbooks (both historical and those currently in use). These papers used comparative analysis to highlight specific design features of textbooks from different countries, different historical periods, or different pedagogical approaches. Using a post-design perspective, such analyses may unveil the design principles and decisions as manifested in the resource from an a posteriori perspective, thus deepening our understanding of resource design. In particular, the contributed studies analyzed comparatively textbooks in China and the U.S.A. (Gao[10], Jia[16], Li[14]), China and Singapore (Li and Dong[17], Ran and Dong[18]); textbooks with different approaches in China (Shen and Bao[15]) and in the Netherlands (van Zanten and van den Heuvel-Panhuizen[13]), and textbooks from different historical periods in Hungary (Wintsche[12]). Finally, Lu and Xu[11] focused on the construction of a framework for analyses of this kind.

Alongside the majority of studies that explored designs from perspective of learning, researchers and designers remain aware that textbooks rarely teach students directly. As a result, the notion of textbook designs, which would not only provide opportunities to learn for students, but also *opportunities to teach* for teachers, surfaced

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[6] *Teaching and learning integers through ICT-based SI UNYIL comics*. Short paper. The paper was not assigned a discussion group, but the authors were able to attend TSG-41.

within the discussions related to this theme. Visnovska et al.[5] reported on the processes of developing resources that would be in service of teachers (as opposed to teachers having to be in service of the resource). They called for the design of the features that would contribute to teachers' re-claiming the agency over both the meanings and decisions associated with mathematics teaching. Similarly, Leshota[4] analyzed educative features in scripted mathematics lesson plans from South Africa and explored the opportunities that the materials presented for the teachers' learning and teaching.

3.3. *Theme 3*

Papers in groups 4, 5, and 6 contributed to Theme 3 (The use and implementation of print and digital mathematics textbooks and other resources and related interactions among resources, teachers, and students). Within this theme, several authors foregrounded the supports for teachers' use and implementation of textbooks and other resources. Kim[23] combined five components, that prior research identified as key for teachers' productive use of resources, into a framework and explored its usefulness with the U.S.A. pre-service teachers who learned how to use resources. Oakes and Lyakhova[21] used interviews, surveys and questionnaires to identify the extent to which postgraduate level resources were seen to support making mathematical connections and whether the users thought that this was improving their teaching and learning. Rafalska et al.[22] explored the linguistic and cultural supports, aiming to develop a deeper understanding of cultural differences in resource systems of teachers from China and Ukraine. Rahimah and Visnovska[19] analyzed an Indonesian textbook from the perspective of the kinds of supports it provided, or failed to provide, for teachers' implementation of a student-centered teaching approach.

Within the scope of investigations of teachers' resource systems and documentational trajectories, Sayah[20] explored how one Algerian teacher adopted and adapted the French Sésamath resources in their professional work, and how these resources fostered the collective work of this teacher. Rocha[24], in turn, analyzed the interaction of the individual and collective resources in the case of one French mathematics teacher, and Similarly, Yan[26] documented how one Chinese expert teacher used the textbook as a resource in her curricular design.

Additional papers explored students' use of curriculum resources and what sense the students were making of the presented content. Wang and Fan[9] surveyed Chinese and English secondary students' use of mathematics textbooks, highlighting striking differences in the perceived purposes for and uses of the textbooks. Mesa and Gerami[8] tracked U.S.A. university students' viewing of a digital textbook in real time and collected students' written narratives about their textbook use and reported on the methodological challenges faced in analyzing this type of data. Silveira and Powell[25] reported on the methodology developed for explorations of elementary students' understanding of visual representations of natural numbers and fractions in select Brazilian and U.S.A. textbooks.

4. Conclusion and Outlook

A number of ideas discussed within TSG-41 appear to be worth carrying forward in the research of future TSG participants.

Many contributions to TSG-41 had shown that textbook and resource designs differ, at times profoundly, for instance in different countries. It is certain that these differences contribute to how teachers can teach and how and what students get to learn and accomplish in mathematics classrooms. However, it is rarely clear which of the documented differences are to be attributed to specific design assumptions, principles, and processes, as those are rarely explicit, and often remain hidden from our views. In other words, for those of us whose research involves resource design, there is a need for more explicit sharing of the accounts and justifications of the principles that guide our current designs, as well as of processes that guide our theoretical and empirical evaluations of these principles in different cultural and institutional contexts.

The latter requirement calls for deeper empirical substantiation of the effects of textbook and resource design. While understanding differences in textbook and resource design as well as particular values and intentions and ways in which these are used to drive designs is important, it is also necessary, and still not common, to investigate whether or to what extent these values and intentions get realized when the textbooks are used in classrooms. Prediger's discussant contribution offered an inspirational example from the KOSIMA project of how research can be structured to intentionally contribute to the production of such accounts. She illustrated how ongoing cycles of empirical evaluations of both the task design and teacher supports were essential in the production of mathematics education resources that positively contributed to students' learning. This links to the need for the pursuit of analytical notions such as resources for teaching, teachers' resources, and opportunities to teach, which were brought up in several paper contributions and small group discussions during TSG-41.

In addition to the design work that is conducted by resource designers and by teachers, Pepin's discussant contribution drew attention of the role of students as designers of their own learning trajectories and thus as co-designers of curriculum. Accordingly, connectivity, in terms of making the connections among mathematical ideas within mathematical curricula explicit to the students, becomes a critical feature of curriculum design, as it plays a role in supporting students in navigating the resources needed for their own curriculum trajectories.

Finally, Cobb's discussant contribution focused on the role that textbooks and other resources (i.e., instructional materials) play within the broader concerted efforts at instructional improvement, especially when attempts are made to orchestrate these at the systems level. Drawing on the MIST project data, Cobb illustrated that the quality of instructional resources made a considerable difference in student achievement scores, even when teachers used the cognitively demanding tasks in somewhat proceduralized ways. This insight is important given that textbooks which are not composed of well-sequenced mathematical tasks of high cognitive demand are still widely available to schools and teachers. It should be concerning that textbooks of this kind directly contribute to portraying students in ways, which do not represent their capacity for mathematical learning, and they do so at scale. Cobb hastened to stress that in order for teachers to thrive while using high-quality teaching resources, teachers must be adequately supported in this use. How textbooks and other resources could be designed to better support teachers' work is worth of concerted research efforts.

To conclude, we would like to extend an invitation to researchers of textbooks and other curricular and instructional resources to join in the events specifically dedicated to this work, including the 4th International Conference on Mathematics Textbook Research and Development (ICMT 4) in Beijing (China) in October 2022, and the next iteration of this TSG (i.e., TSG3.12) at ICME-15 in Sydney (Australia), July 2024.

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https://doi.org/10.1142/9789811287152_0066

Topic Study Group 42

Research and Development in Assessment in Mathematics Education

Abid Sohail[1] and Caroline Long[2]

ABSTRACT TSG-42 provided a forum to share and discuss research and development in the field of Assessment in Mathematics Education. Many interesting and outstanding questions about the nature of interrelationships among assessment and teaching and learning of mathematics, were asked. Recent research has demonstrated the wide range of theoretical and methodological resources that can contribute to assessment in mathematics, including the use of technology. The papers in TSG-42 included reporting on a particular assessment topic or theme, providing the details of an empirical study, giving an exposition of particular assessment practice, or a reflecting on classroom-based assessment.

Keywords: Assessment; Formative assessment; Summative assessment; Large-Scale assessment, Assessment cycle.

1. Objectives

As teachers, practitioners, academics and researchers it is our prime responsibility to conceptualize, debate and formulate learning and assessment systems that prepare our future generations for opportunities and challenges that they may encounter. Assessment is a wide-ranging, multidimensional and vital process integral to teaching and learning. The purposes of assessment can be summarized as being formative, directed at the improvement of teaching and learning, and summative, where the focus is on evaluation of current proficiency, comparability, or evaluating the functioning of an education system as a whole. Various types and formats of assessment support these purposes. Each type of assessment with a well-defined purpose provides specific and useful information to improve standards and quality of teaching and learning. Also, this specific and useful information is beneficial for research. In classroom-based assessment, the interactive teaching, learning and assessment cycle is managed by the teacher, adhering to the perspective provided by policies, procedures and norms of the institutes or states. The assessment cycle may be specifically formulated for a particular cohort of students. In large-scale assessment, this cycle is somewhat extended and generalized to reflect the perspectives and processes which are applicable across countries rather than specific to a certain context. Large-scale assessments have the potential to provide comparative information about a country's curriculum and teaching

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practices generally. The purpose of such assessments, the design and development of instruments and the interpretation of results are factors that affect individual countries and influence internal assessment practices. In the 21st century, we have seen new trends and developments in the field of mathematics assessment, including the assessment of the set of skills that encompass creativity, collaboration, communication and problem-solving. New models have been introduced, many of which have encompassed computer-based testing. Also, in this century, the use of Item Response Theory and Rasch Measurement Theory has influenced the design of tests and the analysis and interpretation of results. For this TSG-42, we invited research based on the recent trends and developments in the field of mathematics assessment which cater to the needs of the 21st century. The research papers, presentations and discussions were such that they were beneficial, communicable and accessible to all the stakeholders, would inform a range of assessment practices, and therefore would contribute to making the teaching and learning of mathematics meaningful. Contributions included studies covering (but not limited to) the following themes:

- Theoretical, philosophical and ethical perspectives and debates concerning the assessment of mathematics proficiency;
- Alternative perspectives, models and practices of assessment;
- Classroom-based assessment (formal or informal assessment);
- Teachers and assessment. What is the role of the teacher in classroom-based assessment? What is the impact of high-stakes assessment? How does the phenomenon of "teaching to the test" play out in various contexts?
- Students and assessment. How do different types of assessment affect student learning and motivation? What is the role of feedback in a learner's life? What is the impact of standardised assessments on learning?

What to test? How is a cognitive focus or cognitive development focus accommodated in a testing programme? How is extended problem solving assessed?

- Test design, construction and administration (theoretical, technical and practical components). How do the underlying assumptions of classical test theory, item response theory and Rasch measurement theory affect the design of testing programmes?
- Technology and computer-based assessment;
- Large scale assessment (perspectives, benefits and limitations);
- Validity and reliability: whether or not a test may report dimensions and types of validity and reliability.

The statement that assessment drives learning was illustrated by many thought-provoking presentations at the conference.

2. Sessions

In TSG-42, there were 45 contributions in total 3 sessions. Sessions 1 lasted 90 minutes. There were 9 short oral presentations of 10 minutes. In the second time slot of 90 minutes, there were 6 long oral presentations of 15 minutes of duration. In the third time slot of 120 minutes, there were 2 long oral presentations of 15 minutes of duration and 9 short presentations. The details are as follows (Tab. 1).

Tab. 1. List of papers presented

Paper and author(s)
Session 1
[1] Students' difficulties in the management of algebraic expression highlighted in large-scale assessment. ***Federica Ferretti*** (Italy).
[2] In-service teachers marking students' answers containing derivation errors. ***Alberto Arnal-Bailera***, *José M. Muñoz-Escolan, and Antonio M. Oller-Marcén* (Spain).
[3] Investigating teachers' awareness of the reasons for students' math errors at primary school level. ***Valentina Vaccaro*** *and Eleonora Faggiano* (Italy).
[4] Cognitive load reduction in math items: performance, gender and socioeconomic status. ***Emiliano Augusto Chagas*** *and Mauricio Urban Kleinke* (Brazil).
[5] Expressions of mathematical proficiency in students' mathematical work. ***Priscila D. Corrêa*** (Canada).
[6] Structural features in classroom level standardized mathematics achievement results. ***Timothy Sibbald*** (Canada).
[7] Philosophical insights into PISA and mathematics education policy issues. ***Ian Cantley*** (UK).
[8] A unique item format to assess attentiveness to students' mathematical ideas. ***Ya Mo***, *Laurie Cavey, Michele Carney, Tatia Totorica, and Patrick Lowenthal* (USA).
[9] Developing preservice elementary teachers' capacity in the design of authentic mathematics assessment. ***Kim Koh***, *Olive Chapman, and Shimeng Liu* (Canada).
Session 2
[10] Evaluating mathematics teachers' professional learning in a PLN: a complex systems perspective. ***Xiong Wang*** (Canada).
[11] Validity of assessments in mathematical textbooks: a study of beginning of primary school level textbook assessments. ***Grapin Nadine*** (France).
[12] Are the stakes the same? A comparison of three types of large-scale assessments in Alberta, Canada. ***Richelle Marynowski*** (Canada).
[13] Factors related to mathematics teachers pedagogic discretion, specifically when evaluating parabolic sketches. *Shai Olsher and Kawthar* ***Nakhash Khalaila*** (Israel).
[14] Assessment based on gamification in Hungarian secondary mathematics classes. ***Marta Barbarics*** (Hungry).
[15] I know all about this mathematical topic, but I cannot answer this question' moment, can I have a clue please? ***Anne D'Arcy-Warmington*** (Australia).
Session 3
[16] Investigating the treatment of missing data in an Olympiad-type test — the case for selection validity. ***Caroline Long***, *Johann Engelbrecht, and Vanessa Scherman* (South Africa).
[17] Mathematics assessment practices of primary school teachers in France. *Nathalie Sayac and* ***Michiel Veldhuis*** (France).
[18] The role of formative assessment experiences in the teaching and learning of mathematics. ***Adri van der Nest***, *Caroline Long, and Johann Engelbrecht (*South Africa).
[19] Assessing math in teacher training; what to learn from our students' research. ***Willem van der Vegt*** (The Netherlands).
[20] Transformative assessment system in mathematics education: engaging mind, body and soul. ***Basanta Raj Lamichhane*** (Nepal).
[21] Analysing students' errors in solving context-based problems in Marwa assessment. ***Ummy Salmah***, *Uki Rahmawati and Bungkus Dias Prasetyo* (Indonesia).
[22] Raw scores or rasch measures? Lessons from Rasch analysis of secondary one mathematics test. *Hairon Salleh,* **Foo Kum Fong**, *and Koh Wei Xun* (Singapore).
[23] Research on the level division of mathematical logical reasoning literacy based on solo taxonomy theory. *Hua Wu,* ***Junhan Liu***, *and Fengqi Zhai* (China).
[24] Quality of mathematical reasoning in a Philippine senior high school's pre-calculus examinations on conic sections. ***Vitus Paul L. de Jesus*** (Philippines).
[25] The Results of large-scale assessment as tools for mathematics activity design. *Alessandro Gambini and* ***Roberto Capone*** (Italy).
[26] Research on the assessment system combining standardization and non-standardization in the mathematics education of top talents. ***Jianren Niu***, *Li Lai, Chaodong Chen, Zhirong He, and Liang Yang* (China).

3. Posters

The presenters from different regions of the globe also shared their learning experiences through poster presentations. These posters provided a chance to look at different practices and innovate ideas from/for practitioners and researchers. These were 10 posters. The details of the posters are as follows (Tab. 2).

Tab. 2. List of posters presented

Paper and author(s)	
[27]	Existing assessment practices: a detrimental factor for the value of cognitive diversity in mathematics classroom. ***Shiva Datta Dawadi*** (Nepal).
[28]	Vertically equating the PSM3 and PSM4. ***Jonathan D. Bostic****, Gabriel T. Matney, Toni A. Sondergeld, and Gregory Stone* (USA).
[29]	Contents-specifics in teachers' assessment of non-cognitive skills in mathematics education. ***Enomoto Satoshi****, Iwata Koji, Sasa Hiroyuki, Nakagawa Hiroyuki, and Aoyama Kazuhiro* (Japan).
[30]	A case study of the assessment process in Japanese math classes. ***Shigeki Kitajima*** (Japan).
[31]	Impact of the standardized test in the classroom: a proposal from the socio-epistemological theory of educational mathematics. ***Beatriz Elena Martínez Díaz*** *and Ricardo Arnoldo Cantoral Uriza* (Mexico).
[32]	Semi-automated assessment for mathematical proficiency: the ultimate time-saver for extensive feedback and reliable grades? ***Filip Moons*** *and Ellen Vandervieren* (Belgium).
[33]	Standardized testing administration time differences on problem-solving outcomes. *Toni A. Sondergeld, Gregory E. Stone, Jonathan Bostic, and Gabriel Matney* (USA).
[34]	Development of mathematics items with dynamic objects for computer-based assessment using tablet PC. *Fumiko Yasuno, Keiichi Nishimura, Seiya Negami, and Yukihiko Namikawa, Jin-ichi Itoh* (Japan).
[35]	Making classroom assessment happen in novice teachers' class through assessment techniques design. *Xiaoyan Zhao and* ***Lingchun Kong*** (China).
[36]	Training and e-assessment of mathematical courses by Xpress-tutor. ***Philip Slobodsky****, Mariana Durcheva* (Israel), *and Alexander Ocheretovy* (Russia).

4. Way Forward

The field of research in mathematics education assessment is broad and exciting. Submissions were received from countries across the globe. The range of papers included research focused on large-scale studies and national studies, on both primary school, high school, and tertiary levels, but what was most interesting is the number of papers dealing with alternate type assessment.

Included in the large-scale assessment were philosophical insights into PISA (Northern Ireland)[7], a comparison of different types of large-scale assessment (Canada)[12], and an investigation into the handling of missing data (South Africa)[16]. We regard the scrutiny of large-scale assessments both for what they can reveal to education systems, and for their limitations, as important themes to be taken up at future conferences.

Highlights of the papers focused on primary school included assessment practices specific to the primary school (France)[11,17], and the validity of the textbook assessments (France)[11]. Focused on the high school were papers on specific mathematical topics. It was most interesting to listen to papers that pushed the boundaries of conventional assessment, and included a number of factors apart from the merely academic that impacted on achievement. A study looking at engaging mind, body and soul, emanated from Nepal[20], a focus on mathematical reasoning emerged from the Philippines[24], and the focus on context-based problems emerged from Indonesia[21]. While the focus on standard assessment and the improvement of this type of assessment is needed, the investigation of assessment which is not in the main stream is most important for the future. We envisage more papers that are looking at alternate forms of assessment, and that assess what has not been assessed previously. Gert Biesta, originally from the Netherlands, challenged the education community to not only value what can be measured. In the case of many of the papers in ICME-14, there is the attempt to measure what is valued, though this is not always easy.

At the tertiary level, the design of authentic assessment was presented (Canada)[9], as well as the assessment of mathematics teachers professional learning (Canada)[10]. A further study looked at the assessment of mathematics in teacher training (Netherlands)[19]. Poonam Batra, working in teacher education in India, stated that if we want change in our educational systems, we have to empower the teachers, and enable professional teacher agency. How assessment practices can assist this is a good question.

Overall the tendency for research in mathematics education assessment is to broaden its scope to include innovative and all-encompassing characteristics. This trend we value as we look to the future. On the other hand, the improvement of standard assessments to ensure validity and reliability should remain a focus on this topic specific group.

As for meaningful engagement with teaching and learning, the role of technology is critical. In future, technology will not only be a tool of learning and teaching rather it will become a part of the process. Henceforward, besides looking at other important aspects, types and functions of assessments, the role of technology in assessment needs our attention.

We look forward to welcoming an equally diverse range of papers that both improves existing practices and offer new practices.

https://doi.org/10.1142/9789811287152_0067

Topic Study Group 43

Research and Development in Testing (National and International) in Mathematics Education

Ivan Vysotskiy[1], Julia Tyurina[2], and Anastasiia Demchenckova[2]

1. Aims of TSG-43

The group based on topics related to the testing as an important aspect of evaluation and also its links in the reform of the mathematics curriculum at the national and international level. The presentations are studies on several interconnected topics.

1) What are the results of development and research of the testing with mathematics content and mathematics competency, their connection?
2) How effective is the use of the LMT measures as a way of comparing teachers' international mathematical knowledge based on various concepts of the necessary knowledge, and hence views on the possibility of measuring them.
3) How typical characteristics of each country can influence the organization of their educational system.
4) What is the evaluation of mathematics education and how examination of the new mathematical knowledge can guide the teaching behavior development along the scientific and rational directions.
5) What is the role of mathematics in educational policies?

1.1. Organizing team

Representatives from different countries were involved in TSG-43 as chair, co-chair, moderators and speakers: China, Japan, the Russian Federation, United States, Brazil, Slovakia, and Norway, that highlights co-thinking and common views not only national, but also international collaborative community of people acting in mathematical education.

The organizing team of the TSG consists of

Chair: Ivan Vysotskiy (Moscow Centre for Continuous Math Education, Russia)
Co-chair: Fumi Ginshima (National Institute for Educational Policy Research, Japan)

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[2] Moscow Centre for Continuous Math Education, Moscow, Russia.

Members:

Richard T. Houang (Michigan State University, USA)

Maria Isabel Ramalho Ortigão (Rio de Janeiro State University, Brazil)

Lidong Wang (Beijing Normal University, China)

1.2. Other participants

Besides these team members, participants also include:

Ekaterina Kuksa (Moscow Centre for Continuous Math Education, Russia)

Tibor Marcinek (Central Michigan University, USA)

Jiangong Dong (Wuhu Institute of Educational Science, China)

Joaquim Pinto (Universidade de Aveiro, Portugal)

Bruno Damien da Costa Paes Jürgensen (State University of Campinas, Brazil)

Yu Fu (Beijing Normal University, China)

2. Sessions and presentations

There were 6 high quality presentations in TSG-43. The ICME organizing committee granted the TSG 2 sessions for presentations, but one of them wasn't presented (Tab. 1). Thus, during the hybrid conference only 5 papers were presented.

Tab. 1. List of the presentations

Paper and author(s)	
[1]	On the eighth-grade mathematics achievement and its effect factors — based on seven areas study. *Chunxia Qi* (China), *Ruilin Wang* (China), *Qi Huang* (USA), *and* ***Yu Fu*** (China).
[2]	International comparisons of teacher knowledge: the case of the LMT measures. ***Tibor Marcinek*** (USA), *Arne Jakobsen* (Norway), *and Edita Partová* (Slovakia).
[3]	PISA assessment of Brazilian students' mathematical literacy. ***Maria Isabel Ramalho Ortigão*** (Brazil).
[4]	On composing distractors for multiple choice problems. ***Kuksa Ekaterina*** (wasn't presented) (Russia).
[5]	How Chinese design mathematics test. ***Jiangong Dong*** (China).
[6]	Reflections on large-scale assessment and the formatting power of mathematics. ***Bruno Damien da Costa Paes Jürgensen and*** *Mara Regina Lemes De Sordi* (Brazil)

In the first 90-minute session, the Chair of TSG-43, Ivan Vysotskiy, introduced the rest of the Team, described the format of the session and conveyed greetings to all participants from the organizers of the Congress and PC, called on the participants to the most informal discussion possible. As a rule, each presentation took 20–10 minutes, followed by a 10-minute collective discussion. Throughout the two days, TSG-43 reviewed all the research related to the topic of testing, discussed the problems that arise during the preparation, testing, and discussed possible solutions.

The majority of the work can be summarized across five connected themes. As part of the work, researchers consistently, step by step, revealed the main topic of research and development in testing (national and international) in mathematics education.

We give a brief overview of the presentations as follows.

2.1. Session 1

First talk[1] given by Fu was on the work collaborated by Qi, Wang and Huang. They tried to design math test and questionnaire for 170748 eighth grade students from seven areas in Chinese mainland. The results showed an inextricable relationship of two parts of the test — mathematics content, and mathematics competency. The mathematics content includes three areas: number and algebra, shape and space, statistics and probability. The mathematics competencies consist of 4 components: knowing, understanding, grasping and application. Eight-grade students achieved high scores, and their pass rate was 92.4%. However, their level of application was lower than the level of knowing, understanding, and grasping. Among all the four components in questionnaire survey, teaching representation had the greatest impact on students' performance, teaching methods and mathematics representation were the second while thinking tendency had a negative effect on students' performance. They also confirmed that development of tools in test booklet and the influence factor in the questionnaire need to be further explored.

The talk[2] given by Marcinek was on his work collaborated by Jakobsen and Partová. It draws on the body of international research conducted in relation to the measures of the mathematical knowledge for teaching developed as part of the Learning Mathematics for Teaching project at the University of Michigan (the LMT measures). There are various conceptualizations of the knowledge teachers need in the work of teaching mathematics. Which means there are different views on feasibility of measuring such knowledge. There were made comparisons between Slovak and Norwegian primary teachers. Three layers of studying were used — Technical layer, Local layer, Global layer. The similarities helped to compare translation-related issues while differences helped to model the issues that likely exist in education systems around the world. The possibility to design LMT forms with close psychometric properties does not imply the meaningful comparisons of teachers' MKT. Local layer provides the most useful information to researchers and constitutes the most important contribution of LMT adaptation literature. Such comparisons might be especially hard to interpret if countries with differences in curriculum, grade bandings, teacher education and certification are compared.

Ortigão[3] presented the results of an investigation focused on analyzing PISA Mathematics items based on Differential Item Functioning (DIF) analysis, to compare results between Brazil and Portugal. Based on descriptive analysis were differences in cognitive skills between the assessed groups. Typical characteristics of each country can influence the organization of their educational system. Knowing these features based on items that favor certain groups, in addition to perceiving the incidence of patterns, is undoubtedly the great contribution from DIF (the identification of items that disregard one of the main assumptions of the IRT) analysis to educational assessment.

2.2. Session 2

On the final session, the participants discussed two topics.

First, Dong[5] called for discussion on the paper of the tests as aspect of evaluation and as an important link in the reform of the mathematics curriculum in middle school. The evaluation of mathematics education includes not only curriculum evaluation but also teaching evaluation, and mathematics testing is an important aspect of the evaluation of mathematics education. The function of testing can be divided into two types — first is to summarize the results of the entire mathematics education stage, to identify the teaching effectiveness or achievement at the end of the whole teaching stage, the other is to gain feedback with purpose to improve the teaching process, to understand the problems and defects in teaching process at the end of each teaching unit. The focus of the study on mathematics evaluation question setting in Dong's paper is on the level of senior high school entrance examination. Improving the quality of question setters and making the tests papers play an evaluation function more scientifically and reasonably is the top priority. Dong suggested certain basic principles for test items setting, and emphasized that the teachers creating the tests questions must firstly understand the structure, and then start from every single question. The new mathematical knowledge has enriched and optimized the mathematical thinking method in the middle school mathematics, further expanded the application space of the knowledge, which will be an important source of original test questions. Examination of the new mathematical knowledge and scientific solving of a certain type of problems step by step, may better guide the teaching behavior development along the scientific and rational directions.

The main goal of another research[6] by Jürgensen and De Sordi was to explore and reflect upon the issues concerning largescale assessments in São Paulo, Brazil, and the market-oriented policies underlying them. It was about the role of mathematics in educational policies, since they have a great impact on teachers, their everyday life in schools and, consequently, on society as a whole and the formation of students. During the research they did Questionnaire answered by 26 mathematics teachers and Interviews with 10 of them on this topic. The state began to adopt differentiated payment of teachers and other state education employees, through a salary bonus according to the achievement of goals set by the government for each school. External and standardized assessments impose a teaching and learning method that does not consider the different realities. Standardization of teaching (in authors' opinion) is flawed. The expense or cost of these assessments is too high in relation to their return. The authors believe that the answers, the results obtained, should change this evaluation system, as it has been in the same way for a long time and they have not observed significant changes. There should have an investment in teacher training and better working conditions to thus develop the skills expected for students, and not just charge in the form of assessment and teacher's penalty through bonuses cancel. More than describing the educational reality, the indicators of quality produced by mathematical models and their subsequent dissemination have altered the work routine of teachers and school teams. Mathematics, in general, goes unnoticed, invisible, in these processes, mainly because it comes "in a package" (Skovsmose and Yasukawa, 2009). Therefore, it is pertinent to open the package and proceed to the questions posed by the authors: "what's in the package?", "Whose package is it?" And "what is done by means of the package?". It is necessary to look beyond technical specifications to reverse this picture and to think of an assessment that addresses the complexity of the educational process (including mathematics) and makes it accessible to the general public. Everyone can and should participate in the definition of what is a "good quality" for education, as long as everyone is involved in this endeavor since its elaboration.

Reference

Ole Skovsmose and Keiko Yasukawa (2009). Formatting Power of "Mathematics in a Package": A Challenge for Social Theorising? In: *Critical Issues in Mathematics Education,* Paul Ernest, Brian Greer, and Bharath Sriraman (eds.). Information Age Publishing.

https://doi.org/10.1142/9789811287152_0068

Topic Study Group 44

Mathematics and Interdisciplinary Education

Carl Winsløw[1], Rita Borromeo Ferri[2], Nicholas Mousoulides[3], and Avenilde Romo-Vasquez[4]

1. Aims of the TSG

The preceding TSG on “Interdisciplinary Mathematics Education” at ICME-13 in Hamburg produced, among other things, a conceptual classification of the general notion of “discipline” as well as the terms used to indicate various degrees of interaction between mathematics and other disciplines (Williams et al., 2016).

The present TSG, as the slight change in title suggests, took a more specific point of departure in mathematics in its current educational and societal shapes, viewed as social realities: namely, mathematics as taught from preschool to higher education, and mathematics as a more widely established set of social practices (such as in academic research) viewed broadly to include also statistics and what is sometimes referred to as “applied mathematics”. The main goal of the group was to share studies and in-depth cases of the ways in which mathematics currently interacts — or is supposed to interact with other educational practices, in part reflecting the role of mathematical practices in society at large. We focused in particular on current or potential contributions of mathematical theory and techniques to elucidate the “big” questions which are increasingly emphasized in general schooling in many countries, such as sustainable development, and the roles and nature of digital technologies in modern society. This concerns how school mathematics functions as preparation for general citizenship, and also for more specific professional specializations. In other words, we focused on ways in which mathematics is or could be taught in a “paradigm of questioning the world”, rather than in a “paradigm of visiting monuments” (Chevallard, 2015).

2. Submissions and Sessions

We received 26 submissions from 19 countries (Europe: 9; North America: 7; Asia: 6; Australia 3; Africa: 1). Out of these 26 submissions, 7 were accepted as long oral presentations, 16 as short oral presentations, 2 as posters, and 1 was rejected. These numbers reflect the total number of papers and posters, submitted mainly at the ordinary deadline in 2020, but with two being added at the extraordinary deadline in 2021.

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The sessions were initially (for 2020) planned with parallel session for short orals. Unfortunately, at the online conference in 2021, not all accepted contributions could be presented due to the authors' choice of not attending the congress. In the actual congress, we ended up with having 6 long oral presentations (LO) and 6 short oral presentations (SO), see Tab. 1, all followed by an opportunity to raise questions and comments in a discussion with the presenter(s). The three sessions were attended by about 25 online participants and 2–5 onsite participants. Posters were presented at the general poster presentation of the congress.

Tab. 1. List of papers presented

Paper and author(s)	
Session 1	
[1]	Interdisciplinary mathematics education: some reflections from the anthropological theory of the didactic. ***Francisco Javier García*** (Spain). (LO)
[2]	Interdisciplinary Inquiry-Based learning with queueing situations: investigating the questions triggering mathematical activities. ***Yuici Nezu*** *and Takeshi Miyakawa* (Japan). (LO)
[3]	A classroom experience: vector concept. ***Viana Nallely García-Salmerón*** *and Flor Monserrat Rodríguez Vásquez* (Mexico). (SO)
[4]	Students' use of geometric cues in an art studio: scaling of artworks. ***Mehtap Kus*** *and Erdinc Cakiroglu* (Turkey). (SO)
Session 2	
[5]	Posing a generating question within the pedagogy of questioning the world: the case of GPS coordinates. *Lenin Augusto Cepeda,* ***Avenilde Romo Vázquez****, and Luis Ramón González* (Mexico). (LO)
[6]	Mathematics and financial education: how do they intersect together? ***Annie Savard*** *and Alexandre Cavalcante* (Canada). (LO)
[7]	Task design features for integrating covariational reasoning with science. ***Debasmita Basu*** *and Nicole Panorku* (USA). (SO)
[8]	STEM projects as didactical situations in mathematics: theoretical frame to construct algebraic institutional meanings. ***Aitzol Lasa****, Miguel Wilhelmi, Olga Belletich, Jaione Abaurrea, and Haritz Iribas* (Spain). (SO)
[9]	Physcial measurements as an environment supporting primary pupils' reasoning about central tendency. ***Lúbomíra Valovičová*** *and Janka Medová* (Slovakia). (SO)
Session 3	
[10]	Questioning interdisciplinarity within teacher education: a module on the evolution of the COVID-19 pandemic. *Laura Branchetti and* **Eleonora Barelli** (Italy), *Berta Barquero, and Oscar Romero* (Spain). (LO)
[11]	A situation of interdisciplinary mathematics education in context of protection of water resources. ***Thi Nga Nguyen****, Thien Thanh Lam, and Minh Dung Tang* (Vietnam). (LO)
[12]	Transdisciplinary and interdisciplinary mathematics in the international baccalaureate. ***Sarah Christina Phillips*** (Canada) *and Jan Mills* (New Zealand). (SO)

3. Paper Topics

Following the call for papers, the papers were classified as far as possible according to the three subthemes of the TSG:

- Subtheme 1: *Mathematics and the study of nature*: here we consider the ways in which mathematics interacts with teaching and learning of subjects such as physics, biology, chemistry etc.
- Subtheme 2: *Mathematics and technology*: interactions with the study and use

of technology in a broad sense, comprising digital technologies, technological innovation and engineering
- Subtheme 3: *Mathematics and the study of human activity and society*, including business and enterprises, economy, creative fields such as art and music, philosophy, history etc.

The specific contents of the presented papers allow to identify several new tendencies and approaches, as is to some extent reflected from the paper titles listed in Tab. 1. The papers certainly included examples and results corresponding to all three subthemes, reflecting a variety of experimental and emergent practices and concerns in mathematics teaching around the world. At the same time, the challenge of mathematics and other disciplines is considered in wider perspectives than has been traditionally the case in this area of research. In particular, 4 of 12 papers refer to the "paradigm of questioning the world" (Chevallard, 2015) and base their experiments and reflections on associated constructs from the Anthropological Theory of the Didactic, such as "study and research paths" and "ecology and economy" (of teaching designs). The range and originality of interdisciplinary phenomena examined by the papers is impressive, and relates to all grade levels including higher education. It is also clear that despite the variation in institutional constraints across the world, the same phenomena (from water resources to satellite based navigation and art design) are increasingly of importance to all of humanity. Several of the studies involve real life data and activities involving data collection, so that the numerical or geometrical aspect of a phenomenon is not merely shown through a table or drawing in a textbook. As always, researchers and special resources are deployed in such experiments, and the question of sustainability arises. In this respect, some of the papers reflect on the fact that teachers face new requirements and challenges if they wish or need to organize interdisciplinary teaching.

4. Discussion and Areas for Future Research

The discussions at the congress allowed participants to get more insight into research methods and teaching practices from around the world. In all countries, interactions between mathematics and other school disciplines seems to be officially encouraged or even prescribed, at least to some degree. However, in some countries, this impetus has come more recently. Both in such countries, but also in other ones, there are still a number of constraints that can inhibit or weaken students' experiences of such interactions. Teachers' educational background is frequently not geared towards independent inquiry across disciplines, or even towards recognizing mathematical components in questions that involve other disciplines as well. Teachers may also be hesitant to orient their teaching towards such inquiry, for many other reasons — including perceived lack of time. Teaching resources for mathematics are, when it comes to relations with other disciplines, often limited to rather sterile examples made to "illustrate" some mathematical technique. On the other hand, experiments as carried out by the authors of several of the papers, do confirm that these constraints can be modified locally. With appropriate support, teachers can be successful with much more ambitious designs for students' experience of the many ways in which mathematical techniques and models appear in other disciplines and, more generally, in descriptions and solutions of important problems for humans and their societies.

There was a strong agreement among participants in the group that teacher education needs to focus more on interdisciplinary learning and teaching of mathematics in school. At present, few if any teacher education programs are geared for interdisciplinary teaching. One reason for this is that teachers need to study more than one discipline to gain some depth in view of the interdisciplinary setting. The study of several disciplines for being a teacher at secondary school is not included in the teacher education programs of most countries, with some exceptions like Germany and Denmark, where two subjects must be studied. But even teachers who studied two disciplines may not teach more interdisciplinary subjects, it does make it easier to combine the two subjects studied, such as mathematics and geography, in the lessons. Doing a project with colleagues from other disciplines makes sense for the learners, but is difficult to implement as a common element in everyday school life. Whether two disciplines are studied or a project is implemented with colleagues, teachers should be made aware of the importance of this interdisciplinarity. Learners benefit especially if they also experience the interrelationships of, for example, the STEAM disciplines on a metacognitive level. This could mean, on the one hand, that teachers have designed their learning unit in terms of the cross-link approach (Borromeo Ferri et al., 2019). One can speak of cross-linking, if at least two (scientific) disciplines are combined during one lesson/project or within the whole lesson-unit/project and are reflected with learners on a metacognitive level (Borromeo Ferri et al., 2019).

It becomes obvious that besides the challenge of interdisciplinarity, there is always the question of which methods can be used in the classroom. Inquiry-based learning, for example, touched upon in several papers presented in this group, is often a promising approach. Reflectively, however, the question should also be asked whether all disciplines are suitable for interdisciplinary work? Which ones fit together particularly well? These questions are useful with regard to the development of interdisciplinary learning environments for specific issues. However, it is important to note that it is through interdisciplinary work that different perspectives should be required of learners, to develop their critical engagement with a topic. Critical engagement or thinking is again one of the central four skills described in the 21st Century Skills, the 4 Cs: Creativity, Critical Thinking, Communication, and Collaboration (e.g., Rotherham et al., 2010). However, creativity, communication, and collaboration can also be promoted and fostered when it comes to issues that learners can solve in an interdisciplinary learning environment.

Furthermore, there is still a research desideratum for more studies that also explore the effects of interdisciplinary learning, for example regarding motivation, or performance in mathematics. If different effects of interdisciplinary learning were investigated more systematically, then one would have a good basis for impact on educational policy including the necessary changes in teacher training.

The topic of interdisciplinarity is becoming more and more prominent in mathematics education, especially for the reasons mentioned above, such as the promotion of 21st Century Skills for the present and future generations. During ICME 14, in the TSG on mathematical modeling, the topic of interdisciplinarity was alo discussed in a separate session in a panel. Real-world problems, which do not necessarily have to be very complex, require solutions from different disciplines. The focus on interdisciplinarity in mathematics learning is interesting for many research areas in mathematics education. The clarification of basic concepts, such as mono-, multi-, inter-, trans- and meta-disciplinarity, has already been well elaborated by Williams et al. (2016). Based on these terms, each research area that focuses on the topic of interdisciplinarity could now work out specifics regarding the topic area in terms of research questions and finally also regarding the practical implementation.

The dialogue between the papers and their authors should continue. The shared character of both constraints and potential resources makes it evident that more, and more international, research is needed in this area — research that goes beyond the local and national contexts considered in the vast majority of studies, here and elsewhere. We also note that only a few of the papers presented here involve, for instance, physics education researchers among the authors. As already mentioned, collaboration between teachers from different disciplines is most evidently needed at schools in order to realize truly interdisciplinary experiences for students. So perhaps more collaboration of researchers from different didactical horizons — not only from mathematics education — is equally needed to engage more efficiently with the issues at the international level?

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https://doi.org/10.1142/9789811287152_0069

Topic Study Group 45

Mathematics for Non-specialists/Mathematics as a Service Subject at Tertiary Level

Burkhard Alpers[1] and Mitsuru Kawazoe[2]

ABSTRACT This contribution describes the themes and programme of the topic study group on mathematics for non-specialist/mathematics as a service subject at ICME-14 and outlines directions of future work.

Keywords: Mathematics for non-specialists; Service mathematics.

1. Working Team, Themes and Questions

The working team for Topic Study Group 45 consisted of the following team members: Burkhard Alpers (Germany, Chair), Mitsuru Kawazoe (Japan, Co-Chair), Marta Caligaris (Argentina), Olov Viirman (Sweden), and Jing Zeng (China). This TSG dealt with the specialties of mathematics as a service subject, i.e. mathematics education provided as service in application study courses. The latter comprise all kinds of study courses in natural sciences, engineering, business and economy where more advanced mathematical terms and models are used but also study courses where mainly statistical methods are applied as in medicine and social sciences. The main educational goal of service mathematics consists of enabling students to understand and use the mathematical concepts, models and procedures as they are needed in their application study courses as well as in later job profiles. Essential questions related to the didactics of service mathematics are:

- Which understanding and competencies are needed in application subjects (like mechanics, national economics, experimental pedagogic) in order to understand the terms and development of models and to work on tasks successfully? How can this information be used to specify a curriculum for a specific study course? By which suitable learning arrangements (e.g. application problems and projects) can such competencies be acquired by students?
- How can mathematics be made relevant for students in application study courses such that students experience mathematics education as integral part of their study course and are thus motivated to undertake the necessary efforts?

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- What are the mathematical transition problems when students enter university in an application study course and what are suitable measures to overcome them?
- What is the influence and role of technology in service mathematics courses? How does the existence of technology embodying mathematical concepts and procedures change the goals of mathematics and the ways of teaching and learning?
- Who teaches service mathematics and how do different backgrounds influence the teaching practices? What are suitable boundary conditions for successful teaching?
- What are promising research designs in service mathematics and how can different roles be integrated: mathematician, mathematics educator, application specialist?

Some of these questions were tackled in the paper and poster contributions to this TSG which will be outlined in the next section.

2. Program Overview

The topic study group's programme consisted of three sessions for paper presentations as well as a small part of the overall poster session. Because of the worldwide Covid-19 pandemic, most participants attended online via a video conferencing system which worked quite well for presentation and discussion. In the video meetings about 14–18 participants attended and there was a lively discussion on topics addressed in the presentations.

The following papers (Tab. 1) were accepted for the conference all of which except for the one by Lehmann[6] were presented at the paper sessions:

Tab. 1. List of papers presented

Paper and author(s)	
[1]	Mathematics as a service subject: historical development and major players from a European perspective. ***Burkhard Alpers*** (Germany).
[2]	A practice report on mathematical modelling education for humanities and social sciences students. ***Mitsuru Kawazoe*** (Japan).
[3]	Flexible content, instruction, and assessment in a university-level quantitative reasoning course. ***Deependra Budhathoki****, Gregory D. Foley, and Stephen N. Shadik* (USA).
[4]	A small-scale implementation of inquiry-based teaching in a single-variable calculus course for first-year engineering students. ***Olov Viirman*** *and Irina Pettersson* (Sweden).
[5]	Sometimes mathematics is different in electrical engineering. ***Jana Peters*** *and Reinhard Hochmuth* (Germany).
[6]	Which mathematics competences are relevant for engineering education? — a mixed methods study. ***Malte Lehmann*** (Germany).
[7]	The attitudes of lecturers and students towards puzzle-based learning: the case of differential equations. ***Farzad Radmehr*** (Norway/Iran), *Faezeh Rezvanifard* (Iran), *and Michael Drake* (New Zealand).
[8]	Can we make mathematics interesting for engineering students? modelling tasks in an ordinary differential equations course. ***Svitlana Rogovchenko*** (Norway).
[9]	Teaching materials on calculus as seen from application to engineering. ***Satoru Takagi****, Kesayoshi Hadano, and Sei-ichi Yamaguchi* (Japan).

The paper by Alpers[1] served to set the scene by giving a historical overview of the development of didactical thinking about the provision of service mathematics. Particularly during the last two decades the topic received more and more attention in research and in communities of practitioners but the area is still under-researched and many open questions need to be investigated. The other papers dealt with mathematics education for non-mathematics majors in different types of study courses. Only Kawazoe[2] and Budhathoki et al.[3] were concerned with mathematics education in a non-engineering study course whereas the other papers investigated aspects of mathematics teaching in engineering where mathematics is well acknowledged as being a fundamental subject. Kawazoe[2] described a concept for making mathematics education more relevant for students of social sciences and psychology where students perform modelling activities in groups. Since the concept has been in use for about a decade, long-term experience and necessary modifications could be reported. The paper by Budhathoki et al.[3] described an entry-level course on quantitative reasoning for non-STEM majors where students were given many opportunities for collaboration in problem solving. Since the course content was not fixed, the instructor could flexibly react to students' needs. The students who had rather negative prior experience reacted positively to this form of education.

The remaining papers were all related to teaching mathematics in engineering study courses. Viirman and Pettersson[4] described a small-scale implementation of inquiry-based teaching in a first-year calculus course where students investigated application problems in group sessions. Since this involved the meaningful usage of concurrently learnt mathematical concepts, students gained a better and deeper understanding of those concepts. The authors used commognitive theory as a framework for their research. Peters and Hochmuth[5] addressed differences in mathematical practices between the usage of mathematics in application subjects and the development of theory in the proper mathematics education classes. They used the Anthropological Theory of Didactics as a theoretical framework for their analysis and made suggestions for dealing with students' problems resulting from the differences. Lehmann[6] investigated how first-year engineering students solved physics problems using the theoretical concept of epistemic games for analysis. He found out that higher mathematical content knowledge was a good predictor for success in physical problem solving. Moreover, the students' problem solving behaviour developed more into the direction of schematic work ("recursive plug and chug game") where they tried to use a well-known schematic algorithm for problem solving which he related to the schematic use of mathematics in their mathematics education.

The next two papers[7,8] addressed the problem of making differential equation courses more relevant and interesting for engineering students. Radmehr et al.[7] investigated the effects of puzzle-based learning on the attitudes of lecturers and students where a puzzle is a non-standard, open question with an entertaining component. Based on a questionnaire and interviews the authors found out that a majority of students enjoyed working on this kind of problems and thought that this work improved their understanding and ability to solve realistic problems. Rogovchenko[8] tried to make mathematics more relevant to engineering students by introducing assessed modelling tasks in a differential equations course. Using the framework of activity theory, she found some contradictions between teachers' and students' goals where teachers strived for deeper understanding whereas students were predominantly interested in getting better grades. Conflicts between students occurred due to different backgrounds, work preferences and mathematical skills. Finally, Takagi et al.[9] developed teaching materials on calculus where they started with an application which made the subsequent concept relevant to the students and enabled them to attach practical meaning to those concepts. They illustrated their approach by providing example applications for introducing partial derivatives and double integrals.

The following posters (Tab. 2) were presented at the poster session.

Tab. 2. List of posters presented

Poster and author(s)	
[10]	Peer-assisted learning in less structured courses: a case study in a first-year course on mathematical modelling. ***William Man Yin Cheung*** (Hong Kong SAR, China).
[11]	Implementation of projects about scheduled in software r in a linear algebra course for students of business computing career at the University of Costa Rica. ***Luis Eduardo Amaya*** (Costa Rica).
[12]	On the mathematical knowledge, skills and related information technology needed to pave the way for students' career development. ***Jiao Liu*** (China).

Cheung's poster[10] investigated the effect of peer-assisted learning on increasing the interest and "sense of belonging" of non-mathematics majors in a course on mathematical modelling. The poster by Amaya[11] was concerned with using the statistics programming language R to get business computing students more interested in linear algebra topics, and Liu's poster[12] described her ongoing work on adapting the mathematics teaching of vocational students to their real needs using spreadsheet technology.

3. Future Directions and Suggestions

A more comprehensive overview of the state of the art in the didactics of service mathematics can be found in the report (Alpers, 2020) which shows that research on this topic has gained considerable momentum. Yet, as can be seen by comparing the questions stated in the first section with the results of TSG-45 presented in the second one, there are still many areas worth further investigation. Since there are many application study courses with special needs and requirements, there is still a plethora of research questions to be tackled in the decades to come.

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https://doi.org/10.1142/9789811287152_0070

Topic Study Group 46

Mathematical Competitions and Other Challenging Activities

Boris Koichu[1], Peter Taylor[2], Ingrid Semanišinová[3], Yijun Yao[4], and Sergei Dorichenko[5]

ABSTRACT The joint focus of TSG-46 on mathematics competitions and other challenging activities is devised in recognition of the fact that all students benefit from studying mathematics through challenging activities but some students do not like to compete. This group gathered mathematicians, teachers, mathematics educators and mathematics education researchers and served as a stage for presentations and discussions related to the following themes: (i) Organizational formats for challenging students mathematically, (ii) Research on students' experiences with mathematically challenging activities, (iii) Characterizing and theorizing mathematical challenge, and (iv) Competition problems as impetus for mathematical research and discoveries.

Keywords: Mathematical challenge; Competitions; Characterization of tasks; Discovery.

1. Background and Agenda

The mathematics competitions movement emerged more than a century ago as a means to engage bright schoolchildren in mathematical activities that would be more challenging than activities traditionally included in regular mathematics curricula. There is overwhelming evidence that all students benefit from studying mathematics through challenging activities, though there are some students within every age cohort who require more mathematically advanced tasks than others do in order to be adequately challenged. In addition, it is well known that many students who enjoy feasible for them mathematical challenge, do not like to compete with other students. Hence, the joint focus of TSG-46 is on mathematics competitions and other challenging activities, within or beyond a mathematics classroom.

The TSG-46 at ICME-14 built upon the work of the previous ICMI-initiated forums, such as the 16th ICMI Study "Mathematical challenge in and beyond the

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classroom", DG-16 at ICME-10, DG-19 at ICME-11, TG-34 at ICME-12, and TG-30 at ICME-13.

In spite of the essential work done in these forums, it still needs to be acknowledged that the mathematical challenge is an elusive notion. For instance, ICMI Study 16 suggests the following conceptualization of challenge:

> "…we will regard challenge as a question posed deliberately to entice its recipients to attempt its resolution while at the same time stretching their understanding and knowledge of some topic. Whether the question is a challenge depends on the background of the recipient; what may be a genuine puzzle for one person may be a mundane exercise or a matter of recall for another with more experience" (Barbeau and Taylor, 2009, p. 5).

This definition puts forward the expectations of the proposers of a challenge regarding actions of its (potential) recipients, but is rather silent about the recipient actual intentions and actions. Accordingly, the following queries are still open and require our attention as a community:

- Why and under which circumstances are our students inclined to accept or not the requests to invest intellectual effort in doing mathematical tasks with which their teachers attempt to challenge them?
- What are characteristics of the mathematical tasks that have a chance to be perceived by the students as engaging and feasibly challenging?
- What is the role of a competitive aspect of a mathematical challenge?
- How can tasks that are initially designed for the use in competitions be used in a regular classroom or in a teacher preparation workshop?
- What are the relationships between engaging students in challenging activities and fostering their creativity and mathematical habits of mind?

These and such queries have been at the heart of the discussions at TSG-46. As in the previous ICMEs, TSG-46 at ICME-14 gathered mathematicians, teachers, mathematics educators and mathematics education researchers and served as a stage for discussions related to the pivotal aspects of the group work: mathematical challenge, competitions, challenging activities connecting school mathematics and mathematics as a research area.

2. Submissions and Presentations

The initial Call for Papers attracted 16 high-quality submissions of different formats. Unfortunately, the changing the year and the format of the conference due to the COVID-19 pandemic did not enable some of the authors to participate in the Congress. Eventually, 11 presentations have been delivered during three 90-minute sessions. Of note is that the group, though small, remained truly international and consisted participants from 8 countries, as follows: Bulgaria, Colombia, Hungary, Israel (2), Portugal, China, Slovakia, and USA (3).

The time of the sessions was distributed between whole-group discussions (45 minutes), two invited talks (IT), one long oral presentation (LO, 45 minutes including

Q&A), and 7 short oral presentations and a poster (SO, 15 minutes including Q&A). See Tab. 1.

Tab. 1. The list of presentations

Paper and author(s):
Session 1
[1] What competitions can tell us about theories in mathematics education. ***Maria Falk de Losada*** (Colombia). (LO)
[2] How to identify multiple solution tasks for mathematical competitions. ***Ingrid Semanišinová**, Ľubomír Antoni, Stanislav Krajči, Daniala Vít'azková* (Slovakia). (SO)
[3] Challenging math tasks for teaching through problem solving approach. ***Hoyun Cho*** (USA). (SO)
Session 2
[4] Unravelling the construct of mathematical challenge based on conceptual characteristics of mathematical tasks. ***Roza Leikin*** (Israel). (IT)
[5] A challenge of deciding who is right and why. ***Reut Parasha**, Boris Koichu, and Michal Tabach* (Israel). (SO)
[6] Students' expected gains from a modeling competition. ***Elisabeth Roan** and Jenifer Czocher* (USA). (SO)
[7] Math trails: Opportunities to learn rich mathematics outside the classroom. ***Rosa Antonia Thomas Ferreira*** (Portugal). (SO)
Session 3
[8] Cutting a polygon: From mathematics competition problems to mathematical discovery. ***Kiril Bankov*** (Bulgaria). (IT)
[9] An introduction of Shanghai grade 11 mathematics competition. ***Yijie He** and Tianqi Lin* (China). (SO)
[10] POSA weekend-camps: A challenging mathematical environment for the highly gifted in Hungary. ***Eszter Bora*** (Hungary). (SO)
[11] Competitions promoting the mathematical science. ***Valorie Lynn Zonnefeld** and Ryan Glenn Zonnefeld* (USA). (Poster)

3. Thematic Overview of the Presentations

3.1. Organizational formats for challenging students mathematically

In line with a well-established tradition, TSG-46 at ICME-14 served as a stage for presenting unconventional formats of competitions and out-of-school activities aimed to challenge students mathematically. He and Lin[9] presented Shanghai Grade 11 Mathematics Competition, which encourages students to use various types of calculators, including graphical calculators. Examples of problems from this competition convincingly show that calculators can be used not only as technical scaffolds but as valuable tools for promoting students' mathematical creativity (e.g., by means of devising computational algorithms) and tools for developing conceptual understanding.

Zonnefeld and Zonnefeld[11] overviewed several types of thematic competitions, including a Math Bee, data-analytics competitions and March Madness competition, in which students design algorithms to select teams for a basketball tournament. Cho[3] described the "I Love Math Day" conducted annually on February 14 in an urban school in New Jersey. The "I Love Math Day" is a celebration that playfully recognizes student long-term mathematical problem-solving effort in small teams.

Bora[10] introduced "Pósa Weekend-Camps" — two-day-long mathematical workshops, in which more than 1500 Hungarian 6–11 grade students have taken part since 1988. A characteristic feature of these camps is that the problems chosen are organized in threads and form a rich network. In this way, students explore mathematics while experiencing a mix of discovery learning and guided learning.

3.2. Research on students' experiences with mathematically challenging activities

Students' experiences with competitions and other challenging activities were in the focus of two presentations[6,7]. In these presentation, organizational formats of challenging the students were described as contextual information, and empirical research was put forward.

Ferreira[7] analyzed the reactions of a class of 6th graders in Portugal on "Math Trail" consisting of a sequence of five stops along a predetermined path from a school to a city center and back to school. At the stops, the students were offered mathematical tasks combining intellectual, social and physical dimensions. The data for the tasks were gathered through direct observation of the environment at the stops. Overall, students' reactions on the entire experience were positive though some of the tasks appeared to be too cognitively demanding for them.

Roan and Czocher[6] presented a post-hoc analysis of pre- and post- survey data from two rounds of the "Challenge Using Differential Equations Modeling" (SCUDEM) competition for high-school and undergraduate students. In the SCUDEM competitions, teams of three work for a week on a modeling problem of their choice. At the competition site, teams are offered an additional modeling issue related to the problem they have handled, and then give a 10-minute presentation of their findings, which is followed by immediate feedback. The analysis revealed profound differences between expectations of students, researchers and problem designers in relation to the gains of the competition. For example, whereas researchers and designers expected the competition to be appreciated as an opportunity to engage in modeling and as a chance for recognition, the participants valued most the experience in modeling and teamwork skills.

3.3. Characterizing and theorizing mathematical challenge

In her invited talk, Leikin[4] considered mathematical challenge embedded in a task as a complex function of the task characteristics, didactical settings in which the task is approached, classroom socio-mathematical norms and mathematical potential of the students who cope with the task. In particular, she focused on such task characteristics as conceptual density, the task's openness, and the complexity of mathematical concepts, mathematical connections and logical relationships required for solving the task.

Semanišinová et al.[2] reported a novel method of analyzing competition tasks by means of "formal concept analysis". The method was illustrated by its application to two multiple-solution problems from the Slovak correspondence mathematical competition. The fine-grained analysis of students' solutions revealed that while expert solution spaces were wide enough for both problems, the collective solution spaces varied. In conclusion, the scholars advocated for the choice of competition tasks that would be less dependent on the knowledge of particular mathematical concepts or specific solution methods.

Parasha et al.[5] introduced a notion of "dialogical challenge", that is, a mathematical challenge associated with collective argumentative activity towards deciding which of contradictory solutions to a problem is right and why. By means of three examples, they argued that a dialogical challenge is two-dimensional: the first dimension is related to understanding and validating the solutions, and the second one — to inventing an argument that would be convincing to the peer students.

Finally, a long oral presentation of Falk de Losada[1] was devoted to theoretical analysis of competition problems by means of the conceptual apparatus suggested by Duval's theory of semiotic registers. She provided illuminating examples of the competition problems that can be solved by changing a semiotic register. Based on these examples, Falk de Losada argued that mathematical thinking is more than representations and treatments, and mathematics is more than a language to be learned, but rather an elastic medium that supports the search of novel and unanticipated connections.

3.4. From competition problems to mathematics discoveries

The invited talk of Bankov[8] elaborated on mathematics competition problems as intellectual products, which can serve as impetus for mathematical discoveries. A classic competition context of "cutting a polygon" was unfolded in the lecture. Inspired by a beautiful problem from the 1968 Saint Petersburg Mathematics Olympiad, Bankov considered a series of follow-up questions and problems, some of which are within the reach of a bright school student, some — require profound mathematical knowledge, and some are still open.

Interestingly, one of the open questions mentioned in the lecture deserved special attention at the summarizing discussion of the group when one of the participants (Sergei Dorichenko) offered an idea for how the question can be answered. The vivid discussion of the Dorichenko idea, which has been formulated on the spot, expressed greatly the creative spirit of TSG-46.

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https://doi.org/10.1142/9789811287152_0071

Topic Study Group 47

Mathematics Education in a Multilingual Environment

TSG-47 Working Team[1]

1. Introduction

Around the world, mathematics education is taking place in multilingual environments, including classroom situations. The environments can be affected by historical and ethnic diversity, and by colonialism, migration, refugee contexts and/or globalization. Research on issues in such environments are growing, and the "problematique" arising is of wide relevance for teachers and students all over the world. The aim of TSG-47 was therefore to examine issues that arise in conducting research on mathematics education in multilingual environments. The following four themes were addressed:

1. Mathematics teaching strategies in multilingual classrooms;
2. Theoretical foundations for the notion of resource in multilingual classrooms;
3. Multilingual students' agency in mathematics classrooms;
4. Design based research in multilingual mathematics settings.

The cooperation in the organizing team worked well over the years to prepare for the conference. We had a thorough review process and ended up with 16 long oral presentations, seven short orals and two posters. The presenters came from all over the world, showing that multilingual issues in mathematics classrooms are studied "everywhere". Multilingualism is studied from different theoretical perspectives and methodologies, such as: semiotics (Raja and Pugalee, 2016), design research (Prediger and Uribe, 2021), code-switching (Hao and Yap, 2022), translanguaging (Ryan, Källberg and Boistrup, 2021), and variation theory (Essien,2021).

2. Sessions and Papers Presented

The TSG met mainly on zoom, but there were Chinese participants on place in a lecture hall. The technique worked well except for a few of the presenters, they had problems

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getting access to the zoom platform. It didn't work to create zoom rooms, so, we had to do that our selves.

The time span from 2020–21, due to Covid-19, made presenters withdraw from the conference. At the time of the conference, a few of the papers presented had already been published, in longer more developed versions, elsewhere. The posters were not presented.

All the presentations are listed in Tab. 1, including two invited presenters: Susanne Prediger[6] and Richard Barwell[10].

Tab. 1. Presentations in TSG-47

Paper and author(s)
Session 1
[1] Code-switching: proposing linguistic relativity as lens in multilingual mathematics education research. ***Lester Cu Hao*** (Philippines).
[2] Practices and functions of colloquial Arabic use to generalize patterns in multilingual classrooms. ***Dibih El Mouhayar*** (Lebanon).
[3] Localised instructional mathematics application programmes: providing access into mathematics in multilingual classrooms. ***Evalisa Miriamu Katabua*** (South Africa).
[4] Fostering mathematics teacher development through experiential learning in multilingual communities. ***Catherine Paolucci*** (USA).
[5] Study on difficulties of math word problems in English-international baccalaureate in Japanese high school. ***Mitsuhiro Kimura*** (Japan).
Session 2
[6] Activating multilingual resources in a superdiverse covariation classroom — a design research study. ***Ángela Uribe*** *and Susanne Prediger* (Germany).
[7] Examining equitable participation and positioning in multilingual classrooms: tasks, language(s), and norms. ***William Carl Zahner*** (USA).
[8] The importance of students' first language as a sense-making resource in multilingual mathematics classrooms. ***Sally-Ann Robertson*** (South Africa).
[9] Exploring the enablement of mathematical proficiency in grade four English second language mathematics classrooms. ***Faith Lindiwe Tshabalala*** (South Africa).
Session 3
[10] Language positive classrooms: an example. ***Richard Barwell*** (Canada).
[11] Towards a framework for understanding the choice and use of examples in teacher education multilingual mathematics classrooms. ***Anthony Essien*** (South Africa).
[12] Impact of an online course of teaching mathematics to emergent bilinguals on teacher perspectives. ***Ji Yeong I*** (South Korea).
[13] Language-related barriers to mathematics learning: an alternative diagnosis. ***Mun Yee Lai*** (Australia).
[14] The problems of bilingual mathematical learners when using mathematics in Arabic. ***Madiha Hassan Mohamed Abd El-Rahman*** (Egypt).
[15] A student may speak with an accent, but no student thinks with an accent in mathematics. ***Clarence Alan Zollman*** (USA).

The organizing team shared the leading of our presentations. Session 1, chaired by Eva Norén, was held at 14:30–16:30 Beijing Time, July 13th; Session 2, chaired by Anthony Essien, was held at 19:30–21:00 Beijing time, July 14th; Session 3, chaired by Eva Norén, was held at 21:30-23:00 Beijing time, July 17th.

All sessions included discussions. The discussions were hold in positive manners. The TSG ended with discussions on how to take results from studies further, and how we could cooperate in the future.

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https://doi.org/10.1142/9789811287152_0072

Topic Study Group 48

Mathematics in a Multicultural Environment

Florence Glanfield[1], Anthony Fernandes[2], Qin Jing[3], and Peter Kajoro[4]

ABSTRACT TSG-48 examined historical, current, and emerging trends as well as issues and experiences in research with/in/on multicultural environments, across four themes: theoretical perspectives, methodological perspectives, emergent perspectives, and knowledge mobilization perspectives.

Keywords: Multicultural environments; Mathematics education; Theory; Methodology; Emergent; Knowledge mobilization.

1. Description and Themes

1.1. Description

Mathematics education occurs in multicultural environments in all countries around the world. The aim of TSG-48 was to examine issues, and explore experiences, that arise in mathematics education policy, practice, and research with/in/on multicultural environments. Research, practice, and policies of/in/with mathematics education are affected by history, colonialism, decolonization, migration, and globalization. There is a growing body of research that is related to Indigenous perspectives, social justice, and equity within these historical and colonial environments. Research in mathematics education arising in such environments is growing and is of wide relevance. Four themes were featured.

1.2. Themes

Theoretical perspectives framing mathematics education with/in/on multicultural environments explored the questions: What theories have been used in conducting research with/in/on mathematics education in multicultural environments and why? What theories have been used to guide the teaching and learning of mathematics

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with/in multicultural environments? What are normative assumptions about mathematics education policy, practice, and research and multicultural environments? How might theory help to challenge normative assumptions? How has theory and research developed in the context of multicultural environments contributed to understanding the learning and teaching of mathematics more generally?

Methodological perspectives engaging mathematics education research with/in/on multicultural environments explored the questions: What new challenges have emerged in the methodological perspectives used in recent years? How might they be addressed? What perspectives have informed research with/in/on mathematics education in multicultural environments in recent years?

Emergent perspectives in framing research, teaching, and learning of mathematics education with/in/on multicultural environments explored the questions: In what ways are new perspectives informing the mathematics education research community about teaching and learning mathematics with/in/on multicultural environments? What are the ways in which these emergent perspectives inform teaching and learning of mathematics equitably? What challenges arise for mathematics teachers, mathematics educators, and mathematics education researchers when working with/in students' and families' diverse multicultural environments?

Knowledge mobilization perspectives explored the questions: How might research with/in/on mathematics education in multicultural environments inform curriculum and/or assessment policy? What challenges and opportunities arise in the interaction of mathematics education research with/in/on multicultural environments and the development and implementation of local, national, and international policy? What insights might be developed in the analysis of such interaction?

2. Program Overview

The Topic Study Organizing Team are the authors of the proceedings. Six contributions to the topic study group remained following the cancellation of ICME-14 in 2020 and the move to a hybrid format in 2021. In this section we firstly describe the format of the Topic Study Group sessions and then provide an abstract of the papers that were delivered.

2.1. Format

Two sessions of papers were facilitated during the ICME-14 meeting, on July 16 and July 17. The beginning of each session started with a welcome, facilitated by members of the organizing team. Following the presentations there was an opportunity for all individuals who presented and those who were listeners to dialogue around the ideas that were featured throughout the presentations. There were two long oral paper presentations and four short oral paper presentations. The long oral presentations were scheduled for 30 minutes and the short oral presentations were scheduled for 20 minutes. The first paper in each session was a long oral presentation.

2.2. Presentations

Papers in TSG-48 are presented in Tab. 1. In the table, LO stands for long oral presentation, SO for short oral presentation.

Tab. 1. The list of presentations

Paper and author(s)	
[1]	Conceptualizing a framework for a new (disruptive) form of culturally responsive pedagogy in mathematics/teacher education. ***Kathleen Nolan*** (Canada). (LO)
[2]	Preservice teachers engaging with traffic stop data to examine issues of bias. ***Anthony Fernandes*** (USA). (SO)
[3]	Intersections of indigenous knowledge systems and mathematics education. ***Florence Glanfield*** (Canada). (SO)
[4]	Taking a strengths-based approach to learning and teaching mathematics. ***Marta Civil*** (USA) *and* ***Roberta Hunter*** (New Zealand). (LO)
[5]	Developing concepts for mathematics teaching units with a focus on migrant and minority students. ***Andrea Ulovec*** (Austria)*,* ***Jarmila Novotná****, and Hana Moraová* (Czech). (SO)
[6]	The use of dominant discourse practices in secondary multilingual mathematics classrooms: A comparison of lessons given by two teachers. ***Michael Alexander*** (South Africa). (SO)

A long oral presentation by Nolan[1] explored the question of how school mathematics and mathematics teacher education might be reframed through critical and culturally responsive pedagogies. By synthesizing perspectives offered by Ethnomathematics (EM), Critical Mathematics (CM), Indigenous Education (IE), Language Diversity (LD) and Equity-based (E-b) approaches to research in mathematics education, Nolan conceptualizes a new (disruptive) form of culturally responsive pedagogy (CRdP). As discussed in this paper, CRdP is pedagogically informed by the EM-CM-IE-LD-E-b collective; it is theoretically informed by Nancy Fraser's (2009) three-dimensional approach to social justice and participatory parity; and it is methodologically informed by discourse analysis.

Fernandes[2] reported on how ten preservice teachers, who were mostly White, engaged with local city traffic stop data to examine issues of racial bias. Police traffic stops are a common occurrence in the United States. Racial minorities, especially African Americans, are stopped at rates that are higher than other races. As data becomes more prevalent in society globally, it is important for teachers to be able to analyze and interpret large data sets. At the same time, teachers need to be familiar with issues that affect students they are going to teach. In the United States students of color are becoming a majority in schools, while their teachers come from different backgrounds and have different life experiences. An analysis of the pre- and post-reflections demonstrates that the preservice teachers associated the disproportionate traffic stops to actions of individual police officers rather than acknowledge racial bias in policing. The study showed that even though studying data can be an important tool to understand structural inequities, a more comprehensive approach to changing dominant beliefs is needed.

Glanfield[3] asked "In what ways might Indigenous knowledge systems shape mathematical understanding and mathematics teaching practices?" This paper described Indigenous knowledge systems and the ways that mathematics exists within Indigenous cultures. The paper illustrated the ways in which Indigenous knowledge systems have the potential to contribute to establishing and building inclusive, and equitable, classrooms. For example, one such aspect of many North American Indigenous knowledges is "we are all related." This aspect points to the ways in which teachers can build on student, family, and community strengths for any classroom that is multi-cultural, not just Indigenous cultures.

Another long oral presentation[4] by Civil and Hunter drew on research with teachers, students and parents to show how strength-based approaches can support culturally sustaining practices in mathematics classrooms. Although there has been a substantial increase in literature around teaching and learning of mathematics in multicultural settings we still have a way to go to gain equitable outcomes for all learners. Civil and Hunter illustrated how repositioning teachers from a traditional role as listened to, to that of being listeners, causes dissonance from which comes change towards culturally sustaining pedagogy. Civil and Hunter showed how authentic two way conversations between parents and teachers provide opportunities for learning with and from each other, not just about mathematics but also about their values and ways of being.

A joint paper[5] by Ulovec et al. shared the work of a Czech-Austrian project team. The multicultural nature of society influences education in many countries. Teachers are usually not sufficiently prepared to deal with a multicultural classroom context. Particularly mathematics teachers feel the need for materials supporting them in teaching in multicultural classrooms. Mathematics teachers' pupils with a migrant background often encounter more difficulties than their native classmates in acquiring basic mathematical skills. Many projects have created mathematics teaching materials in different settings, though these did not take multicultural classrooms into consideration. Few projects have created concrete mathematics teaching materials for migrants, but these were rather closed materials, not concepts and strategies to be further developed by teachers. The project team worked on designing concepts for teaching units based on the analysis of various research studies, examples of concrete teaching units based on these concepts, and guidelines on how to use these concepts. These materials will give mathematics teachers a tool that allows them to create their own teaching units fitting their own classroom needs.

Finally, Alexander[6] explored some of the dominant discourse practices used in multilingual mathematics classrooms by comparing two teachers' mathematics lessons on trigonometry in a Nigerian secondary school. Alexander started with an introduction, and then outlined a theoretical perspective on the nature of classroom interactions which informed the study. The research questions were: What dominant discourse practices were used by teachers as plainly demonstrated in their language (verbal and non-verbal) in multilingual mathematics classrooms? How do teachers use language (verbal and non-verbal) to enact practices as reflected in their discourses in the teaching and learning of mathematics in multilingual classrooms? Data gathering techniques for the study included video observations in the classrooms, and written field notes. The data gathering process covered a period of six months. A total of 6 lessons were observed. Exploration of these practices lead Alexander to discuss a number of the relationships between teaching and learning of mathematics in multilingual classrooms. The analysis showed that teacher G used language to make his identity visible, as well as re-voicing and gesturing mathematically while teacher S used language not only to make his identity visible but also stabilise it during the discourses.

3. Future Directions and Suggestions

The presentations in TSG-48 highlight the diversity and complexity of research in multicultural environments. Though there were different foci, we find some underlying themes that point to future directions in research. It is important to take a strengths-based approach when working in a multicultural environment. Students effectively draw on their language and lived experiences in the mathematics classrooms. Current research around trans-language is also a step towards new understanding of learning and teaching mathematics in a multicultural environment. Further, exposing future teachers and teachers to these approaches is key to future research directions.

Providing future teachers with experiences and opportunities to dialogue is an area that needs more research. Discussions around racism, colonialism, and sexism can remain controversial within a mathematics classroom. Research related to curriculum and classroom implementation remains an important future direction in research.

We also noted that, on terms of methodological approaches, the research presented in Topic Study Group 48 was primarily qualitative. We wondered, what quantitative or mixed approaches might be developed to offer other perspectives of mathematics in a multicultural environment in the future?

References

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https://doi.org/10.1142/9789811287152_0073

Topic Study Group 49

Distance Learning, E-learning and Blended Learning of Mathematics

Marcelo Bairral[1], Tracey Muir[2], and Veronica Hoyos[3]

ABSTRACT Topic Study Group 49 built on current and emerging research in distance learning, e-learning and blended learning. Specifically, we pushed the boundaries of what is known through a deeper examination and discussion of recent research and development in teaching and learning through these modalities, with a focus on primary, secondary, and higher education.

Keywords: Distance learning; E-learning; Blended learning.

1. Aims of TSG-49

Our TSG-49 aimed to build on current research and highlight emerging research into how distance learning, e-learning and blended learning were enacted in the context of mathematics education. Participants were invited to contribute papers for the following sub-themes:

1. The emerging work on the usage of such mobile technologies, as cell phones and tablets, for distance learning or blended instruction.
2. Incorporation of social media in online (or blended) technologically mediated courses.
3. Flipped classroom.
4. Developing the role of the faculty/moderator/tutor in online mathematics education.
5. Exploration of the emergence and sustainability of communities of practice in online environments of collaboration and co-construction of resources.
6. Utilization (Web 2.0, Web 3.0 etc.) and designing tasks, resources or environment in e-learning, blended learning, and distance education modalities.
7. Enabling mathematical collaboration in online mathematics education and orchestrating productive mathematical conversations in an online or in a blended setting.
8. Using distance learning, e-learning and/or blended learning in mathematics

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pre-service teachers' training, professional development of in-service teachers, and/or to promote the collaboration between teacher education researchers.

9. Assessment and evaluating of the effectiveness of distance education, e-learning and blended learning.
10. Research methodologies and paradigms for studying online and blended mathematics education.

1.1. Submissions

We received 23 submissions from authors of eight countries (Australia, Austria, Brazil, Germany, Mexico, Nepal, Russia, USA). Submissions comprised six long presentations, 10 short oral presentations and 7 poster presentations. During the conference, 16 papers were scheduled, and 10 were presented across 4 sessions.

1.2. Sessions

The four sessions included both short and long oral communications. Each session was facilitated by one of our TSG's co-chairs and were well attended by members of our TSG. Discussion prompts were utilized to promote discourse focused on the implications of the study for online learning in the current COVID-19 context and for mathematics education in general. The final session included identification of emerging research themes and identification of opportunities for future research.

2. TSG-49 Themes

The 10 papers presented during the congress were grouped into 3 themes:

- Flipped classroom and hybrid environments
- Mobile devices, task design and other resources
- E-learning and online professional development: Courses, reflections, and interactions

The papers for each theme are listed in Tab. 1 (on the next page).

2.1. Flipped classroom and hybrid environments

The first two papers focused on flipped classroom and hybrid environments. While the flipped classroom approach may seem easily adaptable to cater for online learning as a result of the COVID-19 pandemic, restrictions meant that there was no opportunity to implement the in-class phase of flipped learning. Rothe's study[1] investigated the importance of the in-class time in a flipped classroom, with results suggesting that students experienced difficulties in achieving higher-order learning goals in the changed scenario. Hoyos et al.[2] presented a paper which described the framework, methodology, analysis, and results of an exploratory study around the implementation of a hybrid learning environment designed to address the teaching and learning of

Tab. 1. The list of papers for each theme

Paper and author(s)	
Flipped classroom and hybrid environments	
[1]	Fostering higher order thinking in the flipped classroom — an analysis of students' proof schemes. ***Jennifer Rothe*** (Germany).
[2]	Hybrid environments of learning: teacher efficiency and potential for student learning by collaboration. ***Veronica Hoyos****, Estela Navarro, Victor Raggi, and Sergio López* (Mexico).
Mobile devices, task design and other resources	
[3]	Designing tasks to improve plane transformation using DGE with touchscreen. *Alexandre Assis and* **Marcelo A. Bairral** (Brazil).
[4]	Using free software to implement verification problems with parameters. ***Ilya Alexandrovich Posov*** *and Dmitry Irikovich Mantserov* (Russia).
[5]	Workshop activity in online courses of mathematics education: insights for learning and assessment. ***Niroj Dahal*** (Nepal).
E-learning and online professional development: courses, reflections, and interactions	
[6]	A reflective practice on an online mathematics class. ***Haoyi Wang*** (USA).
[7]	Case study on the change process of a mathematics teacher in an online professional development course. ***Stefanie Schallert*** *and Robert Weinhandl* (Austria).
[8]	Participants' patterns of interaction within and across social networks in a massive open online course for educators. ***Heather Barker****, Gemma F. Mojica, Karen Hollebrands, and James Smiling* (USA).
[9]	The role of the lecturer in facilitating productive mathematical conversations in online mathematics pre-service teacher education. ***Tracey Muir*** (Australia).
[10]	Transforming numeracy professional development for pre- and in-service mathematics teachers and families through e-learning. ***Leicha A. Bragg****, Chris Walsh, and Tracey Muir* (Australia).

functions. The hybrid environment led to improvements in teacher practice, and refinement or validation of students' conceptions.

2.2. Mobile devices, task design and other resources

The next three papers focused on mobile devices, task design and other resources. Assis's and Bairral's research[3] illustrated designed tasks on GeoGebra with touchscreen and reflected about the design of them to improve plane transformation in High School with students without previous instruction in this content. Posov and Mantserov's paper[4], based on computer-aided assessments, rewrote a condition on a function with a logical expression on its parameters, and then proposed ways to implement it using GeoGebra and Sage. And, Noroj's study[5] identified best ways to engage students in the process of learning and peer assessment by using workshop as a learning and assessment tool for MPhil in Mathematics Education for the course Graph and Network.

2.3. E-learning and online professional development: courses, reflections, and interactions

The other five papers focused on e-learning and online professional development (courses, reflections, and interactions). Wang's paper[6] reported on a reflective practice that examines both qualitative and quantitative datasets in understanding the implemented multimedia distance learning environment at an entry-level math classroom at a large state university in the Midwest and its resulting consequences on the math learning and assessment performance of the students. Schallert and Weinhandl's paper[7] presented a case study approach to examine the different elements of a change process of a secondary teacher within an online course for mathematics teacher training. Barker et al.[8], based on social network and sentiment analyses, examined the discussion forum posts of 159 educators from 46 countries who participated in a Teaching Mathematics with Technology MOOCs for Educators. Muir's study[9] provides an example of an online forum, which highlights how pre-service teachers can be engaged in productive online mathematical discussions, particularly when facilitated by the instructor's and other learners' presence. Bragg et al.[10] presented a multipronged approach to transform the provision of numeracy professional development for educators and families through the design of three open-access resources.

3. Areas for Future Research

In our final presentation session, the participants discussed potential future research topics and publication possibilities. Participants in TSG-49 topics' discussion concurred that further research into the nature, purpose, and significance of online, blended, and e-learning in mathematics education was needed, especially in the context of the deep socio-cultural changes brought about globally by the COVID-19 pandemic.

Finally, methodologies to gather data and to analyze online interaction (among students or teachers) using asynchronous or synchronous tools continue to be a field of interest, as well as investigating more on what ways the affordances and limitations of online tools and apps are influencing the task design and/or math's teaching and learning.

https://doi.org/10.1142/9789811287152_0074

Topic Study Group 50

Mathematics Education in and for Work; Continuous Mathematics Education Including Adult Education

Lisa Björklund Boistrup[1] and Geoffrey Wake[2]

ABSTRACT In this report we give account for how TSG-50 was organised, both in terms of structure, and in terms of content of the area of the TSG. The papers presented are summarized and future directions are suggested, including the relevance of a TSG with this theme also for future ICMEs.

1. Organization and Aim

TSG-50 was organised by the following people:

Chair: Lisa Björklund Boistrup, Malmö University, Sweden
Co-chair: Geoffrey Wake, University of Nottingham, UK
Members:
Pradeep Kumar Misra, Chaudhary Charan Singh University, India;
Maria da Conceição Ferreira Reis Fonseca, Universidade Federal de Minas Gerais, Brasil;
Haixia Si, Hangzhou Normal University, China
Gail FitzSimons, Melbourne University (extra member)

Topic study group 50 at ICME-14 had as its aim to exchange ideas and knowledge with regards to two related themes: mathematics in and for work, and continuous mathematics education including adult education. The TSG received eight paper contributions and one poster presentation. Present at the TSG sessions were 15–20 people, where also co-authors took part. In our TSG sessions during the conference, we addressed key questions as an overarching organising structure:

- What issues do we need to consider in designing for mathematics education in and for work?
- What role can theory play in our mathematics education research?
- How can we support the learning of mathematics by adults?

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We allocated time according to the schedule below for contributions to facilitate the discussions around the key questions. In the end of each meeting there was time to discuss the questions across the presented papers.

2. Report from the Sessions of TSG-50

Below we give account for the subthemes and papers of the sessions of TSG-50 (Tab. 1).

Tab. 1. The list of papers for each theme

	Paper and author(s)
Session 1: What issues do we need to consider in designing for ME in and for work?	
[1]	Designing for the learning of mathematics for vocational competence. ***Geoffrey Wake*** (UK).
[2]	Construction of mixed training model for rural mathematics teachers in junior middle school. ***Xiaocheng Li****, Guanghui Zhou, Jun Tao, and Dongxue Tu* (China).
[3]	Sociomathematical norms in vocational mathematics education. ***Trude Sundtjønn*** (Norway).
[4]	Infographics about the world of work: an experience with students of vocational education integrated to high school. *Guilherme Guilhermino Neto,* **Lauro Chagas e Sá***, and Maria Auxiliadora Vilela Paiva* (Brazil). (Poster)
Session 2: What role can theory play in our mathematics education (ME) research?	
[5]	Investigating interfaces between mathematics and vocational content: logos and praxis in education. ***Lisa Björklund Boistrup****, Matilda Hällback, and Divo Racheed* (Sweden).
[6]	"Here we are the boss": numeracy practices as resistance tactics of clothing factory workers in Brazilian northeast. ***Maria da Conceição Ferreira Reis Fonseca*** (Brazil).
Session 3: How can we support the learning of mathematics by adults?	
[7]	Re-thinking the assessment of adults' numeracy skills: new challenges, new responses. ***Javier Díez-Palomar*** (Spain)*, Kees Hoogland* (The Netherlands)*, and Isabelle Demonty* (Belgium).
[8]	Moocs for lifelong mathematics learning of adults in India: promises and strategies. ***Pradeep Kumar Misra*** (India).
[9]	Adults' proportional reasoning in a volume scaling situation. ***Linda Marie Ahl*** *and Lars Ola Helenius* (Sweden).

2.1. Session 1, Tuesday July 13th

The subtheme of this session is: *What issues do we need to consider in designing for ME in and for work?*

This theme was focused by the input of three papers and one poster. These looked at issues relating to developing mathematics education (ME) in preparation for, and in, work. These contributions signaled the wide range of issues to be considered, from overall drivers such as qualifications through to learning based on the use of specific tasks.

Wake[1] developed an argument that we need to consider carefully what we consider to be an appropriate form of the learning which we wish to promote. Consequently, he points to how learners need to get an awareness of how mathematics is developed and applied in vocational domains in ways that are sensitive to the idiosyncrasies that may arise in work settings. He also argued that because application of simple mathematics is important there needs to be time and space dedicated to learn how to do this in ways that are authentic to workplace settings. In this sense he emphasized that learning in and for work is primarily about learners who are involved in identity development as they become members of workplace communities and that a mathematics curriculum should support them in this. Li et al.[2] picked up this theme by illustrating a model of teacher training for rural mathematics teachers of junior middle school students. The paper provided insight into how workplace learning, in the case of teachers promoted a range of carefully designed learning modes that involved them in reflection in different stages that involved them in activity that is both school based and online in a mixed model.

Sundtjonn[3] focuses more closely on students' learning whilst working on a task in a lesson on a vocational education programme in Norway. She considers that even though the task being used is carefully designed to provide some engagement in an authentic activity both teachers and students exhibit behaviours that are those we might expect in traditional school mathematics classrooms. Their focus is primarily on getting mathematical solutions without considering the implications in terms of the real world situation. The study points to how strongly traditional sociopmathematical norms are embedded in the lives of teachers and students and to break through the barriers these present it will be necessary to provide teachers with specialized guidance. The poster[4] by Guilhermino from Brazil likewise looked at students' engagement in a learning activity for students on a Biotechnology technical course. This was found to be motivating as students took ownership of the themes they chose to work with and they drew on learning from other parts of their course as they carried out data collection, analysis and communication through the means of an infographic.

The discussion in this theme pointed to the different levels of support needed if we are designing mathematics education for the world of work. At the strategic levels we have to consider carefully the curriculum and qualifications we design through to the support we give teachers with their teaching at the tactical level of implementation and down to the detail of task design at a tactical level. There is much to do if we are to implement learning mathematics for work successfully.

2.2. Session 2, Friday July 16th

The subtheme of this session is: What role can theory play in our mathematics education (ME) research?

Two papers addressed issues of the use of theory in research with a focus on learning mathematics in and for work. Boistrup et al.[5] used the ideas of Chevallard's Anthropological Theory of Didaactics (ATD) to analyse data from collaborative teaching in mathematics and vocational education in the 'beauty' industries (facial make-up and hair). Their analysis considered both logos (knowledge and theory) and praxis (tasks and techniques) in the vocational tasks that students worked on. They reported, and illustrated, how in two tasks that learners worked on, (symmetry in) facial make-up and (angles in) curling of hair, demonstrated how logos from both mathematics and the world of work were needed and the praxis required was mainly from the vocational context. Their discussion pointed to how their use of ATD led them to confirm the work of other researchers who point to how mathematics can help clarify work practices and vice versa, the context of work can help provide insight into, and explain mathematical ideas.

In a more overtly political paper[6] Fonseca reported ethnographic research that explored the practices that were forged in work, school, and daily life activities of students from Youth and Adult Basic Education (YAE), aged 16. With a focus on the numeracy practices Fonseca's analysis Fonseca identified in the workplace tactics of resistance (Certeau, 2011) to oppressive ways of life and production being delineated, established, and developed. These practices were constituted by diverse discursive practices, including numeracy practices, generally marked by the Cartesian rationality used by capitalist logic. Fonseca suggested that their ways of doing mathematics, the workers were cunningly circumventing the distribution of tasks. They used tricks and expertise to improve and speed up the production — and thus make greater collective gains, not per piece made by each.

These two very different approaches to using theory in general suggest that theorizing our research can provide new insights into what are often complex socio-cultural situations.

2.3. *Session 3, Saturday July 17th*

The subtheme of this session is: *How can we support the learning of mathematics by adults?*

A focus on adults learning mathematics was the common thread in the three papers presented in this session. Again, as in the first theme, the contributions provided a glimpse into the diversity of work that might be considered in research in this area, particularly at strategic, tactical, and technical levels.

At a policy level, Díez-Palomar et al.[7] went back to consider exactly what we mean by the term "numeracy", it's history and its relationship with quantitative literacy. They considered how international studies such as PIAAC are used to measure competence in this area across countries and then argued that we now face new challenges as adults are starting to have to consider solving problems that involve big data, social media, and other new e-social technologies that are deeply transforming social interaction. They point to how there are additional demands being made in terms of employability, citizenship, community participation, etc. and such considerations suggest that we need to rethink how international assessments for adults engage with the new "digital" era.

Picking up this theme, Misra[8] explored the potential of Massive Open Online Courses (MOOCs) as a medium for the education of adults in mathematics. Misra discusses this in the context of India, the worlds' second most populated country, where there are clearly challenges in reaching adult learners in ways that meet their needs in relation to societal, institutional, and governmental needs. The paper suggests that MOOCS offer great potential in this area and is optimistic that this approach can be successful in India.

In a final paper in this theme, Ahl and Helenius[9] considered the learning of a specific mathematical concept, that of proportionality in the specific contexts of volume scaling, speed and cost effectiveness. The fresearch was carried out with adult Swedish students in a prison education programme and contrasted with resuts from upper secondary students in Denmark. It was found that, in the case of volume scaling, the secondary schools students performed considerably better than the adult students. It was suggested that preliminary analysis of the other tasks were pointing to similar results. The authors suggest that proportional reasoning experienced in real situations does not seem to provide useful insight into the scientific ideas that were needed in the tasks used here.

3. Main Outputs and Future Directions

The two themes of the TSG, mathematics in and for work, and continuous mathematics education including adult education, are often related and TSG-50 combined them to consider mathematics education from the different contexts of where mathematics is either related to work (e.g. Wake[1], Sundtjønn[3], Fonseca[6], Boistrup et al.[5]) or other important aspects in the lives of adults (e.g. Li et al.[2], Neto et al.[4], Ahl and Helenius[7], Misra[9]). TSG-50 focused on lifelong mathematics learning and was concerned with mathematics education that takes place in formal education settings, such as formalised adult education; semi-formal settings, such as part of vocational education organized for example by employers or by workers' associations; and informal settings, that may be part of the daily activities of adults in and outside work.

TSG-50 viewed mathematics to be inclusive of the formal academic discipline of mathematics and mathematical processes such as modelling and problem solving in addition to many other informal forms of quantitative and spatial reasoning that arise in a wide range of settings and situations.

The discussions in TSG-50 were lively and focused, with both interactions online, as well as with participant on-site. The contributions pointed towards the future, both in the sense of addressing emerging and current issues, and in relation to the need for joint publications within the area. The plan is to have in process at least one joint publication until next ICME. A conclusion was also the need for a TSG such as TSG-50 also at future ICME's.

https://doi.org/10.1142/9789811287152_0075

Topic Study Group 51

Mathematics Education for Ethnic Minorities

Lianchun Dong[1,2]

ABSTRACT Ten papers were presented in TSG-51, involving nineteen contributors from five countries. Four themes were discussed in these presentations: (1) mathematics teaching practices and strategies in classrooms with ethnic minority students, (2) mathematics teacher professional development practices intended to improve the quality of mathematics teachers in ethnic minority regions, (3) ethnic minority students' performances in mathematics learning, and (4) research methodology in mathematics education for ethnic minority students.

Keywords: Ethnic minority; Teaching strategies; Teacher professional development; Learning results; Research methodology.

1. Introduction

All over the world, members of ethnic minorities (EM) face difficulties with the type of education that is offered to them, particularly in mathematics education. Difficulties are very diverse, including low achievement, dismissal of endogenous mathematical knowledge, mismatch of expectations with school goals, methods, and procedures, and even threats to the cultural and material existence of the minorities. Several educational models and strategies have been proposed to address these difficulties, varying broadly according to the historical, political, and cultural context in which each ethnic minority is immersed.

TSG-51 aims to gather researchers and practitioners from different countries who are interested in share their own experiences, reflections, and concerns about mathematics education for Ethnic Minorities. The TSG is envisioned as an open agora to discuss theoretical or empirical issues of diverse nature, adopting a strengths-based approach that goes beyond deficit perspectives, and is sensitive and respectful of the singularity of the contexts, constraints, and stances of each ethnic minority.

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Members: Aditya Adiredja, University of Arizona, USA.
Lianchun Dong, Minzu University of China, China
Nancy Nui, CEMASTEA Nairobi-Kenya, Kenya

Ten papers were presented in the three sessions of TSG-51. These presentations were given by nineteen contributors from five countries, namely Brazil, China, Colombia, New Zealand, and USA.

2. Main Ideas and Discussions in TSG-51

Four themes emerged in the ten papers (Tab. 1): (1) mathematics teaching practices and strategies in classrooms with ethnic minority students, (2) mathematics teacher professional development practices intended to improve the quality of mathematics teachers in ethnic minority regions, (3) ethnic minority students' performances in mathematics learning, and (4) research methodology in mathematics education for ethnic minority students.

Tab. 1. The list of papers presented

Paper and author(s)
Mathematics teaching practices and strategies in classrooms with ethnic minority students
[1] How does a teacher sustain collective mathematizing among non–dominant students? ***John Griffith Tupouniua*** *and Jodie Hunter* (New Zealand).
[2] The implementation of culturally responsive teaching practices into the mathematics course. ***Hsueh-Yun Yu****, Huey-Lien Kao, and Kuo-Hua Wang* (Chinese Taiwan).
[3] A case study on the application of "situational problems" teaching model in the mathematics education of ethnic primary school students. ***Chang-Jun Zhou*** (China).
Mathematics teacher professional development practices intended to improve the quality of mathematics teachers in ethnic minority regions
[4] Investigation on teacher professional development in minority areas: taking Yao autonomous county of Liannan, Qingyuan as an example. ***Mudan Chen*** *and Ida A.C. Mok* (Hong Kong SAR, China).
[5] Renegotiating recruitment and retention efforts: promoting teacher diversity in mathematics and science classrooms. ***Christine Darling Thomas*** *and Natalie Simone King* (USA).
[6] Investigation and research on mathematical culture accomplishment of primary school mathematics teachers in ethnic minority areas. *Jun Wu and* **Jing Ting** (China).
[7] Preparing the next generation of STEM innovators. Daniela Cabrera, ***Jose David Fonseca****, and Gerardo Lopez* (USA).
Ethnic minority students' performances in mathematics learning
[8] Chinese ethnic minorities students' performance in mathematical problem posing. ***Lianchun Dong*** *and Wei He* (China).
[9] Study on influencing factors of math achievements of ethnic minority senior high school students in Chinese mainland. ***Aoxue Su*** (China).
Research methodology in mathematics education for ethnic minority students
[10] Rethinking ethnography in mathematics education of ethnic minorities. ***Carolina Tamayo*** (Brazil) *and Aldo Parra* (Colombia).

2.1. Mathematics teaching practices for minority students

Tupouniua and Hunter[1] presented their investigation of a teacher's attempt to sustain collective mathematizing among non-dominant students in a classroom that emphasizes collective success. Taking a collectivist stance, they conceptualized the featured classroom as one in which the students function as a single learning organism. They analyzed three roles that the teacher played within a lesson focused on students' engagement with repeating patterns. They also discussed the affordances of the three roles with respect to sustaining three characteristics of a classroom that functions as a single learning organism.

Yu et al.[2] investigated how culturally responsive teaching influenced students' motivations to engage in mathematics learning. They used the approach of action research and the participants are grade eight Bunun students in secondary schools in Taiwan region of China. Followed the culturally responsive pedagogy, researchers designed mathematics learning materials for the participating students. It is found that: (a) students learning motivation had become more positive after experiencing culturally responsive pedagogy of mathematics teaching; (b) culturally responsive pedagogy enhanced students' mathematics capabilities and scores efficiently.

Zhou[3] selected mathematics classrooms from 12 primary schools in Longchuan County, Dehong Dai and Jingpo Autonomous Prefecture in west of Yunnan Province in China, aiming to explore the effects and existing problems of the teaching model of mathematics "situational problems". This three-year study ran from 2014 to 2016, showing that teachers who are culturally sensitive and good at using modern educational methods can use this model to help ethnic students learn mathematics more efficiently and achieve better results. In order to fulfill the effects of this teaching model, attention needs to be paid to we pay attention to the cultivation of teachers' cultural sensitivity and to the improvident of teachers' ability to use modern education techniques.

2.2. Mathematics teacher professional development practices in ethnic minority regions

Chen and Mok[4] adopted the perspective of Mathematics Pedagogical Content Knowledge (MPCK), examining mathematics teachers' professional development in Yao Autonomous County of Liannan, Qingyuan in Guangdong Province. They employed the questionnaire survey and in-depth interview to investigate the current status and teachers' MPCK, and the degree of contribution of different sources to the development of three dimensions of MPCK. They also explored whether characteristic variables have a significant impact on the MPCK development. It is found that hearing the voices of teachers in different contexts and putting forwards schemes for related departments are efficient patterns for accelerating the teacher education development. The implications for education researchers and policy makers were also discussed.

Thomas and King[5] reported evidence-based strategies on how to recruit and retain diverse mathematics and science teachers. Previous studies suggest that many teachers often underestimate the potential of students of color to excel in the STEM disciplines (Brickhouse, Lowery, and Schultz, 2000). These negative perceptions have a tendency to discourage students from realizing their true potentials and perceiving themselves as STEM talent. Although researchers have analyzed various challenges and strategies to decrease the impact of resisting factors, increasing teachers' capacity to create equitable mathematics and science learning spaces within urban settings continues to remain a challenge (Fraser-Abder, Atwater, and Lee, 2006; Kokka, 2016). These realities reify the need to explore innovative ways to prepare and develop culturally competent STEM teachers who can thrive even in the most challenging working conditions. Therefore, Thomas and King attempted to provide potential approaches and solutions so that the relatively homogeneous and static demographic of the teaching workforce (particularly in mathematics and science) can begin to adequately reflect the dynamism and racial and ethnic diversity of U.S. students.

Wu and Ting[6] conducted questionnaire and interview survey of 760 primary school mathematics teachers in ethnic minority areas of Yunnan Province, China. It is reported that mathematics culture accomplishment of primary school mathematics teachers in ethnic minority areas are generally at a medium level. In addition, teachers' mathematics culture accomplishment showed significant ethnicity, gender and urban-rural differences. Based on these findings, Wu and Ting recommended to set up a "primary school mathematics culture" course, establish a primary school mathematics culture teacher community and integrate the national mathematics culture into primary school mathematics teaching.

Cabrera et al.[7] presented a project aimed to address issues of social justice, and the environment in the educational pipeline. To do this, the project incorporated environmental science, math, and cultural elements into hands-on project-based learning activities for 6–12 students in predominantly American Indian and Hispanic communities. Professional development (PD) workshops for the development of a culturally relevant STEM greenhouse project-based learning curriculum was provided for teachers.

2.3. Ethnic minority students' performances in mathematics learning

Dong and He[8] investigated Chinese ethnic minorities students' performance in problem posing tasks. A set of mathematics problem posing tasks in three different situations (respectively free, semi-structured and structured situations) was developed to examine students' performance in mathematics posing. 105 students in year 5 from Xinjiang Uygur Autonomous Region, China participated in this study. It is reported in this study that the number of problems posed by Chinese ethnic minorities students in all three situations is fewer than those by Chinese Han students, but the complexity of the problems posed by Chinese ethnic minorities students in semi-structured and structured situations is not lower than their Chinese Han counterparts.

Su[9] conducted a questionnaire survey of 932 teachers and 1873 senior high students in ethnic minority regions to examine the school factors that impact students' math achievement with the two levels HLM. The results showed: (1) boys' math achievement was significantly higher than girls; (2) students with expectations of learning in mixed classes had a significantly higher math achievements than those who expected to be enrolled in non-mixed classes; (3) SES had no significant influence on math achievement; (4) students' learning strategies and learning self-efficacy had a significant positive impact on their math achievement and learning self-efficacy was the primary factor; (5) teacher job satisfaction had significant positive effects on student math achievement; (6) school location meditated the relationship among the expected mode of class and academic performance.

2.4. Research methodology in mathematics education for ethnic minority students

Tamayo and Parra's presentation[10] aimed to problematize ethnography in research conducted on ethnic minorities. They provoked a movement of deconstruction of the certainties caused by the uses of the method of ethnography, tracing lines of escape to understand that ethnography carries with it a series of assumptions that create limitations of political and epistemological nature for mathematics education research. They argued that some of those limitations end up undermining the possibility of reaching a new understanding of mathematics as a sociocultural practice.

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https://doi.org/10.1142/9789811287152_0076

Topic Study Group 52

Ethnomathematics

Gelsa Knijnik[1], Marcos Cherinda[2], Arindam Bose[3], Cynthia Nicol[4], and Aihui Peng[5]

1. Theme(s) and Description

The "TSG-52: Ethnomathematics" examined issues that consider intersections between the areas of mathematics and culture, but also went beyond to create synergies between them. We used "culture" in a dynamic, emergent, living sense to focus attention on both common traditions and understandings practiced by a group as well as how these understandings and practices shift, vary, and change over time. Our goal was to invite provocative critical engagement in the ideas of ethnomathematics research and pedagogical practices. We explored connections between mathematics, culture, community, politics, and social as well as ecological justice using reciprocal relations while going beyond non-essentialized understandings.

As written in the TSG-52 Invitation, we organized our discussions around the following themes and questions (Rosa et al, 2016):

Cultural self-confidence and reclamation: How can ethnomathematics (or ethnomathematical practices and ethnomathematics research) support transformation of educational systems (from exclusion to inclusion) at local and at global levels (toward regaining "cultural self-confidence")?

Decolonizing and indigenizing: To what extent can ethnomathematics support (or challenge) practices of decoloniality (i.e., challenge how knowledge is constructed across time and place)? How can ethnomathematics be engaged in a political/epistemological level with other systems of knowledge, and in a way that respects self-determination and sovereignty?

Indigenous education and teacher education: What role does ethnomathematics play for Indigenous education and what are the social and cultural impacts of these uses for their own communities? What kind of ethnomathematics experiences are needed for indigenous teacher education? What research exists in this area and what have we learned from these experiences?

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Critical mathematics education: In what ways does ethnomathematics support a critical mathematics education (as conceptualized by Skovsmose) and the problematizing of mathematics epistemologies and mathematics education?

Heterogeneous cultural groups: What can ethnomathematics offer in working with heterogeneous cultural groups (with varied linguistic, ethnic, caste/race diversity) but with access to rich funds of knowledge?

2. Program Overview (format, participants, presentations, posters, main outputs, etc.)

When the submission process of ICME-14 TSGs ended, 37 papers were submitted to TSG-52 Ethnomathematics. The rigorous review process was conducted from an inclusive perspective. As a result, 38 papers were approved: 21 long presentations, 13 short presentations and 4 posters. Due to the global pandemic and the move of ICME Topic Study Groups to a virtual environment our final program included 12 long presentations and 6 short presentations.

TSG-52 met over three sessions. Our initial session was divided in two parts. The first part was held by the TSG-52 organizing team, composed by Gelsa Knijnik (chair), Arindam Bose, Cynthia Nicol and Aihui Peng. Unfortunately, our colleague Marcos Cherinda (co-chair) could not attend the meeting. The TSG team welcomed everyone and participant introductions included everyone locating themselves on a world map to emphasize the varied places/lands/ and political spaces of our shared work. Following introductions, Gelsa Knijnik opened the session with a tribute to and acknowledgement of the vast and deep contributions of ethnomathematician Ubiratan D'Ambrosio. Participants were invited to offer memories and experiences with the Brazilian educator on the TSG-52 Padlet (a virtual visual bulletin board that formed the virtual hub of our group). In the second part of our first session Arindam Bose, Tata Institute of Social Sciences (TISS), Mumbai, India, presented a paper "Revisiting ethnomathematics: another social turn?" to start the TSG-52 discussion.

Each Session began with an Ethnomathematical Riddle posted on our group Padlet where participants could post their responses and strategies such as this Riddle posted on Day 3.

> *nau maun guru, nau maun guruayeen, nau maun ke dunno chela, nau maun bhaar naiya sahela, bari-bari paar karela.*
>
> A teacher's weight is nine *maun* (1 *maun* = 40 kg approx.), teacher's wife weighs the same, two students together weigh nine *maun*, one boat can bear nine *maun* at a time, how do they cross the river.

Discussion during our first session included comments and questions focused on how students' school math learning elucidates students' everyday math knowledge; on how drawing upon funds of knowledge as a theoretical framework could overcome the mechanism of scaffolding that can rework local knowledges to be replaced by prescribed curriculum knowledge; and on whether too much school math can erase our practical mathematics?

During our second session ten papers were presented followed by discussion that included questions around the need for clarity in language and terms used, the need to reduce academic jargon, the role and relationships of language, land and mathematics, and the search for ethnomathematical studies of Indigenous North and Central South America.

Session 3 involved the presentation of seven papers followed by discussion questions such as: the future direction of ethnomathematics; the diversity of ethnomathematics indicating the varied ways of being and doing ethnomathematics; the possible reconsidering of ethnomathematics as research programs (plural rather than singular), and the implications for teaching with an ethnomathematical curriculum to counter colonial understandings of school math curriculum.

Below are the titles of the papers, with the name of their respective authors, in the order they were presented during Session 2 and 3 of the TSG (Tab. 1).

Tab. 1. The list of papers presented in session 2 and session 3

Paper and author(s)	
[1]	A framework for examining the quality of mathematics teaching for mathematical understanding in ethnic minority cultural contexts. ***Aihui Peng*** (China).
[2]	Ethnomathematics and ethnomodelling research: glocalizing educational systems from exclusion to inclusion at local and global levels. *Daniel Clark Orey and* ***Milton Rosa*** (Brazil).
[3]	The ethnomodelling as a math learning strategy in the Ecuadorian educational system. ***Juan Ramon Cadena Villota*** (Ecuador)
[4]	Ethomathematics as pedagogical and political tool in an Indigenous school curriculum. ***Vanessa SenaTomaz*** *and Ozirlei Teresa Marcilino* (Brazil).
[5]	Mexican American Women talking about Graphs: A focus on their lived experiences. ***Fany Salazar*** *and Marta Civil* (USA).
[6]	Regaining cultural signs through ethnomathematical descriptors: artifacts, sociofacts and mentifacts. ***Ma. Elena Gavarrete****, Milton Rosa, and Daniel Clark Orey* (Costa Rica).
[7]	Perspectives of mathematics by traditional P'urhpécha artists. ***Thomas E Gilsdorf*** (USA).
[8]	A study of the Quechua weaving elaboration process and mathematics teaching in basic education. ***Maria del Carmen Bonilla*** (Peru).
[9]	Math trail activity on Machchhindranath Chariot: cultural perspective on mathematics education in Nepal. ***Toyanath Sharma*** (Nepal).
[10]	Ethnomathematical study on cultural artefacts: anethnographic field to classroom practice. ***Jaya Bishnu Pradhan*** (Nepal).
[11]	Coming together, research and desire in the field of ethnomathematics. ***Wilfredo Alangui*** (Philippines).
[12]	Waka migrations: reclaiming cultural traditions and identity. ***Anthony Benjamin Trinick*** *and Tamsin Meaney* (Newzeland).
[13]	Exploring mathematics in the Eskaya tribe: an ethnolearning theory. ***Fe Reston Janiola*** (Philippines).
[14]	Ethno-mathematics of Banyuwangi culture: bamboo woven. ***Mega Teguh Budiarto****, Rini Setianingsih, and Rudianto Artiono* (Indonesia).
[15]	Towards mathematics curriculum recontextualisation: developing a rhizocurrere with Roma students. ***Georgios Kyriakopoulos*** (Greece).
[16]	An international class in Germany: the need for ethnomathematical considerations. ***Marc Sauerwein*** (Germany).
[17]	Ethnomathematics in Ethiopia using glocal approach: the case of Gebeta playing. ***Solomon Abedom Tesfamicael****, Anne H. Nakken, Tirillo, and Peter Grey* (Norway).
[18]	Ethnomathematics constructs of Ibo society in Chinua Achebes "things fall apart". ***Epsi Deme*** (Nigeria).

3. Future Directions and Suggestions

At the end of the work, the participants showed their satisfaction with what we had accomplished. The variety of topics and the depth of coverage were highlighted. The importance of bringing the papers together in a publication was discussed. In fact, the TSG Team has invited all colleagues who had their papers accepted to contribute to an edited book. This volume work is currently underway with an intended publication date before ICME-15 conference so that copies are available during the conference.

Reference

M. Rosa, U. D'Ambrosio, D. C. Orey, L. Shirley, W. V. Alangui, P. Palhares, and M. E. Gavarrete, (2016). *Current and Future Perspectives of Ethnomathematics as a Program*. ICME-13 Topical Surveys. Springer.

https://doi.org/10.1142/9789811287152_0077

Topic Study Group 53

Equity in Mathematics Education

Jayasree Subramanian[1], Darinka Radovic[2], Constantinos Xenofontos[3], and Changgen Pei[4]

1. Themes and Description

The Topic Study Group TSG-53 on Equity in Mathematics Education built on the longstanding practice that ICME had of addressing social justice concerns in mathematics Education. Themes such as gender, disability, indigenous mathematical knowledge, socioeconomic class, culture and language figure consistently and over the years in the activities of ICME and in 2016, ICME-13 introduced a Topic Study Group TSG (33) on Equity (including gender). TSG-33 already acknowledged the need to go beyond gender as a binary and the need to address disability.

1.1. Aim of the TSG

In its description, TSG-53 brought in caste, religion, and race in addition to socioeconomic status, culture, region, ethnicity, physical and mental disablement as factors that contribute to exclusion in mathematics education, noting that these forms of exclusion are historical, structural, as well as interpersonal and individual and they have lasting consequences for the students in accessing higher education and in pursuing mathematics. Instead of making unqualified declarations such as "mathematics for all", the TSG acknowledged the need for a nuanced understanding that every student should have the opportunity to learn mathematics to the extent they feel it is desirable or appropriate for them, both from the perspective of fairness in mathematics education and from the fact that mathematics plays a central role in the geopolitical and globalized technocentric world in which we are living. Moreover, it also acknowledged the need for the realization that the equity question is integrally linked to a critical understanding of mathematics education that is prevalent in schools across the world and a movement towards a more just and humanizing mathematics education for the learner.

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TSG-53 sought to foreground the following concerns and questions.

1) What are the experiences of religious, racial and ethnic minorities and those from marginalized castes in learning mathematics? How do class, caste, race, culture and ethnicity operate to shape opportunities for the learners in the classroom?
2) How do factors such as migration for labour, continued conflict, repeated natural calamities and so on impact teaching and learning of mathematics? What systemic measures (if any) have been evolved to address these issues?
3) How do we address language diversity in mathematics classrooms? Given that English is emerging as the language of power and possibilities and it is the medium instruction in large number of schools in urban locations, what are the challenges involved in teaching and learning mathematics in a language that is not the home language for the teacher as well as the students?
4) How does family and community membership affect mathematics learning and teaching? In what ways can we leverage children's lived experiences and funds of knowledge as resources for learning and teaching mathematics?
5) What do we know about the experiences of students who are placed in the intersection of several categories? More specifically what does mean to be an African American girl, a rural poor boy, an emerging bilingual immigrant child seeking asylum in a different county, a migrant laborer's child living in a multilingual urban slum, a tribal, Dalit or a transgender student from a low/ middle income family and so on in the context of teaching and learning mathematics?
6) How do teacher education programs acknowledge the existing research on equity issues and introduce measures to address the issues in the preservice and in service training programmes?
7) What are some of the new theoretical frameworks evolved to understand the complex contexts in which mathematics education takes place and how we can transform mathematics education to be more humanizing and just spaces to learn and teach mathematics?

1.2. A brief account of the response to the call for submissions

In response to the call, TSG-53 received a fair number of submissions which together addressed the concerns we foregrounded in the call for submissions, except the second one about the impact of labor migration, conflict and natural calamities on the children's education, in particular their mathematics education. However, the complications created by the COVID-19 pandemic and the online mode of the conference made a difference to what finally could be realized.

2. Program Overview

We, the TSG-53 team members decided that we will have a few invited presentations and arrived at a consensus on the names of the scholars we would invite. We decided to invite Luz Valoyes Chavez from Catholoc University of Temuco, Luis Leyva from Vanderbilt University, and Charalou Stathopoulou from University of Thessaly. Stathopoulou wanted to present an invited paper jointly with Peter Applebaum from Arcadia University which we accepted. Including the invited papers, a total of 36 papers were submitted in response to the call for submission from TSG-53. Of these, leaving out the three invited papers, 32 were accepted to be presented as either long oral presentation or a short oral presentation or as a poster. However, as the Congress was going to be held only in the online mode because of COVID-19 pandemic, several authors were not able to participate. This was mainly because most of the participants did not receive financial support from their respective universities for online conferences; and they were eligible to get the Solidarity Funds from ICME. Indeed, this was also the case for one of the invited speakers. As a result, only 20 papers were presented. Of the 20 papers presented, 3 were invited papers, 9 were long oral presentations and 8 were short oral presentations. One of the invited papers could not be presented in person as the speaker did not receive financial assistance to attend the conference from any source. Given the importance of the theme, we got the speaker to record the presentation and send, which we played. Apart from these there were 3 submissions accepted as poster presentations.

2.1. Format of the presentations

TSG-53 had a total of six and half hours (390 minutes) spread across 4 days. On the first days we had 120 minutes and on the remaining three days we had 90 minutes each. As we were informed that more participants would be able to participate in the time zone in which the 90 minutes presentations were scheduled, we decided to allocate one invited talk to each of the three days of 90 minutes.

The time allocation for the presentations is given in Tab. 1 below:

Tab. 1. Time allocation for the presentations

Type of Presentation	Time for presentation	Time for discussion
Invites Talks (IT)	25 minutes	15 minutes
Long Oral Presentations (LO)	12 minutes	8 minutes
Short Oral Presentations (SO)	5 minutes	4 minutes

To the extent possible we tried to group presentations sharing the same or similar themes together. But this was not always possible. We also grouped a few short orals together and took the questions for them at the end to optimize available time for discussion.

2.2. Major themes, participants, and the presentations

The submission and the final set of presentations remained faithful to the central concerns of the TSG. The following Tab. 2 gives the title and authors of the presentation just to indicate the diversity of issues raised in the topic study group.

Tab. 2. The list of papers presented

Paper and author(s)
[1] A framework for detailing White heteropatriarchy in mathematics education. ***Luis Leyva*** (USA). (IT)
[2] Chavez cultural power and the fabrication of race difference in the mathematics classroom. ***Luz Valoyes-Chavez*** (Chile). (IT)
[3] Challenging the abyssal line between Roma and non-Roma in and out of the (mathematics) classroom through common spaces. ***Charoula Stathopoulo*** *and Peter Applebaum* (Greece). (IT)
[4] Education equity in Honk-Kong: factors that contribute to Hong Kong students' performance in trends in international mathematics and science study (TIMSS) 2015. ***Frederick Koon Shing Leung*** (Hong Kong SAR, China). (LO)
[5] Critical mathematics teacher noticing: exploring pre-service teachers' noticing of power, privilege, and identity using online video. ***Theodore Chao****, Melissa Adams-Corral, Youmna Deiri, and Joanne Vakil* (USA). (LO)
[6] Questioning the idea of inclusion of blind mathematics learners in India using the social model of disability. ***Rossi D'Souza*** (India). (LO)
[7] Disentangled narratives: exploring lecturers and students gendered discourses in an engineering faculty. ***Darinka Radovic*** (Chile). (LO)
[8] Gender issues and consequences for undergraduate mathematics women students. ***Weverton Ataide Pinheiro*** *and Vanessa Franco Neto* (Brazil). (LO)
[9] History of whose mathematics for teaching: raising the caste question in mathematics education in India. ***Jayasree Subramanian*** (India). (LO)
[10] From invisible to domestic gender in mathematics textbooks in India. ***Kishor Darak*** (India). (LO)
[11] Teaching practices in diverse mathematics classrooms of the republic of Cyprus: equitable or not? ***Constantinos Xenofontos*** (UK). (LO)
[12] Microexclusions as a challenge to dialogue among deaf and hearing students. ***Amanda Queiroz Moura*** (Austria). (LO)
[13] Gender differences in student-student interactions. ***Desiree Ippolito****, Weverton Ataide Pinheiro, and Jinqing Liu* (USA). (SO)
[14] Socioeconomic differences delimited by gender: students' perceptions about mathematics in Mexican Schools. ***Itzel H. Armenta*** (Mexico). (SO)
[15] Gender differences on specific issue: the case of misconceptions in operating with percentage. ***Chiara Giberti*** (Italy). (SO)
[16] Support for students with mathematics learning dis/abilities on bridging programmes in New Zealand universities. ***Phil Kane*** (New Zealand). (SO)
[17] Coping with the challenges while promoting social justice in mathematics classroom. ***Ram Krishna Panthi*** (Nepal). (SO)
[18] Adapting tasks between including and excluding students. ***Nina Ines Bohlmann****, Ralf Benölken, and Timo Dexel* (Germany). (SO)
[19] Children, dialogue and mathematics education. ***Ana Carolina Faustino*** (Brazil). (SO)
[20] Teacher candidates' perspectives of means to facilitate equitable learning opportunities during a high school mathematics methods course. *Ruthmae Sears, Marilyn Strutchensy, Brian Lawler, Lakesia Dupree,* ***Caree Pinder****, and Cynthia Castro-Minnehan* (The Bahamas). (SO)
[21] How to increase girls' sustaining interest, performance and career choices in mathematics: a high-quality project-based learning approach. ***Lorraine Minette Howard*** (USA). (Poster)
[22] Rural elementary students' mathematics academic performance in china: what are the influencing factors? ***Xiangyi Kong*** (China). (Poster)
[23] Theoretical framework of gendered mathematical identity. ***Yuriko Kimura*** (Japan). (Poster)

The 23 presentations — 20 papers and 3 posters — came from as many as 14 countries which in itself is remarkable, though there were visible absences — there was no submission from any of the countries from Africa for example. The list of countries consists of Bahamas, Brazil, Chile, China (including Hong Kong SAR), Germany, Greece, India, Italy, Japan, Mexico, Nepal, New Zealand, United Kingdom, and United States.

Gender, race, caste, ethnicity, disability, and sexual orientation figured as a common thread running through the presentations. Most of the presentations were very nuanced, brought in fresh perspectives, introduced new theoretical frameworks and raised important questions. In that sense and in terms of the range and the complexity of the themes addressed, it can be said that the TSG realized a significant part of what it set to realize.

However online conference posed several challenges, one major difficulty being not all the papers that were accepted for presentation could be presented. We missed out on 9 more presentations. Many of the authors could not participate because their institution refused to pay the registration fee for attending an online conference, nor did the speakers get fee waiver from ICME. Another difficulty was the difference in the time zone which limited the number of people who were present in each session and contributed to the discussions. Face to face conferences not only bring everyone together at one place, but they also allow for more participations in the discussions and often these discussions continue even after the presentations for the day is over. Online conference deprived us of the opportunities for such interactions.

3. Future Directions and Suggestions

TSG-53 built on what had been achieved with regards to addressing equity issues in ICME, brought in new perspectives and questions. However, there are several more issues that need to be studied and understood and here is a short list of some of them.

a) Even though TSG-53 complexified gender by engaging with heteropatriarchy and with the concerns of gender queer learners, not all presentations looked at gender beyond the binary. This poses the following challenge to us: Can we, as a community of mathematics education researchers, accommodate research that continue to see gender as a binary? We need more studies to understand how young learners who do not identify with the gender assigned to them at birth, learners who identify as gender queer or as transgender persons, cope with (or fail to cope with) mathematics in school.

b) The COVID-19 pandemic and the lockdown has had a devastating effect on the education of learners belonging to socio-economic margins as well as those who live in remote areas with limited or no access to technology. Learners who are socio economically privileged, have had access to technology and parental support at home too have encountered serious shortcomings with online education. The impact of lockdown and online education on mathematics education for diverse learners and the measures taken by the state and voluntary organizations to address these need to be studied.

c) The consequence of migration for work, increasing urbanization, living in regions that face continued conflict, repeated natural calamities etc on education and on mathematics education in particular, need to be studied and reported. One way to ensure that some of these figure in the Topic Study Group on Equity would be to identify scholars working on these themes and invite them to present a paper in the upcoming TSG on equity.

d) Equity can be explored in relation to several social issues (e.g. social class, race, caste, gender, ethnicity, disability etc.). More intersectional research is needed for understanding how various social issues interact with each other, and how these interactions impact mathematics education.

e) There is a lack of studies approaching issues of equity from cross-national/cross-cultural comparative perspectives. Collaborations between researchers, teachers, policymakers etc. working in different educational settings could shed more light on the cultural specificities of educational policies and practices.

https://doi.org/10.1142/9789811287152_0078

Topic Study Group 54

Social and Political Dimensions of Mathematics Education

Paola Valero[1], Kate le Roux[2], Andrew Brantlinger[3], Murad Jurdak[4], and Xuhui Li[5]

1. Aims of the TSG

The broad focus of TSG-54 is to explore how mathematics education practices, research and policy in current societies connect to power. Since ICME-13, when TSG-54 was run for the first time, unexpected world events have drawn attention to a series of deep changes that constitute a new reconfigured context for education, and mathematics in particular. This is a new geo-economic-political configuration of relations, between humans and between human and non-humans that put at stake their conditions of existence, a landscape termed the New Climatic Regime by Latour (2018). These conditions are manifested in the multiple crises of societies, among those the COVID-19 pandemic.

Thus, at ICME-14 we aimed to build on the advances of ICME-13 in the light of the contemporary world landscape, inviting empirical and theoretical contributions offered from different locations and experiences. The following questions guided our task: (1) How do contemporary national and global economic and political interests relate to the changes in material conditions in which mathematical and mathematics education practices take place? (2) What are the relations between policy and the directions they steer and mathematics education practice and research? (3) How does the meaning of key concepts used in sociopolitical research — such as access, equity, quality, inclusion — emerge in particular space and time configurations? How might these be different in a New Climatic Regime at this time, and why? (4) What do theories and methodologies of sociopolitical research offer to understand the articulation of mathematics education and this contemporary landscape? (5) Which forms of activism and action emerge to question and/or promote mathematics at his time?

1.1. Submissions

For the 2019 review process, we received 32 submissions from 20 countries, distributed as follows by continent (Africa: 3; Asia: 6; Australasia: 1; Europe: 13;

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[4]American University of Beirut, Lebanon
[5]California State University, Long Beach, USA

North America: 3; South America: 6). Of these, 13 were accepted as short presentations, and 19 were invited to develop into long paper presentations. In addition, we invited one long paper. Of the 32 submissions accepted in 2019, 24 formed the final TSG-54 at the 2021 online Congress (5 long papers and 19 short presentations).

1.2. Sessions

We chose to prioritize discussion in our three allocated TSG-54 sessions. Informed by our guiding questions, we wished to explore emerging themes across the contributions, with the aim to further develop our thinking together. Thus, prior to the event, we circulated — with the necessary author permissions — the 24 papers put forward for the 2021 Congress to participants. Then, each of the TSG-54 sessions had three main components: paper presentations limited to five minutes per paper and supported with a one-page slide; individual reflection time on common themes and questions recorded in a shared electronic whiteboard; and plenary time to look at individual whiteboard responses and to discuss ideas to take forward. The final two-hour session also included a concluding discussion, reflecting on the contributions and discussions across all sessions, and thoughts on advancing the ideas. To promote inclusion in the online Congress format, the grouping and ordering of presentations was primarily based on the geographic location of the presenters, with common themes considered next.

1.3. Paper topics

A list of the 24 contributions is included in Tab. 1 (on the next page). In total, authors of 23 of these papers presented at the TSG.

2. Common Themes Explored and to be Taken Forward

Participants offered responses to the guiding questions from different areas of mathematics education: teaching, learning and assessment in school and university mathematics classrooms; teacher education; resources such as textbooks and other online materials; research theories and practices; the nature of "mathematics"; and the positioning of the "mathematical citizen". In spite of such variety, there were identifiable commonalities with respect to:

1) Theoretical approaches, in particular posthuman thought and decolonial thought. This is a visible advance in the last years and a contribution from so-named "global South".

2) Attention to context in which mathematics and mathematics education is practised. This means that people research and write *from* the particularities of context rather than adopting an "external" perspective *on/about* context.

3) Acknowledgement that the problematization of mathematics and mathematics education that aims at reimagining it is not completely new. It is highlighted by and exacerbated in the New Climatic Regime, including the health pandemic. Recognition of

the connectedness of *all* places in their experience of crises, albeit in inequitable ways, and the extent of how crises are being experienced in the "global South", make the problematization relevant for all.

Tab. 1. Authors and papers titles in order of presentation

Paper and author(s)	
[1]	Mathematics education and the Anthropocene: Educating in precarious times. ***Alf Coles*** (UK).
[2]	The cultural politics of mathematics education in the "New Climatic Regime". ***Paola Valero*** (Sweden).
[3]	Promised "land" of mathematics education: Towards a sociomaterial tracing of research on children's mathematics. ***Ayşe Yolcu*** (Turkey).
[4]	Thinking about mathematics education and the political with Laclau and Mouffe. ***Dionysia Pitsili-Chatzi*** (Canada).
[5]	Critical, reflexive, justice-informed mathematics education: Troubles of justice and decolonial possibilities. ***Dalene Swanson*** (UK).
[6]	Black holes in Chilean teachers training programs: Mathematics teacher practices and educational policie. ***Melissa Andrade-Molina*** (Chile).
[7]	Governmentality and performativity in the process of making Brazilian mathematics textbooks. *José Wilson dos Santos and* ***Marcio Antonio da Silva*** (Brazil).
[8]	The globalisation of testing and learning outcomes. ***Anita Rampal*** (India).
[9]	Within-school tracking and mathematics learning outcomes: A case study in Yogyakarta. ***Shintia Revina****, Goldy Fariz Dharmawan, and Florischa Ayu Tresnatri* (Indonesia).
[10]	Teacher conceptions on social justice and democracy in mathematical education. *Natalia Ruiz-López and* ***José Bosch Betancor*** (Spain).
[11]	Maths vs. Letters: A systematic delirium. ***Gustavo Nicolas Bruno*** *and Natalia Ruiz-Lopez* (Spain).
[12]	Making mathematical talk possible: A case of teaching calculus in our contemporary world. ***Sabrina Bobsin Salazar*** (Brazil).
[13]	Drawing an aesthetic of mathematics education research. ***Alex Montecino*** (Chile).
[14]	About the mathematics that we teach. ***Yasmine Abtahi*** (Norway).
[15]	Powerful new frontiers: A preliminary exploration of assessment as relational relevance in authentic caring mathematics education. *Paulo Tan, Alexis Padilla and Anette Bagger.*
[16]	Addressing social issues by empowering students using model-eliciting activities and projects in mathematics lessons. *Mulugeta Woldemichael Gebresenbet.*
[17]	The presentation of core socialist values in Chinese junior middle school mathematics textbooks: Based on the analysis of five series of PEP textbooks. ***Jian Li****, Lili Song, Na Tang, Zhentian Mao, Yueyuan Kang, Hong Yan and Han Yu* (China).
[18]	Interrogating the promise of online mathematics instructional programs. ***Lisa Jean Darragh*** (New Zealand).
[19]	Contextual barriers to the integration of problem solving in the Egyptian mathematics classroom. ***Mariam Makramalla*** *and Andreas J. Stylianides* (UK).
[20]	Teaching critical mathematics: Obstacles from the teacher's perspective. ***Daniela Steflitsch*** (Austria).
[21]	Transition of Mozambique's primary mathematics intended curriculum in post-colonial period: A focus on adaptation from exogenous curriculum. ***Satoshi Kusaka*** (Japan).
[22]	Crests and troughs: The use of trigonometric modeling towards a critical and realistic mathematics education. ***Dale Aldrinn Pradel*** *and Catherine Vistro-Yu* (Philippines).
[23]	Mathematics education, citizenship and the "commons" in our "global" world? *Anna Chronaki* (Sweden), *Eirini Lazaridou and* ***Effie Manioti*** (Greece).
[24]	A southern perspective on sociopolitical mathematics education research in the New Climatic Regime. *Kate le Roux.*

4) Concern for an ethical mathematics education that recognises: (a) multiple mathematical knowledges, ways of knowing and doing mathematics, and being a “mathematical” knower, and (b) “the commons”/“relations”: between the human (body and mind), the non-human, material technology, place, and so on.

Yet substantial work is required to understand the particularities of the role of mathematics and mathematics education in the contemporary landscape, in particular how to conceptualise and enact basic common values, aims and conditions for an ethical and responsible mathematics education. Such work requires the challenging tasks of:

- Troubling the strong and dominant narratives of the “power” of a particular place-based “mathematics” (“Western Mathematics”), for example, the use-value and exchange-value attributed to this mathematics (e.g., Williams, 2012).
- Thinking about the role of mathematics and its “use-value” and “exchange-value” in the New Climatic Regime, in particular in the prevalent conditions of precarity and unemployment (absence of “work”), even more urgent in the context of a post-COVID-19 pandemic world.
- Making space in educational practices, curricula and policies to other forms of mathematical knowledges and ways of knowing that can enrich people’s notions and experiences of mathematics beyond the closed standards of school mathematics.
- Navigating the tension of foregrounding and backgrounding mathematics, that is shifting between a mathematics that is considered objective and fixed, and a sensibility for mathematics in the dynamics of context, diversity and power. This requires exploring the extent to which the mathematics education is open to consider “mathematics” content both as unique and at the same time just one of the many equally important elements entangled in the current predicaments of our times. This is important if the aim of mathematics education is providing tools to live in and understand our complex, wicked world.
- Instantiating how “mathematics” and mathematics education might be something else, if at all possible.

Taking such concerns forward, returns us to the notion of *power* that underpins the work of TSG-54. There is a certainly a need for us to consider what theories of power we are using to understand the contemporary world, and to inform our research, activism and pedagogical action. In particular, as a community we need to understand why and how mathematics and its processes of teaching and learning are conceived as powerful in both a positive productive way of empowering, and also as oppressive, selective and excluding from education and society at large. In other words, we need to continue exploring the power relationships at work in mathematics education spaces and what and how practices instantiate both their positive and negative effects. Important questions remain such as what mathematics and mathematics education become valuable? For whom? How do these become valuable? Where do they become valuable? We also need to consider where and to whom we look (beyond the privilege of academia) for such learnings.

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https://doi.org/10.1142/9789811287152_0079

Topic Study Group 55

The History of the Teaching and Learning of Mathematics

Alexander Karp[1]

ABSTRACT This note describes the work of TSG-55, indicating its main organizational stages and the number of reports presented. Brief descriptions of the topics of the presented reports are provided. These may be grouped into several categories: reports devoted to reforms; reports investigating the work of various mathematics educators; reports discussing the history of specific subjects in courses, textbooks, or other handbooks; periodicals, connected with mathematics education; and so on. Lastly, certain conclusions are drawn about the current state of this academic field.

Keywords: History of mathematics education; Reforms; Mathematical journals.

1. Introduction

The Topic Study Group on the history of the teaching and learning of mathematics was formed in 2004 at ICME-10. Since then, this group has worked at all International Congresses. At the Congress of 2021, this group was also active, although the pandemic made it necessary to introduce certain changes into its work. As usual, all submitted proposals were reviewed and evaluated (in this connection, we must note the role of Wagner Rodrigues Valente from Brazil, who was initially chair of the group, but later resigned from participating in its work for personal reasons). At the final stage, when it became clear that the Congress would mainly take place online, all participants were invited to prepare video presentations, which were made available on a specially created website. This made it possible, for example, to become acquainted with presentations that could not be attended due to time differences.

At the Congress itself, three sessions were held, as planned, chaired by Alexander Karp (USA/Russia) and Naomichi Makinae (Japan). In all, eighteen reports were presented; in addition, there was one poster presentation related to the work of the group (Tab. 1 on the next page). Representatives from Belgium, Brazil, China, Croatia, Japan, Nepal, Poland, Russia, Spain and the United States took part.

All participants were also invited to take part in preparing a collection of papers based on the materials of the presentations — the expanded papers were envisioned, substantially greater in size than could be presented at the Congress, but thematically connected with these presentations. From the submitted papers, the best was selected,

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Tab. 1. The list of papers and posters presented

Paper and author(s)
Session 1
[1] Pafnuty Chebyshev as a mathematics educator. ***Vasily Busev*** (Russia) *and Alexander Karp* (USA).
[2] Frédérique Papy-Lenger, the mother of modern mathematics in Belgium. ***Dirk De Bock*** (Belgium).
[3] The history of mathematics education of Tatar Nation. ***Ildar Safuanov*** (Russia).
[4] Mathematics and mathematics education in the 18th century Spanish Journal "Semanario De Salamanca". ***Maria Jose Madrid****, Carmen Leon-Mantero, and Alexander Maz-Machado* (Spain).
[5] Gnomonics in mathematics secondary school education on the territories of Poland in the 17th – 20th century. ***Karolina Karpinska*** (Poland).
[6] The beginning of modern mathematics in Spanish primary education. A look through textbooks and curriculum. ***Antonio M. Oller-Marcen*** (Spain).
[7] Approach of an Early-1940s Japanese secondary mathematics textbook to teaching the fundamental theorem of Calculus. ***Shinnosuke Narita****, Naomichi Makinae, and Kei Kataoka* (Japan).
Session 2
[8] Arithmetic textbooks in Croatia in the premodern period. ***Maja Cindric*** (Croatia).
[9] Missing arithmetic methods: "On the rules for the mixing of analogous things". *Bernardo Gomez-Alfonso and* **Maria Santagueda-Villanueva** (Spain).
[10] The calculation in the first commercialized Decroly's games. ***Pilar Olivares-Carrillo*** *and Dolores Carrillo-Gallego* (Spain).
[11] Mathematical activities focusing on Japanese elementary arithmetic and secondary mathematics textbooks in the early 1940s. ***Yoshihisa Tanaka****, Eiji Sato, and Nobuaki Tanaka* (Japan).
[12] Development history and course setting of mathematics department in early universities in Sichuan Province in modern times (1896–1937). ***Hong Zhang*** (China).
[13] A probe into compiling mathematics textbooks by Christian Missionaries in late Qing dynasty. ***Jun Wei Li*** (China).
Session 3
[14] Building an American mathematical community from the Ground Up: Artemas Martin and the mathematical visitor. ***Sian E. Zelbo*** (USA).
[15] The discarding of the rule of three in the 1960s: changes in elementary education in France and Brazil. ***Elisabete Zardo Burigo*** (Brazil).
[16] Mathematics education for young woman during progressive era: historical overview. ***Yana Shvartsberg*** (USA).
[17] David Eugene Smith (1860–1944) and his work on mathematics education. ***Alexei Volkov*** (Chinese Taiwan) *and Viktor Freiman* (Canada).
[18] College entrance exams in mathematics in Russia before the second world war: development, role, objectives. ***Alexander Karp*** (USA).
Poster Session
[19] Historical development of mathematics education in Nepal. ***Ramesh Prasa Awasthi*** (Nepal).

and consequently a collection was prepared, which was submitted to Springer Publishing (Karp, in print).

2. On the Topics of the Presentations

Below, an attempt will be made briefly to describe the topics of the presentations.

As is usual at conferences on the history of mathematics education, considerable attention was devoted to reforms. Oller-Marcén[6] spoke about the first moments of the introduction of the reform movement in Spain in 1965 and the new features that appeared in elementary school textbooks in Spain. Búrigo[15] also spoke about elementary schools: she compared how the same topic — the rule of three — was studied in Brazil and in France during the 1960s, and what changes occurred in how it was studied. Narita et al.[7] — spoke about a far less well-known reform that was conducted in Japan during the 1940s and involved a radical transformation in how calculus was studied.

In effect, the period of reforms also provided the subject for De Bock's paper[2], which was, however, devoted not so much to the topics studied during this period, as to a figure active during it. This paper discussed the work of Frédérique Papy-Lenger, who was extremely important particularly during the late 1950s–1960s. The paper by Volkov and Freiman[17] was devoted to the early work of a reformer from another era, David Eugene Smith, or more precisely, to his reception of the achievements of German methodological thought and his related early publications. An even earlier period was the subject of the paper by Busev and Karp[1]. This paper relied on recently published materials to discuss the work of the outstanding mathematician Pafnuty Chebyshev in mathematics education.

The history of the teaching of one or another section of the school course in mathematics was investigated in several other papers, as has already been mentioned. Gómez-Alfonso and Santágueda-Villanueva[9] spoke about mixture problems in arithmetic in ancient textbooks. Karpińska (Poland)[5] described how students were taught to tell time by using a sundial, which constituted an important part of the course in mathematics at Polish schools for centuries.

Not a little attention, of course, was devoted to a field that has traditionally attracted researchers' attention: textbooks and other means of instruction. Cindrić[8] discussed the first textbooks in arithmetic used in what is today Croatia during the eighteenth century. Li's presentation[13] was devoted to textbooks written for China by Christian missionaries during the late Qing Dynasty period. Tanaka et al.[11] presented to listeners' attention their analysis of certain mathematical activities and associated instruction materials used in the teaching process in Japan during the 1940s. The paper by Olivares-Carrillo and Carrillo-Gallego (Spain)[10] was devoted to games developed by Ovide Decroly and their adoption in Spain.

In the last decade, it has become popular to study journals that are in one way or another connected with mathematics education. Two presentations were devoted to this topic. Madrid et al.[4] analyzed the eighteenth-century Spanish journal *Semanario de Salamanca*, which devoted considerable attention to mathematics. Zelbo[14] investigated an American mathematics journal, *The Mathematical Visitor*, which was published regularly from 1878 to 1881.

Shvartsberg's presentation[16] contained an analysis of the development of mathematics education for women in the United States between the 1890s and the 1930s, when very many changes occurred both in ordinary life and in education.

Zhang[12] devoted her presentation to almost the same period — 1896–1937 — but focusing on Chinese higher education and its modernization during these years.

Karp's presentation[18] was devoted to entrance exams to higher educational institutions in Russia before 1917.

Lastly, Safuanov[3] delivered an overview of the history of mathematics education in Tatarstan, while Awasthi[19] provided a poster presentation on the history of mathematics education in Nepal.

3. Certain Observations and Conclusions

Summing up the outcomes of the work of the Topic Study Group, we might say that we are witnessing the accumulation of new studies that are based on topics and interests that have already become traditional (Karp and Furinghetti, 2016; Karp and Schubring, 2014). The history of mathematics education as an academic discipline is certainly over a hundred years old — even if we date its beginnings to the appearance of the first doctoral dissertations in the United States (for example, Stamper, 1906), although historical studies in mathematics education had already existed in Europe long before that (Schubring, 2014). Nonetheless, it may be said that the social history of mathematics education, which examines what has occurred in mathematics education as part of a broader social process, is still only at the beginning of its development. The tendency to connect what happened in classrooms or during preparation for class with what was happening in the world can be observed in many, although still not all, studies.

It may be said that many periods, processes, and directions still remain uninvestigated, and in addition that even many extant studies remain unknown and inaccessible to an international audience, if only due to a language barrier. The opportunities this state of affairs offers to researchers are all the greater. Currently, we are in a period of collecting materials about what has not been investigated previously, and thereby making use of new sources and developing new approaches to analysis.

The Congress and other international initiatives make it possible to find out about kindred studies being conducted in different countries. One would like to hope that the work of the TSG, as well as the publication of a volume on the basis of its outcomes, will help historians of mathematics education, enriching them with new facts and new ideas.

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https://doi.org/10.1142/9789811287152_0080

Topic Study Group 56

Philosophy of Mathematics Education

Bronislaw Czarnocha[1], Maria Bicudo[2], and Paul Ernest[3]

1. Introduction

The philosophy of mathematics education can be traced back to the work of Plato. In his Republic Plato considered deeply the role and purpose of mathematics in teaching and learning. His enquiries were founded on ethics, for questions of meaning and purpose within a social context inevitably bring in the Good. At the same time, he was interested in the epistemology and ontology of mathematics and its relations with the Truth and Beauty. Overall, Plato displayed great interest in the subject of mathematics throughout his philosophical work, and he is an inspirational godfather and patron saint of the philosophy of mathematics education.

Current mathematics education research is mostly concerned with two questions, one epistemological and the second methodological. The epistemological questions ask what is mathematical truth and how do we justify and explain it, and above all, how to we come to know it? The methodological questions concern how we can best and most effectively teach and facilitate the learning of mathematics. Research in the philosophy of mathematics education also addresses epistemological questions of mathematics and its teaching and learning, but it does so more explicitly, more theoretically. In addition, it considers the ontological, aesthetic and ethical issues of mathematics with respect to education and society.

The philosophy of mathematics education is an interdisciplinary area of research that incorporates many questions.

- What are the goals and purposes of mathematics education?
- What can we learn from deep analyses of the methods and means of teaching?
- and learning mathematics, as well as from studying the underlying theories and philosophies?
- What new insights are revealed by the application of deep theoretical approaches including Phenomenology, Hermeneutics, Complexity, Embodiment and Critical Theory within research in the philosophy of mathematics education?

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- What are the relationships between and, the mutual influences on each other, of the philosophy of mathematics and mathematics education?
- How do the personal philosophies of mathematics and mathematics education of learners, teachers, teacher educators and researchers impact on practice?
- How are the different actors of interest including students, teachers, researchers, theorists, philosophers and mathematicians linked together professionally within the fields of mathematics education research and practice?
- How do mathematics and the philosophy of mathematics impact on the nature, structure and content of mathematics for teaching?
- What do deep analyses of mathematics itself tell us about its structures, processes and fundamental concepts and about their relationships with its teaching and learning?

2. General Information

Due to the pandemic, there were only 14 presentations out of 24 originally submitted. However, all papers accepted to the original conference have been published in the Special Issue of the Mathematics Teaching Journal Online Vol. 12 N 2 at https://commons.hostos.cuny.edu/mtrj/archives/volume-12-n-2/

Also, many of the authors have been invited to contribute to the upcoming volume Philosophy of Mathematics Education: Work in Progress to be published by Springer in 2023.

The presentations were divided in accordance with the 2020 plans, however we had to fill up a couple of important spots in the program. Scovsmose was especially invited to give an introductory talk on Mathematics and Ethics.

Our surviving presentations have arranged themselves loosely along three themes:

1. Philosophical foundations and approaches, and within them we have several presentations touching upon critical mathematics education; we have an example of phenomenological approach (we had several more examples in the original collection) as well as attempts at epistemological clarification.
2. Philosophical problems; formulating and solving philosophical problems. For example, two presentations following Mathematics and Ethics, address the problem of rigor and the problem of imagination. On Friday we have addressed the problem of the algorithm and the problem of appropriation.
3. Philosophy of mathematics classroom. Here we also have a variety of lenses, math classroom epistemologies, digital games or theory-practice divide. Unfortunately, several other articles addressing modern Internet classrooms have also disappeared.

Meeting of TSG-56 took place on

Tuesday July 13, 19:30 – 21:00
Friday July 16, 21:30 – 23:00
Saturday July 17, 14:30 – 16:30

3. List of Presentations

Tab. 1. The list of papers and posters presented

Paper and author(s)
Session 1
[1] Mathematics and ethics. ***Ole Scovsmose*** (Brazil)
[2] Philosophy, rigor and axiomatics in mathematics: intimately related or imposed? ***Min Bahadur Shrestha*** (Nepal).
[3] Imagination in the philosophy of mathematics and its implication for mathematics education. ***Yenealem Ayalev*** (Ethiopia).
Session 2
[4] Towards a philosophy of algorithms as an element of mathematics education. *Regina D. Möller and* **Peter Collignon** (Germany).
[5] Appropriation mediates between social and individual aspects of mathematics education. ***Mitsuru Matsushima*** (Japan).
[6] Philosophical inquiry for critical mathematics education. ***Nadia Kennedy*** (USA).
[7] Towards critical mathematics. ***Theodore Savich*** (USA).
[8] Recognizing mathematical anthropocentrism. ***Thomas Ricks*** (USA).
[9] Curriculum system of the philosophy of mathematics education for normal students. ***Yaqiang Yan****, Suyue Xue, and Junfeng Ma* (China).
Session 3
[10] Research procedures to understand algebraic structures: hermeneutic approach. ***Maria Bicudo*** (Brazil).
[11] $\mathbf{2+2=4}$? mathematics lost between two pitfalls of essentialism and alternative truths. ***David Kolosche*** (Austria).
[12] Does constructivism tell us how to teach? ***Bronislaw Czarnocha*** (USA).
[13] Teachers epistemology on the origin of mathematical knowledge. ***Karla Sepúlveda Obreque*** *and Javier Lezama Andalón* (Chile).
[14] Mathematical education, body and digital games: play the ball in this way so that it goes, it goes further than the floor. ***Maurício Rosa****, Danyal Farsani, and Caroline Antunes da Silva* (Brazil).
[15] Internet, teaching mathematics: weaving the web. ***Marli Regina dos Santos*** (Brazil).

The first two papers in session 1 studied the underlying theories and philosophies. Scovsmose' study[1] based on the theory of four-dimensional philosophy of mathematics: ontological, epistemological, sociological and ethical. Ethical dimension is explored concentratedly here by showing the broad range of social implications set in motion through bringing mathematics into action. These implications are illustrated in terms of quantifying, digitalizing, serializing, categorizing and imagining. Conclusion draws that the philosophy of mathematics can bring mathematical expertise out of the ethical vacuum. Shrestha[2] examined how philosophy, rigor and axiomatic are related. It seems that philosophy has distant but determining impression on the nature of mathematical knowledge, but rigor and axiomatics seems to be more internal to mathematics.

Most studies in this group focused on specific mathematics education, with different research directions. Ayalev[3] considered "imagination" as a subtopic under mathematics education. Besides the argument whether mathematics was invented or discovered, the construct "imagination" was discussed for the learning of Mathematics.

Möller and Collignon[4] discussed algorithms as well as their roles and importance for mathematics educaiton from a philosophical point of view. The framework of (post-)modernism and a constructivist approach were used. As for social and individual aspects of mathematics education, Matsushima[5] identified five appropriation stages from the discussion of a structural model of social constructivism based on a socio-cultural approach in mathematics education. The result of analysis revealed that a gap in appropriation could occur during the process, and that gap could become the source of creativity. Ricks[8] suggested the benefits of post-anthropocentrism for mathematics education considering that anthropocentric perspectives were inaccurate in lieu of many scientific findings about the mathematical abilities of many non-human entities. The ninth paper is the only one start from teacher education. Yan et al.[9] suggested a curriculum system for the philosophy of mathematics education for Chinese normal students. It is expected to provide "readable materials" for direct application in the practice of future teachers. Czarnocha's study[12] focused on the interface between theory and teaching practice of constructivism. The presentation argued that the research tool, constructivist teaching experiment did define the constructivist teaching methodology and through mathematics teaching-research, it could be introduced into mathematics classroom at large.

There are two presentations about critical studies. Kennedy[6] reported that philosophical inquiry could both offer a space for critical reflection on mathematics, for the development of an epistemological approach, and also a space for the deconstruction and reconstruction of beliefs about mathematics as a form of knowledge, about the social value of mathematical practice, and beliefs about oneself as a mathematics learner/thinker. In Savich's study[7], necessary conditions for arithmetic were expressed as material inferential rules using a normative vocabulary of commitments and entitlements. The explicated critical arithmetic is also related to other projects in critical mathematics education.

The tenth, eleventh and thirteenth paper are about mathematics and the philosophy of mathematics. Bicudo's study[10] was about algebraic and hermeneutic procedures. The openness of abstract algebra may happen through hermeneutic procedures. Kolosche's research[11] started from two examples from popular media to problematize the essentialist epistemologies and relativist epistemologies of mathematical knowledge. Obreque and Andalón[13] argued teachers epistemology attributed to the origin of mathematical knowledge under socioepistemological theory of educational mathematics.

The last two papers of this group are of digital technologies. Rosa et al.[14] conducted a study that high school students conjectured a way mathematically to improve their performance in an the electronic bowling game on Xbox One with Kinect. From that they are drawing up on embodied cognition articulated with the conceptions of perception and body-proper arising from the phenomenological view discussed by Merleau-Ponty. The conclusion is students' perception is shown by the acts of being-with, thinking-with and knowing-doing-mathematically-with-Digital-Technologies. Santos[15] proposed a theoretical and philosophical reflection on the nature of the cyberspace to find the possibilities which the Internet opens to the teaching and learning of Mathematics. The process which educators created resources and spaces for pedagogical practice of mathematics on the Internet is explored.

https://doi.org/10.1142/9789811287152_0081

Topic Study Group 57

Diversity of Theories in Mathematics Education

Angelika Bikner-Ahsbahs[1], Ivy Kidron[2], Erika Bullock[3], Yusuke Shinno[4], and Qinqiong Zhang[5]

ABSTRACT This report presents an overview about the themes of the Topic Study Group 57 on the diversity of theories in mathematics education. Main topics, which were addressed, are the networking of theories in theories related to the use of technology, to design research and beyond. The program, format, contributions, discussions and the main results as well as some future implications are presented.

Keywords: Theory; Networking of theories; Axiology; Epistemology; Ontology; Methodology; Ethics.

1. Themes

1.1. General overview of the topic

Mathematics education is a scientific field with many theory cultures. This diversity can be regarded as richness but it also challenges research as well as communication and cooperation in the field. This is specifically the case when different theories are to be included into research. How the scientific community can cope with this diversity with scientific integrity remains an open question, specifically when research results from different theory cultures are used. Researchers working with the networking of theories (Bikner-Ahsbahs and Prediger, 2014) have started to investigate this question by conducting concrete research. The TSG-57 builds on previous lines of thought on the diversity of theories addressed in various conferences (e.g., ICME-12, ICME-13, CERME 6, 7, 8, 9, 10, 11) and wants to continue this discussion (e.g., Kidron et al., 2018). It aims at exploring how the diversity of theories can be used in mathematics education, how this may influence research theoretically, methodologically and epistemologically and how the diversity of theories may impact on the use of theories and research results in school practice. The TSG-57 wanted to collect concrete research

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examples addressing diversity of theories in order to obtain typical 'argumentative grammars' (structure of argumentation to substantiate evidence) for qualitative and theoretical research, research on technology-based teaching and learning, design research, research addressing different educational levels and networking of theories. The TSG welcomed also further ideas going beyond the subthemes briefly described below.

1.2. Subthemes

The TSG-57 called for contributing to four subthemes where the notion of "theories" not only means grand theories but also theory elements or theoretical models of a restricted scope addressing specific perspectives or phenomena.

Subtheme 1 addresses the diversity of theories in the digital era — using technology and other resources in teaching and learning: Technology use often requires theorizing tools, instruments, hence semiotic resources in connection with theorizing teaching and learning mathematics. That is, diversity of theories is an issue with respect to the use of technology and other resources. Some case studies of networking theories are also discussed.

Subtheme 2 addresses the diversity of theories for design research: Steps in design research often require different kinds of theories, for example normative theories for justifying aims, descriptive or explanatory theories for conducting design experiments and prescriptive theories for deciding about means for the design of instruction, hence, diversity of theories is relevant to consider.

Subtheme 3 addresses the diversity of theories at different educational levels including teacher education. Different educational levels (e.g., pre-school vs. university and teacher education) may require the use of various theoretical perspectives to capture the complexity and nature of its teaching and learning, for instance in the classrooms or for professional development.

Subtheme 4 addresses the networking of theories, which may investigate the relationship and function of theory elements in concrete research cases focusing on specificities of theories and their usages to gain insight on theory cultures in the field and meta-theoretical knowledge about how various theories can be related in research.

2. Program Overview

The TSG-57 had 14 submissions consisting of nine papers, two posters and three invited presentations (Tab. 1 on the next page). All submissions were accepted. Finally, the sessions had six long presentations and three invited ones of 20 min each, two short presentations of 10 min each and two poster presentations of 10 min each. We arranged the program according to the topics of the contributions rather than according to the subthemes.

Each of the three sessions started with an introduction bridging previous work within and beyond the TSG-57 with the presentations in the sessions and the aims and scope for the discussions. In each session, an invited talk addressed the subtheme of the day informing the final discussion.

Tab. 1. The list of papers presented

Paper and author(s)
Session 1: Why do we need a diversity of theories?
[1] Facing the challenge of theoretical diversity: the digital case. ***Michèle Artigue*** (France).
[2] Role of feedback when learning with an artefact. ***Angelika Bikner-Ahsbahs****, Estela Vallejo-Vargas, and Steffen Rohde* (Germany)
[3] Constructing mathematical knowledge by means of analogy: connecting Fischbein's theory on the role of intuition in mathematics and the theory of abstraction in context. ***Ivy Kidron*** (Israel)
[4] Seeking a "Theory" of networking praxeologies in mathematics education: a meta-theoretical discussion. ***Yusuke Shinno*** *and Tatsuya Mizoguchi* (Japan).
Session 2: Methodological approaches to the diversity of theories in design research
[5] Vertical analysis as a strategy of theoretical work: from philosophical roots to instrumental and embodied branches. ***Anna Shvarts*** *and Arthur Bakker* (The Netherlands).
[6] Configuration of the theoretical-methodological construct «the teaching model» by affinity between theories. ***Ulises Salinas-Hernández*** (France and Mexico)*, and Luis Moreno-Armella and Isaias Miranda* (Mexico).
[7] The holistic instructional design model of the unit knowledge structure of elementary school mathematics based on core competencies. ***Shiqi Lu*** *and Wenbin Xu* (China).
[8] Networking theories and methodology: identifying argumentative grammars in design research. ***Arthur Bakker*** (The Netherlands) *and William R. Penuel* (USA).
Session 3: Reconsidering basic commitments in the diversity of theories, specifically ethics
[9] Mathematics teaching and learning as an ethical event. ***Luis Radford*** (Canada).
[10] How can we classify teachers' paradidactic praxeologies in different institutional settings? ***Tatsuya Mizoguchi****, Yusuke Shinno, and Toru Hayata* (Japan).
[11] Theoretical networking in a large-scale Danish and a large-scale Norwegian intervention study: TMTM and PBG. ***Lena Lindenskov*** (Denmark).

2.1. Session 1: Why do we need a diversity of theories?

The chair, Angelika Bikner-Ahsbahs, and the cochair, Ivy Kidron, introduced the TSG-57. They distinguished the notions of foreground and background theory (Mason and Waywood, 1996), clarified the term "theory" based on the work of Radford (2008, 2012) as well as the notion of the networking of theories (Bikner-Ahsbahs and Prediger, 2014).

In her invited talk[1], Artigue focused on her research about teaching and learning in a digital environment and used two conceptual tools for her reflection on issues of theoretical diversity: the landscape of networking strategies and the concept of research praxeology. She presented two research cases where theory diversity was relevant, studies on the instrumental approach and the documentational approach to didactics.

Bikner-Ahsbahs et al.[2] combined Activity Theory and Instrumental Genesis to investigate the feedback of a digital tool designed for the teaching/learning of integers. Reflection on the way the two approaches are related revealed a layering model, which describes how the students in the study developed their knowing in the activity of teaching/learning negative numbers mediated by feedback.

Kidron[3] presented a case of Networking of Theories, which showed how she linked the two originally separated foci into one comprehensive focus, the focus of constructing knowledge by means of analogy.

Based on the concept of research praxeology (Artigue and Bosch, 2014), Shinno and Mizoguchi[4] showed via three case studies of theory networking that theoretical concepts and the language used differ with respect to the kind of study they undertook, empirical study, design study, and theory development.

The main outcome in the discussion was that networking of theories can be a source of tensions between theoretical discourses, but it can also offer the flexibility to establish new theoretical discourses.

2.2. Session 2: Methodological approaches to the diversity of theories in design research

Yusuke Shinno (a member of the TSG-57 team) bridged Session 1 and Session 2 by recalling tension and flexibility in the networking of theories suggesting a dialectical way of working with different notions of the same term, such as scheme. Session 2 was then dedicated to the networking of theories in design research (Bakker, 2019), its gains and pitfalls, relating it to ontological, methodological and epistemological issues.

Shvarts and Bakker[5] pointed to the need of undertaking historical analyses to inform conceptualizing when comparing and contrasting theories. They called this analysis ‘vertical analysis’. Their aim was to identify the grand theories behind local approaches and their ontological and epistemological philosophical presumptions. Anna illustrated this approach by unpacking the roots of action scheme used in the instrumental approach and compared it with embodied approaches distinguishing finally between enactive and mental schemes.

Salinas-Hernández et al.[6] took the idea of “affinity” as an alternative to networking strategies in order to configure the methodological construct “teaching model” by three theories based on a semiotic-mediating view. After identifying related elements (affinities) between the theories and configuring the teaching model, the authors used a qualitative investigation in physics education to analyse the teaching practice of two teachers of grade 11 with different experience.

Lu and Xu[7] presented a holistic instructional design model for elementary school mathematics based on core competencies. The core competence concept relates to the development of mathematical thinking. The basic theory underlying the model is a thick epistemology including five characteristics: unit knowledge structure, learning psychological process, teaching objectives, learning evaluation, learning activities.

Finally, Bakker and Penuel[8] talked about networking theories and methodology. Acknowledging the networking of theories approach they stressed also the danger of this approach of being tied to epistemological justifications of choices in design research and simultaneously ignoring the relevant role of ontology. Because of the transformative nature of design research there is the necessity to include ontological and axiological aspects into justifications of design choices.

Main result from the discussion was the need for an increasing sensitivity to more comprehensively consider methodological as well as epistemological, and ontological issues in justifications of theoretical choices in transformative research in mathematics education.

2.3. Session 3: Reconsidering basic commitments in the diversity of theories, specifically ethics

In her bridging introduction, Erika Bullock (a member of the TSG-57 team) proposed to take up work from the philosophy of science. She referred to Patterson and Williams (1998) when she argued that there is a normative structure to scientific research holding certain commitments. These philosophical commitments involve theories about:

a) The nature of reality and what really exists (ontology)
b) The relationship between the knower and what is known (epistemology)
c) What we value and how we determine that value (axiology)
d) The strategy and justifications in constructing a specific type of knowledge (methodology), as linked to individual techniques (methods) (Daene, 2018, adapted from Patterson and Williams, 1998)

In his invited talk[9], Radford addressed the issue of ethics for teaching and learning in general. For the theory of objectification, he proposed and elaborated a communitarian conception of ethics that joins responsibility, commitment and care in the teaching and learning of mathematics.

Mizoguchi et al.[10] reported about two case studies of networking theories exploring the role of categorisation in a methodological approach, asking: “What is a theory for” and “How does it function”. They used the two case studies to differentiate the kind of knowledge being achieved, in one case they revealed “descriptive” and the other case “explanatory” theory elements (Prediger, 2019).

Lindenskov[11] talked about theoretical networking in large-scale Danish and Norwegian Intervention studies. As teachers were included in these studies, culture-specific views of teachers on theories, goals and organizations played a significant role making visible the contradictions between researchers and teachers’ view of how theories are meaningful.

What we learned from the discussion was the conviction that key commitments in the use of theories should be reflected and made explicit. Ways to do this are vertical analyses, grounding research in philosophical work and merging commitment and responsibility.

3. Future Directions

Networking theory research has led us in the past years to revisit our epistemological assumptions. The recent interest in ontology can complement the advances made on the epistemological side. There are a number of future directions to think about in discussions on theory use in the different kinds of research genres in mathematics education.

There are certain commitments related to theories (i.e., axiological, ontological, and epistemological) and to teaching and learning of mathematics (e.g., ethical). These commitments seem to be relevant for methodological choices in research that influence what kind of knowledge can be achieved about a certain research object. We ask how these commitments become relevant when various theories are taken up e.g., when theories are networked.

Ethical issues are particularly relevant for classroom design. As design research is a transformative way of doing research and development, where do the decisions for or against a specific learning goal come from and what are the criteria in what directions changes of the design are to be made? Here axiological issues come into play affecting the ontology as well as the epistemology of the research conducted. Beyond that, how far is it necessary or even mandatory to take ethical issues of the teaching/learning into consideration for theorizing and what are the relevant concepts of ethics for that? More generally, what are the pitfalls when such commitments go unnoticed?

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Topic Study Group 58

Empirical Methods and Methodologies in Mathematics Education

Christine Knipping[1] and Soo Jin Lee[2]

ABSTRACT In TSG-58 research methods, methodologies, and paradigms related to traditional issues of mathematics education such as instruction, learning, teaching and classroom processes and interactions were discussed. In total twelve papers and two posters were presented and discussed over three sessions. Overall, about 50 scholars participated in this TSG-58.

Keywords: Methods; Methodologies; Paradigms.

Research in mathematics education employs a range of Methods, Methodologies, and Paradigms (M/M/Ps) in the service of key goals. TSG-58 promoted a discussion about diverse strands of M/M/Ps investigating these goals.

1. Methods, Methodologies, and Paradigms in the Service of Key Goals

In the call for TSG-58 six diverse goals central to ongoing research in mathematics education were promoted, but three of them — Mathematics Education and Social Justice, the Role of Culture and Language in Shaping the Teaching and Learning of Mathematics — were not in focus of the submitted papers. Instead, traditional issues as instruction, learning, teaching in general and classroom processes and interactions were explicitly discussed in the submitted methodical and methodological papers of our TSG-58.

1.1. Diverse goals central to ongoing research in mathematics education

TSG-58 was finally organised, in three sessions, around four diverse goals central to ongoing research in mathematics education:

1) Improvement of Mathematics Instruction (e.g., instructional materials, strategies, organization, assessment);
2) Learning of Mathematics;

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3) Teaching of Mathematics (e.g., teacher beliefs, knowledge, decision-making and professional development); and
4) Classroom Processes and Interactions

Each *goal* was addressed using research designs that integrate one or more different Methods, Methodologies, and Paradigms (M/M/Ps). For each *goal*, the contributors of TSG-58 were asked to address the following questions and to discuss which M/M/P combinations help us understand the phenomena at stake in robust and reliable ways.

1) "Suppose you have a hypothesis about this goal. How do you set about evaluating it?" Alternatively,
2) "Suppose you are trying to explain some aspect of individual or group behavior relevant to that goal. How would you characterize and then theorize that behavior?"
3) Or, "How might cultural, historical and political perspectives shape one's understandings of the contingencies related to realizing this particular goal?"

The *goals* of our research clearly in focus, the actual topic of TSG-58 were the different methods, methodologies, and paradigms (M/M/Ps) employed.

1.2. Empirical research methods and methodologies

This TSG was specifically focused on the empirical research methods and methodologies employed to address the four broad goals of research in mathematics education identified above. For our work to be coherent and allow for comparability, each paper identified the specific goal(s) being explored, identified the theoretical frame on which the research design was predicated, and addressed the question of how effectively the research design (M/M/P bundle) addressed the designated goal(s). Participants were asked to

1) Specify the methodology and methods that constitute the research design and identify the particular goal/s that are the focus of the reported research study;
2) Specify the theoretical frame or rationale by which the selection of methodology and methods can be justified, discussing advantages and limitations of methodological choices respectively the identified research goal(s);
3) Further address the appropriateness of the chosen methods in terms of the robustness of the findings generated, their generality or specific domain of relevance, and their capacity to describe, explain or predict phenomena of importance to the field of mathematics education.

The following brief summaries of the papers in our TSG-58 only indicate the theoretical frames, research designs and designated goals of each contribution. Longer papers about the reported research are published elsewhere and indicated to in the references.

2. Program Overview

Tab. 1. List of papers and posters presented

Paper and author(s)
Session 1 (July 13th at 19:30 – 21:00 Beijing Time)
[1] First voyage of the integrated paradigm: The case of an international study on effective mathematics teaching. ***Zhenzhen Miao*** (China)*, and David Reynolds and Christian Bokhove* (UK).
[2] The teaching of mathematical thinking: The conceptualization of a special class teacher in China. ***Na Li*** (China) *and Ida Ah Chee Mok* (Hong Kong SAR, China).
[3] Teaching design of combination from HPM perspective. ***Weiyuan Fan*** (China).
Session 2 (July 16th, 21:30-23:00 Beijing Time)
[4] Understanding the relations between instructional quality and task quality in mathematics classrooms. ***Ann-Kristin Adleff****, Natalie Ross, Gabriele Kaiser, Johannes König* (Germany)*, and Sagrid Blömeke* (Norway).
[5] What is six-questions cognitive model? *Ying Zhou, Xiaofeng Lan, and* ***Tommy Tanu Wijaya*** (China).
[6] Units coordination as a theoretical construct to understand students mathematical activities. ***Soo Jin Lee*** *and Jaehong Shin* (South Korea).
[7] The influence of ICT on the students' science literacy at the national and student level based on ITU IDI Index and PISA2015. ***Zhenrong Xiong****, Ying Zhang, Bo Li, and Na Li* (China).
[8] The effectiveness of teaching mathematics in circle equation by using 5E instructional model in inquiry-based learning. ***Try Kimhor*** (Cambodia). (Poster)
[9] The trend of mathematics teaching method has changed from fragments to systematics. *Yi Lin,* ***Tommy Tanu Wijaya****, and Ying Zhou* (China) (Poster).
Session 3 (July 17th, 14:30-16:30 Beijing Time)
[10] Eye movements and collaborative problem solving: what do long fixations tell about student cognition? ***Markku S. Hannula****, Enrique Garcia Moreno-Esteva, and Miika Toivanen* (Finland).
[11] Examining the phenomenon of interlocutors talking past each other in collaborative proof constructions. ***Ann Sophie Stuhlmann*** (Germany).
[12] Using MRGQAP to analyse the development of mathematics pre-service trainees' communication networks. ***Christian Bokhove*** (UK)*, Jasperina Brouwer* (Netherland)*, and Chris Downey* (UK).
[13] Case study of personalized teaching based on the Q-learning algorithm in the era of big data. ***Lei Wang****, Yong Zhang, Na Li, Bo Li* (China).
[14] Learning research in a laboratory classroom: Advancing methodology and technology. ***Man Ching Esther Chan****, and David Clarke* (Australia).

2.1. Session 1: Teaching

In our first session, Christine Knipping (Germany) and Soo Jin Lee (South Korea) opened the TSG-58 with an introduction. They highlighted the diverse goals of research, listed above, and promoted the diverse strands of the M/M/P bundle as a way to address these designated goal(s), in terms of diverse methods, methodologies and paradigms. As the leaders of TSG-58 they proposed a programme, which is also used in this paper to structure the contributions of TSG-58. The first session started with papers on M/M/P issues around *Teaching*.

Miao et al.[1] reflected on a mixed methods approach as a way of researching teaching in an international setting (Miao and Reynolds, 2018). The title of the submission indicates a quantitative and qualitative methods approach that is designed as a mixed methods comparative approach to gain deeper insights into issues around teaching and learning mathematics.

Li and Mok[2] presented a paper on research issues around teaching of mathematical thinking. Both researchers made overt that understanding (methodologically) teachers' perspectives on mathematical thinking is important to not only describe and analyze their views on the issue, but also the impact on teaching.

In her presentation[3] Fan (China) talked about teaching integrating historical materials into the classroom. Research and methodological issues are related to specific traditions and cultural issues in this paper.

2.2. Session 2: Instruction and learning

In the second session of TSG-58 papers on M/M/P issues were focusing on issues at the intersection of *Instruction and Learning*. How to research learning in the context of specific materials and designed activities were of particular interest in this session.

The presentation[4] by Adleff et al. explored how quantitative research methods can capture and assess the "instructional quality" in classrooms. They also discussed how the "task quality" can be measured and put in relation to the performed "instructional quality" in class (see also Kaiser et al., 2017).

Zhou et al.[5] proposed in their presentation a 6-questions model, which has been developed in China about 10 years ago, combined with technology-based learning media and reflect in how far this cognitive model can foster students' deep learning (see also Lin et al., 2020).

Lee and Shin[6] discussed how far the theoretical construct "Unit Coordination" can be used to understand mathematical activities of students, portrayed as "cognizing subjects" (see also Lee and Shin, 2021).

Xiong et al.[7] also investigated the role of information and communication technology (ICT) in the context of instruction/learning. They wondered what impact ICT developments have on students' scientific literacy.

Two poster presentations, Kimhor[8] and Lin et al.[9], were part of TSG-58. These posters deal with methodical issues around instruction, learning and teaching. The first poster looks into the effectiveness of teaching in the context of inquiry-based learning, based on an instructional model, while the second one proposes a "systematic plan of teaching mathematics", based on the so called "Dick-Carey" model which aims to offer teachers a systematic approach to mathematics teaching. They report in how far mathematics teaching methods have changed in China.

2.3. Session 3: Learning, teaching and the social dimension

In the last session of our TSG-58 even more diverse methodical and methodological facets were brought up and discussed. Students' cognition "as it happens" was looked at, as well as the phenomena of interaction and how these can divert. Also, social networks of peer pre-service mathematics trainees and the methods and theoretical approaches how to research these were presented and discussed. Two further contributions mirrored how diverse and multi-faceted these discussions were. A "personalized teaching intervention" based on a "Q-learning algorithm", conceptualized as a "dynamic optimization problem" was presented and discussed as well as "Learning research in a laboratory classroom", where methodology and technology was designed so that the investigation of social aspects of classroom practice, particularly student-student and student-teacher interactions could be researched.

Hannula et al.[10] discussed how far paper and GeoGebra contexts effect fixation durations in collaborative student activities in geometry. From their observations and categorizations of four different types of long fixations they conclude possible cognitive student activities (see also Hannula and Toivanen, 2019; Hannula, Toivanen and Garcia Moreno-Esteva, 2019).

Stuhlmann[11] investigated students in collaborative proving activities in an undergraduate linear algebra class. Her interactionist methodology and methods allow her to study the diversity of meaning making of students in the same undergraduate class and why it is challenging for the students to come to a consensus during their proving process.

Bokhove et al.[12] the potential of a specific data analysis method (MRQAP) for analyzing longitudinal network data, in order to study the development of peer networks of pre-service mathematics trainees over time (see also Bokhove and Downey, 2018).

Wang et al.[13] presented a personalized teaching intervention, which is aimed to maximize academic performance of students (see also Wang et al., 2013). According to the researchers the latest advances in information technology allow this approach.

Last, but not least, Ching and Chan[14], based on collaboration with David Clarke, presented a multi-theoretic and multimodal research design, implemented in the laboratory classroom at the University of Melbourne (see Chan and Clarke, 2016, 2017). She discussed the complexity of this research design, which focused on student-student and student-teacher interactions.

3. Future Directions and Suggestions

The methodology and methods that constitute the research designs presented in the TSG-58 were not only diverse and multifaceted, but also indicated distinct and specific goals.

For example, *Teaching* was in some contributions not only researched in mixed methods ways, but also with an international comparative focus. Whereas, other research focused consciously on one cultural context only to deeper investigate the rationale of this specific context and historical tradition. Also, related to *Instruction and Learning* different goals were of interest. Established instruction approaches and models were valued in a methodically pragmatic way, i.e. within a new technology-based environment, and when looking at the impact of ICT environments on students' learning. On the other hand, more theoretical stances were taken to better understand mathematical activities of students.

Also *Learning, Teaching and the Social dimension* were studied not only in diverse methodical and methodological settings, but also with different goals. Understanding individual student cognition and academic performance was of interest for some researchers, other scholars focused clearly on the complexity of collaborative settings for students learning, as well as on student-student and student-teacher interactions.

The appropriateness of the chosen methods and theoretical frame or rationale by which the selection of methodology and methods were justified, was overtly discussed in the three TSG-58 sessions. Questions and comments highlighted advantages and limitations of methodological choices respectively the identified research goal(s). This helped to evaluate the robustness of the findings generated, their generality or specific domain of relevance, and their capacity to describe, explain or predict phenomena of importance to the field of mathematics education.

TSG-58 promoted a discussion about diverse strands of M/M/Ps investigating specific goals. Traditional issues as Instruction, Learning, Teaching in general and Classroom Processes and Interactions were explicitly discussed in the submitted methodical and methodological papers of our TSG-58. Extended discussions of Methods, Methodologies, and Paradigms (M/M/Ps) in research in mathematics education, in the service of key goals, will further substantiate the state of our art in the future. But these discussions will have to also include goals of ongoing research in the area of Mathematics Education and Social Justice, the Role of Culture and Language, shaping the Teaching and Learning of Mathematics, which were not in focus of the submitted papers.

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https://doi.org/10.1142/9789811287152_0083

Topic Study Group 59

Mathematics and Creativity

TSG-59 Organising Team[1]

1. Introduction

The major goal of this TSG was to gather educational researchers, research mathematicians, mathematics teachers, teacher educators, instructional designers and other congress participants for the international exchange of ideas directed at better understanding of creativity in mathematics and mathematics education. The TSG gathering was framed by the discussion of the two contrasting perspectives on the nature and nurture of creativity: individualistic vs. social.

The following issues were discussed:

- Does it make sense to perceive creativity?
  1. as emerging from the individual or from dialog, brainstorming and co-construction?
  2. as yielding grand intellectual feats or as an everyday occurrence in each of us?
- How is it best to design situations with dense potential for creativity emergence:
  3. by orchestrating diversity or homogeneity in educational settings?
  4. by focusing within the discipline or promoting interdisciplinary?
- How can we best understand and support creativity in teachers, teacher designs and the teaching process? by addressing the mathematics teacher:
  5. as an individual;
  6. as an actor in a homogeneous community of practice;
  7. as an actor in a diverse community of interest including members from outside mathematics education.

2. Organisation of TSG-59

The TSG activities were organised based upon the theme individualistic versus social' perspectives of creativity and the 30 papers and short orals received. Four sessions are designed, among them two sessions (session 1 and session 4) are simulated (due to the on-line nature of the meeting) "round table" activities, and the other two are oral presentations (Tab. 1).

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Tab. 1. The list of papers presented

Paper and author(s)
Session 1: Opening and Round Tables
Introductory Talks
[1] Introduction: Different faces of creativity: on the program and participants of the TSG-59 ICME-14. ***Roza Leikin*** (Israel).
[2] Opening: Individual vs social perspectives of mathematical creativity. ***Chronis Kynigos*** (Greece).
Round Table 1: Cognitive Perspective of Creativity
[3] Exploring primary students' creativity in hands-on mathematical activities. ***Jiali Xing****, Qiaoping Zhang, and Xuanzhu Jin* (China).
[4] A leap from in school to our school: possibility is creativity development. ***Shin Watanabe*** (Japan).
[5] Creativity in linear algebra through interactions. ***Aditya Adiredja*** *and Michelle Zandieh* (USA).
[6] Students make interactive exhibition experimental mathematics for the museum of entertaining sciences. ***Mariia Pavlova*** *and Maria Shabanova* (Russia)
Round Table 2: Creativity in the World
[7] Mathematical creativity of Filipino and Japanese students: a comparative study. ***Lady Angela Mico Rocena****, Ma. Nympha B. Joaquin* (Philippine)*, and Manabu Sumida, Naomichi Yoshimira* (Japan).
[8] An exploration into Chinese high school students' consciousness of enquiring and innovation. ***Yi Chu*** *and Haiyue Jin* (China).
[9] Promoting creativity in the international baccalaureate diploma programme mathematics. ***Deborah Sarah Sutch*** *and Helen Thomas* (The Netherlands).
[10] A survey of mathematics teachers' perceptions on mathematically gifted learners in Thaba Nchu primary schools in South Africa. ***Motshidisi Gertrude van Wyk*** (South Africa).
Session 2: Collaborative Creativity
[11] Social creativity in a constructionist classroom context. ***Chronis Kynigos*** *and Dimitris Diamantidis* (Greece)
[12] Fostering creativity in a diverse classroom of a community college. ***Malgorzata Aneta Marciniak*** (USA).
[13] Collaborative creation between university students from mathematics and music. ***M. Alicia Venegas-Thayer*** (Chile)
[14] Understanding students everyday play experiences when designing games in the mathematics classroom. ***Erik Ottar Jensen*** (Denmark).
[15] Creative design of digital tools for teaching in a mathematics' teachers' community. ***Chronis Kynigos*** *and Dimitris Diamantidis* (Greece).
[16] Creative art processes to deepen geometrical thinking of middle school mathematics teachers. ***Irina Lyublinskaya*** *and Marta Cadral* (USA).
[17] Beyond Sudoku: creating a course for developing deductive and creative skills. ***Jeffrey J. Wanko*** (USA).
[18] Designing games to foster creativity thinking about randomness. ***Theodosia Prodromou*** (Australia) *and Chronis Kynigos* (Greece).
Session 3: Cognitive abilities and development in connection to creativity
[19] Creativity varies from task to task, doesn't it? — a qualitative view on first graders' individual creativity. ***Svenja Bruhn*** (Germany).
[20] "Rethinking the World" with mathematics: the geometric chess from bauhaus as a basis for creating mathematical ideas and materials. ***Torsten Fritzlar*** *and Karin Richter* (Germany).
[21] Inventing growing patterns by primary school students — a creativity provoking task. ***Daniela Assmus*** *and Torsten Fritzlar* (Germany)
[22] The relation between spatial ability and creativity in geometry in primary school. ***Anastasia Datsogianni*** (Germany)*, and Pantelitsa Eleftheriou, Nektaria Panagi-Louka, and Athanasios Gagatsis* (Cyprus).
[23] How long is half a piece of string — the journey continues. ***Bruce Stuart Ferrington*** (Australia).
[24] Strategy-related and outcome-related mathematical creativity in all as compared to that in gifted. ***Roza Leikin*** *and Haim Elgrably* (Israel).

Session 4: Round tables

Round Table 3: Collaborative and Interactive creativity

[25] Inquiry dialogues in mathematics classroom and mathematical representations and their role in learning mathematics. ***Hanna Zdziarska Slabikowska*** (Norway).

[26] Mathematical creativity workshop to review elements of geometry with high school students. ***Matheus Delaine Teixeira Zanetti**, Mateus G. Fonseca, and Cleyton H. Gontijo* (Brazil).

[27] Comparing social creativity among designers with creativity of mathematical digital resources produced. ***Nataly Essonnier*** (Switzerland), *Mohamed El-Demerdash* (Egypt), *and Jana Trgalová* (France).

[28] Expanding possibilities: a metaphor for the co-construction of students' creative acts. ***Ayman Eleyan Aljarrah** and Jo Towers* (Canada).

[29] Developing mathematical group creativity through mathematical modelling. ***Hye-Yun Jung** and Kyeong-Hwa Lee* (South Korea).

Round Table 4: Evaluation of Creativity

[30] Students and their effects on motivation and performance in mathematics. ***Mateus Gianni Fonseca** and Cleyton H. Gontijo* (Brazil).

[31] Research problems and assessment by students. ***Noriko Tanaka*** (Japan).

[32] Establishment of evaluation index system for primary school students' mathematical innovation competency: investigation and analysis based on Delphi method. ***Yanzhi Wang*** (China).

The first session's activities comprised of two "round table" activities respectively grouping the presentations into the following themes: a) cognitive perspectives of creativity, b) creativity in the world. The round tables were preceded by an introductory talk by Roza Leikin[1] outlining the structure of the meetings followed by another by Chronis Kynigos[2] laying out the frame for the discussions with respect to the TSG-59 theme for the ICME-14 conference.

The activities of the second session comprised of paper presentations grouped around the theme "collaborative creativity". Eight papers were presented and discussed focusing on creativity emerging in a classroom context, stimulated by transdisciplinary approaches to mathematics education and manifested during students' design and tinkering with constructionist digital media and games. To stimulate discussion the session was chaired by Suhy.

The activities of the third session involved paper presentations addressing the cognitive abilities and development in connection to creativity. The papers yielded research on mathematical creativity in individuals' grappling with diverse tasks including patterns, spatial ability, iterative halving of a piece of string. To stimulate discussion, the session was chaired by Kynigos.

The fourth session activities comprised of two simulated (due to the on-line nature of the meeting) "round table" activities respectively grouping the presentations into the following themes: c) collaborative and interactive creativity and d) evaluation of creativity.

The TSG concluded in a discussion around the value and the diversity of the two approaches to creativity concluding that it is worthwhile and necessary to pursue both the individualistic and the collaborative perspectives, to employ research and creativity perspectives outside mathematics education to the extent that they inform and enrich the study of creativity in both the teaching and learning of mathematics, both for high-end performers and in an inclusive spirit as a trait inherent in every student. The group agreed more work is needed especially regarding the nature of every-day mathematical creativity in connection to student activity including the understanding of the role of situational and social context. It was also agreed that more work is needed to understand creativity in teacher classroom activity and with respect to their design of resources as tools to inspire and liberate mathematical creativity in students.

https://doi.org/10.1142/9789811287152_0084

Topic Study Group 60

Semiotics in Mathematics Education

TSG-60 Working Team[1]

1. Aims of the TSG

The TSG-60 aimed at exploring the significance of semiotics and the diverse uses of signs in the teaching and learning of mathematics at all levels. The importance of semiotics is reflected in a large body of literature within mathematics education, an overview of which is to be found in the ICME-13 monograph "Signs and Signification: Semiotics in Mathematics Education Research." The goal of TSG-60 was to expand on prior work, addressing the following themes and sub-themes:

- *Themes*

1. Semiotic perspectives within mathematics education;
2. Sign use and mathematics meaning-making processes;
3. Modes of mathematical narrative through different sign systems;
4. Relationships between sign systems (e.g., natural language, diagrams, pictorial and alphanumeric systems) and transformations between sign systems in mathematics thinking and learning;
5. Inventing and generalizing with visual, alphanumeric, and other sign systems;

- *Sub-themes*

1. Semiotics and Technology (e.g., Design of activities and tasks based on visual-kinesthetic interactions; interplay between physical manipulatives and virtual entities, and roles of animation and video as instructional tools)
2. Semiotics in Specific Areas of Mathematics (e.g., Episodes of sign-use in calculus, geometry, algebra, arithmetic, etc.)
3. Semiotics Inside and Outside Mathematics Education (e.g., differences and similarities between semiotic usages in art, linguistics, or cinema, and mathematics)
4. Semiotics in Relation to Feeling and Expression (e.g., gestures, embodiment, more-than-human agencies, affects, aesthetics, and rituals)

[1] TSG-60 Working team:
Chair: Ricardo Nemirovsky, Manchester Metropolitan University, UK.
Co-chair: Christina Krause, University of Duisburg-Essen, Germany.
Members: Suanrong Chen, Yangzhou University, China.
Francesca Ferrara, University of Torino, Italy.
Kazuya Kageyama, Hiroshima University, Japan.

2. Submissions, participation, and sessions

2.1. Submissions and participation

We received, in 2019, 13 submissions from 8 countries (South America: 1; North America: 4; Asia: 3; Europe: 4; Africa: 1). Of these 13 submissions 10 were accepted as paper presentations and 3 as posters. Of the 13 accepted submissions, only 7 papers were able to be presented during the online conference. We list the papers in Tab. 1.

Tab. 1. The list of papers presented

Paper and author(s)	
Session 1: Embodied aspects, gestures, movement	
[1]	Collaborative gestures among secondary students conjointly proving geometric conjectures. ***Candace Walkington****, Min Wang, and Mitchell Nathan* (USA).
[2]	Conceptualization of co-emergent curriculum in a mathematics lesson. ***Kazuma Kageyama*** *and Masataka Koyama* (Japan)
[3]	Can a movement notation be a mathematical notation? ***Giulia Ferrari*** *and Francesca Ferrara* (Italy)
Session 2: Language, meaning making, social factors	
[4]	Semiotic character and issues in the learning and teaching of linear functions in Japan: The influence of terminology. ***Hiroaki Hamanaka****, Masayoshi Yoshikawa, Hisae Kato, and Mitsunobu Kawauchi* (Japan).
[5]	A semiotic lens on learning math in sign languages. ***Christina M. Krause*** (USA/Germany) *and Annika M. Wille* (Austria).
[6]	Semiotic chaining in Linear Algebra. ***Hamide Dogan*** (USA).
[7]	Interference between artifacts in semiotic chains. ***Andrea Maffia*** *and Mirko Maracci* (Italy).
Session 3: Workshop	

2.2. Themes prominent during the sessions

The themes that became prominent during the three sessions can be outlined along five categories:

2.2.1. Gestures, body, and their annotations

Walkington et al.[1] elaborated on the notion of "collective gestures", in reference to gestural actions bodily coordinated among several students, arguing that they can express important mathematical insights emerging from distributed cognition. *Ferrari and Ferrara*[3] shared a notation for body motion stimulated, in part, by the Laban notation, that they propose to enrich research practices.

2.2.2. Co-emergent curriculum

Kageyama and Koyama[2] distinguish between a hypothetical learning, as it can be traced in a mathematical textbook, and the real learning that incorporates spontaneous contributions from interactions among students and teachers. Their case study focused on the word-usage in a mathematics lesson. They characterize the resultant process as a co-emergent curriculum.

2.2.3. Language and mathematical concepts

Hamanaka et al.[4] discussed the influence of the Japanese phrase for "linear algebra" which, as opposed to the English one, does not connote straight lines. *Krause and Wille*[5] elaborated on different semiotic approaches to analyze the use of sign language among Deaf students in the context of a mathematics lesson.

2.2.4. Semiotic chaining

Dogan[6] traced the emergence of signified-signifier pairs and how they facilitate the emergence of new concepts in linear algebra. *Maffia and Maracci*[7] incorporated a Peirceian semiotic perspective to analyze the enchaining process. They introduced the notion of "interference" to characterize how different artifacts interact in the formation of a semiotic chaining.

2.2.5. Abstraction and mathematics

This was the theme of the workshop that took place in Session 3. The inquiry centered on how semiotics may cast light on the concept of abstraction in the context of two selected video episodes. The complexity inherent in this investigation emerged from the use of physical materials and tools and the expression of ideas in Sign Language, all of which seem to reflect a bodily and "concrete" ground for thinking, allowing for the articulation of abstract ideas.

3. Areas for Future Research and Outlook

The topic of Abstraction and Mathematics led to numerous research questions for future work. These included how the word "abstraction", as uttered in different languages, might lead to distinct ways of conceptualizing it, how to complicate the almost automatic association between "material" and "bodily" with "concrete", and how to question the presumption of abstraction as preventing inclusiveness. Other areas for future research are those that were touched only tangentially, or not at all, by the presented papers, such as semiotics in relation to feelings and aesthetic expressions, the relationship between the uses of sketches and diagrams in mathematics and in other disciplines, or the significance of students' inventing mathematical notations.

The group — TSG leaders and participants alike — agreed that it might be worthwhile to work on a joint publication, for example a special issue of a journal in which will be focused on the topics as they emerged during the discussions. Furthermore, a future seminar and or topic conference/symposium has been considered. Potential contributors met shortly after the conference to elaborate on possible outlets and topics for an open call. The endeavor is still in the planning stage as of March 2022.

https://doi.org/10.1142/9789811287152_0085

Topic Study Group 61

International Cooperation in Mathematics Education

Ui Hock Cheah[1], Masami Isoda[2], Arne Jakobsen[3], Bernadette Denys[4], and Jiwei Han[5]

1. Description of TSG-61

From the perspective of international cooperation, mathematics has always been viewed as an essential literacy which is necessary to address the concerns of globalization. High quality mathematics education has thus become a priority of the reform agenda to achieve the United Nations Sustainable Development Goals (SDGs) (United Nations, 2015). While it is a common goal to progress and improve the status of mathematics education, this aspiration is not easily attained for many countries. As a result, there is a demand for countries and agencies to support and collaborate with each other in international cooperation projects. This demand often provides the rationale for mathematics education to be included in various international education cooperation efforts which go beyond merely adopting successful practices of high achieving countries, to seek appropriate technologies and methods to advance mathematics education. Mathematics educators, teachers, government officials and consultants participate at various levels of international education cooperation, namely, (1) International, (2) Governmental, (3) Institutional, and (4) Personal. Projects also vary in scope from nationwide to provincial and school levels to cater for the demands of both mainstream and marginalized groups. Various aims and purposes of international cooperation in mathematics have been noted: (a) Curricular development which encompasses curriculum reviews, textbook resource development, as well as the enhancement of meaningful teaching and assessment approaches, (b) Professional development at in-service and pre-service levels, and (c) Creating communities to enhance mathematics education. While international education cooperation has been ongoing, there remain many issues and challenges to be overcome (Atweh et al., 2008). Emerging trends in response to these concerns include:

- Renewed emphasis on pre-primary and basic education up to the secondary level.
- Reaching out to special groups, for e.g., disabled, poor and gender groups.
- Re-establishment of higher education as an agenda.

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- Emphasis on assessment for accountability.
- Inclusive involvement and expanding the roles of new partners, donor countries. and agencies, and non-governmental organizations.

The discussions in this group were guided, though not exclusively by the following questions:

1. What were the roles of the cooperating agents in the projects?
2. What were the challenges, and the subsequent methods/solutions/strategies and good practices used to overcome these challenges?
3. What were the views of the various cooperating agents in overcoming these challenges? How were differing views about teaching and learning mathematics resolved?
4. How did the project impact on the quality of teaching and learning mathematics?
5. How did the project plan for sustainability and expansion?

2. Paper Presentations

The papers submitted to TSG-61 were reviewed and a total of 16 papers were selected and presented over 300 minutes, in 3 sessions (Tab. 1, on the next page). All the papers were presented using the online mode. At the end of each session, there was a time for question-and-answers which resulted in fruitful discussions on the papers that were presented.

3. Conclusion

ICME-14 marks the first time the topic of international cooperation has been included as a TSG at the ICME. The papers presented in TSG-61 provided informative glimpses and insights into the current state of practice in international cooperation. The papers presented covered four main areas of international cooperation: Curricular development, professional development, developing communities of practice, and improving environment related to mathematics education. The methods in international cooperation that were reported in the papers included the following strategies: The adoption and adaptation of good practice, capacity building of key personnel/faculties, developing and incorporating local ideas, multilateral dialogue among agents to develop strategic approaches, development of new study programs and certification courses, and the development of curriculum and textbooks.

3.1. Areas for future research and discussion in international cooperation

At the concluding session of TSG-61, three suggestions were highlighted for future research and discussion:

1) Papers in TSG-61 described the success stories in International Cooperation. Issues and challenges are not often discussed. For future projects, reports should also discuss what we can learn from the inadequacies and shortcomings of the projects.

Tab. 1. The list of papers presented

Paper and author(s)
Session 1
[1] Adapting lesson study in Thailand through international cooperation. ***Maitree Inprasitha*** (Thailand) *and Masami Isoda* (Japan).
[2] An experience in developing the regional mathematics curriculum standards. ***Kim Hong Teh*** (Malaysia) *and Masami Isoda* (Japan).
[3] Fostering global citizenship in mathematics classrooms. ***Russasmita Sri Padmi*** (Indonesia) *and Gabriel Matney* (USA).
[4] Development of the national mathematics textbook in primary schools in Papua New Guinea. ***Ileen Palan****, Steven Tandale, Gandhi Lavaki* (Papua New Guinea)*, and Masami Isoda, Satoshi Kusaka, and Akinori Ito* (Japan).
[5] The challenges of improving mathematics education through translated textbook. ***Lambas*** (Indonesia)*, Masami Isoda* (Japan)*, and Wahyudi* (Indonesia)
[6] Developing mathematical thinking through robot programming. ***Wahid Yunianto****, Uki Rahmawati* (Indonesia)*, and Masami Isoda* (Japan).
[7] An electronic assessment workshop for 1st & 2nd year mathematics & statistics course lecturers from East African universities. ***James Musyoka*** *and Michael Obiero* (Kenya)*, and David Stern and Danny Parsons* (UK)
Session 2
[8] Understanding narratives: A pathway towards resolving issues and challenges in international cooperation in mathematics education. ***Ui Hock Cheah*** (Malaysia) *and Masami Isoda* (Japan).
[9] APEC lesson study project (2006–2018) for mathematics education and AI era curriculum project (2019–). ***Masami Isoda*** (Japan)*, Maitree Inprasitha* (Thailand)*, Roberto Araya* (Chile)*, and Sofian Tajul Arus* (Malaysia).
[10] Improving quality and capacity of mathematics education in Malawi through collaboration — lessons from a collaboration between University of Malawi and University of Stavanger. ***Arne Jakobsen*** (Norway) *and Mercy Kazima* (Malawi).
[11] Informal international collaboration and its potentialities: The example of GREMA. ***Bernadette Denys*** (France) *and Jannick Trunkenwald* (Algeria).
[12] Capacity development for mathematics teaching in Tanzania: A follow up of impact on participants. ***Calvin Swai*** (Tanzania)*, Joyce Mgombelo* (Canada)*, Andrew Binde* (Tanzania)*, Florence Glanfield, and Elaine Simmt* (Canada).
Session 3
[13] How El Salvador improved student learning achievement in mathematics: A principle methodology of JICA toward achieving SDGs 4. ***Norihiro Nishikata*** (Japan).
[14] The development of mathematics textbooks in Myanmar: Under the CREATE project. ***Takashi Itoh****, Isamu Imahori, and Koji Takahashi* (Japan).
[15] Impact of APEC lesson study project (2006–2018) in Chile. ***Raimundo Olfos*** *and Soledad Estrella* (Chile).
[16] GUATEMATICS in action. A service learning project for mathematics education between Spanish preservice teachers and teachers from rural schools in Guatemala. ***Elsa Santaolalla Pascual****, Belén M. Urosa Sanz, and Olga Martín Carrasquilla* (Spain).

2) International cooperation should also report what we can learn from different cultures. Knowledge flows both ways. For example, what are the differences in mathematics language, terminologies and concepts that are related to culture? What pedagogical strategies have emerged from local cultures?
3) What can we do about the outcomes of international cooperation?

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https://doi.org/10.1142/9789811287152_0086

Topic Study Group 62

TSG-62: Popularization of Mathematics

Christian Mercat[1,2]

1. Theme and Description

The Popularization of Mathematics Study Group (TSG-62) gathered for the second time in ICME people using interesting and inspiring mathematics to motivate both young people and the general public.

Because of the pandemic situation, we have seen a surge in the use of digital media aimed at teaching mathematics. This technological shift had a profound impact on the way we teach and, because of the format, a lot of teachers have taught in a manner much akin to popularization. Many teachers have turned into directors of educational video clips and borrowed from the narrative springs of Youtubers. You don't teach the same way using the blackboard in the classroom and short videoclips on the internet. You cannot expect students to stay as calm and focused for an entire hour as in the classroom when you are competing with entertainment platforms with enormous means of attraction. You are compelled to be as appealing as possible and this had an impact on the math popularization scene. Teaching then reached almost the same goals as popularizing:

- Democratize mathematics;
- Set a healthier relationship with mathematics;
- Raise performance in math education;
- Share math beauty, power and pervasiveness;
- Justify taxpayer's money in research and education.

2. Activity Overview

The Topic Study Group gathered 30 people, authors of 11 papers and 1 poster that were presented during three sessions (Tab. 1).

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[2] The organization Team of TSG-62:
Chair: Christian Mercat, Université Lyon 1, France.
Co-chair: Clara Grima, Universidad de Sevilla, Spain.
Members: Pan Liu, East China Normal University, China.
Abolfazl Refiepour, Shahid Behonar University of Kerman, Iran.
Patrick Vennebush, University of Maryland, USA.

Tab. 1. The list of papers and poster presented

Paper and author(s)
Session 1
[1] Students make interactive exhibition experimental mathematics for the museum of entertaining sciences. ***Maria Shabanova*** *and Mariia Pavlova* (Russia).
[2] Mathcitymap — popularizing mathematics around the globe with maths trails and smartphone. *Iwan Gurjanow,* ***Joerg Zender****, and Matthias Ludwig* (Germany).
[3] Beyond the classroom and curriculum: the annual maths camp at Bahir Dar University, Ethiopia 2013 – 2019. ***Abdu Mohammed Seid****, Yismaw Abera Wassie, Danny Parsons, Haile Yedeg, and Assaye Walelign* (Ethiopia).
[4] Reconsidering the M in STEM: leaders' conceptions of mathematics to empower girls in gems clubs. *Rose Mbewe, Sue Ellen Richardson, and* **Lili Zhou** (USA).
Session 2
[5] Creating access to engaged views of mathematics and teaching through informal learning spaces. ***Lynn Liao Hodge*** (USA)*, Shande King, and Qintong Hu* (China).
[6] Increasing math appreciation using the upper levels of blooms taxonomy. ***Manmohan Kaur*** (USA).
[7] Math + Origami + Puzzles + Magic → the odds are always in favor of fun. ***Violeta Vasilevska*** (USA).
Session 3
[8] Some suggestions on school-based curriculum construction of mathematics culture for middle school. *Junfeng Ma and* ***Yaqiang Yan*** (China).
[9] Mathematical drama: a new form of popularization of mathematics at East China Normal University, China, 2012–2019. ***Xinyu Liu****, Pan Liu, and Jiachen Zou* (China).
[10] Mutual role of mathematics and culture. ***Abolfazl Rafiepour*** (Iran).
[11] Keeping Popularization of mathematics on track: formative assessment. *Elham Ebrahim Zadeh, Hasan Hoseinpoor,* **Einollah Shokrpourrodbari***, and Younes Karimi Fardinpour* (Iran).
[12] On the impact of popularization oriented assessment: creating excitement. ***Younes Karimi Fardinpour****, Akram Bagheri Gheibi, and Gahimeh Kolahdouz* (Iran).

In Session 1, Shabanova and Pavlova[1] presented the Museum of Entertaining Sciences where students made interactive experimental mathematics exhibitions. Joerg Zender et al.[2] introduced the concept of math trails using the MathcityMap system, a sort of mathematical tourism around the globe. Seid et al.[3] presented their annual math camps since 2013. Zhou et al.[4] empowered girls in STEM clubs with leaders' conceptions of mathematics.

In Session 2, Hodge et al.[5] experimented Informal Learning Spaces, for families and novice teachers to engage with mathematics. Kaur[6], from the Benedictine university, Vasilevska[7] gave an overview of various hands-on outreach programs that demonstrate the fun of math with origami, puzzles and magic that reconciliate students with their own abilities in math.

In Session 3, Yan and Ma[8], investigated how culture in mathematics can be infused into math curriculum at the level of middle school. Liu et al.[9] shared their experience of conducting theatrical popularization through grandiose mathematical drama. Abofazl Rafiepour[10] described how Iranian craftsmanship shows the interrelation between mathematics and culture. Finally, formative assessment was promoted as a way to keep mathematics popular by Shokrpourrodbari et al.[11] and exciting by Farinpour et al.[12].

3. Future Directions and Suggestions

There are many different ways of expressions that fall within popularization of mathematics:

1. Art and science (theater, films, visual arts);
2. Fixed, itinerant, and virtual exhibitions for museums, science centers or non-dedicated spaces. Science or mathematical festival or forums;
3. Competitions in mathematics and computer science Mathematical camps;
4. Contact with research mathematics and mathematicians;
5. Inquiry/research-based projects;
6. Math circles/math clubs;
7. Recreational mathematics;
8. New technologies (apps, websites, ...);
9. International exchanges.

Popularization addresses different audiences and target groups tackling unequal access issues, talent, motivation, gender, social, financial or geographical differences, educational opportunities between countries. We witnessed some of this diversity in this Topic Study Group, but common theoretical grounds to unite all these different views into one perspective still needs some work to be done. Let’s hope that next ICME will see such a vivid community share their enthusiasm for mathematics once again in the flesh.

Part VII

Discussion Groups

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https://doi.org/10.1142/9789811287152_0087

Discussion Group 1

70 Years' Development of School Mathematics Textbooks in China

The Working Team[1]

1. Theme and Description

The discussion group of "70 Years' Development of Mathematics Textbooks in Primary and Secondary Schools in China" aims to show colleagues around the world the development of Chinese mathematics textbooks, methods in school textbooks research, exploration on practical reform of textbooks, and understanding of mathematics and mathematics education.

2. Activity Overview

The chairs of the discussion group were Haidong Li and Xiaochuan Zhou. Group members included Jinsong Zhang, Lili Song and Guozhong Ding. All team members are from the Curriculum and Teaching Material Research Institute, People's Education Press.

We first discussed the theme "the development and characteristics of Chinese primary school mathematics textbooks" in two reports. The first one was *70 Years Development of Mathematics Textbooks in Primary Schools in China* delivered by Guozhong Ding. The report introduced the primary school mathematics textbooks of People's Education Edition in the past 70 years and summarized several characteristics in the textbooks and some experiences in the compiling and revising progress. Xiaochuan Zhou gave the second report with title *Mathematical Thoughts and Methods of Mathematics Textbooks in Primary Schools in China*. Zhou introduced the mathematical ideas and methods in the primary school mathematics textbooks of People's Education Edition.

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We then showed attendees the development and characteristics of Chinese secondary school mathematics textbooks. To begin with, Haidong Li carried out a report titled *Reform and Development of Chinese Secondary School Mathematics Textbooks in the 21st Century.* Li especially introduced the reform and development of Chinese secondary school mathematics textbooks in the past 20 years, summarized and illustrated the characteristics on secondary school mathematics textbooks of People's Education Edition. Secondly, Jinsong Zhang gave a talk on title *Information Technology and Mathematics Teaching Materials*. Zhang analyzed the purpose, function, methods and effects of integrating information technology into mathematics teaching materials in China. Furthermore, he introduced the interesting cases of information technology integration in secondary school mathematics textbooks of People's Education Edition. Lili Song gave a talk on *Mathematical Culture in Secondary School Mathematics Textbooks*. Song firstly gave an overview of the mathematical culture in various mathematics textbooks, followed by focusing the selection and presentation of mathematical culture materials in secondary school mathematics textbooks of People's Education Edition. She ended up her speech with the theory and practice experience on integration of mathematical culture materials with mathematical contents.

The above five lectures, concerning mathematics textbooks in primary and secondary schools, not only introduced the historical evolution but also the characteristics of mathematical thinking and methods, information technology, mathematical culture throughout the past seventy years of development of Chinese mathematics textbooks.

3. Future Directions and Suggestions

In summary, the following points should be paid attention to in the construction of mathematics textbooks in the future: to strengthen the research paradigm of mathematics textbooks, to promote the quality of mathematics textbooks with more standardized scientific research methods, and to further strengthen the international exchanges in the compilation and research of mathematics textbooks.

https://doi.org/10.1142/9789811287152_0088

Discussion Group 2

The Future of Mathematics Education Research: A Discussion Group

Arthur Bakker[1] and Jinfa Cai[2]

ABSTRACT With help of a review study by Inglis and Foster published in *Journal for Research in Mathematics Education*, Jinfa Cai summarized trends in the past 50 years of mathematics education research. Next, Arthur Bakker presented a recent survey published in *Educational Studies in Mathematics* about the future of mathematics education research. Anna Sfard compared this survey with an earlier survey for ICMI. The presentations were discussed in the whole group, after which Jill Adler highlighted a few points she considered relevant.

Keywords: Mathematics education research; ESM; JRME; Research into practice; Online teaching and assessment.

1. Theme and Description

Mathematics education research as a discipline has been celebrating several milestones. One example is that *Educational Studies in Mathematics (ESM)* and *Journal for Research in Mathematics Education (JRME)* have recently celebrated their 50th anniversaries. Fifty years is a small step for human history but a giant leap for mathematics education research journals. To mark this auspicious occasion, this Discussion Group has focused on the future of mathematics education research.

We have used an international survey — conducted before and during the pandemic — as the basis of the discussion (Bakker et al., 2021). The survey is based on one single question before the pandemic (2019): *On what themes should research in mathematics education focus in the coming decade? During the pandemic (November 2020), we asked respondents: Has the pandemic changed your view on the themes of mathematics education research for the coming decade? If so, how?* We have reported the survey results and also provided a list of research challenges that are informed by the themes and respondents' reflections on mathematics education research (Bakker et al., 2021). In particular, we discussed the impact of the pandemic on the shape of mathematics education and mathematics education research, including increased attention to issues such as online assessment and pedagogical considerations for virtual teaching.

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2. Activity Overview

Due to pandemic, the DG was organized both in person and online. On the basis of Inglis and Foster (2018), Jinfa Cai started the DG. Inglis and Foster (2018) analyzed the full text of all articles published in *ESM* and *JRME* (up to 2015). Jinfa Cai quoted the major findings from their analysis, and showed the social turn in mathematics education research. Jinfa Cai has also pointed out the decline of experimental studies in the past two decades in both mathematics education journals. The findings from Inglis and Foster (2018) served as a basis for looking into the future for mathematics education research.

Then, Arthur Bakker reported the results from an international survey (Bakker et al., 2021). The responses from 2019 were summarized in nine themes: (1) Approaches to teaching, (2) Goals of mathematics education, (3) Relation of mathematics education to other practices, (4) Teacher professional development, (5) Technology, (6) Equity, diversity, and inclusion, (7) Affect, (8) Assessment, and (9) Mathematics education research itself. In relation to the pandemic, most respondents considered the importance of their themes to be reinforced. Only few extra themes were added in the 2020 round of the survey. One question raised was whether new theories were needed due to the drastically new situation of large scale emergency remote teaching and online learning.

In 2005, Sfard published results from a survey result on Relations between Mathematics Education Research and Practice. Scholars were invited to reflect on the question of how research has been informing the practice of mathematics education over the last decade. Anna Sfard summarized the findings from the survey and discussed the survey results from Bakker et al. (2021) in terms of teaching, theories, and technology.

The DG were then divided into two groups: one in person led by Jiushi Zhou and one online led by Arthur Bakker and Jinfa Cai. The DG addressed several points, after which Jill Adler addressed the question of what ICMI can learn from the various studies and comments. One point discussed by Adler was the tension between identifying inequity or injustice on the one hand and perpetuating it on the other. For example, it is useful to know about so-called achievement gaps between various student groups to be able to improve education for lower scoring students. However, objectification of differences in terms of gaps can also have negative effects such as reinforcing existing stereotypes (Akkerman et al., 2021). For example, in Sweden Svensson et al. (2014) observed that immigrant parents believed they could never help their children as well as Swedish parents. How do we point to problematic differences in terms of for example gender, race, or ethnicity, without perpetuating the problem? We elaborate on a few directions and suggestions in the following section.

3. Future Directions and Suggestions

The results from the discussion could be summarized into the following three future directions and suggestions. The first is to increase the international collaboration.

While there is an increased international collaboration in the past a few decades, continuous effort is need. Bakker et al. (2021) reported the limitations of their survey:

> The survey results are limited in two ways. The set of respondents to the survey is probably not representative of all mathematics education researchers in the world. In that regard, perhaps scholars in each country could use the same survey questions to survey representative samples within each country to understand how the scholars in that country view future research with respect to regional needs. The second limitation is related to the fact that mathematics education is a very culturally dependent field. Cultural differences in the teaching and learning of mathematics are well documented. Given the small numbers of responses from some continents, we did not break down the analysis for regional comparison. Representative samples from each country would help us see how scholars from different countries view research in mathematics education; they will add another layer of insights about mathematics education research to complement the results of the survey presented here (p. 19).

Through future international collaboration, scholars from different countries can zoom into the critical issues for future research in mathematics education. One theme where we noticed differences in focus was that of equity and justice (commonly mentioned by respondents from North and South America) and diversity and inclusion (more commonly used terms in some other continents).

The second is to improve the contribution of research to educational practice. Both ESM and JRME were founded with a charge to disseminate significant research to improve educational practice in mathematics. Impact of research on practice is a longstanding issue (Cai et al., 2017; Sfard, 2005), and there is discussion whether it is wise to think in terms of impact (Akkerman et al., 2021; Fielding, 2003). Yet participants recognized the continued effort for connecting research and education practices. The participants discussed three possible strategies to make progress of improving the relevance of research to practice: (1) Identify and understand the fundamental reasons for the divide between research and practice in different countries; (2) Identifying successful teaching practice informed by research in different countries; and (3) Examine the successful teacher-researcher partnership in various countries.

The third is to capitalize on aspects of technology for research and practice in mathematics education. Because of the pandemic, lessons have been delivered online and assessments have been conducted online as well. The participants particularly discussed the three issues mentioned in Bakker et al. (2021): (1) The importance of studying the use and influence of low-tech resources in mathematics education such as podcasts, radio, WhatsApp, or WeChat; (2) The need to systematically investigate any possible effect of administering assessments online as researchers have found a differential effect of online assessment versus paper-and-pencil assessment; and (3) The need to rethink social interactions between students and/or teachers in online settings but also study how to engage and motivate students in online settings.

4. Summary

In this changing world, the only thing does not change is change itself. As a discipline mathematics education research is maturing, we need to continuously make efforts for looking into the future: What do we need now and in the future? Akkerman et al. (2021) propose to focus on actuality and generativity. Thus, when the discussion of the future of mathematics education research continues, encouraging and supporting high-quality research that makes a difference for practitioners also continues!

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https://doi.org/10.1142/9789811287152_0089

Discussion Group 3

Capacity and Network Projects: Sustainability and Future Directions

The Working Team[1]

1. Short Description of the Discussion Group

This discussion group will be attractive to congress participants interested in creating networks and communities of practice in challenging and disadvantaged education contexts. Discussion will focus on the Capacity and Network Projects (CANP) of the International Commission of Mathematical Instruction (ICMI) supported by the International Mathematical Union (IMU), UNESCO and the International Council of Scientific Unions (ICSU) as well as regional governments and institutions.

Five CANPs have been organised so far. While each CANP differs in its focus, approach and process the goal is to respond to the challenges in mathematics education that have been documented among other reports in UNESCO 2011. The aims of the Discussion Group at ICME-14 include identifying, sharing and discussing common key issues in creating a critical mass to sustain the network and its activities over long term. Through sharing cross-national regional experiences, we expect to deepen and broaden the understanding of lessons learnt in the process of establishing the CANP and taking it forward.

Discussions will be guided by the following key questions:

- What did the CANP do in 2020? How (if at) were your activities impacted by the pandemic? (Focus on one or two innovations/activities).
- What is planned for the CANP in 2021? Why?
- What new questions arise for the mathematics education community?
- What are the similarities and differences in the opportunities and challenges arising in the CANPs?
- What is the impact of CANP on mathematics education in the region? how could the impact be sustained?

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[1]The working team:
Chair: Anjum Halai, Aga Khan University Pakistan
Team members:
Moustapha Sokhna & Mamadou Sangare (CANP1)
Yuri Morales and Nelly Leon (CANP2)
Maitree, Khamla and Vu Nhu Thu (CANP3)
Alphonse Uworwabayeho and Veronica Sarungi (CANP4)
Augusta, Gabriela and Maria del Carmen Bonilla Tumialan (CANP5)

2. Planned Structure:

Session I — 90 minutes

Planned timeline	Planned activity	Working format /Responsible person
10 minutes	Introductions	ALL
5 minutes	Purpose of CANP	Chair
50 minutes	Presentation by each CANP (10 minutes each)	Representative of each CANP
25 minutes	Discussion	ALL

Session II — 90 minutes

Planned timeline	Planned activity	Working format /Responsible person
45 minutes	Identifying similarity and differences in opportunities and challenges in CANPs across regions and countries	Small groups working on strategies to build on the opportunities and to address the challenges
30 minutes	Report to the whole group on the small group activity	Representative from each group
10 minutes	Identifying ways forward	Representative of each CANP
5 minutes	Closing	Chair

https://doi.org/10.1142/9789811287152_0090

Discussion Group 4

Revisiting Shulman's Notion of Pedagogical Reasoning: Looking Back and Looking Forward

Ban Heng Choy[1], Jaguthsing Dindyal[1], and Joseph Boon Wooi Yeo[1]

ABSTRACT Pedagogical reasoning is not a new concept. More than three decades ago, Shulman (1987) expounded this idea in his seminal paper, well known for its elaboration of pedagogical content knowledge (PCK). Although the notion of PCK has been quite well-understood, the notion of pedagogical reasoning is still under-theorised. Yet, pedagogical reasoning has been seen as an important component of teaching expertise. If teaching actions are based on pedagogical reasoning, then how do we enhance the pedagogical reasoning of teachers to improve teaching? Or more fundamentally, is there a need to reinterpret the components of pedagogical reasoning in the light of the current contexts of teaching and learning? In this Discussion Group, we discussed these questions. More specifically, we critiqued this construct and proposed possible modifications to the framework of pedagogical reasoning. In addition, we also discussed the issues and challenges related to the development of teachers' pedagogical reasoning.

Keywords: Pedagogical reasoning; Teacher noticing; Resources, orientations, and goals.

1. Looking Back at Shulman's Pedagogical Reasoning and Action

Pedagogical reasoning is not a new concept. More than three decades ago, Shulman (1987) expounded this idea in his seminal paper, well known for its elaboration of pedagogical content knowledge (PCK). Following Shulman (1987), teaching begins as an act of reason and continues as a process of reasoning. He also added that pedagogical reasoning forms the basis for all actions by the teacher. In his model for pedagogical reasoning and action, Shulman proposed that teaching begins with the act of comprehending what must be taught, followed by the transformation of that knowledge for teaching the students, which is followed by actual instruction, and an evaluation of the students' learning. Finally, teachers engage in reflections, which may lead to new comprehensions by the teacher. Although the notion of PCK has been quite well-understood, the notion of pedagogical reasoning is still under-theorised (Loughran et al., 2016). Yet, pedagogical reasoning has been seen as an important component of teaching expertise (Choy, 2016). If teaching actions are based on pedagogical reasoning, then how do we enhance the pedagogical reasoning of teachers to improve teaching? Or more fundamentally, is there a need to reinterpret the

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components of pedagogical reasoning in the light of the current contexts of teaching and learning?

As Shulman (1987) had highlighted, new comprehension does not necessarily follow through from reflection. Hence, we argue that new comprehension of content, student learning, and teaching actions occurs when a teacher has a shift of attention, gaining awareness of new possibilities in teaching and learning (Mason, 2002), or simply when a teacher notice critical aspects of teaching and learning (Choy, 2016). These new insights expand the teacher's current cluster of resources, orientations, and goals (Schoenfeld, 2011; Thomas and Yoon, 2013), which in turn becomes the base from which the teacher make sense of instruction. Moreover, as Choy (2016) has highlighted, productive noticing can take place during planning, instruction, and reviewing of lessons. Consequently, new comprehension can occur during any of the activities of Shulman's model of pedagogical reasoning and action.

Building on ideas from both Shulman (1987) and (Schoenfeld, 2011), we adapted Shulman's model of pedagogical reasoning and action to highlight the dialogic processes involved when teacher notices critical aspects of instruction (Choy, 2016) in order to learn from their own teaching as shown in Fig. 1.

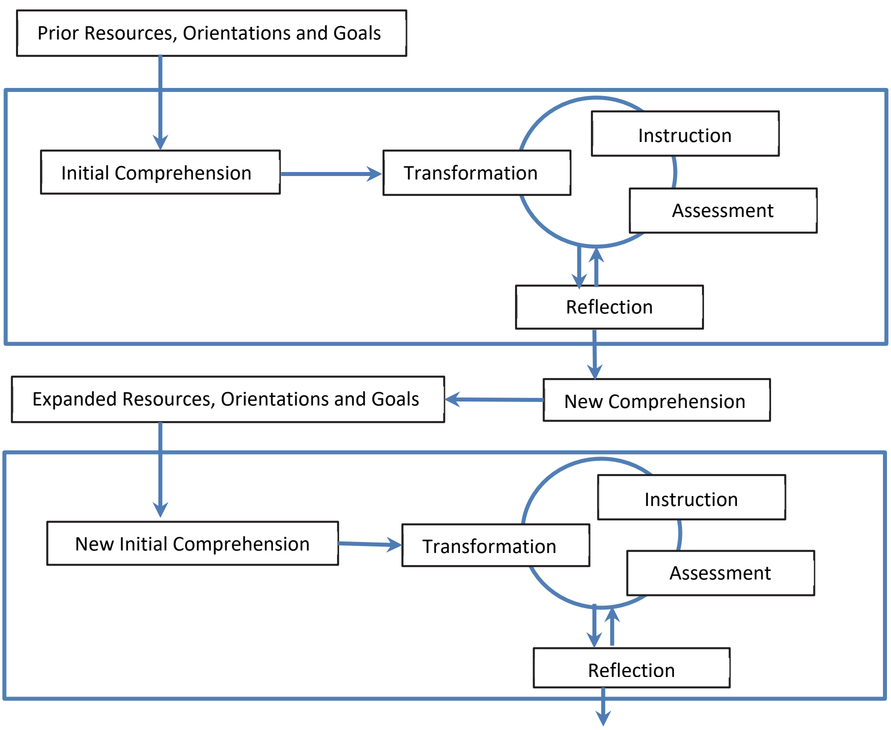

Fig. 1. Adapted model of pedagogical reasoning and action

2. Key Questions About Pedagogical Reasoning and Action

In our discussion group, we got the participants to comment and critique our proposed model, which was developed as part of a larger project. The planned questions and planned activities for the session was summarized in Tab. 1 as shown. However, due to time constraints and the feasibility of discussion, we mainly focused on getting feedback on our proposed model and discussed issues and challenges related to enhancing teachers' pedagogical reasoning. Last but not least, we also briefly discussed the possibility of future collaboration on research related to pedagogical reasoning.

3. Discussion and the Way Forward

To summarize the discussion, participants generally agree that the model strongly resembles what a teacher goes through on a day-to-day basis, and this affords a way to model the teaching activities, while simultaneously focusing on the pedagogical reasoning aspects of a teacher's instructional decisions. Taken as a whole, this model could potentially provide a fine-grained analysis of a teacher's pedagogical reasoning and instructional decisions. It gives a dynamic model of a teacher's pedagogical reasoning and action, which can be used to characterize one's teaching and make teaching more learnable.

Tab. 1. Overview of the planned discussion group activities

Planned timeline	Planned activity	Description
10 minutes	What is pedagogical reasoning and action?	The organisers facilitated the introduction of the participants of this DG and present the key ideas needed in this DG.
20 minutes	What are the components of pedagogical reasoning and what are the roles of each component in teacher education and professional development? What can we say about its relationship to Shulman's notion of pedagogical reasoning?	The participants worked in groups to critique one of the following components: Comprehension, Transformation, Instruction, Evaluation, Reflection, and New Comprehension.
30 minutes		The participants presented their critique and suggest ideas to modify/enhance/clarify the notion of pedagogical reasoning.
10 minutes	What are some issues and challenges with enhancing teachers' pedagogical reasoning?	The organisers summarised the ideas shared by the participants and led a discussion on the issues and challenges.
10 minutes	How can we move forward in our endeavor to enhance teachers' pedagogical reasoning?	The organisers summarised the ideas and discussion to set up possible collaboration opportunities in the future.
10 minutes	Summary and Closing	

However, several issues remain. First, the "individual" components, such as transformation and comprehension, are still perceived as "black boxes". For instance, how do teachers comprehend the materials they need to teach? How do they transform their knowledge into instructional materials? How does their intended design of instructional materials get enacted in the classrooms during instruction? Next, the link between reflection and new comprehension is currently still tenuous. When does a teacher's reflection lead to new comprehension, and why does it not? The mechanisms are perceived to be complex but critical for teacher educators to understand in order to facilitate professional learning. We suspect that teacher noticing (Dindyal et al., 2021; Schack et al., 2017; Sherin et al., 2011) is a useful construct to consider, and in particular, the notion of productive teacher noticing (Choy, 2016; Choy and Dindyal, 2019, 2021) may be helpful for us to understand when reflection leads to new comprehension. Although we do not have any answers to our questions, this discussion group has opened new possibilities for future collaboration and discussion as we continue, as a mathematics education community, to unpack the complexities of teaching practices so that they can be learnable for teachers.

Acknowledgments

The ideas for this discussion group came from a study funded by Singapore Ministry of Education (MOE) under the MOE Academy Fund Programme (AFD 06/17 CBH) and administered by National Institute of Education (NIE), Nanyang Technological University, Singapore. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the Singapore MOE and NIE.

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https://doi.org/10.1142/9789811287152_0091

Discussion Group 5

Roles for Mathematicians in Math Education

Solomon Friedberg[1], Patricio Felmer[2], Carlos Kenig[3], JongHae Keum[4], and Jürg Kramer[5]

ABSTRACT This is a report on the ICME-14 Discussion Group entitled "Roles for Mathematicians in Math Education." The discussion took place on July 14, 2021.

Keywords: Mathematicians in math education; Pre-service teachers; In-service teachers; Math education policy; Advocacy for math education.

1. Theme and Description

Mathematicians have played an important role in K-12 math education for many years; for example, mathematicians Felix Klein (the first President of ICMI), Hans Freudenthal, and Georg Pólya have contributed fundamentally. The roles of mathematicians in K-12 math education today are diverse. They include the training of future teachers in the university, the support of in-service teachers (e.g. helping to promote their on-going engagement with mathematics), roles in public policy such as writing or reviewing K-12 math standards and ensuring that there is a close articulation between pre-collegiate math and university-level math, and roles in advocacy for math education. The mathematicians involved in these efforts have a strong professional connection to the work of K-12 math education.

The goal of this discussion group was to take stock of ways that mathematicians are presently contributing to pre-collegiate math education. We also sought to consider what mathematicians can add to the field of math education *as mathematicians* with their specific training and perspectives, and among these, which contributions by mathematicians are the most important. We noted that in the present landscape, with the emergence of many specialists in education and math education, sometimes grounded in other disciplines, there are other voices and other perspectives regarding

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math education, and that this diversity of perspectives presents both a challenge and an opportunity. We asked what experiences and structures would be most useful in promoting future cooperation among the different professionals involved in mathematics education and in enabling contributions by mathematicians.

2. Activity Overview

The 90-minute discussion was divided into four roughly equal parts, with the following themes.

- Mathematicians and pre-service teachers
- Mathematicians and in-service teachers
- Mathematicians and mathematics education policy
- Connecting mathematicians and mathematics educators going forward: roles, opportunities, obstacles and potential pathways.

For each part, we posed a series of questions and asked those in attendance to respond. We noted that math education in different countries is organized in different ways, and we welcomed explanations about how it is organized in a specific country and how that affects possible roles for mathematicians in that country.

For Mathematicians and Pre-Service Teachers, the questions were:

- How are mathematicians involved in the preparation of future teachers in your country or region?
- How can the preparation of future teachers in your country or region be improved? What roles can mathematicians play in this? How does their expertise as mathematicians affect these roles?
- Research in math education shows the importance of MKT, mathematical knowledge for teaching. How can mathematicians learn about and support the development of MKT in future teachers?

For Mathematicians and In-Service Teachers, the questions were:

- What is the context in your country for interactions between mathematicians and in-service teachers?
- What roles have mathematicians in your country or region played in the support of in-service teachers? How does their expertise as mathematicians affect these roles?
- Please give specific examples that might be models for others to use.
- What have you learned from these examples? What are best practices for such involvement?

For Mathematicians and Mathematics Education Policy, the questions were:

- How have mathematicians been involved in math education policy in your country or region?
- What has gone right in this involvement, and what could be improved?
- How do the goals of mathematicians in math education policy compare to the goals of other math education professionals in your country or region?

For Connecting Mathematicians and Math Educators in the Future, the questions were:

- What are the most important opportunities or roles? What are possibilities for future engagement?
- What are potential pathways for such engagement in your country? In your region? Worldwide?
- What are lessons learned from past engagement that it would be valuable to keep in mind?
- What role can mathematicians play in math education research?
- How can mathematicians assist, regionally or worldwide, in the domain of math education?

The discussion took place in hybrid format, and was fast-paced and energetic.

3. Future Directions and Suggestions

The involvement of mathematicians in K-12 math education is an important topic, and there is considerable experience with such involvement and also variation in the nature of such involvement worldwide. We hope that the discussion here will serve as a catalyst for more extended interactions in the future. We believe that the thoughtful and sustained involvement by more mathematicians in K-12 math education could be a source of improvements to math education and of support for pre-collegiate math educators.

https://doi.org/10.1142/9789811287152_0092

Discussion Group 6

Variations and Series of Tasks, Crossing the Approaches

Katalin Gosztonyi[1], Charlotte de Varent[2], and Luxizi Zhang[3]

1. Description: Organizers, Aims and Underlying Ideas

This discussion group aimed to extend a discussion led by some young researchers from four different countries for some years about variations and series of tasks. Katalin Gosztonyi wrote her PhD (2015) on the comparison of the Hungarian reform of mathematics education led by Varga (pointing out the importance of structuring problems in series and networks) and the French "mathématiques modernes" reform. Charlotte de Varent, wrote her PhD (2018) on the reciprocal contributions of history of mathematics and didactics to each other. One aspect of her PhD was to point out the importance of small numerical variations in Mesopotamian scholarly context. Luxizi Zhang is working on her PhD (Zhang, 2020) towards an analytic model of "teaching mathematics through variation" from the analysis of teachers' documentation work (Gueudet and Trouche, 2009) in China and France, making profit of the variation theory (Gu et al., 2004) and the notion of didactic variable in the theory of didactical situations (Brousseau, 2002).

We entered variation as sequencing and networking tasks and problems. The "variation perspective" appears as an important issue in various traditions of mathematics education, and at the core of teachers' documentation work. International discussions have been launched on this topic for some years. (In our group, there were meetings such as) the "Series of problems" interdisciplinary historical research project (2012–2019) (Bernard 2015), the first (2018, Budapest) and the second (2019 Lyon) "Variations and series of problems" workshop, and the Varga100 conference (2019 Budapest, https://varga100.sciencesconf.org/). The aim of these discussions was to confront different apparitions of this "variation perspective" emerging in different cultural contexts. We would like to advance towards the elaboration of a common model, or a diversity of models, allowing on one hand to develop analytical tools for researchers, and on the other hand to support teachers' design work. We considered the ICME-14, in the country of the Chinese "variations method" and thanks to the

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diversity of the conference's public, a particularly well adapted context for the continuation of this collective work.

2. Activity Overview (Tab. 1)

Tab. 1. 21h30 - 23h, 14th July, 2021

Timeline	Activity	Working format /Responsible person
10 min	Introduction	The coordinators, plenary
20 min	Presentation of the Chinese, Hungarian and French handouts	L. Zhang, K. Gosztonyi plenary
15 min	Analyzing the data, extracting principles with special focus on the structure of the task sequences. Comparing to the participants' teaching traditions	Participants work on the handouts individually
35 min	Sharing the results	Collective discussion
10 min	General conclusions and potential plans for further research	The coordinators, plenary

In addition, a video recorded by Alessandro Ramploud on "Italian adaptation of the Chinese variation" was shared with participants.

3. Questions on Variations Discussed during the DG:

This discussion group aims to extend a discussion on variation activities considered as sequencing and networking tasks and problems. The aim of these discussions was to confront different implementations of this 'variation perspective' emerging in different cultural contexts.

The discussion raised several issues concerning the perspective of variation through this international meeting of traditions on variation studies.

First, participants recalled the difficulty and necessity of classifying varied sets of exercises or problems among different criteria. The discussion started from the question that each participant approaches with his own framework: how to define the variations/how to describe their structure? Underlying questions were asked during the discussion:

1. Do we categorize the series in terms of educational objectives? In terms of expected or resulting effects? In terms of potential/realized interactions? Do we describe them in terms of variation, and/or in terms of problems and numbers used? The possibility of targeting procedural or conceptual variations in problem design was recalled.

Second, participants discussed the description of the tasks given to students in relation to the series, as well as the description of the instrumental context:

2. Are the tasks close together or far apart? How much time is given to the students? How do they interact with the material? Can students also vary the problems? Is the teacher taken into account as a variable and how?

Third, the participants recall that the study is also situated in the present of the teaching situation, they question the existence of interactive variations.

3. Are there also interactive variations, in the classroom, in the moment, and if so, with a final goal in mind or not?

4. Future Directions and Suggestions

In the future, we aim to extend the discussion on variation activities considered as sequencing and networking tasks and problems. We plan to continue to confront different implementations of this 'variation perspective' emerging in different cultural contexts and to advance towards the elaboration of a common model, or a diversity of models. This confrontation seems to us important on the one hand for sharing and developing analytical tools for researchers, and on the other hand to support teachers' design work using variations.

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Discussion Group 7

Mathematics Houses and Mathematics Museums Worldwide

The Working Team[1]

ABSTRACT After a fruitful Discussion Group at ICME-13 in Hamburg (Germany) and the establishment of an International Network of Mathematics Houses [INMH] in 2016 (Rejali, et al., 2017), we worked on the structure of the network and discussed cooperations between mathematics houses and mathematics museums worldwide. Enhancing mathematical awareness inside the communities and impact on mathematics education, as well as the challenges, were the aims of this second Discussion Group at ICME-14.

1. Aims

The Discussion Group aims were to introduce Mathematics Houses and Mathematics Museums and similar institutions throughout the world to a public audience and discuss their importance and effect on mathematics education.

They can foster connections and discuss the important impact of promoting teamwork among the members, as well as seeking new ways to cooperate and exchange the successful experiences as well as their particular activities. They can present studies about their programs and discuss the challenges as well. Finally, they can build an official partnership network and announce it to the greater public.

2. Agenda

The agenda was discussions about the following questions in detail:

1. What are the benefits of such institutes for popularizing mathematics and improving mathematics education?

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2. What are the other roles of Mathematics Houses and Mathematics Museums for the society
3. What are the challenges they face?
4. How can mathematics institutions share activities and cooperate with each other?
5. How can their members benefit from other institutes in other parts of the world?
6. What are the effects of these institutes in mathematics education in the region around these institutes?
7. What is the network [INMH] and what could be its role and what kind of structure would be needed?

3. Discussions

The Discussion Group started with a short welcome statement by Ali Rejali, followed by introducing the Museums of Mathematics by Albrecht Beutelspacher (https://www.mathematikum.de), then Yahya Tabesh discussed the Opportunities for Innovative Multidisciplinary Learning at Mathematics Houses. Albrecht Beutelspacher was the next speaker, who discussed the other roles of Mathematics Houses and Mathematics Museums for the Society.

Abolfazl Rafiepour discussed Challenges facing Mathematics Houses in Iran and also Christian Mercat introduced the French House of Mathematics and explained the challenges (https://prezi.com/p/gcpwtlpucdrg/?present=1), as well. After some discussion between the audience and the speakers, the last speaker was Ali Rejali who introduced his proposal on establishment of an affiliated study group to ICMI with the name of the International Network of Mathematics Houses and Mathematics Museums (INMH) and approved to set up a constitution for this international network.

4. Conclusion

Upon Rejali's proposal, establishment of the International Networks of Mathematics Houses and Mathematics Museum (INMH) as an affiliated study group to ICMI is on the agenda. A constitution will be conducted and will be submitted to ICMI.

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https://doi.org/10.1142/9789811287152_0094

Discussion Group 8

Non-University Tertiary Mathematics Education: An Emerging Field of Inquiry

The Working Team[1]

1. Description

The intent of the discussion group (DG) was to gather ICME-14 participants to engage in conversation about non-university tertiary mathematics education (NTME). Over past ICMEs, it has been a tradition to dialogue about educational matters unique to this area. Over time, with both advances, challenges, and opportunities in tertiary mathematics education, as well as increasing attention, it is apparent that NTME is becoming a critical branch of inquiry in mathematics education. Yet, compared to primary, secondary, and university education, historically NTME has received insufficient attention. Consequently, this DG was to provide an avenue to engage a wider group of mathematics educators, network, exchange ideas, and learn more about NTME practices around the world. The meaning of NTME as well as developing this area as a field of inquiry was explored.

2. Activity Overview

The topic was introduced by Kathryn Kozak. We spent the first 20-30 mins watching a short presentation on related practices in the US (presented by Jim Ham) and Europe (presented by Kees Hoogland). The remaining time was relegated to discussion questions, facilitated by David Tannor and Laura Watkins.

The following questions were discussed:

- What is NTME?
- Why should NTME be a field of inquiry? What is special about NTME to warrant inquiry? How does NTME relate (or is different from) to tertiary and secondary education?
- What are (or could be) some important aims of NTME?
- What is the nature of teaching, learning, and research of mathematics at the non-university tertiary level?
- What are some issues, challenges, and/or opportunities related to NTME?

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[1]The working team:
- Organizers: David Tannor, Kellogg Community College, USA;
- Kathryn Kozak, Coconino Community College, USA;
- Laura Watkins, Glendale Community College, USA

- In light of ICME and our effort toward NTME as a field of inquiry, are we also globalizing (or internationalizing) this area? If so, what would it mean to globalize it? What are some implications/issues that we should be mindful of?

3. Future Directions and Suggestions

The name Non-University Tertiary Mathematics Education (NTME) was coined to aid in conversation about areas of post-secondary education that affect faculty and students unique to this group. It is not clear what exactly defines this group, yet it is understood that there are students and faculty who uniquely reflect this area, and in many ways are different from those typically at the university or secondary levels. Hence, overall discussions were mainly to begin conversation on exploring how to clearly define NTME. It was recognized that the current name has limitations. For example, it is not encompassing or inclusive. Notwithstanding, a model was suggested as a possible guide to examine this field: that is, what is it from students' perspectives; from the perspective staff, faculty, mathematicians, and mathematics educators; and from the viewpoint of society. Given the enormity of students this sector of post-secondary education serves, and an increasing interest to continue to examine mathematics educational matters that affect these students and faculty, DG participants believe NTME is worthy of serious study, and further consideration at future ICMEs either as a discussion group or topic study group.

https://doi.org/10.1142/9789811287152_0095

Discussion Group 9

How do Movements of Bodies and Artifacts Emerge in Mathematics Education?

Anna Shvarts[1], Dor Abrahamson[2], Ricardo Nemirovsky[3], Nathalie Sinclair[4], and Candace Ann Walkington[5]

ABSTRACT The discussion group focused on embodied processes in mathematics teaching and learning. At this discussion group, we aimed to consider the origins of movements performed by students, teachers, and artifacts. We invited group participants to reflect on resources initiating bodily movement and on the agents who perform or share the movement from a theory of dynamic systems, a new-materialist perspective, phenomenological perspective, embodied cognitive science, and cultural-historical approach. We questioned when and how movements become recognized as mathematical activity and discourse; we also discussed the criteria in prompting students to act or suspend enactment and leave room for imagination and articulating prediction of the enactment.

Keywords: Embodiment; Gestures; Artifacts; Embodied collaboration; Theory of dynamic systems; New-materialism; Phenomenology; Embodied cognitive science; Cultural-historical approach.

1. Embodied Interaction: A Variety of Theoretical Perspectives

This discussion group was initiated by an international collective of researchers all concerned with embodied processes in mathematics teaching and learning. Operating from different perspectives that consider bodies as partaking in educational processes, we have been offering theoretical rethinkings of cognitive and affective processes in mathematical practices. Imagine a student who draws the graph of $y = x^2$ on grid paper. From a theory of dynamic systems that Abrahamson uses to argue for his embodied-design framework, this movement emerges as embodied adaptive coordinations in a complex dynamic system bearing agentive, environmental, and task constraints, such as figural features of the paper (Abrahamson and Sánchez — García, 2016). From a new-materialist perspective that Sinclar elaborates in the mathematics education field (de Freitas and Sinclair, 2014), an assemblage of the student with her capacities, the formula, and the paper with the virtual transformation that they imply is actualized towards the graph. From a phenomenological perspective, in which Nemirovsky was engaged for many years (Nemirovsky et al., 2013), objectification of

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a formula includes protention and retention of its usage, where the subject joins both the intentional horizon of the paper and the retentional formula usage, in fulfilling her intentionality of drawing a graph by moving the hand along the paper. From an embodied cognitive science perspective that is within Walkington's expertise, movement is driven by cognitive processing of the formula that is extended beyond the scalp in a distributed system of activity that includes both explicit use of embodied resources and implicit embodied associations (Walkington et al., 2019). From a cultural–historical account, represented by Shvarts in the team (Shvarts and Abrahamson, 2019), the student's drawing is mediated by cultural artifacts — the paper and the formula — and expresses an ideal (cultural) form of action, which the student appropriated in a previous collaboration with a more knowledgeable other.

2. Discussion Group Aims and Proceedings

At this discussion group, we aimed to consider the origins of movements performed by students, teachers, and artifacts. We invited the group participants to reflect on resources initiating bodily movement and on the agents who perform or share the movement. We worked on articulating the difference between motion per se and agential movement as well as when and how movements become recognized as mathematical activity and discourse (language, diagrams, gestures).

The session started with an introduction of corresponding theoretical perspectives by each of the organizational team members and continued with discussions in small groups that each applied a chosen perspective to the analysis of a shared one-minute video fragment. This fragment was filmed in November 2006 at an Aboriginal Headstart Daycare in Ontario, Canada. It presented a 4-year-old child, who, in a collaboration with a teacher–researcher, for the first time used a digital application, TouchCounts (Jackiw and Sinclair, 2014), to explore the operation of addition. After diving into the video fragment from different theoretical perspectives and revealing various aspects of embodied interactions between a child, a technological artifact, and a teacher, we discussed our insights jointly at the plenary discussion. Thirty-two researchers from Australia, Brazil, Canada, Chile, Finland, France, Germany, Israel, Italy, the Netherlands, Peru, South Africa, Sweden, UK, and USA joined the conversation.

3. Outcomes and Future Directions

Discussions brought forth the complexity of explicating the sources of the child's mathematical expressions. Different theoretical perspectives highlighted the role of the artifact's design and the teacher's and student's bodily dynamics in triggering and shaping embodied actions. Mathematical expressions coincided with bodily gestures and poses, being indispensable from materially articulated embodied ideas. Despite exploring various theoretical focuses, participants working in different small groups repeatedly noticed that bodily imitation of the adult's gesture apparently guided the student's performance. Mathematically relevant gestures seemed to occur without strict top-down cognitive regulation based on pre-given knowledge, but as a spontaneous emergent dynamical event enabled by material constraints. Those material constraints included cultural guidance by the teacher through gestures and physical forming of the interactional space. The teacher carefully steered the child to alternate between actively manipulating the digital artifacts and suspending the manipulating to predict the artifact's feedback.

Overall, the discussion highlighted a complementarity of various perspectives that evoked different aspects of embodied interaction in mathematics learning, yet revealing a unified phenomenon rather than providing contradictory visions. Further research questions may concern the emergence of a student's awareness of her mathematical expression and the role and form of pre-knowledge in shaping embodied expressions. When and how does a student come to know their own embodied ideas as mathematical? Another direction of future research might focus on the issue of engaging in physical manipulation versus suspending actual manipulation to form anticipation of the feedback from the (technological) environment. What is the potential of embodied theories in explaining mathematical thinking without direct physical enactment? A final direction concerns the political implications of new ways of sensing/making sense in mathematics, such as the visual and the haptic, and how they remain subordinate to the alphanumerical (language and symbols).

The work of this discussion group will be continued at annual and local conferences, such as the Congress of the European Society for Research in Mathematics Education (CERME) and the annual conference of the International Group for the Psychology of Mathematics Education (PME).

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https://doi.org/10.1142/9789811287152_0096

Discussion Group 10

Computational and Algorithmic Thinking, Programming, and Coding in the School Mathematics Curriculum: Sharing Ideas and Implications for Practice

Max Stephens[1], Djordje M. Kadijevich[2], Qinqiong Zhang[3], and Haozhe Jiang[4]

ABSTRACT Computational/algorithmic thinking, programming, and coding are emerging areas of importance for mathematics thinking and are increasingly being located in the school mathematics curriculum in countries worldwide. This Discussion Group provided an opportunity to examine these international trends for curriculum, and teaching — in the compulsory years of schooling and in the senior high school years — and the need for appropriate policy responses.

Keywords: Computational thinking; Algorithmic thinking; National curriculum changes; Teaching activities; resources; Policy development.

1. Activity Overview

Participation in this DG involved more than 80 scholars/teachers online and offline.

Prior to the ICME14 Conference a website had been set up by one of the coordinators on which contributions to the discussion from more than twelve people had been pre-posted. Two contributions had been posted by members of the host East China Normal University. One prepared by Haozhe Jiang and Xiaoqin Wang described the current state of readiness among Chinese teachers to introduce computational thinking into their teaching using a PowerPoint in English and Chinese. A second contribution by Han Su — Algorithmic thinking in high school mathematics — An instructional design using the Babylonian method — was made available to participants through the website. Other pre-posted inputs to the DG1 are presented in Table 1 below and provided a structure for the discussion group.

Format: Discussion Group 1 had three components:

1. Selective survey of national curriculum documents
2. Presenting some classroom/teaching activities
3. Identifying some resources to support teachers

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Materials for the DG are shown in Tab. 1. For more details see: https://protect-au.mimecast.com/s/q4rSCP7yBls45kKYkSzers7?domain=afrodita.rcub.bg.ac.rs

Tab. 1. Materials for this DG

National Curriculum Documents	Classroom/Teaching Activities	Resources to Support Teachers
Argentina Australia Canada (BC), Coding & CT Canada (O) China England (primary) England (secondary) France Japan South Africa USA	D. Symons and M. Stephens, Australia, Number patterns S. Sardina and M. Stephens, Australia, Sorting activities H. Su, China, AT in high school Q. Zhang and S. C. Wang, China, Sorting + game J. Lu, China, Learning trajectory	Sadosky Foundation, Argentina IDSSP Framework Canada (BC), CT materials: Lighthouse labs, Student sheets, Competencies, Units1&2 W. S. Chang, China D. M. Kadijevich & A. Zakelj, Serbia & Slovenia

2. **National Curriculum Documents** (Coordinated by Max Stephens)

Comments and outputs. The earliest published of the above documents is the English national curriculum published in 2013. Other documents are all relatively recent, giving specific advice for involving computational thinking in the school mathematics programs. What is unknown is the state of readiness of teachers in these countries to undertake specific provisions for teaching computational thinking in their classrooms.

Many countries already have courses for computing, programming, applied computing and algorithmics in the senior high school years providing a transition to university courses in computing and related sciences. The above survey of national documents is focused on the compulsory years of schooling.

Some countries, such as Singapore actively promote computational thinking and algorithmics through intensive teacher development and pilot school programs. There is evidence of a similar trend in China where local educational authorities encourage suitably equipped local schools to implement coding and programming.

Other countries such as Finland have chosen to adopt programming (algorithmics) across different subjects in the school curriculum. Integrated models are not shown is in the official documents above; nor are out-of-school providers that offer weekend courses and study clubs for students in coding and programming in many countries.

Summary. Computational thinking is an emerging and important element of basic education in the national curricula of many countries as shown in Tab. 1. This reflects the prevalence of CT in our social, economic, and scientific environments.

3. **Three Classroom/Teaching Activities** (Coordinated by Qinqiong Zhang and Max Stephens)

Elaborations of these teaching activities are available on the website mentioned above.

3.1. Qinqiong Zhang and Shengchang Wang (Wenzhou University):

This lesson — involving a sorting game for the upper primary school — focused on helping students perceive the different ways to solve a problem through a simulated activity and preliminarily understand computational thinking.

3.2. Duncan Symons and Max Stephens (The University of Melbourne):

How computational/ algorithmic thinking can enrich students' mathematical thinking: Applications of Scratch in number, patterns and geometry in the middle years.

3.3. Sebastian Sardina (RMIT University Melbourne) and Max Stephens (The University of Melbourne):

Five approaches to number sorting activities – where computational thinking is contrasted with human thinking. The YouTube video can be viewed on https://youtu.be/q1TSnkEcQKM An enactment of Bubble sort as shown in the Bubble Sort video: https://www.youtube.com/watch?v=glgnCcjgpek, and also in the general website https://www.cs4all.org/

Comments and outputs. These teaching activities displayed key elements of CT: decomposition, abstraction, pattern recognition, algorithm building and generalization.

4. **Resources to Support Teachers** (Coordinated by Djordje M. Kadijevich)

Comments and outputs. This third component pointed to several resources that may support teachers' professional development and day-to-day teaching activities. These materials are shown in Tab. 1. If we view curriculum as a six-component vector whose dimensions are: goals, content, materials, forms of teaching, student activities, and assessment (Niss, 2016), developing and using good resources to support learning, teaching, and assessments are central issues. The discussion focused on two questions:

1. Are there suitable resources to support teachers' work regarding CT Integration?
2. Are mathematics teachers prepared/equipped to implement this integration?

Preliminary findings revealed that resources rarely focus on CT integration in mathematics. To examine some promising examples, use the hyperlink under Tab. 1. It was also found that various math-related learning activities (not only programming; see Weintrop et al, 2016), comprising and connecting CT stages, are not treated explicitly. In contrast to teachers of informatics, mathematics teachers are reported to have less knowledge and skills to apply CT integration and thus practice CT in their subjects less often (Slovenia). Furthermore, it is usually left to teachers to apply CT key components presented in workshops in subjects they teach (Slovenia, Canada (BC)). These issues generate huge challenges for teacher professional learning.

Summary. The inclusion of CT/AT in school mathematics will entail the search for/presentation of/development of good resources to support work in the classroom. These resources, which aimed at mathematics learning, should include various learning activities. In using resources teachers would not be left to themselves; teacher professional development should support them to do so in effective ways.

5. Future Directions and Suggestions

These elements that are further discussed in Stephens, Goos and Kadijevich (2023):

- *Realize the importance of CT.* Computational thinking is now omnipresent in the sciences, in data analytics and forecasting.
- *Use CT related resources.* Most resources will be on digital platforms, but some activities at all stages of schooling will be *computer-less* or *unplugged*.
- *Resources* provided locally and internationally by educational agencies, private foundations, laboratories, and universities will be essential to support the teaching and learning of CT.
- *Relate CT and mathematical thinking*. Research is needed to elucidate the connections between CT and mathematical reasoning and problem solving.
- *Embedding CT into the mathematics curriculum* will include different dimensions of practice: data *practices*, modelling and simulation *practices*, computational problem-solving *practices*, algorithm design *practices*, and systems thinking *practices*.
- *Develop CT related educational policies*. A key decision is where to split the focus between compulsory education and the later years of schooling where greater opportunities for choice and specialisation can be provided.
- *Building teacher capacity.* Another key policy decision will be determining the rate of change and providing for enhanced teacher professional learning with regard to modes of delivery and assessment.
- *Utilizing in-school cross curriculum models* where teachers of mathematics work in partnership with computer science colleagues.

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https://doi.org/10.1142/9789811287152_0097

Discussion Group 11

Teaching and Learning Linear Algebra

Sepideh Stewart[1], Maria Trigueros[2], and Michelle Zandieh[3]

1. Introduction

This discussion group will draw on the experience of three Linear Algebra researchers and curriculum designers to facilitate discussions around the past and future of Linear Algebra education. Linear Algebra is an important area of study for STEM majors. In a survey paper by Stewart, Andrews-Larson, and Zandieh (2019) the authors summarized some advances in many areas of linear algebra education (e.g., span, linear independence, eigenvectors, and eigenvalues). The survey paper also identified areas that need more research (e.g., systems of linear equations, properties of linear transformations, orthogonality, and least squares), and revealed the gaps (e.g., proof).

This discussion group will provide the opportunity to continue to develop and extend the field. Key questions and issues to be discussed are: What do we know from research about the teaching and learning of Linear Algebra? How can research results be used in the teaching of Linear Algebra? What innovative teaching methods have proved some success in the teaching of Linear Algebra?

2. Planned Structure

Planned timeline	Planned activity	Working format
21:30–21:40	Introduction	The organizers will give a brief overview of their research. Attendees will introduce themselves. The plan for the discussion group as well as a set of questions will be presented.
21:40–22:05	(a) Issues on first-year Linear Algebra topics (b) Teaching Resources (application, technology)	The attendees will break up in small groups to discuss: (a) What are some pressing issues concerning the teaching of first-year courses? (b) What teaching resources do you use to help students to understand the concepts better?
22:05–22:30	(c) Linear Algebra proofs, (d) Second courses in Linear Algebra	The attendees will break up in small groups to discuss: (c) What are some issues surrounding teaching linear algebra proofs? (d) What is the nature of second courses in your institution? The attendees will discuss the pertinence and possible contents of the second courses as a group.
22:30–22:50		Group discussion
22:50–23:00	Closing remarks, supporting new researchers, future work	The organizers will close by summarizing participants' views about future research.

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https://doi.org/10.1142/9789811287152_0098

Discussion Group 12

The Driving Forces Behind School Mathematics Curriculum Change in Asia

The Organizers[1] and the Working Team[2]

1. Short Description

There are many different driving forces behind every mathematics curriculum change around the world including politics, values, and culture. In recent time, one of the driving forces behind mathematics curriculum changes has been international assessment results. For instance, every four years, after the TIMSS results are released, many officials in various education systems tempting to take some remedial measures to improve their countries' ranking by the next TIMSS. The scope of this Discussion Group is to discuss the root causes of such hasty and sudden decisions. The organizers planned to invite the audience to discuss the ways in which, school mathematics curriculum be altered and adjusted in such ways to keep the balance between local and global situations and to use research findings properly to suit different education systems.

2. 1st Meeting (90 minutes)

- *10 minutes:* Introducing the aim and the rationale of this Discussion Group. Responsible person: Zahra Gooya — Co-organizer.
- *60 minutes:* Opened the floor for participants to discuss the controversial or emerging issues and/or dilemmas they have faced in mathematics curriculum in their countries, focusing on driving forces behind mathematics curricula change/reform in some of the Asian countries. Responsible person: Soheila Gholamazad — Co-organizer.
- *20 minutes:* Discussion among team members & Participants. Responsible persons: Zahra Gooya and Soheila Gholamazad, Co-organizers.

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[1] Organizers:
- Zahra Gooya, Shahid Beheshti University, Iran
- Soheila Gholamazad, Organization for Research and Educational Planning, Ministry of Education, Iran.

Both organizers have been involved in mathematics curriculum design and textbook writing at the national level in two decades and in this decade, they have been involved in evaluating new mathematics curriculum in Iran.

[2] The team members:
- Anjum Halai — Pakistan/ ICMI EC
- Yudaria Mohammad Yousef — Malaysia
- Ravi Subramanian — India
- Hamid Faizi — Afghanistan

3. 2nd Meeting (90 minutes)

- *50 minutes:* Challenging participants with the identified issues in the 1st meeting to find the relation between local characteristics and mathematics curriculum reform/change. Responsible persons: Team members and participants.
- *30 minutes:* Examining the development of a framework for studying mathematics curriculum changes in Asia. Responsible person: Soheila Gholamazad, Co-organizer.
- *10 minutes:* Planning for the continuation of the next Discussion Group in the ICME-15 and thinking about a possible publication regarding the scope of the current Discussion Group.

https://doi.org/10.1142/9789811287152_0099

Discussion Group 13

Mathematics Education and Teacher Professional Development System in Jiangsu Province

Xiaoyan Zhao[1], Lianhua Ning[2], Jingya Zhao[3], Jiuhong Wang[4], and Guangming Wei[5]

ABSTRACT In this discussion group, a system at various administrative levels for facilitating professional development of mathematics teachers in Jiangsu Province in China were introduced. By means of setting expert mathematics teacher studio, supporting People Educators in mathematics education, strengthening the cooperation between researchers and in-service mathematics teachers etc., a great progress has been made in mathematics education at primary and secondary level in terms of mathematics teachers' capability of providing instruction and doing research. Two specific examples from primary level were given in order to illustrate what happened in schools and classrooms. In the end, discussion was made between the participants and the audiences about how to make a larger group of both domestic and abroad teachers benefit from such teachers' valuable knowledge and experience.

Keywords: Teacher professional development system; Jiangsu Province; China.

1. Education in Jiangsu Province

Jiangsu Province is located in the middle part of the eastern coast of China, which has been considered as one of the provinces with highest educational development in China. In history, the development of Jiangsu is highly associated with its educational culture. In Nanjing, the capital of Jiangsu, there is the official imperial examination center (Jiangnan Examination Hall) for the Jiangnan region during the Ming and Qing dynasties. In addition, Nanjing Normal University is considered as the origin for teacher education in southern China.

For mathematics teacher professional development in Jiangsu Province, a system at various administrative levels has been established. By means of supporting People

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Educators in mathematics education, setting expert mathematics teacher studio, strengthening the cooperation between researchers and in-service mathematics teachers etc., a great progress has been made. In this discussion group, it will be shared with examples how these supporting systems helps to supporting mathematics teachers' professional development.

2. Activities Overview

As shown in Tab. 1, this one-hour and half discussion group consisted of five parts. The introductory and closing comments were organized in the format of whole group discussion, and in between three presentations were arranged. Around 40 participants in total took part in the discussion.

Tab. 1. Schedule of the discussion group (21:30–23:00, July 14, 2021)

Planned timeline	Planned activities	Format
21:30-21:40	Introductory comments (organizing the group discussion through the key questions)	Whole group discussion
21:40-22:00	System of and strategies for supporting mathematics teachers' professional development in Jiangsu Province	Presentation
22:00-22:25	Suitable for development: Purport of mathematics teaching wisdom under the condition of large class size	Presentation
22:25-22:45	Expert mathematics teacher studios driven by research-based teacher professional development programme	Presentation
22:45-23:00	Closing comments –summarize the presentations and discussions, and identify follow-up questions to investigate	Whole group discussion

2.1. Teacher professional development system in Jiangsu Province

Lianhua Ning and Minjie Chen explained the effective system established for supporting mathematics teachers' professional development in Jiangsu Province. It includes various forms of teacher training at the provincial, municipal and district (county) levels, aiming at improving teachers' professional knowledge and instructional skills (Duan et al., 2017). In addition, many hierarchical reward approaches have been designed by different administrative areas in order to encourage teachers to pay attention to their professional growth in the long run. Among all these measures, the most innovative three highlights are as follows. The first one is Educator Training Project, which focuses on top expert mathematics teachers. By organizing mentor groups, these selected teachers will get concrete and guidance according to their individuals' needs. The second one is Expert Mathematics Teacher Studio. At the center of these organization is a locally famous expert teacher, and the members are mainly the teachers in the neighborhood. Regular workshops and discussions, academic writing, and many other activities are organized within such studio in order to solve the problems encountered in daily teaching. The last one is Curricula Base

Construction, which support schools as a whole to develop their school-based mathematics curricula.

2.2. *Suitable development: The implications of teaching wisdom in large-size classes*

Jiuhong Wang, the headmaster of Nanjing Tianzheng Primary School, presented the research his expert mathematics teacher studio did about how to provide suitable instruction in large-size classes (Wang, 2017). The presentation is divided into four parts: how they came up with such idea, what suitable instruction looks like, how to construct the teaching system for students' suitable development at school level and practical suggestions for teachers to conduct in daily teaching.

In the mathematics curriculum standards of nine-year compulsory education (MoE, 2011), it is pointed out that everyone should receive a good mathematics education, and different people should get different developments in mathematics. This is a difficult goal to be achieved, especially considering that in Chinese mainland the class size is quite large with 45 students as maximum at primary level. This was why Dr Wang and his colleague chose the research topic of "Providing students with suitable development". This requires teachers to have wisdom to help individuals when solving mathematics problems. To be more specific, more attention is supposed to be paid to excellent students and students with learning difficulties. Meanwhile, make the most of the time and space during classroom teaching. In the end, six suggestions for suitable development and instruction were provided, which refer to learning goals, teaching content, students' cooperation, teachers' support, homework design and classroom assessment.

2.3. *Expert mathematics teacher studios driven by research-based teacher professional development programme*

Guangming Wei, the secretary of Experimental Primary School of Jinling High School, explained the construction of Wei Guangming Elementary School Mathematics Expert Teacher Studio and the longitudinal study the had done. In 2008, the studio was officially established with only 4 teacher members from one school; by 2021, the studio has more than 30 members which are from nine schools of four districts in two cities. Then he presented the fact that over the 13 years, studio members have made significant professional progress. When reflecting the reason why the teachers, including the expert teacher himself, pay attention to and benefit from the activities organized by the studio, it came to the conclusion that research-based programme with a clear and concrete research question to be conducted over years is the key. They have tried to applied many scientific research projects as a manner for promoting teachers' professional growth. In total, they had accomplished or were carrying out five projects, and the funding received was more than 600,000 RMB.

Their research focuses on the core mathematics knowledge teaching theory and practice in primary school (Wei et al., 2020), which can be further divided into three aspects: (1) what are the core mathematics knowledge? (2) How to make the core knowledge and skills standing out during the process of teaching and learning? (3) How to support students to make most of what they have gained by learning the core knowledge and transfer to explore other mathematics content by themselves. It turned out that this research-based teacher professional development programme is not only helpful for teachers' professional growth, but also improve students' learning effectiveness.

3. Future Directions and Suggestions

After the presentations, two major issues referring to the primary school practice were raised and discussed in the discussion group. The first one is how to distinguish and deal with the mathematics content except for the core knowledge. Wei introduced shortly how they divide core and non-core knowledge by applying the mathematical abstraction degree proposed by Lizhi Xu, a famous mathematician and educator in China. Teachers are supposed to pay much more attention to design teaching sequences for core knowledge; as for non-core knowledge, students are given more space to explore and reinvent by themselves under teachers' guidance. Another issue for discussion was raised by a researcher from Australia via Internet. He appreciated the valuable examples and experiences shared by the presenters, and considered such research done by in-service teachers and grounded in practice is precious. Also he and other participants in the discussion group, which are mainly Chinese, noticed that language barrier can be a big problem for teachers to share their practical teaching wisdom. Like Wei and Wang, expert mathematics teachers from other countries and districts probably also have precious and effective knowledge and experience including but not limit to professional development. How to make teachers' voice to be heard worldwide and how to promote teachers from different countries to exchange ideas with their foreign counterparts should be put on the schedule of scientific research and international conference.

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Part VIII

Workshops

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https://doi.org/10.1142/9789811287152_0100

Workshop 1

Topological Approach to Game Theory

Giovanna Bimonte[1], Francesco Saverio Tortoriello[2], and Ilaria Veronesi[3]

ABSTRACT We present a laboratory developed in the mathematics activities during the lessons of the research project *Mathematical High School* at the University of Salerno. We consider a continuous location optimization problem, where an optimal location is found in a continuum on a plane, using a topological approach involving the Voronoi diagram and the Delaunay triangulation to find the equilibrium point.

Keywords: Game Theory; Voronoi; Topology; Constructivism.

1. The Research Project

In Italian higher education, the topic of "Game Theory" is not included in the ministerial indications of the mathematics curriculum. Students do not have the prerequisites that enable them to understand and solve multivariable optimization problems.

In order to avoid the impossibility of solving problems of this type using analytical methods, we have chosen to approach them from a geometric point of view. Simple geometry concepts are required, such as the definitions and properties of Euclidean geometry and formulas and solution processes of plane analytic geometry.

Location problems concern the location of resources in a given space. Competitive Localization models also incorporate the fact that some structure is already present in the market and that the new structure will compete for market share. Let us consider a continuous location optimization problem, where an optimal location is in a continuum on a plane. We introduce the Voronoi diagram to solve the location problem, where the number of players is exogenously determined. We use Delaunay triangulation to find the equilibrium point and consider some generalizations of the ordinary Voronoi diagram.

The solution of the problem in the planar case with Euclidean distances and a variety of functions of attraction leads to a finite polynomial algorithm in the number of consumers. Using dynamic geometry software, we construct our case study on the

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Cartesian plane, we check how the results change as the starting conditions vary and we obtain the solutions without even performing the simple calculations required by the Cartesian geometry to find the equilibrium point.

2. The Laboratory: Voronoi Approach for Discrete Competitive Facility Location Model

The laboratory retraces the activities designed and developed in the Mathematical High School Project, which is an interdisciplinary extracurricular path developed by Research group in Mathematics Education of the Department of Medicine of the University of Salerno (Italy) aimed at overcoming the natural didactic division of knowledge in the various disciplinary fields.

The laboratory deals with competitive location models that relate to the fact that some facilities are already in the market and that the new facility will compete for market share.

There are two main approaches to estimating and analyzing the market share captured by facilities such as retail establishments, restaurants, etc., as a function of their location.

We decided to design the laboratory with the Hotelling approach (Hotelling 1929) and many extensions that assume that customers patronise the nearest facility and consider a class of continuous localization optimization problems that can be solved through the Voronoi diagram (Okabe, Suzuki 1987).

2.1. The basic location model

The Basic Location Model is defined as follows:

Consumers are distributed according to a measure λ on a compact Borel metric space $(S;\lambda)$ with S a compact subset of $\mathbf{R}^2$. A finite set $K = \{1,\dots,k\}$ of retailers have located their facilities on S. A new retailer wants to maximize his market share after locating a new facility, depending only on the “distance” variable.

The activity started asking the participants to divide the space according to the rules mentioned above in the simple case of two facilities, positioning in different configurations (symmetric and asymmetric case, with different distances).

2.2. The topological approach: the Voronoi tessellation

To trace the dominant regions of the players (i.e., their payoffs) in the intent of solving this locational optimization problem, we introduced to the participants the Voronoi tessellation of the space, making some reminders on topology concepts.

The location of the new facility was determined by the maximisation of the distance from other existing facilities and the task of determining this location is the largest empty circle problem.

In computational geometry, a Delaunay triangulation for a given set of discrete points in general position is a triangulation such that no point lies within the circumference of any triangle. The Delaunay triangulation of a set of discrete points in general position corresponds to the dual graph of the Voronoi diagram. The circumcenters of the Delaunay triangles are the vertices of the Voronoi diagram.

Thanks to the software, the participants drew the Delaunay triangulation, they built the circumferences circumscribed to the triangles and then they determined the circle of maximum area fully included in the area S.

The largest empty circle is the one centred in *Voronoi vertex*, so the participants draw the new configuration with the location of the new retailer with the maximum area.

In the laboratory activities the participants explored that the Voronoi problem can be approximated by an isolation game with a measure of distance and the solutions of the isolation game correspond to the Nash equilibrium points obtained by solving the maximum distance problem analytically in multivariable functions.

3. Activity Overview

Various techniques and applications of real cases were actively presented during the workshop and participants were involved in finding solutions to common problems. In the introduction, participants were provided with tools and models useful for understanding the problem of optimal localisation. We then made some reminders on topology concepts necessary for understanding Voronoi diagram construction and Delaunay triangulation. The participants were involved in positional games first on the line and then in the plane.

In the main part of the workshop, participants were presented with various positioning problems in the plane, facilitating discussion and an initial solution based on mere observation. During the workshop, geometric and mathematical models and teaching approaches were discussed, the close correlation between the various fields of knowledge was highlighted through examples of reality problems, underlining how mathematics is the transversal language to interpret the real world.

All problems were illustrated and communicated with the use of technology, in particular with the use of dynamic geometry software. Participants were challenged to formulate the solution of the positioning problem in terms of an optimization problem and then in topological terms.

Participants constructed the set of points and solved the problem of finding the optimal placement point in the plane.

In this way, we induced the topological solution, which coincided with the analytical solution without the use of advanced tools such as optimization in $\mathbf{R}^2$.

4. Future Directions

The activity proposed in the laboratory is part of the broader path of the Mathematical High School Project, which aims to provide students with a unified vision of culture. Still in relation to the topic of the activity presented, some paths that retrace the social experiments known in literature are being tested in the economic field, recalibrated through the use of technologies.

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https://doi.org/10.1142/9789811287152_0101

Workshop 2

Linguistic and Logical Methodological Tools to Address Language Diversity in Mathematics Education

Viviane Durand-Guerrier[1], Cris Edmonds-Wathen[2], Faïza Chellougui[3], Judith Njamgong Ngonsap[4], and Jean-Jacques Salone[5]

1. Aim of the Workshop

The aim of the workshop was to share with an international audience the linguistic and methodological tools we are developing in our own multilingual contexts to discuss:

1) the possibility of their generalization;
2) the ways to improve them for wider use;
3) to initiate international collaborations involving a variety of languages.

The main idea underlying this proposal is that in multilingual contexts differing grammatical structures of languages might affect the process of teaching and learning mathematics, whatever the level of instruction. We consider that switching from one language to another in a classroom might be both an obstacle or a resource (Edmonds-Wathen et al., 2016), and that translating even the most straightforward of mathematical statements from a language to another is challenging (Edmonds-Wathen and Bino, 2015). We also consider the issue of translating transcripts for research on language diversity (Chellougui et al. 2016).

2. Organization of the Session

The session held online on July 14th, 2021, 21:30 — 23:00 (UTC+8). After a presentation of the session, we provided two presentations of 15 minutes each about our methodological tools, followed by a ten minutes' discussion. Afterwards, the participants have been invited to fill up an online document with questions and tasks during about 15 minutes. This was followed by the sharing of the results and a collective discussion.

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3. Presentation of Our Methodological Tools

3.1. The logical analysis of statements. A tool for dealing with ambiguities in multilingual context

Logical analysis as a tool for didactic research in mathematics education as shown to be relevant. The tool relies on the assumption that objects, properties, and relations are fundamental logical categories involved in mathematical activities, and that quantificational issues play an essential role. Logical analysis in the frame of first order logic (Predicate calculus) with a semantic perspective allow to make visible, and to overcome logical ambiguities, mainly by considering the respective scopes of connectors and quantifiers in vernacular languages. The example of negation in French highlights the relevance of such analysis. First, the French grammar is not congruent with the logical structure, while the Arabic grammar is. This has didactic impact as shown for example in the Tunisian or the Cameroonian educational contexts (Durand-Guerrier, 2020). Second, the substitution principle *salva veritate* does not apply in French regarding negative sentences. Unpacking the logic of statements permits to overcome ambiguities in case where the context does not. This is useful both for a priori analysis of tasks submitted to students, and for interpretation of students' answers. It seems rather clear that in multilingual context, being able to deal explicitly with such ambiguities would open paths for remediation.

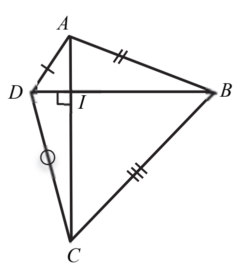
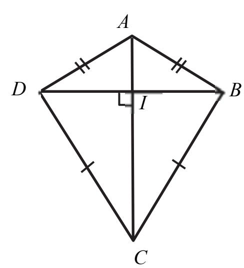
3.2. Linguistic methodological tools for multilingual mathematics education research

This presentation focused on two elements: (1) *Representation of multilingual data*. There are no well-defined conventions in mathematics education research for when language data should be presented in the original language and for how and when translations should be made. Interlinear morphemic glossing was presented as a tool to structure data presentation which makes explicit the process of translation, helping show how meaning is being made and providing more information than direct translation alone (Edmonds-Wathen, 2019; Edmonds-Wathen and Bino, 2015). (2) *A functional typological perspective for conceptualizing mathematical language cross-linguistically*. We know that mathematics registers in different languages are different, but not how different they are and the significance of the differences for learners and mathematicians (Edmonds-Wathen et al., 2016). This element discussed how to investigate mathematical expression/mathematics registers in different languages without privileging one language over another (Edmonds-Wathen, 2019). A typological perspective involves classifying languages according to their structural similarities and differences — focusing on syntactic typology, the different ways that languages structure phrases and sentences, and semantic typology, the different ways that languages structure semantic domains such as time and space. The functional typological perspective enables the mathematical concepts, practices, and affordances of diverse languages to be investigated within a broader mathematical frame. It asks:

- What functions are performed by language features that are used in existing school/academic/formal mathematics?
- How are those functions performed in the target language?
- Can the way that those functions are performed by drawn on or extended for the purposes of school mathematics?

4. Summary of Participants Activity

Eight participants from six different countries attended the workshop. After the two presentations, they worked individually on an online shared document with two parts. The first part on participants' relationships with languages revealed that most of the participants speak fluently only their mother tongue (6/8), even if their research leads them to be interested in other languages. English or French are also a teaching language for most of them (7/8). The second part included mathematical questions out of Ben Kilani (2005):

The 2 figures  Fig. I  Fig. II	1/ definition, existence of quadrilateral with perpendicular diagonals that are not rhombus, definition of "not being a Rhombus" for a quadrilateral, and translations of the answer in another language; 2/ Which of these two drawings represents a quadrilateral in which diagonals are perpendicular and all the sides are not equal? (Answers: Fig. I, Fig. II, both, none); a justification was expected.

The following discussion highlighted the grammatical problems of translating from one language to another. Considering than in multilingual educational context, this is likely to arise during mathematical activities, it appears as a challenge for both teaching and research. The activity on the two figures clearly showed that this is at stake even for elementary mathematical statements, which is generally not recognized by teachers.

5. Conclusion and Perspectives

In this workshop, we intended to share with participants our methodological tools for addressing grammatical issues in mathematics education**,** where the focus is generally on lexis. The next step is to explore other multilingual contexts by developing an international network.

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https://doi.org/10.1142/9789811287152_0102

Workshop 3

Poly-Universe and Lénárt Sphere: Manipulatives from Hungary

Zsuzsa Dárdai[1], István Lénárt[2], János Saxon Szász[3], Eleonóra Stettner[4], Réka Szász[5], and Szabina Tóth[6]

Tools for Participants: In order to enhance the experience, we recommend participants to have the following tools available during the workshop (they are not required): laptop or tablet, two oranges, colored pens or markers that can mark the oranges, rubber bands, toothpicks, bottle caps.

Hungary has a strong tradition of using games and manipulatives to develop concept building and problem solving in mathematics. This stems from the work of Tamás Varga, but there are constantly new developments in this trend (Vancsó et al., 2018). The workshop presents online adaptations of two such tools: the Lénárt Sphere developed in 1986 by István Lénárt (Lénárt, 1996), and the Poly-Universe set developed in 2009 by János Szász Saxon (Saxon and Stettner, 2019). Both tools are used with 6-18-year-old students and in teacher training.

The aim of the Lénárt sphere is to explore analogies and differences between the plane and the sphere, and get a first-hand experience on comparative geometry education for all levels. (Lénárt and Rybak, 2017). Participants will study basic ideas of geometry on the plane and on the sphere, for example straight lines, and circles. Comparison and contrast make concepts understandable and thought-provoking even for those who are indifferent or hostile towards mathematics. Euclidean monologue transforms into a dialogue, a drama between two approaches to geometry. Besides introducing the Lénárt Sphere with spherical rulers, compasses and protractors, we will work with everyday objects such as oranges and rubber bands, which can be used in online and in person classrooms. Independent investigation and peer discussion prevail over lecture, self-made discovery rather than passive acceptance of definitions. Comparing different geometries may help students understand how relative and human all axioms and theorems of science are, and help them develop tolerance and understanding of those who are different in their cultural or social backgrounds.

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The aim of the Poly-Universe session is to explore the various educational aspects of a tool that originates from art, and which connects multiple subjects and mathematical topics. The novelty value of Poly-Universe lies in the scale-shifting symmetry inherent to its geometric forms and a color combination system. As part of the EU Erasmus + PUSE (Poly-Universe in School Education) project 2017/19 an international team of educators, teachers and students designed tasks for using the tool in mathematics education for primary, middle and high school students and in teacher training (PUSE Methodology book, Saxon and Stettner, 2019). These tasks are visual, hands on and analytic at the same time, so they require both right and left brain functions, and they are centred around motivation, experience, interaction, problem solving and creativity. The tasks are connected to the topics of Geometry & Measurement, Combinatorics & Probability, Sets & Logic, Graphs & Algorithms, Complex & Visuality.

In the workshop participants will have the opportunity to try out and discuss the e-learning platform of the Poly-Universe set, which is an online application for manipulating the set and solving visual mathematics tasks: https://www.puse.education.

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Other important links to the polyuniverse:

http://www.poly-universe.com/
http://www.saxon-szasz.hu
http://www.saxonartgallery.com
https://www.punte.eu
http://bsmeducation.com/

Link to the Lénart sphere:

https://iopscience.iop.org/article/10.1088/1742-6596/1840/1/012003/pdf
https://iopscience.iop.org/article/10.1088/1742-6596/1946/1/012005

https://doi.org/10.1142/9789811287152_0103

Workshop 4

Folding for Fractional Understanding

Bjorg Jóhannsdóttir[1] and Heather Ann Coughlin[2]

ABSTRACT The goal of this interactive online workshop was to introduce paper strips as manipulatives to foster understanding of fractions. Attendees gained appreciation for the versatility of the paper strip to visualize concepts, link fractions to the whole numbers, and build arithmetic algorithms. The operations became alive in the participants' hands.

Keywords: Fractions; Folding; Fractional understanding; Number line.

1. Theme and Description

1.1. *The presenters*

Dr. Coughlin and Dr. Jóhannsdóttir are experienced educators of prospective teachers. Dr. Jóhannsdóttir has a doctorate in Mathematics Education and Dr. Coughlin in Pure Mathematics. Together, they have over 25 years of college teaching and 13 years of teaching in junior high/high schools. Dr. Coughlin and Dr. Jóhannsdóttir are co-directors of the Central California Mathematics Projects, where they design and implement professional development workshops for teachers in Central California.

1.2. *Why fractions and paper strips*

Findings from international studies like PISA and TIMSS indicate that students worldwide struggle with concepts related to fractions (Neagoy, 2017). To improve learning of fractions, the presenters model effective practices, using easily accessible manipulatives: paper strips. During the COVID-19 pandemic, presenters successfully used paper strips in all sorts of activities over Zoom to increase pre-service teachers' understanding of fractions, their properties, and operations with fractions.

The goal of this virtual workshop was to introduce paper strips as manipulatives to foster understanding of fractions. The strips were used to model concrete representations to assist in defining and explaining fractions and make sense of basic arithmetic operations using fractions. During manipulations of the paper strips, familiar concepts emerged, such as common denominator, and equivalent fractions. Using a

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paper strip created a tangible way to "extend previous understandings of operations on whole numbers" to fractions, as emphasized in the Common Core State Standards for Mathematics, CCSS-M (California, 2013).

Underlying the manipulations of the paper strips is the definition of fractions as described in the CCSS-M in the United States. "*Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b* (California, 2013)." This definition emphasizes thinking about fractions multiplicatively, instead of additively. Students should see 3/4 as three iterations of 1/4, or 3 (1/4). Attendees took part in creative activities they could take back to their classrooms to build students' understanding of fractions.

2. Activity Overview

The format of the Folding for Fractional Understanding workshop was a synchronous, 90 minutes, Zoom meeting. Each participant came prepared with 10 paper strips approximately 2 centimeters wide cut out along the long side of a piece of paper.

2.1. Background

Following the aforementioned definition of fractions, the CCSS-M, developed in 2009 to coordinate and standardize mathematics instruction across the United States, goes on to direct that students need to "*Understand a fraction as a number on the number line; represent fractions on a number line diagram. Represent a fraction 1/b on a number line diagram by defining the interval 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size 1/b and that the endpoint of the part based at 0 locates the number 1/b on the number line. Represent a fraction a/b on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line* (California, 2013)."

This description of a fraction and its representation on a number line attempt to connect fractions to whole numbers in the "marking off" or counting a number of *a* intervals from *0* of size *1/b*. Intrinsic within this definition, *a* and *b* are whole numbers with *b* not equaling *0*.

The presenters shared how they had noticed that this connection of the definition of a fraction to a location on a number line was difficult for some of their prospective teachers. They shared responses to the activity: *Make five fractions using once each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Place the fractions on a number line. Make a visual representation of each fraction.*

The responses sparked discussions on how the prospective teachers appeared to have discrepancies in their understanding of fractions. Many drew circles as fraction diagrams indicating an acceptable understanding of fractions as parts of a whole. However, representing fractions as numbers on a number line was troublesome and showed little connection to the pictorial representation already made.

To bridge this gap in students' understanding, the presenters recommended using paper strips to build the transition from the concrete area model to the number line model. Either long-edge of a paper strip may be marked in ways similar to a number line. Both the paper strip and number line models emphasize how to work with fractions similarly to whole numbers, and they lessen the need for mixed numbers. Additionally, emphasizing improper fractions is unnecessary. For example, showing *5/4* on a number line requires one to begin at *0*, then count out five lengths of *1/4*.

2.2. *Activities*

2.2.1. Addition and subtraction

Participants were asked to visit a place in time before they had learned how to add fractions and heard of a common denominator. Armed with three paper strips and the definition of a fraction, they were to add 1/2 and 1/3, using the instructions; *You need three paper strips of equal length. First paper strip — show 1/2. Second paper strip — show 1/3. "Add" the paper strips. What is your "sum"? Use the third paper strip to determine your "sum."*

Participants began by showing their 1/2 and 1/3 on two paper strips and were asked to "prove" that those were indeed 1/2 and 1/3, by referring to the definition of fractions. Once it had been established, the parts were combined, see Fig. 1. Now the task was to figure out what fraction of a whole paper strip the combined paper strips parts were. When compared to a whole paper strip, it was easy to see that the combined parts were less than a whole strip, and their length was marked on the whole strip, and the remaining piece colored black. The combined length could not be measured by either 1/2s or 1/3s. After sharing ideas and trying different things, one of the participants suggested that we fold the whole paper strip into pieces the size of the black piece. It turned out that the whole strip could be folded that way into six equal parts, see Fig. 2. Now when compared, the combined parts were of the same length as five of the six equal parts, or 5/6 of the whole paper strip.

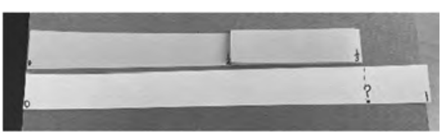

Fig. 1. ½ + ⅓ compared to one whole

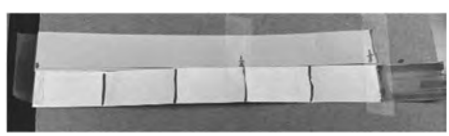

Fig. 2. ½ + ⅓ = ⅚

Participants were asked to subtract (2/3) − (1/2) using a similar process with the paper strips. Adding and subtracting fractions using the paper strips ignited discussions among participants on how these activities paved the way for students' discovery of equivalent fractions and the meaning of a common denominator. This advanced teachers' pedagogy to avoid math tricks and shortcuts, and instead first prioritize conceptual understanding and procedural fluency (Dougherty et al., 2017).

2.2.2. Multiplication and division

Lively discussions arose on how the presented activities could be used in participants' classrooms. This resulted in much less time for the multiplication and division of fractions than planned. For demonstrating multiplication, presenters began with simple problems such as 1/2 times 4/5 and built to more challenging problems. In multiplication the focus is on the definition of fractions, 5/3 is 5 times 1/3. In order to find 5/3 of 1/2, you first need to find 1/3 of 1/2, and then take five such pieces to get the 5/3 of 1/2.

When dividing whole numbers, the fair share model (how many in one group) is more frequently used, but when dividing fractions, the measurement model (how many groups) makes more sense. Using the measurement model, the problem 3/4 divided by 1/4, is asking, how many 1/4 fit into 3/4? The paper strips easily modeled this problem.

3. Future Directions and Suggestions

Being part of ICME-14 was a smooth and fun adventure. Participants from all over the world attended our workshop. It was valuable connecting with colleagues during the difficult COVID-19 times. The organizers of ICME-14 were professional and supplied the support needed to make the workshop successful.

The presenters welcome further exploration of fractions with materials such as paper strips. From here, returning to the paper strips allows students to strengthen their number sense with fractions. Students will have tangible examples to visualize comparing fractions, or answers to situations such as whether the product of a fraction by another fraction will be less than, equal to, or greater than either original fraction.

Acknowledgments

The presenters acknowledge the California Mathematics Project which developed the foundational ideas used in this workshop.

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https://doi.org/10.1142/9789811287152_0104

Workshop 5

From the Power of Intuition to the Beauty of Abstraction

Damjan Kobal[1]

ABSTRACT Contrary to the fact that mathematics many ideas, beauty and inspiration are hidden within simple and intuitive patterns, which are easily noticed and 'intuitively understood', mathematics is considered very abstract. Therefore, the motivation for mathematics teaching and learning should be intuitive and the beauty of abstraction will rise from there. As teachers we need to challenge our sensibility for the importance of the intuitive in mathematics teaching and learning. We introduce these challenges by smartly chosen hands-on (and eyes-on) problems. Like some 'graphic puzzles', which are understood in seconds, but are often harder to formulate then to solve. Through examples we explore how understanding, motivation and challenge often lie within intuitive comprehension and how abstraction (especially on the primary level) only follows later.

Keywords: Intuition; Visualization; Deductive reasoning; Abstract thinking.

1. Introduction

This paper is a short overview, a content summary of the Workshop WS3 entitled *From the Power of Intuition to the Beauty of Abstraction*, which took place at ICME-14 on July 14, 2021, 21:30 — 23:00 at T519 and was conducted on-line. About twenty participants were present physically in T519 and on-line. The content was presented and communicated through provided on-line technology, using *PowerPoint*, *Drawboard*, *GeoGebra* and *Web browser*. To support the presented content and to motivate the engagement of participants a special web page (direct link: http://ko.fmf.uni-lj.si/ICME-14/) had been designed (Kobal, 2021). Careful reading of this paper and thorough study of the inter-active materials referenced in this paper might be rewarded with many useful teaching ideas or even with research insights into what is the meaning of 'understanding'. Throughout the content of this paper, we try to challenge the development of critical and creative approach to the technology use. Several samples of exemplary GeoGebra use are presented. Cases where critical teachers will uncover weak or even counter-productive IT use and content neglect will also be considered. With practical and true hands-on and minds-on activity we

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challenge our reasoning and nourish our intuition to become a valuable predecessor of abstract mathematical thinking.

Presented content is split into five sections. In the first Graphic Puzzles we introduce some intuitive puzzles, which are very easy to understand and often harder to precisely formulate then to solve. In the very short second section Listening to Number Patterns, we intuitively approach some simple mathematical ideas by listening (not by watching). In the third section Basic Examples, we present several elementary ideas where dynamic computer graphic truly helps to motivate and intuitively explain some of the mathematical concepts. In the fourth section titled Motivation vs. Manipulation — Math Rigour vs. Fake News we address some of the dangers of uncritical use of technology. And in the last section with the title Saper Vedere — Saper Raccontare we present a couple of inspiring uses of graphic and intuitive presentations of otherwise very abstract mathematical concepts.

It is worth mentioning that this workshop overview might be hard to follow and understand in every detail for someone, who has not participated at the workshop. Without proper leading, explanations or at least knowledge of the presented ideas, the reader will need to carefully study the interactive animations. Namely, considered ideas and especially prepared interactive animations are designed to speak for themselves and mostly without words, but they do require proper and thoughtful use of its interactive options.

2. Hands-on, eyes-on problems

2.1. Puzzles

- Shikaku puzzle
- Numberlink puzzle

2.2. Listening to Number Patterns

(Link: http://ko.fmf.uni-lj.si/ICME-14/sound/)

2.3. Basic Examples

- Counting (Link: https://www.geogebra.org/m/f92brpez)
- Four (dancing) points (Link: https://www.geogebra.org/m/G3nDp7DE)
- Triangle on the top of a square (Link: https://www.geogebra.org/m/ffp5EBQE)
- Midpoints of a quadrilateral (Link: https://www.geogebra.org/m/p7UwKKkC)
- Regular octagon? (Link: https://www.geogebra.org/m/nwYw4Frd)
- Absolute value formula (Link: https://www.geogebra.org/m/ggh67zxd)
- Circle area (Link: https://www.geogebra.org/m/uphpme6f)
- Line through centroid (Link: https://www.geogebra.org/m/jkvucnuv)
- Intersection of two squares (Link: https://www.geogebra.org/m/twfwnpf5)

- Geometric series formula geometrically (Link: https://www.geogebra.org/m/askxzcwd)

2.4. Motivation vs. Manipulation — Math Rigour vs. Fake News

- Rigour and consideration (Link: https://www.geogebra.org/m/nhhrkxdt)
- “Ingenious” Pitagora’s theorem proofs (Link: https://www.geogebra.org/m/qzcqpyh9)
- Visualizations or illusions? (Link: https://www.geogebra.org/m/qh3jnxuw)
- A geometric paradox inspires thinking (Link: https://www.geogebra.org/m/a3qpvpgq)

2.5. Saper Vedere — Saper Raccontare

- Fixed point theorem (Link: https://www.geogebra.org/m/fzvva3ye)
- Visualizing linearity (Link: https://www.geogebra.org/m/US9XMnmp)
- Visualizing Jacobian (Link: https://www.geogebra.org/m/ZCu2qrcJ)
- Parabola’s section (Link: https://www.geogebra.org/m/Tst5NqdM)
- Parabola and car lights (Link: https://www.geogebra.org/m/udkqcvg3)
- Discrete functions and sound transmission (Link: https://www.geogebra.org/m/VdgTSsrx)
- Three pots problems and trilinear coordinates (Link: https://www.geogebra.org/m/hkfqjjhn)

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https://doi.org/10.1142/9789811287152_0105

Workshop 6

Exploring the Role of an Online Interactive Platform in Supporting Dialogue in Mathematics Classrooms

Qian Liu[1] and Yuan Zhang[2]

ABSTRACT This is a report of an interactive workshop that was collaboratively designed and implemented by a researcher and a group of practitioners. To share theoretical insights and teaching practices concerning the use of an online interactive platform to support classroom dialogue in mathematics classrooms, this article presents the theoretical and research underpinnings of the workshop, the procedure and main activities, the observed outputs, and reflection and suggestions on designing and organising the hybrid format of workshops in this field.

Keywords: Digital technology; Classroom dialogue; Lesson study; Primary mathematics education.

1. Introduction

There is a growing body of research evidence showing that productive classroom dialogue is beneficial for students' mathematical attainment and mathematical thinking development (e.g. Howe et al., 2019; Mercer and Sams, 2006; Webb et al., 2014). Recently, research interest has been increasingly drawn to the role of digital technologies in supporting classroom dialogue. This is mainly because affordances of digital technologies for learning (e.g., multimodality, interactivity, revisability) are argued to have potential in *opening, expanding and deepening dialogic space* (Wegerif, 2007). Within a dialogic space, multiple perspectives are openly shared, critically and creatively linked and synthesised and new meaning collectively constructed (Major et al., 2018). However, the realisation of any perceived potential of digital technologies requires pedagogical intention and practice. A one-year design-based research with Chinese mathematics teachers in a primary school was conducted with the research interest in optimising the technological potential in classroom dialogue. Based on the iteratively developed teacher professional development programme, a group of four mathematics teachers explored teaching strategies for using an online interactive platform, *Zoomabc (全景平台)* to support classroom dialogue. This workshop was not

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only informed by the related theories including socio-cultural and dialogical theories, but also grounded in the exploratory teaching practices in actual classrooms.

The workshop, bridging dialogic theory and classroom practice, aimed to enrich participants' understanding of the role of online interactive technology in mathematics teaching and learning, from the dialogic perspective. Hence, the workshop introduced Zoomabc and focused on illustrating and explaining its potential affordances for supporting classroom dialogue. The second aim was to share the pedagogical framework derived from the research in conjunction with the teaching strategies exemplified by real lesson cases. In addition, the workshop highlights lesson study as a means to professional development to support the development of dialogic teaching with digital technologies. To systematically analyse and scrutinise and reflect on dialogic teaching practices, a coding scheme, the Teacher Scheme for Educational Dialogue Analysis (T-SEDA) (Hennessy et al., 2021) was introduced. It is worth stressing that the workshop was designed and implemented in a dialogic manner, hoping to draw participants into an open, critical, and ongoing dialogue about the workshop theme.

2. Procedure and Main Activities

Based on the aforementioned aims, Tab. 1 outlines the procedure of the 1.5-hour workshop and its activities.

Tab. 1. The workshop procedure and activities

Duration	Activity	Format
10 min	Welcoming and ice breaking: specifying the workshop's main aims and structure. Inviting participants to share their interests, experiences and expectations related to the workshop topic.	Short presentation Publicly sharing ideas on Padlet
20 min	Introducing the theoretical and research backgrounds of the workshop, the lesson study model and the T-SEDA coding tool. The presentation ends with a brief demonstration of Zoomabc in terms of its technical features and potential affordances for classroom dialogue.	Presentation
15 min	Sharing lesson case one: making two-digit numbers using counters on a tens and ones place value chart. Q & A	Presentation and interaction
15 min	Sharing lesson case two: areas of rectangles. Q & A	Presentation and interaction
20 min	Adapting a short lesson episode 'Areas of Parallelograms'. This activity ends with an open-ended question: what factors should be considered and addressed when attempting to foster classroom dialogue with the use of interactive technologies?	Lesson planning and discussion
10 min	Conclusion with the proposed pedagogical framework. Inviting final comments and questions. Inviting participants to share any ideas and questions on Padlet after the workshop.	Plenary

3. Outputs

This workshop was conducted in a hybrid format, aiming to engage participants from diverse backgrounds. The first 20-minute presentation enabled participants to learn about the theoretical and research backgrounds of the workshop and enrich their conceptual and practical understandings of educational dialogue. Some participants posed questions and shared valuable insights online mainly regarding the disciplinary nature of classroom dialogue in mathematics classrooms.

Given the dialogic approach to the workshop, participants asked questions and proposed alternative teaching approaches related to the shared two lesson examples. They agreed that the use of Zoomabc facilitated students' wider and sustainable participation in dialogue, especially contributing to fruitful opportunities for individual students to express ideas publicly and engage with each other's ideas. However, they pointed out that students' online comments may not be so dialogic and lack criticality, and some comments were not relevant to the contributions (e.g., commenting on writing rather than content).

Facilitated by the interactive activity for adapting the lesson episode and the open question, the pedagogical framework (see Fig. 1) and the accompanying pedagogical strategies were discussed both in this research context and in the participants' own contexts.

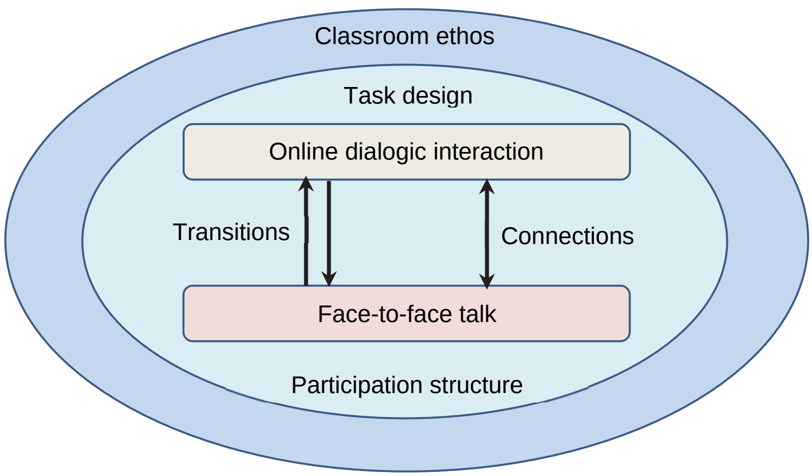

Fig.1. Pedagogical framework for fostering students' dialogic participation supported by using online interactive platforms

4. Reflection and Suggestions

This workshop shared theoretical insights into the role of digital technology in supporting classroom dialogue and focused on the teaching practices concerning the use of Zoomabc, an online interactive platform. Taking the dialogic stance, the workshop enabled participants to share different viewpoints, raise questions, and co-construct understanding and knowledge surrounding this topic.

There are two aspects worthy of our reflection and suggestions. Regarding workshop content, it should be noted that the shared teaching practices and developed pedagogical strategies are situated in the specific context (e.g., the Chinese primary school, technological environment framed by Zoomabc). Hence, applying the strategies to other contexts should be cautious, which requires contextual considerations. The intertwined inquiry between research and practice should be open, diverse and ongoing. Thus, it is hoped that the workshop as a catalyst can stimulate wider interest and encourage researchers from different contexts to further explore and investigate pedagogy for optimising the potential of digital technologies in supporting educational dialogue in mathematics classrooms.

The second aspect relates to the activity formats. To encourage participants' active, critical, and creative engagement, we designed hands-on and interactive activities. The dialogic approach to the workshop enabled us to create and develop an equal, supportive, open-minded, and co-constructive ethos. In corresponding to online interactive platforms, the use of Padlet enabled participants, both in person and online, to have a first-hand and authentic experience regarding how Padlet supported ongoing and cumulative dialogue across time and place. Therefore, future efforts may be needed to strengthen dialogic links between participants who are online and those who are physically present in the hybrid format of workshops.

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https://doi.org/10.1142/9789811287152_0106

Workshop 7

Going Beyond the Numbers — Exploring Social Justice in the Mathematics Classroom through Global Connections

Chadd McGlone[1], Hanna Nadim Haydar[2], and Paola Castillo[3]

ABSTRACT By school years, mathematics in the classroom becomes separated from real life. However, if teachers can bring context back into mathematics, like the kids experience outside of school, math becomes real. Beginning class by teaching students a bit about what it's like to live in another part of the world brings class alive. Global Math Stories (GlobalMathStories.org) is a resource that helps educators make cultural and global connections in the classroom. In this presentation, participants learned about the resource and explored the value of making global connections in the classroom.

Keywords: Culture; Global; Pedagogy.

1. Activity Overview

Participants in this workshop consisted of classroom teachers, education leaders, and teacher trainers from multiple countries who were looking for ways to weave local and global cultures into mathematics lessons.

Mathematics comes alive when teachers make global connections in the classroom. Unfortunately, many classroom teachers lack the time and resources to complete the research required to make these connections.

Global Math Stories (GMS) is a free web-based resource (GlobalMathStories.org) that helps classroom teachers make global connections in the classroom. The site consists of approximately 70, one-page stories written by people from around the World, presented in both English and Spanish, called Mate Mundial (mathkind.org/mate-mundial). Teachers may choose to write their own lessons from the stories or use one of the lessons already developed. Each story is supplemented with resources to further explore the culture and social justice questions to look behind the mathematics. Often, after using a story in their classroom, teachers or students decide to write and submit their own story.

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In this workshop, participants learned about the site and how to use it to make global connections in their classroom. First, they participated in a mathematical task based on the Wagah Boarder Ceremony. In addition to the analysing the mathematics in the tasks, individuals considered social justice questions associated with the story.

Next, participants explored another story, this time based on the island city of Malé, the capital of Maldives. Working as teams, groups developed mathematical tasks they might create based on the story. Additionally, they explored existing and created their own social justice questions that arise from their exploration.

Before choosing a story to presenting themselves, participants reviewed the role of stories in the mathematics lessons. When mathematical tasks are designed to allow students to make meaningful connections, they develop create a context around which to build mathematical concepts.

Progressing along, participants will pick a story from which to develop a task. They presented the story to the audience and describe how a lesson would progress, complete with a social justice question. Some participants developed actual lessons they would use for their classroom.

Finally, everyone will discuss the benefits of making connections of local and global cultures in the classroom. They will be invited to contribute stories to the site and to share it with educators and authors in their communities.

2. Future Direction

Feedback from participants was universally positive. Future work, generally with culture and mathematics, specifically with Global Math Stories, must include ways teachers can fulfil restrictive, mandated objectives while making cultural and global connections. To that end, the producers of the Global Math Stories website is matching specific objectives to each story so that every story will have one specific, identified objective for every grade level. These objectives will be searchable.

Secondly, the social justice questions and dilemmas associated with each story must be more clearly identified and searchable. Each story contains at least two “extension questions”, but they are not searchable. Future work to strengthen the site must make them searchable.

Finally, future research must be conducted to more clearly identify how making global and cultural connections in the mathematics classroom deepens student understanding of the mathematical concepts being taught.

Lessons that teach include global and cultural connections should not be special, one-off experiences. Rather, they must become part of an integrated curriculum that allows teachers to meet the mathematical objectives set forth by the administration or government. Resources like Global Math Stories are a step toward reaching this objective.

https://doi.org/10.1142/9789811287152_0107

Workshop 8

Math for All: Professional Learning to Help Teachers Reach All Students in the Mathematics Classroom

Babette Moeller[1] and Matthew McLeod[2]

ABSTRACT Persistent differences in mathematics performance between general and special education students underscore the need for improving teachers' preparation to better serve the needs of students with different strengths and needs. Math for All is a research-based, intensive professional learning program designed to help K–5th grade teachers provide accessible, high-quality mathematics instruction to ALL students, including students with disabilities. Using a neurodevelopmental framework (NDF), we analyze the lesson, understand the strengths and challenges of a student, and build adaptations that support the student's access to the lesson. Multiple research studies have demonstrated the efficacy of the Math for All approach.

Keywords: Mathematics; Instruction; Students with disabilities; Accessibility; Lesson planning; Neurodevelopmental framework.

1. Theme and Description

Math for All (Moeller et al., 2012; 2013) is a research-based professional learning program for teachers of grades K-5. The program introduces general and special education teachers to a neurodevelopmental framework (NDF) as a tool to analyze a mathematics lesson or task, including all the demands placed on students as they engage in the activity. This same lens is applied to understanding a student's learning profile in an effort to identify their strengths and challenges. Armed with detailed understanding of the lesson and the student, general and special education teachers collaboratively plan adaptations to the lesson that will increase its accessibility while maintaining the goals and rigor of the mathematics.

At the core of Math for All is the neurodevelopmental framework (Barringer et al., 2010). The NDF parses learning into eight functions — memory, attention, psychosocial, language, higher order thinking, spatial, sequential, and motor — that engage whenever we learn or do anything. Focusing on mathematics learning and instruction, participants learn about the NDF through articles and video case studies

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during a portion of the workshops. The remainder of each workshop is devoted to helping teachers apply what they learned through adapting a lesson from their mathematics program. General education teachers work with their special education colleagues and experienced Math for All facilitators to engage in a process of collaborative lesson planning to adapt a lesson that they will teach in the near future. During this planning, teams use the NDF to analyze the demands of the lesson, understand their student's learning profile, and develop adaptations that capitalize on the student's strengths and help mitigate the challenges they might face in accessing the lesson.

2. ICME-14 Activity Overview

In our conference session, attendees sampled a portion of the first workshop in which we introduced the NDF and watched a student as she worked on a lesson about building arrays and finding factor pairs of a number, in this case 24. Participants began by watching a recording of the teacher introducing the lesson and then carried out the task using a set of virtual unifix cubes, followed by creating a list of what they had to know and do in order to succeed at the task — move the cubes, count to 24, form a rectangle from the cubes, etc. We next introduced the NDF and asked participants to categorize each of the demands into the eight constructs. Then we observed a focal student, Jashandeep, while she was working with a partner on the same lesson, and we tried to understand something about her strengths and challenges. Finally, we discussed some potential ways that the teacher did or could have supported Jashandeep to be successful.

We concluded our session by sharing research findings (Duncan et al., 2018) which showed that teachers who engaged in Math for All made positive changes to their instructional practices, felt more prepared to meet the needs of all their students, and saw larger increases in student achievement scores than did comparison group teachers.

3. Future Directions and Suggestions

Several components of the Math for All professional learning model make it particularly successful for changing teachers' understanding of individual students' strengths and needs and for improving student outcomes in mathematics: (1) a focus on learning rather than teaching mathematics and the use of a framework informed by learning science to help teachers better understand how students learn; (2) the collaboration between general and special education teachers; and (3) practice-based professional learning that engages teachers in intentional lesson planning and reflection on their practice. We recommend that mathematics leaders incorporate these components in their work with teachers, and encourage further research into what makes professional learning most effective.

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https://doi.org/10.1142/9789811287152_0108

Workshop 9

Networking Design Approaches: Around the Teaching of Mathematical Proof

Tatsuya Mizoguchi[1], Ignasi Florensa[2], Koji Otaki[3], and Hiroaki Hamanaka[4]

1. Theme and Description

This workshop is based on part of an ongoing research project regarding the cultural and anthropological study on the development of competencies of mathematical proof throughout of secondary school. The workshop focused on the design of teaching, especially for mathematical proof task. For this, various designs based on different theoretical approaches were compared and their characteristics were considered. The key questions of the workshop were as follows:

(1) How teaching of mathematical proof can be designed with each approach;
(2) What characteristics each approach has in the design process;
(3) How does each approach complement the others?

In the workshop, we considered three different approaches: Study and research paths (with Q-A map) in the Anthropological Theory of the Didactic; Japanese problem-solving lesson model (so called open approach); Substantial Learning Environment. Based on the planned structure shown below, the workshop accessed the above questions by designing teaching for a common mathematical task (Tartaglia's triangle) and then comparing/networking them. There are cultural and theoretical differences in teaching design. In the workshop, we searched for the possibility of new approaches on these problems through collaboration with participants.

The workshop was organized by Japan and Spain jointly. Each organizer was familiar with each theoretical approach. In addition, the workshop was conducted in cooperation with the following prospective contributors (alphabetical order): Yoshitaka Abe (Niigata, University, Japan), Terumasa Ishii (Kyoto University, Japan), Hiroyuki Kumakura (Shizuoka University, Japan), Susumu Kunimune (Shizuoka University, Japan), Takeshi Miyakawa (Waseda University, Japan), Yusuke Shinno (Hiroshima University, Japan), and Yuki Suginomoto (Nagasaki University, Japan).

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Study and research paths [SRPs] are the inquiry-based teaching formats proposed by the Anthropological Theory of the Didactic. SRPs are study processes initiated by an open generating question Q stated to a community of study. These processes include moments of study (search for available and relevant information to answer the question) and research (adaptation of the found information to the specific problem, creation of new solutions) (cf. Florensa, et al., 2021). Problem-solving lesson model is a format widely used in mathematics lessons in Japan. It includes the following phases (although each label may vary): problem posing (comprehension); self-solving, refining and elaborating; and summarizing and developing. Stigler and Hiebert (1999) called it “structured problem solving”. In this workshop, we will provide a template which is arranged so that even beginners can use it (cf. Mizoguchi, 2013). Substantial Learning Environment [SLE] is a keyword of Wittmann’s perspective of the didactics of mathematics as a branch of design science. It means a didactic device or organization which fulfills the following four conditions: (1) representing central objectives, contents and principles of teaching mathematics at a certain level; (2) being relevant to significant mathematical contents, procedures and processes beyond this level, and being a rich source of mathematical activities; (3) being flexible and can be adapted to the special conditions of a classroom; and (4) integrating mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research (Wittmann, 2001).

2. Activity Overview

The activities of the workshop were implemented as follows:

Activity	Working format/Responsible person
Introduction and overview of the WS	All participants/T. Mizoguchi gave an overview of the workshop.
Introducing the common task	All participants/H. Hamanaka introduced the mathematical background of the common task: regarding Tartaglia’s triangle.
Short keynotes: Theoretical tools and the teaching-designs	All participants/I. Florensa, T. Mizoguchi, and K. Otaki provided short keynotes from three different approaches as follows. SK1: “The theory of substantial learning environments” by Koji Otaki; SK2: “Study and Research Paths: the ATD proposal” by I. Florensa; SK3: “Problem Solving Lesson Format in Japan” by T. Mizoguchi.
Discussing along with the key questions	All participants/I. Florensa, T. Mizoguchi, K. Otaki, and H. Hamanaka
Summarizing: Reflections and further considerations	All participants/I. Florensa, T. Mizoguchi, K. Otaki, and H. Hamanaka

3. Future Directions and Suggestions

This workshop was organised with the aim of networking the design of didactic practice, in line with recent efforts on networking theories (Bikner-Ahsbahs and Prediger, 2014). Of course, this initiative was still in its infancy and there were many challenges ahead. However, through the experience of this workshop, we have been able to clarify what each approach has in common, and conversely, the differences in what each approach intends to contribute.

So, such workshop is very significant for the scientific design of didactic practice. The outcomes of the workshop will be revealed on another paper.

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https://doi.org/10.1142/9789811287152_0109

Workshop 10

Challenging Ableist Perspectives on the Teaching of Mathematics: A CAPTeaM Workshop

Elena Nardi[1], Irene Biza[2], Solange Hassan Ahmad Ali Fernandez[3], Lulu Healy[4], Érika Silos[5], and Angeliki Stylianidou[6]

ABSTRACT This short paper outlines the main aims and objectives of the CAPTeaM project and the activities that took place during the CAPTeaM workshop at ICME14 on Wednesday 14 July 2021.

Keywords: CAPTeaM; MathTASK; Ableism; Disability; Inclusion; Mathematics.

1. The CAPTeaM Project

The MathTASK and CAPTeaM projects see engaging school and university teachers with challenges they are likely to face in class as an effective professional development approach. We design situation-specific tasks that emulate these challenges (such as: fostering mathematical reasoning; strengthening classroom management; enriching use of digital resources; and, improving the inclusion of often marginalised groups of learners) and we engage teachers with these tasks in reflective workshop settings. In this workshop, we focused on the last of the aforementioned challenges, inclusion. This is the focus of the CAPTeaM project (Nardi et al., 2018), an international partnership and mobility project between institutions in the UK and Brazil and funded by the British Academy (2014–15, 2016–21).

The CAPTeaM project (**C**hallenging **A**bleist **P**erspectives on the **Tea**ching of **M**athematics) sets out from the assumption that, rather than being the consequence of internal, individual factors, disabled students' oft-reported underperformance in mathematics can result from explicit or implicit exclusion from mathematics learning. The project challenges teaching practices that contribute to such exclusion and that may emanate from ableist[7] perspectives on mathematics. The project's aims cohere

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[7] Ableism: "a network of beliefs, processes and practices that produces a particular kind of self and body (the corporeal standard) that is projected as the perfect, species-typical and therefore essential and fully human. Disability then, is cast as a diminished state of being human." (Campbell, 2001, p.44)

with the articles of the United Nations *Convention on the Rights of People with Disabilities* (2006) that both the UK and Brazil have signed up to and the project aims to contribute to a hitherto under-researched, yet growing and highly topical, area of research (Healy and Powell, 2013). CAPTeaM endorses a Vygotskian historical-cultural perspective and elements of embodied cognition (Nardi et al., 2018) and its data consists of written responses and video recorded work on two types of tasks, Type I and Type II, by pre- and in-service teachers of mathematics.

In Type I tasks, participants engage with classroom episodes that evidence mathematical contributions which are made by students with a physical disability (e.g., are visually or hearing impaired), have the potential to shift classroom mathematical discourse towards creatively unexpected turns and may bring learning benefits to all in class. Said episodes are selected from the databases of the Brazil-based (Rumo à Educação Matemática Inclusiva) and UK-based (e.g., Stylianidou and Nardi, 2019) project partners. In Type II tasks, participants engage in small groups with solving a mathematical problem while at least one of them is temporarily and artificially deprived of access to a sensory field or familiar channel of communication. Work on both types of tasks concludes with sharing reflections on the experience and with a brief exposition on the project's hitherto data analysis, findings and plans for the future.

2. The CAPTeaM Workshop at ICME-14

Workshop participants engaged with two tasks, one of Type I and one of Type II. The session lasted 90 minutes and was structured as follows.

Nardi introduced CAPTeaM's aims, objectives, theoretical framework and research design. She outlined the two types of tasks that participants were invited to engage with during the workshop and introduced the first, *André and the pyramid.* In it, participants were asked to consider a mathematical contribution made by André, a blind student: André's description of a square-based pyramid, as an object that can be built from gradually diminishing squares, evokes Cavalieri's (Nardi et al., 2018) description but also diverges from the faces/edges/vertices definition proliferating in textbooks. Participants explored the mathematical affordances of André's proposition and considered the enriching role that such a proposition may play in lessons. They pondered on what constraints and support teachers have for orchestrating the inclusion of disabled learners in mathematics lessons and shared experiences from the very diverse educational contexts each was located in.

Brief findings from data analyses driven by the five themes of *Value and Attuning*, *Classroom Management and Benefit*, *Experience and Confidence*, *Institutional Possibilities and Constraints* and *Resignification* were shared with the participants, before proceeding with engaging with a Type II task. In normal circumstances, this type of task involves the use of several sensory channels, including touch (Nardi et al., 2018). Doing so however is not possible during an online workshop and participants were asked to engage with a COVID-19 pandemic secure adaptation of a Type II task.

This involved working in groups of three (A, B, C), where one participant was A (observer), B could not see, and C could not speak. We communicated a mathematical problem $(354 - 86 = ?)$ to C in a private channel of the chat function on the conference platform. Nardi and Biza collated accounts from the chat and coordinated the discussion of the experience across the entire group. During this, participants shared their experiences of working with limited access to sensory channels. Their coping strategies were then compared with those produced by CAPTeaM participants and the workshop concluded with Nardi outlining project findings so far and mapping current and future CAPTeaM actvities. The workshop concluded with participants asking questions and reflecting on — as well as evaluating — the experience of participating in the workshop.

Throughout the workshop, participants noted substantial differences on inclusion policy and implementation around the globe. Exchanges focussed on how deconstructing the notion of the normal mathematics student/classroom and attuning mathematics teaching strategies to student diversity takes different meanings in different institutional contexts. For example, Nardi and Biza shared the example of a doctoral study in the UK, led by team member Stylianidou, that focuses on exploring and engineering mutual benefits for the mathematical learning of sighted and visually impaired pupils in inclusive elementary mathematics classrooms in the UK, where inclusion is thoroughly legislated but its actualisation supportive mechanisms for teachers is in relative infancy. Other issues, such as mathematical and pedagogical support for specialist teaching assistance staff (e.g., Sign Language interpreters for hearing-impaired/deaf learners) were also raised.

Acknowledgments

CAPTeaM was funded by two British Academy International Partnership and Mobility Grants (2014–15: PM140102; 2016–21: PM160190)

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https://doi.org/10.1142/9789811287152_0110

Workshop 11

Simulation Games for Geometry Learning and the Development of Mathematical Language

Angela Piu[1] and Cesare Fregola[2]

ABSTRACT This report describes a workshop on the structure, teaching/learning process, and functions of collaborative simulation games developed to enhance geometry teaching/learning in primary schools. The online participatory workshop entailed observing an ad hoc video about *Cartolandia*, a simulation game on isometries. The video illustrated the structural and dynamic aspects of simulation games based on a dynamic design model developed within a broader research program. The ensuing discussion with the participants covered: the processes of play, teaching-learning, and mathematization suggested by observing the video and engaging with the simulation game; transferring the game to other school settings.

Keywords: Simulation games; Teaching geometry; Research at primary school.

1. Theme and Description

The workshop was focused on simulation games for primary-level geometry learning and the underlying design model developed within the framework of *Simulandia*, a research and teacher professional development project (Piu and Fregola, 2011).

The research program began in 2011 with *Cartolandia*, a simulation game on isometries. Four other simulation games — on the concepts of isoperimetry, equal area, similarities, and angles, respectively — were subsequently experimentally validated (Piu, Fregola and Barbieri, 2016; Piu and Fregola, 2020).

The games are learning environments organized around a model that represents mathematizable aspects of the real world, including a target mathematical concept, via multiple perceptual stimuli. The participating schoolchildren: act out assigned roles and make decisions in light of the possibilities and constraints dictated by the rules of the game; actively handle structured materials, according to the procedural requirements and aims of the game and the associated learning objectives; interact, communicate, and cooperate with their peers, with the teacher mediating as necessary; and reflect on the meaning of their experience.

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The *Simulandia* games/learning environments have been developed to help primary school students learn key geometry concepts while having fun and acquiring the knowledge/competences required to construct mathematical language.

The theoretical-methodological framework underpinning the design model is informed by: the literature on simulation games; theories and methods from educational science, including the principles of learning by discovery and cognitive load theory (Sweller, 2003; Piu and Fregola, 2020); a semiotic perspective on math learning that emphasizes helping children to attribute meaning to the concepts they are constructing and to the language used to express and share these concepts (Duval, 2017). This implies attending to the patterns of transcoding that underpin the gradual construction of formal mathematical language starting from learners' current everyday language, via an abstraction process that leads to the discovery or definition of geometry concepts (Piu et al., 2016; Vergnaud, 1994).

Research and experimentation with primary pupils and teachers showed that playing *Cartolandia* significantly reinforced children's recall and mastery of concepts: participants in simulation games displayed more robust geometry understanding than peers who took a traditional geometry lesson (Piu et al., 2016).

Based on these results, we developed simulation games on the concepts of similarity, angles, isoperimetry, and equal area, establishing that the structure of the teaching-learning process remained invariant across different games and geometry contents (Piu and Fregola, 2020). All the games share a set of characteristics: rootedness in the literature on simulation games; application of transcoding to the construction of mathematical and formal language suited to the target concepts; the interactional dynamics of children cooperating to solve a complex problem in a simulated real-life scenario; a set game structure and teaching-learning process that facilitate progressive abstraction, representation via multiple semiotic registers (Duval, 2017), and a progressive shift away from the pragmatic language used to describe situations and communicate decisions, discoveries, and outcomes, across the phases of sharing the objectives of the game, initial briefing, play, and debriefing (Piu and Fregola, 2020).

2. Activity Overview

The workshop participants viewed an ad hoc video on the structure and dynamics of *Cartolandia* (Fig. 1, on the next page), a simulation game on isometries for primary school children and teachers.

Participants were briefly introduced to the theoretical-methodological framework underpinning the game's design. They next observed the three-part video, covering: 1) the *Cartolandia* setting (this was to immerse the participants in the experience of the game, and explain its aims and stages; 2) the key interactions among the children during the simulation game; and 3) the debriefing following the game, during which mathematical language is constructed based on the practical experience of playing the game. The final section was narrated by Eledia Mangia, the teacher/researcher who led the game.

Cartolandia is set in a paper city, whose inhabitants learn that a map of the town has been stolen from the city museum. Guided by their teacher, the participants play the role of carto-detectives tasked with retrieving the map, identifying the thief, and reporting to the carto-general on the methods they used to carry out their inquiry.

The children work in a specially designed carto-lab to analyze the carto-carpet — a long sheet of paper with the footprints of all the visitors to the museum — and, later, the visitors' photographs (silhouettes).

In accordance with strict rules, they lay transparent plastic sheets with footprints or silhouettes drawn on them, either over the footprints on the carpet or over the outline shapes of the visitors, to check for matches.

The carto-detectives are invited to discuss the different methods they are allowed to use during the game, and to record their actions on a sheet, so that they can report to the carto-general later. Thus, they are encouraged to invent symbols for the operations they performed with the transparent sheet to match footprints or shapes. They jointly decide how to describe their actions and how they identified the thief, choosing which "code" to adopt in their final report to the carto-general at the end of the game.

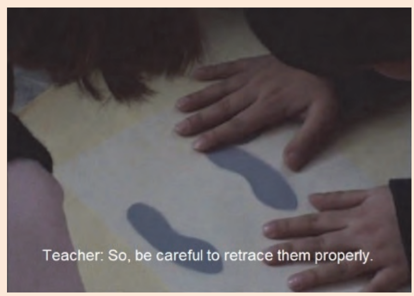

Fig. 1. Cartolandia

The workshop discussion covered the structure/characteristics of the simulation game as well as the teaching/learning processes it elicits. The participants' observations allowed us to explore the three interdependent processes that characterize the design model (and their place within the systemic whole of the simulation game):

- the game process/architecture, which leads to the discovery of concepts via complex problem-solving activities within a simulated scenario with its specific constraints, roles, instructions, and end goal;
- the mathematization process, whereby the children solve the problem via transcoding and the deployment of different semiotic registers, progressively acquiring mathematical language by abstracting, representing, and transforming informal language based on their actions and the rules of the game;
- the teaching-learning process, comprising: opening — eliciting interest/ defining the learning agreement; briefing — presenting the aims/rules of the game; the game itself — experientially and collaboratively discovering

concepts and sharing them with peers via interaction and dialogue; debriefing — systematizing new concepts, facilitating mindful understanding of them, and formalizing them by expressing them symbolically.

Finally, we discussed the feasibility of replicating the simulation game teaching-learning experience in participants' own educational settings, considering differences in culture, teacher training, and methodological approaches to math teaching.

3. Future Directions and Suggestions

Our research program has yielded interesting outcomes, namely: — validation of a dynamic game design model; — identification of the structural constants in the teaching-learning process and its interdependence with a game architecture based on complex problem-solving and the mathematization of reality; — evidence of enhanced understanding and recall of concepts and learning contents. These outcomes are currently being consolidated via experimentation with new games, and extended via investigation of the relational skills of children and teachers from a transactional analysis perspective. In time, this may inform systematic teacher training trajectories and foster the wider deployment of simulation games in schools.

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https://doi.org/10.1142/9789811287152_0111

Workshop 12

Developing Quality Criteria for Creating and Choosing Mathematics Learning Videos

Iresha Ratnayake[1], Eugenia Taranto[2], Regina Bruder[3], and Maria Flavia Mammana[4]

ABSTRACT In this report we present our experience in conducting a workshop on using our catalogues for choosing and creating learning videos for mathematics. On the one hand, the participants had an opportunity to use our catalogues to evaluate videos using two different approaches — by being both users and creators. On the other hand, we had an opportunity to obtain feedback from them based on their own experience of working with pre-service teachers in creating learning videos.

Keywords: Mathematical learning videos; Quality criteria; Video catalogue.

1. Theme and Description

In this workshop, we shared the results of a joint project — *Quality of Learning Videos — Mathematics (QLeV)* — between two universities in Germany and Italy. The project aimed to develop quality criteria to create or choose mathematical learning videos. There are many mathematical learning videos freely available on the internet with many videos being uploaded on various platforms. There are, however, many important factors that need to be considered in creating and choosing a learning video. In our project, we suggested some crucial quality criteria to accomplish the intention expressed above. We started with a catalogue developed under the CAKE project (Feldt-Caesar and Bruder, 2018). This catalogue was a general one including quality criteria for digital learning environments. Following this, the current project was designed to develop quality criteria for learning videos from a mathematical perspective (Ratnayake et al., 2020). The result of our collaboration generated two catalogues: (i) quality criteria for creators and (ii) quality criteria for users (teachers). Our catalogues attended to learning situations, expected prior knowledge, accuracy of the content, learner's expectations, pedagogical consideration and design and technical consideration. During this workshop, we shared the two developed catalogues to stimulate discussion with colleagues in the mathematics education research community about ways in which they might be refined and extended. The refinement of the catalogues can in turn contribute to building a shared understanding of the creation and use of high-quality mathematics learning videos.

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2. Activity Overview

We showed a video (chosen from YouTube) explaining some mathematical concepts. After watching the video, a discussion on following questions was conducted.

- What do you think about the video?
- Would you use this in your lessons?
- If so for what purpose?

The participants were thereafter guided to watch the video again and evaluate the video using our catalogue *Quality Criteria for Teachers as Users.* The participants evaluated the video using the catalogue in the internet portal and saved their evaluations. After the evaluation a discussion was conducted on the following questions:

- What can you say about your judgement?
- Did you change your judgement after you evaluated the video using the catalogue? If so why?
- Do you think the catalogue helped you to make a decision?

Subsequently, we discussed the criteria of the catalogue followed by a discussion on the following questions:

1. Are the criteria in the catalogue helpful in evaluating mathematics learning videos?
2. Is the catalogue helpful in choosing a good learning video that you can use in your lessons?
3. Are the criteria understandable?
4. Do you suggest changing any of the criteria? If so, please provide your suggestions.

In the second part of the workshop, we focussed on the creator's catalogue (https://wwwdid.mathematik.tu-darmstadt.de/videoevaluation/) that we propose using in creating a learning mathematics video. First, participants watched a video developed by a pair of pre-service teachers at the University of Catania as a fulfilment of their assessment of a mathematics course. These pre-service teachers followed our catalogue in developing this video. Participants of the workshop used our creator's catalogue to evaluate the video. We then conducted a discussion based on the following points:

1. Are the criteria in the catalogue helpful in evaluating mathematics learning videos?
2. Is the catalogue helpful in creating a good learning video that you can use in your lessons?
3. Are the criteria understandable?
4. Do you suggest changing any of the criteria? If so, please provide your suggestions.

3. Participants and Their Comments

There were ten online participants and some off-line participants from different countries. Most of the participants' pre-service teachers create videos during their undergraduate programmes. Thus, the audience paid more interest on the creator's catalogue and appreciated its features including the ability to use it as a guideline and as a self-and peer- reflection tool.

4. Future Directions

Currently our catalogue — creator's catalogue — is used for the activity called "5 minutes videos" which is a part of the project "Liceo Matematico" at the University of Catania. This is a national-level competition for secondary high school students. Thirteen schools joined the project. These students use our catalogue in creating videos in groups of 4–5 students. The evaluation of the videos will then also be done using the criteria in the same catalogue to choose winners of the competition.

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https://doi.org/10.1142/9789811287152_0112

Workshop 13

Beyond Financial Literacy and Financial Mathematics: Conceptualizing Financial Numeracy

Annie Savard[1] and Alexandre Cavalcante[2]

ABSTRACT This workshop engaged participants in an inquiry-based environment to conceptualize the role played by mathematics education regarding financial literacy, both in elementary and secondary schools. Our main goal was to introduce a conceptual framework that allows researchers and teachers to move beyond financial literacy and financial mathematics. The international efforts to incorporate financial literacy in schools have also penetrated the community of mathematics educators in several countries, however the role played by mathematics in this topic has been undertheorized, leaving practitioners without proper support to integrate these concepts in mathematics classes

Keywords: Financial Numeracy; Teaching Mathematics; Money.

1. Brief Introduction to Financial Literacy and Mathematics

The workshop started by presenting to participants a short definition of financial literacy. After that, we asked them to answer this question on Menti: *What is the relationship between Financial Literacy and Mathematics?* We discussed their answers before present to them the concept of financial numeracy.

1.1. Financial literacy

Financial Literacy (FL) has been defined over time as a portfolio of financial knowledge and the behavioral use of it (Huston, 2010; Johnson and Sherraden, 2007). Since 2012, The Organization for Economic Cooperation and Development (OECD) tested 15-year-old students regarding FL. Their international assessment program PISA (OECD Program for International Student Assessment 2016) says:

> *Financial literacy is knowledge and understanding of financial concepts and risks, and the skills, motivation and confidence to apply such knowledge and understanding in order to make effective decisions across a range of financial*

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> *contexts, to improve the financial well-being of individuals and society, and to enable participation in economic life*. (p. 85).

Four areas of knowledge and understanding are defined: money and transaction, planning and managing finances, risk and reward, and financial landscape.

1.2. Financial literacy into mathematics

Over the last decade, financial literacy has been introduced to many K-12 school curricula around the world (Lusardi and Mitchell, 2014; Suiter and Meszaros, 2005). At the same time, research indicates that mathematical competencies play a major role in developing financial literacy (OECD, 2017). Such competencies promote the modelling of financial situations, the operationalization of financial concepts, and the development of financial mathematics. As a result, financial literacy is now embedded in some mathematic curricula (Savard et al., 2020), such as Australia, some Canadian provinces such as Ontario and British Columbia, and in Romania.

2. Financial Numeracy

Numeracy is more about how people do mathematics in their daily life rather than what do they know about mathematics. Therefore, numeracy is a social practice (Yasukawa et al., 2018). It is a culturally, historically, and politically situated practice. It conveys the values and visions of the world of social groups. Mathematics can be explicit or implicit. In that sense, numeracy comprises more than simple arithmetic operations (Goos et al., 2019). The concept of financial numeracy refers to the intersection of mathematics and everyday practices in the realm of finance (Camiot and Jeanotte, 2016). We believe that financial numeracy involves the wider scope of mathematical concepts, tools and procedures instead of referring to financial mathematics (which is a specific subfield of applied mathematics).

3. Overview of the Activities Proposed

During the workshop, participants were asked to work in team in breakout rooms for analyzing and categorizing a task coming from a mathematics textbook, in grade 7. We use Padlet for sharing their answers.

Look at each question in the situation.

Categorize them in a way that makes sense to you.

As you categorize them, think about the relationships between financial literacy and mathematics.

We had a whole group discussion about their analysis. At the end of their presentation, we presented our categories for analyzing tasks related to financial numeracy.

3.1. A conceptual framework to define financial numeracy

The categories for analyzing tasks are rooted in a framework developed through our collaborative work to conceptualize the role of mathematics in financial education. Through the activity and the discussions that followed, we were able to present a framework of financial numeracy education that encapsulates three dimensions: a) financial concepts as a means to develop motivation for and understanding of mathematics (contextual dimension); mathematics as a way to create models and understanding of financial concepts (conceptual dimension); and c) mathematical and financial concepts in relation to other epistemological systems such as ethics, politics, culture, values (systemic dimension). This framework was developed through research with secondary mathematics teachers from Quebec, Canada (Savard and Cavalcante, 2021). It encompasses the diversity in perspectives in the literature of mathematics education on financial concepts and situations (Cavalcante, 2020), and contributes to the field by systematizing the relationship between mathematics and financial education.

4. Concluding Remarks

This workshop provided a much-needed opportunity for researchers and practitioners in mathematics education to discuss the future of our classrooms now that mathematics teachers are being asked to teach financial concepts. Curricula all around the world are starting to incorporate financial concepts in mathematics in a wide variety of ways, often leading to inconsistencies and mixed messaging about the connections within our discipline and with other disciplines. We have also found inconsistencies related to the grades in which these concepts are being taught and the profile of students being taught. However, the most important challenge currently lies on the preparation of our pre-service teachers to enter the profession with proper knowledge of financial concepts. This challenge was emphasized during our workshop discussions, which led to the creation of a listserv to keep mathematics educators in contact with one another. We hope to provide future opportunities for events to researchers and practitioners interested in this topic through this listserv. To subscribe, please send a message to:
math-financialeducation-l-request@listserv.utoronto.ca

Acknowledgments

We want to thank l'Autorité des Marchés du Québec for their support. Please note that the opinion reflects only the researchers.

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https://doi.org/10.1142/9789811287152_0113

Workshop 14

International Mathematics Festival: A Fun and Collaborative Event for Students to Discover "Why" and "What If"

Mark Saul[1] and Cherry Pu[2]

This workshop took the form of a wide-ranging discussion of the opportunities afforded mathematics education by a variety of informal and after-school activities. Participants from six countries and five continents contributed their views and experiences. In addition, students from each of these countries participated in a sample activity, which was then analyzed both by the students and their teachers.

What follows is a summary of the main points made by each contributor.

1. Unlocking Student Potential

Many speakers gave examples of how their activities unlocked students' potential in ways that formal classroom activities may not have. Tatiana Shubin (San Jose State University, USA) talked about her experience with math circles (https://mathcircles.org/home-2/), both in her home institution and on the Navajo Indian Reservation in Arizona. She showed video clips of students saying "I'm confused", but grinning in anticipation of learning something. Another student commented that he "didn't know he was smart" until he was given the opportunity to work non-routine problems in a social setting. Maria de Losada (Colombia, Universidad Antonio Nariño) agreed: "The uniqueness of each young person implies that there are myriad ways to unlock her or his potential for creative thought and engaging in mathematical activity. Solving every new and ingenious problem, preparing an exhibit for a math fair, developing strategies and arguments to prove a certain result, participating in situations that are at once fun and challenging, such as those encountered in a math festival, can all have their appeal to young people. with different preferences coming to light according to the richness of opportunities open to each one."

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2. Motivation for Students

At other times, speakers noted that doing mathematics in informal contexts continued to motivate students, even after their potential had been recognized. Often, they remarked that the aspect of mathematics as a joyful experience was something that students learned in and of itself. Hongliang Shi (No. 2 High School of East China Normal University) noted "If students love the subject matter, they take more initiative in the learning process. This is important for students' holistic development." Lynda Phillips (Ridge School, Kumasi, Ghana) commented: "It starts with games, then puzzles. These help the student focus and pay attention to everything they do. Then we can really relate these to the concepts. Gradually they will appreciate the subject, by generating interest and letting them know that math can be fun." Prodipta Hore (Birla Academy, Mumbai, India) noted: We conduct contest for students, and 60 to 70 percent of students sign up for it. Whether they get good scores or not — that's not the question. At least the participation is there. They love it so much that they start participating in it. He further remarked: "Mathematics is not only about numbers. When we teach it to our students, they automatically fall in love with it." Andrey Spivakov (Pelican Study, Russia) likewise commented: "It's simply fun, and keeps members motivated. Our competitions turn games into captivating and unpredictable entertainment." Fanglin Tian (No. 2 High School of East China Normal University) noted: "Math puzzles are great for intellectual development. They can increase students' interest in learning mathematics and broaden students' horizons as well. Therefore, the mathematics and games approach are widely recognized by us."

3. Articulation with Formal Instruction

A number of important points came up linking informal mathematics with more traditional classroom experiences. Hongliang Shi described an effort in his school to produce a classroom-oriented curriculum centered around learning through games, as part of their emphasis on learning based on doing research and developing talent. For the past seven years, he has been developing a mathematics course based on learning through games. The games were used as a gateway to more formal mathematics, expressed in the traditional axiom-and-theorem style. The classroom experience included competition problems and math festival activities, which overflowed 'back' into the informal space. Students developed their own clubs and informal structures to host festivals and administer contests. The enthusiasm was not confined to students: a cadre of five teachers has coalesced around the idea of using informal mathematics to motivate formal learning. A serious point about the relationship between formal and informal education was brought up by Lynda Phillips: "The COVID years have been a challenge. Teachers are overwhelmed, and find it hard to deliver the traditional curriculum. At such times, informal education can fall by the wayside. But even in normal times, parents are concerned about preparing their children for national exams, and may not see the connection between playing games and learning formal mathematics. We have to try to explain to them why we use different approaches."

4. Connections with More General Culture

Aside from linking informal education with the classroom, speakers also made numerous remarks about how informal education contributes to the development of culture more generally. Andrew noted: "High quality 'soft skills' are essential for completing game tasks. That is why our curriculum is geared towards not only learning how to solve math problems, but also developing social skills, such as meaningful discussion and teamwork."

Maria de Losada commented extensively on the effects of her work on more general culture: "We knew we had arrived when a central character in a popular sitcom on Colombian TV was described and singled out as having done well in the math Olympiads…. And yesterday I received an email from a Venezuelan student. Since we were running our summer school online, we opened it up to some Venezuelan students as well as Colombian students. Not only did he thank us for having been in the summer school, but he recounted that he had taken first place in the Venezuelan Math Olympiad for his age group, which was sixth and seventh grades. It's important to understand that things are not normal at all in Venezuela. And the fact that he was able to participate in a school with Colombian students and other Venezuelan students as well. He said it was absolutely one of the most important factors in his being able to do so well in the Olympiad.

5. Connecting Students and Teachers Worldwide

Hector Rosario (CYFEMAT International Network of Math Circles and Festivals) talked about how the network of informal educators keeps expanding. His work in Latin America, where he teamed up with Jeannette Shakeli in Panama, has helped to build networks of informal learning across Latin America. Using the Julia Robinson Mathematics League materials (www.jrmf.org), Rosario and Shakeli explored mathematics with virtual math festivals and circles. Translation of materials into Spanish catalyzed the formation of a community in Latin America that has begun to create its own activities. Rosario described how the use of virtual convening software changed the look of math festivals. Online apps can mimic the experience of playing with manipulatives, and can facilitate communication across vast distances.

The workshop included examples of this phenomenon, as we played clips of students from the six countries involved working together on solving two math puzzles (Wolves and Sheep and Wycoff's Nim). Virtual technology was effective in bringing these students in touch with each other, and their interactions often illustrated some of the points made by their teachers in the conversation. Informal mathematics can forge communities across boundaries, time zones, and oceans. As Tatiana Shubin noted, "Mathematics is the birthright of every human being, and mathematical talent is spread evenly across populations."

The COVID-19 pandemic has changed the world. We have struggled to live with it over the three years it has been with us. One silver lining of the struggle has been that we have found ways to come together without being face-to-face. Indeed, this workshop started out as a face-to-face International Mathematics Festival. Ironically, the COVID-19 pandemic emergency, which has imposed unexpected limits on so many of us, was the occasion for a broadening of the topic of this workshop.

Acknowledgments

Aside from the participants and their affiliated institutions, we would like to acknowledge the help of

- Grigoriy Kondakov, Moscow,
- The Julia Robinson Mathematics Festival, USA,

and

- The MISE Foundation and Joel Mawuenyega Dogoe, Accra.

https://doi.org/10.1142/9789811287152_0114

Workshop 15

Mathematics Learning and Mathematics Games

Hongliang Shi[1] and Fanglin Tian[2]

ABSTRACT Investigating the relationship between mathematics learning and games, we find that games and mathematics are very closely related. In our workshop, we give a brief review of the results to our research, and share our experiences about the series of activities. Then we hold a small Mathematics Festival.

Keywords: Mathematics Learning; Mathematics Games; Traditional Chinese Games.

1. Research on the Construction of Mathematics Curriculum from the Perspective of Literacy

1.1. *Improving mathematics literacy*

A new era calls for educational reform, improving education quality, and STEM education has become a hot topic. The goal of education has changed, with more emphasis on morality and comprehensive educations. In-depth study of subject moral education has been carried out, and the pedagogical approach of subject education have become clear.

The aim of our math education is to improve math literacy. It is the internal literacy that people can observe the world with the eyes of math. It is composed of math knowledge and skills, math thoughts and methods, and math abilities and concepts.

1.2. *Cultivating mathematical thinking*

The core of math literacy is thinking literacy. Dewey proposed a five-step method of reflective thinking and wrote in the book *How We Think*. The whole and only purpose of intellectual education is to cultivate thinking habits. Thinking is Power. It emphasizes multiple perspective thinking and integration of Chinese and Western thinking. Therefore, math modeling and math games have become the best carriers for thought training.

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The key to improving math literacy is to cultivate math thinking. The learning process of math is a process of acquiring math knowledge, methods, application and internalization, and at the same time a process of continuous cultivation and strengthening of the quality of math thinking. People with high math literacy often receive systematic math education, rich in math knowledge, and often show sensitivity and adaptation to numbers in life and work, and can separate math factors from complicated cases, modes, and perform.

1.3. Math games

Math games promote the growth of modeling awareness and higher order thinking. Learning is the process of continuously solving problems and creating meaning in continuous interaction between individual and the situation. Context is the gradual formation of disciplinary concepts, thinking models and inquiry skills, and the basis for continuous structuring of disciplinary knowledge and skills. Project-based activities should be adopted to improve math modeling literacy in modeling practice. The teaching of project-based learning courses such as “Math Modeling” is often experiential learning, cooperative learning, inquiry learning, and constructive learning, which promotes the transformation of students’ learning styles and evaluation methods, which is favorable to the realization of subject education.

2. Mathematics Learning and Mathematics Games

2.1. Math & games in class

Since the fall of 2014, we set an optional course named Mathematics and Games in senior high school. At School of Excellence in 2015, we invited teachers from Stanford to instruct students. Now there are more and more teachers and students involved in Mathematics and Games. Usually, the classroom mode is as follows: Rules and History → Play and Think (individually or in groups) → Modelling → Solution → Play again.

We did inquiries in the latest three terms. The students were from 10th grade students enrolled in Mathematics & Games. Over half of the students select this course for the reason they like playing games or they love math. They prefer to think while playing. Group discussion is also liked by nearly half of the students. One third students enjoy thinking independently. More than half of the students think modeling is difficult in game research. Acquiring playing skills and understanding the principles rank second and third respectively. Talking about harvest, training thinking ability is placed the first. Students thought they learned some new games, ways of solving problems, some abstract questions can be modeled to solve and they also admire math is wonderful. They harvest rich math knowledge and get more practice. Some students were interviewed. Compared with traditional math class, students think math game class is more fun, and they can experience the principles by playing games, it is more understandable.

2.2. Mathematics festivals in school

In the summer of 2017, we start to cooperate with Julia Robinson Mathematics Festival (JRMF), an educational organization from Stanford. We learn a lot from them, especially the activity named Mathematics Festival. Since 2017, we held our own festivals one or two times a year at our campus. In December 2019, COVID-19 broke out. Then we can't go to campus or play together. JRMF opens webinars about games, and we also joined in.

2.3. Student's club

In 2020, a student club on mathematics and games was built. A student from Grade 10 is in charge of it. Students in the club did a good job. They played many games together and did a lot of research on the games. They also share the games to the old people to help them keep sharp in mind. And they went to the community to share their fruit.

3. Mathematics Festival

Mathematics Festival is quite an important part in our workshop. There are 14 games in the meeting room, located at different tables. Each table has one or two table leaders. They guide the attendees to play games. Attendees are free to move to any game they would like to join in at any time. We had a great time during the Mathematics Festival time. People who joined us showed their passion in Mathematics Games teaching. The Chinese traditional games impress the audience very much.

4. Summary

Math changes thinking, math creates the world! The value pursuit of math education in our school includes: strong foundation-laying — a solid foundation of math knowledge for top innovative talents; empowerment — Cultivate students' core literacy in math and enhance their problem-solving ability; enhancing interest — enhancing the learning interest and academic interest of middle school students; quality improvement — comprehensively improve students' math literacy and enhance their thinking ability.

We will continue to develop and change in math and games. And we will do more research to connect math games and math modelling.

Acknowledgments

Thanks to the Julia Robinson Mathematics Festival for the support of technology.

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https://doi.org/10.1142/9789811287152_0115

Workshop 16

Self-Made Automata to Teach Mathematics in Preschool

Oliver Thiel[1] and Piedade Vaz-Rebelo[2]

ABSTRACT The workshop disseminated findings from the European research project AutoSTEM. The project aims to investigate how automata can enrich young children's play to promote a better understanding of Science, Technology, Engineering, and Mathematics (STEM). It aims to provide preschool teachers and other stakeholders of young children's education with tools and materials to build a didactic path, which is simple, replicable, and valuable in terms of (1) promotion of motivation for STEM, especially mathematics, (2) promotion of the development of creative thinking, problem-solving, and comprehension ability, and (3) cultural awareness and transversal values such as recycling.

Keywords: Preschool; Early childhood mathematics; Automata.

1. Theme and Description

Automata are fascinating mechanical toys. They are easy to create in the classroom, suitable for children's age, with simple to complex designs and motions. In the workshop, we presented the "snapping crocodile" developed by the project. We discussed how this mechanical toy could be used to teach mathematics in the early childhood classroom. The goal was to enable ECEC teachers to understand how to use AutoSTEM automata in their teaching practice.

2. Activity Overview

2.1. Presentation of the project's findings

The pedagogical approach developed in the scope of the AutoSTEM project is framed in a play-based pedagogy (Hedges and Cooper, 2018). The AutoSTEM project is distinguished by its potential to approach different disciplines. Besides STEM, it also promotes the development of transversal skills such as problem-solving, creativity and spontaneous cooperation (Thiel et al., 2020). One of the main features of this project is its ability to enhance children's motivation and engagement for STEM learning

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(Santos et al., 2020). Another feature is the ability to increase creativity and well-being in the participating children. A clear example of this is the diversity of the products produced by the children, both the automata and their narratives (Bidarra et al., 2020). Another transversal competence is the spontaneous cooperation emerging in the various AutoSTEM activities (Bidarra et al., 2021).

2.2. Making a 'snapping crocodile'

The activity started with the introduction of the materials. The tutor talked about STEM aspects related to the building process during construction, especially mathematics. The following video explains the mathematical content you can teach using the 'snapping crocodile': https://youtu.be/VPfi2g_t5kg. After the attendees finished their automata, they presented their work by holding the automaton up to their camera, so everyone could offer feedback and compliment them.

2.3. Pedagogical concepts and ideas

Finally, the participants tried out the crocodile. They used it to lift light items and reflected on the following questions:

- How is the crocodile related to the curriculum?
- What learning goals can you reach with the crocodile?
- Which learning activities can you carry out with the crocodile?
- How would you structure a lesson with the crocodile?

When constructing the crocodile, several learning goals can be achieved (cf. Thiel, Vaz Rebelo, et al., 2020):

- The children learn about physics and mechanisms, in particular, linkages.
- They develop engineering competencies of analysis and construction.
- They learn mathematical concepts within the construction and assembly process, including patterns, shapes and numbers.
- They learn biology concepts about the animal and its environment.
- Other soft-learning goals can be included, like problem solving and creativity.

Building the snapping crocodile with preschool children offers a playful approach to introducing STEM concepts, such as linkages, geometry, and mathematical patterns. The construction and assembly process allows children to explore shapes, sizes, and center points while discovering the relationship between aesthetics and functionality. Other mathematical topics are the motion of the part and the length and width of the scissor arm that change in relation to each other when the mechanism is in action.

3. Future Directions and Suggestions

The project AutoSTEM was finished in the summer of 2021. All materials that the project has produced are still available on the project's website https://autostem.uc.pt/.

On the website, you will find an online course that helps teachers to learn how to implement our findings in their early childhood classrooms.

Acknowledgements

The project AutoSTEM has been funded with support from the European Commission, project number 2018-1-PT01-KA201-047499. This document reflects the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

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https://doi.org/10.1142/9789811287152_0116

Workshop 17

Rich Math Activities for a Primary School Class

Albert Vilalta[1], Laura Morera[2], Horacio Solar[3], and Francisco Rojas[4]

1. Description

Innovamat is a math project that develops materials and activities to teach and learn maths at Primary School. Even though we are only three years old as a project, in 2019/20 more than 350 schools around Barcelona, Spain, are using our program.

According to the Catalan curriculum, which is similar to the NCTM Principles and Standards, we define 4 big math processes: "Problem Solving", "Reasoning and Proof", "Connections" and "Communication and Representation". Our activities always seek those children develop content and skills related to these processes, and this is why our teacher's guides focus on class conversation and materials manipulation. Furthermore, we are developing a self-adaptive app that allows every child to follow his or her own content-practice path. In addition to that, Innovamat is making big efforts to educate Primary School teachers in teaching rich maths from a process point of view, and we organize training sessions all around the country. Last September, more than 2500 Primary School teachers attended our Jornades Formatives (teacher training conferences).

Our didactic team is led by experts from the Universitat Autònoma de Barcelona (UAB). We propose rich activities that are designed in order to encourage teaching and learning math from a rich process point of view. The ideas beneath the project are based on research. Specifically, our main research sources are Morera's thesis (directed by J. M. Fortuny and N. Planas), the research from the Freudenthal Institute for Science and Mathematics Education, the Catalan official curriculum, USA Common Core and Mogens Niss' research. In addition to all that, PhD student Vilalta is working on a thesis to study how children take advantage of the learning opportunities generated in a class by the Innovamat project. Finally, Rojas and Solar, from the Pontificia Universidad Católica de Chile (UC), are going to contribute to enrich the activities and the way we communicate them to teachers, mainly thanks to their research experience in continuing teacher training.

At this workshop, we are going to introduce, perform and analyze three examples of activities from our project. Therefore, this workshop might be especially interesting

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for Primary School teachers and any person who wants to know about rich activities and discuss them. The first activity is going to be a geometry activity based on the geoboard. The second one is going to be a numbering activity of productive thinking. The third one is going to be an activity about finding and reasoning patterns. We are going to explain each activity, ask attendants to take part in solving every challenge like if they were students and concurrently we are going to discuss, as teachers, the math didactics and learning opportunities beneath all of it. In addition to that, we are going to project and analyze some short videos from real primary school children working on such activities. We want to focus on the learning opportunities provided by the activities and our classroom management and on how children take advantage of those opportunities (focusing on math processes and contents).

Key questions:

1. Do the proposed activities and class management provide rich learning opportunities from a math process point of view?
2. How do children take advantage of these learning opportunities?
3. How could we make the activities richer?
4. How could we improve the transmission of these ideas to Primary School teachers?

2. Planned Structure

Planned Timeline	Topic Material	Material/Working Format/Responsible Person
00–10 min	Definition of the framework: math processes in the Catalan curriculum.	Computer and projector/Exposition and group discussion/A. Vilalta
10–40 min	Activity 1: Geoboard	Geoboard Computer and projector, paper with geoboard templates/Group discussion/ H. Solar
40–70 min	Activity 2: Productive thinking	Computer and projector, paper/Group discussion/F. Rojas
70–90 min	Activity 3: Patterns	Computer and projector, multilink cubes, paper/Group discussion/A. Vilalta

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https://doi.org/10.1142/9789811287152_0117

Workshop 18

The Felix Klein Project — Vignettes in Practice

Hans-Georg Weigand[1], Michèle Artigue[2], Ferdinando Arzarello[3], Yuriko Baldin[4], Bill McCallum[5], Christian Mercat[6], and Samuel Bengmark[7]

1. Theme and Description

The Klein Project aims to present contemporary mathematics for secondary school teachers. The project wants to transfer the ideas of the legendary books of Felix Klein: "Elementary Mathematics from a Higher Standpoint", written at the beginning of the 20th century, into the present. A collection of Klein Vignettes is found on the website (http://blog.kleinproject.org) in different languages: English, French, German, Spanish, Italian, Portuguese, Chinese, Khmer and Arabian. A Klein Vignette is a short article about a single mathematical topic. Vignettes are intended to give teachers a sense of connectedness between the mathematics of the teachers' world and contemporary research and applications in the mathematical sciences. Modern mathematics could be shown in different ways. E.g., if digital technologies give new possibilities in presenting some "old" mathematics — geometry, algebra, calculus — this could be the basis of a Klein Vignette. Klein Vignettes are for teachers, but they should also motivate them to bring ideas presented in the vignettes to the classroom. In some years of experience, we noticed that the vignettes have to be supported by activities in the frame of professional development, and teachers had difficulties with the transfer of the Klein-ideas into the classroom. They had difficulties in creating adequate classroom materials.

2. Activity Overview

The workshop pursued three aims:

- We wanted to give best practice examples of how the idea of the vignettes could be integrated into the professional development of secondary school teachers;

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E-mail: samuel.bengmark@gu.se

- We wanted to motivate mathematicians to contribute to the Klein project with a new vignette;
- We wanted to motivate especially mathematics educators to think about *Bridging-Vignettes* which bridge the gap between the mathematics explained in a classical vignette and its use in the classroom.

The workshop was attended by 50 participants. It started with a 10 minutes introduction by H.-G. Weigand. Then, M. Artigue and C. Mercat (France) presented a vignette on Entrelacs, its story, and associated resources. They included some practical work of design of entrelacs from selected graphs. Y. Baldin (Brazil) presented material for professional development. She showed innovating didactical sequences while working with Klein Vignettes as teaching strategies in actual classrooms. F. Arzarello (Italy) showed the way from a Klein Vignette to concrete material for the classroom. He reported how he worked with Italian teachers in the frame of a "secret message game". Finally, B. McCallum (USA) gave an example of how a Klein vignette could be adapted into materials for a workshop for teachers on problem-based instruction.

3. Future Directions and Suggestions

There is an ongoing collaboration with other projects like the Snapshots Project (https://mns.co.il) in Israel, the French project "Image des maths" (http://images.math.cnrs.fr), or the project "Mathematical Licei" in Italy. To bring the project ahead, to implement planned projects and to launch new projects, we are looking for cooperation with other projects, we would like to integrate new people into the project team and to motivate others to contribute to the project, writing or translating new vignettes. One main emphasis will be the presentation of contemporary mathematics in classroom situations (Bridging Vignettes), and we want to give examples for this transfer.

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Workshop 19

Exploratory Lessons Using Pop-Up Cards and the Making of Cards

Kazumi Yamada[1], Takaaki Kihara[2], and Anri Yamada[3]

ABSTRACT Static figures are used in the learning of the plane figures. In contrast, it is important not only to use a shape with spatial expanse, but also to present the dynamic movement of that shape when a teacher teaches a space figure. There are advantages in pop-up cards creation as teaching materials. While repeating the card making, various questions will arise and you will discover many properties. For example, they are as follows. "In the blueprint, where are the cut lines? Where are the folds? Are there any secrets to these lines?" The lesson to discover such properties exploratively will be described.

Keywords: Pop-up card; Dynamic movement; Space figure; Exploratory learning.

1. Making of Cards as the Teaching Materials of the Space Figure

1.1. Advantages in pop-up cards creation as teaching materials

We have been continuing workshops on ICME-11, ICME-12, and ICME-13 about teaching spatial figures using pop-up cards. Static figures are used in the learning of the plane figures. In contrast, it is important not only to use a shape with spatial expanse, but also to present the dynamic movement of that shape when a teacher teaches a space figure. There are the following advantages in pop-up cards creation as teaching materials. When making a card, a 3dimensional card is completed by trial and error, making cuts in a plane (card) blueprint, and opening and closing a card repeatedly. In this making process, instruction which connected plane figures and space figures is attained. Especially, the pop-up card called "the origami architecture" is effective as the teaching material from this respect. When you open a card that is folded in two from 0° to 90°, the three-dimensional shaped object appears. When you fold this card, this card is returned to its original state. Work of origamic architecture may look at the home page of Masahiro Chatani. http://www.japandesign.ne.jp/IAA/chatani

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[3] Paper Architect, Student of Tama Art University

1.2. Introduction of origami architecture

1.2.1. How to make

You can make cards in steps from Step1. to Step 6. Step1. Draw the blueprint of this card with cut lines and fold lines. Step 2. Cut the cut lines with a cutter. Step 3. Crease the fold lines. Step 4. Fold complex places with tweezers. Step 5. Fold the card. Step 6. Open the card to 90°.

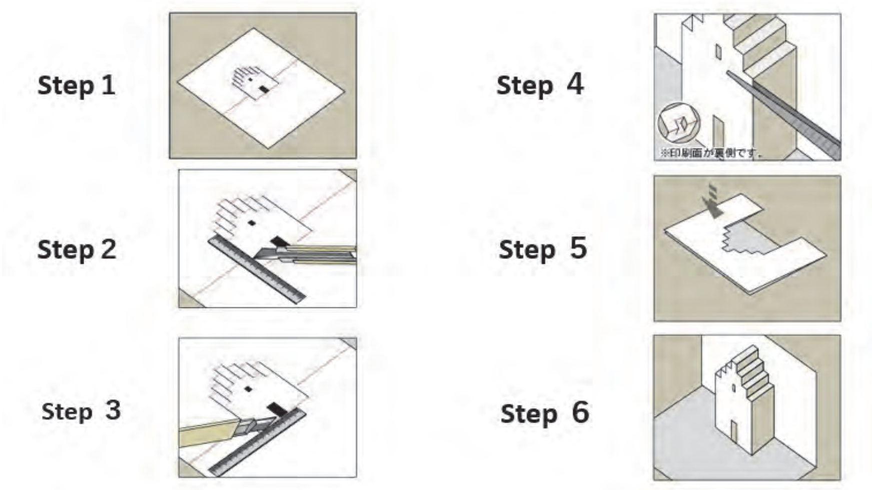

Fig. 1. How to Make

1.2.2. Kihara's collection of works

Mr. Kihara's works can be seen on the Internet.
https://www.facebook.com/media/set/?set=a.127437644000521&type=3.
You can see the cards of the foldable house on
https://www.youtube.com/watch?v=4Q8Nrjlxngg.

Fig. 2. Foldable House

Fig. 3. Foldable Stairs

Fig. 4. Foldable Dragon

2. Exploratory Lessons Using Pop-up Cards

When the card which opened in 180° is being raised to 90° gradually, the blueprint which is one kind of two-dimensional plane figures is changing into a three-dimensional solid gradually. While opening and shutting on a card is repeated, you come to be able to image the continuous movement of the solid and understand the mechanism of the card. The pop-up card is a teaching material effective to bring up the power of the mind operation that dynamically images the transformation of the solid, and to raise the space cognition.

2.1. Basic learning

Let us create the pop-up card as shown in Fig. 6. If you open this card, building blocks as shown in Fig. 5 will appear. For this purpose, you should draw the blueprint of this card on the squares paper. You will notice that you should start by placing building blocks on the squares paper and drawing lines around the bottom and side. As shown in Fig. 7, it is more effective to fill the upward surface of the building blocks.

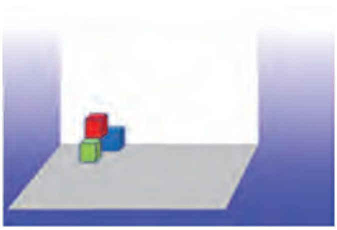

Fig. 5. Building blocks

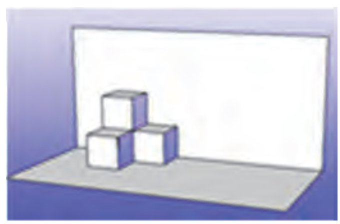

Fig. 6. Pop-up cards

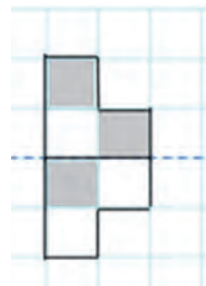

Fig. 7. Blueprint

Through repeated trial and error, you will discover that if you make the cards correctly, you can fold the cards tightly. This is a meaningful discovery that easily checks the correctness of the blueprint.

2.2. Exploratory learning

While repeating the card making by changing the number and position of building blocks, various questions will arise and you will discover many properties. For example, they are as follows. “In the blueprint, where are the cut lines? Where are the folds? Are there any secrets to these lines?” Let us explore the features of the cut lines written in the blueprint (Fig. 10) of the pop-up card (Fig. 8). In this workshop, we will demonstrate a lesson for exploratory discovery of these mathematical characteristics.

Problem Make a work of the origamic architecture such as Fig. 8, Fig. 9.

Think about how to complete the blueprint for this work and pursue your exploration.

Let us examine the regularity of the cut lines. To do this, a teacher should ask the following questions.

Fig. 8. Building blocks

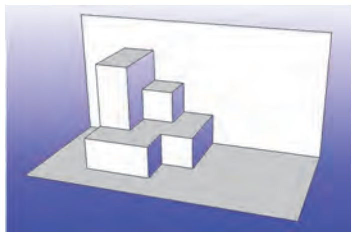

Fig. 9. Pop-up card

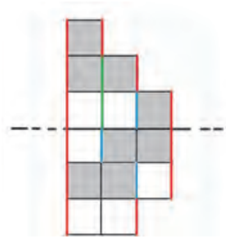

Fig. 10. Blueprint

Question 1: Where are the cut lines that you can easily find?
Answer: The vertical lines of the outer frame are cut lines. These are ① in Fig. 11. Because these parts are parts which be detached from the mount when the card is opened.

Question 2: Where are the adjacent faces that are horizontal and vertical to each other?
Answer: In the squares of the blueprint, it is easier to think that horizontal planes of the work are gray and vertical planes are white. In Fig. 12, when the color of adjacent squares is different, the boundary line is a cut line. Because the square of a white color is a vertical surface, and a gray surface is a horizontal plane, it is necessary to cut it off. These are ② in Fig. 12.

Question 3: Are these all the cut lines? Let us actually make it.
Answer: When the card is opened, if the heights of the adjacent horizontal planes are different, a cut line is required even if the adjacent faces in the blueprint are the same gray. Similarly, if the heights of the adjacent vertical planes are different, a cut line is required even if the adjacent faces in the blueprint are the same white. See Fig.13.

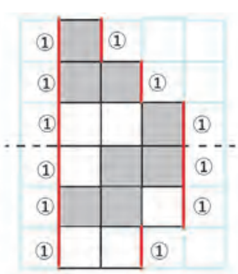

Fig. 11. Question 1

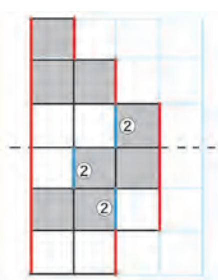

Fig. 12. Question 2

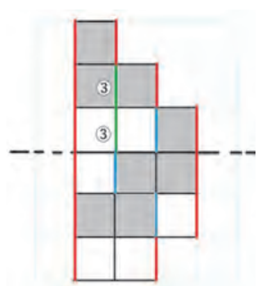

Fig. 13. Question 3

All of the cut lines described above are all cut lines written on the completed blueprint.

Development Problem Find all the blueprints for the card that can be made from the figure of only the outer frame of the pop-up card in Fig. 14.

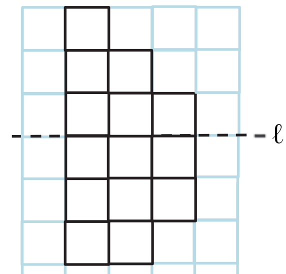

Fig. 14. Development problem

Answer: What is determined only by this figure? It is necessary to discover that the number of grids below the dotted line ℓ of the mount is the number of horizontal planes, and the number of grids above ℓ is the number of vertical planes. It is a surprising discovery that the number of horizontal and vertical planes of the card is determined only by this figure. The top grid of each column of blueprints is always a horizontal plane.

The bottom grid of each column of blueprints is always a vertical plane.

For other planes, the combination of horizontal and vertical planes can be freely determined. You can write all cut lines by using what you learned in "Problem".

3. Activity Overview

We made a presentation using Zoom. We gave the presentation shown in the table below. First, Mr. Yamada gave a lecture on the effectiveness of pop-up cards as teaching materials. Next, Mr. Kihara gave the demonstration of creating pop-up cards. Finally, Mr. Yamada and Miss Yamada made pop-up cards that show the shape of building blocks, and discussed the exploratory approach of various mathematical properties of this blueprint.

Tab. 1. Planned structure

Planned timeline	Topic	responsible person
20 minutes	1. Making of cards as the teaching materials of the space figure 1.1 Advantages in pop-up card creation as teaching materials	Lecture / Kazumi Yamada
30 minutes	Making of pop-up cards	Activity / Takaaki Kihara
30 minutes	2. Exploratory classes using pop-up cards	Activity / Kazumi Yamada, Anri Yamada
10minutes	Question and answer	

4. Future Directions and Suggestions

As mentioned in "2. Exploratory classes using pop-up cards", many mathematical properties are hidden in this type of pop-up cards. The study of exploratory lessons of these properties is profound.

https://doi.org/10.1142/9789811287152_0119

Workshop 20

Mathematical Performance-Based Learning in Hangzhou Yungu School

Jing Yang[1], Fan Zou[2], and Shengwenxin Ni[3]

ABSTRACT Performance-Based Learning is an advanced teaching approach that emphasizes what students can do as a result of instruction. In other words, teachers cultivate and assess students' competencies by requiring them to solve a problem or create something in real-life or simulated scenarios using their mathematical knowledge. Hangzhou Yungu School's math teachers have designed three types of performance tasks: 1) daily-class performance tasks, which are small assignments used in one class, 2) unit performance tasks, which are used during a whole unit of instruction, and 3) multiple-unit performance tasks, which are long-term tasks that last among several related units. Performance tasks focus on competencies acquired in the learning process, assess how well students learn, and guide students to what they can do. Completing performance tasks in mathematics is an excellent deep learning process that meets the expectations of mathematical education. This article presents some cases of the above three performance tasks and provides several suggestions for future research directions.

Keywords: Performance-Based Learning; Mathematics Education; Competency Based Education; Performance Task; Learning process.

1. Theme and Description

1.1. What is PBL?

Performance-Based Learning (PBL) is an approach that teachers let students solve a problem or create something by using their mathematical knowledge in real-life or simulated scenarios. In contrast to traditional paper-and-pencil tests that focus on content knowledge and skills, PBL emphasizes real-world problem-solving, communication, critical thinking, and other transferable competencies. Performance tasks are the most important component of PBL, as they require students to use high-level thinking skills to create or produce something with real-world applications. Research has shown that performance-based assessments provide useful information about student performance to a wide range of stakeholders, including students, parents, teachers, principals, and policymakers (Vogler, 2002). Additionally, performance tasks have been found to propel education systems in a direction that corresponds with how individuals actually learn (Herman, 1992).

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1.2. Why PBL?

As educators, we seek to cultivate "WHOLE PERSON" with practical abilities and innovative spirits to solve real-life problems. However, traditional paper-and-pencil tests are not sufficient to assess students' learning comprehensively. Effective assessment should not only evaluate learning outcomes but also guide students to "what they can do." PBL offers an integrated approach to teaching, learning, and assessment that is aligned with these goals. For instance, our daily-class performance task "Length Measuring by Eyes and Thumb" exemplifies how PBL emphasizes performance-based assessments that evaluate students' ability to apply content knowledge to critical-thinking, problem-solving, and analytical tasks. The task was designed for 9th graders to estimate the length between two sites using their knowledge of similar triangles, while establishing suitable models to specify the principle (Fig. 1). Throughout the task, teachers provided only hints, with rubrics serving as a guide for students. The task enabled students to learn through hands-on experience and collaboration with their peers, promoting active learning and knowledge application to real-world situations.

Fig. 1. Student's output work of this task

Business and education leaders increasingly recognize the importance of comprehensive assessment in students' learning. For example, Fadel, Honey, and Pasnik (2007) suggest that the workplace of the 21st century will require "new ways to get work done, solve problems, or create new knowledge" (p. 1), and that performance-based assessments will be essential to evaluating these skills. Likewise, higher education faculty value "habits of mind" even more than content knowledge, including the ability to think critically and analytically, to independently draw inferences and reach conclusions, and to solve problems (Conley, 2005). Well-designed performance-based tasks have the potential to measure these cognitive abilities more directly than standardized tests of content knowledge.

1.3. How do we perform PBL?

1.3.1. Macro-plan

To design PBL, we start with a macro plan. First, we examine the curriculum standards set by the Chinese Ministry of Education, which are shared by all Chinese secondary schools. Accordingly, we create a preliminary performance task framework for the three-year middle school teaching program (Fig. 2).

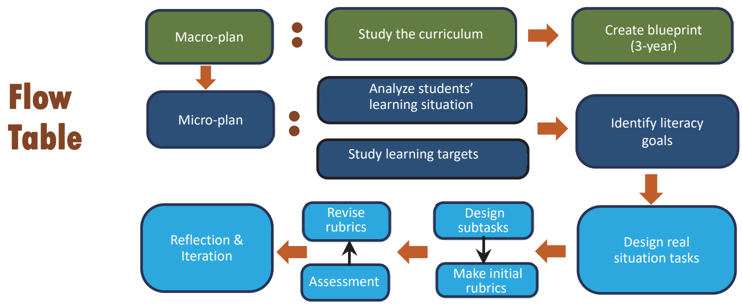

Fig. 2. Framework map

1.3.2. Micro-plan

Next, we design detailed performance tasks for daily lessons, units, or large units according to the framework diagram. We analyze students' current learning progress and goals, identify the competence goals that need to be achieved, and then design practical tasks to match the situation. We develop precise subtasks and make a preliminary rubric. We reflect on and iterate the design after students turn in their assignments. For instance, in the first chapter of the seventh grade of "Zhejiang Education Edition," the curriculum standard requires understanding positive and negative numbers. We designed the performance task of "Stock Market Simulation" accordingly. In Chapter 2, Volume 1, Grade 7, the curriculum standard requires mastering the operation of rational numbers. So, we designed the task of "Shopping Carnival." We help students gain core competency growth by designing specific tasks for corresponding competency goals.

Unit performance tasks play a crucial role in implementing micro-plans. For example, in the Inverse Function unit performance tasks inspired by "Piazza San Marco miracle", students are encouraged to apply their mathematical knowledge to real-world situations. Mathematical problem-solving requires students to apply knowledge, skills and strategies within novel contexts (Lynn et al., 1999). The unit begins with an inspiration video introducing the phenomenon of walking in circles blindfolded towards the tower in the Piazza, which sets the stage for the performance tasks. Students then conduct math experiments in groups, following the teacher's instructions, and participate in individual work, teamwork, and group discussions to achieve a deeper understanding of inverse functions. Throughout the process, the key question "How to explain the phenomenon of walking in circles blindfolded by using mathematical function?" drives students' learning. As it has been revealed, the performance assessment is effective for the teaching concepts and removes misconceptions (Slater,1996). In the second performance task of this unit, students conduct experiments, make conjectures, and summarize their findings to understand the concept of inverse function. They are expected to realize that the product of the radius and the difference in step length does not change, and that it is twice the product of the width between feet and average step length. To promote student engagement, the teacher encourages teamwork and helps students to create an hour-to-hour learning plan. Through this process, micro-plans are implemented step by step, and students are able to develop a deeper understanding of inverse functions and their real-world applications.

1.3.3. Constructing rubrics

A performance task includes three most important sections: learning objectives, task descriptions, and learning rubrics. We construct rubrics using three different methods (Fig. 3): Top to Bottom, Bottom to Top, and using both methods above. Rubrics guide students' learning and enable us to assess students' performance accurately.

Fig. 3. Constructing rubrics

1.4. Conclusion

In conclusion, PBL makes mathematics more approachable to students. Mathematics is not just about numbers, calculations, or logical reasoning, but also an act of thinking that touches our hearts. PBL bridges the gap between the two. All meaningful knowledge is ultimately for action, and performance tasks help children apply meaningful knowledge to solve practical problems in the real world.

2. Reflection and Prospect

PBL has helped students achieve competency goals and consolidate relevant knowledge through specific tasks and real-life applications. Mathematics is not just about numbers, calculations, or logical reasoning, but also an act of thinking that touches our hearts. PBL bridges the gap between the two. Based on years of exploration and practice in Yungu School, valid points in task design have been identified.

Firstly, learning goals should be based on both curriculum standards and learning circumstances. Teachers should carefully study the curriculum standards and determine the goals that require long-term understanding and appreciation, which should be discovered by students through self-exploration. In the inverse function unit performance tasks, the goal was not only about simple problem solving but also the ability to perform mathematical abstraction and establish suitable models to explain the rationale behind experiments. The assessment target determined in this case was suitable for students in the class which are relatively weak in math skills while enthusiastic in math experiment and good at groupwork.

Secondly, tasks should be set up as practical as possible, reflecting the goal of "authenticity". Students should connect the competencies measured by the assessment directly to complex life situations to improve the degree to which acquired competencies transfer to life outside the academic context. The performance task of "measuring distance by eyes and thumb" was designed to help students master the survival skill, and they learned to solve practical problems using the property of similarity. During the process, students used critical thinking skills, worked collaboratively, communicated effectively, and acquired self-learning ability.

Thirdly, competence-oriented rubrics are essential in PBL. Both the teacher's and the students' versions should be designed to match the requirements from the curriculum standards, learning situation, and task situation, and both should be grading rubrics convenient for assessment and providing guidance, especially for students' self-assessment. The rubric design should help students get specific guidance, support them to acquire self-learning ability, and encourage them to perform self-assessment. The rubrics should also be adjusted based on varied learning situations in the face of different students.

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Part IX

Thematic Afternoon

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https://doi.org/10.1142/9789811287152_0120

Thematic Afternoon

Mathematics Education in China: Summary of Thematic Afternoon Activities

Yingkang Wu[1]

ABSTRACT This article summarizes the activities of Thematic Afternoon at ICME-14. There are 13 Thematic Afternoon activities specially designed to show traditions and characteristics of theories and practices of mathematics education in China. These activities are briefly described under one of the four main themes, which are mathematics curriculum standards, textbooks and examinations, mathematics learning and teaching, mathematics teacher learning and teaching research, and mathematics education for ethnic minorities.

1. Overview

The Thematic Afternoon (TA) activities at ICME-14 are specially designed to show traditions and characteristics of theories and practices of mathematics education in China. The TA activities were held in a hybrid mode from 2 p.m. to 4.30 p.m. on Thursday, July 15, 2021, with participants both online and onsite. In particular, more than 800 school mathematics teachers from Shanghai and the surrounding areas physically participated in these activities. There are overall 13 TA activities covering various aspects of school mathematics education in China as shown in Tab. 1 (on the next page). The 13 TA activities were organized by various societies, teaching research institutes, and universities and schools, having a strong practical orientation. In order to highlight features of mathematics education in China as reflected in the TA activities, the 13 TA activities are categorized into one of the four main themes as indicated in the last column in Tab. 1. Although some TAs such as TA-1 are actually cross two themes, they are assigned to the more relevant theme in order to keep their integrity. The TA activities under each theme will be reported briefly. Details of each TA activity can be found on the website of ICME-14 and in the references of this article if available.

2. Themes and Descriptions of the TA Activities

Mathematics teaching is regarded as a cultural activity (Stigler and Hiebert, 1998). In the Chinese culture, teaching and learning integrate with and enhance each other. To be more specific, a teacher' teaching and students' learning promotes each other, and a teacher's teaching and his/her own learning promote each other (Li and Daiqin, 2013).

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Tab. 1. Overview of the 13 TA activities at ICME-14

	Thematic Afternoon activities	Organizer(s)	Theme
TA-13	Forum on Standards of School Mathematics Curriculum in Chinese Mainland	Revision Group on Mathematics Curriculum Standards for General High Schools (2017, 2020) and for Full–time Obligatory Education (2022)	Mathematics curriculum standards, textbooks and examinations
TA-4	12–year Integrated Mathematics Textbook of BNUP: Promoting Well-rounded Student Development	Beijing Normal University Publishing Group	
TA-11	Chinese Mathematics Curriculum, Teaching and College Entrance Examination	Mathematical Education Committee of Chinese Mathematical Society	
TA-2	The Making of a 3-D Mathematic Adaptive Learning System	Shanghai Institute of AI in Education, East China Normal University	Mathematics learning and teaching
TA-3	Domesticating Practice of Primary Mathematics Education in China	Primary Mathematics Teaching Committee of Chinese Society of Education	
TA-8	Mathematics Experiment: A Transformation of Mathematics Learning in Chinese Primary and Middle Schools	Secondary Mathematics Teaching Committee of Chinese Society of Education, Institute of Educational Science of Jiangsu Province	
TA-10	How is the Nature of 'Teaching and Learning Mathematics' Changed During the Pandemic in Shanghai?	Shanghai High School	
TA-12	Mathematical Modeling Inside and Outside Classrooms	NeoUnion ESC Organization	
TA-7	The Chinese Characteristics of Normal Students Training on Primary School Mathematics	Primary School Mathematics Teacher Preparation Working Committee	Mathematics teacher learning and teaching research
TA-1	Demonstration and Discussion of a Plane Geometry Lesson: Teaching "Determination and Properties of Parallel Lines" as a Whole	Secondary Mathematics Teaching Committee of Chinese Society of Education	
TA-5	From "Telling" to "Showing": A Zhejiang Mathematics Professional Development Model for Novice Teachers' Learning from Master Teachers	Teaching & Research Institute of Zhejiang Education Department	
TA-9	The Practice of Teaching Improvement from "Comprehending" to "Exploring"	Qingpu Experiment Research Institute	
TA-6	Reform and Development of Mathematics Curriculum and Teaching for Ethnic Minorities in China	Southwest University	Mathematics education for ethnic minorities

Hence, two of the four themes are "mathematics learning and teaching" and "mathematics teacher learning and teaching research", which correspond to the classroom setting and teacher education setting. Moreover, another theme "mathematics curriculum standards, textbooks and examinations" provides the context for "mathematics learning and teaching" and "mathematics teacher learning and teaching research". The last theme "Mathematics education for ethnic minorities" involves ethnic minorities in China, and therefore is taken as independent.

2.1. Mathematics curriculum standards, textbooks and examinations: Mathematics key competencies oriented

China began its new round of school mathematics curriculum reform in the end of the 20th century. The first Mathematics Curriculum Standards for Full-time Obligatory Education was issued in 2001, and then it was revised almost every ten years. In 2022 the latest version of Mathematics Curriculum Standards for Full-time Obligatory Education was released. The Mathematics Curriculum Standards for General Senior School was first released in 2003, and the latest version was promulgated in 2020. The focal point of the latest versions of mathematics curriculum standards for both obligatory education and for general high school is mathematics key competencies, which is taken as the main line that runs through the objectives, content, teaching, and assessment regarding school mathematics curriculum. TA-13, TA-4 and TA-11 are under the theme of mathematics curriculum standards, textbooks and examinations.

TA-13 was organized by the revision group on Curriculum Standards for General High School (2020) and Mathematics Curriculum Standards for Full–time Obligatory Education (2022). The purpose of TA-13 was to introduce and discuss the latest versions of school mathematics curriculum standards. In fact, the latest version of Mathematics Curriculum Standards for Full-time Obligatory Education was still under discussion and revision when ICME-14 was held. TA-13 comprised four lectures. As the head of the Revision Group on Mathematics Curriculum Standards, Ningzhong Shi from Northeast Normal University talked about the history and developmental trend of school mathematics curriculum in China, with an emphasis on the conceptualization of mathematics key competencies, elective modules and mathematical modelling at high school stage as well as activities of integrated application and practice at obligatory stages. Xiaotian Sun from Minzu University of China explained the connotation of the three mathematics key competencies. The three mathematics key competencies are phrased as "be able to use mathematical insight to observe the real world", "be able to use mathematical thinking to think about the real world", and "be able to use mathematical language to express the real world". Sun further elaborated concrete manifestations of the three "be able to" at different stages from primary level, to secondary level, and to high school level. Shangzhi Wang from Capital Normal University reported an empirical study on assessment and examination of high school mathematics from the perspective of mathematics key competencies. This study developed a series of tests to measure mathematics key competencies from four dimensions, i.e., "contexts and problems", "knowledge and skill", "thinking and expression" and "communication and reflection". Lastly, Changping Wang from Fujian Normal University focused on the five series of elective modules proposed in the latest Curriculum Standards for General High School in order to meet the future development of different students. In particular, he introduced Advanced Placement courses in high school curriculum, including calculus, analytical geometry and linear algebra, and probability and statistics, which provides a feasible way to educate mathematically gifted students in China. Yiming Cao from Beijing Normal University was the chair of the whole session.

TA-4 was organized by Beijing Normal University Press (BNUP for short). BNUP edits and publishes mathematics textbooks from Year 1 to 12 according to the national mathematics curriculum standards. The mathematics textbooks are integrally designed for primary, secondary and high schools. More than 30 million students around the country use BNUP mathematics textbooks, which indicates the influence of the textbooks in Chinese mainland. During the session of TA-4, via various organizational mode including presentations and panel discussions, the overall design and features of the textbooks were introduced, some typical cases on topics such as function, mathematical reasoning, and mathematical modeling from different grade levels were demonstrated, and experiences of textbook editing and compiling as well as future developmental prospect were shared and discussed. Nearly thirty speakers, including the chief editors of the BNUP mathematics textbooks and mathematics school teachers who participated in writing the textbooks, contributed to this session. Information of the speakers and details of TA-4 could be found on the website of ICME-14.

TA-11 was organized by the mathematical education committee of Chinese Mathematical Society. The Chinese Mathematical Society was established in Shanghai in 1935, and the mathematical education committee is devoted to the development of mathematics education in schools and universities. TA-11 consisted of three lectures followed by Q and A. Yufeng Guo from Beijing Normal University and Zhongru Li from Southwest University gave the first lecture entitled profile of mathematics curriculum development in China. They reviewed the progress of mathematics curriculum reform in China, analyzed characteristics of the curriculum development, and taking mathematics textbooks published by People's Education Press as an example discussed mathematics textbook writing and editing. Chunwei Song from Peking University introduced and discussed mathematics paper of college entrance examination in China, the score of which decides the type of university enrolled and therefore is high-stake in nature. He talked about background information of the examination, the coverage and structure of mathematics paper, and analyzed features of the paper by showing sample problems. Fengwen Yang, a mathematics master teacher from Beijing No.4 High School, talked about the practice of mathematics education in China from the view of practitioner using his school as a case. He particularly mentioned the development process of the school-based mathematics textbooks in Beijing No.4 High School from 1920s to the present, highlighting the variation and development of the design principles embedded in.

2.2. Mathematics learning and teaching: Traditions and innovations

Chinese students have outperformed their international peers in large-scale mathematics tests including TIMSS and PISA. Chinese students tended to use symbolic strategies to solve problems and demonstrated abstract and algebraic thinking, and they performed better in solving conventional and process-constrained problems than non-conventional and process-open tasks such as modeling and exploratory tasks (Bao, 2013; Ding, Wu, Liu and Cai, 2022). Students' mathematics learning has been shaped and influenced by its context, in which classroom instruction plays an important role. Mathematics teaching in China has its traditional characteristics such as introducing new knowledge by reviewing the relevant prior knowledge and providing instant feedback based on students' performance in small quiz and assignments (Xu, Kong, Yu, and Su, 2013), emphasizing varied problem solving activities including "one problem multiple solutions", "one problem multiple changes" and "multiple problems one solution" (Cai and Nie, 2007), and teaching mathematics with variations (Gu, Huang, and Marton, 2004), and so on. Five TA activities (TA-2, TA-3, TA-8, TA-10, and TA-12) are under the theme of mathematics learning and teaching, demonstrating some traditions and more innovations in this regard in China.

TA-2 was organized by Shanghai Institute of Artificial Intelligence (AI) in Education at East China Normal University. TA-2 involved various aspects of a mathematics adaptive learning system aiming at integrating AI technology into mathematics teaching and learning. It consisted of four lectures. Xiaoqing Gu from East China Normal University introduced the general blueprint and gave a demonstration of a beta version of this system. Then three lecturers further explained various elements of this system. Specifically, Aiming Zhou from Shanghai Institute of AI in Education and East China Normal University, specialized in computer science and technology, presented the AI techniques including the algorithms for learning material recommendation, learning path planning, and vision recognition. Yan Zhu from East China Normal University, an expert in mathematics learning and teaching, explicated the instructional design for this system in which the knowledge, cognitive and affective aspects of mathematics learning are considered. Lastly, Chanjin Zheng from Shanghai Institute of AI in Education at East China Normal University, a scholar on computerized adaptive testing and cognitive diagnostic assessment, talked about the design, techniques and implementation of various assessments in the system.

TA-3 was provided by Primary Mathematics Teaching Committee of Chinese Society of Education. It was designed to share practical achievements of primary mathematics education in China. It comprised four parts. Part 1 introduced two effective mathematics teaching methods — the attempting method by Xuehua Qiu and Xinlan Ma and another method by Ma Xinlan and Jiawei Sun. Part 2 consisted of four presentations which depicted primary mathematics teaching practices and research aimed at developing students' key competencies. Zhengxian Wu mentioned that mathematics teaching needs to provide "delicious and nutritious" mathematics for school pupils; Yunpeng Ma from Northeast Normal University presented a teaching reform project to promote deep learning in primary mathematics; Dan Zhang from Beijing Academy of Educational Sciences talked about primary mathematics teaching practice guided by children's real questions; lastly, professor-ship mathematics teacher Xiaomei Li from Liaoning Education College classified mathematics lessons into seed lesson, growth lesson, and thematic-activity oriented lesson and elaborated each type with examples. Part 3 offered lesson explaining activity. Four lesson clips on topics relevant to number, measurement, data handling, and relationship were demonstrated by school mathematics teachers Fang Ni, Libing Wang, Rengxuan Liu and Yunhong Xu, and Li Liu, accordingly. Part 4 discussed future prospects of primary mathematics teaching and learning with two mini-lectures. One was provided by Xiaotian Sun from Minzu University of China, who introduced the latest version of Full–time Obligatory Education Mathematics Curriculum Standards that was still under revision at that time, and the other was given by Qiping Kong from East China Normal University, who talked about innovations of primary mathematics education reform under the context of artificial intelligence. Details of TA-3 could be found in Cui (2021) and other articles in the same issue.

TA-8 introduced mathematics experiment as a transformation of mathematics learning in Chinese primary and secondary schools. Linwei Dong and his team have initiated the work of adopting mathematics experiment as an innovative way for students to learn mathematics since early 1990s. They have made fruitful achievements in this field both theoretically and practically. TA-8, consisting of four parts, was chaired by Weikun Zhao and Haiyue Jin from Nanjing Normal University. Linwei Dong first delivered a keynote speech which systematically introduced the process and achievements of mathematics experiment covering its background, initial attempt, findings of theoretical and empirical research, adopting and promoting the approach in scale, as well as its outcomes and achievements. Secondly, school mathematics teachers shared teaching cases of mathematics experiment from various perspectives. Qingsong Guo from the Teaching Research Office of Jiangsu Province, Yuguo Wu, a master school teacher from Nanjing, Zhengsong Liu from the Teaching Research Office of Nanjing City, Xin Li, a master teacher from Suzhou, and Qiong Shen, a mathematics textbook editor, shared how to use mathematics experiment to promote mathematical conceptual understanding; Aiping Zhang from Nanjing Institution of Educational Sciences, Meihua Chen from Changzhou, Lei Wang from Nanjing shared how to use mathematics experiment to explore mathematical rules and theorems; and Weikun Zhao from Yancheng and Zhengyong Zhao from Nanjing shared how to use mathematics experiment to show the application of mathematics. Thirdly, findings of empirical research on mathematics experiment were reported. Ping Yu from Nanjing Normal University analyzed effects of mathematics experiment on students' mathematics learning, and Dinglian Tan addressed the psychological mechanism and its effect on how mathematics experiment promoted mathematics learning. Finally, Detong Xu from the teaching research office of Jiangsu province made a summary. The team has published extensively on mathematics experiment, such as Dong (2020) and Yu (2016).

TA-10 and TA-12 were the only two that school students' voices were heard. TA-10 was organized by Shanghai High School, one of the most well-known schools in China. Two mathematics teachers (Feng Ma and Ziyu Guo) and six students (Ruiqi Jin, Shuo Tong, Alex Jiang, Minhan Zhou, Alyna Tang, and Leo Lu), both from Shanghai

High school, reported secondary school students' experiences of learning mathematics and teachers' experiences of teaching mathematics during the COVID-19 pandemic period in 2020 in Shanghai. They also provided some interesting findings based on questionnaire and interview data collected from 264 students and 332 mathematics teachers in Shanghai, who had experienced online teaching and learning in early 2020. For example, it was found that the participated teachers considered that the playback function was the most attractive advantage of online teaching and uncontrollability of students' online learning was their main worry. Moreover, they found no statistically significant differences on students' performance on preparing lesson before class across grade levels (primary, secondary and high) and between boys and girls, and high school students tended to use online resources more frequently than secondary and primary school students.

TA-12 aimed to promote teaching and learning of mathematical modeling and STEM education in secondary schools. It provided a platform for winners of International Mathematical Modeling Challenge (IMMC) to report their work. IMMC is an innovative contest in mathematical modeling for secondary school students all over the world. Sol Garfunkel, Chair of the international organizing committee of IMMC and president of the American Federation of Mathematics and its Application, gave a brief introduction of 2021 IMMC and congratulated to all the teams participating the contest first. Shangzhi Wang from the Capital Normal University delivered the keynote speech entitled "Let mathematical modeling take rote in mathematics education". Then five teams, from Hongkong Diocesan Girls' School, Branch of Beijing Chen Jinglun High School, Shanghai High School, Hwa Chong Institution of Singapore, and IES Thader School of Spain, presented different solutions to the same problem in 2021 IMMC, fully demonstrating the openness and creativity of mathematical modeling activity. In addition, Xue Tan from Beijing Academy, Baoqi Yang from Hong Kong Diocesan Girls' School, and Xiaming Chen from Shanghai Experimental School shared their teaching experiences on mathematical modeling. Junfeng Yin from Tongji University summarized the whole session. Details of TA-12 could be found at Qiao (2022).

2.3. Mathematics teacher learning and teaching research: Systematic and continuous

Teachers are highly respected and regarded as engineers of human souls in China. Mathematics teacher learning and education in China has its unique features. Generally speaking, both primary and secondary mathematics teacher preparation programs consistently and highly value the importance of mathematics content knowledge and start to emphasize the development of pedagogical content knowledge and filed experience in recent years. For in-service mathematics teachers, teaching research system provides a strong support for their continuous professional development. Teaching researchers from various teaching research offices or institutions, most of whom were previously experienced and master teachers, are committed to guide and improve teaching practice, advance curriculum reform, build curriculum and teaching resources, promote teacher professional development, and cultivate and popularize excellent teaching achievements (Ding, Wu, Liu, and Cai, 2022). There are four TAs under the theme of mathematics teacher learning and teaching research, with one (TA-7) on preservice primary mathematics teachers' preparation and the other three (TA-1, TA-5, TA-9) on in-service primary and secondary mathematics teachers' professional development.

TA-7 was organized by primary school mathematics teacher preparation working committee under China Association of Higher education. The working committee is devoted to the preparation and development of primary school mathematics teachers all over the country. TA-7 consisted of five lectures demonstrating characteristics of primary school mathematics teacher education in China. Zhigang Wang from Nantong Normal College introduced the beginning of primary school mathematics teacher preparation and its developmental stages in China; Shuping Pu from Chongqing Normal University proposed features of preparation mode of primary school mathematics teachers, which were consolidating mathematical foundation, strengthening teaching skills, emphasizing practical teaching and highlighting competence development; Mingxiang Liu from Yangzhou Polytechnic College shared curriculum and textbooks used in preparing primary mathematics teachers, and he pointed out that textbooks for mathematics content courses need to reveal the nature and basic principles of mathematics and textbooks for mathematics pedagogical courses need to reflect ideas, objectives and methods emphasized in the latest Full–time Obligatory Education Mathematics Curriculum Standards; Shuhong Zhou from Harbin University discussed teaching strategies in working with preservice primary mathematics teachers and she pointed out that university, government, teaching research institution and primary school need to make collaborations to better prepare school teachers; finally, Xiaohui Liu from Xiamen University summarized achievements of the efforts in preparing primary school mathematics teachers.

TA-1 was organized by Secondary Mathematics Teaching Committee of Chinese Society of Education. It demonstrated and discussed a plane geometry lesson "determination and properties of parallel lines", so as to reveal the underlying principle of plane geometry teaching and to illuminate how such a lesson was prepared, implemented and improved with the full support of teaching researchers and master teachers. During the session, the classroom teaching video of the lesson was played and self-explained by the teacher, Jianhao Chen from Shanghai Soong Ching Ling School. The lesson took an integral approach of logical reasoning and visual experiences to deal with determination and properties of parallel lines. Secondly, two very experienced teaching researchers Zengshen Wu and Xuan Zheng evaluated the lesson from different perspectives. Wu interpreted the lesson from the basic idea of geometry teaching, theoretical considerations of classroom teaching and students' mathematical cognition, while Zheng analyzed the lesson by focusing on the interactions between the teacher and the students and highlighted the kind of mathematical thinking behind the interactions and dialogues. Thirdly, Da Liu, Caifeng Xiao, Shuangshuang Chen and Xiaodong Mu explained how to make such a quality lesson and elaborated the kind of support provided by teaching researchers. Fourthly, Yudong Yang, from Shanghai Academy of Educational Sciences, examined features of Chinese lesson study exhibited in this lesson development. Finally, Jianyue Zhang, president of the Secondary Mathematics Teaching Committee of Chinese Society of Education, summarized the whole session. TA-1 not only illustrated features of geometry teaching in China but also presented how the teaching research activities ensured the quality of the lesson and promoted teachers' professional development. Details of TA-1 could be found at Zhang (2021) and the related articles in the same issue.

TA-5 was organized by Teaching and Research Institute of Zhejiang Education Department. Its purpose was to share a successful model which has been implemented in Zhejiang province for years. This model is based on the theories of situated learning, improvement sciences, and learning from masters. In this model, novice teachers, teaching researchers, and master teachers form a teaching research community, where novices learn directly from masters how to teach via the collective and collaborative activities of "master teachers' observing and discussing a lesson, and then novice teachers' immediate revising and delivering the lesson again". The entire process usually lasts from two to three days. Due to time limitations, TA-5 demonstrated five major steps after Miaoer Si, the teaching researcher at the province level, gave a brief introduction of this model. These five major steps are as follows. A lesson video on the topic of addition and subtraction of fractions with different denominators, delivered by a novice teacher Chengyu Wang, was played first. Then the participated teaching researchers and master teachers including Zhengqiang Yu, Shanna Liu, Xiangyang Zhu, Xiaoping Yuan, Minmin Liu, and Xiping Zhu provided feedback about the lesson and discussed with the novice teacher, along with other observers, about how to revise the lesson (Telling). Afterwards, Zhengqiang Yuan, Xiaoping Yuan and Xiping Zhu actually delivered some episodes of the lesson (Showing). Next, another two novice teachers were randomly selected from the observers to revise and redesign the lesson and then teach the lesson again. Finally, Chengyu Wang reflected on the whole learning process, and Qinqiong Zhang from Wenzhou University, Peixun Chen a teaching researcher from Shanghai, and Haixia Si from Hangzhou Normal University commented on this model. Details of TA-5 could be found at Liu (2021).

TA-9 was organized by Qingpu Experiment Research Institution which is located at Qingpu District of Shanghai city and with Lingyuan Gu as its founding president. Gu gave a plenary speech at ICME-14 involving a 45-year experiment on mathematics teaching reform conducted in Qingpu District, an urban-rural fringe area in the west of Shanghai. TA-9 was actually an extension of the plenary speech, focusing on the practice of teaching improvement from comprehension to inquiry. Lianyun Zhu, the current president of Qingpu Experiment Research Institution, made an opening introduction to the background and ideas of this session. Secondly, a lesson video on the topic properties and applications of linear function delivered by Haiyan Qian from Qingpu Experiment School was played and discussed. Jie Wang from Shanghai Normal University commented on the lesson and highlighted the whole process of action education which included three focuses (focus on personal experience accumulation, focus on new ideas and experiences, and focus on the real gains of students) and two reflections (looking for the gap between oneself and the others, looking for the gap between plan and reality) which supported the development and improvement of the lesson. Binyan Xu from East China Normal University analyzed the three levels of learning mathematical modeling, with acquiring skills that could be directly or indirectly related to mathematical modeling, developing competencies in mathematical modeling and completely solving a mathematical modeling task as the first, second and third level, accordingly. Next, a lesson video on the same topic but delivered by Wei Yu from the Middle School Affiliated to Qingpu Teachers Training College was played to demonstrate how to popularize the outcome of the teaching reform at a regional level. Hua Wang shared his view on how to disseminate and promotion of Qingpu Experiences. TA-6 ended in a warm discussion among online and onsite participants.

2.4. Mathematics education for ethnic minorities: Diversity in culture and improvement in research and practice

China is a multi-ethnic country with a long history. In addition to Han ethnic group with its population more than 90% of the whole population in China, there are 55 ethnic groups, including Tibetan and Mongolian, referred to as ethnic minorities. Mathematics education for ethnic minorities in China features diversity in culture and has made great improvement in research and practice.

TA-6 has three 40-minute lectures and one 30-minute video playing. Daiqin from Inner Mongolia Normal University used Mongolia as an example and talked about the educational value of mathematics culture from ethnic minorities in China and how to integrate mathematics culture from ethnic minorities into mathematics teaching. Wei He from Minzu University of China addressed issues on in-service mathematics teacher education in ethnic minorities' area. She investigated the current state of mathematics teaching and learning at school levels via questionnaires, tests, classroom observations and interviews, and found there were significant difference in mathematics performance between students from primary schools in ethnic minorities' areas and their counterparts from Beijing and there were challenges in conducting in-service teacher professional development activities in ethnic minorities' areas as well. She further proposed that teacher professional development mode in ethnic minorities' areas should have its own specificity. Xinrong Yang from Southwest University reported an empirical study on knowledge and beliefs among mathematics teachers from ethnic minority area and the relationships among these teachers' knowledge and beliefs, their teaching quality and their students' mathematics performance. In addition to the three lectures, the organizer also played a 30-minute video prepared in advance, which displayed mathematics culture of ethnic minorities using a large number of artifacts, and classroom teaching clips to illustrate achievements and research outcome of mathematics teaching and mathematics curriculum reform for ethnic minorities. Details of TA-6 could be found at Chen and Qin (2021).

3. Concluding Remark

The 13 Thematic Afternoon activities at ICME-14 show to the world, the Chinese theories and practices of mathematics education at school level. They shared experiences and wisdom from perspectives of mathematics curriculum, mathematics learning and teaching, mathematics teacher learning and teaching research, and mathematics education for ethnic minorities. Of course, there are also challenges and issues in the field of mathematics education in China. More efforts and works are deserved for a better future.

Acknowledgement

The author is grateful for Jiansheng Bao, Yufeng Guo, Weikun Zhao, Feng Ma, Alfred Cheung, Shuhong Zhou, Jianyue Zhang, Miaoer Si, Kaiyu Huang, and Ting Chen for the valuable information and materials provided.

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Part X

Early Career Researcher Day

https://doi.org/10.1142/9789811287152_0121

Early Career Researcher Day

Supporting Early Career Researchers in Mathematics Education: An Overview of Early Career Researcher Day at ICME-14

Lianghuo Fan[1]

ABSTRACT Primarily designed for supporting early career researchers in mathematics education, the event of "Early Career Researcher Day" was first created for the 13th International Congress on Mathematics Education. The tradition was continued at the 14th International Congress on Mathematics Education or ICME-14. This article presents an overview about the event of ECRD for ICME-14, including its general aims, focuses, activities, and other related information.

Keywords: Early Career Researcher; ICME; Mathematics Education Journal; Mathematics Education Research;

1. Introduction and General Background

The event "Early Career Researcher Day" (ECRD) is a whole day program related to ICMEs. ECRD was first created in the 13th International Congress on Mathematics Education (ICME-13) held in Hamburg, Germany in 2016, for supporting early career researchers in mathematics education. It was attached to the congress, but not an integral part of the congress itself.

Like ICME-13, the event for ICME-14 was also organized by the Local Organization Committee of the congress with Lianghuo Fan (the author) as its program chair; it was held on the day of, but before, the opening ceremony of the congress, 12 July 2021.

The general aims of the ECRD for ICME-14 were threefold. They were as follows:

1. To provide the participants (early career researchers) with opportunities to develop their research competencies in related fields.
2. To provide the participants with opportunities to establish new contacts and build new networks.
3. To provide the participants with opportunities to work with and learn from internationally renowned scholars and experts in related fields.

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In terms of the content, the focus of the ERCD program for ICME-14 was mainly on three aspects about mathematics education research: (1) research conceptualization and methods, (2) writing research proposals and implementing research projects, and (3) academic writing and publications.

2. The Program of ECRD

The program of the ECRD for ICME-14 consisted of 9 parallel workshops, 2 plenary sessions, and 7 parallel interactive discussions. Due to the impact of the COVID-19 pandemic, all these activities took place in a hybrid mode, i.e., both onsite — in the physical venue of the congress, i.e., the Putuo Campus of East China Normal University, Shanghai, China and online — via zoom video conferencing to all registered participants in the world, simultaneously.

The ECRD program was opened in the morning by a welcome message from the program chair, Lianghuo Fan, East China Normal University, China. That was followed by 9 parallel workshops, each lasting two and a half hours. Those nine workshops focused on "research conceptualization and methodology" with specific themes such as qualitative research, mixed methods, video-based research, large-scale assessments, ethnographic studies, argumentation analyses, participatory action research, etc. More detailed information can be found in Tab. 1 below.

Tab. 1. Titles/themes and presenters in parallel workshops at ECRD for ICME-14

	Title/Theme	Presenter (, and Co-presenter)
Workshop 1	Qualitative Research Methodology	Marcelo Borba (São Paulo State University, Brazil), and Liliane Xavier Neves (Universidade Estadual de Santa Cruz, Brazil)
Workshop 2	Design-Based Research	Andreas Stylianides (University of Cambridge, UK), and Gabriel Stylianides (University of Oxford, UK)
Workshop 3	Mixed Methods	Susan Prediger (TU Dortmund University, Germany), and Kirstin Erath (TU Dortmund University, Germany)
Workshop 4	Video-Based Research	Ida Mok (University of Hong Kong, Hong Kong SAR, China), and Wenjun Zhao (Beijing Normal University, China)
Workshop 5	Large-Scale Assessments	Christian Bokhove (University of Southampton, UK)
Workshop 6	Naturalistic Paradigm and Ethnographic Methods	Judit Moschkovich (University of California, Santa Cruz, USA)
Workshop 7	Argumentation Analyses	Christine Knipping (University of Bremen, Germany), and Fiene Bredow (University of Bremen, Germany)
Workshop 8	Participatory Action Research	Julie Amador (University of Idaho, USA)
Workshop 9	Methods of Textbook Research	Sebastian Rezat (University of Paderborn, Germany)

The afternoon program started with the first plenary session, moderated by Jinfa Cai, University of Delaware, USA. Editors of eight major journals in mathematics education introduced their journals in sequence, all with a focus on "academic writing and academic publishing" in relation to their respective journals. The plenary presentations were followed by seven parallel interactive sessions for questions, answers and discussions between the interested participants and the journals.

Tab. 2 provides the information of the seven parallel interactive discussions.

Tab. 2. Journals presented at parallel interactive sessions at ECRD for ICME-14

No.	Journal	Editor/Editor-in-Chief
1	*Educational Studies in Mathematics*	Arthur Bakker
2	*Journal for Research in Mathematics Education*	Jinfa Cai
3	*Journal of Mathematics Teacher Education*	Despina Potari
4	*Mathematical Thinking and Learning*	Lyn English
5	*Mathematics Education Research Journal*	Peter Grootenboer
6	*Research in Mathematics Education*	Jenni Ingram
7	*ZDM Mathematics Education*	Gabriele Kaiser

Note: The editor(s) of *Journal of Mathematical Behavior* participated in the plenary session, but were unable to join the interactive discussions.

The final activity of the ECRD was the second plenary session, moderated by Lianghuo Fan. The three invited panelists were Alan Schoenfeld, University of California, Berkeley, USA (recipient of the 2011 Felix Klein Award), Gert Schubring, Bielefeld University, Germany (recipient of the 2019 Hans Freudenthal Award), and Anna Sfard, University of Haifa, Israel (recipient of the 2007 Hans Freudenthal Award). During this 90-minute session, the three panelists each shared their research and publication work including research results, methods and experiences, etc. which were well received and appreciated by the participants.

More information about the event can be found in ECRD Program Organization Team (2021) and Li et al. (2021).

3. Concluding Remarks

In total, there were 424 participants who registered for the ECRD, and 32 invited presenters who presented in a range of the activities programmed as described above. The number of participants was considerably more than expected, an indicator of the popularity of the event for researchers, especially those at early stages of their career as researchers in mathematics education.

Due to the COVID-19 pandemic, most of the presentations to the participants and interactions between the presenters and the participants had to be in pre-recorded videos or online, which was a challenge for the organizers and participants (including presenters). Nevertheless, the event was overall well received and highly commended by both onsite and online participants.

Based on the feedback received from the participants, there is a strong belief that the ECRD merits continuation in the future ICMEs.

Acknowledgments

As the program chair and the chair of the ICME-14 Advisory Committee appointed by Jianpan Wang, convenor of the congress, I wish to place on record my appreciation to all the other members of the Advisory Committee: Ferdinando Azarello (University of Torino, Italy), Jinfa Cai (University of Delaware, USA), Bernard Hodgson (University of Laval, Canada), Peng Yee Lee (National Institute of Education, Singapore), Frederick Leung (University of Hong Kong, China), and Anna Sfard (University of Haifa, Israel) for their support and advices, especially at the early stage of the preparation for the program. My appreciation also goes to all my colleagues and assistants for all the help they provided for preparing and organizing this ECRD event.

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Part XI

The Closing Ceremony

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https://doi.org/10.1142/9789811287152_0122

The Closing Ceremony

The Closing Remarks from the Local Organizing Committee

Jiansheng Bao[1]

Dear participants, both online and onsite,
Ladies and gentlemen,

Good morning, good afternoon, good evening, and good night!

During ICME-14, we have heard such greetings many times. This is a very special set of greetings for ICME-14.

While in many aspects, ICME-14 follows the ICME tradition, contains all kinds of academic activities, including plenary lectures, plenary panels, survey teams, awardee lectures, invited lectures, Topic Study Groups, Discussion Groups, Posters, Workshops and Special Sessions for Affiliated Organizations, National Presentations and ICME Studies, ICME-14 has made many new records in the history of ICME.

First of all, ICME-14 is held in a hybrid mode and this is the first time in the history of ICME. Consequently, ICME-14 for the first time runs the main academic activities from afternoon to midnight in Beijing time.

Secondly, the number of formal registrations is 4055, among them there are 3156 people who attended the congress. ICME-14 is the first ICME with more than 570 thousand people all over the world online and another over one thousand onsite participants watching the opening ceremony.

While the onsite participants are mostly local Chinese, the number is twice higher than that of all the Chinese participants who have ever attended the former ICME. This is, of course, related to the fact that this is the first ICME held in China.

ICME-14 is a big news these days (Fig. 1, Fig. 2, and Fig. 3).

For the variety of academic activities, ICME-14 held four plenary lectures, three plenary panels, four survey teams, five ICMI awardee lectures, more than 60 invited lectures, 62 Topic Study Groups, 15 Discussion Groups, more than 300 posters, 27 Workshops, and the sessions for Affiliated Organizations, National Presentations, ICME Studies, etc.

In addition, a total of 13 Thematic Afternoon activities showing the characteristics of Chinese mathematics education were specially set up in the Congress. More than one thousand onsite participants and 15.4 thousand online participants attended these

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Fig. 1. ICME-14 has received extensive media attention since the opening ceremony on July 12th, 2021 (UTC+8)

Fig. 2. There are many full-page and front-page reports, including *China Daily*

activities. These 13 Thematic Afternoon activities are intended to illustrate the characteristics, experiences and achievements of Chinese mathematics education to the world. The Organizing Committee planned 13 city tour routes for participants to understand Shanghai's unique culture and latest development trends. (It is a pity that online participants were unable to take part in this activity.) Furthermore, the Symphony Orchestra of East China Normal University presented a wonderful symphony concert for ICME-14, which was broadcasted live globally (Fig. 4).

Fig. 3. Many topics concerned by the media

Fig. 4. The symphony concert at ICME-14 (conductor: Runyu Hou)

Back to year 2016, when Shanghai won the bid for ICME-14, we were extraordinarily delighted and proud, because it is a very important event in the history of Chinese mathematics education. While we did know this could be a very challenging task, we had profound understanding of how difficult it is till we started to prepare for the Congress. All the preparation work for the Congress, including but not limited to fundraising, IPC meetings, a great deal of documents and endless online communication, logistics planning (for example, Congress venue, public transportation, accommodations) were all targeted to the Congress held in 2020. However, due to the severe global pandemic COVID-19, the Congress was postponed to year 2021, the preparatory work of the Congress then almost returned to its original point, and many of the manpower and material resources were gone with the wind. In fact, the preparation work for the 2021 Congress were carried out under the circumstance of many “uncertainties”. We do not know whether COVID-19 will become a history in this summer; We do not know whether overseas participants can come to Shanghai for the Congress; we do not know how many participants are willing to accept such a hybrid attending mode; we do not know what difficulties and challenges such a hybrid mode will face.

Till this moment, in the closing ceremony, I can say, we have accomplished this almost impossible mission.

Therefore, today, what I want to express the most is gratitude!

On behave of the Local Committee of ICME-14, I would like to extend my sincere thanks to many people.

First of all, I’d like to thank all of our online participants. We owe you an onsite visit. China is an ancient and charming nation with a lot of sightseeing places and special cultures, so as for the city of Shanghai. However, the COVID-19 broke the way for many of you to come, but I am sure and sincerely hope you could come to China, to Shanghai in the near future. The door of China and the city of Shanghai is always open to you.

Secondly, I’d like to thank all of my Chinses colleagues, both onsite and online. In order to make Congress time convenient for the largest number of participants in the world to participate in the Congress, the most important academic activities of ICME-14 were arranged to start at 7:30 p.m., Beijing time. This is usually a resting and sitting-out time for the Chinese people. Many Chinese colleagues told us that the first thing they need to do with ICME-14 is to adjust “jet lag”, because every night, they can’t go to bed until after midnight. When the Congress comes to the end today, the first thing for them is again to adjust “jet lag”. We feel very sorry that this Congress has disrupted your biological clock. Thank you! Thanks for your full support. Whenever we need help, it is you who give us new strength; When we encounter difficulties, it is you who stand firmly beside us and become our most reliable backing. Sincerely thank all of you!

Here, I would also like to thank all the leading departments for their strong support, from the State Council, the Ministry of Foreign Affairs, to the Chinese Association for Science and Technology; from the Chinese Mathematical Society to the Shanghai Mathematical Society; from the city of Shanghai to East China Normal University to the School of Mathematical Sciences. In the process of ICME-14’s bidding, preparation and hosting, the guidance and support you have given is unparalleled. Thank all of you!

Finally, I would like to give special thanks to our team. In order to host this Congress, we have a large team, including the leaders of East China Normal University and all related functional departments, teaching staff and all administrative staff of the School of Mathematical Sciences, East Star Event Management Company, and of course, our Local Committee members and volunteers. I am here particularly grateful to our volunteers. Among them are leaders and administrators of fellow universities and colleges, as well as young students. After the daily Congress, they still have a lot of work to do, and many people need to take about one-hour bus to return home or dormitory, and the next day, one hour before the Congress starts, they need to return to the Congress in vigor. The gratitude to you is beyond words, thousands of words can only be combined into four words: Thank you very much! 谢谢你们!

Thank you! Thank you all with my best wishes!

https://doi.org/10.1142/9789811287152_0123

The Closing Ceremony

Reports from the ICMI EC

Abraham Arcavi[1], Jean-Luc Dorier[2], Frederick Leung[3], and Lena Koch[4]

Lena Koch:

ICME-14 is like every ICME held under the auspices of the International Commission on Mathematical Instruction. It's a tradition that the ICMI secretary general reports about the activities of the ICMI EC since the last ICME. As you all know, ICME-14 must be postponed for one year due to the pandemic. Therefore, the past and the current secretary general are reporting today.

The Secretary General for the term 2017 to 2020 was Professor Abraham Arcavi from Weizmann Institute, Israel. Since 2021, Professor Dorier from the University of Geneva in Switzerland is the new Secretary General.

I have the pleasure to work with Professor Arcavi for 8 years, and I want to take this opportunity to thank him for the wonderful and intensive collaboration during this time.

Since 2021, I have the pleasure to work with Professor Dorier, who would have reported about the activities of the new EC since January 21. And it's still overshadowed by the pandemic and its applications.

I can confirm first hand that the workload of ICMI's secretary general and also of ICMI's president is really immense.

Now I would like to show you the impressive numbers of ICME-14.

ICME-14 had more than 3988 registered participants from 129 countries. We had 592 scholars who received the Solidarity Grant/waiver of the registration fee. More than 417 teachers are registered and participated. There were 4 plenary lectures, 3 plenary panels, 4 national presentations, 65 invited lectures. There were 62 T.S.G. with almost 2,000 presentations. There were almost 1500 oral presentations. There were 334 posters, 16 discussion groups, 27 workshops and 13 thematic afternoons.

I would now like to give the word to Professor Arcavi and Professor Dorier.

Abraham Arcavi:

Thank you very much, Lena. It is an honour and pleasure to be here for the closing

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ceremony, in which it is a tradition that the Secretary General gives a report of the activities carried out by ICMI in the preceding 4 years.

This is a joint presentation with my colleague Jean-Luc Dorier. I present the data on the 2017–2020 period, and Jean-Luc will present the ICMI activities of the first 6 months of his term as Secretary General.

The following photo of the Executive Committee (EC), in office during 2017–2020 under the leadership of President Jill Adler, was taken in Geneva on the occasion of the first meeting (Fig. 1). Fig. 2 is a screen shot of the last meeting which was online and took place on September 22, 2020.

Fig. 1. 2017–2020 ICMI Executive Committee

Fig. 2. Online ICMI Executive Committee in 2022

Jean-Luc Dorier:

And this is actually the first meeting online of the new ICMI Executive Committee. We have to face the new situation because unlike the preceding ECs, we weren't able to meet face to face. So, we decided to have a 2-hour meeting because of time differences, since one of our EC members Marta Civil is in Arizona and our vice president Merrylin Goos is in Brisbane in Australia. So, two hours' meeting means that Marta has to start at 05:30 in the morning and Merrylin has to finish it at midnight. We have so far four 2 hours' meetings, which is great because we've managed to do quite a lot of things.

Abraham Arcavi:

I divide my report into the two main types of the ICMI activities: organization and policy on the one hand, and on the other hand, education, research and dissemination.

My first remark regarding organization and policy is to remind that ICMI is a commission of the International Mathematical Union (IMU). The IMU President, the IMU Secretary General, and usually one more representative of the IMU Executive Committe are ex-officio members of the ICMI Executive Committee. During our four-year term, we have the honor and pleasure to work with two IMU Presidents, Professor Shigefumi Mori from Japan and Professor Carlos Kenig from the United States, and with Secretary General Helge Holden from Norway. The IMU, ICMI's overarching umbrella, provides institutional support and the spirit of international collaboration and inclusion. Moreover, IMU provides ICMI with most, if not all, of the funding needed for ICMI's functioning, either in a direct way or through CDC, another IMU Comission. IMU also provides advice and guidance during the EC meetings and beyond them.

Jean-Luc Dorier:

Actually, if ICMI celebrated its first hundred anniversary in 2008 in Rome, IMU is about to celebrate its first hundred anniversary. Actually, the conference was scheduled for last September, and now is rescheduled for next September in Strasbourg due to COVID situation (Fig. 3). You can actually read the very nice historical vignette about all the stories of ICMI in IMU in our last ICMI Newsletter by Bernard R. Hodgson. So now your turn again, Abraham.

Abraham Arcavi:

Organizationally, ICMI also relies on colleagues who represent the countries, which are its members. These country representatives are in contact with the ICMI EC in order to exchange ideas, disseminate information and to make suggestions. In a sense, this is our "senate", since they also vote in the election of the ICMI EC every four years at the ICMI General Assembly. We tried to keep a fluent contact with all the country representatives, involving them as active participants in all ICMI activities. Most of the 83 countries which are ICMI members (except for 7 of them) are IMU members. The roles and responsibilities of the country representatives are stipulated in the documents that can be found in the ICMI website.

ICMI has affiliated organizations, which are autonomous. The EC reorganized the

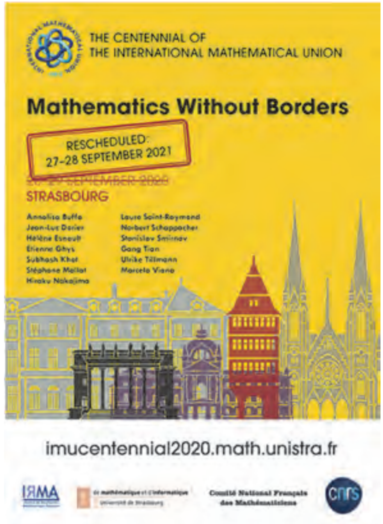

Fig. 3. The Centennial of the IMU in Strasbourg

affiliated organizations into two groups: the thematic affiliated organizations and the regional affiliated organizations. The thematic affiliated organizations are devoted to topics and themes of interest. During our term, the International Society for Design and Development in Education, applied to become and ICMI affiliated organization, and this status was granted in 2017. The eight regional affiliated organizations are distributed all over the world from Africa to Asia to Latin America to Europe. During last 4 years, two new such organizations joined the ICMI.

A main organizational task is the ICME. Every four years, ICMI publishes a call for proposals to countries that would like to host the congress. An intensive selection process follows, which ultimately results in choosing one of the submitted proposals. Australia was selected to host the ICME 15 in 2024. The other task related to ICME is to appoint its International Program Committee (IPC) of about 20 members. The EC produced a revised and extended version of the guidelines related to the many aspects of the functioning of an ICME, its mains activities and the work of the IPC. Towards 2020, the EC had to take the difficult decision to postpone the ICME 14 to 2021, which finally had to be conducted mostly online. This included conducting online as well the General Assembly (GA) in July 2020, which usually takes place during one whole day prior to the ICME, and this time, about a year before the conference. A main activity of the GA is to elect the new ICMI EC, which took office in January 2021. Fortunately, 55 country representatives could attend, which is the largest number ever. Due to time differences, the GA meeting lasted only one hour, which included a report of the ICMI activities and the election process, conducted electronically by an external organization specialized on this.

Another organizational activity are the ICMI awards. The ICMI President selects the Awards Committee, and the names of its members (except for the head of the committee) are kept undisclosed to the public at large during the process of their deliberations. In order for the committee to work optimally, the EC produced a document on Conflict of Interest for the Awards Committee members to sign, alongside guidelines for their work. Among other things, the differences between the Felix Klein Award and the Freudenthal Award were explicitly explained. Similarly, the eligibility for the Emma Castelnuovo award was restricted to individuals or small groups and was excluded for established and large institutions.

The second type of ICMI activities consists of education, research and dissemination. A main activity are the well-known ICMI Studies. The ICMI EC took the important decision to publish the ICMI Studies as open access, namely, to make them available free of charge to everybody. This implied to allot considerable amount of funds for that purpose, and given the success of this experience, it was decided to fund the publication as open access of three of the latest ICMI Studies (Fig. 4). Two ICMI Studies are now on the way, and are close to the final stage of publication: Study 24 on School Curricula and ICMI Study 25 on Teachers' Working in Collaborative Settings.

ICMI decided to conduct a survey in order to evaluate the effectiveness and the impact of the ICMI Studies, the ways they are used and regarded by the international community. The first component of the survey was quantitative, it was completed and its insightful results were made public. The second component is qualitative consisting on interviews and other types of information and is now being processed.

Regarding the future of ICMI Studies, I will let Jean-Luc talk about this.

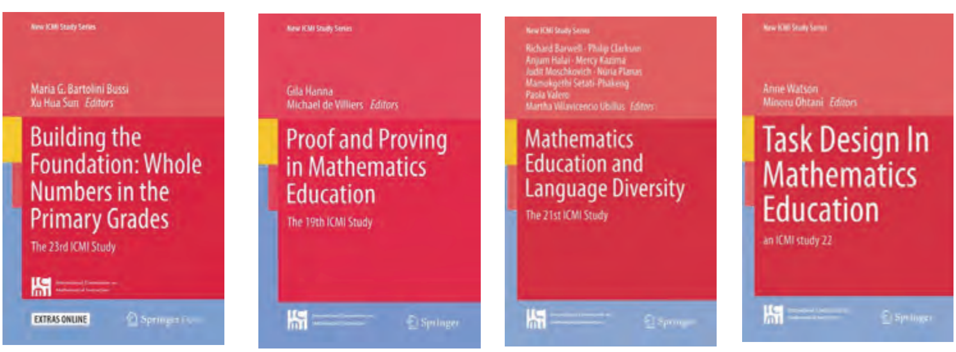

Fig. 4. Recent ICMI studies in open access

Jean-Luc Dorier:

Yes, thank you, Abraham. Yesterday in its meeting the new ICMI executive committee was very interested in ICMI Studies. The thing is that we are very much in favour of continuing being the last EC concern of having all the ICMI Studies to be open access. So, the new ICMI Studies should be open access straight away. We're also working with Springer to have old ICMI Studies to be open access as well. And we are planning for the future to our new ICMI Studies during our term, even though the 4 years of a term for EC is not enough to do the whole of the process, but for the first time in the ICMI history, because before that, the choice of the team for new ICMI Study either came from the ground position from people spontaneously or from the EC members. This time, we decided to put a call for proposals for new ICMI Study which has been published in the special issue of our Newsletter in last June, so I really think that you should go to the website, and see for these calls. In these calls, we see that the proposal should address the following essential 3 elements. The 3 elements are that: the theme is of broad international interest, representing either a mature or emerging field; the second one is that there is sufficient substance in terms of research, literature and practice in a diversity of contexts and cultures, to ensure productive work and to provide a coherent and useful vision of the theme at stake; and the third is that there is a critical mass of scholars of renowned expertise in the theme who can provide leadership, vision and experience and are committed to invest the effort involved in the production of a Study over a 4-to-5-year time frame. So, I hope that we're going to get a lot of proposal and choose new themes for a new ICMI Study. OK, Abraham, your turn again.

Abraham Arcavi:

The second main activity under education, research and dissemination is the Capacity and Networking Project (CANP) devoted to encourage and support the creation and functioning of communities in developing countries all over the world to consolidate and coordinate activities among mathematics teachers, mathematicians, and mathematics educators. So far, five regions are successfully functioning (Fig. 5), and I hand out to Lena to talk more about this.

Fig. 5. CANP in five regions

Lena Koch:

So, a major resource of the CANP project is the publication of three SpringerBriefs about mathematics teacher education in three CANP regions (Fig. 6). And the books have been made available open access last year and we do recommend you to have a look at them. In 2016 and 2017, a survey was made and showed that the project is very successful, but it also showed that all 5 networks still need some further support. Therefore, the EC from 2017 to 2020 decided in 2017 to focus on consolidation and expansion of the 5 regions instead of having a sixth account. During this period, various activities took place, but they were slowed down by the pandemic. At this ICME, representatives of all five CANP regions met as a discussion group and they discussed mainly about the impact of the pandemic in their regions and also to math education. The new EC has already decided to continue its support for the five existing networks. Now I give back to Abraham.

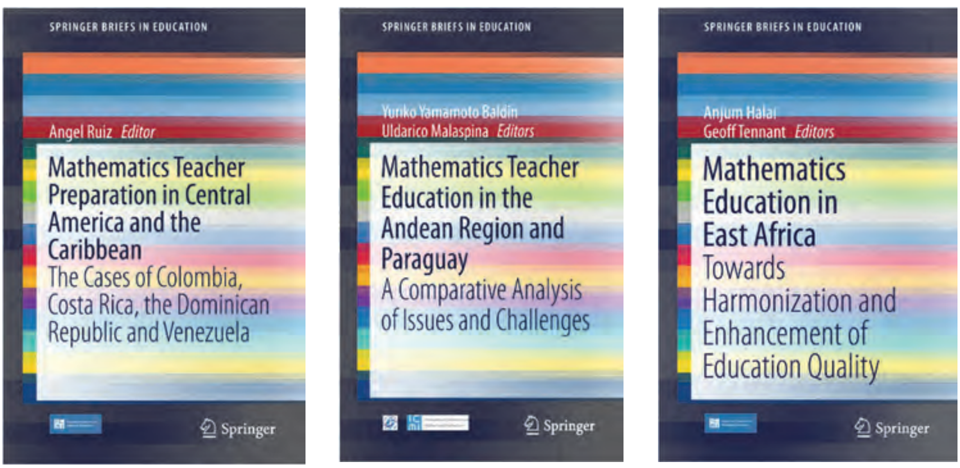

Fig. 6. Three SpringerBriefs about mathematics teacher education in three CANP regions

Abraham Arcavi:

Thank you, Lena. The third major project that was born out of suggestions of former IMU President Ingrid Daubechies about conducting MOOCs. Jean-Luc Dorier took the responsibility to lead the development of this project. So over to Jean-Luc to explain about this project.

Jean-Luc Dorier:

Thank you, Abraham. It's actually not quite MOOC, because MOOC needs a lot of processes in everything. We call it AMOR for Awardees Multimedia Online Resources, which means that there are all resources that can be used in the MOOC but we are not doing the MOOC resources. So, the idea was to post some online resources about different aspects of mathematical education as beginning with more into research. So we decided to use the works of the awardees and we started with a French team by

Michele Artigue, and Claire Margolinas, Annie Bessot for Brousseau, and Marianna Bosch for Chevallard. And then I worked with Anna Sfard, and with Abraham as well. And Anna started a unit. And now that we've started with a new EC, Núria Planas has joined us, and she has made Celia's Unit start. So, we have, at the moment, actually 4 awardees who have started their unit. We are planning quite a few more in the next years.

It's going to take a lot of time, but we are doing it slowly, but we are going to there. Marta Civil has also joined us, and she's working with Alan Schoenfeld to do some things, and we are thinking of course of Ubiratan D'Ambrosio. The units are made this way in each unit there are modules, starting with module 0 which shows the background. Then in each module, you have a short text of presentation, and then you have the main part of the module which is the video. You can see module 1 of Michele Artigue (Fig. 7).

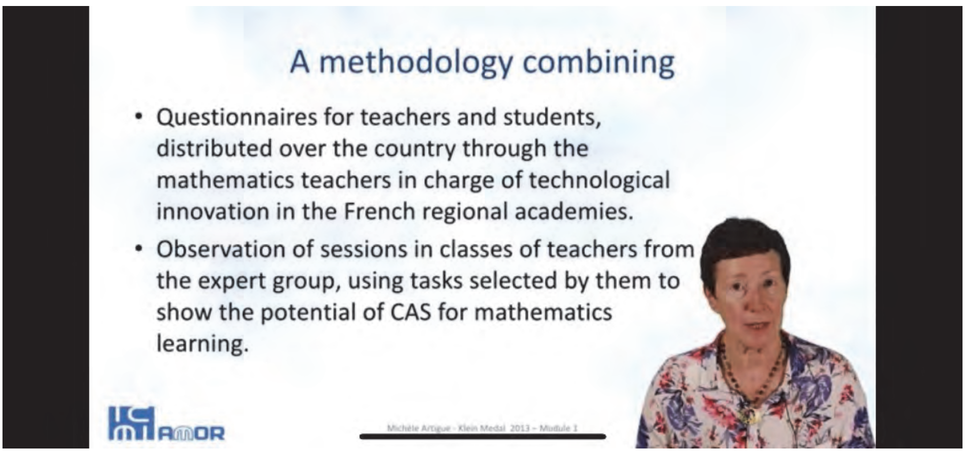

Fig. 7. Module 1 of Michele Artigue

Michele Artigue is actually the only unit which is completely finished. Yves Chevallard's unit is nearly finished, only one module is missing, Brousseau is two-third, Anna Sfard is about a half, and Celia has only started. And then, after the video, which is about 20 minutes, you have a list of texts, most of which are open access, so you can download them from the website of AMOR.

And you have those texts which are referring to the video, and it's a way to have access to the works of the awardees and to the idea of what is a research of mathematics education around the world. We are also planning now to have something for the Emma Castelnuovo Award, and we are working with Hugh Burkhardt on his unit. Thank you, Abraham.

Abraham Arcavi:

The Klein Project consists of the development of a collection of short mathematical vignettes in which mathematicians communicate cutting-edged issues of mathematics in a way that is accessible to teachers not necessarily for them to teach these subjects in class but for the enhancement of their own mathematical knowledge. There is a rich blog with all the vignettes translated into several languages, open for everybody to use, and the EC appointed Professor Weigand to lead the Project, to expand its Advisory Committee and to develop further vignettes.

Jean-Luc Dorier:

Thank you. This is also quite a new initiative from IMU mostly, but we are both in it. The International Day of Mathematics is planned every year on March 14, because March 14, as you know, is Pi Day. So, you have a very nice website where you can see all the events that were planned last year for the International Day of Mathematics all around the world. As you can see on this map, there are masses of things everywhere in the world, and we are glad to be part of that (Fig. 8). Thank you.

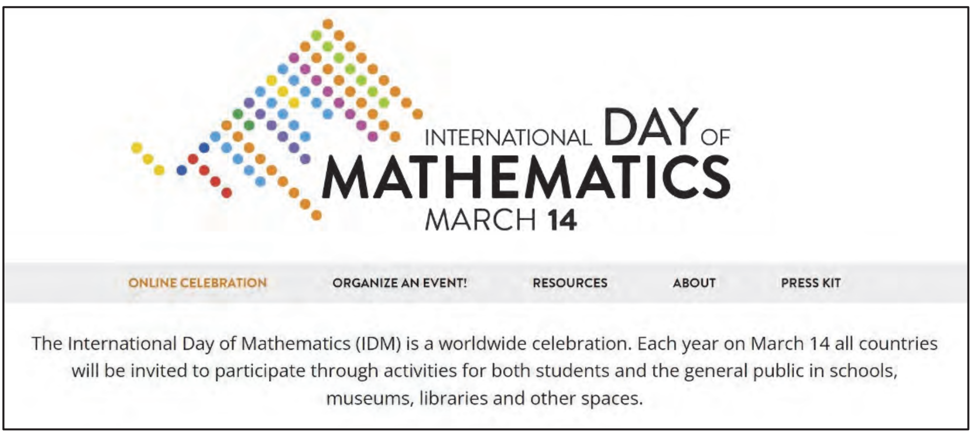

Fig. 8. International Day of Mathematics

Abraham Arcavi:

ICMI undertook the creation of an online large database of curricular materials from all over the world. The country representatives contributed to this database by submitting official curricula documents at all levels and all ranges of mathematics education in their countries. This database is being continuously updated.

Former ICMI Secretary General Bernard Hodgson collects historical and other documents. He also works for the archives of the IMU, and he has started the work in ICMI Archive. Some products of his work in the Archive are reflected in his vignettes that appear regularly in the ICMI Newsletter.

I would like to end with some personal words. In spite of this unusual circumstances, which made us to deliver online the closing words of an ICME, I would like to stress the continuity of the work of one executive committee to the other, as you could see in our joint presentation. I would like to express my deep gratitude to Jean-Luc and Lena for their excellent work and for adding new initiatives. And finally, Jean-Luc, your closing words.

Jean-Luc Dorier:

Well, I will return the thank you to you, because you are here today, six months after the end of your term and you are still in charge, and it has been quite a deception and a lot of work for you to have to face this situation. It wasn't easy for nobody, and especially for you and Jill, I think it has been a very frustrating situation, and I really appreciate the fact that we are still working together. I must say that Abraham is a very dear good friend, and we have been good friends for many years. I'm really very thankful to him and very proud that he trusts me to be the successor. Thank you.

Abraham Arcavi:

Thank you.

Jean-Luc Dorier:

Well, so please stay tuned which means that you should be in contact with ICMI all the time. We have many ways of being present with you. The website is actually one of the biggest issues we want to refresh (Fig. 9).

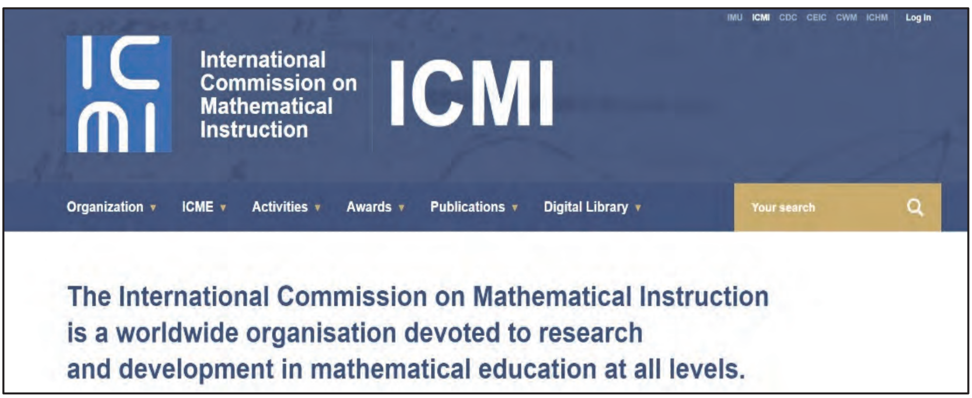

Fig. 9. Website of ICMI

We are working with Merrylin Goos, Lena and Susanne Prediger. And we also have some more fancy ways of being in contact with you with Facebook and there is, of course, the ICMI Newsletter, and you can find that on the website.

Lena has made great job of renewing it completely in form. The last ICMI Newsletter published on July 1 is the first issue of that new form. And you can also write to your country representative, and we are working with them to be present

Fig. 10. ICME-15 in Sydney

everywhere in every country, in every part of the world. One thing in ICMI is that we want to be inclusive, and we want to be part of your work everywhere in the world especially in the countries which are more difficult to reach on the planet. And so anyway, we are looking forward to meeting you all face to face in July 2024 in Sydney (Fig. 10).

Even though it's completely close to any travelling at the moment, but we are working on the preparation of this conference which will be the first post-COVID conference. I wish we are able to offer this friendly and very nice conference in Sydney. See you then, bye.

Frederick Leung:

Thank you very much, Abraham, Jean-Luc and Lena, for presenting this report of ICMI. As all of us can see, ICMI has been doing a lot of work in the past four years. I am sure some of you already knew about this work, or some of this work. But I am also sure that not all of you know all of this work done by ICMI. So, I hope that the report will help you understand the work of ICMI better.

ICMI is devoted to the development of mathematics education in all areas and at all levels around the world, and to promote international cooperation. For that we need the involvement and the support of the whole mathematics education community, including the ICME participants here today. My hope is that after learning about the work that ICMI does, your attention and involvement are not just confined to your own research area or to your own geographical area or your own country. I hope you will think of yourself as part of the worldwide mathematics education community, and support the work of ICMI to make mathematics education better for the whole world.

There have been already many thank you words spoken earlier in this closing ceremony. But I really want to thank the local organizing committee — they are doing a marvelous job. In organizing a physical plus an online conference, or a hybrid conference, one plus one is much greater than two as far as the organization workload is concerned. You are doing the equivalent of more than five conferences in the efforts that you have been putting into this!

I know that there are many people involved in the organization of this conference and ICMI wants to give a present to each of you. But if I buy a present for each of you, ICMI will be bankrupted! So what we have done is we have bought three presents. I will call upon the names of three people representing all of you who have worked so hard for the conference.

The first one is Professor Jiansheng Bao, co-chair of the local organizing committee.

The second one is Professor Binyan Xu, co-chair of the local organizing committee.

Last of course is our Honorable Professor Jianpan Wang, chair of the conference.

These are Switch watches. Let me explain. Usually it is the ICMI Secretary-General's job to do this, presenting a present to the organisers. But because Jean-Luc cannot join us here — he is in Switzerland at the moment. So he flew in some watches from Switzerland — to a shop in Shanghai! and then I went to buy these watches for the three. This is just a token of appreciation.

Thank you very much. Thank you.

https://doi.org/10.1142/9789811287152_0124

The Closing Ceremony

Welcome Address from ICME-15

Kim Beswick[1]

My name is Kim Beswick, and I'm proud to be the convener of ICME-15, to be held in 2024 in Sydney, Australia, and I'm looking forward to welcoming you all to the Congress (Fig. 1). I want to begin by thanking the organizers of ICME-14 for the opportunity to tell you a little about ICME-15. So, ICME-15 will be just the second ICME Congress to be held in the southern hemisphere (Fig. 2). And we've been planning for ICME-15 already for number of years, and it's a huge chain method. The Congress is supported by eight organizations with an interest in mathematics education. We are planning an innovative Congress with a uniquely Australian view of the world. It also builds on well-established Sydney programs and traditions. So, our aspirations for ICME-15 are for a Congress that allows delegates to enjoy their time in Sydney, but it also legacies that stand well beyond the Congress, and it has impacts on local, regional, and global levels. So, we're planning a Congress as accessible as possible for delegates from developing countries. We're working on ways to make the Congress as

Fig. 1. Kim Beswick made a welcome speech as the convener of ICME-15 at the Closing Ceremony

[1] Professor, University of New South Wales, Australia; Convenor of the 15th International Congress on Mathematical Education. Email: k.beswick@unsw.edu.au.

Fig. 2. ICME-15 will be held in Sydney

affordable as possible. We also want the Congress to support the emergence of the next generation of Australian mathematics educators. ICME-5 was held in Adelaide, Australia in 1994.

Several delegates held the first ICME have gone on to bright things in mathematics education. So, I hope that 40 years on, there will be among the delegates of ICME-15 that the fabulous young people who will lead mathematics into the future. We are also planning a Congress that has a central focus on indigenous mathematics and the ways in which that can inform all of our efforts to improve mathematics teaching and learning. The Congress will focus on mathematics learning at all ages, including a focus on undergraduates' mathematics and statistics. And my major issue in Australian education is the disparity between China students in metropolitan and rural areas. We know this is also an issue in other parts of the world. So, we are looking forward to being able to share experiences and learn together about the ways to tackle this issue. We're planning a Congress that will influence classrooms, so we're going to make sure that there is plenty of value in the Congress for teachers of mathematics as well as researchers. ICME-15 will provide an opportunity for teachers and researchers to interact and learn from one another. Of course, every great Congress facilitates and supports people to make connections with others that can lead to ongoing collaborations in friendships. This will be supported by a scientific program and a wonderful social program. We also want to build into the Congress as many options as we can for Australian public to engage with mathematics. So, we're really excited about all the things we can achieve to hold this Congress in Australia.

We have, in conjunction with the ICMI executive committee, established the international program committee whose names you can see on the screen. And our local organizing committee is already had done a lot of work. They are a truly wonderful team. The venue of the Congress will be the International Convention Centre which is a fabulous building located right on the harbour, a short walk from the centre of Sydney. There are lots of hotels and other accommodation options to suit all budgets. It is situated on the land of Gadigal people of Eora Nation. It's a truly beautiful venue for us to meet in 2024. Australia is really easy to get to from all parts of the world. You might want to fly straight to Sydney, or you want to come in via another city and see more of Australia on the way. You might also want to take in some other conferences that happen around the same time in Sydney. And we anticipate it will be in the region in 2024. For example, the Annual Conference of the Mathematics Education Research Group of Australaisa (MERGA) and the Annual Conference of the International Group for the Psychology of Mathematics Education (PME). Coming to ICMI-15 will be a great opportunity to see more value and uniqueness of the country. And finally, I'd like to thank our Congress partners. Our vision for ICME-15 in Sydney is for a vibrant meeting characterized by accessibility and embracing diversity. See you in Sydney.

https://doi.org/10.1142/9789811287152_0125

The Closing Ceremony

The Closing Remarks from the Congress Chair

Jianpan Wang

Distinguished guests,
Dear participants on-site and online,
Ladies and gentlemen,

The 14th International Congress on Mathematical Education is coming to an end. Eight years have slipped away since we submitted our proposal for bidding to ICMI in 2013. Over the eight years, we've been through many zigs and zags and have received much warm help and support. Today, I'd like to extend our sincere thanks to all the people who have kindly offered us their help and support.

From our bidding till today, ICMI has seen three sessions of executive committee, and all the three sessions have high expectations of us and have offered us their utmost support. Therefore, I'd like to thank the three presidents of the three committees. They are (Fig. 1)

- Mr. Ferdinando Arzarello,
- Ms. Jill Adler, and
- Mr. Frederick Leung.

Fig. 1. Three Presidents of ICMI
Left: Ferdinando Arzarello, President 2013−2016, on the ICMI site-visit to Shanghai
Middle: Jill Adler, President 2017−2020, on the first IPC meeting of ICME-14
Right: Frederick Leung, President 2021−1024, on the opening ceremony of ICME-14

It is the three presidents and the three committees under their leadership that make ICME-14 possible.

I'd like to thank International Program Committee of ICME-14. This is an IPC on super-long standby. Since 2016 when it was formed, IPC has held three formal meetings, of which 2 are on-site and 1 online. IPC plays a crucial role regarding our academic decisions. Whenever we ran into questions, IPC members would always give their response promptly and help us out. This kind of interaction between IPC and us last all the way until the opening of the Congress. We are deeply grateful for the professional devotion of IPC!

I'd like to thank Local Organizing Committee and its working team! With a general layout, they do everything carefully and earnestly during the preparation of the Congress, which is arduous and complicated. It is their devotion and great effort that guarantee the completion of the preparation work of the Congress. All the IPC and LOC members present here, please come over to the stage!

The preparation and holding of the Congress have received strong support from central and local government departments. China Association of Science and Technology and the Ministry of Education gave us support from the very beginning. Shanghai Municipal Government has been supporting us at long-term base. Shanghai Municipal Education Commission and Foreign Affairs Office have played critical roles in particular. Putuo Distric Government of Shanghai has also given us support in many aspects.

East China Normal University has been a good host. Starting from the bidding of the Congress, ECNU has been fulfilling its duty of guidance and management over the whole process of the Congress. It is worth mentioning that when the pandemic added more variables to the preparation of the Congress, ECNU adjusted the schedule of teaching and other activities and volunteered to offer the venue for the Congress. Furthermore, ECNU has made necessary repairs and renovation on the venue, and provided support in facilities and human resources.

The functional departments of ECNU have made important contribution to the Congress: during Congress preparation, they offered much help in decision-making, consulting, program management, financial services, and external contact; during the holding of the Congress, they provided all-round guarantee in many aspects including network communication, Congress security, health and epidemic prevention and logistic services. Now let's invite the leaders of ECNU and the heads of functional departments to the stage!

My warm thanks go to ECNU Symphony Orchestra for their high-level concert and ECNU Student Art Troupe for their vivid artistic performance.

Meng Xiancheng College of ECNU gave us support in various forms during the preparation of the Congress. I'd like to thank all my colleagues there. My special thanks go to Ms. Nan Shi, who is the designer of Congress logo and the artistic designer of bidding document and a series of Congress prints as well.

I also want to thank the School of Foreign Languages of ECNU, especially Ms.

Honghong Li, who provided us continuously with language advice and assistance from the very beginning of the bidding up to now.

School of Mathematical Sciences of ECNU is the direct support unit of the Congress in bidding, preparing and holding. It is impossible to enumerate all the work they have done. During the holding of the Congress, a number of faculty members and students adhere to their posts industriously and contribute greatly to the success of the Congress.

Let's invite the leaders and faculty members of School of Mathematical Sciences, School of Music and Meng Xiancheng College to the stage!

I'd like to thank all the volunteers and the graduate students who volunteer for the Congress! Let's invite the representatives of volunteers and graduate students to the stage.

I'd also like to extend our thanks to the companies who have worked for the Congress. They are

- Shanghai DLG Exhibitions & Events (Group) Company, East Star Event Management Company,
- Shanghai Hua-shen Sino-Foreign Cultural Exchange Service Company,
- Shanghai Huada Electric Appliance Company,
- Shanghai AMRTang-Plus Technology Company.

Special appreciation goes to the following units who have supported the Congress financially:

- The International Commission on Mathematical Instruction
- Shanghai Municipal Education Commission
- Chinese Mathematical Society
- East China Normal University
- National Natural Science Foundation of China
- Shanghai High School
- People's Education Press
- East China Normal University Press
- Shanghai Educational Publishing House

The proceedings of the Congress will be published by World Scientific Co. Ltd. of Singapore. We'll make further announcements about the collection and publication of the proceedings. The representatives of the World Scientific are also present today. Let's invite them to the stage.

Dear guests,
Dear participants on-site and online,
Ladies and gentlemen,

Let's extend our most sincere thanks to all the people on the stage and the units they represent! Let's declare the closing of ICME-14 in the music of "Auld Lang Syne" (Fig. 2). May our academic connections and friendship established via ICME-14 bear rich fruit in the future!

Fig. 2. "Auld Lang Syne"

Appendices

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https://doi.org/10.1142/9789811287152_0126

Appendix 1

Hosting Organizations

1. Chinese Mathematical Society

As a member of International Mathematical Union (IMU) and a member of International Commission on Mathematical Instruction (ICMI), the Chinese Mathematical Society (CMS) is the bidding and hosting body of ICME-14, 2020, in Shanghai.

Founded in 1935 in Shanghai, the Chinese Mathematical Society (CMS), a member of the Chinese Association of Science and Technology, is a national academic association for Chinese mathematicians and mathematical educators. CMS aims to unite mathematicians and mathematical educators to promote the research and popularization of mathematics in China and enhance both the development of China's science and technology and the modernization of the country.

The Chinese Mathematical Society has a long tradition of caring about and supporting the research and practice of mathematical education. As early as in 1980, Professor Lokeng Hwa, a world-famous Chinese mathematician and then president of the Chinese Mathematical Society, attended at invitation the 4th International Congress on Mathematical Education (Berkeley, CA, U.S.A.) and gave a plenary lecture. Nowadays affiliated to the Society, there are several committees related to mathematics education, such as the Committee of Popularization, the Committee of Mathematics Olympic Competitions, the Committee of Elementary Education, and the Committee of Higher Education. Besides the mathematics research journals such as Acta Mathematica Sinica, Acta Mathmaticae Applicatae Sinica, and Advances in Mathematics (China), the Society also publishes journals serving mathematics education such as Shuxue Tongbao (Mathematics Bulletin) and Zhongxuesheng Shuxue (Mathematics for Secondary School Students).

2. East China Normal University

Together with the Shanghai Mathematical Society, East China Normal University (ECNU) will take the responsibility of organizing ICME-14.

Aiming to cultivate thousands of qualified teachers for the new-born republic, ECNU was founded in Shanghai in October 1951 by merging several then well-known private or Christian universities such as Great China University, Kwanghua University, and St. John's University. After 65-year's development, in particular, after the latest 38-year's rapid development benefited from the China's reform and opening-up policies, ECNU now is a comprehensive research university. ECNU boasts two State Key Labs, one National Engineering Research Center, one National Field Observation and Research Station, as well as a great number of key labs and research bases approved by the Ministry of Education or Shanghai Municipality, including the Key Lab of Mathematics and Application and the Research Base of School Mathematics, both approved by the Shanghai Municipality. Currently, the university employs 4,331 staffs, academic and non-academic, and enrolls 36,311 students, among them 21,928 being graduate students. Keeping in mind its original goal, ECNU still pays great attention to teacher education and education research, and it is commonly regarded as one of the leading powers of teacher education in China.

Having two campuses located in Minhang and Putuo Districts with a total area of over 207 hectares, ECNU has long been reputed as a Garden University for its beautiful campus scenes.

3. Shanghai Mathematical Society

Established in 1950, the Shanghai Mathematical Society (SMS) is a local academic organization of mathematicians and mathematical educators in Shanghai. The Society provides a platform for academic exchanges for mathematicians and mathematical educators in Shanghai, allowing them to enhance the level of mathematical research and education in Shanghai, as well as to vitalize the economy of Shanghai and to promote the quality of Shanghai citizens.

SMS enjoys good reputation nationally and internationally. It makes great contributions in the fields of pure mathematics, applied mathematics, applications of mathematics, and mathematics education. Chaohao Gu, an academician of Chinese Academy of Science and a former president of SMS was awarded the State Supreme Science and Technology Award in 2009, and the Minor Planet Center of the International Astronomical Union named the Minor Planet 171448 after his name. The Society publishes research journals such as Chinese Annals of Mathematics.

Focusing on mathematical education is an important task of the SMS. For example, it deeply involves in the school mathematics curriculum reform. The society also holds a spare-time school of mathematics, and organizes mathematics competitions on mathematics for school students and for college students. The All these have made very positive influence on the society.

With regard to international academic exchanges, the society not only encourages its members to be more active in relevant activities, but also organizes important academic events. For example, in recent years, the “Chinese-French Symposium on Applied Mathematics”, the “International Conference on Dynamical Systems and Differential Equations” and the “International Conference on Representation Theory” and the “Sino-US Symposium on Mathematics” have been held in Shanghai with the direct support of the Shanghai Mathematical Society. To organize ICME-14 together with East China Normal University is another important international academic event for it. The Society will make all efforts for it.

https://doi.org/10.1142/9789811287152_0127

Appendix 2

Committees

1. International Program Committee (IPC)

Chair:

- Jianpan WANG (ICME-14 Congress Chair, China)

Members:

- Jill ADLER (ex-officio member, ICMI Ex-president, South Africa)
- Abraham ARCAVI (ex-officio member, ICMI Ex-secretary General, Israel)
- Jiansheng BAO (ICME-14 LOC Co-chair, China)
- Daniel CHAZAN (USA)
- Faïza CHELLOUGUI (Tunisia)
- Marta CIVIL (USA)
- Alicia DICKENSTEIN (Former Vice-President of IMU, Argentina)
- Jean-Luc DORIER (ex-officio member, ICMI Secretary General, Switzeland)
- Yufeng GUO (China)
- Anjum HALAI (Pakistan)
- Gabriele KAISER (ICME-13 IPC Chair, Germany)
- Caroline LAJOIE (Canada)
- Frederick K. S. LEUNG (ex-officio member, ICMI President, Hong Kong SAR, China)
- Celi Espasandin LOPES (Brazil)
- Thomas LOWRIE (Australia)
- Maria Alessandra MARIOTTI (Italy)
- Takeshi MIYAKAWA (Japan)
- Frode RØNNING (Norway)
- Ewa SWOBODA (Poland)
- Luc TROUCHE (France)
- Catherine VISTRO-YU (Philippines)
- Binyan XU (ICME-14 LOC Co-chair, China)
- Ivan YASHCHENKO (Russia)

2. Local Organizing Committee (LOC)

Co-Chairs:

- Binyan XU (East China Normal Universty)
- Jiansheng BAO (East China Normal University)

Secretary General:

- Yingkang WU (East China Normal University)

Members:

- Yiming CAO (Beijing Normal University)
- Jun CHAI (East China Normal University)
- Yifei CHEN (Chinese Mathematical Society)
- Yuelan CHEN (East China Normal University)
- Jing CHENG (East China Normal University)
- Lianghuo FAN (East China Normal University)
- Zhigang FENG (Shanghai High School)
- Fuzhou GONG (Chinese Mathematical Society)
- Yijie HE (East China Normal University)
- Hua HUANG (Teaching Research Department, Shanghai Municipal Education Commission)
- Qiping KONG (East China Normal University)
- Honghong LI (East China Normal University)
- Di LIU (East China Normal University)
- Xiaoli LU (East China Normal University)
- Ming NI (East China Normal University Press)
- Naiqing SONG (Southwest University)
- Shengli TAN (East China Normal University)
- Jialu WANG (East China Normal University)
- Xiaoqin WANG (East China Normal University)
- Bin XIONG (East China Normal University)
- Yijun YAO (Shanghai Mathematical Society)
- Jianyue ZHANG (People's Education Press)
- Jinyu ZHANG (Minhang Institute of Education, Shanghai)
- Yan ZHU (East China Normal University)
- Jiachen ZOU (East China Normal University)

https://doi.org/10.1142/9789811287152_0128

Appendix 3

Logo of ICME-14[1]

The original version of the logo of ICME-14 is shown in Fig. 1.

Fig. 1. Original version of the logo of ICME-14

The basic idea comes from Hetu (The River Map) in ancient China. Hetu, together with Luoshu (The Luo Writing), is commonly regarded as the origin of the Chinese civilization. The *Book of Changes* (*I Ching*) indicated that “The Yellow River gave forth the Map, the Luo River produced the Writing, and from them Saint Fuxi got the idea of the trigrams.” Many notions in Chinese traditional culture such as Taiji, Eight Trigrams, Fengshui may all originate from them. Hetu and Luoshu include mathematical contents such as the classification of numbers by their parity, the arrangements of numbers with equal differences or equal sums, as well as magic squares.

They are essentially the plain understandings of mathematics by people at that time. Hetu is also drawn as a round-styled picture, as shown on the left part of the ancient pot in Fig. 2.

The round-styled Hetu is used as the mould for the logo. In the logo, the chordal graph in the center can be regarded as the combination of five small geometric figures (four congruent right triangles and one square), replacing the five points in the center

[1] The logo was designed by Jianpan Wang and Nan Shi.

of the Hetu. The circle outside the chordal graph stands for the circle with ten dots in the Hetu. The two helix-shaped cantilevers in blue and red respectively circumscribed to the circle represent the ranges of yin numbers (even numbers, 2, 4, 6, 8) and yang numbers (odd numbers, 1, 3, 7, 9) rotating clockwise starting from the south (up) direction and north (down) direction respectively. Here, we only highlight and draw the yin dots representing number 2 and the yang dots representing number 7 in the south (up) direction.

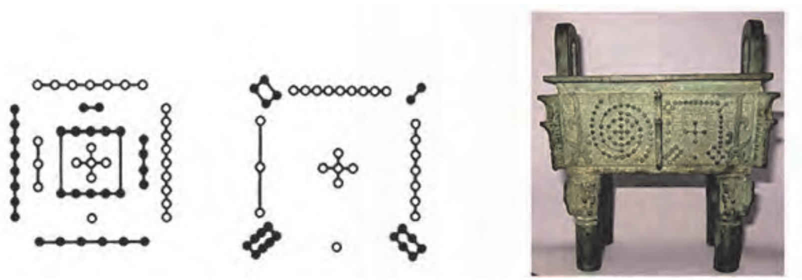

Fig. 2. Hetu (Left), Luoshu (Middle), Ancient pot with Hetu and Luoshu (Right)

The chordal graph is a perfect proof of the Gougu Theorem (known as Pythagorean Theorem in the west) proposed by Zhao Shuang, a mathematician during the Three Kingdoms period. It is now the logo of Chinese Mathematical Society. Therefore, it represents both the tradition of Chinese mathematics and mathematics education, and Chinese Mathematical Society, the hosting body of the Congress.

The two cantilevers symbolize that China is opening her arms to embrace participants from all over the world. It also shows China’s opening-up attitude.

The product of 2 and 7 is 14, indicating the session of this Congress.

At the lower right corner of the centerpiece picture under the “ICME-14” four Chinese traditional trigrams (guas) are used to write down number 3744 in octal system, which is 2020 in decimal system, indicating the year in which the Congress will be held. In addition, the binary code of “2020” can be read from the four trigrams: (0)11111100100. The octal system and the binary system connect the brilliant civilization of ancient China with modern science and technology.

Mathematical elements are extensively disseminated in the logo: the Gougu Theorem, even and odd numbers, the octal number system and the binary system, and so on. They are not only the achievement of ancient China, but also the content of teaching in modern elementary and secondary schools. The design is very geometric, particularly the centerpiece picture that consists of circles and helixes is centrally symmetric.

The use of helixes also represents the concept of spiral rise in modern teaching theory.

The centerpiece picture assumes the shape of “S”, meaning Shanghai, the city where the Congress will be held. Its momentum of moving forward indicates our proactive attitude.

Due to the pandemic of COVID-19, ICME-14 had to be postponed from 2020 to 2021, and was changed from a complete physical conference to a hybrid conference. For this reason, we changed the bottom line of the rightmost trigram from “Break” to “Connect”, and indicated in red to emphasize that this was a necessary change. In this way, the four trigrams in the lower right corner of the new logo represent the octal 3745, or the binary (0)11111100101, which thus representing 2021 instead. The final version of the logo of ICME-14 is shown in Fig. 3.

Fig. 3. The final version of the logo of ICME-14

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https://doi.org/10.1142/9789811287152_bmatter

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Proceedings of the

14th International Congress on Mathematical Education

Volume II: Invited Lectures

Editor

Jianpan Wang

East China Normal University Press

World Scientific

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14^{th} International Congress on Mathematical Education

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Proceedings of the

14th International Congress on Mathematical Education

Volume II: Invited Lectures

The 14th International Congress on Mathematical Education

Shanghai, China, 2021

Editor

Jianpan Wang

East China Normal University, China

East China Normal University Press

World Scientific

Published by

East China Normal University Press
3663 North Zhongshan Road
Shanghai 200062
China

and

W

5 Toh Tuck Link, Singapore 596224
USA 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Control Number: 2024002523

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

PROCEEDINGS OF THE 14TH INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION
(In 2 Volumes)
Volume I
Volume II: Invited Lectures

ISBN 978-981-12-8937-8 (set_hardcover)
ISBN 978-981-12-8716-9 (set_ebook for institutions)
ISBN 978-981-12-8719-0 (set_ebook for individuals)

ISBN 978-981-12-8714-5 (vol. 1_hardcover)
ISBN 978-981-12-8715-2 (vol. 1_ebook for institutions)

ISBN 978-981-12-8717-6 (vol. 2_hardcover)
ISBN 978-981-12-8718-3 (vol. 2_ebook for institutions)

For any available supplementary material, please visit
https://www 142/13700#t=suppl

Printed in Singapore

https://doi.org/10.1142/9789811287183_fmatter

Editor's Notes

This is the second volume of two-volume Proceedings of the 14th International Congress on Mathematical Education (ICME-14), held in Shanghai, China, from July 11−18, 2021.

This volume collects 50 papers on invited lectures. Invited lectures (called regular lectures on the 12th and earlier International Congress on Mathematical Education) were selected by the International Program Committee (IPC) of an ICME according to lecturers instead of themes of lectures.

In the 1st IPC Meeting of ICME-14 (September 11–16, 2017), 83 invited lecturers were nominated. However, only 70 of the nominees accepted the invitations before the 2nd IPC Meeting (March 27–29, 2019), 10 invitations were rejected while 3 did not receive reply. The 2nd IPC Meeting decided to send another invitation to the three nominees who did not respond, while considering geographical balance, adding a new nominee. Four new invitations receive 3 positive replies. Thus, there were 73 invited lecturers in total after the 2nd IPC meeting.

Unfortunately, the sudden pandemic of COVID-19 disrupted the preparatory process of the Congress, and the Congress had to be postponed for one year to the summer of 2021. Some of the invited lecturers were unable to attend the postponed ICME-14 due to various reasons. As a result, we had 60 invited lectures present, on site or online, in the ICME-14, according to the conference video recording.

Among the 60 invited lecturers, 50 submitted their papers to the Proceedings of 14th International Congress on Mathematical Education. The 50 papers collected in this volume were arranged in alphabetical order based on authors' last names.

Though the themes of invited lectures are not decided by the organizers of the Congress, this volume is a good collection of papers on mathematics education with broad academic coverage and deep academic concerns. In addition, in terms of geographic distribution, it is also satisfactory. The numbers of speakers from different continents are: Africa 4, Asia 18, Europe 14, North America 9, South America 3, and Oceania 2.

I, as the Congress chair and IPC chair of ICME-14, deeply thank all scholars from around the world who involved themselves to the event of Invited Lectures, whether or not you have ultimately presented lectures or submitted papers. I also thank all the participants of ICME-14. No matter which stage or activity you participated in, your participation is an important guarantee for the success of the Congress. May the special experience of ICME-14 leave a deep impression on you, and I also hope that you would have the opportunity to visit or revisit China to make up for the regret of not being able to attend ICME-14 in person and attending online instead.

Jianpan Wang
Shanghai
January 2024

https://doi.org/10.1142/9789811287183_fmatter

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Appendix

https://doi.org/10.1142/9789811287183_0001

01

Embodied Design: Bringing Forth Mathematical Perceptions

Dor Abrahamson[1]

ABSTRACT Embodied design is a proactive educational research program that promotes and investigates humans' universal capacity to understand STEM concepts. The program's empirical work is centered on design-based research projects that contribute theory to the Learning Sciences through the practice of building, implementing, and evaluating experimental pedagogical architectures that inform instructional practice. Using both historical and emerging technologies, embodied-design activities are typically two-stepped: (1) draw on students' evolutionary inclination for purposeful sensorimotor engagement with the natural environment; and only then (2) introduce heritage symbolic artifacts that students initially adopt to enhance the enactment, evaluation, or explanation of their intuitive judgments and actions, yet, in so doing, find themselves adopting normative disciplinary forms, language, representations, and solution procedures. Embodied-design researchers apply mixed methods — from ethnomethodological conversation analysis through to multimodal learning analytics and cross-Recurrent Quantification Analysis — in analyzing empirical data of learning process, including records of students' motor actions, sensory behavior, and multimodal utterance in conversation with peers and instructors. Several decades of projects across numerous mathematical content domains have increasingly implicated *perception* — a hypothetical Psychology construct believed to govern sensorimotor and cognitive behavior — as pivotal in explaining students' capacity to first solve challenging motor-control coordination problems and then bridge through to discursive articulation of their movement strategy. As they attempt to operate the educational technology according to an unknown interaction regimen, new information patterns, e.g., an imaginary line connecting their hands, come forth spontaneously into students' perceptual experience as their cognitive means of managing the enactment of the activity's targeted movement forms. These emergent, proto-mathematical, multimodal, dynamical ontologies are then languaged and entified into consciousness, grounding the meaning of conceptual terminology and procedural routines. The embodied-design framework has been applied in building technologies for students of intersectional diversity, including populations of minoritized epistemic — linguistic practices and atypical neural, cognitive, and sensorial capacity.

Keywords: Attentional anchor, Enactivism; Mathematics Imagery Trainer; Movement; Technology.

[1] Graduate School of Education, University of California Berkeley, 2121 Berkeley Way, Berkeley, CA 94720–1670, USA. E-mail: dor@berkeley.edu

1. Introduction to Embodied Design

1.1. Objectives, disciplinary foundations, inspirations, and ethical positionality

Embodied design (Abrahamson, 2009, 2012, 2014, 2015) is a research quest to understand what it means to learn a mathematical concept. We ask the ontological question, what is a mathematical concept that we can understand it, and we ask the epistemological question, what is the mind that it can understand a mathematical concept? Operating from a broad reading of the cognitive sciences, we address these grand questions through examining how people teach and learn together in activities centered on technological artifacts we build and develop. In our theorizing, design craft, and data analyses, we are chiefly informed by the embodied paradigm shift in the cognitive sciences, which has foregrounded the formative role of situated sensorimotor interaction in the phenomenology of conceptual reasoning (Newen et al., 2018).

The disciplinary affiliation of embodied design is the Learning Sciences, a field of study developed in the 1980's by cognitive scientists wishing to apply their theories and methods to empirical investigations of educational practice. A premise of the Learning Sciences is that researchers should assume a transformative orientation toward educational problems — they should not only document, diagnose, and denounce these problems (e.g., the "misconceptions" genre) but dismantle and ameliorate the phenomena by way of building and evaluating theory-based alternatives. This quest to engineer better educational practices was called *design experiments* (Collins, 1992). With time, the name evolved into *design-based research* (Cobb et al., 2013) or, variably, just *design research* (Bakker, 2018). In its ethical foundations to improve extant cultural practices, design-based research aligns well with revisionist readings of foundational tenets driving Lev Vygotsky's cultural — historical psychology: Culture is taken not as a *status quo* but in its very essence as a system in flux that necessarily requires continuous adaptation to avail of envisioned opportunities and counter emergent contingencies (Stetsenko, 2017). Embodied design work is always conducted as design-based research studies (Abrahamson, 2015).

Embodied design is inspired by educational visionaries, from Friedrich Fröbel, Maria Montessori, Caleb Gattegno, and Hans Freudenthal through to Seymour Papert, Mitch Resnick, Uri Wilensky, and Ricardo Nemirovsky, whose pedagogical artifacts creatively utilize technology to empower young learners. Originating in the University of California Berkeley at the Embodied Design Research Laboratory, embodied design is now pursued by collaborators and colleagues worldwide (Abrahamson et al., 2020). While most embodied design projects to date have addressed mathematical concepts, its framework caters more broadly to STEM domains (Abrahamson and Lindgren, 2014). The embodied-design framework has been applied to a range of concepts (Alberto et al., 2021) to serve students of intersectional diversity, including populations of minoritized epistemic — linguistic practice (Benally et al., 2022), atypical neural, cognitive, and sensorial capacity (Lambert et al., 2022; Tancredi et al., 2021), and at remote locations (Shvarts and van Helden, 2021).

1.2. Grounding conceptual meaning in perceptual phenomenology

Embodied design is a design-based research effort that includes a design framework mobilizing its research agenda. The embodied-design framework informs the creation of learning environments, where students construct the meaning of mathematical concepts and procedures. The embodied-design research agenda is to understand how students construct mathematical meanings in these environments.

By "construct" we draw on the theories of genetic epistemology (a.k.a., "constructivism," Piaget, 1971), radical constructivism (Steffe and Kieren, 1994), and enactivist cognition (Varela et al., 1991) to stipulate students' active role in making sense of the world through goal-oriented embodied engagement. As Piaget (1971) writes,

> Knowing does not really imply making a copy of reality but, rather, reacting to it and transforming it (either apparently or effectively) in such a way as to include it functionally in the transformation systems with which these acts are linked (p. 6).

By "meaning," in turn, we refer to a presymbolic notion (Radford, 2014) — a phenomenological orientation toward engaging the world purposefully that lends a sense of understanding for a mathematical sign, such as the notation "+" symbolizing the arithmetic operation of addition. The meaning of "+" might be experienced as bringing the hands toward each other to accumulate substance, whether one actually enacts this movement form or imagines doing so. This bimanual "image making" (Pirie and Kieren, 1989) or "concept image" (Tall and Vinner, 1981) associated with the notation "+" is experienced as a non-linguistic dynamic bodily feeling of acting on the world — the embodied experience grounds the mathematical symbol in sensorimotor phenomenology (Harnad, 1990). Put colloquially, the meaning of a mathematical concept is not inside the signs we read or write — it's what we experience when we first sense that we got its core idea, it clicked for us, we grasp it, we have a grip on it, we own it, we can improvise on it. But embodied design maintains that we can develop a new grip on the world even before we appreciate that it will become mathematically meaningful (Bartolini Bussi and Mariotti, 2008; Nathan, 2012; Vogelstein et al., 2019). This makes sense developmentally — we learn to add stuff with our hands long before we know the word "add" (L. B. Resnick, 1992); years before doing so lends meaning to the arithmetic idea of addition (Silverman, 2021).

Embodied designs are necessary, because mainstream education may not occasion opportunities for students to develop canonical dynamical image perceptions as the core proto-mathematical meanings grounding their conceptual understanding. The research program of embodied designs is motivated by a concern for students' general "absence of meaning" (Thompson, 2013) for mathematical concepts, which we diagnose as the absence of enactive capacity to understand the concepts. Embodied designs create the socio-material conditions for students to learn a mathematical concept by developing capacity to enact the movement form that later becomes the meaning of the targeted concept's inscriptional markings. For example, what might be a presymbolic enactment of "proportion" that would be analogous to the bimanual enactment of "addition" discussed above? Most people are absent an enactive meaning for "proportion" — they are hard pressed to enact the concept, to gesture it. How do you grasp a proportion and mobilize it? What might be a dynamical invariant that you enact and maintain as you move in proportion to think through it, talk about it, teach it? And how about a parabola? A sine function on the unit circle? As we now explain, to develop enactive capacity is to develop new ways of attending perceptually to the world for organizing the enactment of movement forms.

1.3. Perception: The cognitive pivot from phenomenology to language

According to Varela et al. (1991), "the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (p. 173).

Embodied design emulates the enactive maxim by presenting students with motor-control problems whose dynamic solution requires discovering sensorimotor patterns from whence emerge proto-mathematical cognitive structures. For example, a cognitive structure grounding the mathematical concept of proportionality emerges from solving a motor-control problem whose solution is raising the hands simultaneously at different constant speeds above a common surface. Try this. Place both hands palms down on your desk, and now raise them both at once, perhaps with the right hand moving double as fast as the left. Appreciate the strangeness of this movement form and the challenge of enacting it. How are you accomplishing this task? What sensory modality are you attending to? What are your criteria for maintaining the dynamical form? What have you figured out? We submit that learning to enact movement forms is where meaning is potentiated for mathematical concepts. Still, what exactly is the role of perception in performing this bimanual movement? Why do developmental psychologists and enactivist philosophers implicate our natural perceptual faculty as soliciting the mental construction of new cognitive structures from recurrent sensorimotor behavior? And how could doing all this become mathematics?

Empirical research in the movement sciences has demonstrated that the human capacity to enact challenging bimanual movements, such as lifting the hands at different speeds, is achieved by developing new perceptual orientations towards the activity situation (Mechsner, 2004). In the absence of appropriate perceptual orientation, a task may appear daunting, even insurmountable, and yet once the perceptual orientation has been established — through exploration, guidance, or some mix thereof — the impossible task becomes manageable. What more, one is often able to articulate how one is orienting perceptually toward a situation, such as when we teach a novice how to parallel-park, flip an omelet, or finesse a crochet stitch. Teachers are particularly good at explicating their expertise (Newell and Ranganathan, 2010; Shulman, 1986), and doing so often involves highlighting for the novice within a shared situation certain embedded forms that the expert discerns but the novice does not (Flood et al, 2020; Goodwin, 1994).

A given bimanual movement form may be perceptually guided in a variety of ways. For example, to raise your hands at different speeds, you might attend to the vertical spatial gap between the hands and keep increasing that gap; you might ensure that the right hand is always double as high above the surface as the left hand; or that the left hand is always half as high as the right; and so on. That is, the phenomenology of performing a bimanual movement form may vary, and each variation instantiates a different mathematical model of the movement form. Calling movement forms “polysemous,” Abrahamson et al. (2014) demonstrated that coordinating among these models may lend new conceptual insight.

Note the ontological difference between the movement form as described by a third person, for example, “Dor is raising his hands such that his right hand is moving twice as fast as his left hand,” and the individual’s first-person experience, for example, “The vertical gap between my lower left hand and my upper right hand should always be equal to the height of my left hand over the surface.” Embodied designers are interested in foregrounding the first-person experience — soliciting, characterizing, and documenting its variability across students — because we believe that talking and gesturing about these experiences can improve both research and practice (Abrahamson et al., 2022).

Embodied design begins from our species’ universal capacities for thriving in natural and cultural ecologies. Being the biological organisms that we are, we are evolutionarily inclined to solve the existential problem of learning to perform new movements, whether walking, waltzing, or weaving, by discovering task-effective sensorimotor patterns — the how of attending to a situation. This natural neural proclivity to develop new perceptual orientation toward the environment as a means of operating on it can be solicited in fields of promoted action (Reed and Bril, 1996), social interventions that foster the development of culturally valued movement forms. Once novices figure out how to move in a new way, they can be encouraged to verbalize how they are perceiving the situation, which, under appropriate pedagogical settings, may lead to normative disciplinary discourse, including performing various inscriptional routines. Thus, cognitive structures that enable perception to guide action emerge as ontologies grounding mathematical concepts. It is in this sense that embodied design enables students to construct mathematical meaning from perception.

Having explained the rationale and theoretical underpinnings of embodied design, we now turn to discussing findings from research studies that evaluated activities built according to the framework. At the center of these activities is a type of pedagogical interaction architecture called the Mathematics Imagery Trainer.

2. The Mathematics Imagery Trainer

2.1. Rationale and build

The Mathematics Imagery Trainer (hence, the Trainer) is an activity architecture designed to serve as an instrumented field of promoted action (Abrahamson and Trninic, 2015) — a technological apparatus for administering embodied-interaction activities, in which students learn to participate in the physical enactment of an epistemic practice. Students are invited to solve a motor-control problem, where they manipulate virtual objects in an attempt to change the state of the environment, for example to cause a red screen to become green and then stay green as they keep moving the objects. That is, students learn to move in a particular way that is coded into the Trainer's digital feedback regimen, for example to lift their hands at the speed ratio of 1:2, where the right hand rises double as fast as the left (see Fig. 1). Learning to move in this new dynamical form is challenging, because the feedback regimen frustrates students' existing repertory of sensorimotor schemes for interacting with the environment. For example, they may try to raise their hands at the same speed, only to be repeatedly countenanced (red screen), so that they must readjust their hands' positions. To assimilate the feedback regimen of the obdurate environment, students must accommodate their schemes (Abrahamson et al., 2016). They learn to move in a new way — an ecologically coupled way (Abrahamson and Sánchez–García, 2016).

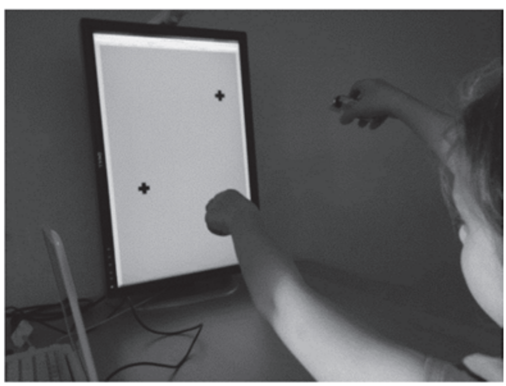

Fig. 1. The Mathematics Imagery Trainer for Proportion set at 1:2. A child is manipulating two virtual objects. The right-hand object is twice as high above the bottom of the screen as compared to the left object. This spatial configuration of the two objects relative to each other satisfies the task of making the screen green. To move her hands in constant green, the child would need to keep this ratio. She will learn to move in a new way by attending to a new information structure.

The new movement forms that students learn to perform have been designed as "conceptual micro-choreographies" (Abrahamson and Sánchez–García, 2016), in the sense that these dynamic forms bear semiotic potential (Bussi and Mariotti, 2008) to become mathematically meaningful through quantitative modeling. The semiotic consolidation of movement as mathematics is then ushered by making available to students a variety of symbolic artifacts (Sfard, 2002), such as a grid of lines laminated onto the activity space (see Fig. 2). Students recognize in these new resources potential instruments for enhancing the enactment, explanation, or evaluation of their effective movement strategy. Consider students who'd been raising the two virtual objects simultaneously while attending to the vertical gap between the objects (Fig. 2b). They say, "The higher the hands go, the bigger the distance between them" (Abrahamson et al., 2011). When the grid is flashed onto the screen (Fig. 2c), the students initially attempt to replicate this same strategy for enacting the movement form that had been satisfying the task conditions. Yet, as they raise their hands now, a horizontal line affords a convenient specified location to "park" one of the virtual objects, while the other hand searches for its complementary location that makes the screen green. As such, the sensorimotor pattern that had solved the motor-control problem of making the screen green becomes distributed over the environment, so that the students find themselves drawn into a new sensorimotor pattern, where the hands are moving sequentially, ratcheting up the lines. They say, "For every 1 line I go up on the left, I go up 2 lines on the right" (Abrahamson et al., 2011). Thus, as they engage the utilities that they discern in the new accessories to improve their grip on the world, students transition into enacting new movement forms that incorporate the symbolic artifacts. In so doing, the students appropriate quantitative frames of reference, so that their utterance takes on the linguistic forms of normative disciplinary discourse (Abrahamson and Bakker, 2016).

2.2. Attentional anchors

How does an invisible spatial interval between two virtual objects suddenly avail itself as a perceptual means of managing a challenging motor-control task? The embodied-

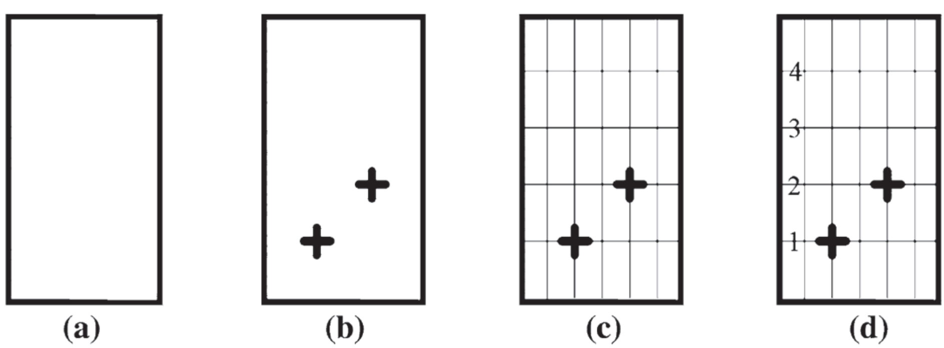

Fig. 2. From movement to mathematics — interpolating symbolic artifacts into students' activity space brings about transitions in acting, thinking, and speaking: (a) when the screen shows no virtual objects, students focus on their hands; (b) introducing virtual objects draw students' attention to the screen, where they explore for movement forms that sustain the favorable feedback; (c) supplementing a grid changes the activity space from continuous to discrete — students incorporate the lines as a frame of reference and develop a unitized movement form; and (d) further supplementing numerals solicits students' arithmetic skills, enabling them to calculate and predict right–left locations satisfying the feedback regimen.

design learning process depends on this intriguing perceptual phenomenon — if students didn't "mind the gap" between the objects, they could not develop a new cognitive structure that would enable them to enact the movement form that solves the problem; they could not articulate their solution; and they could not then transition to mathematical models. Yet how should we theorize this figment of perception for coordinating the motor actions of two independent limbs?

It turns out from Movement Sciences that: (a) the sensory and motor faculties are neurally intertwined and mutually constraining — we constantly grope for a better grip on the world by moving to sense, sensing to move (Fiebelkorn and Kastner, 2019); (b) sensorimotor activity is a complex system in flux, with new dynamic stabilities self-organizing adaptively to changing environmental contingencies (Chow et al., 2007; Kostrubiec et al., 2012); and (c) the mind relentlessly yet tacitly searches for, and generates new Gestalt structures to serve as perceptual means of organizing dexterous manipulation (Mechsner, 2004). That is, *we are evolutionarily inclined to complement our raw sensation of the phenomenal world with imaginary auxiliary constructions that facilitate its manipulation*. These Gestalts are the cognitive structures that emerge by and for task-driven, explorative actions, enabling action to be perceptually guided.

Enactivist philosophers call these emergent cognitive resources attentional anchors. *Attentional anchors* are perceptual orientations toward the environment that come forth through exploration and guidance as our means of accomplishing the sensorimotor enactment of complex movement forms (Hutto and Sánchez–García, 2015). Attentional anchors are information structures that we groom forth from the lived environment as affording our task-effective action; once detected, we thereafter iteratively adjust our actions to maintain our perceptual hold of those structures that, reflexively, enable us to act on the world (Abrahamson and Sánchez–García, 2016).

The very type of emergent structures that let us ride a bicycle, pole-vault, juggle props, or play a viola arpeggio could serve us in getting a grip on mathematics (Abrahamson, 2021; Hutto, 2019), albeit it takes an appropriate learning environment (Abrahamson and Sánchez–García, 2016; Hutto et al., 2015). It is thus, we believe, that theories of embodied cognition may inform the practice of mathematics education (Fugate et al., 2019; Shapiro and Stolz, 2019). We now take a closer look at practice.

2.3. Learning with the Trainer: from movement to mathematics

Drawing on research conducted by Utrecht University researchers of embodied design (Bongers, 2020; Bongers et al., 2018; Duijzer et al., 2017), this section elaborates on Trainer learning trajectories.

The activity begins by presenting the student with a bimanual motor-control problem. Here the student is working on an Orthogonal Proportion task. She is guided to manipulate the orthogonal dimensions of a rectangle, which initially is red (see Fig. 3a):

Fig. 3a. A Mathematics Imagery Trainer tablet activity. Initially, the manipulated geometrical figure, a rectangle, is colored red, because its selected dimensions do not comply with the yet-unknown specifications.

Fig. 3b. Reconfigured at a 1:2 height-to-width ratio, the rectangle turns green. Next, both hands must move simultaneously to keep the rectangle green while changing its dimensions.

Her left-hand (LH) index finger slides the rectangle's top-left vertex up/down along the y-axis to change its height, and her right-hand (RH) index finger slides the rectangle's bottom-right vertex right/left along the x-axis to change its width. The student is tasked first to make the rectangle green and, once that is accomplished, to keep moving the two vertices at the same time whilst keeping the rectangle green. The rectangle is green when the quotient of its height/width measured values is some yet-unknown constant number, for example 5 (see Fig. 3b). As such, once a green rectangle is generated, moving forward its dimensions must be adjusted simultaneously to maintain the rectangle continuously in its particular preset green aspect ratio.

In the course of solving Orthogonal Proportion problems, study participants typically develop some new Gestalt to coordinate moving their LH–RH fingers simultaneously at different rates along orthogonal paths. For example, Lars (see Fig. 4a) worked on a variant problem, where he was tasked to move cursors along orthogonal axes in the absence of a rectangle. When Lars achieved fluent movement in green, he was asked to explain his method. Lars said he was attending to an imaginary diagonal line connecting the cursors. The color blots in the images are post-production data-visualization overlays marking the location of Lars's foveal eye gaze. Soon after (see Fig. 4b), Lars demonstrated how he moves the diagonal line to the right.

Fig. 4a. Lars, a 14 years-old low-tracked prevocational-education student, gestures an imaginary diagonal line he perceives as connecting his LH and RH points of contact on the axes.

Fig. 4b. Lars uses his emergent attentional anchor to guide proportional bimanual coordination: He moves sideways the imaginary diagonal subtended between his fingertips.

The eye-gaze markers indicate that he is no longer foveating on his fingers but, rather, near the center of the LH–RH diagonal lines. As you scan the five photographs in Figure 4b, note the successive locations of the eye-gaze marker: Curiously, Lars's gaze path, as he imagines the successive LH–RH diagonals, runs along a different diagonal line — a diagonal trajectory from the origin (on the bottom left) and up to the right that describes a y = .5x function. This emergent foveal trajectory is a secondary attentional anchor. Lars's diagonal solution was quite typical. Yet, across participants, we found evidence for a variety of attentional anchors, such as gazing at the imaginary top-right corner closing a rectangle composed of two axial segments subtending between each fingertip and the origin and two lines from the fingertips to the imaginary point (see Duijzer et al., 2017, for the array of attentional anchors recurring across participants).

Once students have achieved a pre-specified criterion of minimal performance level, the activity proceeds with the teacher — who may be either a human or a virtual pedagogical avatar (Abdullah et al., 2017) — introducing onto the activity space supplementary resources designed to steer students to develop quantitative re-articulations of their movement forms. For example, Fig. 5 shows the presentation of a grid (Fig. 5a) and then numbers (Fig. 5b) onto the tablet interface.

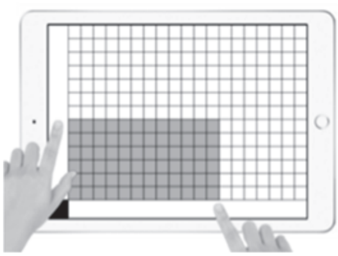

Fig. 5a. A grid is overlaid onto the movement space. The continuous space thus becomes discrete, affording the enumerative quantification of, and reference to uniform spatial intervals.

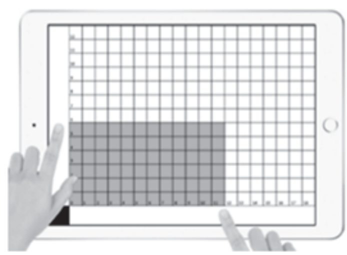

Fig. 5b. Numerals are supplemented onto the grid. Strategies of iterative manual incrementation are substituted by explicit arithmetic functions enabling multiplicative prediction of green rectangles.

Undirected, students count grid lines or units corresponding to their actions and, thus, are able to: (1) describe their strategy quantitatively; (2) draw on their arithmetic skills; (3) confirm the veracity of their strategy; (4) determine with greater precision the location and trajectory of the attentional anchor; (5) enact the movement form correctly independent of the color feedback; and (6) predict properties of yet-unenacted geometrical shapes satisfying the interaction regimen.

Students are now equipped with quantitative rules derived from the tablet activity, so that, given a new "green" geometric shape, they are able to calculate a set of additional "green" shapes. The lesson activity now disengages from the tablet and turns to paper. Fig. 6 demonstrates a paper-and-pen activity, where the geometrical form presented to the students "materializes" the imaginary diagonal attentional anchor, which they had generated imaginatively on the tablet as their means of solving the tablet interaction problem of moving in green. Students are asked to use the pen to show what would be other "green" triangles. As students engage the paper-and-pen offline tasks (see Fig. 6a), they no longer have recourse to immediate real-time interactive feedback on the quality of their performance. Nevertheless, the students now have a formalized rule for generating additional instances of the new equivalence class, which has yet to receive a mathematical name. Figures 6b–d demonstrate students' creative technical strategies, using available resources, for creating new lines running between the *y*-axis and *x*-axis parallel to the hypotenuse of the given triangle.

Fig 6a. A sheet of paper showing a starter shape is placed directly on the tablet screen.

Fig. 6b. Anna places an available sheet of paper alongside the triangle's hypotenuse.

Fig. 6c. Anna slowly slides the page away, keeping it parallel to the hypotenuse.

Fig. 6d. Using the sheet of paper as a straightedge, Anna draws a parallel line.

In Fig. 7, Bongers et al. (2018) deftly illustrate the study participants' typical quantitative strategies for generating further "green" diagonals on paper. Both the with the virtual grid. With that, the tablet-based perceptual strategy of handling animaginary Gestalt has materialized as a paper-based geometrical strategy of generating a set of "green" diagonals. The lines' mutual affinity — what makes them satisfy the tablet-based task. Yet, now on paper, the lines' setness in turn draws also on new perceptual criteria — their salient parallelism and the similitude of the triangles they configure.

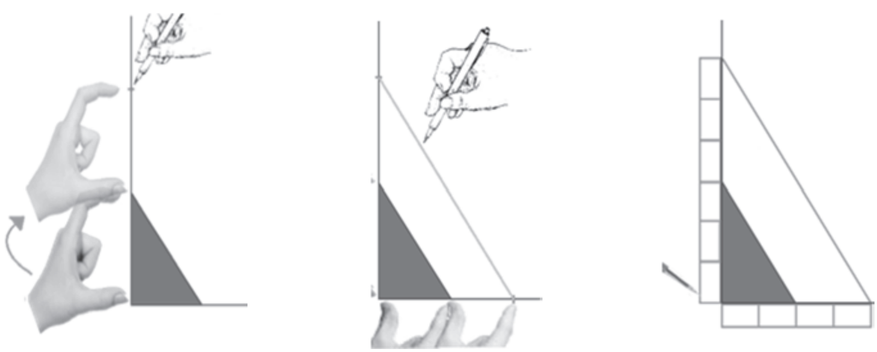

Fig. 7a. A participant gauges a vertical span, transports it upwards to form an equivalent concatenated span, and marks its reach.

Fig. 7b. The participant next performs analogous actions along the horizontal span.

Fig. 7c. The participant draws units alongside the triangle legs, then extends 3 and 2 units, respectively, along the vertical and horizontal legs.

A set of triangles thus produced through rule-base iterated co-expansion of the legs (e.g., 3-per-2 in Fig. 7) is named as bearing the mathematical quality of "proportionality."

With that, we have demonstrated the evolution of a mathematical concept grounded in an attentional anchor: (1) from a personally experienced *ad hoc* imaginary percept that emerges spontaneously to organize the sensorimotor enactment of a movement solution in an assigned motor-control problem; through guided discourse, (2) into a publicly evoked qualitative ontology in the form of co-speech hand gestures adumbrating the imaginary line for the interlocutor; then becoming (3) a quantitative ontology pinned onto a frame of reference that enhances performance, calculation, and prediction; next, denoted (4) as the contour of a sheet of paper that indexes the prospective location and form of a linear inscription; and then materialized (5) as an actually inscribed line on paper, along with diagrammatic and symbolic labels and multimodal quantitative explanations for the diachronic and contextual meanings of this line. As such, we wish to detail the cascade of semiotic actions by which subjectively experienced perceptual structures that come forth to facilitate motor action are endorsed into mathematical discourse that imbues and articulates the structures with conceptual meaning by implicating their quantitative invariance.

3. Closing Words

The embodied-design research program speculates that "If you can't move it, you don't get it." That is, one's understanding of a mathematical concept begins at the point where one can enact a movement form that, per experts, instantiates the concept. Yet to enact a new movement form, one must attend in a new way to the environment, including one's body. That is, to perform a conceptual choreography, we must detect in the environment an information structure whose maintenance facilitates, enhances, and regulates our grip on the world (Abrahamson, 2021; Abrahamson and Sánchez–García, 2016). In turn, our mimetic capacity to reflect on our own actions (Donald, 1991; Piaget, 1971) enables us to surface these tacit forms in multimodal language and formalized inscription (Donald, 2010; Malafouris, 2013). The Mathematics Imagery Trainer constitutes an instrumented field of promoted action guiding this micro-genesis of movement into mathematics.

Trainer studies have generated empirical data enabling researchers to investigate, corroborate, and extend with unprecedented precision longstanding tentative tenets from seminal theories of cognitive development, including Varela's enactivist cognition (Hutto et al., 2015), Piaget's reflecting abstraction (Abrahamson et al., 2016), Vygotsky's zone of proximal development (Shvarts and Abrahamson, 2019), Araújo's ecological dynamics (Abrahamson and Sánchez–García, 2016), and Vérillon and Rabardel's instrumented activity theory (Shvarts et al., 2021). Quantitative analyses of students' motor and sensory activity have enabled the research collaboration to pioneer the demonstration of conceptual phenomenology as perceptual assembly of sensorimotor behavior (Abdu et al., 2023; Tancredi et al., 2021).

Embodied-design activity architectures are “pan-media,” in the sense that they can be implemented in a range of human — computer interaction platforms. As such, the Trainer can cater to students of diverse sensory capacities and needs. For example, Trainers have been built for sighted students’ remote-action (Howison et al., 2011) or hands-on tablet manipulation (Abrahamson et al., 2011) yet also for enhanced accessibility (PhET, 2021), including haptic devices for students who are blind or visually impaired (Lambert et al., 2022).

As we enter the systemic era in theorizing mathematics education (Abrahamson, 2015), we foresee increasing adoption of constructs and methods from dynamic systems theory. The Mathematics Imagery Trainer, while supporting student development of deep conceptual understanding, could furnish the empirical context for investigating the pivotal epistemic role of learning to move in new ways.

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https://doi.org/10.1142/9789811287183_0002

02

Openness of Problem Solving in the 21st Century: Mathematical or Social?

Takuya Baba[1]

ABSTRACT Mathematics is one of the oldest disciplines in the world. Bishop (1991) expressed its value regarding human relationships and social institutions as Openness — that is, mathematical constructs such as propositions and ideas are open to human deliberation. Even before such systematization, many problems were solved and simultaneously created since earliest civilizations. This effort became the foundation for further endeavors.

What is the "Problem" in problem-solving? It has various types. Especially, the open-ended approach (Shimada, 1977) has been developed in Japan as a method to evaluate and develop mathematical thinking. Furthermore, problem posing can be an extension of problem solving. While posing various problems we may notice the patterns among those problem variations. In this sense, problem posing itself can be a problem. What is "Solving" in the problem-solving? It is dependent on the type and characteristic of problem. For example, the open-ended problems provide more than one solution. Socially open-ended problems provide solutions together with values. Problem posing requires developing problems and such development itself can be a solution. Therefore, importantly, the meaning of solving a problem is extended beyond traditional problem solving.

This paper explores the idea of problem-solving in mathematics to appreciate the value of openness under the Open Science movement (OSF, 2021). Open science is a movement accommodating experts and non-experts to have access to the outputs of scientific research and can participate in the research activities. This is essential for future citizens and is related to the ethical dimension of mathematics education (Ernest, 2012).

Keywords: Openness; Problem solving; Sociality; Mathematicality; Meta-problem.

1. Mathematical Value "Openness" (Bishop, 1991)

1.1. Background

Mathematics is one of the oldest disciplines in the world. Bishop (1991) described a set of values for mathematics and one of them is openness, which is related to human relationships and social institutions. It means that mathematical constructs such as propositions and ideas are open for human deliberation and they can be discussed

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among themselves. Even before the invention of such mathematical constructs, many problems were solved and created since the earliest civilizations. These efforts became the foundations for future endeavors.

During the Greek period, they considered paradoxes. "Zenon paradox regarding the infinity numbers of points on a line, threatened the certainty of mathematical science (Wilder, 1980, p. 11)." This mathematical problem solving includes both "mathematicality" and "sociality". Mathematicality refers to things related to content and method of mathematics. They include mathematical concepts such as algebra, calculus, and numbers, and mathematical process and method such as algorithm, proof, and calculation. Sociality means things related to context and method of society. They include social problem contexts such as purchasing at the market, surveying fields, and sharing food, and method such as discussion, debate, and voting. We analyze whether mathematical problem-solving involves sociality, in addition to its mathematicality.

Many mathematical problems started with the needs of society. For example, the mathematics in ancient civilizations such as Egypt and Babylonia was connected to weather forecast, cultivation, surveying and so on (Wilder, 1980; Cajori, 2015). Therefore, many problems do contain sociality. "Mathematics is what human beings create, and the form of mathematics, which human beings create is just a function of cultural need at that time, just like any other adaptation systems (Wilder, 1980, p. 5)."

In the process of problem-solving, their interests shift from a specific problem to the general solution to solve problems with similar nature. Then how to get the general solution became the object for further consideration. Mathematics as scientific efforts began to use it to explore the nature of such objects. The most important thing in this (genesis of scientific concept) is that we can imagine any bigger numbers than those ever known in this physical world and proceed with studying on nature of such numbers once they are created (Wilder, 1980, p. 9).

1.2. Contents of problem-solving

As we have seen, many mathematical problems have their root in society. From this perspective, problems in problem-solving contain sociality as a starting context. Eventually, some problems are even theoretically considered as they become an object of thinking through symbolization and formalization. It is surprising yet very natural that societies developed various numerals and later unified them through interaction. Following these symbols and number concepts, millions of similar contexts had been practiced and abstracted.

Besides the context in the beginning, sociality can also appear in the end — that is, application of mathematics into solution of social problem. In this highly technological society, mathematics is inseparably connected to science and technology; therefore, most social phenomena can be described through mathematical models. Mathematical models are an object of thinking, a starting point of consideration for the next stage. Such problems provide an opportunity for integrating mathematicality and sociality.

Therefore, the mathematicality and sociality of mathematical problem-solving can be discussed as starting contexts and application as its end. They are a part of content of problem-solving.

1.3. Method of problem-solving

Generally, solving a problem is goal of the problem-solving. However, if we further consider getting a general solution, we need to explore the process of problem-solving or how it was solved. "Thales (BC 640–546) is an outstanding philosopher because he reformulated what many people accepted as truth into a theorem and further provided a proof for it (Cajori, 2015, p. 76)."

Difficult problems such as Geometric problems of antiquity made us think of a general solution and system of logics such as Euclidean Geometry developed. Whether a certain statement would hold true or not for all cases is beyond the necessity of everyday life, because the people simply want to solve a specific problem in everyday life. However, philosophers at that time, called Sophists, discussed the need for such complete accuracy. "Athenians valued liberty and fairness in their life and Pythagorean custom of secrecy disappeared ... They wanted to prove themselves to be excellent through public debate regarding philosophy and science (Cajori, 2015, p. 83)."

Here, a solution (method) is an object of consideration. From this perspective, method is called method knowledge. This age-old knowledge is being used even today. It is valuable social infrastructure for problem-solving, although we no longer remember the original context of such problem-solving.

1.4. Openness, mathematicality, and sociality

In the above mathematical problem-solving, not just one solution is sought, rather a general solution or how to solve problems is considered. In the latter, the developed general solution creates method knowledge such as "algorithms", "proof" and so on.

When general solutions and theorems were made, they might not be directly related to the necessity of the daily life. Later they might become social infrastructure and support scientific development. For example, quadratic equations were meant to calculate simply relation of areas, but later the general solution for it was developed due to theoretical necessity. Such a general solution further has become foundation for the theory of polynomial equation and description of ballistic path. In this sense, they later became a social necessity and were involved in society's daily life.

In the modern times, the school education has been systematized and the subjects have been established. The subject, social studies, was established much later to learn about the society. Distinction of mathematicality from sociality may strengthen the subject boundary. However, such distinction may result in refraining mathematics from perceiving society through a mathematical lens (mathematical literacy).

As noted earlier, the value openness represents human relationship and social institution. It is important to consider how the relation between mathematicality and sociality is established through openness as mathematical culture was practiced in the old days. "[T]he educational imperative is clearly there to demonstrate, and critically evaluate, the value of openness as represented by Mathematical knowledge (Bishop, 1991, p. 77)."

2. Historical Development of Problem-Solving: the Case of Japan

Since this paper focuses on problem-solving in mathematics education, and sociality and mathematicality, and sociality may vary from one society to another, we pay attention to a specific society, Japan. We briefly review the history of mathematics education in Japan. Reference materials are Baba et al. (2012) and Ueda et al. (2014).

2.1. Problem-solving (empiricism)

After World War II, education reform was implemented. In senior secondary mathematics education, the "central idea" was proposed to cut across different fields within mathematics and later developed into mathematical thinking. On the other hand, in primary mathematics education, the life unit learning based on John Dewey's philosophy used life events for the context of problem-solving.

For primary and junior secondary education during this period, the problem was presented based on a life event. For example, multiplication of fraction is introduced based on an episode of rice planting. At that time, Japan was predominantly an agricultural country and rice-planting was very prevalent across the country. The description of the context is very long and rich. Thus, during this period, the focus of problem-solving is placed on sociality. It seemed natural for Japan to have such focus due to a scarcity of natural resources. However, there was a criticism regarding as lowering the achievement of students (Kubo, 1951) and this approach was suspended suddenly.

2.2. Open-ended approach

In 1958, the national curriculum, the Course of Study, was published for the first time. The term "mathematical thinking" first appeared as an objective at the primary school level (Baba et al., 2012). During this period, the focus shifted more to mathematicality. Soon after, because of the influence of modernization of mathematics in the USA, focus on mathematicality was further strengthened. The issue at that time was how to evaluate this mathematical thinking.

The open-ended approach (Shimada, 1977) had been developed in 1970s as a method to evaluate and later grow mathematical thinking. This open-ended approach uses an open-ended problem (Fig. 1), which has more than one solution. Shimada (1977) provides a theoretical background and a compilation of examples of such problems.

Open ended approach is to set an open-ended problem as the task, to utilize proactively its various solutions, to combine in various ways the previous knowledge,

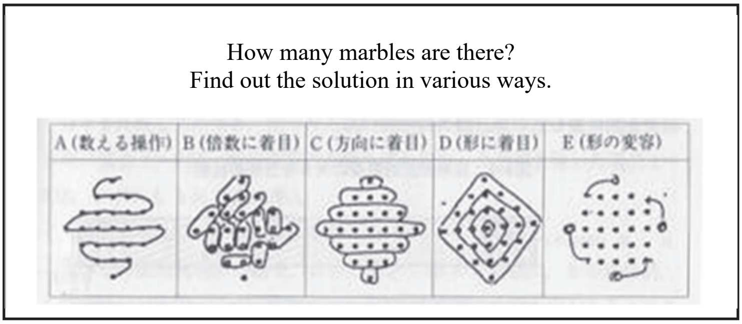

Fig. 1. Example of open-ended problems (Shimada, 1977)

skills and ideas …, to give students an experience in which they have found new things (Shimada, 1977, pp. 9–10).

In 1980s, the systematization of an open-ended approach, which was called Learning through Problem solving (Kadai-gakushu), was proposed. This clearly separate "finding a solution to the problem" and "learning something through solving the problem". In other words, it is important whether finding a solution is the purpose or method of problem-solving. If it is the method, the purpose should be set appropriately; thus, mathematical thinking reappeared here again.

Treatment of more than one solution was an important topic. Koto (1990) summarized the treatment of various solutions into four types. Not only Kadai-gakushu and the treatment but also other approaches became extensively practiced and studied. Problem-solving has become an integral part of mathematics lesson in Japan. Such efforts have developed a unique characteristic of lesson in Japan as a "structured problem-solving lesson" (Stigler and Hiebert, 1999).

2.3. Problem posing

Further, problem posing can be an extension of the problem-solving. While varying problems systematically, we may be able to pose as many problems as we want, and realize the patterns among such variations. In this sense, asking for problem posing can be a problem in itself and posed problems are a solution to it.

Walter and Brown (1983) proposed the approach called "What if not" for problem posing. A part of the original problem is changed by asking "what if not". This approach consists of steps such as a starting point, listing attributes, "what-if-not"-ing, question asking or problem posing and analyzing the problem. Takeuchi and Sawada (1984) proposed another approach called "From a problem to a problem". Takeuchi (1976, pp. 11–12) employed the theory of scientific knowledge growth by Popper and approached this issue from the perspective of the nature of mathematical activity. This played an important role in shifting the research from the open-ended approach to "extensive treatment of problems."

This assumes that "Existence of problem causes cognitive activity. Cognition develops knowledge. Progress of cognition and knowledge is brought by self-proliferation. In other words, it is a chain reaction of from a problem to a problem (Takeuchi and Sawada, 1984, p. 15).” In practice, they segmented the original problem into parts and alter each part with other phrases. For example, the original problem is “How many diagonals are in the regular octagon?” The part “diagonal” can be replaced with “side”, “angle”, and so on.

Nohda (1983) summarized the openness in problem-solving into three types. The first one is “End products are open.” This type has multiple correct answers. Shimada (1977) and his colleagues have been developed this type of problems. The second one is “Process is open.” This has multiple ways of solving the original problem. It is needless to say that all mathematical problems are inherently open in this sense. The last one is “Ways to develop are open.” After students solved the problem, they can develop new problems by changing the conditions or attributions of the original problem (Takeuchi and Sawada, 1984).

2.4. Mathematical literacy

Mathematical literacy is defined as an individual’s capacity to reason mathematically and to formulate, employ and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It helps individuals know the role that mathematics plays in the world and make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st century citizens (OECD, 2012, p. 100).

Here, the students are expected to make an interpretation using mathematical modelling. Sociality appears explicitly once again in the history.

Lesh and Zawojewski (2007, pp. 783–784) describes “the problem solver will engage in ‘mathematical thinking’ as they produce, refine, or adapt complex artifacts or conceptual tools that are for some purpose and by some client.” This kind of problem-solving is called “model eliciting activities (MEA)”, includes traditional problem-solving tself, and makes mathematical sense of problem solution (Fig. 2).

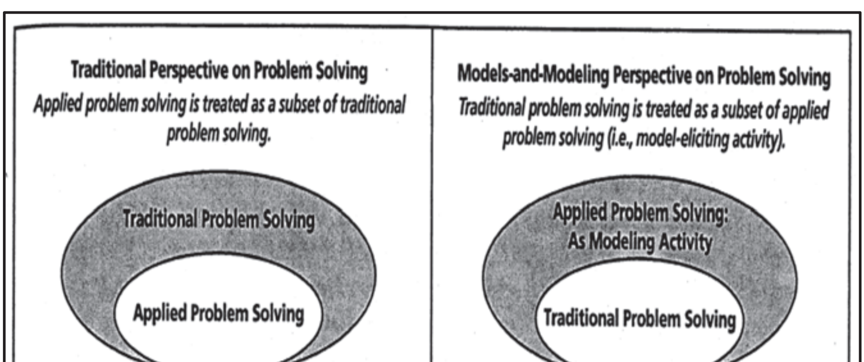

Fig. 2. Modelling (Lesh and Zawojewski, 2007, p. 783)

2.5. Shifting between mathematicality and sociality

Going through the history of problem solving in mathematics education and, we can see the shift of focus between mathematicality and sociality from time to time. Simultaneously, due to theoretical development of mathematics education research, both sociality and mathematicality are treated more sophisticatedly than before. Especially, MEA involves both in an integrated way. There are two important points to consider.

First, integration of both mathematicality and sociality. As we have seen, the idea of MEA plays such a role. Besides, there are many variations. Our research group (e.g. Baba, 2007; Shimada and Baba, 2015; Hattori et al., 2021) also has tried to extend the open-ended approach by paying attention to the values and the sociality. We call them social open-ended problems, and the students provide solutions with various values. This is further discussed in the chapter 4.

Second, the societies in 1950s and in 2020s are considerably different even in some countries. This is important because we deal with sociality, and it varies from one society to another as well as over time. Thus, it is necessary to consider time and space.

3. What is a Problem and Problem-Solving in Today's Society?

3.1. Traditional problem-solving and new problem-solving

Through reviewing the historical development of mathematics education in Japan, we realize that problem solving has occupied a central position throughout its history and has changed its approach and focus in and of itself. Therefore, although we use the term "problem solving" extensively, it may not mean the same thing. Thus, it is crucial to be conscious about the meaning of the word.

In the traditional problem-solving (in the left of Fig. 2), solving the problem is the purpose, while in a new way of problem-solving (in the right of Fig. 2), "mathematical ideas and problem-solving capabilities co-develop during the problem-solving process" (Lesh and Zawojewski, 2007, p. 783).

Here, we would like to consider the etymological origin of the word "problem". Since "pro" means "forward" and "ballein" means to "throw", the word means "anything thrown forward". It does not necessarily have a negative connotation. Especially the solution in the new way of problem-solving may open a new way of interpreting the situation. If it would lead to a later development, identification of the problem is a first step of development.

The current society is called a highly technological society and/or highly information-oriented society. In this society, various advanced technologies and ICT, which connect such technologies, occupy a central position. Unimaginable things may be becoming reality with innovation of technology. However, such technology may generate problems at the same time. For example, we manifest some cases in which an incurable disease become curable due to an advancement in medical science. Certainly, simple extension of our life does not necessarily mean good. There are people who do not appreciate prolonged life if that entails their staying in bed longer. This is related to the life which is very fundamental for us all. Certainly, life at individual level may be considered and decided by the individual person.

There are many more problems which we face due to technological advancement. It includes leakage of private information, control of freedom of expression in SNS (Social Networking Services), and so on. They are not related to life but still significant in our life. They are called trans-scientific problem (Kobayashi, 2007). Most of us are non-scientists or non-experts but should find out some solution. We should decide among some alternatives by negotiating different views and need to establish a new system of deliberation among citizens.

Thus, in the new problem solving — MEA, "the solution (artifact, tool) problem solves create embodies the mathematical process they constructed for the situation (Lesh and Zawojewski, 2007, p.784)." We can realize and interpret emergence of various model eliciting activities such as socio-critical modelling (Barbosa, 2006; Dede, Akcakin, and Kaya, 2020), and ethno-modelling (Rosa and Orey, 2010) due to the complexity of this highly technological society. They form a bridge between mathematicality and sociality.

3.2. What is the range of problems in problem-solving?

It is impossible to understand all advanced technologies. We may be overwhelmed with intricate details of them. Therefore, it is important for us to grasp an essential problem surrounding them. This may be an ability of problem-posing in the broader sense.

What does "solving" mean in the problem-solving? As there are different types and characteristics of problems, the meaning of "solving" also depends on their types. If the problem has only one correct answer, then solving problem means to get exactly that answer. If the problem is open-ended, solving the problem may mean to get all answers which satisfy the condition. If the problem is to pose problems by changing the original problem, solving the problem may not necessarily mean to get a problem and problems as an answer but to understand the characteristics of the original problem structure and to pose as many problems as possible through creativity. An important point here is that the meaning of solving a problem is extended beyond traditional problem-solving.

Then, an important question is "how do we deal with the extension of problem-solving?" Here openness may be keywords. We may have to go back to the trans-scientific problems.

[social openness] Considering today's society as highly technologically advanced and highly information oriented, the idea of "trans-scientific" problem (Weinberg, 1972; Kobayashi, 2007) indicates a crucial relation between science and society. This is the problem which "can be asked of science and yet which cannot be answered by science" (Weinberg, 1972, p. 209).

[evaluation openness] The trans-scientific problem requires not finding only one logically correct answer but evaluation of alternative solutions.

It is important to think subjectively and engage with society through viewing the society mathematically and thinking mathematically in solving social problems. Therefore, it is impossible to teach all necessary mathematics concepts and skills in advance. Of course, it is a minimum requirement to master basic mathematical ideas and acquire skills of applying them into a problem.

The model eliciting activities are a new mode of problem-solving. Solving a problem in the model eliciting activities means to deal with problems, to develop mathematical models based on the given conditions and to evaluate the alternatives. It can include trans-scientific problems. Here to-and-fro motions are evident for bridging between mathematicality and sociality. Then how far should we deal with, if the range of problems to be dealt with is ever-expanding?

To grasp the situation holistically, content-based mathematical thinking (e.g., proportional reasoning, functional literacy, linearity, exponential and logarithmic thinking) is necessary. To see the situation and make a decision, we need to acquire such a mathematical way of viewing supported by mathematical thinking. More importantly, deductive and logical thinking is another important asset of mathematics, while natural science and statistics are basically inductive. Thus, it is important for students to understand deductive reasoning which is invented by human beings and its difference from inductive reasoning.

Henceforth, the ability of dealing with big data is necessary. However, if the people are not cautious enough, they simply believe in the result which the computer software gives. Or, once the people are given a percentage such as 95% and 99.9%, they may believe it is high and it is perfect. We cannot easily say "it is absolutely ..." and may be confused by the expression "it is significant with 95%." We may call this as a logical contextual thinking because not just logical thinking but also judgement based on the context are important.

3.3. How to deal with the problems

The model eliciting activities are to solve a problem, using logical contextual thinking. In this sense, we are able to see varieties of model eliciting activities as stated previously. Here are two to-and-fro motions:

(1) A to-and-fro motion between mathematicality and sociality

Fig. 3 shows the statistical problem-solving and deals with the real world through problem and data. Fig. 4 shows a cycle of mathematical problem-solving and deals with the real world more directly. They develop, analyze and discuss mathematical model as a dynamic activity. This implies an important point when children learn to acquire mathematical knowledge and skills and to apply to solve the problems. It is crucial to create cycle between knowing why mathematical thinking functions as it is and creating how well such thinking is applied to solve the problem.

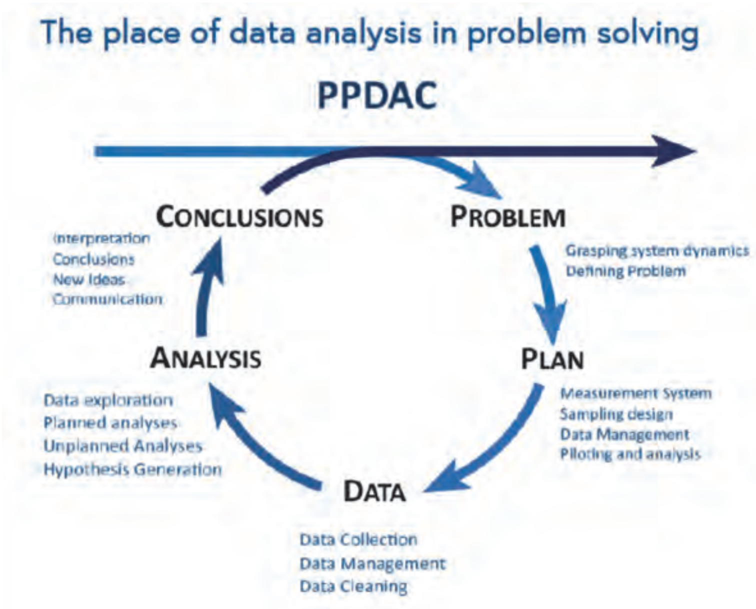

Fig. 3. Statistical problem solving (Wild and Pfunkuch, 1999)

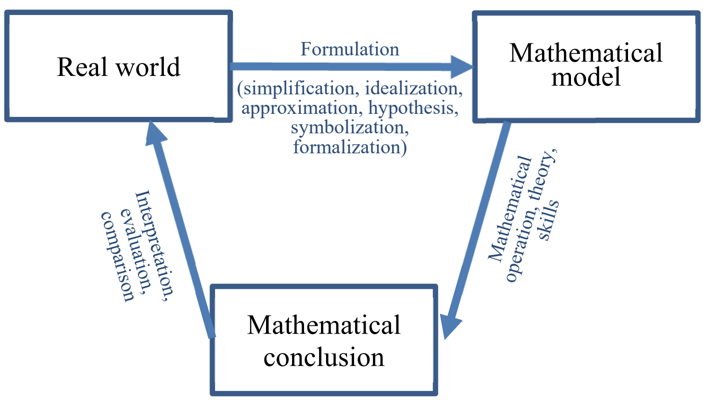

Fig. 4. Mathematical problem solving (Miwa, 1983)

After graduation from formal education, there is no longer a framework of subjects for thinking through. That is why the children need to master not only subject knowledge and method bounded by the subjects but also methods and attitudes to think and solve problems beyond subject borders.

(2) A to-and-fro motion between product and process knowledge

In both, we see the real world through mathematical lens (process) and manipulate mathematical model (product). Relation between process and product has been emphasized such as "procept" (Gray and Tall, 1994) and "objectification of method" (Hirabayashi, 1978).

Katagiri (1988) classified mathematical thinking into those related to methods and contents. Furthermore, mathematical thinking plays more significant role not as distinct entities of methods and contents but as an integrated form of them.

Based on these two to-and-fro motions, what role does mathematics play in relation with society? Who is required with how much mathematics? What kinds of social problems do we deal with mathematically?

What is the problem in creating these motions and to facilitate proactive and deep learning? It does not automatically guarantee that to solve mathematical problems related to society will create such learning. Rather, it is our task to think how to ensure such learning intentionally and systematically as an extension of problem-solving.

3.4. Problem and meta-problem

Therefore, it is important to ask what kind of problem can promote such learning. This is a kind of meta-problem that is "a problem about a problem" (Chalmers, 2018). Examples of meta-problems are "what kind of problems do we deal with?", "Why do we deal with them?" and so on.

Here are two levels in relation with problem solving. One level is called an "object level of solving a problem." This is usually the level of problem solving. Solving a given problem belongs to this level. Although the meaning of solving may vary depending upon the types of problems, at least problem and solution correspond each other. On the other hand, the other one is "meta-level of solving a problem." (Fig. 5). Thinking about the meaning and reason of solving a problem belongs to this level. We expand the range of problems in relation with society and consider what problems and why. This thinking facilitates students to acquire not only problem-solving skills but also viewing the real world through problem-solving.

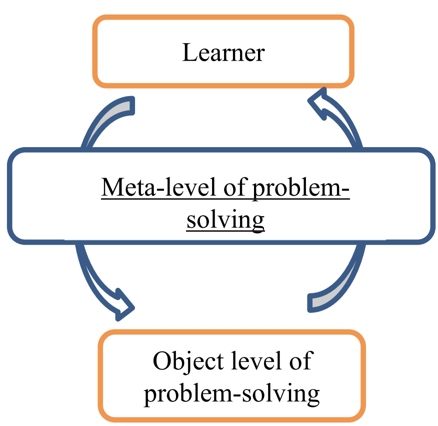

Fig. 5. Problem and meta-problem (Author created)

4. What will Be a Problem and Problem-Solving in Future?

4.1. Openness in open science movement

This chapter introduces the idea of Open Science movement (OSF, 2021) and explores its relation with problem-solving in mathematics. Open Science represents “a new approach to the scientific process based on cooperative work and new ways of diffusing knowledge by using digital technologies (European Commission, 2015, p. 33).” Because the impact of science on the society is getting more serious, it is more important not only for experts but also for non-experts to participate in the research activities.

Here, participate does not mean the same thing for experts and non-experts. And openness means different aspects of science such as openness in methodology, source, data, access, peer-reviewing, and educational resource. One example is methodological openness. Citizen science is scientific research conducted, in whole or in part, by amateur scientists. It is sometimes described as “public participation in scientific research”, participatory monitoring, and “participatory action research by improving the scientific communities” capacity, as well as increasing the public's understanding of science. Another example is open access journals are to ensure everyone’s access to the research. Since the journal publication becomes huge industry, it becomes too commercial and sometimes impacts on the research ethics. As a result, the interests of society and citizens may be at risk.

On the other hand, openness in mathematics means “With rationalism as an ideology and progress as the goals, individuals are liberated to question, to create alternatives and to seek rational solutions to their life’s problem (Bishop, 1991, p. 76).” These practices of mathematics contain open discussion and alternatives.

Here openness shows the social aspect of problem-solving. Social aspect is not only related to the problem content but also solution method and reason. “… they (Greeks) develop the skills of articulation and demonstration in Mathematics (Bishop 1991, p.75).” It is important to explain and discuss rationally. This concerns openness, “relationships between people, and within social institutions (Bishop, 1991, p. 75),” and an ethical dimension of mathematics education (Ernest, 2012; Atweh and Brady, 2009).

4.2. A case study: Hitting a target

Here we take one case from our recent efforts. The “Hitting a target” given to Grade 4 students at school (Shimada and Baba, 2015) is a socially open-ended problem (Fig. 6).

They understand the problem and develop following mathematical models and values after the individual problem-solving activity. Some of them focuses on kindness because the player of the game is the first grader, and they tend to be kind to small children. Others focus on fairness, but such values did not appear explicitly at first. It became explicit only after being compared with kindness group (Tab. 1).

At a school cultural festival, your class offers a game of hitting a target with three balls. If the total score is 13 points or more, you can choose three favorite gifts. If you score 10 to 12 points, you get two prizes, and if you score 3 to 9 points, you get only one prize. A first grader threw a ball three times and hit the target in the 5-point area, the 3-point area, and on the border between the 3-point and 1-point areas. How do you give the score to the student?

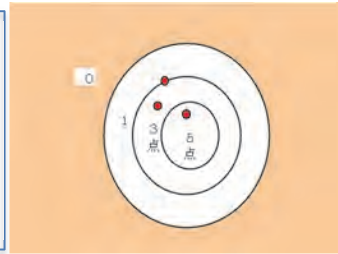

Fig. 6. Matoate problem (Shimada and Baba, 2015)

Tab. .. Mathematical models and values (Shimada and Baba, 2015)

Mathematical model	Expected value type	Explicit value (%)	Explicit value (%)
a. 5+3+3	Kindness to the first grader (Specific person)	92.9 (13/14)	94.4 (17/18)
b. 5+3+(3+1)		100.0 (1/1)	
c. 5+3+3+1+1		100.0 (1/1)	
d. 5+3+2		100.0 (2/2)	
e. 5+3+2	Fairness and equality to the whole class (all students)	0.0 (0/9)	0.0 (0/20)
f. 5+3+1		0.0 (0/10)	
g. 5+3+3		0.0 (0/1)	

After these mathematical models and values are presented to the whole class, the discussion started among the students. They ask questions to understand others' ideas and others explained their opinions. After discussion among the whole class, all students were asked to choose a model and a value again at the end. Some students have changed their opinions and while others maintain their opinions. Among those who changed their opinions, a few of them polished mathematical models.

This case is reflected from the perspectives of openness and open science. Regarding the value openness of mathematics, three points can be considered.

(1) Open-ended problem generally ensures multiple answers and solutions. This case stimulates students to have mathematical models based on their own values.

(2) The problem contains social context and promotes students to think more realistically. Thus, the solutions may contain mathematical models based on some values. This can be referred to as the social openness of the problem.

(3) The discussion is open to all students. They enjoyed mathematics and some of them even changed their opinions by agreeing with the others' explanations. This openness polished their models as well. It creates a culture of mathematics classroom.

Regarding open science, especially methodology, this case contains two meanings of openness.

(1) The first is to share and discuss their own mathematical models and values in a classroom. Through comparing and discussing, they realized the existence of different ideas and agree/disagree with the different ideas. This is a foundation for open discussion as a method.

(2) The second is to explore different ideas for oneself. Self-reflection enhances fluency, uniqueness, and originality to develop creative ideas. This self-exploration and self-reflection can facilitate students being aware of different ideas and appreciating the value of those ideas.

From these, individual exploration and mutual discussion are materialized in this classroom and the to-and-fro motion between sociality and mathematicality is ensured in the process. Especially, Matoate problem contains values such as kindness to those not proficient and impartiality. This is related to ethnical dimension of mathematics education (Ernest, 2012; Skovsmose, 2018) and is essential for future citizens. Thus, sociality appears both in method and content of problem solving.

5. Concluding Remarks

Mathematical value and open science are connected to each other via "openness." At the base of mathematical problem-solving, there is a connection between mathematicality and sociality and thus openness. Here MEA is a new type of problem-solving using these. Relation between problem (object level) and meta-problem (meta-level) can offer a theoretical discussion about openness and problem-solving in mathematics education. This relation is related to the to-and-fro motion between mathematicality and sociality and another motion between content and process. Students appreciated this motion during the problem-solving although sometimes they feel it is beyond their mathematical problem-solving.

One solution to the meta-problem is the category of rulemaking which this Matoate problem belongs to and involves judgement and calculation. We may further ask if there are any other problems in this category and different categories from rule making. Problem and meta-problem are connected in this way.

Because society and time have sociality and mathematicality more intermingled, integration of both, rather than separation, should be carefully examined by avoiding their careless mixture. "… the educational imperative is clearly there to demonstrate, and critically evaluate, the value of openness as represented by Mathematical knowledge (Bishop, 1991, p. 77)."

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03

Mathematical Argumentation, a Precursor Concept of Mathematical Proof

Nicolas Balacheff[1]

ABSTRACT This lecture offers a reflection on the challenge posed by the current trend of curricula and standards to recommend starting the learning of proof from the very beginning of the compulsory school. This trend pushes on the fore the notion of argumentation, it is here discussed as well as its relations to proof as a convincing and an explaining legitimate means to support the truth of a statement in the mathematics classroom. Eventually, a didactical concept of mathematical argumentation is discussed and elements of its characterization are proposed.

Keywords: Mathematical argumentation; Early learning of proof; Epistemology.

1. Early Learning of Mathematical Proof

While "mathematical proof"[2] disappears from the mathematics teaching challenges of the 21st century compulsory school, learning how to back the truth of a statement in the mathematics classroom is still on the fore with the concept of "proof":

> The notion of proof is at the heart of mathematical activity, whatever the level (this assertion is valid from kindergarten to university). And, beyond mathematical theory, understanding what is a reasoned justification approach based on logic is an important aspect of citizen training. The seeds of this fundamentally mathematical approach are sown in the early grades (Villani and Torossian, 2018, pp. 25–26 — free translation).

Since it is meant to cover all grades, "proof" is used here with its vernacular meaning. The expression of this objective takes different form in curricula, using a variety of expressions: *deductive reasoning*, *proof*, *justification*, *mathematical argumentation,* etc.

Since 2003, the TIMSS[3] assessment frameworks provide a picture of the way proof and proving have evolved since the beginning of the 21st century. They distinguish

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[2] "Mathematical proof" is used to translate the words used by Roman language which etymology is the Latin "demostratio" (e.g. démonstration in French). "Demonstation" was used by Anglophone mathematicians until the beginning of the 20th century.

[3] "Trends in International Mathematics and Science Study" which gives a consensus picture of the common core competencies for 4th and 8th graders. https://timssandpirls.bc.edu/timss2003i/frameworksD.html

content domains (the mathematics subject matter) from *cognitive domains* (the behaviours expected from students). Issues related to validating a mathematical statement are addressed in the sub-domain "reasoning" of the cognitive domain:

> Reasoning mathematically involves the capacity for logical, systematic thinking. It includes intuitive and inductive reasoning based on patterns and regularities than can be used to arrive at solutions to non-routine problems. […] reasoning involves the ability to observe and make conjectures. It also involves making logical deductions based on specific assumptions and rules, and justifying results.

Reasoning includes several skills among which *Justify*; this keyword was associated to *Prove* in the 2003 assessment framework, it disappeared from the following assessment campaigns. Then, the reference to mathematical proof being abandoned, the keyword which is chosen is "justification" with specific requirements: "reference to mathematical results or properties" (TIMSS 2007, 2008, 2011). Then comes back the key expression "mathematical argument" (TIMSS 2015, 2019) in a short and allusive statement.

Researchers in mathematics education are fully seized of the problems of teaching proof, witnessing its fundamental character for the learning of mathematics. The number of articles and conference communications has impressively increased since the pioneer work of Alan Bell (1976). One of the first collective book "Theorems in School" (Boero, 2007) deserves a special attention. Its idea was born in the context of the 21st PME conference which demonstrated "the renewed interest for proof and proving in mathematics education" and that of "important changes in the orientation for the curricula in different all over the world" (ibid. p. 20).

In 2007, ICMI launched its 19th study on "Proof and proving in mathematics education" (Hanna and de Villiers, 2012/2021). As learning to justify/prove was to be addressed since the early grades, the idea of proof had to be extended. The study introduced the idea of "developmental proof" as "a precursor for disciplinary proof (in its various forms) as used by mathematicians" (ibid. p. 444). The introduction of this idea intended (1) to provide "a long-term link with the discipline of proof shared by mathematicians", (2) to provide "a way of thinking that deepens mathematical understanding and the broader nature of human reasoning", (3) to "gradually developed starting in the early grades" (ibid.)

The introductory discourse of these initiatives reflects a complexity we know since the seminal exploratory study of Harel and Sowder (1998), covering a large spectrum from "external conviction proof schemes" to "analytical proof schemes" (ibid. p. 245). I will not address all this complexity here, instead I will focus on the educational project aiming at developing the early acquisition of the competence of arguing (to convince) and of proving (to establish the truth). *The didactical project is to teach how to respond to the question of truth and to understand the role of proof in mathematics.*

Proof is a difficult concept per se. We discussed it at length. But, not surprisingly as mathematicians, we didn't discuss the concept of truth. Maybe we should have.

I will address this issue in the next section. Then, I will consider some terms of the related vocabulary with the objective to propose elements for a characterization of the concept of *mathematical argumentation* as a precursor for the transition to *mathematical proof*.

2. Are We Sure of What "True" Means?

Proof and *truth* are inseparable concepts, yet discussions on what can count as proof proceed as if the meaning of the word *truth* were clear. This may seem an irrelevant issue in mathematics where *true* and *false* are just the elements of set where propositions or predicates take value. But mathematical logic is not the logic of mathematics insofar as the activity of mathematicians is not reduced to carrying out a formalism. "Actually, the criterion of truth in mathematics is the success of its ideas in practice; mathematical knowledge is corrigible and not absolute; thus, it resembles empirical knowledge in many respects", wrote Hilary Putnam (1975, p. 529) in a brief paper entitled *What is mathematical truth?*. This position is rather radical, but it is relevant for our topic: *more than a science, in the K-9 mathematical classroom, mathematics is a practice*.

In school, the words *true* first borrows its meaning from the vernacular culture. If students in higher education maintain a difference between the mathematical meanings of true and its meaning in everyday life, this is not the case for K-9 students. In thinking about this problem, I wondered whether we share the meaning of true and truth? To get a glimpse of an answer, I looked at the case of writing in English something thought in French: is the direct translation of the French *vrai* by the English *true* without consequence?

The etymology[4] of *true* goes back to the word *tree*, which denotes firmness or faithfulness. Its evolution incorporated other meanings among which the mathematical one (i.e. logical necessity). Still, the contemporary use puts sincerity and reliability ahead of veracity. The etymology of *vrai* goes back to the Latin word *veritas* whose paradigm is normative: it refers to the legal truth that a legitimate institution locks and preserves. The evolution has introduced the producer of the statements claimed true, of his or her sincerity, but the normative meaning still dominates.

This issue concerns all the languages and background cultures of our research projects. The epistemological differences silently shape research. Eventually, the investigation which started by noticing possible translation issues ends up inviting us to consider the vernacular epistemology. The tension between vernacular languages and mathematical language should lead to *questioning the culture that proof and truth* carry with them.

Davidson (1996) warned us that *it is folly to try to define truth*. But the word-concept *proof* is inseparable from the word-concept *true*. In agreement with Durand-Guerrier (2008, p. 373), I turn to Alfred Tarski's solution to chose "a definition which

[4] According to the Vocabulaire Européen des Philosophies (Cassin, 2004)

is materially adequate and formally correct.", he defines "truth and falsehood simply by saying that a sentence is true if it is satisfied by all objects, and false otherwise"[5]. But he adds the condition that sentences are elements of "[a language] whose structure has been exactly specified." (Tarski, 1944, pp. 341, 347 and 373).

In order to take the consequence of this condition, let us introduce the distinction made by John Langshaw Austin (1950) between *statement* and *sentence*. The utterance of a *statement* requires words and a good command of linguistic rules to produce a *sentence* appropriate to the communication objective which underpins it. This objective includes semantic adequacy and formal correctness. Furthermore, Austin introduces a speaker and an audience, in other words *the intentional character of the speech act uttering truth, and its social dimension*. Hence, aside from coherence and correspondence, the hypothesis of sincerity and steadfastness of the speaker and of the audience must be included.

Although somewhat limited, this discussion sheds light on the difficulty of comprehending the meaning of truth when taking a step beyond mathematical logic while remaining within the mathematical territory. Mathematics as a scientific discipline is universal. Mathematical activity is diverse, it embraces the cultural and historical characteristics of the society in which it develops. This is even more so for its learning and teaching, which are situated mathematical activities framed by institutions and political projects of a society.

Then, I propose four conditions to consider *the truth of a sentence*:

- to be ethically minded (sincerity, reliability)
- to be linguistically appropriate (statement vs sentence)
- to be semantically adequate (correspondence)
- to be formally correct (coherence)

These conditions will not have the same importance within the transition from argumentation at the earliest learning stages to mathematical proof. Nevertheless, we ought to take on such epistemological and didactical perspectives to revisit the classical issue of defining proof in mathematics fitting the needs of mathematics teaching.

3. Reasoning, Explanation, Argumentation and Proof

3.1. Reasoning

The general framework within which problem solving and proving are studied is under the common umbrella of the word "reasoning", which often denotes the mental process of making inferences. I used such a definition for my early work. On reflection, this formulation was awkward because it directed attention to the modelling of mental processes, whereas the problem posed to the teacher is that of *the mathematical interpretation* of observed behaviour and productions. Then, I turned myself to

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[5] Tarski's definition grounds the deduction theorem which bridges syntax and semantic, truth and validity.

Raymond Duval's definition which refers to tangible expressions of thought. It makes the analysis of reasoning a work on discourses and texts whose contextualisation by the state of knowledge, the levels of language and the constraints of the situation are considered:

> Reasoning is the organisation of propositions which is directed towards a target statement in order to modify the epistemic value that this target statement has in a given state of knowledge, or in a given social environment, and which, as a consequence, modifies its truth value when certain particular conditions of organisation are met (Duval, 1992, p. 52 — free translation).

By "particular conditions of organisation", Duval refers to both the logical structure and the particular norm of the proof discourse. This definition, on the one hand, satisfies our theoretical needs, on the other hand, it does not introduce contradiction with a psychological approach.

3.2. Explanation

Gila Hanna pioneered the discussion on the distinction between *proof that proves* and *proof that explains*. It refers to *the question of why* a statement is true, which is that of the link between proof and knowledge.

Duval, did not miss these distinctions: "once the question of epistemic value has been resolved, the question of the construction of coherence or belonging of the new production to the system of knowledge arises" (ibid. p. 40). At the end of the problem-solving process, the explanation is thus the explicit system of relations of the stated result with the available knowledge of the problem-solver. The related proof will have an explaining value if this system is congruent with the knowledge of the interlocutors. This approach is reasonable and productive in our domain, but it induced Duval to assert a division between explanation and reasoning (to justify). The former, he wrote (ibid. pp. 37, 39 and 51), gives one or more reasons to make a result understandable, whereas for the latter the role of the reasons put forward is to communicate to the statements "their strength of argument"; that is to say: their role is to convince.

In claiming the existence of such a division, Duval induces one between explanation and proof that Hanna rejects:

> A proof becomes legitimate and convincing for a mathematician only if it leads to a real mathematical understanding. (Hanna, 1995, p. 42).

To deepen this issue, it is interesting to return to the term "argumentation".

3.3. Argumentation

One always comes to argumentation with a substantial knowledge of what argumentation is, remarks Christian Plantin. In addition to the common-sense conceptions of argumentation, several disciplines contribute to its meaning, among which philosophy, logic, cognitive sciences, linguistic. For the issue addressed here, I will focus on the contribution of linguistic. Within this discipline, there is not a single approach of argumentation, it is therefore advisable to specify this word in order to have an effective characterisation and move forward without creating insurmountable conflicts.

In common use, the term argumentation designates both the action of arguing and its product. The associated process implements linguistic and representational means to make possible interactions (actual or potential) between protagonists who seek to ensure the validity of a statement or, on the contrary, oppose and confront their positions. The outcome takes the form of a discourse that materializes the reasons for agreement or disagreement. In order to distinguish between the process and the product, I will use the verb "to argue" to refer to the former, and the noun "argumentation" for the latter. Drawing on Plantin (1990) synthesis, I suggest the following characterisation:

Argumentation is a discourse

- *Oriented*: it aims at the validity of a statement;
- *Critical*: it analyses, supports and defends;
- *Intentional*: it seeks to modify a judgment.

Arguing is a process

- Which instruments the language;
- Which changes the epistemic value of a judgment;
- Which changes the relationship to knowledge[6].

This distinction is congruent with that made by Duval between rhetorical argumentation and heuristic arguing (ibid. p. 51). The former aims at convincing an interlocutor, whereas the latter emphasizes the role of arguing in guiding problem-solving. This distinction makes it possible to bring the common understanding of argumentation closer to one that is congruent with the requirements of a mathematical activity. Then, an argumentation is accepted or rejected according to two criteria: its relevance (semantic coherence) and its epistemic value (strength of a belief).

Moreover, the concept of epistemic value facilitates shaping the difference between mathematical argumentation and mathematical proof. The reference to the epistemic value induces the idea of its dependence to an author, whereas the value of a mathematical statement depends on the mathematics not on the mathematicians[7]. There is a possibility of thematizing this opposition (Hanna, 2017) by taking up the distinction made by the philosophers Frans Delarivière and Bart van Kerkhove between epistemic value, which implies the existence of an agent, and ontic value, which is independent of any agent. For these authors, it was a question of qualifying the intrinsic or relative character of the explanatory value of a proof. Here is what they write:

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[6] Knowledge refers here to the pair {statement, argument}

[7] I don't ignore the pragmatic limit of such a claim since mathematics is the product of a human activity.

> A mathematical proof can be thought of as an argument by which one convinces oneself or others that something is true, so it may seem difficult to go beyond epistemic discourse about an explanatory proof. However, if the content of any particular piece of evidence is the product of one person's epistemic work, it can be separated as an object independent of a particular mind. Other people can read this evidence and be convinced of it. This brings us to the question of whether showing why a theorem is true is a feature of the proof itself or a feature of communicative acts, texts or representations. (Delarivière et al., 2017, p. 3)

This is to be compared with the criterion for recognising the heuristic or epistemic character of an argument, "[which] is either due to the existence of a theoretical organisation of the field of knowledge and representations in which the argumentation takes place, or to the absence of such a theoretical organisation." "A heuristic argumentation requires the existence of a theoretical organisation of the field of knowledge and representations in which the argumentation takes place" and "that one is able to understand or produce a relation of justification between propositions that is deductive and not only semantic in nature" (Duval, 1992, pp. 51 and 52).

Thus, the distinction between rhetorical argumentation and heuristic argumentation comes down to the evaluation of the epistemic value and the ontic value of statements. We can then argue that *an argumentation will be admissible in the sense of mathematics if the epistemic value of its statements is conditioned by their ontic value*. It is this criterion that will allow it to be recognised as a proof in mathematics. The mathematical normalisation of proofs is a technical means of carrying out this evaluation.

3.4. Proof

We have learned that the epistemological journey from argumentation to mathematical proof is long and full of pitfalls. The first issues were on proof and logic, then on the relation between explanation and proof, and between proof and mathematical proof. Argumentation became later a research theme with the idea of a fundamental conflict between argumentation and proof. The former could be seen as an epistemological obstacle to the latter. I support this idea. But I see a solution to the problem it raises, which is to give room to the concept of *mathematical argumentation* in the that of *developmental proof*. The distinction between rhetorical and heuristic argumentation, and between epistemic and ontic value, makes it possible to progress in this direction.

Following Duval, the tension between argumentation and mathematical proof originates in the nature of inferences which *could be* of a semantic in the first case and *must be* of a logical in the second case (Fig. 1). It suggests a shift in the learner's position from a pragmatic stance to a theoretical stance (Balacheff, 1990). An adequate characterization of mathematical argumentation should be a tool to facilitate this evolution.

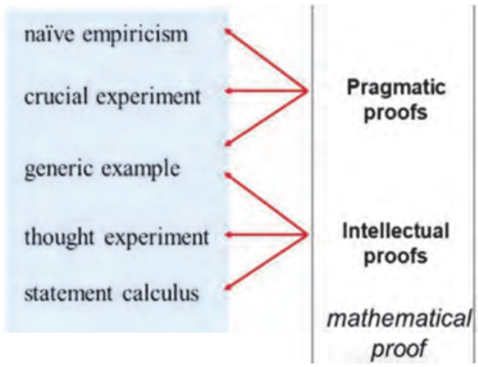

Fig. 1

My research questioned what could be considered as a proof for students *before* they were formally introduced to the Euclidean norm of mathematical proof. It led me to distinguish between pragmatic and intellectual proofs, and within each category to identify different types of proof. The outcome of this initial research was that the type of proof is determined in the first place by the nature of the students knowing and their available semiotic representation. In the private space, the effort is to construct an argumentation which at is both convincing and meaningful. It is in the context of a social interaction that argumentation may take precedence over explanation. The split between them could cover the large range of the possible proof schemes. Indeed, social interaction cannot be avoided; it is a source of complex phenomena that the teacher has to manage.

Then, what comes first is an "explanation" of the validity of a statement from the student's own perspective, without prejudging *what counts for her or him as an explanation*, whether in terms of content or of form of the text which expresses it. The rationale for this postulate is that the explaining power of a text is directly related to the quality and density of its roots in the learner's knowing. So, the key issue of an approach of the learning of proof is that of the nature of the relation between the students' knowing and their argumentation supporting the validity of a statement.

The passage from explanation to argumentation is imposed by the need to communicate reasons and their organisation. Having others accept that an argumentation establishes the validity of a statement changes its status, it becomes public and gets the status of proof.

The important point is to highlight the existence of a boundary between the private and public spaces. In the private space, explanation works on objects and their relations, it is the basis for the construction of the explanation which backs the validity of the solution of a problem, whether or not this work ensures the submission of epistemic value to ontic value. Crossing this boundary implies the search for a consensus. This social process, by its very nature, cannot guarantee that the protagonists individually recognise the explanatory character of the collectively accepted argumentation — *the proof*. This uncertainty is even greater in the case of mathematical proof because of its normative character which takes precedence over its rhetorical characteristics.

4. Three Short Stories and One Lesson

This section presents three examples intending to illustrate aspects I will later consider in order to characterize mathematical argumentation from a didactical perspective. They deal with the relation between knowing, semiotic resources and controls as tools in a validation process. I start with the case of a famous mathematician, so that we realize that the issue is not only that of beginners but in a way intrinsic to mathematics.

4.1. Short story 1, where rationality and cognitive maturity are not the issues

In his *Cours d'analyse*[8] published in 1821, Augustin Cauchy formulated a first version of a theorem on the convergence of series of continuous functions:

> Let (I) "$u_0,\ u_1,\ u_2, \dots, u_n,\ u_{n+1}, \dots$" be a series, then the theorem states:
>
> When the various terms of series (I) are functions of the same variable x, continuous with respect to this variable in the neighbourhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighbourhood of this particular value. (trans. Bradley and Sandifer 2009 p. 90)

As we now know, this statement is false. Cauchy recognized its refutation by other mathematicians. He modified it and published a new modified statement in the *Comptes rendus à l'Académie des Sciences*, thirty years after the first edition of the course, in 1853. Why such an outstanding mathematician didn't realize the error he was making once refutations were known, and why was it so difficult to overcome it?

Gilbert Arsac (2013) studied this episode paying attention to avoiding anachronisms which could introduce the rewriting of Cauchy's writings with the formalization of the contemporary mathematics. Such rewriting would have hidden the conceptual difficulties mathematicians met, especially with the notions of *function* and *variable*.

Arsac first points that the variable x is not explicit in the expression (I), although the modern notation $f(x)$ was used in the course. In fact, in this expression, $\mathrm{u_n}$ and x are two variables, x being the independent variable on which depends the functions $\mathrm{u_n}$. Second, he reminds us that the dominant concept image of *limit* is cinematic, reinforced by the role drawing the curve of functions played. Then validity of the theorem was established using a narrative which expressed a qualitatively the reasoning. Here is an extract:

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[8] http://gallica.bnf.fr/ark:/12148/btv1b8626657

> [let s_n be the partial sum as rank n, r_n the reminder and s the limit, these] three functions of the variable x, the first of which is obviously continuous with respect to x in a neighbourhood of the particular value in question. Given this, let us consider the increments in these three functions when we increase x by an infinitely small quantity α. For all possible values of n, the increment in s_n is an infinitely small quantity. The increment of r_n, as well as r_n itself, becomes infinitely small for very large values of n. Consequently, the increment in the function s must be infinitely small." (Bradley and Sandifer, 2010, pp. 89–90)

However, Cauchy did not present this text as a mathematical proof as he did for other theorems in his course, but as a *remark*. This remark invites the reader to imagine with the mathematician the monotonous movement of x and the effect it causes on the functions at each step of the reasoning. Things happen because they "*must*" happen.

The 1853 proof introduced the criterion of uniform convergence:

> $s_{n'} - s_n = u_n + u_{n+1} + \cdots + u_{n'-1}$ becomes infinitely small for infinitely large value of the numbers n and $n' > n$.

But still, this proof has the style of a narrative dominated. The order of the statements and the appearance of the terms driven by the rhetoric of argumentation is not congruent with the logical order of the formal n/ε proof. As it were, it hides the dependence of n on ε and not on x, as it is evidenced by the modern algebraic expression[9], is. The style of the Cauchy's revised version is still to that of the initial remark, however he now calls it a mathematical proof.

The will to be rigorous is undoubtedly present throughout Augustin Cauchy's work, but it encounters obstacles: the definitions of *variable* and *function*, the absence of *the sign* $<$ and hence the formal manipulation of inequalities, the absence of a notation for *absolute value* (introduced by Weierstrass in 1841) and of the *quantifiers* (introduced at the turn of the 20th century). Eventually, the natural language is infused by a cinematic concept image of *convergence* and the Leibnizian "*lex continuitatis*" (law of continuity).

Gilbert Arsac analysis evidences the tight relation between *representation*, *language* and the *reasoning tools* on the one hand, and on the other hand the limits due to the cinematic conception of *continuity* and *limit*. The difficulty of Cauchy was not due to his underlying rationality and his cognitive maturity.

4.2. Short story 2, where it is a question of semantic control

It is common to observe that students' early learning of geometry meets difficulties with the concepts of perimeter and area, and their relations. I studied some of these difficulties met by 7th and 8th graders, using a classical task about the perimeter and the area of a rectangle which is a familiar object for them. They know a lot about it, either

[9] $\forall\varepsilon\ \exists N\ \forall n > N\ \forall n'\ [n' > n \rightarrow \forall x\ |s_n - s_{n'}| < \varepsilon]$

as a geometrical object or as a shape for which they can calculate the area and perimeter. The task consists of asking students working in pairs what they think of certain claims attributed to other students. I take the case of a pair, A&C, about two of these claims:

> *Serge:* if you increase the area of a rectangle its perimeter also increases.
> *Brigitte:* all rectangles that have an area of 36cm² have a perimeter that is not less than 24cm.
> What do you think of what each of these students say: do you agree or disagree? Explain why.

A&C judged positively Serge's proposition, but students did not see at once how to explain it: *"It's silly because it's obvious [...] how can we prove it?"* They return to this question after considering Brigitte's claim which induces to use the area and perimeter formula. Without changing their initial judgment, they invoke arithmetic properties:

> When you increase the perimeter, the numbers you increase them ... there, the numbers that multiply ... that add up [...] well yes, because when you increase the perimeter, the length and width increase. So when you multiply them both, it increases too.

The A&C case illustrates an area-perimeter conception that develops within the framework of symbolic arithmetic in which formulas provide a representation whose manipulation and interpretation is under the control of their referent (i.e., what they model). The principle of a monotonously increasing covariation of area and perimeter is strong enough to impose itself and control the manipulation of the formulas. In both cases, students were not limited by the semiotic tools needed to achieve the proposed task, nor by logical skills. Their search was bounded by their conceptions.

4.3. Short story 3, where the issue is the restructuration of knowledge

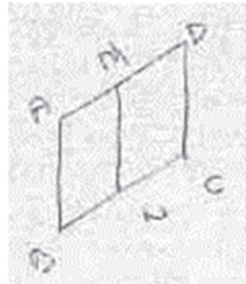
Although students seem to master some mathematical tools, the way they use them in different situations may reveal inconsistency which could leave the teacher wondering. The following vignettes come from a study of the relation between proving and knowing of 9th graders (Miyakawa, 2005, p. 225). The two students, L&J, are solving construction and recognition reflective symmetry tasks:

> *Problem:* construction of the symmetrical of a segment:

28. J: it's ok there.
29. L: a right angle…, then, we take the compass like that… you see?
30. J: yes.
31. L: oups, wait… if we fold it like that… yes it fits, it's ok

32. J: hum.

Problem: to recognize a relation of symmetry



Given hypothesis:
$ABCD$ parallelogram
M middle of $[AD]$
N middle of $[BC]$

L&J Proof:
A is the symmetric point of D with respect to M, because A and D are at the same distance to M and the 3 points are aligned.
B is the symmetric point of C with respect to N, because B and C are at the same distance to N and the 3 points are aligned.
Conclusion: AB and DC are symmetrical with respect to the line MN

148. J: yes, that's the same as before. If, if M is the middle of AD, and N is the middle of BC. MN ...
[...]
153. L: that means that somewhere, the right angle and all that, it doesn't exist anymore.
154. J: hum...
155. L: so, so, wait, M is ..., shit, A is the symmetrical of D.
156. J: ah, yes, we say the same thing.
157. L: well yes.

The case of L&J evidences the critical role of controls on the decisions and actions. There are both visual controls related to symmetry as paper folding and controls associated to the use of instruments (problem 1), and controls based on geometrical properties and based on a global common sense of symmetry (which obliterates the geometrical control). The issue that is illustrated there is not a lack of logic or the absence of knowledge but that of the restructuration of knowledge. Even a mathematician, in everyday life, first assess perceptively and globally the symmetry of an installation, before using his mathematical competencies.

5. Conception, Explanation and Argumentation

We have personal and daily experiences of using the same piece of mathematical knowledge in different ways depending on the situation and on the context. Without noticing, we could use decimal numbers as a pair of integers when they represent a price to be paid, or as integers equipped with a dot depending on the choice of the unit. Both are not congruent to the mathematical meaning of the corresponding concept. In the case of students, it can lead to errors in certain situations. We used to see there the evidence of "misconceptions". However, these errors more often than not are the result of the extension of procedures and knowledge valid within a certain domain but faulty beyond it.

I proposed to unify the facets of a same piece of knowledge within a model constructed on the notion of "conception" to denote an understanding which has the properties of a piece of knowledge within a certain domain of validity. Once *a set of problems* has been specified as being its domain of validity, a conception can be characterized by three joint and linked sets: a *set of semiotic tools*, a *set of operators* and a *control structure* that allows one to assess, choose and decide (Balacheff, 2013).

Control structures regulate problem-solving processes from its very beginning until the final decision of its successful end. Thus, the validity of a solution is fundamentally dependent on the conceptions. At the early stages, students may rely on a combination of pragmatic and knowledge-based criteria, which is not in line with the mathematical norms. But we know that these norms evolved over history, as they evolve with the learning of mathematics. Then, we may agree on the following claims:

- The validation of a statement, depends on the means of representing, linking and processing the objects at stake, as well as on the associated means of control.
- The rationality of students is built up from the very first activities in the mathematics classroom, which enable them to enter into a validation approach well before the complete formalisation of mathematical objects.

Then, the collective activities in the classroom, regulated by the teacher as a mathematical referent, imposes a *socio-mathematical norm* (Cobb and Yackel, 1996) which may not comply to the canonical ones, but which can be accepted provided it respects minimal conditions (Pedemonte, 2005, p. 17):

- Availability of theorems corresponding to the operators;
- Existence of a mathematical framework that can be substituted for the conception and provide the theoretical basis — i.e. objects and a system of deduction and accepted principles.

6. Proving and Knowing, A Dialectic Interaction

6.1. Empirical and intellectual proofs

The mutual dependence of representation systems and control structures makes it necessary to distinguish different types of proof in order to account for their differences and their evolution. The classification I proposed at the end of the 1980s had this objective. It is often interpreted as a sequence of "stages", which it is not. The observations, on which it was based, evidenced that students accept a type of proof according to their conceptions *and* according to their perception of the situation.

This dependence is particularly obvious when dealing with counterexamples. Different validation approaches can be identified in the course of solving a problem or in the course of a contradictory debate. The stakes of the social interactions or those of the situation may even lead to the obliteration of argumentation in favour of persuasion. Eventually, a type of proof is less an information on the student than on *the student in a situation at a given moment in his/her mathematical history.*

In the early grades, problems preferably deal with familiar or concrete experience. The more the students advance in their schooling the less such a context is available, mathematics becoming more and more abstract. But, having or not having access to a concrete referent is a characteristic of a learning situation that play a central role in setting up the problem of validation. The possibility to execute a decision or to satisfy an assertion give access to pragmatic validations. When this access is not possible, validations are necessarily intellectual. So, the production of intellectual proofs requires, among other things, the linguistic or semiotic expression of objects and their relations.

The passage from naive empiricism to mathematical proof can, as it were, describe the movement of the learning of proof in the mathematics classroom. This movement is that from a pragmatic approach to a theoretical one, and thus of an evolution of *the reading of the learning situations* in which the mathematical activity unfolds and the status of the mobilized knowledge evolves.

6.2. The pivotal role of generic examples

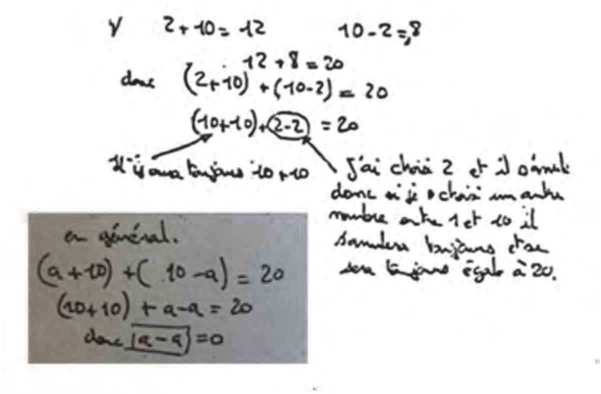
The generic example consists in the elicitation of the reasons for the validity of a statement by the realization of operations or transformations on an object present not for itself but as a representative of a class of objects sharing the same characteristics. The formulation puts highlights and structures these characteristics of the class while remaining attached to the exhibition of one of its representatives without depending on its singular properties. This the process by which we see the general in the particular.

The generic example is on the border between pragmatic and intellectual proof, which crossing is brought about by the awareness of the generic character of the case.

Here is a vignette illustrating the generic character of the example used by the student is attested. This come from a replication of the work of Alan Bell which I replicated at the beginning on my research (Fig. 2).

What is written completes the movement towards a representation that gives an account of generality, while at the same time retaining a control over the thread of the writing that reflects that of the construction of the solution; thus, one can understand the strange “therefore $a - a = 0$”.

The challenge for the teacher who may use examples in his teaching, is being precise in making the generic character of the case. As a probationary means, a generic example is not just an example.



"*donc*" translation "so"

– There will always be $10 + 10$

– I have chosen 2 and it nullifies itself, so if I choose another number between 1 and 10, it always nullify itself and always equal.

In the grey box the final version of the proof.

Fig. 2. Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right. (Bell, 1976, p. 40)

6.3. The didactical challenge

The early learning of proof in the mathematics classroom requires the creation of a situation in which students are likely to make a problem their own in order to take responsibility for the solution they propose. Research projects have explored various approaches from open inquiry-based learning situation to designing specific situations. They imply a demanding commitment of the teacher to implement them and to maintain a mathematical meaning of the activity while stepping back in order to respect the students' autonomy. The weak point is wrapping up the situation moving from a debate on the validity of a statement, to a debate on the nature and structure of the argumentation itself as an object whose explicit characteristics condition its admissibility as a proof. In other words, the question of the validity of the solution of the problem precisely at stake must be surpassed to leave room for that of the criteria of truth, which is nothing other than laying the foundations of the production of mathematical knowledge.

The validation of a mathematical statement does not get its legitimacy from the compliance to logic and from the sole status of the statements mobilized, but from that of the set of statements to which they are linked within a structured whole: a theory that must be recognized as such.

In effect, the reference to an explicit theoretical framework as a context for mathematical activity is present in many researches but has not been thematized until Alessandra Mariotti's (2001; 1997) proposal to define a "theorem" as the system of mutual relations between three components: a statement, its proof and the theory within which this proof makes sense.

Designing situations that allow to realize these conditions is the main problem we are facing. Among them is taking argumentation, the heart of problem solving, as an object for understanding and learning what a proof is in mathematics.

7. Mathematical Argumentation

7.1. The complexity of the epistemological genesis of mathematical argumentation

There are various forms of validation which weights change along a continuum from the statement of a problem to the communication of its solution according to a norm in force. Their interactions with and their dependencies on the underpinning conceptions a system whose nature determines that of mathematics itself.

During the last two decades, educational decision makers have sought to establish a relationship with mathematics that is closer to the epistemological characteristics of the discipline. Thus, the acquisition of knowledge was completed by that of “competences” among which curricula designate reasoning and mathematical communication. Could the rather broad definition of these prompts the emergence of an activity that gives depth to the mathematical discourse and thus bring to life in the classroom a real little mathematical society? Of course, there is no clear-cut answer.

Proof situations must have the characteristics of situations of validation with the additional constraint of creating an intrinsic need for the analysis, certification and institutionalization of the means of proof in the collective framework of the class. But while we know rather precisely what a proof should be in terms of a learning objective at the end of the compulsory school, there is no shared characterization that can serve as a reference in the course of the schooling that precedes it. Thus, a major theme is the characterisation of *mathematical argumentation* as a legitimate means of establishing truth and as a precursor to the learning of mathematical proof.

A mathematical argumentation must be potentially admissible with respect to the norms of the mathematics classroom, i.e. be accepted as proof by the class and confirmed by the teacher. This is a minimal condition taking into account the social dimension. I propose to start from the Andreas Stylianides definition (Stylianides, 2007, p. 291):

> A proof is a mathematical argument, a connected sequence of statements for or against a mathematical assertion, with the following characteristics:
> 1. It uses statements accepted by the class community (set of accepted statements) that are true and available without further justification;
> 2. It uses forms of reasoning (modes of argumentation) that are valid and known to the class community, or within its conceptual reach;

3. It is communicated using forms of expression (modes of argument representation) that are appropriate and known to the classroom community, or within its conceptual reach.

For the most part, this proposal is congruent with the common definition of proof. Its interest lies in highlighting three characteristics which correspond to three problems that need to be solved in teaching. The first one is the problem of the creation of a reference, the form of which must be modelled and the conditions of creation specified. The second and third distinguish two aspects of argumentation, its nature (types of argumentation) and its expression (modes of representation of arguments). These two characteristics are in fact intertwined in the process of producing argumentation: reasoning and argumentation are constrained by the means of representation, the language skills, and the level of the conceptions mobilize and shared (e.g. the case of the generic example).

However, although the historical roots of mathematical proof would give it legitimacy, *the concept of mathematical argumentation will be a didactic concept* and not the transposition of a mathematical one, unless we consider that the "social" function of the latter, within the scientific community, is constitutive of it. This would be an epistemological as well as a theoretical error: although being the product of a human activity that is the object of a certification at the end of a social process, a mathematical proof is independent of a particular agent. The normalization of proof in mathematics, besides the institutional character of its theoretical reference, has required its depersonalization, decontextualization and timelessness. On the contrary argumentation is intrinsically carried by an agent and is dependent on the circumstances of its production.

The characteristics of mathematical argumentation must not only allow it to be distinguished from other argumentation practices and norms in order to guarantee the transition towards the norm of mathematical proof, it has also to be effective when it comes to arbitrating the students' proposals. Moreover, the mathematical argumentation must satisfy the requirements of institutionalization. It is a difficult and delicate problem at the elementary levels, the recognition of its mathematical character cannot be reduced to assessing its form. How, for example, to arbitrate a generic example which puts in balance the general and the specific, whose equilibrium is found at the end of a contradictory debate seeking an agreement as little as possible tainted by compromise?

Proof is both the foundation and the organizer of knowledge. It contributes to reinforcing its evolution and to providing tools for its organization. In teaching, it legitimizes new knowledge and constitutes a system: knowledge and proof linked together make up "theory". The institutionalization of proof places explicit validation under the arbitration of the teacher who is ultimately the guarantor of its mathematical character. This social dimension, in the sense that scientific functioning depends on a constructed and accepted organization, is at the heart of the difficulty of teaching proof in mathematics.

7.2. When is an argumentation mathematical?

One engages in looking for a proof of a statement if there are reasons, based on her or his conceptions, to support its truth. This condition being verified, the statement deserved its recognition as a conjecture. This observation led me to propose a characterization of conjecture which mirror the characterization of theorem:

$$\text{Conjecture} = \{\text{conception}, \text{statement}, \text{argumentation}\}.$$

Establishing the validity of the conjecture requires reasoning and its formulation including shaping a sentence to express its statement. These constructions evolve along with the problem-solving process up to the point where *an explanation of the truth* is established *in the eyes of the problem-solver*, individual or collective, which could work at least as an argumentation for others, possibly even being accepted as an explanation.

In a proper mathematical activity, the expected future of a conjecture is to be transformed into a theorem. But in early grades the knowledge of reference is not organised into a theory and the structure of the proof does not conform mathematical norms. Moreover, in mathematics teaching not all true statements become theorems: a theorem is in the classroom an institutionalized statement which can be used without producing again its proof. For this reason, at the grade levels considered, I suggest to refer to the validated conjecture as valid statement, and to characterize it by the triplet:

$$\begin{aligned}&\text{Valid statement}\\ &\quad = \{\text{knowledge base}, \text{sentence}, \text{mathematical argumentation}\}.\end{aligned}$$

This puts on the fore the role of the knowledge base which is meant to be the same as the role of theory in the case of theorem, that is the reference where it is legitimate to take statements for constructing the argumentation. More often than not, this reference exists but it is left as an implicit clause of the didactical contract; it is the statements which have been stamped as such in previous lessons. This is more a tool-box (Reid, 2011, p. 26) but it could play a role analogous to that of a theory, congruent to the Hans Freudhental (1973, p. 390) idea of a *local organization* which can be regarded mathematical if it is limited enough so its consistency and its domain of validity can be pragmatically ensured. An example of such a reference being explicit to the students could be the quasi-axiomatized[10] geometry of 8th text books in Japan based on a “deliberate choice” of fundamental properties, and their local organization as a system (Miyakawa, 2016). Another example, could come from the use of microworlds which have the specific property to evolve from a few tools and primitives to complex objects with the knowledge of the student (Mariotti, 2001).

Associating different semiotic registers, a *mathematical argumentation* is a multimodal text which does not stand alone: it is built around a sentence and

[10] “Quasi” means that certain properties are introduced by observation or accepted.

contextualised by a state of knowledge. Its characterization requires that of each of these components:

- *A knowledge base* — explicit, established by and for the classroom community;
- *A sentence* — linguistically appropriate, semantically adequate, of a general stance;
- *An argumentation* — ethically minded, formally coherent, congruent to students' conceptions linking the sentence to the knowledge base.

Generic examples and thought experiments are candidate forms of such argumentations. Becoming a sociomathematical norm, mathematical argumentation shall turn the elementary classroom into a mathematical society, although situated and provisory. It will prepare the K-9 graders to move from the position of practitioners to that of a theoretical approach of mathematics as a science. However, reaching this objective is a challenge for the mathematics education community. One of its aspects was highlighted by Patricio Herbst as we were co-authoring a paper, it deserves the concluding words:

> Classroom activities are not mathematical performances just because the classroom is a mathematics classroom and not only when their performance is faithful to a mathematically vetted score, yet the observer needs means to support the claim that a classroom activity is a mathematical performance even when they may not have used an accepted definition, a conventional symbol, or a syntactically valid proof. (Herbst and Balacheff, 2009)

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https://doi.org/10.1142/9789811287183_0004

04

Why Language Diversity Matters in Mathematics Education

Richard Barwell[1]

ABSTRACT I examine the question of why language diversity matters in mathematics education, offering four responses, illustrated with examples drawn from my research. The four responses look at the nature of language diversity, its role in learning and teaching mathematics, its connection with social stratification, and its connection with the ecological crises faced by our planet.

Keywords: Mathematics learning; Mathematics teaching; Language diversity; Sociolinguistics; Ecojustice.

1. Introduction

There is now a broad understanding that mathematics classrooms often feature learners who speak more than one language or who may be learning the language of instruction. In fact, despite widespread evidence (including our own experience), mathematics education as a field has not yet recognized that language diversity is the more common state: that is, in the vast majority of mathematics classrooms around the world, some degree of language diversity is present. Nevertheless, a growing body of work has examined different features of teaching and learning in mathematics classrooms in the context of language diversity (see Barwell et al., 2016; Barwell et al., 2017). This work has, for example, examined the challenges experienced by mathematics teachers, the different language practices used by learners and teachers for mathematical meaning-making, and the impact on learners' performance. I have made some contribution to this literature over the past 20 years, through ethnographic studies of learners' participation in mathematics classroom interaction, particularly in second-language contexts (i.e., those in which the language of instruction is a second or additional language for some or all learners).

My interest in language diversity in mathematics classroom arose first as a mathematics teacher. I grew up in the UK and completed my mathematics teacher education in west Wales, where there are many Welsh speakers. My university offered the mathematics teacher education program in both Welsh and English. Later, I went to work in northern Pakistan. I worked in English-medium schools, but my students spoke a local language, Burushaski, at home, as well as Urdu, the national language. I was impressed at how my students could learn mathematics through a combination of

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three (or more) languages and I spent much time thinking about what it meant for my teaching. I now work in the Ottawa region of Canada, in a bilingual (French-English) university in a society that features Indigenous languages, two official languages (French and English), as well as the languages of many immigrant communities. Language diversity is the norm in mathematics classrooms in this part of the world, as in most places.

For this paper, then, I return to the question I first thought about as a mathematics teacher in the 1990s: How does language diversity matter in mathematics education? I answer this question four times from different angles. In my first response, I examine in more depth what language diversity means and why it is relevant for mathematics educators. Second, I review key findings from the literature about the difference language diversity can make in mathematics classrooms. Third, I show how language diversity links mathematics classrooms to broader social forces. And fourth, I connect language diversity in mathematics education to the future of Planet Earth. Throughout these responses, I include examples from my research to illustrate the different points.

2. What is Language Diversity? What does It Have to Do with Mathematics Education?

Language refers to self-organized systems of semiotic interaction used to coordinate human activity. Humans use combinations of sounds, symbols, gestures and so on, to coordinate social action, whether it be organizing a party, running a political campaign, bringing up children or teaching and learning mathematics. Language enables information, values and learning to be shared across time and space. A language, meanwhile, is a form of language shared by a group of people. Urdu is a language (shared by most people in Pakistan, many people in India, as well as their diasporas). Cyrillic is not a language (it's a writing system). From this perspective, mathematics is not a language (it is embedded within languages and uses a symbolic system).

My thinking about language is influenced by the writing of the literary theorist M. M. Bakhtin (e.g., 1981), for whom language is fundamentally and inherently diverse, captured by the term heteroglossia. Think of the many forms of language and the many different languages that might be heard in a street market (the languages of vegetable sellers and fishmongers and snack stalls), or in hip-hop (from different countries, by different artists, in different traditions), or on the television news (the language of political reporting, economics, sport, the weather). The infinite variety of language means that even within a market, the variety is constantly changing: language changes over time, and individual speakers have their own versions of a language.

In similar vein, we can see that there is no single, unified language of mathematics (or mathematics register or mathematical discourse). Apart from the fact that mathematics may be conducted in many different languages, we must also note the diversity of languages within mathematics (of geometry, of algebra, of probability, etc.) and in different contexts (in school, in an undergraduate class, in an academic seminar, in a popular media article). Again, each individual speaks their own version of mathematical language, which changes over time. Language diversity is diverse.

Once we accept this view of language diversity, it is apparent that the category 'language' is analytically unhelpful: language is hard to pin down, even if, of course, there are commonalities that can be discerned. As an example, consider the following observation, taken from fieldnotes from an upper elementary school mathematics class for new immigrants to the province of Quebec, Canada, in 2010, in which the language of instruction is French, a language the learners are in the process of learning:

> Luis ajoutait qu' «un losange ce n'est pas un carré qui tourne, je le sais déjà en anglais. Les angles d'un losange sont plus grands (grandé) à deux côtés.»

In English, this says:

> Luis added that "a diamond is not a square that turns, I already know it in English. The angles of a diamond are bigger (grandé) on two sides."

In this brief fragment, we see that Luis, who is a Spanish speaker, talks about mathematics in French, clearly influenced by Spanish in his accent and word choice, while referring to English, a language with which he also has some familiarity.

If language is always diverse, there is also a tendency to standardize different features, such as in the idea of correct speech. This ideal of standard forms of language, known as unitary language in Bakhtin's work, is ideological in nature and is related to European enlightenment ideals of unified peoples in nation states. This ideology informed European colonialism not only in their language policies but even in the way in which peoples and nations were identified (Makoni, 2011). In mathematics, the idea of standard formal mathematical discourse, also known as the mathematics register, reflects a unitary language perspective. Hence, while the notion of language has weak analytical power, it is politically powerful:

> The traditional idea of "a language" is an ideological artifact with very considerable power — it operates as a major ingredient in the apparatus of modern governmentality; it is played out in a wide variety of domains (education, immigration, high and popular culture etc.), and it can serve as an object of passionate personal attachment. (Blommaert and Rampton, 2011)

In contemporary sociolinguistics, heteroglossia is linked to superdiversity: the condition of multiple affiliations, identities, and hybridities in the context of global migration and communications. This work has led to some commonly used concepts being challenged. For example, the notion of native speaker proficiency is problematic, since the idea that a standard form of a language can be determined is ideologically loaded. The notion of native speaker proficiency instead serves deficit-based evaluations, with speakers being seen as reproducing with greater or worse fidelity some standard form of language. An alternative view is to see that speakers draw on complex repertoires of language practices. Different parts of an individual's repertoire are brought into play according to the needs of the situation. We all operate with broad repertoires, usually drawing to some extent on several different natural languages.

A final powerful idea introduced by Bakhtin (1981) is that there is constant and inherent tension between unitary language and heteroglossia. This tension is described in terms of the metaphor of centripetal and centrifugal forces. Centripetal forces represent the drive to standardize language, while centrifugal forces represent the opposing tendency for language to diversify through use. For Bakhtin, the tension between these two forces shapes every interaction. This idea explains well many of the findings in mathematics education research that expose tensions and challenges for teachers, learners, policymakers and families (Barwell, 2012). For example, research in South Africa has shown how an official policy of using one language in mathematics

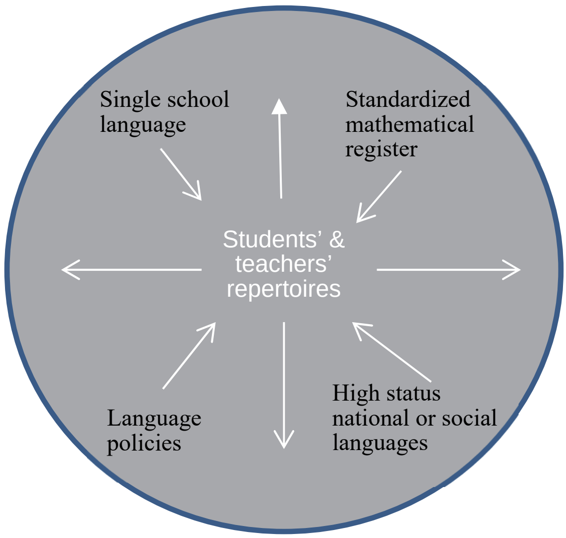

Fig. 1. Centripetal and centrifugal forces in mathematics classrooms

classrooms is not reflected in classroom interaction in which many languages may be used (e.g., Setati, 2005). Or teachers have reported challenges relating to the fact that learners use many diverse informal ways to talk about mathematics while the teacher aims to ensure that the use 'standard' mathematical discourse (Adler, 2001). These tensions are illustrated in the diagram in Fig. 1.

To sum up, language diversity is itself diverse. Multiple language practices are present in any given milieu. Mathematics classrooms feature speakers with diverse repertoires. Language practices are constantly changing across time and space. Hence language diversity matters in mathematics education because:

- Every mathematics classroom is a site of language diversity;
- Every mathematics learner has a different repertoire of language practices;
- Every mathematics teacher has a different repertoire of language practices;
- Every mathematics teacher must adjust their teaching in relation to the different repertoires of their learners;

- Interaction in every mathematics classroom is shaped by the tensions between a unitary language ideal of standard mathematical discourse and the heteroglossia of all language.

3. What do We Know about Language Diversity and Learning and Teaching Mathematics?

What difference, then, does language diversity make to learners and teachers? Research on language diversity and attainment in mathematics shows that in many situations, learners of mathematics who are learners of the classroom language under-achieve. In some situations, however, learners of mathematics who are learners of the classroom language match or exceed expected levels of attainment. A key factor is the learner's proficiency in the different languages they speak (see, for example, Clarkson, 2007). For example, learners who have high proficiency in any language (e.g., the classroom language, or a language they use at home) will have comparable mathematics attainment to monolingual learners, while learners who have high proficiency in two languages will tend to out-perform monolingual learners.

A major problem with this kind of research is its basis in a unitary language ideology. Research that compares multilingual learners with monolinguals assumes that monolingualism (usually in a globally dominant language like English) is the norm. It tends to assume that populations and speakers are homogenous in how they use language in mathematics. Moreover, such work relies in written tests which tend to favor 'native speakers' of the official classroom language (see Barwell, 2003a, for a discussion). We can conclude, therefore, that language diversity does make a difference to learners' performance in mathematics assessments, but these differences are relative to the assumptions about language embedded in the assessment instruments.

Research on language diversity has also examined learners' different practices in mathematics classroom activities to show how they may draw on varied and complex repertoires of languages and language practices to participate in mathematical meaning-making. Some of the language practices that have been identified in the research literature include (Barwell et al., 2017):

- Using multiple languages
- Code-switching, translanguaging
- Drawing on the features of different mathematics classroom genres (e.g. worksheets, proofs)
- Gesturing (pointing, tracing shapes or curves)
- Working with diagrams
- Using deixis (context-based words like "this" or "there")
- Using mathematical discourse features (ways of explaining, justifying, questioning, etc.)
- Using mathematical vocabulary
- Using everyday forms of expression

Much attention has been given to theorizing these practices and their role in learning mathematics. The prevailing theoretical approach is based on the idea of language as a resource (e.g., Planas, 2018; Planas and Setati-Phakeng, 2014). Code-switching, or gestures, for example, have been cited as resources that learners may use. I have argued that this perspective is based on a view of language that is too static. As an alternative, I have proposed the idea of language as a source of meaning (Barwell, 2018).

An example of learners' diverse repertoires of language practices is provided by in a study conducted in the UK, conducted in a language diverse Year 5 primary school class (Barwell, 2003b). The study revealed four significant sources of meaning (as I would now call them) for learners in relation to mathematical word problems:

- Mathematical structure (i.e., the arithmetic relationships involved in a word problem);
- Narrative (in the form of learners' accounts of their own experience);
- Genre (i.e., the expected features and organization of word problems);
- Language features (such as verb tense, spelling, in this case in English).

Learners drew on these four sources of meaning to construct and interpret arithmetic word problems. In the following exchange, for example, Cynthia and Helena are devising a subtraction word problem (from Barwell, 2003b). Cynthia first went to school in Hong Kong SAR and speaks Cantonese. Helena grew up in the UK, speaks English.

Helena	Cynthia has thirty pounds for/
Cynthia	no/ not for her mum/ if I bought/ for my mum
Helena	for her mum's present
Cynthia	if give my mum thirty pound I bought nothing from her/ that not make sense
Helena	no/ I won't writing for you mother/ I said Cynthia has thirty pounds for her mother's present
Cynthia	thirty pound/ I gave thirty pound for my mum present
Helena	no/ I didn't say give it to her
Cynthia	then how why you
Helena	you have thirty pounds [for your mum's present
Cynthia	[no but/ I think this make sense/ Cynthia has thirty/ pound/ thirty pound/ she bought err something something something/ it's cost something something/ from her mum present/ and how much she left?/ is that make sense little bit

In this exchange between Cynthia and Helena, all four of these sources of meaning are apparent. With respect to *language features*, they negotiate the use of words like "for", "from", "give" and "have". Cynthia provides a concise *generic* summary of how a word problem should be organized, complete with blanks ("something something something") where the numerical values can be inserted. She relates their word problem to the *narrative experience* of buying a gift for her mother. In subsequent discussion, she refers to what her mother likes to determine what the gift could be. Finally, the problem has a clear *mathematical structure* and in the full discussion, they discuss how to ensure a mathematically coherent subtraction problem.

Another strand of research has focused on how mathematics teachers deal with language diversity in their classrooms. Research has shown, for example, that teachers may struggle with working with language diversity, facing various challenges and dilemmas (e.g., Adler, 2001). Faced with such challenges, teachers may initially focus on mathematical vocabulary more than other aspects of language. The findings referred to above, however, suggests that vocabulary is just one source of meaning for learners. Additional research shows that mathematics teachers can and do develop skillful and supportive practices. Promising approaches include: those which draw on students' full repertoires and help them to make connections between different parts of these repertoires; those which have a strong focus on mathematical meaning and thinking; and those which integrate language learning with mathematics learning (Moschkovich, 2018). Design-based approaches hold much promise as a systematic way to identify and develop productive teaching strategies along these lines (Prediger, 2019).

Taken together, research clearly demonstrates that multilingual learners, including those learning the classroom language, can participate successfully in mathematics classroom activities in the right conditions. I call them *language positive classrooms*, since they feature practices that acknowledge and incorporate learners' languages. I recently described some of the potential characteristics of such a classroom, based on findings from an ethnographic study of four second language mathematics classrooms in Canada (Barwell, 2020). The four classrooms were: one of Indigenous learners, one of recent arrivals to Canada, one in a French immersion program, and a mainstream classroom featuring some second language learners. The study examined socialization practices across the four classrooms resulting in a distinction between language positive and language neutral classrooms. In language positive classrooms, I observed the following common practices:

- Students' home languages were regularly heard; students reference or use home languages during mathematical discussion.
- "Non-standard" accents, pronunciation, spelling or punctuation were present and explicitly related to norms.
- Explicit attention was given to features of mathematical discourse: learners actively participated in socialization into mathematical discourse.
- Relations between more formal and more informal mathematical discourses were made visible.
- Working with mathematics classroom genres was inclusively supported through specific socialization practices.
- Gestures in mathematical interaction were actively used by learners and teachers; explicit links were made between gestures and other aspects of mathematical discourse.
- Explaining mathematical thinking was actively supported.

- Learners participated actively, taking extended turns or sequences of turns and initiating exchanges.

To sum up this section, language diversity matters in mathematics education because:

- It can affect positively or negatively learners' mathematics attainment;
- It affects how learners participate in mathematics class, including with respect to specific subdomains of mathematics;
- Learners bring diverse repertoires of language practices;
- It affects how teachers teach mathematics, including with respect to specific subdomains of mathematics;
- Some mathematics classroom conditions are more favorable in relation to some kinds of language diversity.

4. How is Language Diversity in Mathematics Classrooms Connected to Wider Social Structures?

Research on political aspects of language diversity and mathematics has shown how the relative status of different languages influences the choices of students and teachers in mathematics classrooms. Setati (2008), for example, showed how families in South Africa often preferred English as the classroom language even though it could make learning mathematics more challenging, since English was perceived as valuable for accessing better educational or employment opportunities. It is also clear that learning and teaching mathematics are organized in relation to the politics of language. In Canada, for example, there is still no widespread support for the use of Indigenous languages in education. In many countries, language diversity is not represented in mathematics policy or curriculum, which often privilege one dominant language. It is also worth noting that the discourse of mathematics education is itself also organized in relation to the politics of language. The papers in this proceedings, for example, are written in English, which favors researchers used to working in that language (see Barwell, 2003a).

The political organization of language diversity can be theorized using Bakhtin's ideas. As noted above, centripetal language forces represent the societal pressure to standardize language. These forces tend to align with dominant groups or ideologies. Meanwhile centrifugal language forces indicate the opposing tendency for language to diversify, apparent in the heteroglossia of human interaction. Centrifugal forces tend to be driven by the whole of society and as such create the space for its marginalized members. These two forces are in tension and produce a stratification of language in society, whereby accepted forms of unitary language are preferred over divergent forms, and hence speakers who use these accepted forms are able to access the dominant groups. This theorization explains the tension that Setati (2008) recorded in South Africa. Stratification occurs through, among other things, patterns of indexicality, which refers to the symbolic value of different forms of language:

> Ordered indexicalities operate within large stratified complexes in which some forms of semiosis are systemically perceived as valuable, others as less valuable and some are not taken into account at all, while all are subject to rules of access and regulations as to circulation. That means that such systemic patterns of indexicality are also systemic patterns of authority, of control and evaluation, and hence of inclusion and exclusion by real or perceived others. (Blommaert, 2010, p. 38)

Thus, learners who can display valued forms of mathematical discourse are more likely to be seen as knowledgeable. Conversely, many learners may be marginalized by discourses found in mathematics classrooms.

As an example, consider the discussion shown below, recorded in the class of Indigenous Cree learners as part of the ethnographic study mentioned in the previous section. Most of the students are from the James Bay region, far to the north of the city in which they attend school. The discussion took place between me, Curtis and Ben, as they worked on a problem that included the following text (see Barwell, 2014):

> Every year Ottawa holds a world-renowned tulip festival in the month of May. There are different gardens in various locations, one of which is on Parliament Hill. The Canadian Tulip Festival was established to honour Queen Juliana of the Netherlands, in 1953. It is the largest tulip festival in the world, making this flower the International symbol of friendship and the beauty of spring. This festival receives thousands of tourists every year from North America, Europe, and Asia and has an economic impact of approximately $50 million on the Ottawa region.

In the following extract from the discussion, we read through part of the problem, which positions the reader as a gardener and asks them to complete a pattern of increasing square designs composed of tulips:

RB	You’re going to draw poppies = well its about tulips haha ok (.) so: maybe we can read out that part again (.) ^what do you think^ Ben can you read it out alright Curtis?
Curtis	you are the gar(.)dener (.) for tulips
Ben	tulips
Curtis	ah what’s that?
RB	**bulbs**
Curtis	bulbs (.) for Canadian tulip festival in may
RB	so you are a **gardener** (.) do you feel like a **gardener**?
Curtis	^what^
RB	are **you** a gardener Ben?
Ben	^what?^
RB	are you a gardener (.) do you know what a gardener is=yeah it says in this problem you have to think you are a gardener okay (.) to plant tulip bulbs (.)

In a second extract, from a few minutes later, I support Curtis to write a suitable account of his reasoning:

RB	so (.) that's a good beginning (.) but you need to explain like the calculations that you did (.) you need to say what kind of calculations you did
Curtis	times
RB	yup but precisely what did you times what did you add
Curtis	I timesed seven (.) times seven (.) six times (.)
RB	right right
Curtis	seven plus thats it
RB	so like when you worked out for purple
Curtis	I did five times five
RB	uhum
Curtis	plus one
RB	right so I would write purple and then exactly what you just said

My analysis of the full text of the tulip problem, as well as of the interaction relating to the problem, shows that it indexes certain standard ways of languaging mathematics in relation to the world and thereby alienates the two students in particular ways, as shown in Tab. 1.

Tab. 1. Indexicality and alienation in relation to a word problem

Indexed by the Tulip problem	Alienated by the same problem
A nation (Canada)	The students' Cree nation
A region (Ottawa)	The students' home region of James Bay
An event (the tulip festival)	Events in the students' communities
Speakers of a language (English)	Speakers of Cree
A register (school mathematics)	Informal mathematical expression in English and Cree
A genre (word problems)	Everyday and Cree genres
Some mathematics (geometry and arithmetic)	Mathematics in Cree activities

This example illustrates how the tension between centripetal forces, as demonstrated by the left-hand column of Tab. 1, and centrifugal forces, apparent in both columns, result in the marginalization of the two students. A very concrete instance of this marginalization is the fact that they are unfamiliar with tulips, since they do not grow in James Bay.

In this section, I have shown how language diversity in mathematics education matters because:

- Mathematics learners whose diverse repertoires do not fit the prevailing dominant standard forms of language in mathematics classroom risk being alienated; i.e. if you don't talk and write in the right way, you won't succeed;
- Mathematics classrooms risk maintaining a social hierarchy based on (forms of) language.

5. How is Language Diversity in Mathematics Education Connected to the Future of Planet Earth?

Although much of my research has focused on language diversity in mathematics classrooms, I have also written about environmental sustainability and climate change in mathematics education (e.g., Barwell, 2013; Barwell and Hauge, 2021; Hauge and Barwell, 2017). This work is a response to the multiple crises facing our planetary ecosystem, such as climate change, mass extinction, pollution and ecosystem degradation. It turns out that there is a connection between language diversity and

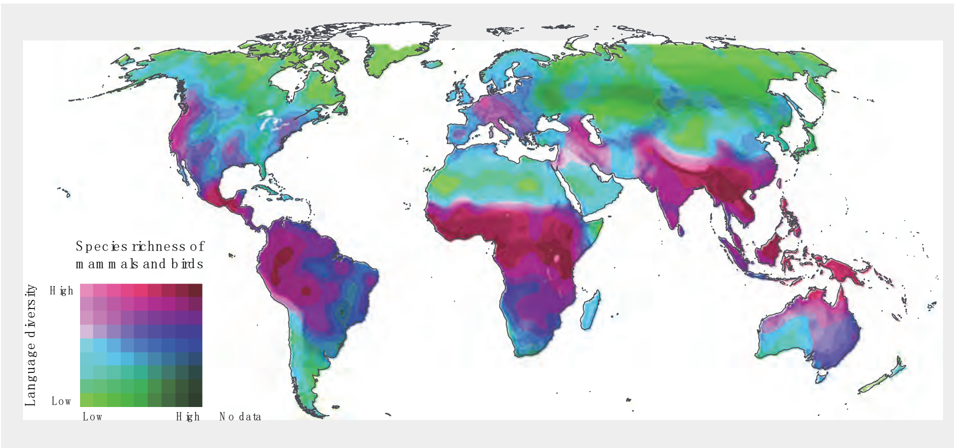

Fig. 2. Language diversity and biodiversity are spatially associated (IPBES, 2018, Assessment report on land degradation and restoration, p. xxxiv. Copyright © 2018, Intergovernmental Science-Policy Platform on Biodiversity and Ecosystem Services)

biodiversity: a recent report shows how regions of the world with a high degree of language diversity are also regions with a high degree of biodiversity (see Fig. 2) (IPBES, 2018).

The report highlights evidence that loss of language and culture, specifically Indigenous language and cultures, is associated with loss of ecosystem knowledge, including knowledge about different species, the ecosystem and its various interconnected relations, as well as sustainable practices for maintaining a healthy ecosystem. For example, the report states:

> Alienation of indigenous peoples and local communities from the land often leads to the irreversible loss of accumulated knowledge on how to manage land. In most cases, land management practices based on indigenous and local knowledge have proven to be sustainable over long time periods and offer alternative models to the currently dominant human-nature relationship. (IPBES, 2018, p. xxxv)

There is also a clear association with economic and political marginalization, as well as individual and community well-being:

> Land degradation causes a loss of sense of place and of spiritual connection to the land, in indigenous peoples and local communities as well as in urban residents living far from the affected areas. (IPBES, 2018, p. xxxiv)

The report refers to the many Indigenous or local knowledge systems which recognize the interdependent relationship between humans and other species, and that assume a stance of relational ethics rather that one of technological progress of economic growth (p. xxxv). Research in mathematics education is beginning to incorporate similar perspectives (e.g., Boylan, 2016; Coles, 2017; Gutiérrez, 2017).

A perspective of interdependence and relationality stands in contrast to prevailing ways of thinking in mathematics education and in dominant economic systems such as neoliberal capitalism, which tend to be based on ordering, hierarchy and dominance. As I discussed in the previous section, mathematics classrooms are connected to social stratification. Such orderings in relation to language diversity in mathematics education are also apparent at a macro scale. For example, looking across Canada as a whole, there is a clear ordering in terms of which languages are valued in mathematics classrooms: 1. English (the official language of the majority of Canadians); 2. French (the official language of a minority of Canadians); 3. Languages of immigrant communities, such as Chinese or Spanish; 4. Indigenous languages, such as Cree or Inuktitut. This kind of ordering contributes to the disappearance of minority languages around the world. These hierarchies can be translated into the specific orderings of language in mathematics classrooms. In Canadian mathematics classrooms, the ordering is: 1. English mathematics register; 2. French mathematics register; 3. other mathematics registers. Of course, there are regional variations (in Quebec, French is dominant, followed by English, for example), but overall, there is a hierarchy, in which learning and teaching mathematics is inscribed and which it reinforces. Similar orderings can, I suggest, be found in most mathematics classrooms in the world.

Hence, mathematics education, perhaps inadvertently, reinforces an ordering way of thinking. This way of thinking has been identified as a "root metaphor" (Bowers, 2001) underpinning many aspects of the environmental crises we currently face (Martusewicz et al., 2014). In relation to biodiversity, common scientific, mathematical and societal discourses are related to orderings (of species) and alienation (of humans from other species). For example, one ordering of species might be: 1. humans; 2. farm animals (e.g., chickens, sheep); 3. agricultural crops (e.g., rice, wheat); 4. wild animals (e.g., wolves, seals); 5. wild plants (e.g., forests); 6. insects; 7. viruses. Such orderings are generally in relation to how useful a species is to human exploitation. In a similar way, orderings of languages are often justified in terms of how useful a language is within the dominant economic system, as found in Setati's (2008) research, for example. Such orderings, as shown in the previous section, also alienate; in particular, they alienate speakers of languages that are positioned as marginal or peripheral, with respect to the dominant, unitary language forms. These unitary language forms are related to colonial, often Eurocentric, ideologies that privilege Eurocentric ways of thinking and make it difficult to build relational practices and ways of thinking. Teaching mathematics most often involves reinforcing these ways of thinking, and requires innovative, decolonizing approaches to disrupt the prevailing order (see Parra and Valero, 2022, for an example of such work in Columbia).

Language diversity therefore matters in mathematics education because:

- Ordered patterns of thinking about languages and ordered patterns of thinking about biodiversity use the same underlying way of thinking, and result in the same pattern of alienation;
- Ordered hierarchies reflect a Eurocentric, colonial, unitary perspective on language and on species;
- The ordering way of thinking makes ethical relationality difficult;
- Many languages and many species are threatened with extinction.

6. Conclusion

In this paper, I have examined several different reasons why language diversity matters in mathematics education. First, and fundamentally, language diversity matters because mathematics classrooms around the world are diverse. Moreover, the language diversity that is present in mathematics classrooms is itself diverse and constantly changing. Second, language diversity matters because learners of mathematics draw on many aspects of their repertoires of language practices to participate in mathematical meaning-making. Teachers need to adopt strategies that are responsive to language diversity, such as those found in language positive classrooms. Third, language diversity matters because it is one of the links between mathematics classrooms and wider society. Mathematics classrooms are sites of social stratification, linked to the stratification of language. Failure to acknowledge such links risks mathematics classrooms reproducing patterns of marginalization and alienation that are present more widely. Finally, language diversity matters in mathematics education because the way we think about language diversity often reflects deeply embedded ordering discourses that reflect Eurocentric, colonial ways of thinking. These discourses are also implicated in the ecological crisis facing our planet, through a dominant ordering of humans with respect to other species.

There are some clear implications arising from these points. First, language diversity should be more widely recognized in mathematics education research and in mathematics curriculum and policy. Most research in the field, for example, ignores the presence of language diversity among research participants and does not consider the possible implications of this diversity on research design, data collection or analysis and interpretation of data and findings. By extension, it would make sense to incorporate language diversity into theories of learning and teaching mathematics.

Similarly, strategies for teaching mathematics in contexts of language diversity should be the basis for any general guidance for teachers, since all contexts potentially feature such diversity. While centripetal and centrifugal language forces are always present and the tension between them shapes every utterance, research, policy and teaching can all work to ensure that unitary language ideologies are made visible and that there is space for heteroglossia; in effect, a dialogic approach to language diversity is more likely to avoid rigid stratification and consequent alienation. Another way to say this is to argue for an approach informed by ethical relationality with respect to learners, teachers, languages and other species that recognizes the interdependence between them (and us) all.

Acknowledgments

Some of the research referred to in this paper was funded by the Social Science and Humanities Research Council of Canada. I am grateful to the many children, teachers and schools who have invited me into their mathematics classes in the course of the different studies referred to in this paper. I also thank the many research assistants who have contributed to different aspects of these studies. Finally, this work is only what it is thanks to interactions with and influences of many inspiring friends and colleagues.

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05

Seeking Social Justice in Mathematics Teaching and Learning

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ABSTRACT This session article unpacks mathematics teaching and learning focused on racial equity and social justice. Specifically, the session will explore the intersection of mathematics teaching and learning with racial equity and social justice across four critical reasons: a) Building an informed society; b) Connecting mathematics to cultural and community histories as valuable resources; c) Confronting and solving real-world mathematics as a tool to confront inequitable and unjust contexts; d) Use mathematics as a tool for democracy and creating a more just society.

Keywords: Social Justice; Mathematics; Pedagogy.

1. Teaching Mathematics for Social Justice

1.1. Four critical reasons to teach mathematics for social justice

Teaching mathematics for social justice supports situating mathematics content and concepts in contexts that allow students to use their cultural, social, and contextual resources to deepen their understanding of mathematics. By deepening students' understanding of mathematics, teaching mathematics for social justice provides opportunities to use mathematics to critique the world, understand the connections between social and cultural issues that impact people's lives, and advocate for social changes (Berry et al., 2020). To teach mathematics for social justice, teachers must first appreciate students' cultures, understand the development of knowledge within students' cultural frameworks, and recognize that the interpretation of information and mathematics happens within students' cultural and experiential frameworks (Rubel, 2017). Teaching mathematics for social justice goes beyond stating the importance of connecting mathematics to lived experiences and interests; it positions students as actors in their world. Teaching mathematics for social justice is critical for four reasons:

- *Builds an informed society.* To build an informed society, students must become better informed about their lives and the lives of others who may be

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different from their own. Mathematics serves a role to inform teachers and students about people's lives, contexts, and conditions that may be different from their own (Ladson-Billings and Tate, 1995).

- *Connects mathematics with students' cultural and community histories.* Too often, students' mathematics experiences in school are detached from meaningful contexts. Teaching mathematics for social justice creates opportunities for deepening mathematical knowledge by connecting mathematics teaching and learning to cultural and communal histories (Ladson-Billings and Tate, 1995).
- *Empowers students to confront and solve real-world challenges they face.* Empowering students requires identifying unjust issues and using mathematics as a tool to analyze, critique, and confront unjust issues (Ladson-Billings and Tate, 1995).
- *Helps students learn to use mathematics as a tool for social change.* The potential for education is to support students to better their lives and better society. When we use mathematics to explore, understand, and respond to social injustices, we learn to use mathematics as a tool to transform inequities and create social change (Ladson-Billings and Tate, 1995).

1.2. Creating lessons for teaching mathematics about social justice

Teaching mathematics for social justice includes the National Council of Teachers of Mathematics (2014) eight effective mathematics teaching practices and requires educators to understand and demonstrate pedagogies associated with four bodies of work associated with equitable teaching practices in a nested relationship (Picha, 2019). Fig. 1 demonstrates the nested relationship of equity-driven mathematics teaching frameworks: Standards-Based Mathematics Instruction (NCTM, 2014), Complex Instruction (Featherstone et al., 2011; Horn, 2012), Culturally Relevant Pedagogy (Ladson-Billings, 1994), and Critical Mathematics Education (Frankenstein, 1983; Freire, 2000; Powell, 1995; Skovsmose, 1995).

Fig. 1. Equity-driven mathematics teaching frameworks — a nested relationship (picha, 2019)

- Standards-Based Mathematics Instruction emphasizes learning mathematics for understanding over attending primarily to fluency with algorithms and facts (NCTM, 2014).
- Complex Instruction values many different ways of being mathematically "smart" (Featherstone et al., 2011).

- Culturally Relevant Pedagogy ensures that equitable instruction draws on students' cultural practices, experiences, and assets to build academic excellence and critical consciousness (Ladson-Billings, 1994).
- Critical Mathematics Education extends the tenet of critical consciousness from CRP to explicitly attend to power, fairness, and social justice (Freire, 2000).

The teacher plays a critical role in students' educational experience by bringing forward important mathematics and social issues to be learned. Student voices are elevated in the classroom are critical to implementing a social justice mathematics lesson. The intersection of these experiences and questions begins a social justice mathematics lesson (Fig. 2). Often, the "challenging social and mathematical question or concern" generated by students, along with the "action and public product", extends outside the classroom into the school community and continues to evolve based on previous actions and students' power to respond to social justice issues. However, during the classroom teaching episode, the teacher can create opportunities that deepen students' mathematical and social understanding through purposeful investigations that encourage reflection to develop their critical consciousness.

As students complete a social justice mathematics lesson, it is important to note that three inner elements seen in Fig. 2 are not mutually exclusive, likely realized throughout a lesson at different points for different students. Allowing students to grapple with the lesson's social and mathematical goals should be handled carefully. While some students may be wrestling to make sense of data or understand a mathematical analysis, others may be confronted with data or mathematics that dispels a former belief. Teachers should be attentive to the intersection of mathematics and social injustice and establish and attend to goals specific to each domain — math and social justice (Teaching Tolerance, 2016).

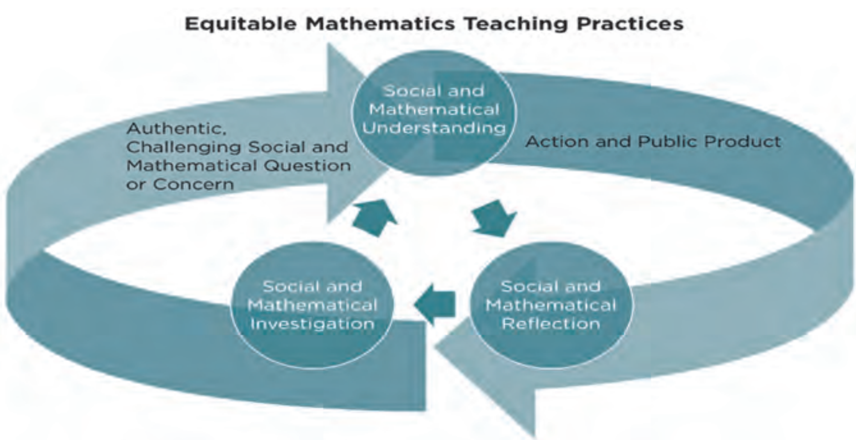

Fig. 2. Equitable Mathematics Teaching Practices (Berry et al., 2020)

Authentic, Challenging Social and Mathematical question or concern: A social justice mathematics lesson must be grounded in a question or concern that could arise from students, allowing for authentic and challenging learning. Examples of social justice topics include civil rights, laws, environmental rights, identity issues, health, immigration, and racism. These contexts can help students observe patterns, critique information, ask questions, and reflect.

- Social and Mathematical Understanding: A social justice mathematics lesson must identify what students need to know and understand mathematically and socially. A social justice mathematics lesson identify and provide opportunities to assess three goals: a) students' understanding mathematics content, b) engaging in mathematics practice for students to show what they know and can do, and c) social justice for students to demonstrate their understanding of and response to social justice issues.
- Social and Mathematical Investigation: Because tasks emerge from students' questions or concerns, a social justice mathematics lesson needs to be grounded in a mathematically driven investigation of a social justice issue.
- Social and Mathematical Reflection: A social justice mathematics lesson should promote reflection about the mathematics, social justice issue, and how the two inform one another.
- Action and Public Product: A social justice mathematics lesson must include an opportunity for students to take action or develop a public product.

1.3. Enacting a vision of teaching mathematics for social justice

Incorporating social justice in the mathematics classroom points to students' need to design and take action on what they have learned. A teacher's practices and students' responses are "founded on the belief that mathematics is the tool to be used to challenge the status quo that is adversely impacted by the lack of social justice" (Berry et al., 2020; p. 1). This can and should be the natural cycle for teaching mathematics for social justice, a process launched by student's authentic and rich questions or concerns about their school, community, world, and lives that through mathematization — investigation, understanding, and reflection — they are compelled to take action or create a public product (Fig. 2; Berry et al., 2020). Once students have mathematized and investigated a social justice issue, more in-depth understanding and awareness is a personal growth outcome that might be expressed in the way an individual interacts with others through deeper learning about identity, diversity, and justice (Berry et al., 2020):

- *Identity* — how we view ourselves;
- *Diversity* — how we view others and their perspectives; and
- *Justice* — how we view fairness and unfairness, unequal power relations, and the impact of bias.

However, unless some form of action is included in a lesson, the work to teach mathematics fo*r* social justice misses a key component — for students to see themselves as able to have an impact on their world, as both "an actor and author of history" (Garcia, 1974, p. 16). The Social Justice standards developed by Teaching Tolerance (2016) provide age-appropriate learning outcomes in four domains — identity, diversity, justice, and action. Below is an overview of the anchoring standards for learning outcomes for the action domain.

- Students will express empathy when excluded or mistreated because of their identities and concerns when they experience bias.
- Students will recognize their responsibility to stand up to exclusion, prejudice, and injustice.
- Students will speak with courage and respect when they or someone else has been hurt or wronged by bias.
- Students will make principled decisions about when and how to stand against bias and injustice in their everyday lives and do so despite negative peer or group pressure.
- Students will plan and carry out collective action against bias and injustice globally and evaluate the most effective strategies.

These anchor standards can help teachers provide some framework and guidance to students' ideas about what to do with their mathematical analysis and a more in-depth understanding of the social injustice being studied.

Examples of actions identified in social justice mathematics lessons are (Berry et al., 2020):

- Develop and present an infographic.
- Design and post informative social media posts.
- Begin an informational campaign, including a variety of public service announcements (posters, flyers, other creative media).
- Organize a letter-writing campaign.
- Present to a school council meeting or school board meeting.
- Invite a panel of community members to discuss the topic in a public forum.
- Start a community-based reading club.
- Conduct a household inventory/analysis.
- Arrange a meeting with a local, county, or state government representative.

A social justice mathematics lesson must ensure the opportunity for reflection and action. As you design your lesson, consider what options you might provide for students to reflect on what they've learned and to discuss possible actions they can take to make the first steps toward addressing an injustice. In addition, the lesson must consider ways students share what they have learned about social injustice and ways they use mathematics to bring greater insight into the issue. Mathematics has great potential to empower students, not only to analyze complex situations but also to develop confidence and a positive identity. Taking action and engaging in social justice curricular experiences empower students to stand up to the exclusion, prejudice, and bias in many contexts of their lives. By supporting them in deciding upon and designing an appropriate and effective response to social injustice, grounded in a mathematical rationale, they are rehearsing their future work as uniquely empowered activists against social injustice.

2. Time for Action

Responding to social justice issues requires a commitment to serving our global society. Mathematics teachers and teacher educators can respond by infusing social justice into mathematics teaching and learning. Now is the time to determine how we will teach mathematics about, with, and for social justice so that the goal of facilitating authentic, meaningful relationships between students becomes a lived reality.

First, commit to reading the position papers below. Then, reflect on how they inform your understanding of social justice in mathematics teaching and learning and what questions or wonderings you might need to explore further.

- TODOS: Mathematics for ALL. (2020). The mo(ve)ment to prioritize antiracist mathematics: Planning for this and every school year. https://www.todos-math.org/statements
- Benjamin Banneker Association, Inc. (2017). Implementing a social justice curriculum: Practices to support the participation and success of African-American students in mathematics. http://bbamath.org/wp-content/uploads/2017/11/BBA-Social-Justice-Position-Paper_Final.pdf
- National Council of Supervisors of Mathematics & TODOS: Mathematics for ALL. (2016). Mathematics education through the lens of social justice: Acknowledgment, actions, and accountability. https://www.todos-math.org/socialjustice

Next, determine a starting point by envisioning what a classroom may look like and sound like that is ready to tackle the injustices of students' lives. Then, identify a goal and list the steps you will take the next 3-, 6-, and 12-months to make the vision become a reality.

Finally, share your vision with others and invite them to hold you accountable and support you as you bring social justice to your mathematics classroom (Staley, 2018). Accountability partners hold one accountable for their actions, words, and beliefs. Teaching mathematics for social justice requires shifts in teaching and mindsets.

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https://doi.org/10.1142/9789811287183_0006

06

Challenging Tasks: Opportunities for Learning

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ABSTRACT Challenging tasks are essential in developing and demonstrating mathematical understanding. They provide opportunities to learn and the motivation for student to engage with learning. This chapter highlights how the real-world and digital technologies provide many opportunities to design and implement challenging tasks for all learners. The affordances of technology-rich teaching and learning environments need more attention if teachers and their students are to be better enabled to maximise opportunities for learning mathematics. A range of tasks are presented and discussed. Planning by teachers for varied student responses is critical in enabling 'as needed' in-the-moment scaffolding to keep students engaged with mathematical thinking.

Keywords: Affordances; Challenging tasks; Cognitive demand; Digital technologies; Engagement; Real-world.

1. Challenge

Jaworski (1992) described mathematical challenge as part of what is required, for students, to learn mathematics. She argued that mathematical challenge can only be realised where attention also is given to the supportive learning environment that fosters learning. Challenge "involves stimulating mathematical thought and enquiry, and motivating students to become engaged in mathematics thinking" (p. 8). It influences task design and implementation and the environment where learning occurs. Opportunities to engage in mathematical thinking, "cannot be taken up if it is inappropriate, or if strategies for handling it have not been created … but challenge is required to get mathematics done" (p. 14).

A focus on the learning environment was explored by Wood (2002) and others with a particular focus on how the expectations correlated with the level of mathematical thinking. Wood showed how elements of a classroom culture that fostered the development of mathematical thinking included shifting the responsibility for thinking and participation in discussion from the teacher to the students. Critically, she describes the need to enable students to become active listeners and explainers. This notion that the focus on mathematical thinking in the classroom should be undertaken by learners, facilitated, and scaffolded by the teacher as needed, with the mathematical thinking of the teacher occurring beforehand (i.e., during planning), has been well explored by Smith and Stein and colleagues (e.g., Smith and Stein, 2011;

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Stein, Engle, Smith, and Hughes, 2008). These expectations, of learners in class, and teachers as part of planning, are critical to challenge, and hence mathematical learning (Jaworski, 1992) being realised. The importance of the teacher's role is further detailed by Baxter and Williams' (2010) discussion of discourse-oriented teaching.

2. Challenging Tasks

A *challenging* task is, by definition, high in cognitive demand. However, what teachers say or do as students begin tasks, typically reduces the cognitive demand faced by students (González and Eli, 2017; Smith and Stein, 2011). Interactions between the teacher and student, particularly, but not limited to, the beginning stages of task solving can be planned for or involve in-the-moment decisions. Challenging tasks should engage students in their learning of mathematics. Motivation to learn and to value mathematics can be influenced by the task itself and how both teacher and students interact with the task. Participation by countries in PISA suggests it is a clear expectation that students do engage successfully with challenging tasks.

Several frameworks have been developed to consider the level of challenge of tasks and their implementation. Stillman (2001) developed a cognitive demand profile to analyse applications tasks, particularly those intended for upper secondary mathematics students. She notes, "the cognitive demand is related to the interaction between the mathematical demand of the task and the extent to which the mathematics needed to model the situation is embedded in its description" (p. 459). Stillman, Edwards and Brown (2004) extended this tool to consider the task implementation, that is, what the teacher and students "do" with the intended task. They developed a framework "for engineering the cognitive demand of tasks, lessons and lesson sequences" (p. 489) that considered three mediators of cognitive demand, namely, task scaffolding, task complexity, and complexity of technology use. They found that by "orchestrating the interplay between degree of task scaffolding, task complexity, and complexity of technology use, teachers are able to craft lesson sequences involving tasks of appropriate level of challenge for their students" (p. 500).

Challenging tasks can be of a variety of types. They include problem solving tasks of varying length, mathematical modelling where the focus is on solving a real-world problem, and investigative tasks. They involve "doing mathematics", as distinct from memorisation or procedures without connections, and are of high cognitive demand (Smith and Stein, 2011). Barbeau describes a challenge as, "a question posed deliberately to entice its recipient to attempt a resolution, while at the same time stretching their understanding and knowledge" (2009, p. 5). Noting that challenge depends on background and interest, he argues that good challenges involve explanation, questioning, multiple possible approaches, and evaluation of the solution.

Following Francisco and Maher (2005), learners often develop rich understandings of key ideas through solving complex, challenging tasks, particularly when given the opportunity to explore inter-related tasks. For example, students learn to model as they engage with multiple modelling tasks over time. Students need time to develop modelling competencies through modelling, in the same ways students "need time to develop and come to understand the importance of a way of working based on sense making and justification of ideas" (2005, p. 368).

Challenging tasks are critical learning activities at all levels of schooling. Without mathematically challenging tasks being a normal part of the teaching and learning environment, opportunities to learn are limited. Real-world tasks are by their very nature challenging. By real-world tasks, I refer to task solvers solving real problems. This includes making sense of the real-world context, making decisions about what is relevant and important, and mathematising the problem — bringing the problem into the mathematical world so that it can be solved. Once solved, the mathematical solution must be interpreted in terms of the real world to ascertain what the real-world solution is and if this is acceptable. Typically, in mathematical modelling, the initial real-world solution would be explored further, perhaps by relaxing some of the simplifying assumptions, accounting for additional factors, or revisiting estimates of important factors, to seek an improved real-world solution. At the very least, students should reflect on their solution and consider varied assumptions or estimates or approaches.

Teachers play a critical role in providing students with challenging tasks, and also ensuring the cognitive demand remains high during task implementation. A challenge for many teachers is to maximise the mathematical thinking done by the students, rather than themselves during the lesson. A technology-rich teaching and learning environment also poses a challenge for teachers. Affordances of such an environment must be both perceived and enacted by students during task. Prior to this occurring, teachers must provide opportunities for students to experience such affordances and consider their applicability or usefulness in different mathematical situations.

2.1. Issues implementing challenging tasks

According to Sullivan et al. (2015), teachers typically view mathematics as procedural (p. 124). This view of mathematics as a set of disconnected skills, results in the belief that challenging tasks have no place in the teaching and learning of mathematics. Furthermore, teachers consider students "reluctant to engage with challenge … and unwilling to persist" (p. 124) when faced with challenge. Furthermore, teachers typically reduce the cognitive demand of tasks when planning. Earlier, Stein et al. (2008) found that teachers tend to overexplain tasks when implementing them, thus reducing the cognitive demand for students as some of the mathematical thinking is undertaken by the teacher rather than students. Russo and Hopkins (2019) identified "issues related to teachers' self-perceived capacity to teach with such tasks" (p. 760).

Consideration is also needed when it comes to student engagement with challenging tasks, when they do experience them. Williams (2014) makes an important distinction between persistence and perseverance. She describes perseverance as occurring when task solvers, find ways to proceed toward successes when situations are unfamiliar, and a clear pathway is not apparent. Williams (2014) found that elements of perseverance underpin creative problem-solving. In contrast, persistence is when the task solver keeps on trying, no matter the quality of the attempt, when difficulties are encountered (p. 420). Recognising this difference will enable teachers and students to consider alternative pathways when blockages (Stillman, Galbraith, Brown, and Edwards, 2007) or dead ends are reached. Furthermore, anticipating solution pathways (e.g., Stillman and Brown, 2014) might also see task solvers more likely to persevere rather than simply persist, or give up. Teachers might shift their focus from encouragement to prompting about considering alternative pathways.

3. Factors Impacting on the Level of Challenge

In this section three factors that impact on the level and extent of challenge possible, and being realized, are discussed. Initially each of these, the real-world, affordances and digital technologies, are carefully defined. This is critical as, in particular the first two are used in multiple ways in the literature and too often without definition. Interactions between the factors are then discussed.

3.1. Real-world

Mathematical modelling tasks in school involve a genuine link with the real-world. The real-world is critical at the beginning and end of the task. It is also necessary to keep in mind throughout the solution process although at times to a lesser extent than others. This is referred to by Stillman (1998) as context as tapestry rather than context as wrapper or context as border. In the latter two categories the real-world can be thrown away after initial consideration or ignored altogether as it is superficial to the problem. (See also Brown, 2019). This is not the case in genuine modelling tasks.

Solving modelling tasks necessitates some level of complexity of mathematical thinking, that is, higher order thinking, by task solvers (Brown and Edwards, 2011). Higher order thinking is "taken to mean instances where there is evidence that a student appropriately *makes choices* about the solution path; … makes links across representations; expects to verify a conjectured solution; appreciates the value of, or need for verification" (Brown and Edwards, 2011, p. 190). Resnick (1987) noted that whilst defining higher order thinking may be problematic, recognising it is not.

3.2. Affordances

Following Gibson (1979) who invented the term, *affordance*s are opportunities for interactivity between actors and their environment expressed in a linguistic form (i.e., < … > -ability) to indicate this *opportunity*. Brophy (2008) describes the affordances insightfulness-ability, understand-ability, information processing-ability, problem-solving-ability and decision-making-ability as important in increasing motivation to learn and valuing of mathematics as a discipline by students.

Some see affordances as a property of an object, but this is not the interpretation taken here. Following Gibson (1979), affordances are part of an actor-environment system. "If an affordance exists, in order to avail oneself of the opportunity for interactivity, the actor must act (on or with the object). The precursor to acting is *perceiving* — without which the actor cannot act when the affordance is for teaching or learning" (Brown and Stillman 2014, p. 112).

3.3. Digital technologies

Access to digital technologies can change the complexity of tasks student explore. More complex or larger and hence more realistic data sets can be explored. Calculations beyond one's by-hand capabilities are accessible. Graphing calculator or equivalent computer-based technologies allow for the making and testing of conjectures, and exploring ideas, that would not otherwise be possible. CAS-enabled technologies provide upper secondary students the capabilities to apply calculus related ideas to any functions. Digital technology use provides opportunities for students to deepen and expand their mathematical knowledge. In many educational jurisdictions, digital tools are available and expected to be used for teaching and learning mathematics. It does not follow necessarily that challenging tasks become the norm. In other jurisdictions, for various reasons, digital tools are less accessible or absent.

Often when opportunities exist for teachers to deepen understanding or use more complex problems, afforded by the presence of available digital tools, they often do not do so even when possessing the relevant knowledge (Brown, 2013b, 2015a, 2015b, 2017). Similarly, students are reluctant to use their knowledge of the available digital tools to expand or deepen or demonstrate their mathematical understanding. Teacher expressed intentions to allow students to make decisions with respect to task interpretation and understanding and choice of digital tools and the ways they are used is often not realised in practice (Brown, 2017). Knowledge of teacher tactics and expected responses (Brown, 2013b) and potential student strategies (Brown, 2015a, 2015b) play a critical role in teacher preparedness for successful task implementation where the mathematical thinking is predominantly undertaken by the students.

3.4. An Illustration

Consider the *Platypus Task*, which is briefly described here. This task was implemented where digital tools, namely graphing calculators were expected to be used. The *Platypus Task* was solved by Year 11 students (aged 15–16 years). Their school was located close to the Yarra River where platypus have lived for many millenia. The platypus, an Australian monotreme (i.e., an egg laying mammal) is in danger of local extinction. The task context was used to either already be of interest to students or to encourage an interest in their local environment.

The task presented students with 'local area' data before and after an intervention project. Two questions posed were: What was predicted for the platypus 'local' population before intervention? What is the situation post intervention? The graphing calculators used by the students allowed multiple affordances to be perceived and enacted in successful solution of the task.

The selected analysis of one student's engagement with this challenging task illustrates that affordances of the technology-rich environment need to be both perceived and enacted. The affordance of interest here is *multiple function view-ability*. Enactment of this affordances allows the task solver to view their plot and function model for the two data sets on the same set of axes. This is critical in order to easily see the effect of the intervention. In this case, the student, Sali, did not perceive, nor enact, the available affordance during task solving.

In her post-task interview, Sali suggested it was *not possible to compare* the *before* and *after* models. Looking at her by-hand sketches, she said,

> Well, they both have different scales so you can't really tell which one. [pause] Obviously, this one looks like it is more steadily decreasing than the other one but I wouldn't be able to tell you which model is exactly better than the other one.

When asked why she had used only two lists (in her calculator she had deleted the "before intervention" data, in order to consider the "after intervention" data), it immediately occurred to her that four lists would have allowed her to see all the data together and compared the models for each data set as required. Note the default setting for the graphing calculator used was six lists and more could be added.

> *Sali:* Oh, no, I should have done that because then you could have seen them both together then you could have put the line in … I should have done that. …
>
> *Sali:* Yeah, you would look at it and they would be on the same, roughly the same scale so you would be able to have a more accurate look at them and see which one was decreasing faster or leveling out or whatever. Awh!!

Consider the difference in opportunities to analyse the effect of the intervention shown by Sali's views of the data and models (viewed sequentially, different data set) (see Fig. 1) with those of Chris — all data and two models visible simultaneously. Differing viewing domains and ranges, added to the difficulty of analysis.

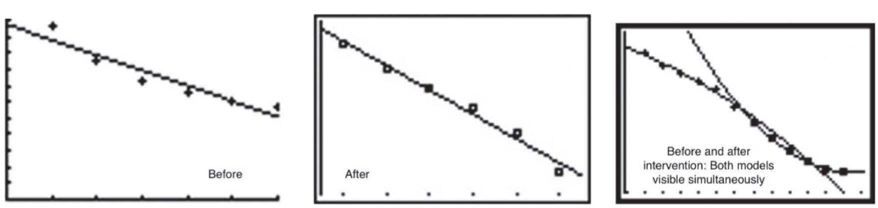


Fig. 1. Sali's (a) pre- and (b) post-intervention data and model view, (c) Chris's simultaneous views

3.5. Perceiving and enacting affordances in a TRTLE

In unpacking the complexity of the task and its solution in a technology-rich teaching and learning environment, several factors need to be considered as students perceive and enact affordances. The primary issue for teachers as observers of classroom learning experiences on which to base judgements about enactment of affordances of technology-rich teaching and learning environments (TRTLEs) (e.g., Brown, 2015a) in function situations is how students manage their competence in enacting affordances as they attempt to solve a function task. Fig. 2 shows that in students' perceiving and enacting affordances of a TRTLE to solve function tasks, both mathematical content knowledge (of functions) and technological knowledge (of digital tools) are required by the teacher. In addition, both perception and enactment of affordances by students need to occur if independent student use of the technology in solving functions tasks is to occur. In a TRTLE, the affordance bearer is broadly speaking some feature of the digital technology (e.g., Window, Zoom) whereas the affordance is the interactivity between the user for some particular purpose (Function View-ability).

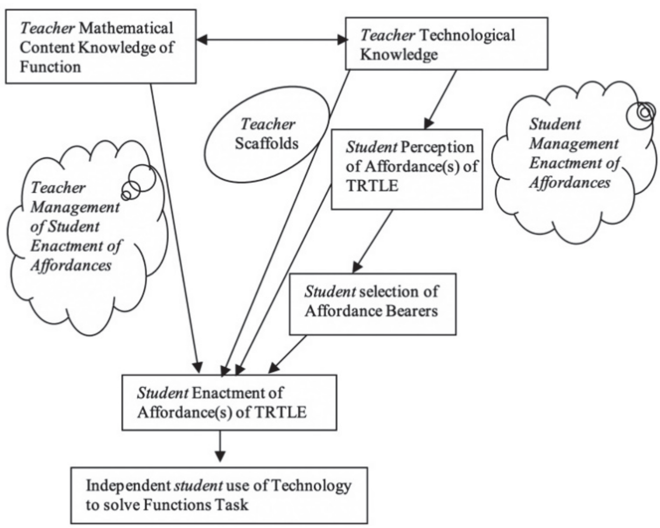


Fig. 2. Factors involved in enabling student independent technology use

Knowledge of teacher tactics and expected responses (Brown, 2013b) and potential student strategies (Brown, 2015a, 2015b) play a critical role in teacher preparedness for successful task implementation where the mathematical thinking is predominantly undertaken by the students. Teacher management of students' engagement with challenging tasks is complex. See also Baxter and Williams (2010). This is even more so in a technology-rich teaching and learning environment.

4. Modelling, Mathematics Content, and Digital Technologies

Galbraith, Stillman, Brown, and Edwards (2007), researchers and a teacher in the RITEMaths project, looked at the interactions between modelling, mathematics content, and technology. A similar approach is taken here. Fig. 3 illustrates the possible interactions between two or three of mathematical modelling (MM), digital technology

(DT), and mathematics content (MC). Interactions can occur between two or three of these. In brief, interactions include *Interactions between* MM *and* DT: MM ∩ DT (Strategy decisions. Choice of technology, selection of a particular technological function); *Interactions between* MM *and* MC: MM ∩ MC (How is the problem represented? What is the goal?); *Interactions between* MC *and* DT: MC ∩ DT (Use of calculations, plots, graphs, approach to verifications, varying specific case to more general); and *Interactions between* MM *and* DT *and* MC: MM ∩ MC ∩ DT: (Carrying through to solution: formulation … approach, use of representations, interpreting results (algebraic, numerical, graphical) → interpretation → implications).

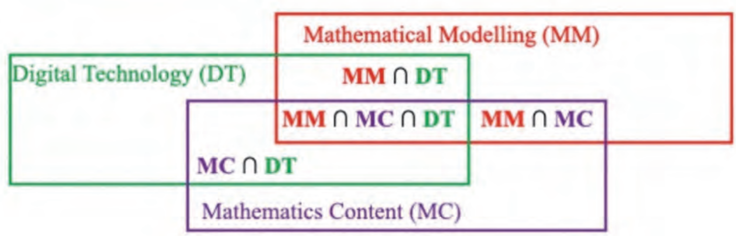


Fig. 3. Interactions between mathematical modelling, mathematics content, and digital technology

4.1. Interactions between Teacher, Students, and Learning Activities

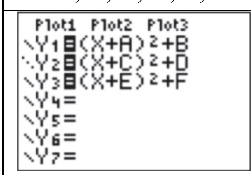
In addition, we can also consider interactions between teacher, students and learning activities. Work by the Learning Mathematics for Teaching Project (LMTP, 2011) team, which included Hill, Ball, and Bass, makes it clear a focus on *teachers, students, content,* and the *interactions* among these is essential for quality teaching and learning. The teacher is responsible for determining lesson content whether this be a task or learning activity, albeit influenced by the intended curriculum. During class, teachers manage the dynamic interactions including what they say and do, and "what students say and do, and what a curriculum affords" (p. 30). A task may have potential, but if students do not engage with it, this potential is not realised. Similarly, a highly motivating task that usually engages students may miss the mark if the teacher fails to notice teachable moments and focuses on trivial or irrelevant mathematical features.

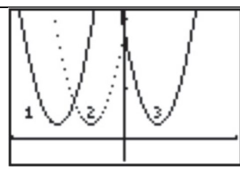
4.2. Digital technology: Opportunities for improved task design MC ∩ DT

Consider the two tasks shown. These tasks were written by the author in the late 1990s. The digital technology here is the TI-83 graphing calculator. These tasks are part of a larger set of tasks designed for two classes of Year 11 students in a curriculum context where graphing calculator use was mandated. Student were expanding their knowledge from simple polynomials to include power and transcendental functions and beginning to learn calculus where algebraic, graphical and numerical representations play an important role. Irrespective of whether the technology was available during task solving, the TRTLE provided enhanced opportunities to develop and demonstrate conceptual understanding. These tasks illustrate interactions between mathematical content and digital technologies. What is possible depends on the technology available, the teacher, and the students. In this circumstance, the availability of DT allowed the teacher-researcher to rethink her teaching approach. Task design provided opportunities to (a) focus on generality, (b) consider functions using a multiple-representation approach, and (c) explore, conjecture, test, and check. Although the tasks presented were originally used to assess student understanding of functions, they can be used as learning activities. It was an important point in time where the teacher-researcher saw ways to make learning accessible to more students.

Example Task 1

Given the equations and graphs below what can you say about the values of A, B, C, D, E and F? Can you suggest a possible set of values for A, B, C, D, E, and F?





Example Task 2

The following table shows some values of $y_1(x)$ and $y_2(x)$.

X	Y1	Y2
0	0	1
1	1	0
2	4	1
3	9	4
4	16	9
5	25	16
6	36	25

Write down the equation of $y_2(x)$ in terms of $y_1(x)$.
Explain how you did this.

4.2.1. Multiple representations: algebraic, graphical, and numerical

The tasks were specifically designed to provide opportunities for students to consider translations *within* and *between* representations (Fig. 4). Note the bi-directional arrows, drawing attention to translations, for example *from* the graphical *to* the algebraic representation and *from* the algebraic *to* the graphical representation. The single headed arrows indicate translations within that representation, for example considering two different graphical representations of the same function.

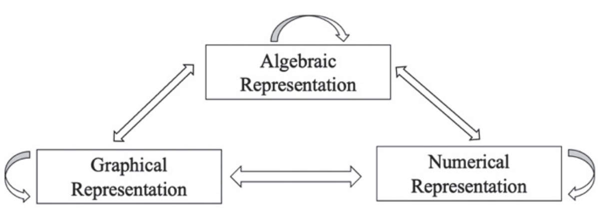


Fig. 4. Translations within and between representations

Kaput (1989) suggests that multiple linked representations allow learners to combine understanding from different representations and build a better understanding of complex ideas and apply these ideas and concepts more effectively. In secondary school mathematics curricula, specifically focusing on functions Romberg, Fennema and Carpenter (1993) drew our attention to the notion of multiple representations of functions, the importance translations among representations, and three critical representations, referred to here as algebraic, graphical, and numerical. The numerical representation appears to still be undervalued (Bannister, 2014).

Whilst Example Task 2 appears to focus the task solver's attention on the numerical and algebraic representations, opportunities exist to consider the graphical representation. The numerical representation provided shows that $y_2(x)$ has the same value as y_1 for the previous x value (i.e., when x is 1 less). This can be considered as graphically y_2 is the same shape as y_1 but shifted 1 unit right. Zazkis, Liljedahl, and Gadowsky (2003) have shown that this transformation is typically less well understood than others — by teachers and students, and still is a current focus in mathematics education research (Sudihartinih and Purniati, 2018).

The collection of tasks, from which these two are selected, gave equal value to each representation, and focused student attention on generality. A study by Brown (2003), including the collection of items found more attention was given, by students, to across, rather than within, representation notions, albeit there are more possibilities for the former. Not only was greater attention given to across, rather than within, representation movement, but also there was not and even spread across representational pairs. Perhaps not surprisingly, connections between representations, where the initial representation was the numerical, received the least attention.

4.3. Real-World Problems and Primary Students MC ∩ MM

In this section, we consider challenging tasks with a focus on the real-word and mathematical content. Three tasks (*Letters, Brass Numerals, Packing Truck*), from two research projects[2] are briefly presented and their learnings from their implementation discussed. All three tasks are research designed, one by the author, one co-designed with a colleague, and the third modified from Swan (2015).

Mathematical modelling and applications provide opportunities for teachers to teach in engaging ways and students to become increasingly confident in working with challenging mathematical tasks. Yet their use remains less common in the primary years of schooling. Both research projects had a focus on improving the quality of teaching and learning. Design and implementation of the task formed part of that approach. Teacher observations provided opportunities for them to see what level of challenge their students engage with.

4.3.1. *The Letters and Brass Numerals tasks*

The Letters Task involved preparing and communicating advice to a local toy store owner who was considering a new product. The items of interest were painted wooden letters and can be used to decorate a bedroom door. Harvey might want only his initial

[2] CTLM: Contemporary Teaching and Learning Mathematics Project — funded by the Catholic Education Office Melbourne. TALR: Teacher as Learner Research Project (unfunded).

H, whereas Darcy might prefer 5 letters to display his full name. How many of each letter should the bookstore owner order, with reasons, so future orders can follow the same strategy (Brown, 2013a). Although students engaged with the Letters Task, few interacted with the real-word context. Many groups saw the task as a (challenging) division problem. Most groups made no use of data, although easily accessible. Teacher — student interactions may have directed focus away from the real-world and to the mathematical world. Had students felt more comfortable using, a four-function calculator — they may have considered the context more important.

The Brass Numerals Task involved determining how many brass numerals the local chain hardware store (Bunnings) should have in stock, to satisfy customer needs for identifying their house (12) or unit number (2|302) so mail was delivered to the correct location (Brown, 2013a, Brown and Stillman, 2017). This task — perhaps similar at a surface level, saw increased engagement with the challenging real-world problem. Task design meant students could no longer throw away the real-world aspects and treat the context as wrapper (Stillman, 1998). In-the-moment the teacher thought students needed more time and an extra lesson was allocated for students to continue working on the task. This additional time enabled some students to take notice of numbering in their neighbourhood and increased attention to the frequency of each digit. This new contextual knowledge was shared with their group the next day.

Implementing these two different tasks, highlights the importance of task design impacting on whether students saw the task as realistic (Brown, 2013a). It is clear that, "tasks that required students to reflect … and make their thinking explicit can contribute to … students perceiving themselves as playing an important role in interpreting the real-world problem situation and relating it to the world of mathematics" (p. 304). The requirement to communicate the solution to an outsider (e.g., toy store owner, ordering manager) had proved particularly helpful in scaffolding students to clearly communicate their solution.

4.3.2. Packing truck task

The previous two tasks were implemented with Grade 6 students (aged 10−11), whereas the *Packing Truck Task* (Brown, 2021; Swan, 2015) was for youngers grade 3–4 students (aged 7−9 years old). In-task scaffolding was included for these young novice modellers. The task began by considering packing inside a box, then shifted to stacking these boxes. Finally, the task considered how many of these boxes could be packed in a truck (dimensions specified). To successfully solve real-world tasks, students must notice what is relevant, and decide how to act on this to progress their solution. Teachers must also discern what is relevant and nurture student capacity to notice. What teachers do matters and is critically important. In the *Packing Truck Task* most students attended to real-world aspects of the task. However, *little evidence* was found of teachers attending to this as they observed. Improved teacher noticing of real-world considerations and hence asking questions of students when stuck, or off-track, that better support student use of realistic considerations would have increased student success with the task. Teachers need to focus attention on explaining reasoning — not why a particular calculation gave a particular result, but where the numbers come from — what they represent and the operation and/or result making sense? Students need more experiences with such tasks as argued by Francisco and Maher (2005) and others.

4.4. *Quality teaching*

For teachers and researchers, it is *a challenge to design or select* a task where students will engage with the real world. Task design depends, in part, on typical approaches to teaching and learning. Blum (2015) suggests the continued overuse of "dressed-up word problems" (p. 83), rather than genuine applications or modelling, is because teaching mathematical modelling and applications is demanding and related to teacher quality. He argues that high quality teaching is particularly necessary for effective teaching of mathematical modelling and applications. Recommendations from Blum (pp. 83–86) include: effective student-centred classroom environments, group work; activating learners cognitively and meta-cognitively; using a broad range of contexts; teacher encouragement of solutions different to their own; a systemic approach involving regular and long-term use of modelling; assessment valuing modelling; and beliefs and attitudes that value modelling (acknowledging these take time to develop).

4.5. *Real-world problems MC ∩ DT ∩ MC*

Stillman and Brown (2021) revisited a task from Riede (2003) with Year 10 students in the Victorian component of the Enablers Research Project[3] Weightlifting is a particularly interesting sport. When the body weight categories are revised, as occurred in July 2018, all world records are nullified. The International Weightlifting Federation then needs a model to establish *reasonable minimum lifts* that can be considered as a record in each body weight category. This led us to develop the *Weightlifting Task* for students to consider in the lead up to the planned 2020 Tokyo Olympics.

The *affordance: data plot-ability* was important here to allow student to visualise the data. The plot affords task solvers greater insight into the relationship between the variables weight category (WC) and weight lifted (WL). This also brings up the importance of teachers being familiar with the digital technology. In this task, with TI-Nspire calculators used by most students, "missing data" creates no issues (Fig. 5). The data are simply ignored as shown. For a spreadsheet used by some students, this was problematic, resulting in error messages. This in turn impacted on whether, or not, students viewed that data as relevant or not. Digital technology that automatically sets a viewing window such that the Viewing domain and Viewing range are inclusive of

[3] Australian Research Council's *Discovery Projects* funding scheme (DP17010555).

all data can help and hinder. Some students who decided a linear model was needed, viewed the "outlier" as something that could be removed from further consideration.

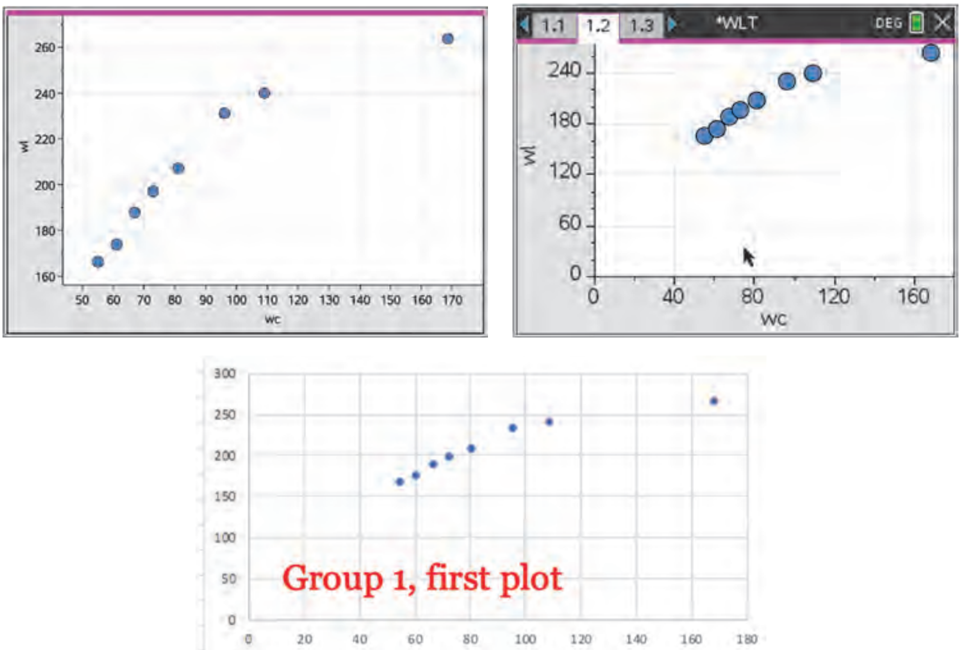


Fig. 5. Different views of the data (weightlifted v weight class)

During task implementation, it emerged that some students applied different mathematical "standards" to non-linear functions. They seemed able to ignore parts of the function beyond the domain of the data (i.e., related to extrapolation) when the function was linear, but not otherwise. This clearly impacted on their choice of model(s). Extrapolating "to the right" was prioritised over "to the left". More attention was given to behaviour of the model for larger values and in contrast, little or no attention given to behaviour of the model for small values of the domain. Furthermore, some students incorrectly "believed" a model must include the origin.

Other differences arose where different function choices led to different affordances being available. For example, if and only if, considering a linear model using the TI-Nspire, the affordance: line move-ability could be perceived and enacted. This allows the user to directly manipulate the graphical representation of a straight line model so as to "best fit" the data. Many digital technologies allow enactment of the affordance: regression calculate-ability thus allowing users to select from a given set of function types to produce a line that statistically — not necessarily realistically — "best fits" the data. There is no doubt the regression capabilities [often blackbox] increase the student "toolbox" but this is often at the expense of real-world and mathematical considerations. There is no need for this to be the case as teachers could explain the process of regression, irrespective of where this sits in the local curriculum.

This would allow students to be better able to decide when to use this and, if used, how the output might be interpreted. As with the Platypus Task described earlier, the affordance: multiple function view-ability enables students to visually compare models under consideration. From a modelling perspective, there is a big difference between deciding a quadratic model is appropriate and using regression to identify such a model and finding two regression models and selecting between them on the basis of fit only (by–eye or available statistics). Students (and some teachers) do not understand nor value this distinction.

5. Concluding Thoughts

It is well known and evidenced by mathematics education researchers that challenging tasks support mathematical understanding. We know students enjoy engaging with challenging tasks, they learn and use mathematics in doing so, and increase their valuing of mathematics and motivation to learn. The challenge for mathematics educators and researchers continues. How can we increase the number of students given substantial opportunities during schooling to experience, and being enabled to be successful, with challenging tasks? We need more students to echo, the thoughts of Tabitha when asked: What do you think of tasks such as the Platypus Task, compared to other tasks you have done?

> I actually found it enjoyable to do this kind of thing. It is challenging and it puts to work the ability to decide where [pause] like you have got so many mathematical tools at your disposal and to be able to find out how you can apply them and how to know when to use them and that kind of thing.

Acknowledgments

Aspects of the research reported here were supported fully by the Australian Government through the Australian Research Council's funding scheme: The University of Melbourne, *RITEMaths* Linkage Project (LP0453701) — Chief Investigators K. Stacey, G. Stillman and R. Pierce, Texas Instruments Australia was an Industry Partner, and The Australian Catholic University (ACU), *Enablers* Discovery Project (DP17010555) — Chief Investigators were V. Geiger, G. Stillman, J. Brown, and P. Galbraith. M. Niss was a Partner Investigator. The views expressed herein are those of the author and are not necessarily those of the Australian Government or the ARC.

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07

Chinese Mathematics Curriculum Reform for Compulsory Education in the 21st Century

Yiming Cao[1]

ABSTRACT Curriculum reform is a fundamental factor in pushing forward educational development and reform. The aim of this study is to present the current situation of mathematics curriculum reform for compulsory education in Chinese mainland since 2000. In this study, we examine the development and implementation of Chinese mathematics curriculum standards for compulsory education. Based on mathematics curriculum standards, this study introduces the reform of mathematics textbooks, classroom instruction and mathematical achievement assessment.

Keywords: Mathematics curriculum reform; 21st century; Compulsory education; Chinese mainland.

1. The Background of New Century Chinese Mathematics Curriculum Reform

From an international perspective, we live in an age witnessing a rapid development of science and extraordinary changes in people's lifestyles. New knowledge, innovative technology, socialization, and globalization have related modern mathematics closely to all areas of human existence (The Research Group of Mathematics Curriculum Standard, 1999). Since the 1980s, many countries around the world have hoped to improve the mathematics literacy of their own citizens through various efforts. Many of the world's major countries and regions have implemented new rounds of mathematics education reform, including the Principles and Standards for School Mathematics in the United States (National Council of Teachers of Mathematics [NCTM], 2000) and the National Curriculum in Great Britain (Cockcroft, 1994).

Social and economic development in China (especially the development of information technology, digital technology, life-long learning, and democratization (The Research Group of Mathematics Curriculum Standard, 2002) have raised the bar for mathematics literacy. New demands for modern citizens have required corresponding changes in public schools, especially in mathematics curriculum and instruction.

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From June 1996 to 1997, the division of basic education in the Ministry of Education organized a survey to investigate the status of the implementation of compulsory education in all subjects, including mathematics, across the nation. The data and facts collected from this survey demonstrated that the curriculum used at that time achieved certain goals (e.g., basic knowledge and basic skills training); however, many problems were identified. For example, the old curriculum was characterized as "complex, difficult, partial, and old." Students suffered from rote memorization and drill practice. At the same time, teachers struggled with "draining students in a sea of problems" (Liu, 2009). There was too much emphasis on using test scores to screen students. The old curriculum was highly centralized, with little flexibility for local adaption, and it did not meet the different social and economic requirements of a diverse student body (Liu, 2009). To address the above issues, the current mathematics curriculum reform in China began at the beginning of the 21st century.

2. The Mathematical Curriculum Reform Since 2001

2.1. 2001: The Mathematics Curriculum Standards for Full-Time Compulsory Education (Trial Version)

The Mathematics Curriculum Standards for Full-time Compulsory Education (draft) was completed and put forth for extensive comments from the community in March of 2000. The mathematics standards research group mentioned above consisted of mathematics and mathematics education scholars, researchers and staff members from local provinces (cities), and school teachers. About 70 percent of the research team members worked in higher education institutes and about 30 percent of them worked in public schools. This research group developed new mathematics standards by studying the research results and best practices from both the Chinese and the international mathematics education communities. The research team members also solicited comments from scholars and experts in various fields including mathematics, psychology, mathematics education, and school teachers. The comments received by the team ranged from discussions of the nature of mathematics and educational goals to issues about methods for handling the definition of multiplication. The development process adopted a procedure of open discussion so that the resulting curriculum policy could benefit from the wisdom of different parties, with a careful consideration of diverse values (Song and Xu, 2010, p. 121). The Ministry of Education formally promulgated and implemented Mathematics Curriculum Standards for Full-time Compulsory Education (Trial Version) (MCSFCE 2001) in June 2001.

2.1.1. The new characteristics of the MCSFCE2001

The MCSFCE differed from the products of previous curriculum reform in several fundamental aspects, such as the basic curriculum ideas, curriculum objectives, curriculum implementation (including guidance on textbook development), teaching suggestions, evaluation recommendations, and even curriculum management. It provided detailed descriptions in some dimensions. For example, the traditional syllabus only provided a brief description of teaching content and objectives. Most of the descriptions of teaching objectives were included in the textbook developed by the state. MCSFCE changed both the scope and depth of the role that the state plays in the curriculum by providing descriptions of learning content, learning processes (special attention), and teaching recommendations (including several cases for some content). This provided a standard for the transformation from one single national textbook policy to a policy of diversity; a national committee certificated and authorized the different versions of textbooks, according to the curriculum standards.

To examine some of the differences between the old Syllabus and MCSFCE 2001 in more detail, consider the following descriptions of how students and teachers should approach the Pythagorean Theorem (see Tab. 1).

Tab. 1. The Pythagorean Theorem in the old Syllabus and the MCSFCE 2001

	The old Syllabus		**The MCSFCE 2001**
(1)	Master the Pythagorean Theorem. (Students) know how to use the Pythagorean Theorem to solve for the third side given the measurement of the other two sides.	(1)	Explore the proof process of the Pythagorean Theorem.
		(2)	(Students) know how to use the Pythagorean Theorem to solve simple problems.
		(3)	(Students) know how to use the converse of the Pythagorean Theorem to determine if a triangle is a right triangle.
(2)	(Students) know how to use the converse of the Pythagorean Theorem to determine if a triangle is a right triangle.	(4)	The recommendations for textbook development suggest introducing several well-known proofs (such as the Euclidean proof, Zhao Shuang's proof, etc.) and some well-known problems so that students are aware that mathematical proof can be flexible, beautiful and sophisticated.
(3)	Conduct patriotic education by introducing the research on the Pythagorean Theorem done by ancient Chinese mathematicians.	(5)	Students should also be aware of the Pythagorean Theorem's rich cultural connotations.
		(6)	Some teaching suggestions include guidance on the teaching activities and teaching process of the Pythagorean Theorem.

2.1.2. The basic reform idea in the MCSFCE 2001

As mentioned above, the MCSFCE 2001 proposed a basic reform idea: "Mathematics for All." In other words, "Everyone can learn valuable mathematics; everyone can learn the necessary mathematics; different people benefit from different mathematical development" (Ministry of Education of the People's Republic of China, 2001). This concept was totally different from the underlying idea of the old Syllabus (Zhang and Song, 2004). The MCSFCE suggested following the psychology of learning mathematics and using real-life experience to motivate student development. Students were to experience the process of mathematical modeling, which would allow for the interpretation and application of the problem-solving process. Thus, as was the hope of mathematics education reformers elsewhere in the world, students would be enabled to grow in mathematics understanding, mathematics thinking ability, attitudes towards mathematics, and appreciation of mathematics (National Council of Teachers of Mathematics [NCTM], 2000).

2.1.3. The curriculum objectives in the MCSFCE 2001

Even though, in terms of curriculum objectives, MCSFCE inherited qualities from traditional Chinese mathematics education which emphasized training in basic knowledge and basic skills ("The Two Basics") (Zhang et al., 2005), the MCSFCE also emphasized learning goals for the growth of mathematical thinking ability, problem-solving skills, attitudes towards mathematics, and the appreciation of mathematics.

2.1.4. The nature of mathematics and the "non-formalized aspect"

The MCSFCE highlighted the nature of mathematics and the "non-formalized aspect" of mathematics content, including applications of intuitive geometry and a spiral curriculum (Zhang and Song, 2004). At the same time, emphasis was placed on the cultural value of both pure and applied mathematics, real world applications of mathematics, the importance of human development, the technical attributes of mathematics, and the connections between mathematics and calculators (and computers). MCSFCE 2001 defined mathematics as a language to describe the real world. It was considered a process of theory abstraction from nature using qualitative/quantitative methods that also involved the application of theories to solve real world problems.

2.1.5. The curriculum content in the MCSFCE 2001

In terms of specific curriculum content, the MCSFCE was arranged in several sections, including "Number and Algebra," "Space and Figure," "Statistics and Probability," and "Practice and Synthetic Application." The focus was on the development of students' number sense, symbol sense, space concepts, statistical concepts and the application of awareness and reasoning abilities. In the number and algebra section, the MCSFCE added the concept of negative numbers and applications of calculators, and strengthened the role of estimation. The emphasis on the use of the abacus, complicated operations, and the use of simple numbers was decreased. In terms of geometry (Space and Shape section), the topics of translation, rotation, symmetry and other geometric transformations were increased to a certain extent to replace the traditional Euclidean geometry system. The coverage of topics in orientation, measurement, space and shapes was also increased, as was emphasis on the real world application of measurement and estimation, and the application of mathematics topics in everyday life. The MCSFCE especially increased attention to probability and statistics, reflecting the basic mathematical literacy requirements for citizens in modern society.

2.1.6. Critical-thinking skills, inquiry, and cooperation

The MCSFCE proposed the use of critical-thinking skills, inquiry, and cooperation in mathematics teaching and learning (Zhang, 2008), pointing out that the mathematics learning process is full of observation, experiment, simulation, inference and other exploratory and challenging activities. One emphasis of the MCSFCE was that textbooks should make connections with other disciplines by incorporating science, social studies, and other relative subjects to teach mathematics. The textbooks should also provide space for student investigations and communication. Accordingly, teachers were urged to use concrete examples and demonstrations to guide students in the learning process and encourage them to communicate ideas via discussions. According to the MCSFCE, teachers should encourage students to think critically and independently. Also, they must recognize individual differences generated by the culture, learning environment, family background and different thinking styles.

2.1.7. Mathematics learning assessment in the MCSFCE 2001

The MCSFCE additionally put forward clear guiding principles for development and evaluation by focusing on the process and different assessment methods, notably recommending that assessment should be used to inform teaching (Kong and Sun, 2001). It also provided recommendations for evaluation according to grade bands. For example, the evaluation schema for grades 1−3 emphasized the assessment of students' mathematics learning processes, mastery of basic knowledge and basic skills, and their ability to identify and solve problems. In particular, it was felt that multiple evaluation methods should be used.

2.1.8. The Implementation of the MCSFCE 2001

Before the release of the MCSFCE a set of textbooks based on the idea of the new curriculum had been designed by a research group for experimental use (the majority of the members were to part in the development work of the MCSFCE later). Since 1994, this group had conducted two rounds of experiments; more than 60,000 students from more than ten provinces (including both well-developed school districts to undeveloped school districts) participated, which provided abundant empirical experience for the later implementation of the MCSFCE.

The Ministry of Education started a national curriculum reform conference to convene the implementation of the new curriculum in July 2001. Several decisions were made at the conference. First, the overall objectives and strategies for the implementation of the new curriculum in public schools were determined. Second, the strategies to spread the curriculum reform to all Chinese public schools were developed. Third, professional development and teacher training programs were set up. The positioning of the trial version of the curriculum standards necessitated a multi-stage process for spreading the new curriculum. The first stage was to set up the goals, then to conduct preliminary experiments before the nationwide implementation, and finally to broaden the experiment gradually.

In the initial round of experimental implementation of the curriculum, school participants were recruited on a county basis in 2001. First, applications to be volunteer schools were submitted by counties and were examined before being approved by the Ministry of Education. Forty-two regions (3,300 elementary schools, 400 secondary schools) participated in the first round of the national curriculum reform with about 270,000 first graders (1% of the population of first graders nationwide) and about 110,000 seventh-grade participants (0.5% of seventh graders) in 2001. Starting in 2002, each province developed a curriculum reform plan at the province level and determined their experimental regions. There was a total of 570 experimental regions with 20% of Chinese first graders and 18% of the seventh graders participating in the new curriculum. Subsequently, more schools from an additional 1,072 counties became experimental regions at the province level, bringing in about 40%–50% of the student population of each grade. Including the earlier participants in 2001 and 2002, there were 1,642 experimental regions with about 35,000,000 students participating in the new curriculum in 2003. Based on the results of these pilot tests, the new curriculum entered the phase of nationwide promotion. By 2004, 90% of the school districts in China were using the new curriculum. As of 2005, except for a few places, the new curriculum had been implemented all over Chinese mainland.

2.2. The Revision of Mathematics Curriculum Standards: from MCSFCE 2001 to MCSCE 2011

Since the implementation of the MCSFCE (Trial Version), the work of developing it has never been interrupted. After the first round (3 years) of mathematics curriculum reform, the revision process began. Based on the experience, account was taken of the problems arising from the implementation of the standards, as well as comments from society (including severe criticism from some mathematicians). In May 2005, the Ministry of Education organized the revision group for mathematics curriculum standards for compulsory education, and officially began the revision process.

There were 14 members in the revision group, from different backgrounds including universities, coaching offices and primary and secondary schools. About half of them had worked on the design of MCSFCE (Trial Version). Through the process of surveys, situation analysis and discussions of special issues, the Mathematics Curriculum Standards for Compulsory Education (2011 Version) (MCSCE 2011) were finished in 2010, and approved in May 2011. The standards were published officially in December 2011 (Ministry of Education of the People's Republic of China, 2012, p. 34).

2.2.1. The new characteristics of the MCSFCE 2011

MCSCE 2011 was developed from the trial version; several revisions were made (Zhu, 2012), such as the basic curriculum ideas, curriculum objectives, content standards and suggestions for curriculum implementation. The following several paragraphs summarize several aspects of the important revisions, such as the structure, the expression, the concrete content and suggestions for curriculum implementation (Ministry of Education of the People's Republic of China, 2012, p. 34).

1) For the value of mathematics and the function of mathematics education, MCSCE 2011 discussed the research object of mathematics and the

relationship between mathematics and human society, and then gave the fundamental characteristic of mathematics, which were different from the statement of the trial version.

2) MCSCE 2011 expanded the 6 core concepts (Sense of Number, Sense of Symbol, Idea of Space, Idea of Statistics, View of Application and Ability of Inference) into 10 core concepts (add Perceptual Intuition of Geometry, Idea of Modeling, Operations Ability, and changing the Idea of Statistics into View of Data Analysis.). The new concepts were very important in mathematics education research.
3) For the curriculum objective, MCSCE 2011 used the "Four-Basics" (Basic Knowledge, Basic Skill, Basic Idea and Basic Activity Experience) to expand the "Two-Basics" (Basic Knowledge and Basic Skill).
4) The Basic Idea generally included the Idea of Mathematical Abstraction, the Idea of Mathematical Inference, and the Idea of Mathematical Modeling. Basic Activity Experience refers to the individual experience the students gain by experiencing mathematical activities personally. The Basic Activity Experience was the one of the characteristics of MCSCE 2011. This issue was considered by Chinese scholars since the 1980s, but did not receive due attention. After the introduction of MCSCE 2011, many scholars began to explore this issue.
5) Revisions were made to the concrete contents and the requirements, across all the domains (Shi et al., 2012). The content domains of "Space and Figure" and "Practice Synthetic Application" were revised into "Space and Geometry" and "Synthetic and Practice". The word "Geometry" emphasized the abstraction of concrete figures and space, and also explained the general laws behind figures and space. The word "Synthetic" emphasized that an important stage of learning was knowing the relationship between the knowledge and concepts that students learned, and "Practice" was a higher requirement. Some concrete content was omitted, such as the requirements of the trapezoid and position relationship between circles.

2.2.2. The implementation of the MCSFCE 2011

With the base established by the implementation of the MCSFCE (Trial Version), MCSCE 2011 was implemented at one time. Since the autumn semester, all beginning grades (for primary and middle schools) began to implement the new curriculum standards (not only mathematics).

Some changes appeared in the high-risk examinations. For example, the entrance examination to high school in Beijing adapted the concrete content and new rubrics were introduced focusing on the Mathematical View, Mathematical Activity Experience and Mathematical Ability.

MCSCE 2011 discussed the relationship between plausible and deductive reasoning, and the relationship between the real-life world and systems of knowledge.

Its objectives highlighted the development of students' creative and application abilities, and added the ability to discover and raise problems (Ministry of Education of the People's Republic of China, 2012, p. 84).

The two versions of standards consolidated and perpetuated the achievements of the new century mathematics curriculum reform and played an important role in giving impetus to the healthy and continuous development of mathematics education in China.

2.3. Reform of mathematics textbooks

The curriculum reform has led to many new ideas for developing textbooks. The reform also supports the transformation from one single national textbook to authorizing different versions of textbooks, according to the curriculum standards.

The curriculum standard provides guidance and principles for the mathematics textbooks. Based on the guidance, Chinese new century mathematics textbooks have some common features.

1) These mathematics textbooks place emphasis on the relationship between knowledge learning and its applications.
2) These textbooks pay attention to knowledge development, heuristics, and investigation, which can give students more chances to explore and discover knowledge.
3) These textbooks increase the content presentation to inspire students' interest in mathematics.
4) These textbooks provide mathematics context knowledge to embody mathematical cultural value.
5) These textbooks stress the integration of information technology and mathematics curriculum to improve the effectiveness of mathematics teaching and learning.

2.4. Reform of classroom instruction

Teachers and classroom teaching are the critical factors to maintain the effective implementation of new curriculum reform. As mentioned before, the new mathematics standards provide some teaching recommendations. Teachers are advised to change and improve their teaching methods based on the MCSCE 2011.

1) teachers should take account of students' learning styles in classroom teaching.
2) teachers are advised to provide students more opportunities and guidance for independent, cooperative, and inquiry-based learning.
3) teachers should consider and meet students' psychological needs for cognitive development.
4) teachers should try to arouse the students' desire to learn and inspire them to think actively. Teachers need to help students establish specific learning aims and strong learning motivation, and guide them to explore knowledge actively.
5) teachers are advised to integrate classroom teaching with information technology.

2.5. Reform of mathematical achievement assessment

It is well known that China has a large population but relatively scarce high-quality educational resources. Under such conditions, primary and secondary mathematics selection examinations play an increasingly important role. The examination process has been described vividly as "crossing a single-plank bridge", which demonstrates the fierce competition in the examination in China.

It is worth noting that a lot of changes have occurred in the ideas, content methods, and evaluators in mathematics learning assessment in the new curriculum reform.

1) The mathematical achievement assessment has changed from a traditional identification assessment to a developmental assessment based on modern teaching concepts.
2) The traditional mathematical achievement assessment mainly focuses on basic knowledge and abilities. Based on the new curriculum objective in MCSCE 2011, the new assessment methods pay more attention to learning processes and methods.
3) The traditional mathematical learning assessment mainly relies on the written examination. The new mathematical learning assessment expands to various methods including class observation, homework analysis, and student files.
4) The traditional mathematical learning assessment is mainly based on teachers' evaluations. The new methods encourage students to assess their learning performance and process by themselves and their classmates.

Overall, the assessment under the current reform emphasizes the evaluation of students' overall abilities, the application of mathematics in real life and interdisciplinary context, mathematical culture, and the history of mathematics.

3. Looking Ahead to the Next Decade

As the implementation of the curriculum reform, the revision of curriculum standards never stops. In 2019, the Ministry of Education began a new round of curriculum standard revision[2]. There are some important directions in the new round revision.

1) The new curriculum standards will pay attention to developing key competency for the compulsory education stage.
2) The new revision will promote the integration of interdisciplinary knowledge.
3) The new curriculum standards will advocate unit-based teaching design, project-based, collaborative and inquiry learning.
4) The new curriculum standards will highlight the incorporation of technology in teaching and learning.

[2] A new version of the curriculum standard — *Compulsory Education Mathematics Curriculum Standard (2022 Edition)* — has been issued and textbooks based on the new curriculum standard are currently being compiled. For the new standard (Chinese Version only), please refer to: Ministry of Education of the People's Republic of China (2022). *Compulsory Education Mathematics Curriculum Standard (2022 Edition)*. Beijing: Beijing Normal University Press. — The editor

5) The new curriculum standards will focus on competency-oriented assessment.
6) At present, the newest curriculum standard is still under revision and consultation. The new curriculum standards will be officially released in 2022.

The curriculum reform in the last two decades led to changes in ambitions, curriculum content, teaching methods, textbooks, and assessment methods. These changes played an important role in promoting the development of mathematics education in China.

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https://doi.org/10.1142/9789811287183_0008

08

Online Cognitive Diagnostic Assessment with Ordered Multiple-Choice Items for Grade Four Topic of Time

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ABSTRACT It is a great challenge for the teachers to practice differentiated instruction in the heterogeneous mathematics classroom because there is a great demand for a valid and reliable diagnostic assessment. To address this issue, this study sought to develop and validate an online Cognitive Diagnostic Assessment (CDA) with Ordered Multiple-Choice (OMC) items for Grade Four Topic of Time. However, this paper only focuses on the results of six cognitive models for conversion between time units. Each cognitive model was measured by an assessment comprising seven OMC items. The quality of the online CDA with OMC items was assured with robust psychometric properties, convincing reliability, and satisfactory model-data fit. Perhaps this instrument could support the teachers in diagnosing pupils' cognitive strengths and weaknesses, followed by practicing differentiated instruction in the mathematics classroom.

Keywords: Cognitive diagnostic assessment; Ordered multiple choice; Time.

1. Introduction

Students' diversity is a critical issue to be addressed for promoting equal opportunity in mathematics learning in a heterogeneous classroom (Csapó and Molnár, 2019). Thus, teachers are encouraged to practise differentiated instruction in the mathematics classroom to tailor to students' needs. However, it is a great challenge for the teachers to practise differentiated instruction because there is a great demand for a valid and reliable diagnostic assessment (Brendefur et al., 2018) that could provide detailed information on students' skill deficits. The teachers could only identify students' learning needs by conducting informal assessments through classroom interactions, evaluating the students' learning artefacts, or assessing students' understanding of a narrow scope using the teacher-made instrument, which might have quality concerns (Csapó and Molnár, 2019).

By integrating educational measurement with learning psychology, cognitive diagnostic assessment (CDA) emerged as an alternative assessment that could support teachers' classroom assessment practice. The CDA consisted of three components:

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(i) cognition, (ii) observation, and (iii) interpretation. Different from the diagnostic assessment commonly used in the classroom, the CDA was developed based on a cognitive model (cognition component) which illustrates the skill acquisition hierarchy. While each item (observation component) in CDA was designed to elicit the students' response on the subskill (attribute) included in the cognitive model, the use of measurement model (interpretation component) to analyse students' responses could reflect their knowledge states, which composed of attribute mastery combinations and corresponding misconceptions or systematic errors (Kuo et al., 2016). Thus, the diagnostic inference such as students' weakness in skill acquisition could be made based on the results obtained from the CDA. Based on this imperative information, the teachers could adjust the instruction to support the needs of the struggling students in mathematics learning (Ketterlin-Geller et al., 2019). In short, CDA could be essential to support teachers in implementing differentiated teaching in the mathematics classroom.

In fact, CDA is rarely being used in the mathematics classroom (Wu, 2019) due to practical constraints. To locate students' skill deficits, the teachers would have to apply a measurement model on the binary pattern of students' responses, after scoring the answer script dichotomously. This eventually discourages the practical implementation of CDA in the mathematics classroom because the application of highly technical measurement models might barely be understood by teachers. To address this issue, this study sought to develop the CDA as a web application with an automated scoring feature. Following this, the complex scoring procedure of CDA could be mechanised and thereby increase the practicability of the CDA to be used in the mathematics classroom.

Several CDAs have been developed in the past using multiple-choice questions (e.g., Ketterlin-Geller et al., 2019; Roberts et al., 2014) or constructed response questions (e.g., Sia and Lim, 2018; Sia et al., 2019). Rather than developing a CDA with multiple-choice questions or constructed response questions, we developed the online CDA with a novel item format, named ordered multiple-choice (OMC) items, which was introduced by Briggs et al. (2006). Apparently, OMC items and multiple-choice questions look quite similar (Briggs and Alonzo, 2012). However, each option of OMC items is linked to the developmental level of students' skill acquisition as depicted in the construct map (Briggs et al., 2006). Thus, OMC items have a higher diagnostic value compared to typical multiple-choice questions, while retaining their scoring efficiency advantage (Briggs et al., 2006).

In this study, we only focused on the topic of "Time" which is important in daily life yet hardly being mastered by the students (Kamii and Russell, 2012, Tan et al., 2017). This topic is included in the domain of measurement (National Council of Teachers of Mathematics [NCTM], 2000; Malaysian Ministry of Education [MOE], 2016). In grade four, the Malaysian students (average age of 10 years old) learn about the conversion between the time units, performing an arithmetic operation on the measurement with time units and solving problems involving time units (MOE, 2016). To support teachers' classroom assessment practice, we developed an online CDA with

OMC items for the Grade Four topic of "Time". As part of the larger research project, this paper only discusses the development and validation of the online CDA for conversion between the time units.

2. Development of Online CDA with OMC Items

To promote evidentiary coherence for enhancing the validity of result interpretation and use (Nichols et al., 2017), the online CDA with OMC items was developed based on the principled assessment design adapted from the assessment system of Berkeley Evaluation and Assessment Research (BEAR) Centre (Wilson and Sloane, 2000). The usefulness of the BEAR assessment system to guide the development of OMC items has been illustrated in several studies (i.e., Briggs et al., 2006; Hadenfeldt et al., 2013; 2016). By adapting the BEAR assessment system, the online CDA with OMC items was developed sequentially following the five building blocks as described in the following sections.

2.1. Building block 1: Construct map

The first building block involves the development of construct maps, which serve as the basis for OMC items construction. Due to the absence of a substantive theory on conversion between time units, two associate professors in the field of mathematics education were invited to specify the attributes by conducting the expert review and task analysis. With a sophisticated understanding of the curriculum, the two experts listed out the six skills related to conversion between time units, based on the review of the Grade Four mathematics curriculum document and textbook. Then, the experts selected a task for each skill and conducted the task analysis to specify the attributes required to master the skill. The example of task analysis for conversion of the unit of time involving day and hour from a larger unit to a smaller unit is as shown in Fig. 1. The process began with solving the task and showing the working in a detailed manner.

Working	Attributes	Cognitive Model
3 days 7 hours = ________ hours		
1 day = 24 hours ①	A1: State 1 day = 24 hours	A1
		↓
24 hours + 24 hours + 24 hours = 72 hours ② or 3 × 24 = 72 hours ②	A2: Covert the unit of time from days to hours by repeated addition or multiplication	A2
		↓
72 hours + 7 hours = 79 hours ③	A3: Covert the unit of time from days and hours to hours by repeated addition or multiplication	A3

Fig. 1. Example of task analysis and cognitive model derivation

This was followed by specifying the attributes based on each step by considering the measurability of the attributes (Alves, 2012). Then, the attributes were arranged in a hierarchical order to form the cognitive model. The construct map was then derived based on the cognitive model, where each level indicates an accumulation of attributes mastered by the students. This was supported by the claim made by Szilagyi et al. (2013) in which mathematics learning is conceptualised as an accumulation of knowledge and skills. The six construct maps developed are as shown in the Appendix.

2.2. Building block 2: Item design

The second building block involved the construction of the stem, key, and distractors of the OMC items. To ensure the alignment between the OMC items developed and the cognition component of CDA, the second building block started with generating the Q-matrices which outline the item-attribute relationship in the CDA developed. The Q-matrices are generated sequentially as shown in Fig. 2.

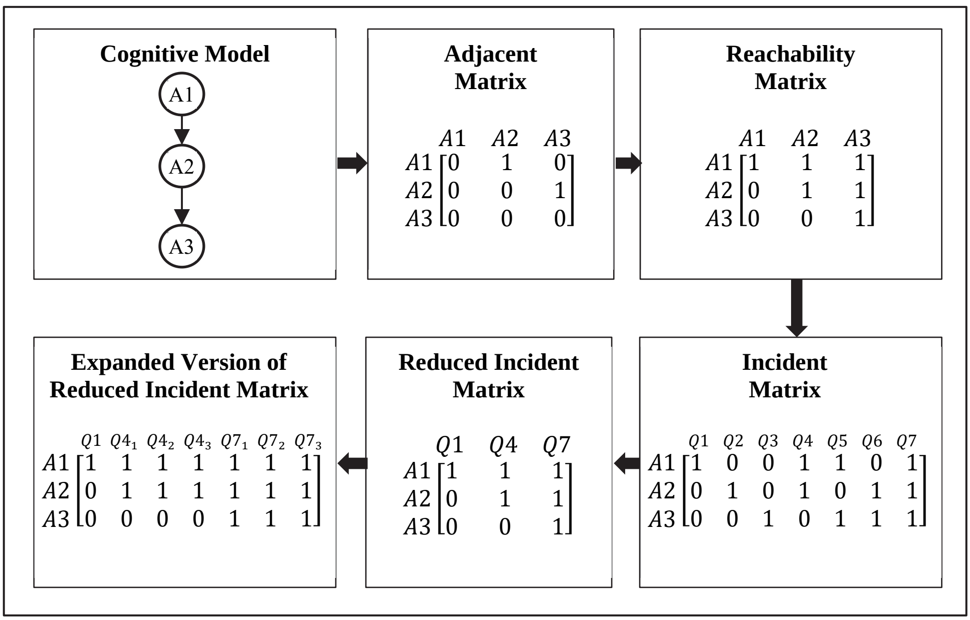

Fig. 2. Generation of Q matrices

The first step is converting the cognitive model which is illustrated as the directed network into the adjacent matrix (***A*** *matrix*) by positioning the value ‘1’and ‘0’ at the respective entry to indicate the presence or absence of a direct prerequisite relationship between the attribute pair. As shown in Fig. 2, the value ‘1’ is positioned at Row 1, Column 2 and Row 2, Column 3 because A1 is the direct prerequisite attribute for A2, while A2 is the direct prerequisite attribute for A3. The ***A*** matrix is used to represent the direct prerequisite relationship of the attributes in CDA (Tatsuoka, 1990).

The second step is deriving the reachability matrix (***R** matrix*) by performing Boolean addition and multiplication using the formula $\boldsymbol{R} = (\boldsymbol{A} + \boldsymbol{I})^{n}$, where $\boldsymbol{A}$ refers to the $\boldsymbol{A}$ *matrix*, $\boldsymbol{I}$ refers to the Identity Matrix and n refers to the integer required to reach invariance, where $n = 1, 2, 3, \ldots, c$. In other words, ***R** matrix* is obtained if the matrix remains the same when the integer, *n* is substituted with two subsequent integers. The value of "1" or "0" in the ***R** matrix* indicates the presence or absence of the direct and indirect relationships between the attributes.

The third step is deriving the incident matrix (***Q** matrix*) that illustrates the involvement of the attributes in each item of the potential item pool. The number of potential items, *i* can be determined by using the formula, $i = 2^{k} - 1$, where *k* is the number of attributes. Since there are three attributes in the cognitive model as shown in Fig. 2, the CDA might consist of seven items with different involvement of attributes. The attributes involve in each item was determined based on the subsets of attributes (i.e., A1, A2, A3), such as {A1}, {A2}, {A3}, {A1, A2}, {A2, A3}, {A1, A3}, {A1, A2, A3} and {}. Except for the empty set, each subset illustrates the attribute(s) involved in each potential item in the ***Q** matrix*. Then, the value of '1'or '0'is positioned at the row entry for each column in the ***Q** matrix* based on the involvement of attributes.

The fourth step is deriving the reduced incident matrix ($\boldsymbol{Q_r}$ *matrix*) from the ***Q** matrix* by imposing the prerequisite relationships among the attributes. Based on the cognitive model as shown in Fig. 2, attribute A2 has a prerequisite attribute (i.e., A1), while attribute A3 has two prerequisite attributes (i.e., A1 and A2). Notably, items Q2, Q3, Q5, and Q6 do not comply with the prerequisite relationship among the attributes. Thus, they were removed from the item pool. This brings about the reduction of columns Q2, Q3, Q5 and Q6 in the ***Q** matrix*. Following this, the $\boldsymbol{Q_r}$ *matrix* formed only consists of three columns, namely Q1, Q4, and Q7. Thus, the $\boldsymbol{Q_r}$ *matrix* serves as the basic item pool of CDA.

The last step is deriving the expanded version of the reduced incident matrix (expanded version of $\boldsymbol{Q_r}$ *matrix*) which serves as the test blueprint of CDA. To increase the reliability of the CDA, each attribute is measured by three parallel items (Gierl et al., 2009). Following this, a total of nine items (3 attributes × 3 items per attribute = 9 items) should be constructed for the three-attribute cognitive model as shown in Fig. 2. However, only seven items were constructed based on the cognitive model because attribute A1 could only be elicited by one item. To accommodate all items and the respective attributes required to solve the CDA items, the $\boldsymbol{Q_r}$ matrix is expanded as shown in Fig. 2.

Upon the generation of the expanded version of $\boldsymbol{Q_r}$ *matrix*, the construction of OMC items began. The process started with the construction of the OMC item stem based on the expanded version of $\boldsymbol{Q_r}$ *matrix*. A total of 42 items (6 cognitive models × 7 items per cognitive models = 42 items) were constructed for the six cognitive models specified. The English written OMC items were then being translated into Malay, Chinese and Tamil languages to match the medium of instruction in the three types of primary schools. This was followed by the content validation of the OMC item stems by two subject matter experts from each school type with at least eight years of Grade Four mathematics teaching experience. After the validation process, the OMC item stems were piloted to a total of 192 grade four students from each school type to collect the students' common mistakes for each item. For each OMC item, the correct answer would form the key, while the incorrect answers associated with the common mistakes would form the three distractors. It is worth noting that the pupils seldom made mistakes when answering the item which measured the basic attribute such as A1. As a result, the distractors could barely be extracted from the mistake made by the pupils. In this situation, the distractors could be derived from the common mistake reported in the literature (Sadler, 1998).

2.3. *Building block 3: Outcome space*

The third building block involved the construction of the outcome space, which specifies the relationship of the OMC items with the construct map (Wilson, 2005). In this study, the outcome space refers to the relationship of each option of OMC items with the level of mastery of attributes as shown in the construct map. Two mathematics education experts were invited for assigning the level to each option of the OMC items based on the construct map. Specifically, the correct option of each item was assigned to the mastery level which includes the corresponding attribute being measured. On the other hand, each incorrect option was assigned to the lower mastery level based on the incorrect working and associated with the common mistakes presented to the experts. A sample of the OMC item with both correct and incorrect workings associated with each option is illustrated in Fig. 3.

Stem	88 hours = ______ days ______hours			
Option	A. 1 day	B. 3 days	C. 3 days 16 hours	D. 8 days 8 hours
Working	24 hours = 1 day	24 hours = 1 day	24 hours = 1 day	10 hours = 1 day
		3 → 3 days $24\overline{)88}$ $\underline{72}$ 16	3 → 3 days $24\overline{)88}$ $\underline{72}$ 16 → 16 hour	8 → 8 days $10\overline{)88}$ $\underline{80}$ 8 → 8 hours
Mistake	State 24 hours = 1 day but do not proceed with the conversion of days into days and hours.	Perform correct long division but do not write down the remainder as the number of hours.	None	Divide the number of hours with the wrong divisor
Level	Level 1	Level 2	Level 3	Level 0

Fig. 3. Assigning the level to each option of the OMC items based on the construct map

To ensure the validity of the level assigned, each option and the corresponding level was then validated by an associate professor and a senior lecturer in the field of psychometric and educational measurement with mathematics teaching experience. After the validation process, the English version of the OMC items was translated into Malay, Chinese and Tamil languages to match the medium of instruction in the three types of primary schools.

2.4. Building block 4: Online CDA web application

The fourth building block involved the development of an online CDA web application by the web developer. The online CDA is a cohesive and comprehensive online assessment system that could be used to assess pupils' understanding of the topic of "Time" and to profile their skill acquisition. To match the medium of instruction in the three types of primary schools, the online CDA developed possesses a language switching function. Specifically, the Malay, Chinese, and Tamil language versions of the online CDA would be used by the pupils from National Primary School (NPS), National Type Chinese Primary School (NTCPS) and National Type Tamil Primary School (NTTPS), respectively.

The pupils, teachers and researchers were the three main users of the online CDA. The pupils would be able to take the assessment and view their scoring report if they logged in to their account in the online CDA. The teachers would be able to view the content of the assessment and the pupil's scoring report at both individual and class levels in the online CDA after their pupils had answered the assessment. The researchers would be able to manage the item bank of the online CDA, manage the users, access, and extract the scoring reports at the assessment level. Upon the completion of the development process, the researchers created the item bank by keying in all the attributes, followed by the OMC items and the respective answer key. After entering all the attributes, the OMC items, and the answer key in the item bank, the Online CDA was ready to be used in the classroom.

2.5. Building block 5: Measurement model

The fifth building block involved applying the measurement model to determine the psychometric properties of each item, to evaluate the reliability of the assessment, to assess the model-data fit, and to map the pupil's responses onto the construct map. The responses collected during the pilot test were analysed by using the measurement model, named Classical Test Theory (CTT) to determine the psychometric properties of each item such as item difficulty (proportion of correct response [p-value]) and item discrimination (point-biserial correlation [r_{pb}]). This is because the result of item analysis could be communicated easily to the teachers during the item revision stage.

Likewise, the reliability of the dichotomously scored CDA was determined using Kuder Richardson 20 [KR-20] coefficient, which is rooted in CTT. In this study, the CDA was developed to measure students' attribute mastery. In other words, the students' results would be reported at attribute level, besides total score. Thus, the consistency of the attribute-level measurement was determined using the attribute reliabilities coefficient (α_{AHM}) in the measurement model, named Attribute Hierarchy Method (AHM). The α_{AHM} is derived from the Cronbach's Alpha reliability coefficient by imposing the prerequisite relationships among the attributes. The formula of α_{AHM} is as follows:

$$\alpha_{AHM_k} = \frac{n_k}{1-n_k}\left[1 - \frac{\sum_{i\in S_k} {W_{ik}}^2 \sigma^2 x_i}{\sigma^2 \sum_{i\in S_k} W_{ik} x_i}\right], \quad (1)$$

where n_k = number of items which involves attribute k, S_k = a set which consists of items which involve attribute k, i = an element in the set S_k, $W_{ik} = P(X_i = 1| A_k = 1) - P(X_i = 1| A_k = 0)$, where $P(X_i = 1| A_k = 1)$ is the conditional probability that an examinee to answer item i correctly, given that he has mastered the attribute k (i.e., percent score for attribute k more than .50), $P(X_i = 1| A_k = 0)$ is the conditional probability that an examinee who has not mastered attribute k (i.e., percent score for attribute k at most .50) can answer item i correctly, $\sum_{i\in S_k} {W_{ik}}^2 {\sigma^2}_{X_i}$ = sum of the weighted variances of the observed score for item i, and ${\sigma^2}_{\sum_{i\in S_k} W_{ik} X_i}$= variance of the weighted observed total score (Gierl et al., 2009, p. 300). To explain the result of the attribute reliability, the effect of adding parallel items to the CDA was determined using the attribute-based Spearman-Brown formula as follows:

$$\alpha_{AHM-SB_k} = \frac{n_k \alpha_{AHM_k}}{1 + (n_k - 1)\alpha_{AHM_k}}, \quad (2)$$

where, n_k is the number of additional parallel items involving attribute k added into the assessment, and α_{AHM_k} is the reliability of attribute k (Gierl et al., 2009, p. 300).

Then, the model data fit was used to describe the extent to which the students' responses match the expected response derived based on the cognitive model. Following this, the evaluation of model data fit would provide validity evidence on the CDA developed. In this study, the model data fit was determined based on the Hierarchical Consistency Index (HCI) in AHM. The formula of HCI is as follows:

$$HCI_i = 1 - \frac{2\sum_{j\in S_{\mathrm{correct}_i}} \sum_{g\in S_j} X_{i_j}\left(1 - X_{i_g}\right)}{N_{c_i}}, \quad (3)$$

where S_{correct_i} is a set that consists of the items that are correctly answered by student I, X_{i_j} is the score (1 or 0) of student i for the item j, where item j is an element in the set $S_{\mathrm{correct}_i} S_j$ is s set which consists of the items which required subset of attributes measured by item j, where item $j \notin S_j$, X_{i_g} is the score (1 or 0) of student i for the item g, where item g is an element in the set S_j, N_{c_i} is the total number of comparisons for all the items that are correctly answered by the student i (Cui and Leighton, 2009, p. 436).

After analysing the HCI, the pupils' percentage subscore for each attribute was calculated. Then, the attribute mastery pattern of each cognitive model was determined by categorizing each attribute probability estimated into "Mastery" and "Non-mastery"-based on the cut-off score proposed by Bradshaw (2017). Since the attribute mastery is presented as a row vector, the attribute probability exceeding the minimum threshold of .50 would be categorised as "Mastery" and coded as "1", whereas the attribute probability less than or equal to .50 would be categorised as "Non-mastery" and coded as '0'. For each assessment, the attribute mastery was then mapped onto the corresponding levels in the construct map based on the guidelines given in the Appendix.

3. Reliability and Validity Study

Reliability and validity are the important facets of assessment that should be evaluated during assessment development for ensuring the consistency of the measurement and meaningfulness of test score interpretation. Since Malaysia practices the vernacular school system, the reliability and validity study of online CDA developed were conducted in a Malay-medium NPS, a Chinese-medium NTCPS, and a Tamil medium NTTPS. The sample of the study consisted of 90 Year Four pupils from NPS (30), NTCPS (48) and NTTPS (12) in Penang, Malaysia. The findings of the reliability and validity study are reported in the following sections.

3.1. Psychometric properties of OMC items

The psychometric properties of OMC items were evaluated based on CTT. The range and mean of item difficulty index (p-value) and item discrimination index (r_{pb}) of OMC items are tabulated in Tab. 1. Although the three versions of the assessments consisted of the same items, the findings indicated that the item difficulty of the assessments was not the same. Specifically, all assessments were considered as very easy for the pupils from NTCPS with the mean difficulty index ranging from .91 to .94 (>.80) (Tavakol and Dennick, 2011). However, only four out of the six assessments and three out of the six assessments were considered as very easy for the pupils from NPS and NTTPS respectively (Tavakol and Dennick, 2011). Nonetheless, all items in the six assessments were still considered as very good discriminating items with the minimum mean r_{pb} of .52 (>.40) (Ebel and Frisbie, 1991), regardless of the difference in terms of language. In other words, these items were good in differentiating pupils from high mastery level and the low mastery level across the school type.

Tab. 1. Psychometric properties of OMC items

	NPS				NTCPS				NTTPS			
	p-value		r_{pb}		*p*-value		r_{pb}		*p*-value		r_{pb}	
CDA	range	*M*	range	*M*	range	*M*	range	*M*	range	*M*	range	*M*
CDA 1	.83−.97	.88	.45−.89	.61	.85−.96	.92	.26−.82	.56	.83−.92	.86	.47−.93	.79
CDA 2	.63−.97	.82	.39−.74	.63	.88−.98	.94	.33−.75	.58	.42−.92	.70	.29−.89	.69
CDA 3	.57−.97	.80	.31−.90	.56	.85−.98	.93	.38−.82	.54	.58−.92	.79	.39−.93	.61
CDA 4	.50−.97	.79	.38−.81	.65	.77−.98	.91	.35−.83	.58	.50−.92	.79	.28−.93	.64
CDA 5	.63−.97	.84	.20−.87	.61	.81−.98	.93	.23−.76	.52	.75−.92	.83	.48−.94	.72
CDA 6	.47−.97	.83	.39−.85	.61	.81−.98	.92	.37−.75	.54	.58−.92	.81	.33−.90	.66

3.2. Reliability of assessment

The reliability of the assessment was evaluated based on AHM (i.e., α_{AHM}) and CTT (KR- 20). The attribute reliabilities (α_{AHM}) and KR-20 of each assessment are as shown in Tab. 2. Attributes A1 and A2 in each CDA for NPS, NTCPS and NTTPS showed moderately high reliability, with the alpha coefficient ranging from .51 to .93 (Hinton, McMurray and Brownlow, 2014). Compared to attributes A1 and A2, the reliability of attribute A3 in each CDA for NPS, NTCPS and NTTPS were found to be lower with the range of .17 to .70.

Tab. 2. Attribute reliabilities and KR-20 of each assessment

	CDA 1				CDA 2				CDA 3				CDA 4				CDA 5				CDA 6			
School-Type	α_{AHM} *CM1A1*	α_{AHM} *CM1A2*	α_{AHM} *CM1A3*	*KR-20*	α_{AHM} *CM2A1*	α_{AHM} *CM2A2*	α_{AHM} *CM2A3*	*KR-20*	α_{AHM} *CM3A1*	α_{AHM} *CM3A2*	α_{AHM} *CM3A3*	*KR-20*	α_{AHM} *CM4A1*	α_{AHM} *CM4A2*	α_{AHM} *CM4A3*	*KR-20*	α_{AHM} *CM5A1*	α_{AHM} *CM5A2*	α_{AHM} *CM5A3*	*KR-20*	α_{AHM} *CM6A1*	α_{AHM} *CM6A2*	α_{AHM} *CM6A3*	*KR-20*
NPS	.72	.66	.48	.69	.77	.72	.49	.73	.73	.71	.70	.73	.82	.73	.61	.78	.74	.71	.55	.71	.77	.74	.63	.74
NTCPS	.69	.70	.61	.68	.68	.59	.47	.65	.62	.59	.47	.59	.70	.66	.36	.66	.60	.60	.67	.58	.58	.51	.17	.53
NTTPS	.93	.89	.66	.90	.84	.78	.77	.84	.76	.73	.68	.74	.79	.74	.56	.76	.87	.84	.55	.84	.83	.78	.59	.80

This is because the number of items that measured attribute A3, directly and indirectly, is relatively less compared to that of A1 and A2 (Alves, 2012). Due to the prerequisite relationship among the attributes, the simple attribute (i.e., A1) will be measured indirectly using the items which elicit the response for the more complex attributes (i.e., A2 and A3) in the attribute hierarchy. Thus, attribute A1 is measured by 7 items (1 item$_{A1}$ + 3 items$_{A2}$ + 3 items$_{A3}$), attribute A2 is measured by 6 items (3 items$_{A2}$ + 3 items$_{A3}$), and attribute A3 is only measured by 3 items (3 items$_{A3}$).

Although the attribute A3 of some CDA for NPS and NTCPS was found to have low reliability with α_{AHM} ranging from .17 to .43 (Hinton et al., 2014), the result was acceptable as Gierl et al. (2009) argued that the short diagnostic tests with less than 12 items per cognitive model could hardly yield satisfactory attribute reliability. Based on the attribute-based Spearman-Brown formula, these unsatisfactory attribute reliabilities could be increased to at least .50 by increasing the number of parallel items by two-fold. Nonetheless, it comes with the price of an increase of time allocation for each CDA (Gierl et al., 2009). This might cause the participants with low performance to be opted out from the main research project as nearly 30 assessments had been developed in total.

In general, all CDAs were reliable with the values of KR-20 ranging from .54 to .86 surpassing the minimum threshold (KR-20 = .50) for an assessment with less than 15 items (Kehoe, 1994). This indicates that the CDAs could provide a consistent measurement of students' understanding using the total score.

3.3. Model data fit

The model data fit of each assessment was evaluated based on AHM using HCI. The HCI of the cognitive model corresponded to each assessment is as shown in Tab. 3. The mean HCI for NPS pupils and NTTPS ranged from .70 to .80 for the six cognitive models. This indicates that the pupils from NPS and NTTPS exhibited a moderate fit (.60 ≤ mean HCI ≤ .80) for the six cognitive models (Roberts et al., 2014). Compared to NPS and NTTPS, the six cognitive models for NTCPS were found to have a better model-data fit with the mean HCI ranging from .86 to .88. This might be due to the higher mathematical proficiency of the pupils from NTCPS compared to their counterparts from NPS and NTTPS as reported by Ghazali and Sinnakaudan (2014).

Tab. 3. HCI of the cognitive model corresponded to each assessment

	NPS		NTCPS		NTTPS		Overall	
CDA [CM]	M_{HCI}	Fit category	M_{HCI}	Fit category	M_{HCI}	Fit category	M_{HCI}	Fit category
CDA 1 [CM 1]	.79	Moderate	.88	Excellent	.72	Moderate	.83	Excellent
CDA 2 [CM 2]	.77	Moderate	.86	Excellent	.72	Moderate	.81	Excellent
CDA 3 [CM 3]	.80	Moderate	.88	Excellent	.73	Moderate	.83	Excellent
CDA 4 [CM 4]	.72	Moderate	.86	Excellent	.70	Moderate	.81	Excellent
CDA 5 [CM 5]	.76	Moderate	.88	Excellent	.71	Moderate	.82	Excellent
CDA 6 [CM 6]	.75	Moderate	.88	Excellent	.70	Moderate	.81	Excellent

Notes. CM indicates a cognitive model. M_{HCI} less than .60 indicates poor fit, M_{HCI} between .60 and .80 indicates moderate fit, and M_{HCI} more than .80 indicates excellent fit (Roberts et al., 2014)

In general, the six cognitive models that corresponded to each assessment were found to be excellently consistent with the student's responses with the mean HCI ranging from .81 to .83 (Cui and Leighton, 2009). This implies that the mathematics experts correctly identified the relevant attributes and their ordering through the task analysis (Robert et al., 2014). Following this, the attributes used by the pupils in solving the tasks were consistent with the prediction of the mathematics education experts. Thus, the diagnostic inferences made based on the six cognitive models would be valid.

4. Concluding Remarks

CDA is an alternative assessment that can provide a clear picture of the pupils' learning process to education stakeholders so that instructional strategies can be designed to tailor to pupils' needs. In this paper, we describe the process of development and validation of the online CDA with OMC items for conversion between the time units. The findings of the validity and reliability study indicated that the OMC items developed were of good quality with high discrimination power even though most of the OMC items were considered very easy. Besides that, the online CDA with OMC items developed in this study was found to be reliable at both attribute level and assessment level. With the satisfactory model-data fit, the inferences made about pupils' attribute mastery based on their performance in the online CDA with OMC items were valid. Perhaps this instrument could support the teachers in diagnosing pupils' cognitive strengths and weaknesses, followed by practising differentiated instruction in the mathematics classroom.

However, we identified some limitations in the process of developing the online CDA with OMC items. Due to the practical constraints, the sample size of the study was quite small. This could affect the generalisability of findings. To address this limitation, future studies are recommended to be conducted with a larger sample size. Since the online CDA with OMC items was translated into three different languages, future studies are suggested to analyse the Differential Item Functioning (DIF) to identify the potential item bias which might be present in the multi-lingual online CDA. Future studies should also explore the practical use of the online CDA with OMC items in the classroom setting.

Acknowledgments

This study was made possible with funding from the Research University Grant (RUI) Scheme 1001/PGURU/8011027 of Universiti Sains Malaysia (USM). The authors would like to thank all the subject matter experts, mathematics educators and psychometric experts who were involved in the development and validation process of the online CDA with OMC items. The authors also would like to thank the computer science experts, the web developer, and the technician from Centre for Knowledge, Communication and Technology, USM who were involved in the Online CDA web application development process. Lastly, the authors would like to thank all the teachers and pupils who voluntarily participated in this study.

Appendix

CDA/ Construct Map [CM]	Attributes [Code]	Descriptors [Attribute Mastery Pattern]
CDA 1 [CM 1]	1. State 1 day = 24 hours [CM1A1] 2. Convert the unit of time from days 3. to hours by repeated addition or multiplication [CM1A2] 4. Convert the unit of time from days and hours to hours by repeated addition or multiplication [CM1A3]	Level 0: Do not master any attribute [0 0 0] Level 1: Master attribute CM1A1 [1 0 0] Level 2: Master attributes CM1A1 and CM1A2 [1 1 0] Level 3: Master attributes CM1A1, CM1A2 andCM1A3 [1 1 1]
CDA 2 [CM 2]	1. State 24 hours = 1 day [CM2A1] 2. Convert the unit of time from hours 3. to days by repeated subtraction or 4. division [CM2A2] 5. Convert the unit of time from hours to days and hours by repeated subtraction or division [CM2A3]	Level 0: Do not master any attribute [0 0 0] Level 1: Master attribute CM2A1 [1 0 0] Level 2: Master attributes CM2A1 and CM2A2 [1 1 0] Level 3: Master attributes CM2A1, CM2A2 and CM2A3 [1 1 1]
CDA 3 [CM 3]	1. State 1 week = 7 days [CM3A1] 2. Convert the unit of time from weeks 3. to days by repeated addition or multiplication [CM3A2] 4. Convert the unit of time from weeks 5. and days to days by repeated addition 6. or multiplication [CM3A3]	Level 0: Do not master any attribute [0 0 0] Level 1: Master attribute CM3A1 [1 0 0] Level 2: Master attributes CM3A1 and CM3A2 [1 1 0] Level 3: Master attributes CM3A1, CM3A2 andCM3A3 [1 1 1]
CDA 4 [CM 4]	1. State 7 days = 1 week [CM4A1] 2. Convert the unit of time from days 3. to weeks by repeated subtraction or division [CM4A2] 4. Convert the unit of time from days to weeks and days by repeated subtraction 5. or division [CM4A3]	Level 0: Do not master any attribute [0 0 0] Level 1: Master attribute CM4A1 [1 0 0] Level 2: Master attributes CM4A1 and CM4A2 [1 1 0] Level 3: Master attributes CM4A1, CM4A2 andCM4A3 [1 1 1]
CDA 5 [CM 5]	1. State 1 year = 12 months [CM5A1] 2. Convert the unit of time from years 3. to months by repeated addition or multiplication [CM5A2] 4. Convert the unit of time from years 5. and months to months by repeated addition or multiplication [CM5A3]	Level 0: Do not master any attribute [0 0 0] Level 1: Master attribute CM5A1 [1 0 0] Level 2: Master attributes CM5A1 and CM5A2 [1 1 0] Level 3: Master attributes CM5A1, CM5A2 andCM5A3 [1 1 1]
CDA 6 [CM 6]	1. State 12 months = 1 year [CM6A1] 2. Convert the unit of time from months 3. to years by repeated subtraction or division [CM6A2] 4. Convert the unit of time from months 5. to years and months by repeated subtraction or division [CM6A3]	Level 0: Do not master any attribute [0 0 0] Level 1: Master attribute CM6A1 [1 0 0] Level 2: Master attributes CM6A1 and CM6A2 [1 1 0] Level 3: Master attributes CM6A1, CM6A2 andCM6A3 [1 1 1]

Notes.
CM 1: Conversion of the unit of time involving day and hour from a larger unit to a smaller unit
CM 2: Conversion of the unit of time involving day and hour from a smaller unit to a larger unit
CM 3: Conversion of the unit of time involving week and day from a larger unit to a smaller unit
CM 4: Conversion of the unit of time involving week and day from a smaller unit to a larger unit
CM 5: Conversion of the unit of time involving year and month from a larger unit to a smaller unit
CM 6: Conversion of the unit of time involving year and month from a smaller unit to a larger unit

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https://doi.org/10.1142/9789811287183_0009

09

The Masters: Speculative Assemblages of Mathematics Education as Enclosures and Commons with Ursula K. Le Guin's Feminist Utopian Anarchism

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ABSTRACT The present lecture engages a speculative reading of "The Masters", a science-fiction novel written by Ursula K. Le Guin to narrate a state where citizens are governed by the law of negating mathematics education. In this oppressive context, Le Guin crafts a collective whose desire to practice mathematics subverts the fear for death used as punishment for mathematical heresy. This allows to ponder into thinking as "negation" and "affirmation" and, consequently, to speculate two interweaved assemblages of mathematics education; *first,* the assemblage where negating mathematics enforces masculine knowledge enclosures and *second,* the assemblage of affirming the practice of mathematics as knowledge commons. The chapter contributes by rethinking of mathematics education as/for/with the commons and by discussing about speculation as an act of thinking.

Keywords: Speculative thinking; Science-fiction; Mathematics education; Enclosures; Commons; Le Guin; Marcuse; Deleuze; Haraway; Stengers; Bakhtin.

1. To Enter: Becoming Masters at the "End of the World"

Scene I: The Oath

Ganil, a young mathematician, stood "alone, naked, holding a smoking torch" in immeasurable darkness and cold wind seeking to become a Master in the state of Edun, a place facing severe ecological decay where, even, the "current of time had stopped" (p. 40). Its atmosphere denotes alienation pictured in empty streets, shut shops, and idle lonely people. One of the Priests commanded him gently to "walk forward", to "lie in the Grave of Knowledge", to drop the torch of 'Human Reason' and to "walk in the Light of Common Day!" (p. 41). In coerced obedience, Ganil walked toward a spacious hall where an old bold man, raising a silver cross, shouted: "Swear then, Masters of the Rite!" ordering the Oath rehearsal: "Under the Cross of the Common Day I swear never to reveal the rites and mysteries of my Lodge (…) To live well, to work well, to think well (…) To avoid all heresies, to betray all necromancers to the Courts of College (…) To obey the High Masters of my Lodge from now forth till

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my death (…) And I swear never to teach the Mysteries of Machinery to any gentile, I swear this beneath the Sun" (p. 42). Whilst most men loudly repeated the oath, some purposefully stuttered and mumbled. Mede, a Master in one of the lodges, whispered to Ganil's ear: "Don't Swear" (p. 43). In response, Ganil muttered a word or two and, then, stood silent opening, already, a tiny crack for dissensus to the Priest's totalitarian governing by means of negating mathematics and machines.

Scene II: The Law

"Behold the Light of Common Day" cried in triumph the old man closing the ritual of the newcomer's initiation by saying: *"Welcome, Master Ganil, to the Inner Rite of the Mystery of the Machine"* (p. 42). He was, thus, officially pronounced Master based on performing the Oath of swearing to obey Edun's law — a strict law of mathematical negation for all citizens that comprised the following principles:

(1) The Masters must abstain from teaching the mysteries of machinery.
(2) The Masters must teach the apprentices only some basic mathematics without access to new techniques, symbols or language, not inquiring about the stars or not questioning the state of things.
(3) All Citizens must betray those who seem suspects of mathematical heresy (see above 2) or those who simply practice mathematics in varied ways.
(4) Suspects of mathematical heresy are those who practice mathematics but also those who know, meet, or make friends with a necromancer, a mathematician, a mechanical or an artisan.
(5) Suspects are imprisoned and they are tortured to testify what they know until they become judged by the High Courts.
(6) When suspects are judged guilty, they must serve a Death sentence.
(7) All Citizens in Edun must live the light of the Common Day.

Scene III: Dissensus

Labor at Lee's lodge is shared with Master Mede, the young apprentices, and Lee's daughter Lani. Ganil's everyday work with them reveals the effects of this total mathematical repression on people's labor and life. Since everybody must obey the law, the requirement from apprentices is to utilize only basic and redundant mathematical tools such as the comparing stick for simple measurements or the Roman numbers for counting or operating quantities. Since methods for measuring or practices for algorithmic calculations are forbitten, apprentices depend on memorizing operation tables. But, remembering is quite a difficult task for most of them. Ganil dissents the effects of this law enactment in their working life context when, he impatiently says: "*XVI plus IXX …. what the devil, boy, can't you add?*" or when he emphasizes the lack of method at times of using the thumb as a measuring unit: *"(v)ery interesting. But it doesn't matter how long your inch is so long as you use it consistently"* (p. 45). Mede joins Ganil for dissenting the Priests' violent imposition of negating mathematics. In this context, they discuss quietly, but with passion, about the starts, the sun, the mechanics or even the circle as being the figure for the sun and the symbol for nothing. They spoke in low voices, standing close to each-other, taking care of not being noticed by anyone who could betray them as suspects. Such cracks of dissensus were also traced during the initiation ceremony when Mede insisted for Ganil to fake his Oath performance.

Scene IV: The Encounter

Despite this austere mathematical repression, Ganil becomes introduced by Mede to a circle of mathematics connoisseurs, formed in secrecy, at the margins of Edun. In this encounter, mathematicians, artisans, engineers, apothecaries, weavers, necromancers, and lay people were assembled around their common desire for practicing the art of mathematics. Mede insisted: "*And yet there are things to be learned Ganil ... Outside"* (p. 47). Coming together meant that the risk of becoming suspects of heresy and, in turn, taken to court, being tortured, or sentenced to death was shared through their solidary existence in the encounter. In their meetings, they open time and space for practicing mathematics, inquiring about cosmos, stars, and the planets, questioning the status of knowledge, proposing the change of methods. They, even, dare discuss forbitten matters such as the sun's shape or its distance from earth and query for complex issues like what is nothing, how the figure of nothing compares to that of circle. And, moreover, they find comfort and joy with each other through sharing knowledges, experiences, wine, and ale: *"Some wine, young Masters, or ale? My dark ale came out first-rate this year. So, you like numbers, do you, Ganil?*" (p. 50) said Yin the old necromancer. This gesture notes that people are assembled with care for each other and for their desire to learn mathematics in diverse needs and interests. Despite the risk of being betrayed as suspects of heresy and sentenced to death is, always, at proximity, pursuing their desire for mathematics becomes a space for dissensus, a way for living their mortal life including a way for learning to die.

Scene V: The Death

One day, after many years of draught, the skies open, rain comes, the sun appears again and people in Edun run out in streets, squares, chimneys or further out in the fields to watch this scarce phenomenon. Suddenly, the evening star appeared. At that night, Mede was arrested. He was betrayed for pointing an instrument at the Sun in the sky trying to measure the distance between Earth and God. He was accused of the heresies of invention and computation. Soon, Ganil was also arrested for the heresy of knowing Mede. Both were sent to the Court and tortured so that to betray and speak about others' practices. Ganil, upon refusing to testify, was tortured severely, and was left with an injured hand. Mede accepted all heresies addressed to him and, thus, was sentenced to death. In solidarity for his friend and in hope for saving his life, he proclaimed that Ganil knew no mathematics. By the noon, Mede was brought to the square and a gold-robbed Priest set the fire. Soon, the smoke suffocated him. Ganil could hear the soft voice of his friend: "*What is the Sun? Why does it cross the sky? ... Do you see how I* *need your numbers?... For XII, write 12 ... the figure of nothing"*. At this moment, although Yin, the old necromancer asked Ganil to leave away with him escaping Edun, he opted to stay and confront the risks of death.

2. Speculative Thinking in Mathematics Education

2.1. A carrier bag theory of science fiction novels for countering heroes

Donna Haraway (2020) dedicates her chapter "Carrier bags for Critical Zones" to Ursula le Guin for closing the much-celebrated volume "Critical Zones: *The Science and Politics of Landing on Earth*" edited by Latour and Weibel (2020). The tangible metaphor of "*carrier bag"* was coined by Ursula le Guin (1986) to theorize the literary art of fiction grounded not on the grant theories that prevailed the male dominated world of science and science-fiction but on the modest creations of women, animals, aliens and other minoritarian species. Inspired by Virginia Woolf's (1938) feminist troubling of language in "Three Guineas", where "hero" and "heroism" are renamed to "bottle" and "botulism" to emphasize masculine power as containers holding things invisible.

Taking us back in time, when *"hominids evolved into human beings"* and the activity of gathering seeds, roots, fish, rabbits, or other species was core for staying alive, she explains how the "carrier bag" as *"... the tool that brings energy home"* (le Guin, 1987 p. 6) becomes vital. For her, the bag as a humble and easily carried tool supports her strivings against; the disciplined loyalty to fixed forms of culture, identity, and civilization to govern people, "*the killer story"* of the Hero for mediating competition, wars, laws, exploitation and, the dominant scope of science-fiction to mythologize or condemn modern science and technology for conquering earth, space, and alien species as the end of the world must be confronted and new futures must be quickly invented. Walking the steps of Bakhtin (1981) she rejects heroic epics and opts for novels telling stories from below because as Le Guin (1986) argues: *"... instead of heros they have people in them"* where the narrative avoids *"... the linear, progressive, Time's (killing)-arrow mode of the Techno-Heroic and redefines technology and science as primarily cultural carrier bag rather than weapon of domination" (p. 8)*. The novel, rather, affirms phenomena as being complex, polyphonic, situated in space and time, non-teleological and in need of troubling. Similarly, her novels reflect events that matter for scientific, indigenous or artisan communities. She writes eloquently:

"So, when I came to write science-fiction novels, I came lugging this great heavy sack of stuff, my carrier bag full of wimps and klutzes, and tiny grains of things smaller than a mustard seed, and intricately woven nets which when laboriously unknotted are seen to contain one blue pebble, an imperturbably functioning chronometer telling the time on another world, and a mouse's skull; full of beginnings without ends, of initiations, of losses, of transformations and translations, and far more tricks than conflicts, far fewer triumphs than snares and delusions; full of space ships that get stuck, missions that fail, and people who don't understand. I said it was hard to make a gripping tale of how we wrested the wild oats from their husks, I didn't say it was impossible. Who ever said writing a novel was easy?" (ibid, p. 7)

It is due to Ursula Le Guin's "carrier bag" theory of science fiction that Donna Haraway emphasizes that "*it matters what stories we tell"* urging us for *"staying with the trouble*" — both well quoted phrases. Haraway (2016) argues for speculative fabulation as a *"... mode of attention, a theory of history, and a practice of wording"* (p. 230) that disrupts habitual knowing and norms. Denoting the need for transversal articulations across diverse modes of thinking in scholarly work, Haraway (2013) uses the SF acronym referring to: *speculative fabulation, science fiction, situated feminisms, string figures, scientific facts, so far.* The "so far" conveys the political cry for the feminist work that must continue in our scholarly worlds for creating onto-epistemic cultures as a matter of care for both male and female as observed by Isabel Stengers (2021) who argues that: *"So Far is the very cry of resistance against the normality claimed by states of affairs. They have established themselves as "normal," but only so far!"* (p. 126).

To enter the lecture, the five scenes (oath, law, dissensus, encounter, death) were created based on the Le Guin's novel "The Masters". Written in the early 60s this début novel of hers was published in science fiction magazine Fantastic and was adapted and staged as Hidden Sky between 2000–2010 in a musical theatre opera by Peter Foley and Kate Chisholm. Le Guin narrates life in the dystopic state of Edun where "mathematical prohibition" is law. Mathematics education is banned for all people by the Priests of Edun, and failure to obey is punishable by death. Despite fear in this totalitarian anti-math regime, some opt for encountering a math collective in which they share and practice mathematics in common. A first speculative reading in Chronaki (2018) has argued that Le Guin's story not only offers a contrast to a "maths for all" call but also an opportunity to unfold a thought experiment asking: *"what if mathematics became forbidden?'.* This "what if" experiment allowed to interrogate the polarized dialectic amongst *"maths for all"* or "*no to maths"* discourses present in mathematics education and to ask for the need to move beyond this binary. In this lecture, speculating with Le Guin allows to inquiry alternative assemblages.

2.2. But, who is Le Guin?

Ursula Kroeber Le Guin (1922–2018) is a world celebrated fantasy novelist who, in the after WWII male dominated science and science-fiction communities of scientists, authors readers, artists, politicians or theorists, became known for her utopian speculative fiction. Her stories cut across worlds governed by totalitarian state regimes or self-organized communities based on anarchist visions of no-hierarchy, no-property, no-development or no-gender. In these worlds the needs for technoscientific methods and artefacts for communication, invention or inquiry are weaved with society's problems, needs and interests. Her writing moves purposefully beyond the masculine subgenre of endless technological growth of spaceships, ray guns and heroes crafting worlds of "electrifying political intervention". Instead, she writes for technoscientific practices as non-unified but subjected to ecological questions shared with local indigenous communities and anti-capitalist anarchist economic collectivism and problematizes hierarchical relations, property ownership and endless growth. For example, *The Word for World is Forest* (1976) unfolds imperialism's horrors for the ecology of local indigenous communities, whilst The *Left Hand of Darkness* (1969) narrates a world of genderless and gender-fluid human beings in planet Gethen who choose to change gender or have sex at specific spacetimes. And *The Dispossessed* (1974) tells the story of Shevek, a young scientist from the anarchist community of Anarres, who travels back and forth to the capitalist world of Urras for developing and sharing a novel scientific devise (i.e., a tool called ansible) for interstellar communication.

Her science fiction has produced divergent responses amongst feminists. On the one hand, it has been critiqued by liberal feminists who insist for the fight against "*men or male institutions as a major cause of present social ills"*. They argue for resolving patriarchy through separatist politics amongst men and women representing women as equal to men (Marcellino, 2009). On the other hand, SST feminist scholars and transgender activists find her narratives of science and gender worldviews as radical and comforting. In relation to Le Guin's narratives of gender-fluid societies, Smilie (2017) acknowledges the comfort produced because: "*This world of malleable gender, free from racialized differentiation offers a respite from the rigidly defined and violently enforced systems of racism and binary gender currently found in the United States and in many cultures on Earth"* (p. 2). Despite the fact, that these worlds might not always depict the real-life experiences of transgender people, her speculative interventions of troubling gender binaries have been appreciated as courageous, creative, and radical at a time of scarcity. In relation to issues permeating gender and science, Stengers (2021) notes that her speculative fiction is grounded on a situated feminism vision that affirms both the arts of magic (also used by the so-called witches or indigenous healers) and the logics of onto-epistemic cultures (also anchored in technoscientific practices). Le Guin distrusts polemic scientific arguments that prevent potential openings in science communities and distances from science denials. At the same time, she strives to open binary problematics concerning nature/human, proofs/refutations, or scientific/everyday phenomena and emphasizes the reproduction of scientific facts in collectives that counter power hierarchies and capitalist relations.

Le Guin (1976/1987) acknowledges her critiques by admitting that in the early 60s she did not have the tools, the language and, even, the experience, to tackle matters of gender or race in more critical or creative ways. Although, her feminist science fiction started without necessarily crafting female worlds for breaking gendered stereotypes or creating female heroines, two issues must be noted. First, Le Guin's genre has managed to engage a wide male readership of science fiction in the 60s and 70s to visualize complex gendered, genderless, and male dominated communities. And second, by troubling masculine strategies even within male worlds, she embraced the difference across male-masculine and female-feminine poles. This was done *first* by insisting on her opposition for heroes, heroines, and heroic acts and *second* by engendering principles from the anarchist traditions of Bakunin and Kropotkin to reimagine community organization and life. By and large, as Le Guin's subversive utopias question authoritative power and create anti-capitalist relations, they allow for active inquiry of anti-hierarchical, horizontal relations, solidarity, labor, resources, or property ownership allying with the valuable insights from anarchist philosophy that contemporary activist collectives find important (Kallis and March, 2015; McKay, 2018, Brown, 2021). The plot characters of the "The Masters" are mostly male apart Lani and her mother who not only appear very little but enact the stereotype of obedient and reserved female longing for love and home but not for science. Still, the novel creates minor, but radical, cracks in this male world of Edun offering assemblages that exemplify disruptive connections by complexifying masculine hierarchies and power relations as we will try to show.

2.3. Speculative assemblages for mathematics education

Herbert Marcuse (1978) discusses the role of aesthetics in Marxist critical theory arguing that it must engage the political so that to identify the revolutionary transformative potential embedded in cultural sites that would, in turn, be deliberatively utilized to subvert oppressive and repressive capitalist forces. Whilst orthodox Marxists focus on transforming proletariat class consciousness by excluding other social sites, Marcuse argued for the importance for including cultural arts in the struggle to negate modernity and overcome societal challenges, contradictions, conflicts, and antagonisms in a civilized and technology dependent society. In this convergence, Deleuze and Guattari (1987) insist "*…for a revolutionary rupture owing a theoretical and political debt to Marxism*" through the arts and especially the narrative arts of literature, cinema, and theatre as mode to produce sensations and insights for alternative worldviews, they distrust the "labour of the negative". Deleuze (1995) in conversation with Toni Negri asserts that *"... any society is defined not so much by its contradictions as by its lines of flight*" (ibid, p. 171) emphasizing the dynamic forces of assemblages created in the presence of rigid molar structures.

The concept of assemblage (*agencement* in French) has a long history of development in Deleuze and Guattari's work. It is coined to emphasize the constant interplay amongst structure and contingency, organization and change, stability, and fluidity as a processual continuum. An assemblage is described as the fluid collection of heterogenous elements, things or ideas grounded in the milieus (or the social and natural environments in which they work and function) coming together in specific diagrammatic relations or connections that counter the rigidity of structures. They argue: "*We will call an assemblage every constellation of singularities and traits deducted from the flow — selected, organised, stratified — in such a way as to converge (consistency) artificially and naturally; an assemblage, in this sense, is a veritable invention"* (Deleuze and Guattari, 1987, p. 406). As such, an assemblage is neither a predetermined nor a random arrangement. Instead, it expresses a particular event around qualities and affects aiming to address uncertainty by avoiding causality. Moreover, assemblage elements include lines of three distinct types allowing them to exist in connections that adapt, change, regenerate and create territories.

First, *the molar or rigid segmentary lines* construct the assemblage as a structured territory, second *the molecular or supple segmentary lines* create necessary revisions for its resilience and third *the lines of flight* allow to reach outside the constructed assemblage escaping its structure and reaching with what is outside. Although interweaved, each one of these lines have different roles to play. Specifically, whilst the molar lines form assemblages of power as the ones found in institutions (e.g., state, church, school, family) organized around distinct segments of norms, the molecular lines are fluxes creating border thresholds inviting the becoming of many things in diverse rhythms and producing micro-ruptures as minor changes in the social plane. These formations could work as two distinct poles: a striated pole highly coded and territorialized (seeking stability) and a diagrammatic pole decoded and deterritorialized (seeking becoming). Between the two, there are infinite intermediate states and forces contributing to its continuous heterogeneity and functioning.

For Deleuze, the novel as a literary art event cannot remain an epistemological resource to represent a particular object or a canon for thinking. Instead, it must serve as an ally for thinking through assemblages of life events, as the ones described above (see the scenes I to V), that produce ruptures with intensities of affects and immanent spacetimes. The lines of flight dramatize conditions for thinking to move beyond or escaping the unavoidable construction of poles created with molar and molecular lines. Resorting to Nietzsche, Deleuze (2006) appreciates dramatization as the act of giving "birth of thought" through virtual surfaces that determine the actualisation of novel ideas. Dramatization moves thinking beyond the boundaries of humanly known worlds compelling to embrace the unexpected in its immanent plurality. So, by introducing lines of flight, the novel can form major ruptures creating the movement of a journey to the unexpected and the possibility of social novelty:

"At the same time, again, there is a third kind of line, which is even more strange: as if something carried us away, across our segments, but also across our thresholds, toward a destination which is unknown, not foreseeable, not pre-existent. This line is simple, abstract, and yet is the more complex of all, the most tortuous: it is the line of gravity or velocity, the line of flight and of the greatest gradient" (Deleuze and Parnet, 1987, p. 125).

Le Guin's science fiction as argued by Stengers (2021) supports imagination by creating lines of flight, as the ones described above, for *"... changing planes to envision thick worlds"*. The question of how to read these worlds and how such reading might permeate mathematics education remains. The speculative fiction in novel "The Masters" can be read as dramatizing two interweaved assemblages that present interest to mathematics education; *first,* the assemblage where the negating of mathematics education crafts the molar and molecular lines of masculine knowledge enclosures and *second,* the assemblage of affirming mathematics education where lines of flight create mathematics as knowledge commons. Speculating the potential of these assemblages, allowing mathematics education to be conceived beyond heroic acts, will be discussed below by resorting both to Le Guin's novel and theories of the commons.

3. Negating Mathematics Education: *Or, Enforcing Assemblages of Masculine Knowledge Enclosures*

3.1. Negating mathematics at "the end of the world"

Ursula le Guin crafts Edun as a dystopic chronotope of earth decay. A counter paraphrase of biblical Eden's paradise garden in Genesis, Edun portrays "solar catastrophe" (Brassier, 2003) where the living resources of sun/light and rain/water are in severe scarcity. People live with fear as the "end of the world" approaches. Phenomena such as the death of stars including Sun and Earth, often narrated as apocalypse by theologists, are studied in astrophysics and cosmology as recurrent events for life creation in multiple universes (Grammatikakis, 1990). Current urgencies of climate change effects facing Gaia as noted by Latour (2017) or the fast-accelerating mass extinction concerning equally species and earth systems (e.g., water cycle, floods, draughts, heat absorption, ocean acidity, soil moisture) as argued by Kolbert (2014) hold modern human behavior accountable and responsible for disturbing Earth's biodiversity, balance and interconnectivity. Blaming modernity's arrogant reliance on "reason', "logic" and "science" for ecological imbalance, catastrophes and extinctions can be traced back in the 60s and 70s critical theory advanced mainly by the Frankfurt School with Adorno, Horkheimer and Marcuse amongst others. Such claims have become reappropriated in mathematics education mainly through the programs of ethnomathematics, critical mathematics education and sociology of mathematics knowledge (D"Ambrosio, 1985, Mellin-Olsen, 1987, Restivo, 1992, Skovsmose, 1994) but also through the work of scholars active mainly in Mathematics Education and Society community (http://mes.community.org/).

Taking such environmental decay into account, one might sense that the Priests impose the law of negating mathematics for all as policy for reversing Gaia's extinction. They create an assemblage of restrictions, controls, laws, fines, fearful talk etc. geared to prevent more causes of further catastrophes. By enacting the law as a molar segmentary line that forbids citizens' engagement with the tools of modernity (i.e., mathematics and machines) they construct mathematics as heresy. In this politics, to become a "Master" (i.e., educator, teacher or trainer for apprentices in a Lodge) requires to swear the Oath (scene I) that bans mathematics education: *"... never to teach the Mysteries of Machinery to any gentile"* (le Guin, 1975, p. 42). This means that, in everyday life and work, only some basic and, even, redundant forms of mathematical knowledge are permitted for teaching whilst using, for example, Hindu-Arabic numerals instead of roman symbols, algorithms for calculating instead of memorizing result tables, measuring instruments instead of comparing sticks are considered heresies against the Oath.

Negating the heresy of mathematics as the language of “reason”, “logic”, “number” and “machine” becomes a prerequisite for people to live, work and think well in what the Priests call the “Light of the Common Day”. Edun’s citizens must not focus on creating any new concepts or machinery but, ought to pursue a common leisure life liberated from the dangers of mathematics. If citizens disobey, they risk being betrayed as necromancers and referred for death sentence (scene V). In parallel, the oath and the law ask them to betray others that means accepting the role of traitors. They must refrain from mathematical practices that allow the production of machines and technological artifacts and disturb the “common day”. And they must betray anyone who is not disciplined with this condition — a harsh moral for living life based simultaneously on modes of epistemicide (loss of knowledge) and genocide (loss of people). A fascist ethics is inscribed through a biopolitical fabrication of citizens’ living, working, and thinking as subjects who must fear mathematics as something heretic and dangerous for their state.

3.2. Mathematics and machines as threat to society

The call for returning to the “light of the common day” is tightly linked with the law for negating mathematics as seen above and in scenes I, II where the citizens are obliged for idle leisure. This can be read as critiquing, resisting, or refusing technoscientific investments in a world that already faces continuous crises of ecological and economic catastrophes requiring immediate action. The plot could be read as an exaggeration of Marcuse’s seminal work “One-Dimensional Man”, inspired by critical philosophical work with scholars in Frankfurt School. In that book, Marcuse (1964) interrogates industrial societies for undermining traditional cultures when machines and technologies become the principal modes of social control that, eventually, create the paralysis of critique, make individuals incapable to resist oppressive power and, finally, construct an obedient society without opposition — critiquing the catastrophic effects of machines and technoscience for creating the One-Dimensional civilisation. Kellner (1991) cites Marcuse (1964) arguing that:

“The chief characteristic of this new mode of thought and behaviour is the repression of all values, aspirations, and ideas which cannot be defined in term of the operations and attitudes validated by the prevailing forms of rationality. The consequence is the weakening and even the disappearance of all genuinely radical critique, the integration of all opposition in the established system” (ibid, p. xii).

Herbert Marcuse (1964) denotes the risky outcomes of a massive invasion of everyday life and culture by an epistemic culture heavily based on scientific instrumentalism and grounded on assumptions of mathematical logic, objectivity, and neutrality. He is not alone of noting such threat. Jacques Ellul, in the same year, wrote “The Technological Society” arguing that technology and overreliance on technoscientific expertise presents fatal societal risks that erase humanity and destruct ecosystems through “technological buffs” creating unintended natural catastrophes and deskill workers (Ellul, 1964). Currently, Rossi Braidotti (2019) notes that as the fourth industrial revolution (Schwab, 2015) advances augmented digitalised life with biotechnology and biodata and converges with severe biodiversity extinction (Kolbert, 2014) and advanced capitalism promoting modes of commodified life, labour exhaustion and democracy fatigue, the threats are not easily reversible. Today, we become more and more aware of life being subjected to digitalised datafication practices since decisions of civic, economic, and personal importance are, already, automated by systems managing large amounts of biodata creating what Cathy O'Neil (2016) calls "weapons of math destruction" amplifying inequalities and placing democratic societies out of control. In a mathematical knowledge monopoly, digital machines become means of bioproduction but also apparatus of biocapitalist relations alienating people from their immediate life-worlds. Herbert Marcuse (1964) argued that citizens become easily imprisoned in this logic and they freely subject themselves to its power. And based on the critical philosophies of Hegel and Marx, he engages the dialectic power of "negative thinking" for countering this deadening path. Ursula le Guin eloquently offers a speculative fabulation of Marcuse's suggestion by narrating a state that negates mathematics and mathematics education. But what is negative thinking?

Marcuse (1941) endorses Hegel when argues that: *"Thinking is, indeed, essentially the negation of that which is immediately before us"* (Marcuse on Reason and Revolution, 1941, p. 64). For him, this is inevitable because the societal world contradicts itself in at least two levels: *first,* the social reality itself is contradictory since extreme and uncontrolled forms of poverty and wealth perpetuate vast injustices, and *second,* the society could be identified as a free democratic civil body even though not every subject is, really, free to participate for decision making. Referring to the multiple risks and exclusions in west democratic states, dialectic thinking is proposed as a step in the long route for social justice:

"The function of dialectical thinking is to expose these contradictions. Things in themselves are dialectical. Therefore, dialectical thinking simply requires seeing things as they are. To see things are they are is not only to see them as established facts but, rather, to see them in terms of their unactualized potential. Hence, dialectical thinking is negative thinking as it must negate the established social facts so that their emancipatory potential may be realised" (Farr, 2008, p. 235).

The above quotation clarifies his view of dialectic thinking as, not about technoscience denial, but as articulating the thesis-antithesis contradictions toward creating a dialectic synthesis of "is" (i.e., how we see things) and "ought" (i.e., to see their actualised potential). Dialectical thought must understand the critical tension between "is" and "ought" as an ontological condition that recognizes being in a concrete practice where the given facts are taken as a priori false and negative before becoming true. When Marcuse (1964) in chapter entitled "From negative to positive thinking" discusses the case of mathematics, he resorts to Husserl's phenomenology stating the construction of two "life-worlds" that need dialectic thought: "*To be sure, algebra and mathematical logic construct an absolute ideational reality, freed from* *the incalculable uncertainties and particularities of the* Lebenswelt (i.e. specific mode of seeing or lifeworld) *and of the subjects living in it"*(ibid. p. 166, my explanation in parenthesis). In other words, mathematics offers the theory and techniques for making certain ideational construction that serve to idealise the new lifeworld:

"The result was the illusion that the mathematisation of nature created an "autonomous" absolute truth, while in reality, it remained a specific method and technique for the Lebenswelt. The ideational veil of mathematical science is thus a veil of symbols which represents and at the same time masks the world practice" (ibid, p. 166).

And further citing Husserl: "In the mathematical practice, we attain what is denied to us in the empirical practice, i.e., exactness. For it is possible to determine the ideal forms in terms of absolute identity … As such, they become universally available and disposable (ibid, p. 167).

The above problematisations concerning nature and mathematics occupies mathematics education research within the milieu of phenomenology, dialectical thought, and critical mathematics education even though Marcuse's work has not been thoroughly discussed. Ursula Le Guin, in her novel, troubles the dialectic call for "negative thinking" by crafting a legally enforced mathematics negation as a condition that whilst is set to govern the citizens of Edun, it does not affect the authority of the High Masters and Priests who still can practice mathematics as experts protected by the Court of Colleges. This fictional line of flight allows a move of our thinking in "negating mathematics" as not a process, necessarily, for the citizens to engage with dialectics, but, instead, for the few to create extensive mathematical knowledge enclosures as will be outlined below.

3.3. Or, assembling mathematics with knowledge enclosures?

Historian Peter Linebaugh (2014) traces enclosures in several attacks of violent expropriations concerning open commonly shared lands and their resources (i.e., forestry fields and pastures) during feudal Europe by the wealthy nobility marking what, later, discussed by Marx as primitive accumulation for capital. Such procedures of land theft were legalised by launching laws and constructing prisons for those who resisted exploitation. Early land enclosures were followed by forceful capitalist antagonism amongst European nations enacting expeditions toward conquering, colonising, and enclosing the lands of Asia, Africa, and America in the realm of mining common resources (e.g., gemstones, minerals, coal, oil) creating new profitable markets (e.g., the cotton trade). Practices of enslaving indigenous and aboriginal people and of taking possession of not only their lands and bodies but also their minds through catechism and education were, again, legalised. These become visible in our contemporary times through subsequent enclosures waves. The political imagination of enclosures has become today synonymous of privatising, commodifying, and marketing (Harvey, 2003; De Angelis, 2007; Caffentzis and Federici, 2014) and exemplified in the continuous mining practices for lithium or digital data — both for digital industries' interest. Means et al. (2017) argue for the continual perils of educational sites through current modes of patriarchal, colonial and capital enclosures.

Linebaugh (2014) explains how "… (t)he organization of this "brute force" requires an army and navy, a centralized taxation system, public debt, a state bank, and international financial understandings" (p. 68) noting that enclosures' expansion was legitimized by parliamentary acts protecting proprietors' interests through lawful rights for profitable production. Entrepreneurs for legalising enclosures during the late 18th and early 19th century became the known "gang of four" who advanced capitalist ideology at the convergence of science, law, and politics. Specifically, jurist Jeremy Bentham (1748–1832) worked for property laws, and introduced the panopticon idea for society control, agronomist Arthur Young (1741–1820) created economy laws and advocated earth development as capital asset, criminologist Patrick Colquhoun (1745–1820) organised the police law and institution, and demographer Thomas Malthus (1766–1834) established population studies. Their efforts for societal political administration and organization were influenced by the rise of natural philosophy and experimental science with mathematics playing a pivotal role. This has been exemplified by the work of William Petty (1623–1687) on "political arithmetic" to discuss political, religious, and economic organisation at a time England's colonial empire was expanding its home markets with commodities across the Atlantic. Political arithmetic was considered a new social enterprise or an ambitious art of demographics. It applied quantitative methods to analyze human and natural resources through an explicit political program for governing population shifts in the realm of improving infrastructure, agriculture, and trade.

Theodore Porter (1995) discussing the role played by numbers, modelling, and statistics or other mathematical fields for the administrative purposes of a state-nation in the 19th century, cites state engineer Jules Dupuit who in 1844 argues: *"Mathematics ... is a machine that ... can think for us; we derive as much advantage from its service as from machine in industry that work for us"* (p. 33). Appreciating the doxa of mathematics in modernity, Porter discusses the blind "trust in numbers" as an absolute pursuit of objectivity in science and public life. He indicates the perils of a growing expertise relying mainly on applying quantitative knowledge for making public decisions in areas such as engineering, economics, accountancy, or technology and for relying on standardized measures of a general validity that was valued as reliable, true, and fair despite the risks for creating impersonal false knowledge. Mathematics and mathematics education continue to work as a biopower fabricating people into "industrious" subjects as Michel Foucault indicated (Foucault, 1975, 2008; Hacking, 1986/2002). And Linebaugh argues that: *"(t)he incessant accumulation of "industrial" subjects required their enclosure from the cradle to the grave. To be ruled the population of civil society had to be confined and to be confined it had to be brought under complete surveillance"* (p. 38). The literature on enclosures further denotes that core in capital relations remains the issue of separating workers from their means of production and reproduction something that is central in Le Guin's novel. The Masters are crafted as subjects who know mathematics but are forbitten to teach it and this exemplifies a case of "accumulation by dispossession" that matters not merely for mathematics but for mathematics education itself.

3.4. Separating workers from their means of re/production?

Enclosures during the 15th and 17th centuries of agrarian capitalism (i.e., based on the primitive accumulation of agricultural lands and their resources for profit) in Europe and its colonies came together with dispossession strategies that served to remove forcefully peasant populations from their soils, to demarcate property ownership with fences or walls, to exploit land workers to slavery and to extent agriculture to limitless growth of natural goods (Linebauch, 2014). Throughout the centuries the word "enclosures" signifies *"... the complete separation of the worker from the means of production"* (Linebauch, 2014, p. 32). Harvey (2003) proposes "accumulation through dispossession" to emphasize how, today, brutal legalised neo-colonial acts of privatising require people's dispossession from their lands, resources, and knowledges. Dispossession from the means of production and reproduction becomes, thus, a primary condition for profit concentration in the hands of few landowners (active in markets for profit making) whilst the land laborers (active in farms for land working) or the women (active for home and children) become further impoverished.

Fischer (2022) discussing critically the making of agricultural "enlightenment" argues that in this early capitalist period, known as agrarian capitalism, land enclosures were accompanied with knowledge enclosures. Specifically, books for farming (also called husbandry denoting male domination) served not merely to store, produce, transfer, and legitimize customary knowledge of land cultivation and improvement, but to produce codified knowledge for exercising greater managerial control over workers' land labor and has strengthened social class divisions between laborers, managers, and landowners. Despite the agricultural books' ambiguous status (i.e., praised as vehicle of a scientific language for improving land cultivation whilst critiqued for cultural decay through distrusting practical handwork) and rural farmers' opposition to the enclosure of knowledge, the books progressed steadily. Alongside circulating the discourse that *"the master should know more"* (ibid, p. 235) so that to govern and organize agriculture with expertise, social class divisions and antagonism was gaining grounds. In short, knowledge enclosures through books constructed the Master's theoretical expertise more important than the workers' operational skills and at the same the reading practices more valuable than the oral, embodied shareable traditions utilized by the commons. Current, knowledge enclosures include the privatization and marketisation of information as intellectual property, personal data, software, digital and scientific products but also the extreme racial, ethnic and gender related knowledge divisions amongst so-called expert, novice or ignorant (separating the ones who know from the ones who do not know). We note here how these strategies form assemblages of rigid structure by imposing molar segmentary lines with restrictive divisions and allowing molecular supple lines that strive for corrections and change within that context.

Separating workers from their means of production was analysed by Karl Marx as core capitalist strategy for profit making when machines were introduced in the early industrial period of the factory. Specifically, in his chapter entitled "Machinery and Modern Industry" (i.e., fifteenth chapter of Capital, volume one, published in 1867) Marx questions mechanical inventions for lightening workers' labor. Starting with how mathematicians and engineers discuss machine as a complex tool based on series of combinations amongst simple motive powers, he analyzed the machine as an organized mechanical system (e.g., steam or weaving machine). He noted how this system is set to perform the operations formerly done by workers but it, now, supersedes human force by implementing labor in a mode that requires *"... the conscious application of science, instead of rule of thumb*" (Marx, 1867, p. 7). And argues that the "*… separation of the intellectual powers of production from the manual labour, and the conversion of those powers into the might of capital over labour is, as we have already shown, finally completed by modern industry erected on the foundation of machinery"* (ibid, p.24) is what creates social class divisions and produces surplus value. He further denotes that *"... the gigantic physical forces, and the mass of labour that are embodied in the factory mechanism and, together with that mechanism, constitute the power of 'the master'"* (ibid, p. 24).

Following this line of argument, Marx (1867) explains how the "masters" tend to monopolize the intellectual labor of machinery by separating its value from the factory workers' operative labor that is appreciated as low skilled. In turn, such skills will be required upon appropriate education and training valorising "the masters" machinery far more important for the production process than "the workers" operative skills. In this, Marx argues the technical subordination of workers to: "*… the uniform motion of the instruments of labour ... give rise to a barrack discipline ... elaborated into a complete system in the factory ...dividing the workpeople into operatives and overlookers, into private soldiers and sergeants of an industrial army"* (ibid., p. 23*)*. He concludes his description of factory life by noting that: "*The main difficulty ... lay ... above all in training human beings to renounce their desultory habits of work, and to identify themselves with the unvarying regularity of the complex automaton.*" (ibid, pp. 24–25) — allowing to appreciate the gap between machine's automated labor and worker's manual operative skills. For Marx this complex context of divisions in the factory signifies Industrial Revolution, "primitive accumulation" and social class antagonism as core principles of capitalist enclosures. Further, the continuous separation amongst people as possessors and non-possessors, knowers and non-knowers creates a mass of people freely and habitually subjected to commodity production of surplus value and designates how capitalist economy relations are being formed and sustained. It is worth noting that although Marx did not focus on how patriarchy was played out through industrialism, his analysis of the factory foregrounds the harsh conditions suffered by both children and women.

3.5. Enforcing masculine knowledge enclosures?

The masculine character of enclosures at the dawn of capitalism becomes central for Sylvia Federici (2004, 2014) who argues that, historically, through transitions from feudalism to capitalism, the patriarchal power played by the institutions of church, state and the family served to repress more and more women's lives by exploiting their reproductive labour as *"... necessary conditions for the existence of capitalism"* (Federici, 2004, p. 13). This was evident by strategically limiting women rights, denying them access to education, controlling female bodies as reproductive domestic labour and, even, concealing their contributions to science. In this context, specific acts were enacted for demonising certain knowledgeable women as "witches" and sentenced to death. Federici explains how these "witch-hunts" followed the peasants' revolts in the 15th and 16th century Europe addressing women who were active in solidary communal work but also shared knowledges of plants and nature for creating remedies or medicines and for curing diseases. These women were experts not only for offering medical care but, mainly, for keeping their communities resilient and reproductive. But, their knowledges became a threat for the economic interests of the newly developed medical profession around principles of modern science that, already, had exploited female knowledge. As land enclosures expropriated peasants from communal land, the witch-hunts served to separate women from common knowledge skills concerning the caring of bodies and lives. Today we experience the effects of these early enclosures of care in education including mathematics education where the caring practice of teaching becomes more and more commodified and privatised.

Federici (2014) argues that whilst women were mostly depicted as socially invisible assuming they only could have a private life of reproductive labour that was righteously exploited by men who could work in the public sphere, witches' presence embodied what had to be denied and destroyed: "*the heretic, the healer, the disobedient wife, the women who dared to live alone, poisoned the master's food and inspired slaves to revolt"* (Federici, 2014, p. 11). Such deadening of local female knowledges served to construct the need for "experts". As such, the affirmative potential of witchcraft had to be defeated. The sheer absence of women in Edun's public sphere confirms a similar masculine dystopia of knowledge enclosures. And, in parallel, the witch-hunts in Europe come in analogy with the math-hunts in Edun where mathematicians are demonised as necromancers (using magic powers). Edun's law demanded that the suspects of practicing maths should be betrayed and, then, sentenced to death in the same way as women who practiced sciences were demonised as "witches" in Medieval Christianity at the dawn of colonialism, capitalism.

Summarising the above section, it can be noted that as Le Guin seems to espouse Marcuse's call for "negative thinking" and creates a world for "negating mathematics education" she offers us a virtual spacetime to speculate how this "negation" function for the social organisation of the society. Besides Marcuse's dialectics (e.g., negative thinking, reason, logic), the novel engages heavily with the vocabulary, concepts and description offered by Karl Marx in his detailed description of factory life. All these elements and connections create an assemblage around rigid molar and supple molecular lines around the Oath, the Law and the Death sentence enforcing a society of fear, control and surveillance functioning as a security "state of exception" (see Chronaki, 2018). This speculative thinking takes us a step further to appreciate the rigid assemblage as also an enclosure of mathematics education that engulfs dispossession from key practices including the loss of technics and knowledges. But, Le Guin within this rigid assemblage crafts a third line, a line of flight, that allows people to affirm mathematics education as will be seen below.

4. Affirming Mathematics Education: Or, Encountering Assemblages of Knowledge Commons

4.1. Affirming mathematics despite the "end of the world"

The violent loss of mathematical knowledge due to Edun's oppressive law of negating mathematics becomes a concern for Ganil in his everyday work with apprentices and co-master Mede. He worries for young apprentices being, tenaciously, dispossessed from learning mathematical techniques, tools, language, and methods that could, otherwise, permitted them work better and enjoy learning. This legalised mathematics education repression becomes an enforced epistemicide (i.e., knowledge loss) by cutting any teaching and learning potential and, thus, limiting the reproduction of common knowledge such as; performing calculations, creating artefacts, moving from Roman to Hindu Arabic symbols, learning about numbers and algorithms, making measurements with tools and methods beyond the comparing stick, expanding memory and perception by calculating and computing instead of remembering or memorising facts, inquiring the unknown by asking "what is nothing", "how to measure the distance from earth to sun", inventing concepts and symbols for uncertain or dynamic concepts such as zero, circle, nothing, infinity, limit etc. Often, today, possibilities for interesting mathematical inquiry become lost or disappears within curricula praxis that emphasize decontextualised or tedious content, enact oppressive pedagogies or performs intensified high-stake testing. Such contexts restrict access to knowledge and/or make youth to refuse mathematics learning in school (Tate, 2008; Chronaki, 2018; Chronaki and Kollosche, 2019; Chronaki and Yolcu, 2022).

Ganil, knowing the risks of death, dissents, as seen in scene III, by questioning the use of tedious or false techniques and by insisting for the value of intellectual inquiry. Such minor dissensus acts are core for individuals and communities seeking democratic emancipation and justice (Ranciere and Corcoran, 2010). The history of science and mathematics narrates times when access to mathematical knowledge was limited or forbitten. The stories of Socrates, Galileo, Hypatia, the medieval long wars amongst abacists and algorists, the focus of modern science in rigorous axiomatisations and decontextualised models at the expense of other onto-epistemic mathematical cultures and, even, the expropriation of data analytics, biometrics and algorithms serving the ideals of either eugenics or surveillance policies are indicative of knowledge enclosures (Restivo, 1992; Hacking, 2014; Chronaki, 2023). In the historical context of these events, dissensus was conveyed by oppressed subjects such as: the early medical doctors or nurses called witches (Federici, 2004), the female white and black mathematicians (Bullock, 2019; Gholson and Martin, 2014; Tamboukou, 2022) working at the margins of their communities, or the indigenous people performing complex mathematical operations in the context of artefacts making, language and spirituality (D'Ambrosio, 2006; Gutteriez, 2022; Trinick, 2016; Khan et al, 2022). These issues are of core importance for rethinking mathematics education and continue to occupy scholars during the last decades.

Separating people from skills such as mathematics denotes their dispossession from key tools that allow the reproduction of artefacts, concepts, or ideas and thus as seen in the previous section. By being deprived from learning to practice reason, intuition, and imagination or to perform technics for complex activity, people lose not only specific knowledges but, moreover, their capacity and autonomy to reproduce. As such, they become increasingly dependent on capitalist relations where everything (things, relations, knowledges) becomes a matter of consuming commodities or creating human capital. Simondon (1958; 2017) warned for the risks of technique loss including the risk of losing the capacity of transforming a technique to technicities for resolving social problems. And Stiegler (2018) argues that, today, bio consumerism creates the new proletarian as a global subject dependent on digital industry's communication capital resorting on biotechnologies, biodata and biocapitalism.

Coming back to the novel, Le Guin seems to craft a line of flight through Ganil's dissensus to consider the people who perform tedious tasks in the name of living the "light of the Common Day" as becoming gradually deprived from their capacity not only to create new things, but also to critique the very meaning of "common day". Upon dissenting the Law of negating mathematics, Ganil and Mede seek the company of each other to come-in-common so that to talk about their common desire for mathematics, to perform inquiries and techniques but also to challenge knowledge ownership (see scene IV). For example, they interrogate the "mystery" status of concepts like nothing or zero and problematise mathematics knowledge accumulation as property by the Priests when ask: *"... whose knowledge is the figure of nothing"* (Le Guin, 1975, p. 46). In this Mede's critical reply: "*No one's. Anyone's. It's not mystery"* (ibid, p. 46) becomes a radical dissensus that affirms the learning of mathematics by disturbing (and dissenting) the capitalist principle of property.

Affirming mathematics education becomes evident when they encounter others in the circle of mathematicians (see scene IV) a vital force for learning as re/producing modes of thought, sensations and affects that matter for their life. Mede expresses the need for sharing learning when he says: *"... there are things to be learned ... outside"* (ibid, p. 47) followed by suggesting Ganil to encounter a collective of people who practice mathematics. Despite their diverse backgrounds and skills in being apothecaries, weavers, masons, artisans, mechanics, or machine masters, they come together at Yin's place (the necromancer) for sharing, asking questions, and storying their strivings (see scene IV). Yin welcomed Ganil as newcomer: *"... come freely ... and go freely. If we're betrayed, so be it. We must trust one another. Mystery belongs to no man; we're not keeping a secret, but practicing an art. Does that make sense to you?"* (ibid, p. 50). In this encounter, fear for death (see scene V) is confronted with compassion for living a life free from punishment and control. Their sense of freedom is experienced through a diverse modes of learning mathematics that connects them with inquiries about land, nature, planets, or stars. Such learning-in-freedom does not depend on ownership but, instead, sustains solidary and entrusting relations despite heterogeneity in a collective where it is celebrated as common good. These are all core ethics assembling the ethos and horizon of actualizing "the commons" (Haiven, 2017) and are embedded in strivings to unlearn subjection to enclosure strategies and, at the same time, to learn reclaim what is lost — be it land, languages, cultures, resources, or mathematical knowledge.

4.2. Mathematics as threshold for learning and becoming

Encountering the circle of mathematicians interweaves with strivings to counter mathematics knowledge enclosures by reclaiming mathematics as learning. As noted previously, the etymology of mathematics in ancient Greek (i.e., μαθηματικά) shares the prefix with the word learning (i.e., μάθημα) meaning that mathematics involves not only the art of number, arithmetic, geometry and astronomy but also the art of learning itself (see Liedell Scott and Konstandinidi, p. 75). This comes close to how Whitehead (1929) approaches mathematics not as method for complete understanding but as the "anarchic art of life" (cited in Chronaki, 2018, pp. 29–30) and with Deleuze's view of mathematics as inherently entwined with thinking life (Duffy, 2013; de Freitas, 2016). Deleuze's interest on creating concepts for thinking life with mathematical theories (e.g., topology, analysis, algebra, fractals, differential calculus) is evident in his philosophy of difference and repetition, in his seminal work concerning the logic of sense and in his co-authoring with Guattari for creating a political philosophy that counters capitalist relations by affirming the intensive force of desiring machines (Deleuze, 1994, 1990; Deleuze and Guattari, 1987).

Mathematics education, today, runs the risk of losing its capacity to act for the art of rethinking life and, even, the art of learning to rethink life as it faces severe enclosures in international policy guidelines and curricula driven simultaneously by state, market, and digital industries scope (OECD, 2013). Such curricula enclose its potential within intensified national and international high stakes testing procedures that feed antagonism, sustain patriarchal, colonial capitalist ideals, and emphasize its human capital (Tate, 2008; Chronaki, 2023). These are traces of an implicit dis/appearing of mathematics or, even, an epistemicide realized as a process of deadening local cognitions (Chronaki, 2018; Chronaki and Lazaridou, 2022; Chronaki et al., 2023). In our work, it has been important to value local knowledges as diverse modes of thought and practices including the immanent potential of classroom work with teachers, learners and researchers and to note strivings for troubling essentialism

(Chronaki, 2009, 2011), refusing fixed mathematical identities or language-use (Chronaki and Kollosche, 2018; Chronaki and Planas, 2019), creating affective bodying that embraces difference in pedagogic relations (Chronaki, 2018), encouraging dialogicality (Chronaki et al., 2023) and re/making spacetimes for a radical pedagogy of mathematics for the commons (Chronaki and Lazaridou, 2017, 2022; Chronaki et al., 2023). We are not alone in this but work in alliance with the work of several scholars in mathematics education within the milieus of ethnomathematics, critical mathematics education, critical post-humanities, or various arts-based or culturally responsive mathematics. Despite divergences, endeavors to rethink mathematics education from below requires affirming mathematics and mathematics education. But what is affirmative thinking?

Deleuze discussing "negating" and "affirming" as gestures for a political philosophy of emancipation explains that although they both involve difference, they approach it in opposed ways. Whilst Marcuse (1964) proposes the dialectics of negating to expose the difference between "is" and "ought" aiming to recover difference through a thesis-antithesis-synthesis work, Deleuze insists for affirming difference in itself and accept its differential potential. He argues:

"Negation is difference, but difference seen from its underside, seen from below. Seen the right way up, from top to bottom, difference is affirmation. This proposition, however, means many things: that difference is an object of affirmation, that affirmation itself is multiple, that it is creation but also that it must be created, as affirming difference, as being difference in itself. It is not the negative which is motor. Rather, there are positive differential elements which determine the genesis of both the affirmation and the difference affirmed." (Deleuze, 1994, p. 55)

Deleuze does not agree with the demands of dialectic thought for a "unity of opposites" (i.e., is and ought, thesis-antithesis-synthesis). Instead of focusing on the power struggles of negating, he emphasizes the force of becoming. This force becomes regenerated through encountering assemblages that strive for immanent spaces for creating connections across heterogenous elements and lines of flight that deterritorialize and territorialize acts and desires beyond the molar and molecular lines that perpetuate enclosures within the rigidity of structured organizations. Such lines of flight support the re/making of spacetimes for learning — a learning to create common notions (i.e., concepts, practical ideas, affects, relations) and collective practices of emancipation and transformation. For Isabelle Stengers, a feminist science philosopher, thinking with Whitehead and Deleuze, interested in questions facing science and "the intrusion of Gaia', learning is vital. Contributing to the heated debates of science wars and critical theorists debates of the 60s but also navigating the polarised arguments amongst natural scientists and the postmodern readership of the 90s, her intervention destabilises claims for science's neutrality, rationality, and objectivity but without espousing the paralysing effects of negating science through generalised accusations. Insisting that "another science is possible", Stengers (2018) suggests that the aims, goals and practices of scientific work cannot be described in abstract terms (e.g., objective, rational, neutral) but they must be situated within contexts giving birth to specific modes of thinking by allowing learning of something new to happen.

Stengers argues that the practice of science must be driven thoroughly by the question of learning. A question that urges scientists to rearticulate the workings of their scientific practice rather than hide themselves behind general method claims, the axiomatics of "real" science, or the norms of a scientific community (Stengers, 2018). Her gesture involves an affirmation of science not for its own sake, but through appreciating science's vulnerability. Specifically, Hoppe (2020) explains how Stengers stresses both the importance of careful work with science (i.e., affirmation) and the significance of situating science in contexts where the problematics of intruding life are not ignored (i.e., vulnerability). So, instead of affirming a science that portrays an enterprise of rigor, objectivity and norms (i.e., royal or major science as coined by Giles Deleuze), science is described as a careful (and caring) learning process that exists in unknown territories and is focused towards addressing the doubts and hesitations being expressed by scientists when they are engaged with material and immaterial entities or strive to use techniques and methods in complex situations. And, instead of denying science or opting for a negative thinking dialectics, Isabelle Stengers allies with Ursula Le Guin who affirms the collective practice of mathematics and with Donna Haraway's call for "staying with the trouble". The problematics of affirming scientific and mathematical practices is core in their thinking of situated ethics in endeavours for reproducing knowledge in the context of learning. The speculative fiction of "The Masters" creates the spacetime where learning is realised as situated within a collective of people — the circle of mathematicians also called necromancers. Despite being an heterogenous group with diverse skills and backgrounds, people interrogate intellectual property and hierarchies and desire freedom. In this, Le Guin crafts a utopian world that encounters knowledge commons for countering the deadening life experienced in knowledge enclosures. Following her, this worldview will be examined below.

4.3. Encountering the knowledge commons

Claiming that "the commons is invisible until it is lost", Linebauch (2014) argues that the violent acts for enclosing lands, bodies and knowledges have served, paradoxically, for making visible the commons that must revive. Linebaugh traces the movements for the commons in people's efforts to resist massive land enclosures such as the Peasants' Revolt in Ireland during 14^{th} century followed with the 15^{th} and 16^{th} centuries peasant movements in Europe, the Luddites' machine-breaking in north England during the industrial period of early 19^{th} century for reclaiming the working means of craftspeople (stockinger, cropper, weaver) and the current social movements against the continuous colonial expropriation of Indigenous and First-Nation lands, cultures, languages and knowledges allying with occupy movements and street protests against recurrent forms of crises including economy, ecology, racism and sexism. We know from anthropology that human societies, based mainly on oral local traditions, have been always depended on intricate commons relations to organize social reproduction and production of food, objects, or artefacts in rituals of exchange or gift economies (Graeber, 2001).

The commons resurgence in contemporary times is related with: *first*, an increased realisation that basic commons of a material or immaterial substance (e.g., land, forests, rivers, soils, water, air, energy, knowledge, cultural heritages, languages, urban and rural spaces, code, software, information, biodata) are being stolen, destroyed, killed, or placed in danger and *second,* a growing recognition for the need of creative responses. As such, the commons today remain, as has been always in the past, a political issue for people and their communities giving birth to a nexus of local and global political movements (e.g., Zapatistas, Occupy Wall Street protests, Idle No More, global south struggles in India, Brazil or South Europe) along with philosophical thinking concerning the concept of the commons. Engagement with the commons in the urban public sphere is evident in current creations of fab labs, hacker spaces, makerspaces, jamming sessions, sharing and solidary economy, kitchens, clinics, gardens that, also, become activist spaces. Such self-organised empowerment struggles, linked with the urge to actualise the commons, create affective relations across bodies, knowledges, and space (Stavridis, 2016; Haiven, 2017; Federici, 2014, 2018).

The commons as a nexus of movements and actualised practices in the public sphere have been primarily concerned with the co-organisation and co-sharing of natural goods (i.e., water, air, soil etc. also referred by Marx as "free gifts of nature": cited in Hard and Negri, 2009, p. viiii) among individuals and collectives. Elinor Ostrom's original research on "Governing the Commons" (1990) questioned dominant models of managing and sharing natural and human-made resources and her sociological empirical approach in local economies supported the argument that several communities can manage their commons by maximizing quality, wellbeing, sustainability, and resilience over the centuries. Her work turned attention to collective acts of producing, managing, sharing, and distributing and thus governing the commons. In short, assembling the commons as social fluid systems that resist rigid structural organisation comprise two sets of elements: *first,* the material and immaterial components constituting what is shared as common; and *second,* the social relations framing the commoning practices around values, rules, and rituals grounded in the communities. Bollier and Helfrich (2015) observe certain patterns that characterise commoning practices such as; making community-based decisions for action, production or operation, working at a modest local scale, distancing from hyper-growth ambitions, sharing knowledge practices and valuing traditional knowledges. When examining knowledge commons, emphasis is placed on procedures that decentralise production and ownership of information including personalised digital resources, biodata, technologies, or intellectual products (Ostrom, 2008; Bauwens and Kostakis, 2017). Knowledge as closed, commodified, privatised and marketized is opposed to legal alternatives through open collective practices of which Wikipedia, Open-Source Software and Creative Commons Licences are only indicative examples.

Haiven (2017) discussing the actuality of commons in the context of educational commons indicates three interweaved categories — the ambient (i.e., natural goods such as rivers, soils, neighbourhoods threatened by enclosures), the built (i.e., the commons we collectively built such as solidary kitchens, clinics or schools and are in need of continuous rebuilt, defend, reclaim) and the cognitive (i.e., the realm of ideas, processes, methods, intellectual and cultural products, technologies). All these three categories interrelate at sites where the cognitive knowledge commons become reproduced in multiple shared potentials for communicating and constructing products and relations amongst ambient and built commons. And, as the knowledge commons generate ideas and ideals for imagining and creating future worlds, they engulf accelerated potential interest for profit making enclosures. "*This is why the struggle over the enclosure of ideas like the commons is so vital ... Without them, we lack the shared cognitive material to name and advance struggles"* (Haiven, p. 27). Here, the work of Caffentzis and Federici (2014) with global South communities highlights the commons as remaining in autonomy from the market and the state so that to focus on processes in anti-capitalist organisation of social needs and desires. For this, the commons as collective resources of life face simultaneously threat for enclosure and become strategic sites for envisioning a horizon for emancipation for all. They argue that just as for Marx the commodity is the cell for capitalist production, so the common good is the cell form of post-capitalist wealth, wealth-in-common, shared wealth. For Federici (2004, 2018), the reproductive nature of the commons (including caring relations) should be emphasized by discussing how primitive accumulation relates closely to women's labor, knowledges, and bodies.

Taking into consideration that education including mathematics education is a site where the reproductive nature of the commons (e.g., material and immaterial commons, relations, affects) takes place through the caring relations of parents and teachers, a dual point unfolds: while subjects are being reproduced as laborers for knowledge commons, at the same time their desires and capabilities exceed that role (Barbagallo et al., 2019). In other words, locating the commons in formal or informal educational sites (e.g., school, classroom, leisure sites, family) the complex dynamics across micro and macro levels of working, caring, living and learning require transversality (De Angelis, 2019). Moreover, Hard and Negri (2009), based on Deleuze and Guattari's political philosophy denote the prominence for an immanent ontology of the commons where humans cohabitate the world with other species of a more-than-human nature by disdaining from exploitation and separatist practices. In this, they argue that the production of commons becomes a biopolitical process emphasizing the need for affective integration of diverse living lives including forms of cooperation for sharing not only natural resources (i.e., the free gifts of nature) but also habits, values, desires, ideas, languages, and knowledges. Means et al. (2017) acknowledging the importance of pedagogic and learning relations for reproducing the commons embrace educational commons by focusing on: *first,* the contemporary struggles for countering education as a site of capitalist enclosures or as an abstract neoliberal capture that encloses subjectivity through modes of privatisation, standardisation, commodification that sustain patriarchal, colonial, and capitalist relations. And *second,* the importance of considering the centrality of creating immanent spaces for pedagogy and learning as core political relations that understand the commons as a terrain that is always divided and contested. As Means et al. (2017) argue: *"… just as the literature of the commons pushes educational theory in new directions, understanding the commons as an educational theory yields new insights for enacting the global commons more broadly"* (ibid, p. xx). In this junction, the potential of assembling mathematics education as the commons and for the commons could be inquired.

4.4. Assembling mathematics education as/for/with the commons

Tracing the commons in their diversity across time, Linebauch (2014) stresses several quality elements and connections as principles for assembling the commons where education as creating spacetimes for learning and pedagogic relations is core. *First,* solidarity is a principal foundation in children's games and people's struggle against disasters (fire, floods, earthquake, wars) but also becomes fundamental for combating hierarchical relations (e.g., antagonism, individualism). *Second,* the commoning practices are primary for life denoting active engagement in practices where people become reconnected (instead of being separated or alienated) with their means of production and where they do not aim for profit. *Third,* learning about commoning begins in small communities such as the family or the school and retains a spiritual emphasis recognizing habits for sharing habits starting from a meal or drink up to values, languages, resources, ideas, knowledges, and tools. *Fourth,* commoning practices remain local and depend on customs, memory, and oral traditions requiring continually teaching and learning. And *fifth,* the commons strive for reversing inequalities amongst the haves and the have nots, but they work beyond class division struggles focused on affirming difference where it exists (Linebauch, 2014).

Taking into consideration the above, the potential re/assembling of mathematics education as/for/with the commons could be imagined as a process of reclaiming spacetimes for a radical pedagogy for learning itself. Such learning could align with principles noted above as creating and sustaining the ethos and horizon of actualizing the commons (see Haiven, 2017) within the context of mathematics education in formal and informal spaces of classrooms, schools, families, and leisure sites. Moreover, such learning is enacted through a gesture of affirming mathematics and mathematics education by "staying in trouble" and addressing its vulnerability (see Stengers, 2018). This could translate into a mathematics education as "being" and "making" in common where any attempt to practice mathematics in context (e.g., to describe, prescribe, conjecture or even model natural and social phenomena) encounters seriously the embedded doubts and hesitancies. Whilst, *mathematics as the commons* could be seen in processes of actualizing the practice mathematics as a learning process for espousing the ethos and horizon of "the commons" as expressed by Haiven (2017) and embracing the principles of commoning practices as indicated above by Linebauch (2014), *mathematics as/for/with the commons* could be seen as assembling a nexus of human and more-than-human elements (concepts, ideas, languages, tools etc.) and relations that support individual and collective strivings to counter enclosures but also to actualize and preserve the commons of a mathematical or a non-mathematical practice.

Seeking for the potential of mathematics as/for/with commons toward re/creating connections with life, one wonders what might be the ontology to align with. For this, Deleuze's ontological perspective of mathematics becomes vital as it opens for developing modes of sensing and thinking life itself (Deleuze, 1994; Deleuze and Guattari, 1987) and its potential has been discussed for mathematics education (de Freitas and Sinclair, 2004; de Freitas, 2016). Specifically, Deleuze argued for thinking about life through a close encounter with specific theories and practices of mathematics that depart from the axiomatics of a major or royal science. Taking seriously the history, philosophy, and theory of mathematics, they distinguish between two currents namely the problematics and the axiomatics (Deleuze and Guattari, 1987). As de Freitas (2016) explains, *on the one hand,* a major axiomatic mathematical science is founded upon the royal route of "rigor" criteria aiming for constructing a generative theory (e.g., Bourbaki school) that has become contextualized in state mathematics education curricula creating a prevailing norm of mathematical literacy through core standards. *On the other hand,* mathematical theories exemplify a genius concern for learning (i.e., "mathesis universalis" ibid, p. 2) as an open and free process for resolving universal problematics focusing on inquiring "quantitability" (e.g., measurement, number) not as a standard route but as a virtual dimension for studying the material or immaterial elements of a situation and thus creating a minor, nomadic and less rigorous science that accounts for the paradox, the nonsensical, or the monstrous. As such they embrace differential calculus, topology, fractals amongst other mathematical fields and examine the work of several mathematicians (Duffy, 2013).

Although the conjecture of assembling mathematics education as/for/with the commons as presented here requires further thinking and actualization in specific contexts, one could consider the work that is already happening in the areas of natural sciences and social commons. Specifically, Bazzul and Tolbert (2017) discussing the potential of educational commons, they argue for a continuous need to reassemble the natural and social commons for combating "*uneven distribution of natural resources and wealth between the global elite and the impoverished majority"* (ibid, p. 55) and for countering the effects of educational reforms as *"apparatus for social control and human capital reproduction"* (ibid, p. 56). Resorting to Deleuze and Guattari's (1987) concept of assemblages they apply diagrammatic methods to envision specific elements and connections that unfold the immanent potential of science education in school classrooms and activist movements striving for socioecological justice in a world where agency is being captured by bio capitalist processes of governance and control (i.e., subjects' needs and desires are driven through biodata). In this, they ally with Hard and Negri (2009) to rethink education as a process for "making the common".

Recognising the reproductive nature of education (including science and mathematics education) any process for "making the common" risks to oppress or ignore a multitude of singularities working at the intersections of difference across race, gender, culture, class, or spirituality and demands to consider the decolonial horizon of the commons (De Lissovoy, 2017). At the same time, we risk actualising and delivering the commons in contexts where the interests, values and desires of a market-oriented economy absorbs the collective commons and transforms them into profitable products. This requires to consider the anti-capitalist horizon of the commons (Caffentzis and Federici, 2014; Berlant, 2016).

In summary, the above section started with Le Guin's speculation for a mathematician's collective that, knowing the risks, affirm the practice of mathematics and moved toward discussing the affirmative gesture in scientific and mathematical practices by espousing mainly the work of Deleuze as an affirmative ontology of mathematics. Ganil and Mede by affirming the learning of mathematics they encounter a circle of mathematicians where they can practice mathematics as being in common. By crafting this line of flight, the relation with "the commons" and with educational commons was crafted so that to speculate a social organisation of learning and pedagogic relations that counter knowledge enclosures. In this context, the assembling of mathematics education as/for/with the commons was conjectured.

5. To Close: Afterthoughts for Mathematics Education

Mikhail Bakhtin (1981) by asking "what is a novel?" argues that a "novel is never given". Instead, as a literary event, the novel forms and transforms with the readers specific configurations of space and time (or chronotopes in his words) in which the world is being rethought, prevailing discourses are troubled, marginal voices are heard allowing society to examine itself and even to create previously nonexistent meanings. By "chronotope", he assembles time and space in close relation where time takes spatial flesh and space falls into rhythm in both plot and history. In a similar tone, Giles Deleuze (1987, 1990) approaches the literary event not as a representation of content but as a way for dramatizing concepts and crafting lines of flight that provide escapes to new worldviews. In both Bakhtin's and Deleuze's work, the novel is not only what occurs in the moment of writing, but mostly how the literary event allows, equally, for the readers a spacetime dramatization to consider aleatory life relations that matter for them and their communities, to offer ways to invent concepts that disrupt predetermined logics and to invent new ethical and aesthetic forms of appearance.

The speculative fiction of Ursula Le Guin allies with both of these two philosophers, who being resorted to a minor literature move us beyond heroic acts in their philosophical thinking, as she strives to engage the readers with her science-fiction novels as life chronotopes. In these, she invites the readers to transverse fictional and real spacetimes where technoscientific events have multiple roles to play. Revisiting "The Masters" novel and reflecting on her experience as science-fiction author, she argues that although this was her "*first published genuine authentic real virgin-wool science fiction"* it contains *"a story in which or to which the existence and the accomplishments of science are, in some way essential"* (Le Guin, 1975, p. 40). But, her affirmative gesture for science embraces not only the vulnerability of the scientific practice itself as argued by Stengers but also the ethos of "the commons" so that to encourage reproduction of mathematical knowledge in anti-capitalist and anti-hierarchical collectives that counter knowledge enclosures. She writes: *"Some science-fiction writers detest science, its spirit, method, and works; others like it. Some are anti-technologists, others are technology-worshippers (...) The figure of the scientist is a quite common one in my stories, and most often a rather only one, isolated, an adventurer, out on the edge of things. The theme of this story is one I returned to later, with considerably better equipment. It has a good sentence in it, though: "He had been trying to measure the distance between the earth and God." (ibid, p. 40).*

Walking with Le Guin, but also with Deleuze and Stengers, Haraway, Marcuse, Bakhtin amongst others, my speculative reading of "The Masters" offered me, and hopefully to the readers of this chapter, multiple entries to several conceptual issues that matter for a philosophical anthropology of mathematics education as an ontic and epistemic immanent plane that counter knowledge enclosures and embrace the commons. Despite the importance for attending more in depth how "the commons" might permeate diverse theories and practices of mathematics teaching, learning and researching, I consider the speculative rethinking of mathematics education crafted in this chapter as offering a modest contribution toward this opening.

Acknowledgments

I would like to thank the ICME-14 conference organizers for the invited lecture and especially the editors of this volume for their patience and care throughout a process that included the pandemic amongst other precarities.

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(Re)Assessing Mathematics Education in The Digital Age

Alison Clark-Wilson[1]

ABSTRACT Digital technologies have been evident in the field of mathematics education since the late 1970s, a time of great optimism and enthusiasm for how emerging technologies would impact on school mathematics as a subject — and how mathematics would be taught and learned. Some fifty years on, whilst the pace of educational technology design accelerates, the parallel global transformation of school mathematics curricular — and associated high stakes assessment systems — lag noticeably behind. Within the mathematics education research field, there is general agreement about the barriers to systemic change: an underestimation of the professional needs of the teaching workforce; insufficient and inequitable access to suitable technologies; an unrealistic or ill-defined vision for students' digitally-enhanced mathematics learning experiences; challenges in the design and "at-scale" uses of (mathematical) technologies in classrooms and its role within high-stakes assessments (Clark-Wilson, Robutti, and Thomas, 2020; Hoyles, 2018). The (re)emergence of computer programming, which was commonplace in UK mathematics classrooms of the 1980s has prompted some rethinking but, to date there are no widely accepted definitions of what a student's school mathematics educational experience in the digital age should comprise. The coronavirus pandemic prompted a global upskilling of students', parents' and teachers' digital skills within all phases of education and put technology, in its most general sense, on the map. In this invited lecture, I will offer a vision for how students' experiences of learning school mathematics *with and through* (mathematical) technologies might be reconceived. Alongside this, how the parallel assessment processes might be designed to enable a more student-centric approach that takes account of multiple sources of evidence. Most crucial to this is the role of teachers, whose expertise is more vital than ever as they support students to actively engage with substantive *dynamic* mathematical tools that make core mathematical ideas more tangible. The lecture concludes by highlighting how a deeper understanding of the theoretical construct of the "hiccup" (Clark-Wilson, 2010; Clark-Wilson and Noss, 2015) might underpin wider understanding of the process of teachers' classroom-based learning concerning the adoption of mathematical technologies towards this vision.

Keywords: Dynamic mathematical technology; Hiccup; Cornerstone Math; Landmark activity; Mathematics teacher professional learning.

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1. Introduction

I am a former school mathematics teacher and a mathematics teacher educator who now works as a researcher at UCL Knowledge Lab within the IOE Faculty of Education and Society. I begin by thanking the ICME-14 Committee for inviting me to give this lecture.

In my lecture, I stand on the shoulders of the giants in our field to argue why, in the context of the fourth industrial revolution and its accelerating technological advancements, alongside the huge challenges that humanity faces across the globe, there is an urgent need to reassess what mathematics is taught in school — and how it is taught. In parallel, the critical examination of the nature and focus of associated high stakes assessments is necessary to ensure that we test the mathematical content and processes that we value. Furthermore, the education sector's interest in technology, which has been fuelled by the need for remote and hybrid teaching of mathematics during the 2020–21 pandemic period, has many now thinking seriously about the role that technology might play in the future of education around the globe. However, any changes to the mathematics curriculum, and its high stakes assessments will require the use of innovative and dynamic mathematical tools. The research field has documented and established over the last 20 years just how complex, and demanding it is for teachers to adopt and adapt the more epistemic digital tools into classroom practice. I'll expand on these challenges later in my lecture. My lecture includes many video images of dynamic mathematical tools, which can be accessed via the hyperlinks that are provided in the footnotes.

My lecture will be structured as follows. I'll begin with some personal reflections on how I began to use technology, initially as a learner of mathematics in the late 1970s and later as a teacher. I'll then move on to present my argument for why curriculum and assessment reform in mathematics is so urgently needed. This will be followed by an outline of the challenges relating to scaling and sustaining the integration of more dynamic mathematical tools, for which I will use the Cornerstone Math project from the UK as an example. My lecture will conclude with a focus on some key theoretical components of teachers' professional learning experiences and trajectories, and in particular, my own contribution; the construct of the lesson 'hiccup'.

2. A Personal Reflection on Learning and Teaching Mathematics with and through Technology

In my own mathematics education, I was a member of the very first cohort of school students in England who were permitted to use a digital calculating device — a calculator — in the high stakes' pre-university examinations (the Advanced level, or A-level). My newly acquired calculator replaced the need for me to use a paper booklet of tables of logarithms, trigonometric ratios and exponentials that had been in the mathematical toolkit of teachers and learners for the preceding 100 years.

Interestingly, my 1976 model Texas Instruments Calculator (Fig. 1) was marketed originally as an “electronic slide rule calculator”, which suggests that the calculator was the digital disruptor of its day, as it was positioned to replace the established tool of choice for many engineers and mathematicians, the slide rule.

Fig. 1. The Texas Instruments Electronic Slide Rule Calculator (1976)

As a school student, I fully embraced this calculator. I read its manual, I worked out what every button did, and I was genuinely interested in how it worked. How was it calculating square roots? How was it doing it so quickly? And I was particularly interested in the memory store and memory recall functionalities, finding all sorts of ways to help myself to be able to solve the problems at hand — to take advantage of its affordances. My A-level mathematics teacher, however, did not share the same enthusiasm. She saw my possession of a calculator as a disruption. I was one of only a handful of students in the class who were fortunate enough to own one, and my teacher was not at all curious about how this new technology would impact her teaching, and her students’ learning. Consequently, she left us to work out how to integrate the tool into our own learning experiences.

Ten years later I found myself beginning my professional life as a secondary school mathematics teacher, and the technologies had evolved to include: the LOGO programming language; spreadsheet and graphing softwares; and portable graphing calculators. These technologies were all being explored in the schools in which my teacher training placements took place. I observed many teachers using these technologies, and slowly began to plan and teach lessons that integrated technology. In the early 1990s, even though there were no school computer networks, data projectors or interactive whiteboards, we found ways to exploit the resources that we had. In this period, my personal curiosity and creativity was fuelled by these new mathematical tools. At first, I found myself re-examining my own mathematical knowledge and associated understandings, much of which I had acquired by using a very different toolkit. I became inquisitive about my teaching, and I began to develop as an action researcher. I was beginning to question how the affordances of these new tools were going to change not just the way I could teach, but perhaps the order in which I would approach the teaching of topics, and the ways in which I would think about student learning.

3. The Need for a (Re)Assessment of Mathematics Education

As I write this text in 2021, it is highly apparent that the speed, connectivity, interoperability, representational infrastructures and automation of the digital technologies that are now available demand us to question every aspect of the way in which we design, teach, assess and evaluate mathematics curricular around the world.

So let me present my argument. The vision offered in the OECD's manifesto *The Future of Education and Skills: Education 2030* highlights the need to prepare young people for lives and futures that will require them to be more adaptable, collaborative, critical, self-directed and resourceful than any previous generation (Organisation for Economic Co-operation and Development, 2018). But how do we prepare young people to solve problems for which there is no known, accepted, or even correct solution? One thing we do know is that mathematical ways of thinking will be a critical component. Secondly, the available technologies have moved on at great pace around us. We already have the technologies that enable rapid, accurate and a more natural mathematical user experience that will make some standard written algorithms redundant, or at least less important. In this context, how do we collectively *re-vision* the mathematics curriculum, and its assessment, to retain the integrity of the subject and its relevance to humanity? This may mean shaking off the shackles of international comparisons and the competitiveness of the international *race to the top*.

Let me start by highlighting some freely available mathematical tools that are shaking some of our foundations. Fig. 2 shows a very typical looking question from a timed written examination that is designed to be answered only using paper and pencil tools — in this case from a high-stakes examination for 16-year-olds in England.

Work out the value of $\frac{3^7 \times 3^{-2}}{3^3}$

Fig. 2. A typical high-stakes examination question from England

Using a web-based tool, such as *Microsoft Math Solver* (Microsoft, 2019), I can scan the question with my iPad camera (Fig. 3) and solve the problem in just a few seconds (Fig. 4).

I can also take a more advanced typical question, for example to solve the pair of simultaneous linear equations, $5x + y = 21$ and $x - 3y = 9$. Again, a simple text scan (Fig. 5) can provide the numerical solutions in a range of different number representations. It also offers a choice of solution steps and the graphical representation (Fig. 6).

Handwriting recognition has also advanced greatly so, whereas previous computer algebra systems required us to learn and teach syntax — the specific language to talk to a computer — we can now use our more natural handwritten mathematical notations to the same output (Fig. 7).

Fig. 3. Scanning the text

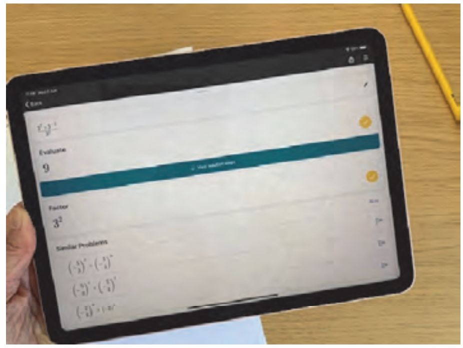

Fig. 4. The solution

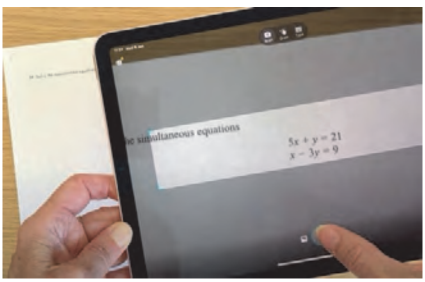

Fig. 5. Scanning the question text[2]

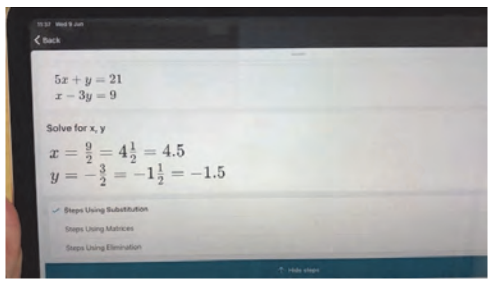

Fig. 6. The auto-generated response that gives multiple equivalent solutions as both products and processes

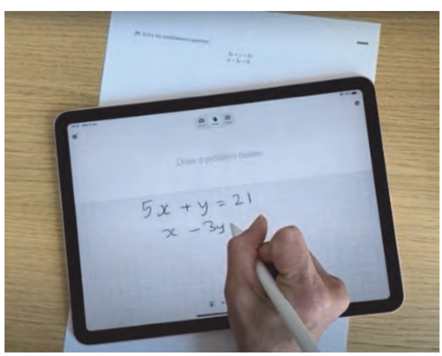

Fig. 7. Handwriting the problem[3]

[2]Video: Scanning and solving the problem.
https://drive.google.com/file/d/1qp60FdT9eD8X5wbLS12XkjWOqBZsPKnH/view?usp=sharing
[3]Video: Handwriting and solving the problem.
https://drive.google.com/file/d/1diP8xIK-dV52CF1Gru8Jb-8H1ZEgnN72/view?usp=sharing

Now, whilst *Microsoft Math Solver* offers a more transactional “input-output” environment, other mathematical technologies such as *Math Whiteboard* (Fluidity Software, 2019) enable handwritten mathematics *and* embed functionality that affords more natural explorations of mathematics — and perhaps offer opportunities for more pedagogical uses of such tools (Fig. 8). I urge you to view the video clip for this example. Tools such as *Math Whiteboard* also offer real-time cloud-based collaboration opportunities. We can be working on the same problems remotely and collectively.

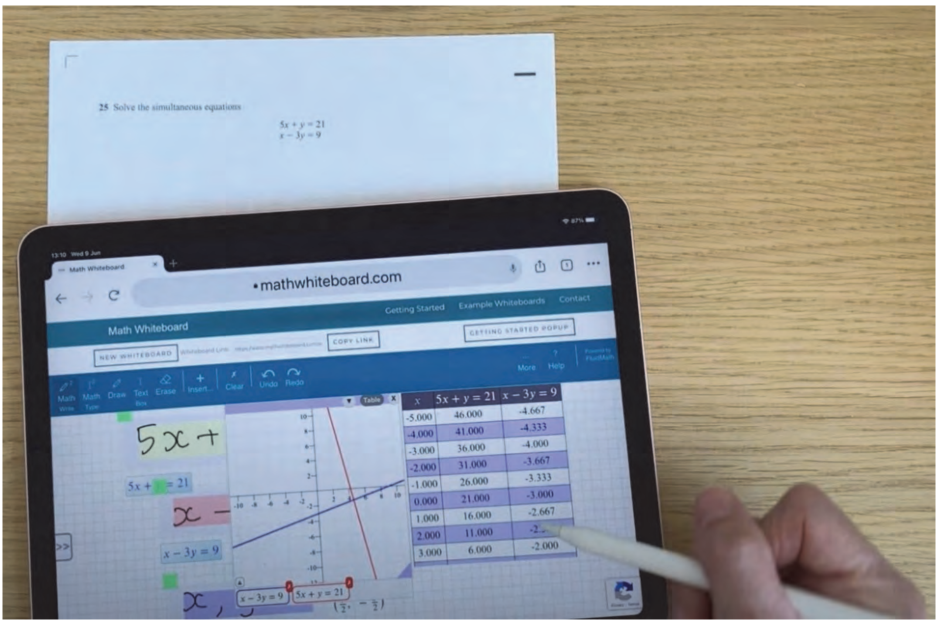

Fig. 8. Integrating handwritten and digital representations[4]

The examples above align very closely with the type of mathematics that is perceived to be the end goal for the teaching and learning mathematics for the majority of school students in many countries around the world. However, the availability and ease of use of the technologies shown above, may be seen to undermine the mathematics taught in school and be perceived by the community as facilitating *cheating*. An alternative perspective embraces such efficient and accurate tools in the same way that the calculator has now completely replaced paper and pencil methods for the calculation of square or cube roots. Hence, the mathematical work of the classroom becomes less about finding the answers to classical problems using drilled techniques, and more about the range, diversity, correctness, aesthetics and inherent mathematical beauty of different approaches to solving more contextualised problems that have wider relevance to society.

[4] Video: Integrating handwritten and digital representations.
https://drive.google.com/file/d/1fUYulU-gW3qY4jrvkKzbRRF6H8EW8XRH/view?usp=sharing

4. Towards a More Exploratory Approach to the Use of Digital Technologies in School Mathematics

So how do we move towards an integration of digital technology for mathematics that enables a more exploratory and inquiry-based approach for our learners? It is widely accepted that there are four main drivers of change within education systems that impact the way in which technology is taken up in mathematics classrooms, which are:

(1) *School mathematics curricula*, particularly where the jurisdiction has also mandated national/regional curricula, textbooks and (digital) resources, pedagogical approaches, etc.

(2) *Assessment processes*, which range from high-stakes testing to more teacher- or student-centric classroom or school-based approaches.

(3) *Support for teachers' lifelong professional learning*, to use digital technologies which, in the case of mathematics teachers, includes the more epistemic mathematical tools, which by their nature both *embed and represent* mathematics. The pace of technological developments mean there will always be a need to teachers to rethink and adapt such tools for classroom contexts.

(4) *National education strategies*, which attend to the accessibility and inclusivity of teachers' and students' uses of technology in all its forms. The related policies need to be continually updated such that the technology serve the needs of education rather than drive its use.

What follows is a case example of a country-wide project that began in 2013 to support more exploratory approaches to mathematics using the *Cornerstone Maths* digital mathematics curriculum units in England (UCL Institute of Education, 2017).

A case example: Cornerstone Math in England

The *Cornerstone Math* project was a multi-year collaboration between Stanford Research International (SRI) in the United States and colleagues at UCL Institute of Education, led by Richard Noss and Celia Hoyles. The project aimed to build on a series of earlier research projects that had developed dynamic mathematical technologies and had generated good evidence of improved students' learning outcomes. However, in each case, the digital resources required further research and development to enable them to be accessed, and used more easily, by teacher and (lower secondary age) students. Such research would aim to facilitate their use to be scaled into hundreds of schools in England (Hoyles et al., 2013).

Each of the *Cornerstone Math* curriculum units (Fig. 9) focused on an area of mathematics that was known to be *hard to teach* in lower secondary mathematics: early algebra and the notion of variable; linear functions; and geometric symmetry (to include trigonometric ratios).

This case example focuses on the unit on Linear Functions unit (*Designing Mobile Games*), the design of which was built on the very solid foundation of Jim Kaput and colleagues' work in the US. This unit embraces Kaput's seminal vision for the use of multi-representational technologies within context-based problem-solving tasks that aimed to democratize access to school mathematics for all learners (Hegedus and Lesh, 2008; Hegedus and Roschelle, 2013; Kaput, 1989, 2001).

Fig. 9. The *Cornerstone Maths* Curriculum Units

Each *Cornerstone Maths* unit embed the internet-based digital resources alongside *paper and pencil* task books for students. Alongside, teachers are provided with extensive guidance and professional support, which can take place both within and away from their schools. The *Cornerstone Maths* resources were developed over a period of 5 years using design-based research methodologies. Each iteration of the designs enabled both the technology (and the materials that were developed alongside) to evolve, and also for the research lens to be directed first towards student learning, then teacher learning, then scaling within England (and now in Chinese Taiwan and Indonesia).

All *Cornerstone Maths* curriculum units are framed by three theoretical ideas:

(1) *Transformative technology*, "computational tools through which students and teachers (re-)express their mathematical understandings" (Clark-Wilson et al., 2015; Hoyles and Noss, 2003).

(2) *Scaling technology use in STEM education*: The processes and products that bring innovative technologies to most mathematics classrooms on a national level (as elaborated by Hung et al., 2010).

(3) *The "landmark activity"*: One that is indicative of a rethinking of the mathematics or an extension of previously held ideas. It is our assumption that disruptive but carefully designed technologies lead to a cognitive breakdown, or a "situation of non-obviousness" (Winograd and Flores, 1986, p. 165). We devised the landmark activity as a methodological tool that

enabled us to maintain focus when working with large groups of teachers and many hours of curriculum materials.

Now, I introduce you to one of the landmark activities from the curriculum unit on linear functions (Fig. 10), which you may have seen before, as it has been extensively researched, and features in many of our research group's publications and presentations! If not, I strongly encourage you to view the hyperlinked video to appreciate the dynamic nature of the multi representational environment.

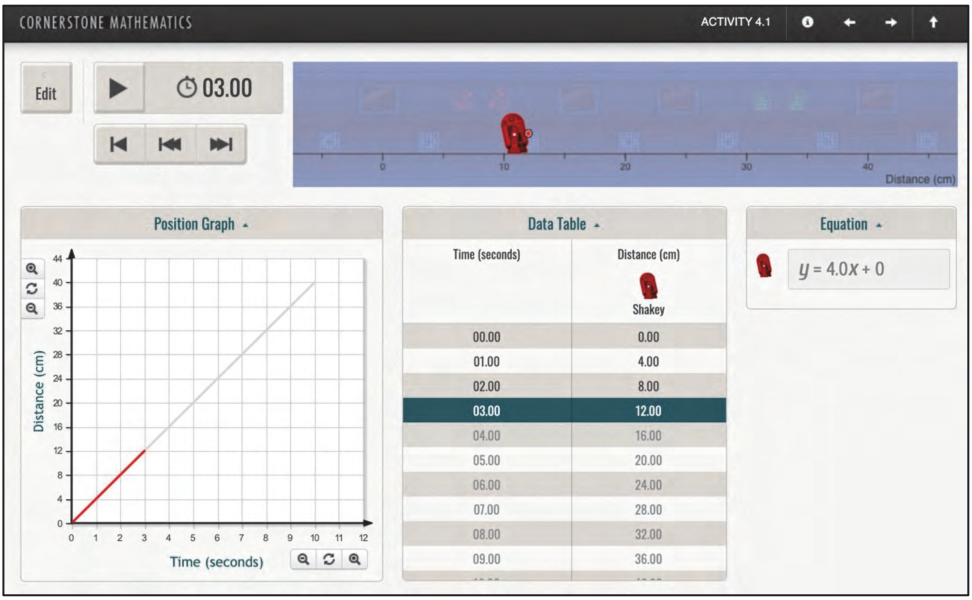

Fig. 10. Landmark activity: Shakey the Robot[5]

As you watch the animation, take time to look at the various representations on the screen. As Shakey begins to move, let your eyes wander around the screen. In a pedagogical setting I would encourage learners (which includes teachers in professional learning contexts) to carefully observe the different representations: a graph pane, a table of values, an equation, a character positioned on a number line and some control buttons that enable the animation to be played, stepped through (forward and back), stopped and rewound. As the character moves, you might also notice representations that are changing such as, the colour filling on the graph or some values being highlighted in the table. Did you also notice that some representations did not change? For example, the all-important invariant equation, which tends to go unnoticed by learners as they encounter the environment for the first time. The affordances of this environment enable some key mathematical ideas to be connected. For example, stepping through the animation second by second allows us to focus on how the characters motion is being represented in the different environments.

[5] Video: Landmark Activity — Shakey the Robot.
https://drive.google.com/file/d/18o2hFuzjmsPMG8aKOBSsIeE0jFpgQxdC/view?usp=sharing

Selecting the *Edit* button unlocks this initial scenario, by revealing a number of editable objects, which take time to make sense of in the professional learning context. A number of objects, which we call “hotspots” can change the model, and each in a different way. A deep understanding of how varying the position of each hotspot impacts the resulting new mathematical scenario is crucial for teachers as they begin to consider (and support) the classroom discussions they (might) have with learners. Although the teacher guide and pupil workbooks provide many prompts for such discussions, our research findings concluded the importance of teachers’ classroom discourse, underpinned by a level of tool fluency that fully exploits its dynamic and epistemic nature (Clark-Wilson and Hoyles, 2017; Simsek, 2020).

The above activity is the fourth of a series of twelve. It is the activity where students *first* meet the algebraic representation of a function/equation. The previous activities focus on the relationships between the animation, graph and table of values. Our approach contrasts with the traditional way that the graphing of functions is introduced, where students are expected to learn to use the function to produce the table of paired values, followed by plotting these values as coordinate points to create the graph. The Cornerstone Maths technology enables this topic area to be approached in reverse. Students are given time to make sense of the animation, the position graph and the table of values prior to meeting the algebraic syntax of the function. The language of the character (i.e., describing how Shakey is moving) supports students to be able to explain and articulate their models and explore different scenarios (i.e., making Shakey disappear or move backwards). Hence the equation is first experienced as a context-specific syntactic representation that generalises the situation at hand.

The big mathematical ideas that are addressed within the complete unit are: the coordination of algebraic, graphical, and tabular representations; the use of speed as a context to introduce rates of change; explorations of $y = mx + c$ as a model of constant velocity motion (the meaning of m and c in the motion context); and the definition of velocity as speed with direction; and the concept of average velocity. Alongside, the design principles for the technology include: offering dynamic simulations and linking between representations; the facility to ‘drive’ the simulation from the graph or the function; and the ability to show or hide representations, as appropriate.

I now move to outline some of the more effective teaching practices that were revealed by our many observations of the previously described landmark activity. These include:

- Emphasising strongly the need for pupils to make sense of the ‘hotspots’ that facilitated the graph to be edited.
- Emphasising the multiplicative/additive relationships in the table to justify the meaning of the equation.
- Highlighting the invisible variant m within the table for equations with a non-zero value of c.
- Paying attention to the graph’s axes and promoting discussion of the effect of changing the scale.

- Gathering back the students' multiple responses — as particular cases — in order to support the overarching generalisation that the greater the value of m, the faster that Shakey will move.
- Extending Shakey's journey time such that his final position could not be read from the graph nor the table — to provoke pupils to use the equation to calculate its position after a given time and thus highlighting the power of the mathematical equation as a generalisation.

Although these are all promising practices, they were collectively observed across a small number of classrooms by fewer than ten teachers (Clark-Wilson and Hoyles, 2017). The vast majority of teachers *did not interact with the technology at all* when they were discussing the students' task outcomes in either small groups or when leading whole-class plenaries. We conclude that the development of classroom teaching that embeds the use of dynamic, epistemic digital tools such as *Cornerstone Maths* requires teachers to have professional learning time alongside, and opportunities to reflect individually and with others, over years, rather than weeks or months (Clark-Wilson and Hoyles, 2019).

5. How do Teaching Practices with Dynamic Digital Technologies Evolve?

Professional programs, training events, collaborative projects, and self-directed learning, are all fundamental for teachers to come to know new technologies that might support the teaching and learning of mathematics. Ideally, driven by teachers' own curiosities, and supported through collaboration with others, new teaching ideas emerge, which are sometimes shared within different communities. The nature of the technology that is selected, brings different challenges — adopting a generic online quiz technology is a very different technology to a more epistemic dynamic geometry or graphing application. Similarly, the type of technologies that are suited to younger learners differ greatly than those for older learners. I focus what follows on the more epistemic mathematical technologies within the context of mathematics teachers in England within the secondary phase (11–16 years).

My own doctoral research sought to understand and theorise the way that teachers' learning evolved through their classroom use of a new technology (the TI-Nspire handheld and software). The teachers' common purpose for the use of the technology was to support their learners to develop mathematical generalisations through teacher-designed tasks that promoted explorations of related variant and invariant properties. The study was framed by Verillon and Rabardel's (1995) *instrumental approach*, which had been elaborated for mathematics education by the seminal work of Guin and Trouche (1999). Hence the processes of *instrumentation* (coming to know the tool and its affordances) alongside *instrumentalization* (learning to exploit the tool for a mathematical/pedagogical purpose) would underpin my analysis of the teachers' actions in the classroom.

My research concludes the theoretical construct of the "lesson hiccup", which I define as "the perturbations experienced by the teachers during the lesson, triggered by the use of the technology that illuminated discontinuities in their knowledge." (Clark-Wilson, 2010, p. 138). Hiccups are phenomena that are: "highly observable" and "cause the teacher to hesitate or pause" (ibid). The analysis of 14 lessons taught by two teachers over a period of nine months resulted in 66 hiccups, which were each categorized as one of seven hiccup types, which are described in Tab. 1.

Tab. 1. The seven types of lesson hiccups experienced by teachers (Clark-Wilson, 2010)

Hiccup type	Relating to the:
Task design	Choice of initial example. Sequencing of examples. Labelling of objects. Pedagogical approach.
Interpretations of the mathematical generalisation	Specific ⇔ general case. Range of permissible responses. Failure to notice.
Unanticipated student responses	Students' prior understanding. Student's misinterpretation of activity objectives. Students' own approaches.
Student perturbations	Unanticipated outputs from the digital tool. Doubting the authority of the digital tool
Students' instrumentation issues	Making inputs to the digital tool Grabbing and dragging dynamic objects Organising on-screen work Navigating between representations Accidental deletion
Teacher's instrumentation issues	Forgotten (or not yet learned) techniques
Unavoidable technical issues that are out of the teacher's control	Classroom network failure Technology failure

Appreciating the nature of the hiccups that teachers experience in their classrooms for particular mathematical technologies could be key to designing research-informed professional learning opportunities for teachers, alongside supporting the communities' reflective processes. By definition, a hiccup only exists if the teacher has experienced the technology-triggered perturbation and it is my hypothesis that lessons that involve dynamic mathematical technologies provoke many hiccups for most teachers — particularly during the early lessons. Professional learning models such as Lesson Study, and its many cultural and contextual adaptations, offer promising scenarios for the hiccup construct to be adopted as a lens to support teachers' collective lesson design, classroom enactment and associated reflection processes.

6. Rethinking the Assessment of School Mathematics

I would like to return to the topic of assessment. If we have redesigned mathematics curricular in which students' technology-enhanced classroom activities require them to produce mathematical work that embeds the use of dynamic mathematical tools, how does that change what and how we might assess? I propose that, rather than grappling with ways to manage the integration of digital technologies to synchronous exam room settings (which requires considerable resources), we look to how digital tools might altogether transform the way that we conceive the body of knowledge that comprises school mathematics.

One such digital tool is the Cambridge Maths Framework[6], a conceptual mapping tool which has been over 6 years in development. It comprises a searchable network of key mathematical ideas and the relationships between them, that spans the domain of school mathematics and is informed by over 1400 research sources and expert consultation. Evident within the framework, is the key construct of "waypoints", the mathematical ideas that have emerged as being multiply connected and/or crucial for the progression of later ideas (Fig. 11).

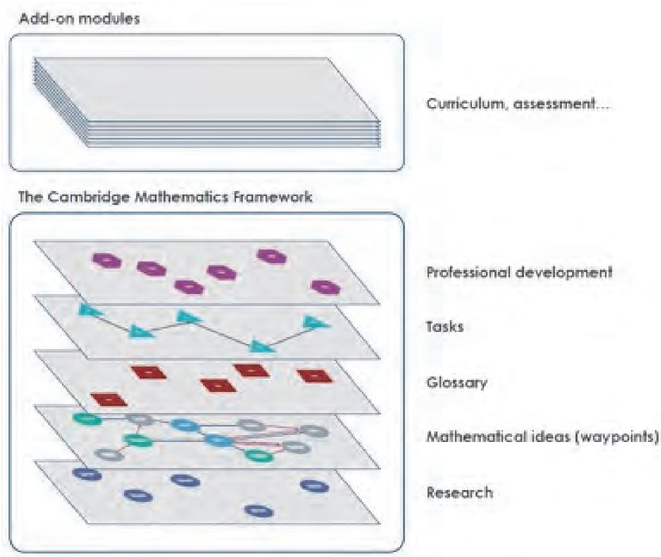

Fig. 11. Connected layers within the CM Framework and external add-on modules (Koch et al., 2021, p. 128)

The dynamic digital network graph format of the CMF allows external content from curricula or resources to be manually "mapped on" to CMF content so that it can be analysed according to the conceptual orderings, interdependencies and justifications from the research literature represented in the CMF. There are two potential applications of such a tool for the redesign of assessment goals and associated processes.

(1) The development of more naturalistic assessments of waypoints that can be taken by students "when ready" and involve the use of digital technological tools, where appropriate. Students' assessment outcomes could be recorded on their personal and portable record of mathematical achievement that could have a more universal application and understanding within global education

[6] https://www.cambridgemaths.org/

contexts. Such an approach might lead to a (re)visioning of teachers' professional roles as assessors of their students' mathematical understandings.

(2) Designers of digital technologies that already collect or collate students' assessment outcomes relating to waypoints might develop interoperability with locally contextualized versions of the framework to enable a more real-time approach to assessment. This approach might reduce the status of high-stakes examinations to account for alternative forms of assessment that support education systems to evolve.

I do not envisage the CMF as a tool to support the micro-assessment of all aspects of a learners' mathematical journey. Whilst it helps us to think about the smaller processes that might underpin conceptual development, it is more helpful as a tool to help us determine what to assess, in what sequence and how.

Returning to the examination question that assessed simultaneous equations that I highlighted earlier (Fig. 5), how might this concept be assessed within an environment where students have access to dynamic mathematical tools? Fig. 12 shows another scenario from the Cornerstone Maths curriculum, which includes three characters, each of which are designed to move at different speeds.

Fig. 12. Modelling simultaneous equations

The scenario provides multiple opportunities for students to create and justify models that might evidence their understanding of linear functions in general, and the notion of simultaneous functions, in particular. Again, the context of the scenario (characters in an animation game that move according to some criteria), offers a familiar environment to mathematicise *coinciding* or *overtaking*. This increases the accessibility of the mathematics, whilst also offering contextual language to support both explanations and applications of the associated knowledge.

7. Concluding Comments

To conclude, I'd like to introduce two ideas to stimulate the community to think about the nature of learners' mathematical experiences in a world where they have increasing access to a blend of digital and non-digital resources in, and away from the classroom. I'll address this first at a macro level, by considering the overall picture for any particular learner.

My "healthy learning plate" (Fig. 13) is elaborated for school mathematics from Laurillard's pedagogic theory, "the Conversational Framework", which was designed to consider digital transformation of students' learning experiences within the context of higher education (Laurillard, 1993, 2002).

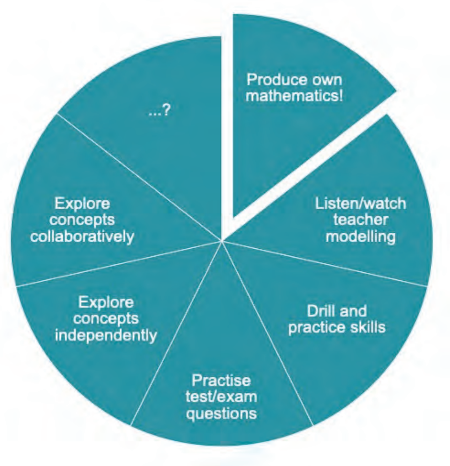

Fig. 13. A digitally enhanced school maths "healthy learning plate" — Adapted from Laurillard's pedagogic theory — the "Conversational Framework" (1993, 2002)

The plate shoes the main forms of learning activity that a learner might experience, which I offer as a tool to reflect on the balance of these experiences for learners at different stages of their mathematical development. In doing so, we might consider:

- Which of these experiences might be added, or taken away?
- What proportions of the different experiences are desirable?
- Which of these experiences might take place in, or away from the classroom?
- How do we ensure that, as the stakes get higher, and the balance of their experiences might shift, learners do not lose out on the most mathematically nutritional elements?
- How balanced are our current approaches?

My second idea is focusses attending more carefully to what we really mean when we talk about *doing mathematics*. In our field, this idea is very widely and broadly addressed. If we seek to collate lists of terms that capture human experiences when doing mathematics, we arrive at many lists that have traditionally been associated with different mathematical content domains. However, as the mathematical digital tools merge the boundaries between these domains, it is time to take a more holistic approach. Fig. 14 includes my own list on which I have highlight those processes that were exemplified in the Cornerstone Maths tasks earlier.

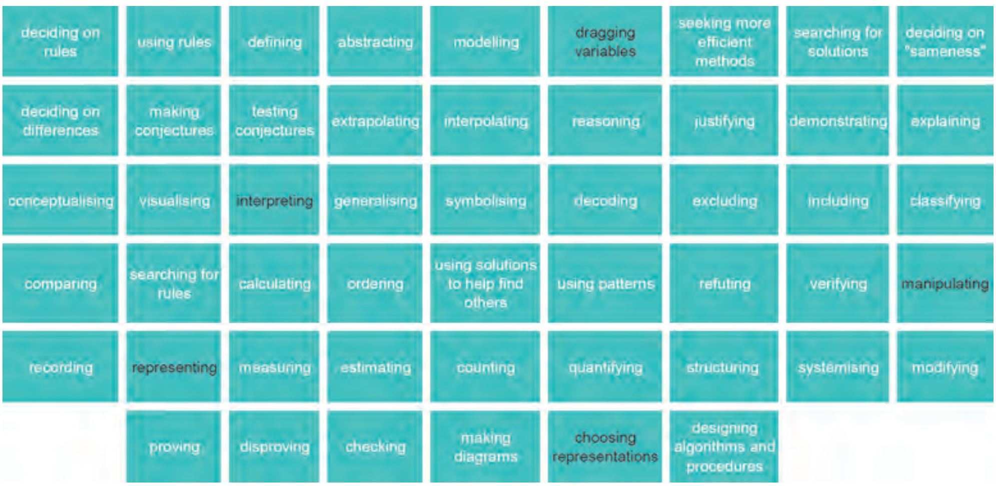

Fig. 14. A digitally enhanced school maths "healthy learning plate" — Adapted from Laurillard's pedagogic theory — the "Conversational Framework" (1993, 2002)

It is these processes that might be:

- noted and reflected upon by the learners themselves.
- observed by teachers and parents.
- captured and logged by the digital tools to inform the design of more appropriate assessments.
- used by all to enrich task and resource designs (including textbooks).

In my view the increasing access to, and use of technology within mathematics education around the world should prompt us to (re)assess how we can move towards mathematical experiences for all learners that promote rich mathematical experiences. Alongside, we must take advantage of the affordances of the digital and non-digital resources to design more authentic assessment approaches that align with nurturing all children to develop confidence and competence in mathematics.

Acknowledgments

The data collection carried out during the TI-Nspire Research Evaluation project was funded by Texas Instruments as part of two phases of research, subsequently reported in Clark-Wilson (2008, 2009). The Cornerstone Maths project was funded by the Li Ka Shing Foundation (2012–2014) and Nuffield Foundation (2014–2017). In addition, I would also like to acknowledge the many teachers, colleagues, students, industry colleagues, advisers, policy makers and other educational partners whose work and insights have been formative to my thinking and actions.

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11

Beyond Procedural Skills: Affordances of Typical Problem for the Teaching of Mathematics

Jaguthsing Dindyal[1]

ABSTRACT In this paper, I will focus on how teachers can use typical problems to develop both conceptual fluency as well as procedural skills. These typical problems will be considered as mathematical tasks that teachers can opt to select, reformulate or create new ones for use in class. Teachers may find it easier to make sense of the mathematics and pedagogical considerations in the implementation of these tasks, as they are readily available in textbooks and past examination papers as compared to rich tasks. I will demonstrate that typical problems do have affordances for developing conceptual fluency and in that sense are equally good, if not better, better than the so-called rich tasks. With limited time at their disposal, teachers have to be strategic in noticing the affordances of typical problems and in optimally using their available time for selecting and using relevant tasks for their lessons. As such, the gist of the paper, connecting to my previous work, will be on how teachers can use typical problems in their day-to-day practice to enhance the learning of their students, against a backdrop of teacher noticing.

Keywords: Typical problems; Rich tasks; Affordances; Teacher noticing.

1. Introduction

1.1. Mathematical tasks

The teaching of mathematics is no doubt a complex activity. In his model for pedagogical reasoning and action Shulman (1987) stated that prior to instruction, there are two important stages that the teacher sequentially goes through. First, the teacher has to comprehend the content related to the topic to be taught and second, he or she needs to transform that comprehension to delineate individual concepts and skills and to package these into suitable tasks for students to demonstrate their learning. Mathematical tasks are central to students' learning because "tasks convey messages about what mathematics is and what doing mathematics entails" (National Council of Teachers of Mathematics [NCTM], 1991, p. 24). Accordingly, one of the daily concerns of mathematics teachers is whether to use existing tasks, create new tasks or reformulate existing tasks for use in their day-to-day practice.

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The teachers' role is to set appropriate mathematical tasks to elicit certain anticipated learning outcomes. Accordingly, Henningsen and Stein (1997) claimed that the nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of the subject matter with which they are engaged. So, what constitutes a mathematical task may depend on the perspectives of both the teacher and the student. The teacher has to select the task and the student has to agree to work on the task. Stein, Grover and Henningsen (1996) described a task as a "classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). The mathematical idea could be a particular concept which the teacher wishes to teach to the students. On the other hand, Watson and Thompson (2015) refer to a task as the written presentation of a planned mathematical experience for a learner, which could be one action or a sequence of actions that form an overall experience. As such, a task could consist of a single problem or a sequence of problems, a textbook exercise, or even a sequence of more complex problems that may be interdisciplinary in nature.

Lester (1983) considers mathematical problems as tasks. He describes a problem as a task for which: (1) the individual or group confronting it wants or needs to find a solution; (2) there is not a readily accessible procedure that guarantees or completely determines the solution; and (3) the individual or group must make an attempt to find a solution. He further adds that, "Posing the cleverest problems is not productive if students are not interested or willing to attempt to solve them." (p. 232) An important point comes to the fore: a task on its own may not make much sense. Of prime importance is the willingness of the individual or group to work on the task.

1.2. Embeddedness of tasks-lessons-units

It is to be noted that in any unit of study in mathematics, there is a sequence of lessons used by a teacher and within each lesson, the teacher uses a sequence of tasks. Sometimes the teacher uses a single task in the lesson but commonly uses more than one task in a specified sequence in the lesson, as shown in Fig. 1 (see Choy and Dindyal, 2021). It is through mathematical tasks that mathematics teachers "transform" (see Shulman, 1987) what they comprehend about the subject into appropriate chunks for eliciting expected learning outcomes from the students.

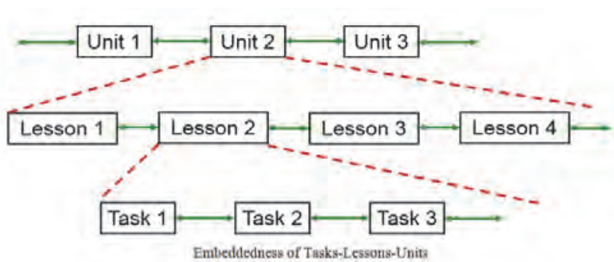

Fig. 1. The embeddedness of tasks, lessons, and units

1.3. Mathematical task as a classroom activity

Fig. 2 illustrates sequentially how a mathematical task is represented in instructional materials, how it is set up by the teacher in the classroom, and how it is implemented by students in the classroom leading to students' learning outcomes (see Henningsen and Stein, 1997).

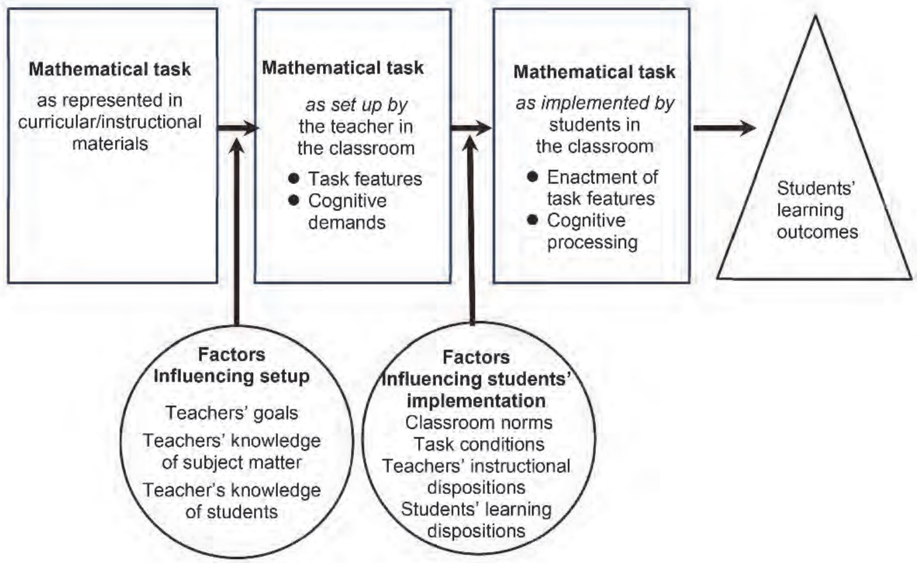

Fig. 2. Mathematical task as a classroom activity (from Henningson and Stein, 1997)

2. Types of Tasks

2.1. Rich task or high-level task

Consider the following Task A (see Dindyal, 2018):

> *Task A* Unit cubes are used to make larger cubes of other sizes. The surface area of each of the large cubes are painted and then disassembled into the original unit cubes. For each large cube, investigate how many of the unit cubes are painted on three faces, two faces, one face, and no faces? Describe the patterns you observe.

Without a manipulative, this task involves the students visualising the larger cubes and systematically recording their observations to find useful patterns. Students may approach the task by using simpler cases or what Mason et al. (2010) called specialising. Considering $2 \times 2 \times 2$, $3 \times 3 \times 3$, $4 \times 4 \times 4$ and $5 \times 5 \times 5$ cubes, and generalising to the $n \times n \times n$ cube, several patterns can be observed. There are eight unit cubes with paint on three faces; $12(n-2)$ cubes with paint on two faces; $6(n-2)^2$ unit cubes with paint on one face; and $(n-2)^3$ unit cubes with paint on no faces. We note that the patterns generate constant, linear, quadratic and cubic functions respectively. The teacher can explore this task in various ways with the students by providing some guiding questions: For a $3 \times 3 \times 3$ cube, which unit cubes will be painted on three faces? Which unit cubes will be painted on only two faces? Which unit cubes will be painted on only one face? Which unit cubes will not be painted at all? It is to be noted that not every student will benefit from these scaffolding questions. Some questions that the teacher may additionally ask at the implementation stage: can the number of cubes with paint on one face be 1800 for any large cube? How many unit cubes will you need to make a large cube that will have 729 cubes with no paint on any face?

Task A can be considered as a rich mathematical task (see Grootenboer, 2009). It has those task features that mathematics educators have identified as important considerations for the development of mathematical understanding, reasoning and sense making (see Henningsen and Stein, 1997). Also, the cognitive demand of Task A is high. Similar tasks have been used quite extensively in the literature but termed differently. Amongst others, we have worthwhile mathematical tasks (NCTM, 1991), challenging tasks (Sullivan et al., 2014), high-level tasks (Henningsen and Stein, 1997), and open-ended tasks (Zaslavsky, 1995). While acknowledging the benefits of using such tasks, research has also surfaced some shortcomings. These high-level tasks are often more complex and take longer for implementation and may even evolve into less demanding forms of cognitive activity (see Henningsen and Stein, 1997). Such tasks are generally not meant for developing procedural skills but rather to enhance conceptual understanding. Choy and Dindyal (2017) have added that despite the affordances of challenging tasks in enhancing learning experiences, there are at least three obstacles that hinder the prevalent use of these tasks in the classrooms: (1) These tasks may be too difficult for many students, and so additional prompts or supports are needed (Sullivan et al., 2014), (2) It is time-consuming for teachers to select, adapt, or design challenging tasks to use, and (3) The inherent complexity of the tasks would involve mathematics from across the curriculum and such tasks are best implemented across several lessons, or after a few topics are taught. Henningsen and Stein (1997) have also cautioned that the mere presence of high-level mathematical tasks in the classroom will not automatically result in students' engagement in doing mathematics.

2.2. Typical problem

Consider the following Task B (see Dindyal, 2019)

> *Task B* Mary puts six identical red balls and four identical blue balls in a bag. She then takes out two balls at random from the bag, without replacement. Find as a fraction in its simplest form that she draws one red ball and one blue ball.

This task is based on elementary concepts of probability and as a problem can be solved by applying a standard procedure. A student working on this task may or may not use a tree diagram. Similar problems are quite common in examination papers and in standard textbooks. The focus here is not so much on solving the task but much more on the fact that this task is what can be termed as a typical problem, as it has a fairly straightforward answer with a very moderate difficulty level, that students have practised previously.

Typical problems are used by teachers very often on a day-to-day basis. Choy and Dindyal (2017) describe typical problems as:

> … standard examination-type questions or textbook-type questions which focus largely on developing procedural fluency and at times, conceptual understanding. These questions can be solved more quickly than challenging tasks and are used frequently in mathematics lessons. Given the omnipresence of such questions in textbooks and other curriculum materials, we see typical problems as an untapped resource that can be used to orchestrate daily learning experiences. Using tasks developed from typical problems to orchestrate learning experiences would position mathematical learning experiences as an integral part of mathematics lessons, and not just reserved for occasional "enrichment" lessons. (p. 158)

Typical problems need not always be contextual. A non-contextual problem such as the one shown below can also be considered as a typical problem.

Solve the quadratic equation, $x^2 - 4x - 5 = 0$,

Typical problems can be solved in a procedural manner where the focus of the teacher might be on the development of certain specific skills. To summarise, we can say that typical problems

1. are standard examination-type questions or textbook-type questions, contextual or non-contextual, which focus largely on developing procedural fluency;
2. can be solved by students in less time;
3. are omnipresent as compared to "rich tasks" or "challenging tasks";
4. are easier for teachers to access, modify, adapt and use in class as compared to rich tasks and so are used more often in class as compared to rich tasks.

Are there ways in which teachers can engage students in mathematical tasks for developing their procedural skills as well as their conceptual fluency while using simple day-to-day tasks? The type of tasks that are readily available to teachers are typical problems, which include regular textbook problems and examination-type questions. Do typical problems have affordances (see Gibson, 1986) for developing conceptual fluency besides the expected development of procedural skills?

3. Affordances

3.1. Theory of affordances and typical problems

Henningsen and Stein (1997) have highlighted that even high-level tasks or challenging tasks have affordances to be implemented routinely by teachers. On the other hand, a legitimate question to ask, is "to what extent do typical problems, usually used for developing procedural skills have affordances for developing conceptual fluency?". What does the term affordances mean in this context of using mathematical tasks?

The perceptual psychologist, Gibson (1986) coined the term "affordances" in his famous book, *The Theory of Affordances: An Ecological Approach to Visual Perception*. Gibson claimed that, "The affordances of the environment are what it offers the animal, what it provides or furnishes, either for good or ill." (p. 127) Gibson came up with three important ideas about affordances: (1) an affordance of an object exists in relation to an observer; (2) the affordance of the object does not change as the need of the observer changes; and (3) the observer may or may not perceive or attend to the affordance according to his needs, but the affordance being invariant is always there to be perceived. He also added that an affordance is not bestowed upon an object by the need of an observer and his act of perceiving it. This idea of affordances can be applied to typical problems, if we consider the teacher to be the observer and the typical problem or task to be the object. We can state that, (1) an affordance for using a typical problem to develop conceptual fluency exists relative to the action and capabilities of the teacher; (2) the existence of the affordance is independent of the teacher's ability to perceive it; and (3) the affordance does not change as the needs and goals of the teacher change. Thus, we can say that to perceive the affordances of a typical problem means to be able to notice the characteristics of the task in relation to the particular understandings of the related concept in order to adapt the task for use in classrooms. According to Gibson, affordances can be perceptible or hidden. In the context of typical problems, we may consider the use of typical problems in a routine way to develop procedural skills as a perceptible affordance, because that is how they are expected to be used. On the other hand, the use of typical problems in a non-routine manner to develop conceptual fluency may be considered as a hidden affordance as this type of use of typical problems is not expected. Gibson had added that affordances that exist in relation to an observer could be positive or negative which in the context of mathematical tasks may mean a more productive or less productive use of the task in class by the teacher. To realise the positive affordance of something, Gibson suggested that we need to magnify its optical structure to that degree necessary for the behavioural encounter. How do we magnify the positive affordances of typical problems? One way is to reformulate or modify the typical problems.

3.2. Alice and the reformulation of a typical problem

In an earlier paper (Choy and Dindyal, 2017), we described how Alice, an experienced teacher enhanced the positive affordances of a typical problem on matrix multiplication (her first problem in the lesson) by reformulating the problem (modifying the problem). In this class she used four typical problems. She basically reworded the original matrix problem (not shown here) for the students avoiding the use of any matrices and thus opening up the solution space.

The reformulated typical problem:

> Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. Teresa obtained 29 Gold, 10 Silver, and 5 Bronze awards.
>
> Robert obtained 30 Gold, 6 Silver, and 8 Bronze awards. They gained 5 points from each Gold award, 3 points for each Silver award, and 2 points for each Bronze award.
>
> Find the total number of points that Teresa gained.
>
> Find the total number of points that Robert gained.

As such, the students could provide both arithmetic solutions and solutions with matrix multiplication. After the students worked on the problem, Alice orchestrated a mathematically productive discussion (Smith and Stein, 2011) by carefully attending to students' answers sequentially during the whole class discussions. Alice's orchestration of instructional activities differed from the five practices in two important ways. First, Alice used a selection of four contextual tasks on matrix multiplication, taken from past-year examination papers. This stands in contrast to Smith and Stein's idea of using a single rich task for the lesson. Second, although Alice's way of orchestrating discussion seems to reflect the five practices, Alice interjected to explain the connections in between the different solutions, instead of connecting the solutions at the end of the presentation. This provided opportunities for her to emphasise the connections between matrix multiplication and arithmetic to provide meaning to matrix operations in between students' presentations (see Choy and Dindyal, 2018). Hence, Alice kept the concept in focus and ensured coherence in the discussion by co-constructing the explanations for the different approaches with her students (Lampert et al., 2010). We can also consider Alice as using mini-cycles of the five practices by Smith and Stein (anticipating, monitoring, selecting, sequencing, and connecting), one cycle for each of the four typical problems that she used in the class.

4. Modifying Typical Problems

To notice the affordances of a typical problem, a teacher should be able to modify the typical problem in various ways, just as Alice did. The modification of typical problems can surface approaches to implementation of these problems in class either for developing procedural skills or for developing conceptual fluency. To modify typical problems, I suggest the following procedure: (1) focus on the given typical problem; (2) focus on the kind of modification that suits your implementation needs; (3) focus on the implementation of the problem in your class; and (4) after the implementation in your class, review the problem and its use and decide whether to keep it or to further modify it for subsequent use. Let's further look into these ideas as detailed in Dindyal (2018).

4.1. Focusing on the given problem

The teacher needs to check, carefully the unit on which the typical problem is based and be aware of the expected learning outcomes. All skills and sub-skills for solving the problem should be considered and the use of any results, techniques, or conventions should be highlighted. Another important idea to check is the concepts on which the problem is based, and the teacher should ask the questions: Do my students have the necessary resources to solve the problem? Is the problem within the requirements of the syllabus? Additionally, the teacher should check for the kind of structure that is already present in the statement of the typical problem and specifically look at supporting diagrams, charts, graphs, tables, parts, and subparts, etc. The wording of the problem should be scrutinized for the language, in particular, the teacher should check all action verbs (find, state, calculate, etc.), key words (all, some, at least, at the most, etc.), technical/mathematical terms, connectives, etc. The teacher should solve the problem in multiple ways, if necessary. In the end, the teacher should look out for the kind of affordances that the problem provides for enhancing the students' learning experience.

4.2. Focusing on the modification of the problem

The teacher needs to bear in mind the kinds of skills that he or she wants to elicit from the students and ask some questions such as: Do they have to draw something? Do they have to calculate, solve or prove? One idea to consider is whether to make the problem harder or simpler. The teacher should think about reducing or increasing the structure in the problem by drawing or removing diagrams, adding or reducing the number of parts and sub-parts, using or not using a table, graph or chart. The teacher should decide whether to keep the same action verb and key words as in the original problem or to select new ones. Strategies to modify the problem include: changing the numbers in the problem, changing the context of the problem, changing what is given and what is to be found, creating an open-ended problem, or integrating with another topic in mathematics. It is also worthwhile to focus on the procedural skills and the conceptual knowledge that would be elicited through working on this problem in the lesson. The use of the problem should be considered both as a stand-alone problem or embedded in a sequence of other problems by the teacher.

4.3. Focusing on the implementation of the problem

The implementation of the problem in the classroom is another aspect that should be considered carefully by the teacher. When and how will the problem be used? How will the lesson be orchestrated using this problem? Anticipate all possible solutions from the students to the given problem and which ones would be highlighted for the whole class. Using the problem in the classroom involves continual assessment throughout the lesson. One idea to focus one is: What kind of questions to ask to deepen the students' understanding? During the implementation process, the teacher may consider which other problems and in which sequence, could he or she use to enhance the students' learning.

4.4. Reflecting on the use and modification of the problem

No problem used in the classroom is perfect. There is always room for improving either the statement of the problem itself or its implementation. Accordingly, it is important to review the use of a typical problem either on its own or in conjunction with other problems. Questions to ask might include: Was this problem used optimally to enhance my students' learning? What other affordances does the problem provide that can be explored further? So, what kind of skills or dispositions should teachers possess to be able to reformulate and use typical problems?

5. Teacher Noticing

5.1. Teacher noticing and typical problems

Whether teachers select, modify or create tasks for use in class, they have to see and make sense of the mathematics and pedagogical considerations in the implementation of these tasks. Mason (2002) has previously stated that, "Every act of teaching depends on noticing: noticing what children are doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be said or done next" (p. 7). Ball (2011) has cautioned that although noticing is a natural part of sense making, the kind of noticing required in teaching is not a natural extension of being observant in everyday life. We can consider mathematics teacher noticing as the process of attending to students' mathematical ideas and making sense of the information to make decisions in an instructional context (Jacobs et al., 2011; van Es and Sherin, 2008). This specialised seeing, sense making, and decision making is a set of three inter-related skills referred to by researchers as teacher noticing (Mason, 2002; Sherin et al., 2011). More specifically the three processes are attending, interpreting and responding. For a more detailed focus on the conceptualisations of noticing see Dindyal et al. (2021).

Referring to Alice (Choy and Dindyal, 2017), we note that Alice noticed the different solutions of her students: the arithmetic solution, the solution from two separate matrix products, and the solution from a single matrix product. Alice interpreted the solutions of her students based on her experience as a teacher and in line with the learning about matrix multiplication. She interpreted each solution on its merit; she did not say that the answers were correct or incorrect; she interpreted which solutions were most important for later discussion; and she interpreted which sequence about the solutions would be most meaningful for discussions. Regarding her responding, Alice decided to send a student who had an arithmetic solution to write it on the board first. She then asked for the student who had two separate matrix products to write his solution on the board. And finally, she asked the student who had a single matrix product to write his solution on the board. Perhaps, the most important part in the discussion was when she asked if there was a different way to represent the solution as a single matrix.

Referring to Fig. 3, we can say that besides noticing the mathematics embedded in the typical problems (teacher-task-mathematics), Alice also noticed her students' thinking as they worked on the task (students-task-mathematics) and harnessed their answers as she orchestrated a whole-class discussions (teacher-students-mathematics) based on her interpretation of students' thinking as captured in their answers to the task (teacher-task-students). Connecting Alice's noticing and the affordances of the task, we note that Alice used shorter cycles of Smith and Stein's (2011) five practices—Anticipating, Monitoring, Selecting, Sequencing, and Connecting; one for each of the four problems she used in the observed lesson. She attended to students' possible confusion about using matrices to represent information in different ways (different order of matrices and matrix multiplication) and she recognised the affordances of a typical task and used it to lead students gain new insights (both conceptual and procedural fluency).

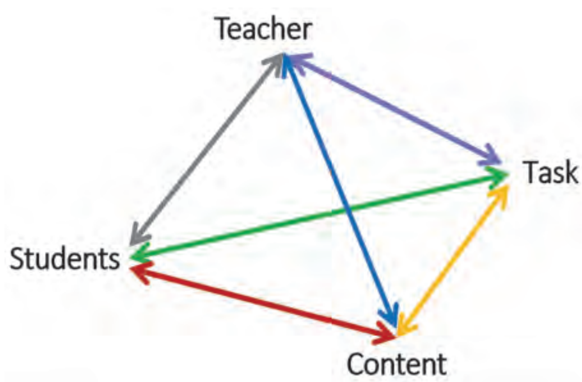

Fig. 3. Socio-didactical tetrahedron for using tasks (adapted from Rezat and Sträßer, 2012)

6. Conclusion

Henningsen and Stein (1997) have claimed that the nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of the subject matter with which they are engaged. Do the tasks have to be the so-called "rich tasks" or "high-level tasks"? I do see the value in using such tasks, as they do have a place in the curriculum. Such tasks have a high cognitive demand because they test Higher Order Thinking (HOT); make connections between different concepts and topic areas across the curriculum and are best implemented across several lessons or after a few topics have been covered. However, the methods of solution for such problems are not obvious and are generally harder for students, for which some students need prompts and cues. Besides such problems are quite difficult for teachers to select, adapt and design for classroom practice. Also, they are generally, used for developing conceptual fluency and not developing procedural skills.

On the other hand, typical problems are standard examination-type questions or textbook-type questions which focus largely on developing procedural fluency and at times, conceptual understanding; they can be solved by students in less time; they are omnipresent as compared to rich tasks; can be used for developing procedural skills, but they can also be used for developing conceptual fluency; and they are easier for teachers to access, reformulate or modify, adapt and use in class (Choy and Dindyal, 2017; Choy and Dindyal, 2018). It is to be noted that teachers can reproduce several such problems from their own example space. These problems can be used individually or in a planned sequence and are more user friendly to students as compared to rich tasks. Furthermore, typical problems can be used to develop students' reasoning (Dindyal, 2019).

The important role of teachers in using typical problems should not be underestimated. Henningsen and Stein (1997) have stated that, "Not only must the teacher select and appropriately set up worthwhile mathematical tasks, but the teacher must also proactively and consistently support students' cognitive activity without reducing the complexity and cognitive demands of the task." (p. 546) When using typical problems, mathematics teachers progressively increase the complexity and the cognitive demand of the task. It is clear that a teacher's own resources (knowledge and skills) which can largely be characterised by not only the teacher's content knowledge (CK) but also by his or her pedagogical content knowledge (PCK) as well as his or her pedagogical reasoning (see Shulman, 1987), are crucial elements in the teachers' ability to confidently use typical problems in the class. With limited time at their disposal, teachers have to be strategic in optimally using time for selecting and using relevant tasks for their lessons. As such, my point is that the use of typical problems has to be examined against a backdrop of teacher noticing that can be useful to teachers in their day-to-day practice. As humans, we are all involved in acts of noticing and more so in acts of not noticing. We can confidently say, "so do teachers". Some teachers can reformulate and use typical problems in pedagogically more meaningful ways while others cannot. Schoenfeld (2011) has cautioned that noticing is consequential "… what you see or don't see shapes what you do or don't do." (p. 228)

There is enough evidence to suggest that just as rich tasks or high-level tasks can be used in cognitively less demanding ways, typical problems do offer affordances to be implemented in cognitively more demanding ways. It is difficult for teachers to create new rich tasks or high-level tasks or even modify them for their specific use. On the other hand, typical problems are easily available, can be reformulated easily and provide a fairly smooth entry point for most students, even the lower performing students. It is worthwhile for teachers to consider the use of typical problems beyond the development of procedural skills and explore the affordances to develop conceptual fluency.

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https://doi.org/10.1142/9789811287183_0012

12

Textbook Transformation as a Form of Textbook Development: Approaches, Issues, and Challenges from a Social and Cultural Perspective

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ABSTRACT In this article, I use the term "textbook transformation" to refer to the development of a textbook or a series of textbooks based on another selected pre-existing textbook(s), which leads to the formation of a new textbook(s). By mainly drawing on my own mathematics textbook research and development experiences, particularly in transforming a popular Chinese mathematics learning resources series, *One Lesson One Exercise*, or *Yi Ke Yi Lian* in Chinese, to the English learning resource series the *Shanghai Mathematics Project Practice Books* and developing the *Zhejiang Secondary Mathematics Project Textbooks* over the last two decades, I argue that the means of textbook transformation can be classified into five types: translation, adaptation, revision, rewriting and a combination of them, based on the selected pre-existing textbook(s). Following this classification, the article analyzes and discusses the approaches, issues and challenges in textbook transformation by using concrete examples from available textbooks, and illustrates in particular how social and cultural factors play an essential role in textbook transformation and its significance in international exchange and collaboration in the development of school mathematics textbooks.

Keywords: One lesson one exercise; Shanghai Maths Project; Textbook research; Textbook development; Textbook transformation; Zhejiang Secondary Maths Project.

1. Introduction

1.1. Textbook research in mathematics education

Over the last two decades, issues in mathematics textbooks have received increasing attention internationally. This trend can be partly seen from the fact that many academic and research conferences which focused on mathematics textbooks have received a considerable amount of attention from international mathematics education community. In 2011, the International Conference on School Mathematics Textbooks

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(ICSMT) was held at East China Normal University and chaired by Jianpan Wang, focusing on the comparison of different textbooks across different countries. In 2013, The journal, *ZDM — International Journal of Mathematics Education* (hereafter "*ZDM*"), published its first special issue on the theme of "textbook research in mathematics education" (Fan et al., 2013). The First and Second International Conferences on Mathematics Textbooks Research and Development (ICMT), held in the UK in 2014 and Brazil in 2017, respectively, attracted a large number of participants from over 30 countries (Jones et al. 2014; Schubring et al., 2018). Subsequently, a second *ZDM* special issue on the theme of "recent advances in mathematics textbook research and development" was published in 2018 (Schubring and Fan, 2018). The Third International Conference on Mathematics Textbook Research and Development (ICMT-3) was held in Germany in 2019. The third *ZDM* special issue with the theme "Mathematics Textbooks as Instruments for Change", will be published soon, and the Fourth International Conference on Mathematics Textbook Research and Development (ICMT-4) will be held in Beijing next year[3].

In a comprehensive review article I and my co-authors published in the *ZDM*'s special issue in 2013, I proposed the following five directions for future textbook research (Fan et al., 2013).

1. To establish more solid conceptualization and theoretical underpinning of the role of textbooks and the relationship between textbooks and other variables in a wider educational and social context, and view the existence of textbooks from a broader perspective.
2. To have more confirmatory research about the relationship of the textbook and students' learning outcome.
3. To have more research directly focusing on the issues about *the development of textbooks* [emphasis added].
4. To employ more advanced and sophisticated methodology in textbook research.
5. To have more research on the use and development of electronic textbooks in mathematics.

For ICMT-3, the conference themes included content and its presentation in textbooks, both traditionally printed or more recently digital, use of textbooks, historical perspective on textbooks, comparative studies of digital textbooks or traditional textbooks, textbook and policy, research on textbooks, and the development of textbooks. Here, the development of textbooks includes concepts, task design, learning-teaching-trajectories, methodological approaches, quality, design-based research, etc.

However, as researchers have pointed out, among the available studies in this area, there has been notably a lack of research directly addressing the issues concerning the

[3] [post-congress note]: The third *ZDM* special issue on mathematics textbooks was published in 2021 and ICMT-4 was held in 2022. The proceedings of all the ICMTs, from ICMT-1 to ICMT-3, are available at https://acme.ecnu.edu.cn.

development of mathematics textbooks or how textbooks are produced, and more research in this direction is highly needed (Fan, 2013; Johnsen, 1993). This article (presentation) focuses on the development of mathematics textbooks. More specifically, by mainly drawing on my own mathematics textbook research and development experiences, particularly in transforming a popular Chinese mathematics learning resources series, *One Lesson One Exercise*, or *Yi Ke Yi Lian* in Chinese, to the English learning resource series, the *Shanghai Mathematics Project Practice Books* (Collins, n.d.), and developing the *Zhejiang Secondary Mathematics Project Textbooks* (Zhejiang Education Publishing Group, n.d.) over the last two decades, my aim is to propose and look into a new form of textbook development: textbook transformation, with a main focus on a social and cultural perspective.

1.2. The Shanghai Maths Project

The Shanghai Maths Project is an international collaborative effort between HarperCollins in the UK and East China Normal University Press in China. It is based on the latest edition of the award-winning series of *One Lesson One Exercise* (or *Yi Ke Yi Lian* in Chinese) published in China (Editorial Team, 2014). After adaptation following the English National Curriculum, its English series *Shanghai Maths Project Practice Books* were published by HarperCollins in the UK (Fan, 2015, 2016), and there are 11 books in total.

The project and its main product, i.e., *Shanghai Maths Project Practice Books*, has attracted much attention both in China, the UK, and internationally (e.g., see Farrington, 2015; Fan, Ni, et al., 2018; Fan, Xiong, et al., 2018; "WITMAS International School", n.d.[4]).

Since I started working on the project as its director, I have been frequently asked a variety of questions and I summarized them into the following five key questions:

1. Is it a direct translation?
2. Is it an adaptation?
3. Is it a rewriting?
4. Is it related to a Shanghai maths learning series?
5. What is it?

Here come some simple answers based on my experiences. Firstly, the project is not completely a direct translation; some questions and ideas were indeed a direct translation, but many are not. Secondly, the project is partially an adaptation, and partially a rewriting, as about 30% of the tasks were virtually brand new. The project is, essentially, a form of textbook development, related to and based on the Shanghai maths learning series, *One Lesson One Exercise*. Therefore, I eventually named it a "transformation", which is across different social and cultural settings.

[4] The website was valid at least until the end of 2018. Also see Westcoast International Primary School: https://www.wipschool.com/teaching-learning/maths/. This primary school provides another example of Shanghai Maths Project in Africa.

1.3. The Zhejiang Secondary Maths Project

The Zhejiang Secondary Mathematics Project is aimed to develop a series of secondary mathematics textbooks from Grade 7 to Grade 9, which have been published by Zhejiang Education Publishing Group and widely used by students in Zhejiang province of China, probably about 90% or 95% of the student population.

From the initial edition developed in 2004 to the current latest edition, which is still being used, two editions of the textbooks were developed following different Chinese primary and secondary mathematics curriculum standards issued by the Chinese Ministry of Education at different times. In addition, the later edition was slightly revised in 2019, which was necessary and helpful.

The development of the *Zhejiang Secondary Mathematics Project Textbooks* from its first edition to the latest is certainly not a translation, adaptation, or rewriting. It can be also viewed as a kind of transformation, although within the same social and cultural backgrounds.

2. Definition of Textbook Transformation

I use the term "textbook transformation" to refer to the development of a textbook or a series of textbooks based on another selected pre-existing textbook(s) based on the pre-existing textbook(s), which leads to the formation of a new textbook(s). So, it is not a start from scratch nor is the textbook(s) a brand new work.

Based on my experience in research and textbook development, at least five types of textbook transformation can be identified in order to understand the different means of textbook transformation. Those means can be translation, adaptation, revising, rewriting, or a combination of the above approaches. Tab. 1 shows a framework about textbook transformation. The left column is added to describe the transformation of textbooks from an international perspective.

Tab. 1. A conceptual framework of textbook transformation from the older to the newer

Means	Description	Languages before and after
1. Translation	Direction translation; the difference between pre-existing and the newly formed are in the languages used only.	Different
2. Adaptation	The pre-existing and the newly formed are tailored to the different target users with different requirements, conditions or backgrounds, no matter whether the pre-existing and the newly formed use the same language or different languages.	Different or the same
3. Revision	Usually for the same target users using the same language, but for different requirements or conditions of users and often at different times.	Same
4. Rewriting	It is for different requirements or conditions of users in the same language. It can be for the same or different targeted users.	Same
5. Combination of the above	It is a combination of two or more of the above means to produce a new textbook based on a pre-existing one.	Different or the same

It should be emphasized that all the textbook development activities listed in the table are based on selected pre-existing textbooks and will lead to the formation of new textbooks. It is a formation of the newer by transformation of the older.

There are abundant historical examples of textbook transformation both within and across nations and educational settings for mathematics textbook development. For example, Mathematics 1 shown below (Fig. 1) was edited by the University of Chicago School Mathematics Project or UCSMP, which was a translation of Japanese textbook for grade 10 from Japanese into English and published by UCSMP. Russian Grade 2 Mathematics shown below (Fig. 2) was a translation of a Russian mathematics textbooks from Russian to English, also published by UCSMP. More information about the UCSMP textbook translation can be found in its website (UCSMP, n.d.).

The *Shanghai Maths Project Practice Books* were based on Shanghai's *One Lesson One Exercise*, as mentioned earlier, were jointly published by Collins and East China Normal University Press, as said earlier. The most recent example I have been involved, within the same educational setting, is the *Zhejiang Secondary Mathematics Project Textbooks* (see more information below).

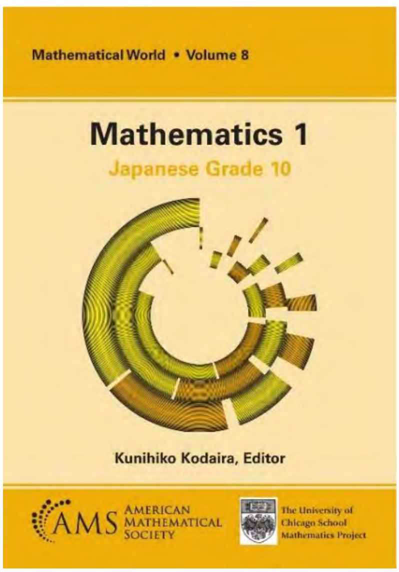

Fig. 1. Mathematics 1: Japanese Grade 10 (Source: https://bookstore.ams.org/mawrld-8/)

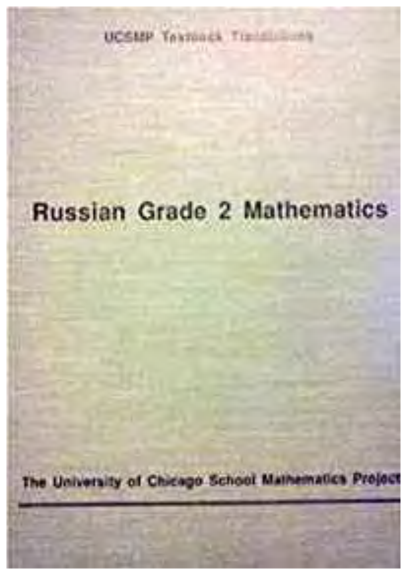

Fig. 2. Russian Grade 2 Mathematics (Source: https://www.amazon.com)

3. Cases of Textbook Transformation: Issues, Challenges and Approaches

In this section, I introduce two studies I have conducted in relation to textbook transformation, with some concrete examples when appropriate, to illustrate the issues, challenges and approaches concerning the transformation of mathematics textbooks.

3.1. Case 1: Shanghai Maths Project

One of the basic principles of the transformation from the Shanghai series to the English series in the Shanghai Maths Project was to follow the English national curriculum. It is not simply a reduction or an adaptation. While basically retaining the original pedagogy in Shanghai, which is a main reason for the existence of the project, there are clearly changes made in various aspects from the original Shanghai series to the English series. The issues and challenges I encountered can be, to a large extent, classified into the following categories: curriculum-related, language-related, culture-related, and context-related.

3.1.1. Curriculum-related issues

There are similarities as well as a considerable number of differences in compulsory mathematics curriculum between Shanghai and England.

Tab. 2 provides an overview about where different topics of statistics are introduced in the two series, both of which must follow their different curricula.

Tab. 2. Where the topics of statistics are introduced in the Shanghai and English series

Topic in Statistics	Shanghai series	English series
Statistical tables	Grade 2	Grade 2
Bar charts	Block diagram without coordinates in Grade 2. Bar chart with vertical axis in Grade 3.	Bar chart based on the knowledge of pictogram and block diagram in Grade 2. Bar chart with concepts of horizontal and vertical axes in Grade 3
Broken line charts	Broken line chart, with concepts of horizontal and vertical axes in Grade 4.	Grade 4.
Mean number and its application	Grade 5, Term 1.	Grade 6.
Pie diagrams	Grade 6, Term 1.	Grade 6.

From the table, we can see that the Shanghai series introduced bar charts in Grades 2 and 3, which is similar with the English series. However, the English national curriculum introduces block diagram and pictogram, while the Shanghai curriculum does not. Another example we can see is that the mean number and its application in the English series are introduced in Grade 6, while they are introduced in Grade 5 in Shanghai.

In addition, in Shanghai, the initial introduction to the idea or concept of fractions is in the 2nd semester of Grade 3, while in the English series it is in Grade 2 and Grade 3. The comparison of fractions, the addition, and the subtraction of fractions are also introduced earlier in the English curriculum than in the Shanghai's, which might be beyond many people's expectation.

Understanding the differences in the coverage and sequence of mathematics contents in the two curriculum is certainly a challenge for the textbook developers involved. I would argue that having experienced and knowledgeable team members or external consultants proved to be most helpful to tackle the challenge.

3.1.2. Language-related issues

There are tons of examples of language-related issues. For example, considering “1001” in English, we pronounce it as “one thousand and one”, while in Chinese the pronunciation is “yi qian ling yi” or “one thousand zero one” in English literally. So, in the Chinese textbooks, students were asked “Do you have to pronounce zero when you read 1001?”. But if it is directly translated into English, it does not make any sense.

Another example is dengshi (equation). In Chinese, a dengshi is a statement of equality, e.g., $7 + 3 = 10$. That is different from a fangcheng, which contains at least one variable or unknown, e.g., $7 + 2x = 10$. But in English, a dengshi and a fangcheng can both referred to an equation. So, communication will be a problem if one does not distinguish the difference between a dengshi and a fangcheng.

By the way, I noted that some Chinese textbooks or research articles translate zhengshi into integral expressions, which is incorrect and could not be understood for English readers as it was expected to understand. The reason is that, in Chinese, a zhengshi as a mathematical term means either a monomial or a polynomial expression. However, there is no equivalent term in English.

In short, a direct translation from one language to another language, like in this case, does not make sense, which is a real challenge, and one has to either rewrite or drop such questions for the English series.

3.1.3. Culture-related issues

There are also culture-related issues or challenges, as mathematics teaching and learning are often contextualized, and teachers and students all have certain culture imprints and are not culture-free. The examples in Tab. 3 depict such a difference and the transformation made.

Tab. 3. Two related questions in the Shanghai series and the English series

Chinese series [for the 2nd year]	**English series [for year 3]**
Xiao Qiao went to the *Shanghai Book City* with *200 yuan*. She bought a *Xinhua Dictionary* for 79 yuan and a set of the Fairytales World series for *114 yuan*. ① Please estimate whether *Xiao Qiao* had enough money. ② If she had enough money, how much change should she get? If she did not have enough money, how much was she short?	*<u>Joan</u>* went to a *<u>bookstore</u>* with *<u>£200</u>*. She bought *<u>two dictionaries</u>* for *<u>£79</u>* and a set of fairy tale books for *<u>£114</u>*. (i) Estimate whether *<u>Joan</u>* had enough money for the dictionaries and books she bought. (ii) If she had enough money, how much change should she get? If she did not have enough, how much was she short?

(Source: Fan, et al., 2018)

In the example, seven items relating to cultural factors, as highlighted, were changed from the Shanghai series to the English series. Those changes include from the popular Chinese name Xiao Qiao to the popular English name Joan, from Shanghai

Book City, which is a bookstore familiar to Shanghai students, to a bookstore, and from the most popular Chinese dictionary *Xinhua Dictionary* to two dictionaries, and so on.

Another example is that, in China, there are many national holidays, such as the Tomb Sweeping (or Qingming) Festival, Dragon Boat Festival, and National Day, introduced as question contexts in mathematics textbooks which I think is not suitable to use as background for English students to learn mathematics, as an English student would not understand nor need to understand them as a learner of mathematics. Therefore, it is necessary to replace the traditional Chinese festivals with those familiar to English students in textbook development.

In the Shanghai Maths Project, we changed, for example, International Labor Day, Dragon Boat Festival, Woman's Day, etc. which are celebrated in the Chinese culture to New Year's Day, Good Friday, Christmas Day, Boxing Day, and so on in the English series. So students in the United Kingdom or other English-speaking countries could understand the meaning of the question without cultural-related challenges in questions' backgrounds.

A case study I and my colleagues conducted on the manifestation of cultural influences in the formation of mathematics textbooks suggested that culture-related differences should not be underestimated, and the results revealed that most adaptations between the Chinese series and the English series are related to "ways of behaving and customs" and "artifacts, flora and fauna", followed by "identities" and "geography", and the least are related to "organisations" and "history" (Fan et al., 2018).

3.1.4. Context-related issues

In addition to culture-related differences, there are also context-related issues, often more complex than one might expect. It should be pointed out, not everything is culture, and some are more context-related, though culture and social context are sometimes intertwined.

For example, there is a question in the Shanghai resource book series:

> There are 205 yellow cattle. There are 4 times as many water buffalo as there are yellow cattle. How many water buffalo are there? [In the Shanghai series, Grade 3 book]

As yellow cattle and water buffalo are not commonly seen in English, for helping students learn better it is beneficial to change them to some more popular animals, such as goat and sheep, which are more familiar to English students. Below is the "transformed" question in the English series:

> There are 205 goats on a farm. There are 4 times as many sheep as there are goats. How many sheep are there? [In the English series, Year 3 book]

Another example is related to buildings. Many landmark buildings are used as problem contexts in the measurement topics in the Shanghai series, such as Shanghai Centers (632 meters), Global Financial Center (492 meters), East Pearl TV Tower (468 meters), and Jinmao Tower (428 meters). For Shanghai students, it is familiar and helpful. But for English students, it may not be so helpful to learn mathematics using the Shanghai contexts. So, those contexts in the questions were changed into Canary

Wharf Tower (244 meters), One Churchill Place (156 meters), and Broadgate Tower (178 meters) with the help of my English colleagues.

Nevertheless, there are some other problems in the Shanghai series for which we could not find comparable contexts for English students. For example, we could not find any buildings in London or England which are more than 400 meters high. So, to help students learn big numbers in the English series, we had to change the questions or use other types of contexts for the questions.

3.2. Case 2: Zhejiang Secondary Maths Project

The Zhejiang Secondary Mathematics Project, in which I was also privileged to serve as the project director, is another typical case reflecting the idea of textbook transformation, although within the same educational settings, as aforementioned.

A basic principle from the first edition to the second edition in the project is that the level of standards needs to be maintained the same. Apart from this, there have been systematic changes which have been necessary and helpful. The dimensions of the transformation from the first edition to the second edition were reflected in, and can be classified into, curriculum-related, content-related, pedagogy-related, and context-related issues, but basically not language-related or culture-related, as it is in the same language, i.e., Chinese, and the users are from the same cultural background.

There are a great number of changes which are curriculum-related. According to the national educational policy and the law, school mathematics textbooks in China except Shanghai which had its own city-wide official curriculum, must follow the national curriculum. In the first edition of the Zhejiang textbooks published in 2004, we introduced “Special parallelogram and trapezium” for the 2nd semester of Grade 8. However, the topic of trapezium had to be removed in the second edition, since the newer curriculum no longer required students to learn the topic. Besides the curriculum requirement, some contents in the textbooks also need to be changed to reflect the development of mathematics and the society, in addition to the curriculum.

By integrating teachers’ feedback, some improvements were made to the pedagogy in the second edition. For example, some basic-level questions designed as Group A exercises for students’ homework to develop their basic knowledge and skills in the first edition were reclassified and moved to Group B in the second edition because according to the teacher’s feedback, they were too difficult for many students. In addition, the second edition also reflected, to some extent, the new development of pedagogy in mathematics teaching and learning, for example, by providing more opportunities for students to do hands-on activities and solve more open-ended problems in their learning of mathematics.

Many changes, probably more than any other kinds, were context-related from the first edition to the second edition, which also reflected the fast social, economic and scientific development in China during the period of time.

Adopting PISA’s well known “Personal, Societal, Occupational and Scientific context framework” (OECD, 2019), I and my colleagues examined the context-related items including worked examples and exercise questions in the textbooks for Grades 8 and 9. More specifically, we compared the differences between the second edition published in 2013 and 2014 respectively and it is revised edition[5] published in 2019 in terms of the number of context-related items. The results revealed there were some changes, but not as significant as one might expect, as shown in Fig. 3 for Grade 8 and Fig. 4 for Grade 9, with inter-rater reliability being 0.93 and 0.89, respectively.

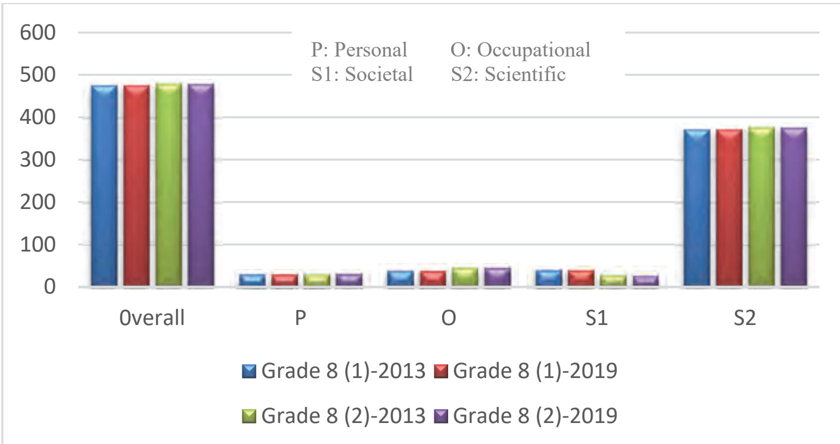

Fig. 3. A comparison of context-related items for Grade 8

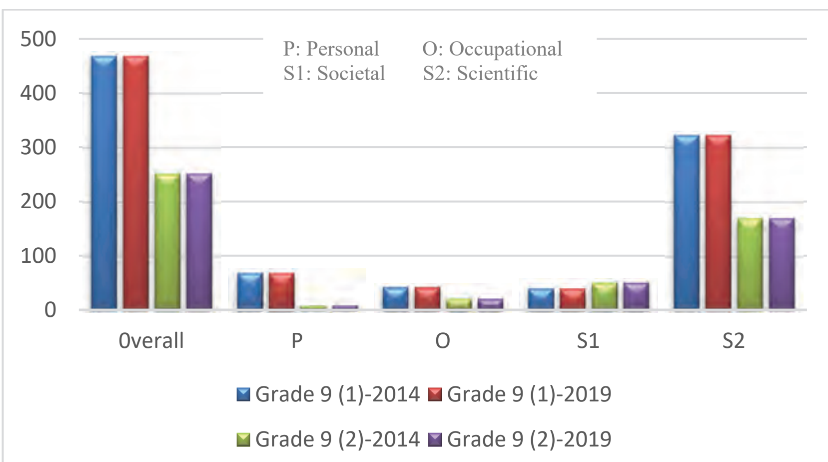

Fig. 4. A comparison of context-related items for Grade 9

More important changes were found when we further examined the contexts of the questions themselves between the first edition, the second edition and its revised edition. Below are two examples for illustrating the change of contexts in the categories

[5] It is not called the third edition as it was a relatively "minor" revision, which can also largely explain why the distributions were quite similar.

of societal and scientific contexts using the PISA framework. The reasons for such changes are virtually self-explanatory (Note: the main changes were highlighted).

Societal context: Hangzhou Bay Bridge (In Section 1.1, "From natural numbers to fractions", First semester, Grade 7).

First edition (2004):

> Please read the following: The longest cross-sea bridge in the world, the Hangzhou Bay Bridge, was laid on June 8, 2003, and it was planned to be completed and opened to traffic in 5 years. This 6-lane highway cable-stayed bridge with a designed daily traffic volume of 80,000 vehicles, with a designed speed up to 100 kilometers per hour and a total length of 36 kilometers will be the first cross-sea bridge in the Chinese mainland. What numbers did you see in the above text? What kinds of numbers do they belong to?

Second edition (2012/13):

> Please read the following: The Hangzhou Bay Bridge was opened to traffic on May 1, 2008. This 6-lane highway cable-stayed bridge has a designed daily traffic volume of 80,000 vehicles, with a designed speed to 100 kilometers per hour and a total length of 36 kilometers, and a service life of 100 years. It was the longest cross-sea bridge with the largest engineering workload in the world at that time. What numbers did you see in this report? Please find out these numbers and explain which of them are for counting and measurement, and which of them are for labeling or sorting.

Scientific context: Supercomputer (In Section 3.1, "Multiplying powers with the same base", Second semester, Grade 7).

Second edition (2012/13):

> The measured calculation speed of the "Tianhe-1A" supercomputer in China has reached 2.566 million billion per second. If it works at this speed for a whole day, how many times of calculations can it perform?

Revision of second edition (2019):

> The measured calculation speed of China's "Shenwei·Light of Taihu Lake" supercomputer has reached 93 million billion times per second. If it works at this speed for a whole day, how many times of calculations can it perform?

It should be mentioned that about the Hangzhou Bay Bridge, when we wrote the first edition, the bridge was only a plan to be the longest bridge in the world. But for the second edition, the bridge was already completed in 2008, and moreover, many longer bridges were built or planned to build, so it was no longer the longest at all after it was completed. Therefore, it must be changed for the second edition. A similar approach was taken for the context of supercomputer to reflect the rapid scientific development in China during those years.

There are also many changes related to personal contexts and occupational contexts. For example, the textbook for the first semester of Grade 8 in the first edition used the price of the residential water rate in a city as 1.2 yuan/m^3, while in the latest version of the textbook revised in 2019, the price was changed to 2.9 yuan//m^3 to reflect the increase of the water price. Another example is that the hourly wage was 6 yuan/hour in the first edition while it was changed to 25 yuan/hour in the second edition. This change was also necessary as the standard of monthly minimum wage is increased yearly, and the minimum hourly wage was increased from 4.2 yuan/hour in 2005 to 12 yuan/hour in 2013 (Zhejiang Province Government, 2012). Without change, a wage of 6 yuan/hour would be not only unrealistic but also illegal.

4. Summary and Concluding Remarks

To summarize, in this article, "textbook transformation" is defined to refer to the development of a textbook or a series of textbooks based on another selected pre-existing textbook(s), which leads to the formation of a new textbook(s).

By mainly drawing on my own mathematics textbook research and development experiences, particularly in transforming a popular Chinese mathematics learning resources series, *One Lesson One Exercise*, to the English learning resource series, the Shanghai Mathematics Project Practice Books, and developing the *Zhejiang Secondary Mathematics Project Textbooks* for its different editions, the article argues that the means of textbook transformation can be classified into five types: translation, adaptation, revision, rewriting and a combination of them, based on the selected pre-existing textbook or textbooks.

Furthermore, the main issues or challenges in textbook transformation can be classified into curriculum-related, language-related, culture-related, context-related, content-related, and pedagogy-related, to different extent and depending on the types of textbook transformation.

A related question is, why should we take on the challenges? From the macro-level or external factors, I would argue it is because of social development, mathematical development, pedagogical or educational development. Furthermore, because of globalization and international exchanges, we all want to develop ourselves by learning from others. From the micro-level or internal factors, it is one of the most efficient and effective ways to develop textbooks by not only learning from others but also learning from the past. This is particularly clear when we consider that developing a set of textbooks starting from scratch is truly complex and time-consuming.

Finally, regarding approaches to tackling the challenges, I would argue collaboration of textbook developers with different expertise and backgrounds is a key to ensure the quality of the transformation, and exchanges in terms of research are also important. Unfortunately, there are not many studies available yet in this line, and I hope this article can make a meaningful contribution to it.

Acknowledgments

Part of the work reported in this article was presented as a progress report at an education research seminar at the University of Oxford, December 2018. This article is a final report including new data that were not available for the earlier report. I wish to thank two of my doctoral students Na Li and Jiali Tang for their assistance in preparing my lecture at ICME-14 and this chapter for the monograph of ICME-14. The work described in this article was jointly funded by East China Normal University, from its "Happy Flowers" Strategic Research Fund (Grant number: 2019ECNU-XF2H004) and a grant from the Asian Centre for Mathematics Education of the same university (Award No: 92900-120215-10514).

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13

The Roles of Learning Trajectory in Teaching Mathematics Using RME Approach

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ABSTRACT This paper discusses the critical role of a learning trajectory in teaching mathematics using realistic mathematics education (RME) approach. The first part will present a brief history of RME and how RME was adopted into the Indonesian context. It is followed by a discussion of some of the principles and characteristics of RME. Furthermore, it is explained the learning trajectory, hypothetical learning trajectory (HLT), and how the principles and characteristics of RME are integrated inti HLT in learning mathematics. In the next section, the development of HLT through design research is discussed, and in the last section, some examples of HLT are given and their impact on students' mathematical abilities.

Keywords: Learning trajectory; HLT; RME; PMRI; Design research.

1. Introduction

Mathematics learning at schools in Indonesia generally tends to take place mechanistically (teachers convey information and methods, give examples, and then ask students to do exercises such as examples) (Widjaja et al., 2010; Turmudi, 2010; Fauzan et al., 2013). Surprisingly, most teachers believe it is the best way to learn mathematics (Webb et al., 2011; Fauzan, 2013 and 2015; Rangkuti, 2015). Students are less interested and unmotivated to learn mathematics (Fauzan, 2013; Fauzan and Yezita, 2016). Several results of the TIMSS and PISA studies show that the mathematical ability of Indonesian students tends to be low (http://nces.ed.gov/timss/), (Stacey, 2011; Fauzan, 2013; OECD, 2013 and 2015).

One of the causes of the above conditions is because teachers tend to follow the sequences of material presented in the textbooks (starting with definitions, then followed by several examples). In addition, the more frequently used teaching method is chalk and talk (Fauzan, 2002; Fauzan et al., 2013). As a result, most students think that the mathematical concepts they learn are complicated to understand because they are unrelated to their daily lives. It contradicts Freudenthal's idea that mathematics is

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a human activity and that learning mathematics means doing mathematics (de Lange, 1987; Gravemeijer, 1994).

Observing the problems that occurred, it is argued that the Realistic Mathematics Education (RME) approach has the potential to overcome these problems. It is because RME aims to change mathematics education so that most children can do and enjoy mathematics, solve math problems, and develop mathematical knowledge and skills (Sembiring et al., 2010). Learning mathematics with the RME approach focuses on how mathematics is taught and how students learn mathematics in class in a meaningful way. To realize the principles and characteristics of RME in teaching a mathematics topic, it is necessary to design or to develop a learning trajectory (LT).

1.1. RME in the Netherlands

RME is a didactic approach or a domain-specific instruction theory for teaching mathematics that wa*s developed in the Netherlands*. RME has its roots in Freudenthal's interpretation of mathematics as a human activity (Freudenthal, 1991; Van den Heuvel-Panhuizen, 2003; Van den Heuvel-Panhuizen and Drijvers, 2014; Treffers, 1987; Jupri, 1998; Gravemeijer and Cobb, 2013, Van den Heuvel-Panhuizen and Wijers, 2005). The beginning of the emergence of RME theory was since the reform of the mechanistic approach to mathematics learning which had previously been used in the Netherlands (Van den Heuvel-Panhuizen and Drijvers, 2014). It resulted in mathematics as rigid knowledge that is reproductive. As an alternative to this mechanistic approach, modern mathematics, which was currently trending in the world, almost affected the Netherlands.

RME theory is heavily influenced by Freudenthal's ideas. He says that the main idea of RME theory is that mathematics is introduced as meaningful knowledge for students, and mathematics is a human activity. Therefore, in the learning process, mathematics is not studied as a closed system but must be studied as an activity to mathematize reality and mathematics itself. RME is a term for realistic, which comes from the Dutch term zich REALISEren, which is meaningful to imagine. The term realistic does not mean that it is directly related to the real world but rather to the use of problems that students can imagine. The emphasis is on making things honest in mind. It means that RME does not always have to use real-life problems. However, abstract mathematical problems can be made real in students' minds, so the mathematics material taught needs to be accurate for students. It is what underlies so-called Realistic Mathematics Education.

Freudenthal suggests that mathematics education has to be organized as a process of guided reinvention, where students can experience a similar process to the process in which mathematicians invented mathematics. Gravemeijer (1999) sees the guided reinvention principle as long-term learning process in which the reinvention process evolves as one of gradual changes. The stages always have to be viewed in a long-term perspective, not as goals in themselves, and the focus has to be given on guided exploration. To realize this view, a learning route (learning trajectory [LT]) has to be mapped out (by a developer or instructional designer) that allows the students to find the intended mathematics by themselves.

1.2. RME in Indonesia

Indonesian Realistic Mathematics Education (PMRI) was initiated by a group of mathematics educators in Indonesia. The initial motivation was to find a replacement for modern mathematics, which was abandoned in the early 1990s. The birth of RME was applied in general in public schools. Approximately 30 years later, RME entered Indonesia under Indonesian Realistic Mathematics Education (PMRI).

The history of PMRI started with the efforts to reform mathematics education carried out by the PMRI Team (initiated by Prof. Sembiring et al.), which was officially implemented in 1998. Furthermore, the initial trials of PMRI started in late 2001 in eight elementary and four Islamic elementary schools. Then, PMRI began to be applied simultaneously from first grade in Surabaya, Bandung, and Yogyakarta. The number of schools involved, in this case, called LPTK partner schools, is not less than 1000 schools (Sembiring, 2010).

2. RME's Principles and Characteristics

2.1. RME's principles for instructional design

There are three key principles of RME for instructional design (Gravemeijer, 1994 and 1997), namely *guided reinvention through progressive mathematization, didactical phenomenology,* and *emerging models.* In the guided reinvention principle, the students should be given the opportunity to experience a process similar to that by which mathematics was invented (Gravemeijer, 1994 and 1999). For this purpose, a learning route has to be designed by a developer or instructional designer to allow the students to find the intended mathematics by themselves.

Through the didactical phenomenology principle, the developer or instructional designer has to provide students with contextual problems taken from phenomena that are real and meaningful for them (Gravemeijer, 1994). The real phenomena will facilitate students to experience the process of horizontal and vertical mathematization.

Related to emerging models, developer or instructional designer have to give the opportunity to the students to use and develop their own models when they are solving the contextual problems. At the beginning the students will develop an informal model which is familiar to them. After the process of generalizing and formalizing, the model gradually becomes an entity on its own. Gravemeijer (1994) calls this process a transition from *model-of* to *model-for*.

2.2. RME's characteristics

There are at least five RME's characteristics of RME which are summarized from Treffers (1987), Zubainur et al. (2020), Wewe and Juliawan (2019), Paredes et al.

(2020), Sampoerno and Meliasari (2019). Teaching and learning using RME approach are required to apply these characteristics:

a. Using contextual problems (the use of context); learning begins by using contextual problems, not starting from the formal mathematics concepts. Contextual problems raised as the initial topic of learning must be simple problems that students recognize. It means that students are given realistic problems and begin developing their thoughts to find solutions to the problems.
b. Using models (use models, bridging by vertical instruments); students pass through the levels of mathematical understanding: from understanding that is informal, semi-formal, to formal stages. To bridge it, it is necessary to model related to situational and mathematical models developed by students. Students solve problems using mathematical models (tables, graphs, pictures, equations). Solutions at this stage can be informal or formal. Teacher guides and directs students to solve the problem so that students rediscover ideas or concepts and can form a structured mathematical model of the material in their way.
c. Using student contributions (student contribution); significant contribution to the teaching and learning process is expected to come from students, meaning that all students' thoughts (construction and production) are considered. Students also compare and discuss answers in study groups which are used to train students' courage to express their opinions even if they differ from the results or income of friends or teachers.
d. Using interaction (interactivity); in the learning process, students actively discuss, and express ideas both in-class activities and group activities, so that there is an interaction between students and teachers, students and students. Interactions such as negotiations, explanations, justifications, approvals, questions or reflections are used to achieve forms of formal mathematical knowledge from informal forms of mathematical knowledge discovered by students. The teacher optimizes students' competence in the use of ideas. To optimize the interaction between students and students or with the teacher as a facilitator, students find formal mathematics to solve the problems given.
e. Using intertwinement; mathematical structures and concepts are interrelated. Mathematical problems facilitate students, and linking mathematical topics such as numbers, algebra, and geometry are not seen as separate but as interrelated and integrated topics.

2.3. The process of mathematization in RME

Linguistically, the word mathematization comes from the mathematization of mathematization. The words mathematization and mathematization are nouns from the verb mathematize or mathematize, meaning math. Treffers (1987) distinguishes mathematization into two types: horizontal and vertical. According to Van den Heuvel-Panhuizen and Drijvers (2014), the idea of horizontal and vertical mathematization in the mathematical process which was originally initiated by Treffers was taken over and refined by Freudenthal (2002), Kabael and Deniz (2016), Araújo and De Lima (2020), Amala and Ekawati (2020), Jupri et al. (2021), Widada et al. (2020). In horizontal mathematization, students use mathematics to transform realistic problem situations into mathematical situations in the form of mathematical models. In vertical mathematics, students work in the world of symbolic mathematics by reorganizing the model until a problem is found.

Freudenthal defines horizontal mathematization as an activity to convert contextual problems into mathematical problems (symbols). Horizontal mathematization is related to the generalizing process. The horizontal mathematization process begins with identifying mathematical concepts based on regularities and relationships found through visualization and schematization. While vertical mathematization is a form of a formalizing process where the mathematical model obtained in horizontal mathematization becomes the basis for developing more formal mathematical concepts, the vertical mathematization process is interrelated. The horizontal and vertical mathematization process cannot be directly separated into two large parts sequentially. Namely, the vertical mathematization process takes place after the entire horizontal mathematization process occurs. The processes of horizontal and vertical mathematization tend to occur gradually like steps of a staircase, often alternating with each other. Fig. 1 shows the process of horizontal and vertical mathematization.

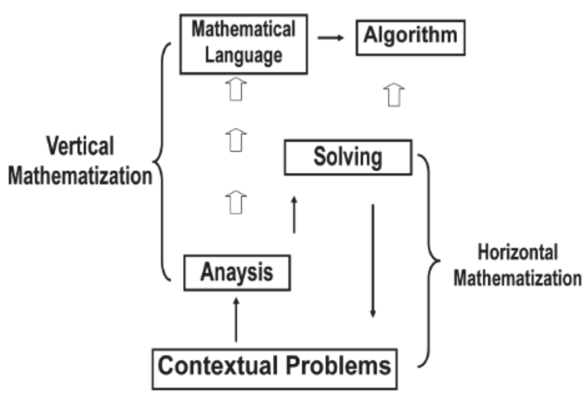

Fig. 1. The process of horizontal and vertical mathematization
(Source: Gravemeijer, 1999)

3. Learning Trajectory

Learning trajectory (LT) is the sequences of activities or tasks to guide students to achieve a specific instructional goal. LT are based on "hypothetical learning trajectories" (HLT), a concept proposed by Simon (1995) from a constructivist point of view. Simon uses this concept to describe a teacher's prediction of the trajectory that can be taken in the learning process, defining it as the teacher's predicted trajectory for student learning (Simon, 2004; Daro et al. 2011; Akdeniz and Argun, 2021). It can be seen from this description that HLT includes learning and teaching practice, which is focused on the instructions given by the teacher (Clements, 2011; Sarama and Clements, 2019). The LT concept identifies student developments hierarchically and moves hypothetically with implementation and validation through empirical study.

> … descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain.

Experts define the concept as consisting of three components: the developmental process, instructional activities, and instructional goals, which are also the main ideas of mathematics. The learning trajectory describes students' thinking levels, mental ideas and actions required, thought processes to express those ideas and actions, and instructional activities and strategies based on these processes (Clements and Sarama, 2014; Daro et al., 2011). It is supported by Fauzan (2016), which states that the learning trajectory is a series of learning activities (activities to solve contextual problems) that will facilitate students to reinvent (reinvention) formal mathematical concepts by optimizing their informal knowledge.

3.1. Hypothetical learning trajectory (HLT)

Hypothetical Learning Trajectory (HLT) is defined as three essential components, namely goals for meaningful learning (goals for meaningful learning), a series of tasks to achieve these goals (learning activities), and hypotheses about student thinking and learning (hypothetical learning process) (Simon, 2004; Cárcamo Bahamonde et al., 2017). A learning goal is a goal to be achieved in the learning process. While learning activities are a design of learning trajectories that students will pass in achieving the learning objectives set. These activities are given as mathematical tasks (mathematics tasks) in the form of contextual questions. The final, hypothetical learning process, namely predictions or assumptions about the understanding and reasoning of students that will develop in the learning process, and there is anticipation or feedback from the teacher to help students achieve learning goals (Simon and Tzur, 2004; Sumirattana et al., 2017; Dickinson, 2020; Gravemeijer, 2020).

In its implementation, HLT does not aim to want the results obtained by students but how the process of these students rediscovering mathematical concepts. The process of rediscovery is divided into several levels of models that will appear in the learning process (Gravemeijer, 1999). In this principle, the model is understood as to how students produce each observable activity (Zandieh and Rasmussen, 2010). In designing the learning trajectory in HLT, four activity levels will appear situation, model level, model level for, and formal knowledge. The situational level is the basic level that gives rise to situational knowledge and strategies used in conjunction with the context of the situation. The referential level is the use of models and strategies at this level to show the situation described in the problem. The referential level is also known as the model-of level.

Meanwhile, at the general level, the for-model appears in mathematical knowledge, focusing on strategy dominating the reference to the problem context. Finally, the formal level requires that reasoning with conventional symbolization does not last long to support the for-model mathematical activity. These four activity levels aim to enable students to solve problems with an informal approach and gradually construct their knowledge into formal knowledge (Gravemeijer, 2020). Meanwhile, at the general level, the for-model appears in the form of mathematical knowledge with a focus on strategy dominating the reference context of the problem. Finally, the formal level requires that reasoning with conventional symbolization does not last long to support the for-model mathematical activity. These four activity levels aim to enable students to solve problems with an informal approach and gradually construct their knowledge into formal knowledge (Gravemeijer, 2020). Meanwhile, at the general level, the for-model appears in mathematical knowledge, focusing on strategy dominating the reference to the problem context. Finally, the formal level requires that reasoning with conventional symbolization does not last long to support the for-model mathematical activity. These four activity levels aim to enable students to solve problems with an informal approach and gradually construct their knowledge into formal knowledge (Gravemeijer, 2020).

In addition, Hans Freudenthal emphasized that a realistic mathematical approach, "mathematics should be connected to the reality," must be included in every divided into several learning activities in HLT (Van den Heuvel-Panhuizen and Drijvers, 2014; Dickinson, et al., 2020). Learning activities involve the process of horizontal mathematization and vertical mathematization. In designing the HLT, one must also pay attention to the key principles of RME, namely 1) Use of context, 2) Use of models for progressive mathematization, 3) Utilization of student construction results, 4) Interactivity, 5) Linkage (Swidan, 2020; Fessakis, et al., 2017).

3.2. Local instructional theory (LIT)

Local instructional theory (LIT) is a theory in a learning process that tells a learning flow about a complete learning topic in a set of activities that support it (Gravemeijer, 2009; Andrews-Larson et al., 2017). According to Simon (2018), calling it a (domain-specific) theory, the theory only discusses a specific domain or topic of learning. By using instructional design in designing HLT to find strategies and ways of thinking, students anticipate formal concepts; identify design principles for learning activities that can be used to generate these strategies and ways of thinking; identifying design principles for learning activities can be used to leverage these strategies and ways of thinking to support formal concept development. In the end, a Local Instructional Theory will be obtained.

3.3. Developing LT through design research

As mentioned in the previous part, at the beginning, a LT is designed in form of an HLT. After a series of the try out in the classroom and revisions, the HLT become a LT. The process of designing, try outs, and revisions of HLT suit to the idea of design research proposed by Gravemeijer and Cobb (2013) which characterized by a cyclic process of preparing for the experiment, conducting the experiment, and retrospective analysis. The cyclic process is described in Fig. 2.

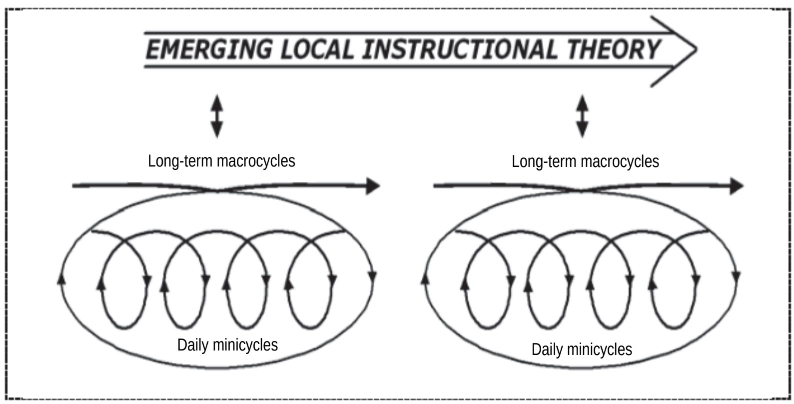

Fig. 2. The cyclic process of design research (Source: Gravemeijer and Cobb, 2013)

In preparing for the experiment, it is determined the end points of the instructions, followed by designing the series of activities of solving contextual problems. These activities are designed so that they would facilitate students to do horizontal and vertical mathematization as well as stimulate students' thinking and reasoning. In addition, we also formulated the predictions of students' thinking and solutions, and the anticipations. In the experimental phase, the HLT is tried out in the classroom. After the retrospective analysis and re-design processes, we will get a LT for teaching a mathematics topic using RME approach. Some examples of LTs that have been developed through design research will be mentioned in the next part.

3.4. Examples LT from the previous research

The local instructional theory has been successfully applied in recent years at the university level in several fields of science, such as abstract algebra material (Larsen, SP, 2013), in calculus (Bos et al., 2020; Swidan, 2020; Gilboa et al., 2019), in statistical material (Syafriandi et al., 2020), in linear algebra (Andrews-Larson et al., 2017; Cárcamo Bahamonde et al., 2017) and ordinary differential equations (Yarman et al., 2020). These studies show that students can construct informal knowledge into formal knowledge with a series of activities provided by the lecturer.

Similar research was also successfully applied to junior high school students in measurement and geometry materials (Sumirattana et al., 2017; Fauzan et al., 2020), in mathematical literacy (Arnawa and Fauzan, 2020). Other studies were also applied to high school children on the material of sequences and series (Azizah et al., 2021).

Several other researchers have also applied local instructional theory in elementary schools to the material for adding whole numbers to dyscalculia students (Fauzan et al., 2022), material abstract algebra is the use of variables for children aged 3–7 years (Ventura et al., 2021), in social arithmetic material in Grade 3 students (Stemn, 2017), in measurement and geometry materials (Möhring et al., 2021; Akdeniz et al., 2013), in geometry material using mobile computing device technology (Fessakis et al., 2017).

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14

Experimentations in Mathematics Education with Art and Visuality

Claudia Regina Flores[1]

ABSTRACT "Art and Mathematics" has been considered in mathematics education primarily for the possibility of teaching and learning mathematics through art. Many reasons are implied in this: to give meaning to mathematics; to motivate or contextualise teaching; to broaden mathematical visualization, among others. However, neither colonizing art by mathematics nor instrumentalizing mathematics by art, we have been considering this pair for the experimentations that can happen in the exercise of thinking. Taking this into account, in this presentation, first, I introduce the idea of visuality in differentiation with the concept of visualization in mathematics education to point out some theoretical concepts of the research. Then, I present some research works I have been developing, especially those that have been effects of the production of a methodological stance that occurs at the interface between paintings, visuality and mathematics education. Finally, I draw some conclusions, outlining an ethical, aesthetic, and political stance for teaching mathematics with arts.

Keywords: Philosophical perspective; Historical critical attitude; Experimental ethos.

1. Introduction

This paper reports on an empirical study that aimed to legitimize the visuality's perspective for mathematical visualization (Flores, 2013; Flores, 2012), as a frame for research on visualization in mathematics education. This emerges as an analysis strategy and, consequently, as an extension of research on visualization and art, discussing how our look was historically constituted, and thinking about other relations between art and mathematics for teaching.

One of the connections between art and mathematics has been considered in mathematics education for the possibility of teaching and learning mathematics through art. Many reasons are implied in this: to give meaning to mathematics; to motivate or contextualize teaching; to learn about geometric terms and basic shapes; to improve retention of key concepts and vocabulary; to connect math with other disciplines; to develop mathematics skills, such as mathematical visualization. Actually, the point is: Once you see the relationship between the two, art and

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mathematics, you will — no doubt — begin to see opportunities everywhere to use art in your math classroom.

So, the question is no longer: What connects mathematics and arts for mathematics teaching? It seems that there is more to it than the fact that mathematics underlies patterns and perspective, for instance.

Having this in mind, I have divided my presentation (and this paper) into three parts. First, I am going to talk about the idea of visuality in contrast with the concept of visualization in mathematics education to point out some theoretical concepts of the research. I find this point of the visual interesting because it puts us in another relationship with art.

Then, I present some research works I have been developing, especially those that have resulted from the production of a methodological stance that occurs at the interface between paintings, visuality, and mathematics education. Briefly, those works are developed as workshops with elementary school children, carried out by our research group, GECEM[2].

Finally, I draw some conclusions, outlining an ethical, aesthetic, and political stance for teaching mathematics with arts.

2. Visualization and Visuality

The role of visualization in mathematics teaching has been considered important for learning since the 1990s. Many researchers have emphasized this importance of visualization and visual reasoning for mathematics teaching and learning (Presmeg, 1989; Zimmerman and Cunningham, 1991; Dreyfus, 1991; Arcavi, 1999); others explored concrete examples of visualization and visual reasoning in the context of problem solving (Zimmerman, 1991; Goldenberg, 1991; Tall, 1991); some others defended the idea that mathematical technologies and software play a fundamental role in the process of visualization, contributing to the development of the student's ability to visualize in mathematics (Nemirowsky and Noble, 1997; Borba and Villareal, 2005); and others considered teachers and their beliefs on the role of visualization (Biza, Nardi and Zachariades, 2010).

Taking this into account, Flores, Wagner and Buratto (2012) analyzed Brazilian research dealing with visualization in mathematics education from 1998 to 2010. The authors found that most of those research works use the term visualization to refer to a visual capacity that would be necessary to foster, encourage, teach and educate through visual activities. Among these activities, the arts have served as a powerful mechanism to meet the expectations of learning and teaching mathematics.

Also, art in mathematics education has been deemed a powerful instrument to be associated with aspects of visualization in mathematics as a whole, for learning

[2] GECEM: Group of Contemporary Studies and Mathematical Education. Group registered at the Portal of the National Council for Scientific and Technological Development — CNPq and at the Federal University of Santa Catarina. [www.gecem.ufsc.br]. Leadership by Prof. Dr. Cláudia Regina Flores.

concepts. Flores and Wagner (2014) analyzed research conducted in Brazil on art and mathematics education from 1987 to 2013. The authors found that most research works are based on constructivist models of cognition and representation. In this case, visualization has been considered as a set of cognitive activities that lead to the understanding of information from the images. As such, teaching and learning mathematics through arts is either to repeat experiments already carried out, or to study the rules and postulates discovered by science. Consequently, art has served as a mere support to a very well-defined end, which is to teach mathematics.

2.1. From the perspective of visuality

When operating on a theoretical — conceptual displacement, the term visualization opens up to the questions of visuality which, transported from the studies of visual culture (Brennan and Jay, 1996; Sturken and Cartwright, 2001; Dikovitskaya, 2005), comes to be seen as the result of the sum of a multiplicity of visual and discursive practices in the scope of history and culture. The image is considered a focal point in the process through which meaning is made in a cultural context (This subject is neither independent nor autonomous; rather, it is a deeply interdisciplinary activity).

Therefore, this means that, when we learn to see socially, i.e., when our visual retina is articulated amid experiences and codes of recognition, we are part of a system of visual discourses that organize and indicate how we should see and how we see. For example, if we are given the world in perspective, it is because the technique of the perspective appears to us as a model of vision that produces a three-dimensional space, as much as a rationalized view of the perspective space.

From this vantage point, the art is considered a poetic component, an opening of creativity that produces an intimate connection between thinking and being. Thus, with art, instead of identifying and repeating mathematical concepts, one thinks with it. Thinking about concepts, about education, about teaching, about learning, where thinking would be a type of learning that happens by experimenting with mathematical formulations, mathematical thinking and mathematical language.

Conceptually comparing both terms, visuality and visualization, we could say that the first leads to a deconstruction of the founding principles of the sense of sight and perception, instead of being mostly concerned with the learning of concepts of geometry and visual skills.

Tab. 1. Conceptual differences between visualization and visuality

Visualization	Visuality
Process of construction and transformation of mental images	It is the sum of the discourses that inform how we see
Learning concepts and developing visual skills	Discusses visual practices in the context of history and culture

Going further in this comparison, I could add that visualization implies representation, cognition, perception, identification, visual abilities, while visuality implies historicity, narrativity, the process of creation, problematization, imagination, and so on (see Fig. 1).

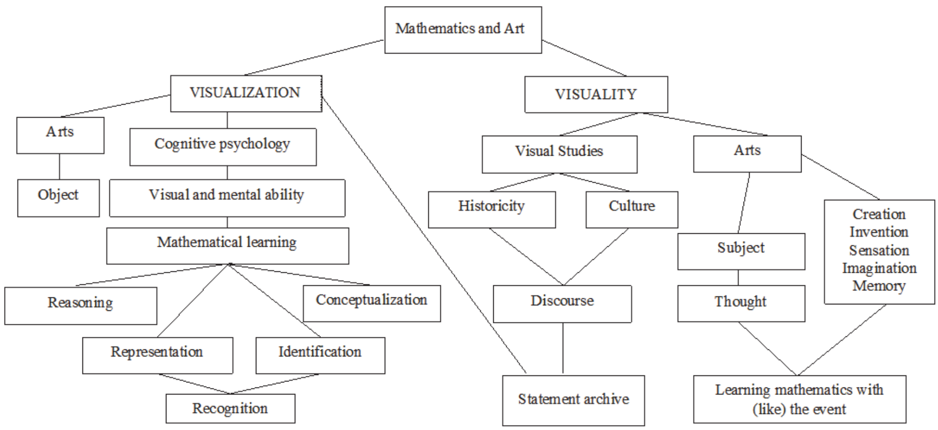

Fig. 1. Map of comparison between visualization and visuality

Therefore, in my research, I have proposed a perspective of visuality for mathematical visualization (Flores, 2013; Flores, 2012), which analyzes briefly how paintings have effects on a way of looking, specifying that we are engaged by mathematical thinking. It also provides the study of further forms of educational practices that could be possible by understanding the role of art in shaping a particular mathematical gaze. Then, my works invite mathematics education researchers to open spaces of reflection on research about visualization in mathematics education, particularly in relation to the mathematical thinking, but also suggests that bodily experience of mathematics may be fundamental for learning.

To put it simply, it will be necessary to question about how, with the arts, other ways of teaching and learning mathematics could be possible. For this purpose and inspired by the visuality point of view, together with my group of students, I've mentioned in the beginning of this paper, we have been working on practices of looking and exercises of thinking about mathematics with arts at the Elementary School.

3. Laboratories Settings: Experimenting with Mathematics and Art

By the perspective of critical educational research, I have found that the main problem addressed above could be developed since explains "the way in which we could help students to arrive at a more open, better, more critical, emancipated or liberated view." (Masschelein, 2010, p.274). According to the author, the critical education research is about educating the gaze, and

> it is related to this idea of being "present in the present". It is related to an understanding of education not in the sense of "educare" but of "e-educare" as leading out, reaching out. E-ducating the gaze is not about getting at a liberated or critical view, but about liberating or displacing our view (Masschelein, 2010, p. 277).

Furthermore, the basis of this theoretical perspective is aligned with Foucault's idea that we must try to proceed with the analysis of ourselves as beings who are historically determined, opening up a realm of historical inquiry. It means that

> The critical ontology of ourselves […] has to be conceived as an attitude, an ethos, a philosophical life in which the critique of what we are is at one and the same time the historical analysis of the limits that are imposed on us and an experiment with the possibility of going beyond them. (Foucault, 1984, p. 47).

By means of a historical-critical attitude in the work of philosophy and critical educational research, and considering the perspective of visuality to mathematics education, workshops, that are laboratory settings, have been created and developed with Elementary School children, by our research group, in Brazil, in order to analyze critically both mathematical visuality practices and also mathematics research under a philosophical perspective. A laboratory can be conceptualized:

- "As an experimental system that should allow for (new) things to happen, to appear as such, […], emphasizing the practice of making as trying to call into presence." (Masschelein, 2012, p. 367).
- "It is about registering, seeing, illuminating, bringing into play, penetrating, inviting, inspiring, experimenting; it is about exposing oneself and trying the words and verbs again." (Masschelein, 2012, p. 368).
- They are a place for exposing and registering, while gathering students, teachers and researchers around the questions of our present time (how mathematical thinking organizes the way we represent and look at the world). (Masschelein, 2012).

As an example, let me consider three works carried out by our research group. For all of them, the activities were developed with children from the 5th grade of Elementary School, from the College of Application of the Federal University of Santa Catarina, Brazil.

3.1. *The case of experiencing mathematics with cubist art*

Inspired by Cubist art, a workshop was carried out. The first one, was developed by Bruno Francisco (2017), a master student. The workshop that he carried out was formed by both the photographs of the children and the cubist painting of portraits. The experience itself was constituted by putting clippings of photographs that were taken from each of the children inside yellow and blue boxes. Each one received their own little box containing pieces of their photos. The pieces were cut by the researchers in various shapes, which could be regular but also irregular, an invitation to strangeness, exercise and study. The question was: How could they redo their own photographed images if these were not a jigsaw puzzle of geometric figures?

Well, a normal photo was not possible because not all the parts of the same photo were there. The solution was to make a portrait similar to cubist portraits. By doing that, children experienced an inventive process of assembling, disassembling and reconstructing their own image. The discomfort, the de-regulation, the strangeness and the disorganization are some of the statements that were reverberated from that experience, and which co-emerge in a cubist way of thinking. Thus, such performances, which expand mathematics education's habitual use, contribute not only for a contingency of knowledge, but also for an exercise of thought. To get an idea of that, look at Fig. 2 which is a set of images from a video of Francisco's work.

Fig. 2. Set of images from a video of Francisco's work. Available in <https://youtu.be/Q286va8Sung>. Accessed on Jan 14, 2022

3.2. The case of experiencing mathematics with surreal art

Starting from a larger project, entitled "De-dramatizing Education (Mathematics): Experiences with Art Workshops in Elementary School"[3], we created a workshop[4] inspired by the art of Salvador Dali. Jéssica Souza (2018) invented a story in which narrated characteristics of a surreal world. After listening carefully to this story, the children were invited to build their own world, making collage from clippings from various magazines in egg cartons. During the workshop, narratives of children linking the deformation of the figures to something wrong or ugly, and attempts to non-deform

[3]Developed with support from CNPq, Brazil, from March 2017 to February 2020, aiming to recognize art workshops as spaces of freedom to experiment with mathematics.

[4]Developed and applied by the scientific initiation scholarship holder, CNPq, Jéssica Juliane Lins de Souza, from July 2017 to June 2018.

the figures through different strategies were observed. The collage on the corrugated surface of the box caused disquiet and strangeness for not being a flat surface.

From the analysis of this workshop, that:

(i) The real has to do with proportion: something too big or too small is not part of the real world; it needs a different function or name to be part of some invented world.
(ii) The real has to do with form: and form has to do with beauty. It is beautiful and real the thing that maintains its original shape, without deformations. Deformed and twisted figures are weird, crazy and ugly.
(iii) The real has to do with organization and method: things seem to be more real when they are organized and categorized. Scrambled and mixed images make the world seem confused and weird.
(iv) The real has to do with reason: and reason has to do with the correctness. What escapes our reason and our sense causes strangeness and is associated with something wrong, that needs to be corrected, done otherwise or undone.
(v) The real has to do with a model: the real is the representation of what we see, reproduction of the world as it presents itself to us. Anything that we don't recognize escapes reality.
(vi) The real has to do with Euclidean geometry: to represent the real, the use of a flat surface that does not cause disturbances in the images is indicated. Objects represented in another geometry are not part of reality. (SOUZA, 2018, p. 62)

Look at Fig. 3 which is a set of images from a video of Souza's work for getting an idea that.

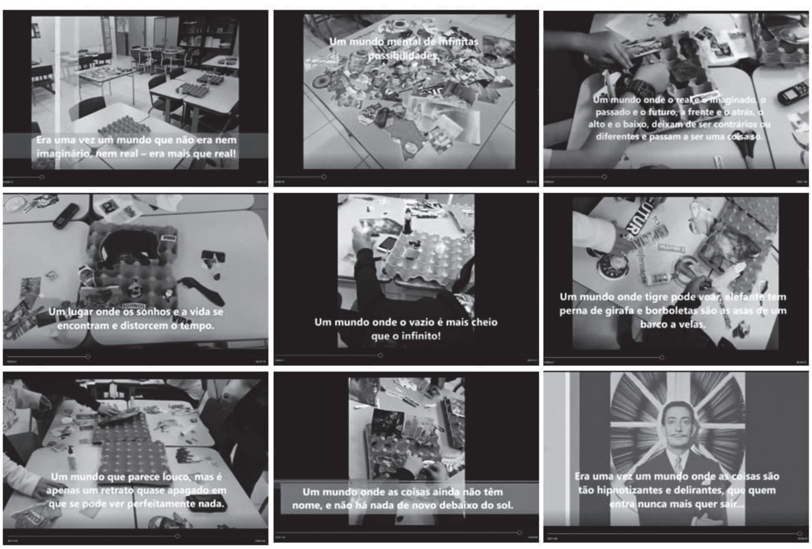

Fig. 3. Set of images from a video of Souza's work. Available in <https://youtu.be/jTa8I74jcj4>. Accessed on Jan 14, 2022

3.3. The case of experiencing mathematics with abstract art

How Euclidean geometry gives way to topology, that is, a topological plan that has no inside and no outside, has no beginning and no end? From this question Mônica Kerscher (2018), a master student, developed a workshop taking the geometric abstract art. With the workshop the children experienced knowledge of Euclidean geometry, such as geometric shapes, division of equal parts of a two-dimensional plane and so on, but also, the idea of continuity and the infinite came up.

One of the activities was: the children received colored ribbons that took on different shapes: a drop, a wheel, a zero, an eight turned, a Möbius ribbon, or even, the shape of infinite. From these forms of tape, a hole-line was made on the plane, whereby the child would cut the length of the tape, but they must follow the order of not being able to divide it in two parts. The question raised was: how far could you cut the tape without dividing it? To have an idea of this work, look at Fig. 4 which is a set of images from a video of Kerscher's work.

Fig. 4. Set of images from a video of Kerscher's work.
Available in <https://youtu.be/TRBs47FuSoc>. Accessed on Jan 14, 2022

With these three examples, I can summarily say that the space created by the workshop made it possible to teach and learn mathematics from the senses of experience, problematizing others connections with art and mathematics. A space not to see what we think, but to experiment with: mathematics, learning, culture, visuality, history.

4. Final Remarks

From these work that I have showed, in relation to what is proposed to teach mathematics with art, apparently, for some it could be considered as a brilliant or innovative perspective, perhaps an unregulated attempt to teach mathematics, where mathematics appears here or there without substance, or foundation. However, I would like to draw attention to the fact that the space created by the workshop with art and children, made it possible to teach and learn mathematics from the senses of experience, problematizing connections with the two areas of knowledge: mathematics and art. Then, a space not to see what we think, but to experiment with mathematics, learning, culture, visuality, history.

Apart from this, they may think that if there are suspended rules, the place is occupied by habitus or repetitive practices. However, the way that we are connecting art and mathematics takes us to see not what a painting means, but ask about how it works: in the way we look, we create thoughts in which mathematics appears as an effect and agent of a way of being in the world and, based on this, re-create or re-think teaching and learning practices in which visual and mathematics are connected.

Finally, I would like to stress that all of this leads us to outline another ethical, aesthetic, and political stance for teaching mathematics with arts. It implies letting oneself be affected by resonances, dismantling devices, creating and experimenting with others, and ethically, aesthetically and politically questioning the truths. That is, an *ethos* that asks more than is affirms, and makes of the research practices as if it is multiple and multiform spaces of producing collective knowledge with: the child, researcher, mathematics, art.

Acknowledgments

I gratefully acknowledge the Brazilian agency CNPq for the financial support.

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https://doi.org/10.1142/9789811287183_0015

15

Designing Student Learning: A Practical Research Case Study of a Mathematics Professional Development Program

Keiko Hino[1]

ABSTRACT This paper purposed to examine the process employed by two lower secondary school teachers in designing a lesson for students through discussion with three university researchers. The teachers conducted practical research at a professional development program in mathematics and collaboratively designed a lesson for Grade 7 on the geometrical transformation of figures. The results revealed the incorporation of considerable changes to the lesson objectives and development, indicating the emergence of different perspectives. The paper employs the study results to discuss the significance of designing lessons that enhance student problem-solving to the professional learning for teachers.

Keywords: Lesson study; Perspective; Professional development.

1. Introduction

Researchers have synthesized the features of effective mathematics professional development (MPD). These elements include a focus on content and student learning, the provision of active learning opportunities for teachers, or the fostering of collective participation (e.g., Sztajn et al., 2017). However, the details of teacher learning beyond such broad features remain unclear (Goldsmith, et al., 2014). Scholars have also recommended the facilitation of professional development through the interplay between theory and practice (Huang and Shimizu, 2016; Sztajn et al., 2017); however, the means of realizing such interplay vary. More detailed information is needed on teacher learning in specific MPD contexts. The present paper examines an MPD case in Japan in which teachers conduct practical research (Miyakawa and Winsløw, 2019) with university scholars and clarify the process and product of designing student learning pathways through problem-solving lesson. This paper attends to such learning by mid-career teachers who have accrued mathematics teaching experience of 15–20 years. The present investigation is intently focused but intensive in its analysis of one aspect of Hino, Makino, and Kawakami's study (2020): designing a research lesson.

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2. Study Framework

2.1. *Researching: A critical aspect of lesson study*

Lesson study is a practice-based, research-oriented, and collaborative style of professional development (Huang and Shimizu, 2016). The aspect of research activity by teachers is critical to the process of lesson study. Fujii (2016) specified five components of lesson study (Fig. 1), asserting that research lessons and post-lesson discussions denote the two most visible components and attract attention from many teachers and educators in other countries. He insisted that other components are nonetheless critical. According to him, the autonomous formulation of questions by teachers is especially essential, and this component distinguishes lesson study from teacher training (see also Takahashi, 2019). Miyakawa and his colleagues (Iwasaki and Miyakawa, 2015; Miyakawa and Winsløw, 2019) stressed the importance of teachers performing practical research for professional learning. The phrase *practical research* is "a broader term that denotes the study and research on teaching practices, carried out mainly by an individual teacher or a group of teachers for the purpose of improving their teaching practices" (Miyakawa and Winsløw, 2019, p. 283). All the above stated concepts have emphasized the research activities of teachers as a critical dimension of lesson study.

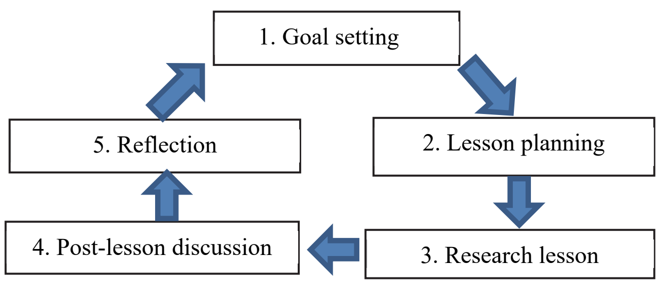

Fig. 1. Five components of lesson study (cited from Fujii [2016])

2.2. *A lens to capture teacher learning*

This paper references Lin et al.'s work (2018) and builds on their idea of perspective to apprehend learning by teachers. Lin et al. explain that perspective means to "specifically describe how one deals with a situation by coordinating a set of ideas and actions" (p. 198). They explore perspectives relating to the use of theory by mathematics teacher educators and researchers to facilitate the development of teachers. In the present study, the notion of perspective is expected to contribute to the exploration of the core ideas attained by teachers so they can view and deal with a situation occurring in their teaching practice. The study aims to capture the growth of the perspectives of the participating teachers during the course of their practical research activity: how newer perspectives emerge, how a certain perspective is extended, or how different perspectives can be related or even merged.

2.3. Research questions

— How can practical research activities teachers can perform be designed for a MPD?
— What teacher perspectives on the teaching/learning of mathematics emerged in the goal setting and lesson planning phase, and how did they materialize?

3. Study Context

3.1. The "Naichi Ryugaku" program: The studied MPD

Numerous professional development (PD) programs exist in Japan at national, prefectural, and municipal levels. Varied types of university-associated PD programs are delivered for practicing teachers, including the traditional master's, a newly developed professional degree, and PD in collaboration with prefectural boards of education. The program described in this paper represents the third type mentioned above and denotes a joint project undertaken by Utsunomiya University and the Tochigi Prefectural Board of Education. Participants are recommended by their principals and examinations are conducted by the Board of Education. Participants are granted continuous leave from their jobs for the entire six-month period (one semester) to focus on PD activities. One or two teacher(s) participate in the studied MPD every semester. The selected teachers are usually experienced, having accrued 15–20 years of teaching practice in primary or lower secondary schools.

The studied MPD encompasses three major activities: attending courses at the university, participating with researchers in several lesson studies in local public schools, and performing practical research. The practical research activity includes the formulation of a research theme, the design and analysis of a problem-solving research lesson on the established research theme, and continuous dialog between the participating teachers and the three researchers in the department of mathematics education (Hino, Makino, and Kawakami). Further, the lesson study cycle presented in Figure 1 is adopted for the research theme, highlighting the components of goal setting, lesson planning, and reflection.

3.2. The two teachers and their practical research activities

Mr. T and Mr. M worked in different lower secondary schools. They participated in the studied MPD during the second semester (October — March) of 2018. They began their practical research activity according to the cycle of lesson study. Initially, each developed a research lesson plan for a geometry chapter. However, Mr. M could not conduct the research lesson because of the circumstances at his school. Therefore, he participated in the development of Mr. T's lesson plan. Thus, the two teachers collaboratively developed a research lesson delivered by Mr. T to his seventh-grade class on December 20. After the research lesson, the two teachers engaged in a post-lesson discussion and also interviewed several students. Thereafter, the two teachers and three researchers took around two months to examine the effects of the lesson by analyzing the data obtained from students. The teachers finally summarized the contents of their inquiry in the form of a report.

The team of teachers and researchers met 20 times to discuss the contents of the practical research performed by teachers. These gatherings included general meetings attended by all five members (GM), small meetings between some of the members (SM), and on-site meetings at Mr. T's school. This paper focuses on the goal setting and lesson planning discussions and thus describes the first eight meetings (sessions 1 to 8) to grasp how the perspectives of the teachers emerged. There are no video or audio records of the meetings. The paper's descriptions of the contents of the eight meetings are based on summaries prepared by the teachers. Memos inscribed by the researchers during the meetings are referenced to outline the contents of the discussions that transpired.

4. Goal Setting and Lesson Planning by the Two Teachers

4.1. Session 1 (goal setting)

At the first meeting, the three university researchers, Mr. T, and Mr. M met and discussed the theme of practical research that would be undertaken during the semester. Tab. 1 summarizes the reflections noted in their reports by Mr. T and Mr. M vis-à-vis mathematics teaching and their expectations of conducting practical research.

Tab. 1. Reflections on teaching, issues, and expected research themes in the summaries prepared by the two teachers

Content	Brief description
Mr. T	
Reflections on my teaching	I have changed my teacher-centered pedagogy to a more participatory method of mutual learning among students, targeting that no one should be left behind.
Issues of mathematics education	The connection between mathematics and everyday life; individual differences in understanding and motivation; difficulties students face in explaining their ideas.
My teaching objectives	The joy of learning mathematics; developing autonomy in students toward learning that creates competencies; nurturing the personalities of students during lessons by inculcating the importance of cooperation with other people.
My research theme	To foster the competencies of thinking, decision-making, and expressing ideas through mutual learning; to imbibe a sustainable method of teaching mathematics that I can implement in my daily lessons.
Mr. M	
Reflection of my teaching	I have tried to foster thought articulation by students by devising a way of board writing, questioning, providing opportunities for mutual teaching, or recommending explanatory aspects.
My teaching difficulty	Substantial individual differences exist among students. Hence, it is inadequate to merely impart awareness of foundational concepts and principles of mathematical skills.
My interests and research theme	How to incorporate the processes of problem-finding and problem-solving into the lesson; how to foster the knowledge necessary for mathematical problem-solving in students; to develop a teaching plan for an entire unit.

The discussion revealed that Mr. T and Mr. M shared concerns about fostering the abilities of thinking, decision-making, and expressing ideas (TDE) in their students through their mathematics lessons. It was thus determined that the research activity would begin by establishing a common theme: investigating mathematics lessons that foster TDE abilities in students.

4.2. Sessions 2–8 (lesson planning)

The lesson plan proposed by the two teachers was discussed from sessions two to eight. Both teachers decided to conduct a research lesson on the subject of geometry. Mr. T planned a lesson on the textbook chapter on transformations of geometrical figures (Grade 7). Mr. M's lesson plan concerned the chapter encompassing proof. Each teacher prepared a summary report and presented the specified lesson plan for each meeting. The synopsis essentially comprised lesson objectives and outlined the progression of the lesson. In the meetings, ideas were freely expressed using the summaries as objects of discussion. The summaries also served as artifacts that mediated the boundaries between the teachers and the university researchers.

4.2.1. The transitioning of lesson objectives

Tab. 2 (on the next page) overviews the objectives established in Mr. T's summary for the research lesson. Students were expected to apply their conceptual knowledge of the parallel, symmetric, and rotational transformation of geometrical figures (TGF) to achieve their activity goals.

Tab. 2 displays the changes in lesson objectives through the eight sessions. Mr. T's lesson objective was rather general in the summary report prepared for session 2 (R2) [2]. He began to assert specific objectives only in R3 in which T stated his aim for students to "express their ideas mathematically." Besides, as evidenced by his notation of "today's task," Mr. T conceived of students creating personal emblems that others could recreate. The keyword "recreate" first appeared in the "today's task" section of R3 and this notion appears to have emerged during the session 2 discussions. Mr. T first used the phrase "modify the explanation" in the lesson objectives inscribed in R4 in which he also explicitly inscribed the term "make an instruction manual." These statements reflect the discussions that occurred in session 3, during which the notion of students creating and modifying instruction manuals as an activity was introduced. The report R5 incorporated the term "using mathematical expressions" into the idea of "modifying the explanation." The lesson objectives further clarified the connection to geometrical proof in the Grade 8 curriculum and appeared to reflect Mr. M's interests. Pertinently, Mr. M joined Mr. T's lesson planning efforts around session 5. Moreover, the lesson objectives were described in detail in R5 to establish their connections to the four evaluation standards set by the Ministry of Education (MEXT). However, these associations disappeared in R6, which also introduced a change from the R5 lesson

[2] R1, R2, R3, … R8 are used for the summary reports prepared by Mr. T for all eight sessions.

objectives by adding the word "evaluate." Another modification in R6 involved the addition of the word "friends." Both insertions reflect the discussions held in session 5. Specifically, one researcher asked the question, "who are "others"?" (Makino), and the meaning of the term "others" apropos the student activity was made explicit. The lesson objectives became longer in R7, which introduced new phrases: "critical thinking" and "self-evaluation." These inclusions indicated the contemplation of session 6 discussions. However, the phrases disappeared in R8.

Tab. 2. The objectives established in Mr. T's summary for the research lesson

Session	Type	Lesson objectives
2	SM	Students can (i) solve a problem by integrating previously learned knowledge and (ii) think autonomously by appreciating multiple views and ideas.
3	GM	Students understand specific situations in which TGF can be used and make personal emblems. Through such a process, students can acquire the ability to express their ideas mathematically and to think and make decisions. Today's task: make an emblem that others can recreate, and explain its structure
4	SM	Use interactive discussions to explain the process of creating a motif and modify the explanation for better effect so students can enhance their TDE abilities and acquire problem-solving skills through the application of previously acquired knowledge. Today's task: Use mathematics to create a motif and make an instruction manual so that others can recreate your design.
5	GM	Students use TGF to create a motif and collaboratively elucidate their process. In so doing, they enhance their logical thinking abilities and produce geometrical proof; they also improve their articulation skills by modifying their explanations using mathematical expressions. Therefore, the lesson adopts two objectives: students can use TGF to create a design that others can recreate (skill) they can devise a better way of drawing the pattern and can explain the process so that others can recreate it (mathematical viewing and thinking)
6	SM	Students take the viewpoint of TGF to capture the components of traditional patterns or insignia and make an instruction manual. They also evaluate and revise their manual so their friends (others) can recreate their design.
7	SM	Students employ the standpoint of TGF to grasp the components of traditional patterns or insignia and produce an instruction manual. They further evaluate and revise their manual so their friends (others) can recreate their design. In so doing, students apply acquired knowledge of TGF, acquire critical thinking skills, and imbibe the perspective of self-assessment by utilizing the evaluations made by others.
8	SM	Using TGF, students can generate better instruction manuals to help their classmates recreate their designs. (mathematical viewing and thinking)

4.2.2. The transition of lesson development

Scrutiny of the lesson outlines inscribed in the reports reveals considerable transformations through the eight sessions. However, space constraints allow the description of only two transformative aspects. First, the major cognitive processes to be experienced by students during the lesson were modified. At the beginning (R2), Mr. T thought students would be provided a ready-made emblem and would be asked to apply their knowledge of TGF to discover its structure. In R3, he first introduced the activity of students creating an emblem on their own for others to recreate. Then the student activity described in R4 included the acts of producing and revising an instruction manual that others could follow to recreate the emblem. These processes of production and modification denoted major aspects of the student experience of the lesson. Nevertheless, it was still not apparent how these facets would be incorporated into the lesson. In R4, Mr. T proposed the procedure for the creation of an instruction manual, its revision, and a presentation on how the manual could be amended delivered by both the teacher and the students. He planned two consecutive lessons for these activities, but the discussions in session 4 pertained to how a primary process could be realized from the intricate lesson flow. In R5, Mr. T separated the activities into two parts after contemplating the abovementioned discussions: first, students would be asked to create an instruction manual (lesson 1); subsequently, they would be required to think of a better version (lesson 2). This proposal by Mr. T in session 5 caused all members to further discuss the processes that students should experience from the lesson. It was decided that students should concentrate on the design they created, and not the teacher. The team members confirmed that the production and revision of the manual should both be designated as principal processes for all students. It was also determined that group activity should contribute more to the realization of the desired processes than individual work. Mr. T's lesson plan in R6 explicitly included an activity in which a student group would recreate a motif by reading the manual generated by another peer group.

Second, the task and teaching materials used in the lesson were altered. In R5, Mr. T thought students would freely create an emblem by using parts that were given. In session 5, this issue was debated and it was ultimately decided that students would be provided certain figures to help them focus on the activity of creating and modifying their instruction manuals. Mr. T deliberated on this discussion and stated in R6 that TGF could be used to create two figures (Fig. 2). R6 also included an introductory activity of recreating a simpler figure as pair work. This activity was aimed at motivating students to generate an effective instruction manual because it was assumed that students would find it difficult to communicate how the figure could be made to a friend who would have to draw the figure without looking at it. Mr. T also decided that the introductory phase of the lesson would use a well-known fretwork typical of the Kanuma area in which the school is located (R7).

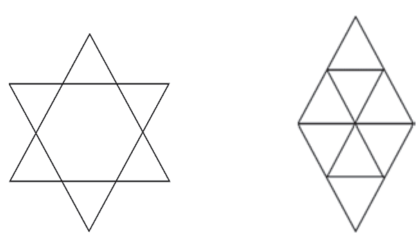

Fig. 2. Two figures used for instruction manual

Tab. 3 outlines the flow of the two consecutive classes implemented by Mr. T for his research lesson.

Tab. 3. The flow of the research lessons

Activity	Brief description	Form of activity
<u>Lesson 1</u>		
Introduction to the Kanuma fretwork	The teacher introduced the traditional and well-known fretwork of the Kanuma area.	
re-creation game	Students worked in pairs. One student was shown a figure and was asked to explain its construction to the other student. The other student then attempted to recreate the figure.	Pair
Presenting today's task and goal of the lesson	The lesson objective was: "Using transformations of geometrical figures to create a better instruction manual to help classmates recreate an original figure."	Full class
Make instruction manual	Six groups of students were formed, with four students per group. Three groups were assigned the pattern shown on the left in Figure 2; the other three groups were assigned the motif on the right. Members of each group worked together to create an instruction manual. The diagram parts were also provided in concrete form in case students needed to manipulate them to attain additional ideas.	Group
Reflection of the lesson	The teacher and students shared information about the events of Lesson 1.	Full class
<u>Lesson 2</u>		
Re-creation of the diagram by using the instruction manual	Student groups exchanged instruction manuals and recreated the diagram created by another group by reading the instruction manual. The groups were then shown the original motifs to check for accuracy.	Group
Evaluation of the manual and provision of feedback	Each group thoroughly studied the instruction manual it had used. Members commented on it and returned it to the group that had created it. Students summarized the elements that should be included to make the instruction manual more effective.	Group
Revision of the manual	Each group utilized the comments of their peer group to reassess and modify their instruction manual. Students noted important points that would help them produce better instruction manuals that could help others recreate their motifs.	Group
Summarizing the lesson	Students summarized the aspects to be incorporated to create a better instruction manual.	Full class

4.3. The emergence of diverse perspectives

The above-described examination of modifications made to the lesson objectives and development allows the extraction of three core ideas (perspectives) attained by Mr. T and the other members. These viewpoints aided in the consideration and handling of situations occurring in their teaching practices. In particular, they helped resolve the circumstances of designing an activity for students in a TGF lesson for Mr. T's class.

The first perspective involves "using the language of mathematics." Mathematics offers a common and universal language that enables the clear transmission of one's intentions to others. The lesson objectives outlined in R2, and more explicitly in R5 and R6, indicated that Mr. T aimed to enhance the use of geometrical figure transformations by students to enable clearer, more specific, and more accurate communication. The lesson development sessions involved discussions of aspects that would constitute a “better manual” from the point of view of the language of mathematics (R5). Mr. T further thought about the extent to which students were requested to use TGF words such as “rotation” (R7). These observations in Mr. T’s reports suggest that this perspective had existed from the beginning of the sessions because it is closely connected to his motivation for practical research. This perspective thus seemed to underpin the emergence of other viewpoints.

The second perspective is described as “revising the initial method to create a better version.” Amending the manual to create a more effective version constituted a core activity of the research lesson. It first appeared in the R4 lesson objective and became the predominant standpoint from its conception. Session discussions entailed how this activity could be actualized and ensured for all students during the lesson. It also became evident that this activity would offer students the opportunity to reflect on their reasoning.

The third perspective denotes “thinking with doubt”. It is vital to question phenomena instead of taking them for granted. A statement of this ability first appeared in the R7 lesson objectives. The inclusion of the terms “critical thinking” and “self-assessment” in the lesson objectives appear to mirror an opinion tendered by a researcher (Kawakami) in session 6 that the student activity of evaluation and revision of the instruction manual could be considered relevant to critical thinking, an aptitude that increasingly attracted Japanese mathematics educators. This perspective was thus borrowed from the researcher’s point of view and was foreign to the standpoint of the teachers at first. It thus disappeared in R8. Nevertheless, critical thinking ability became a focal point in the sessions conducted after the research lesson because of its theoretical and research-related association with mathematics education (Hino et al., 2020). The evolving nature of this perspective indicates that teachers came to comprehend the significance of theories for their practical research activities.

5. Concluding Remarks

This paper examined the process of designing student learning via the development of a lesson plan for the Grade 7 topic of the geometrical transformation of figures. The lesson objectives and development changed because of the discussions that transpired through the eight described sessions. The analysis showcased in this paper relied on written materials developed by the teachers and the personal notes taken by the author during the discussions. Nevertheless, three perspectives attained by Mr. T (and all other members) could be discerned. These perspectives aided the contemplation and handling of the situation of designing a student activity for the research lesson.

The three perspectives did not develop simultaneously. The perspective of language emerged in the earliest session and underpinned subsequent discussions. Initially, this core view represented the lesson objective. Later, it became the object of discussions on lesson development. The perspective of revision also became evident in the early sessions and continued to be debated and stated both in the lesson objectives and the development of the lesson outline. The notion of doubt appeared later and appeared to have been instigated by a researcher. Discrete but related perspectives were thus evoked in different contexts and at diverse times. Fujii (2016) insisted that lesson planning is a critical component of lesson study. This study's discernment of the three perspectives validates Fujii's (2016) claim of the vital role of lesson planning.

The analysis of changes in lesson objectives and outline development reveals the critical functions discharged by the university researchers in the discussions (see also Iwasaki and Miyakawa, 2015). The researchers proposed individual ideas and interests that contributed substantially to the development of perspectives in the teachers. Moreover, the study's analyses indicate the significance of the problems and interests vis-à-vis mathematics teaching the teachers brought into the MPD (see also Makino and Hino, 2018). They remained pivotal as they engaged in their practical research.

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https://doi.org/10.1142/9789811287183_0016

16

Chinese Lesson Study in Mathematics: A Local Practice or a Global Innovation?

Rongjin Huang[1]

ABSTRACT This chapter aims to provide a holistic portrayal of the features of Chinese lesson study (LS), the mechanisms of Chinese LS, and its recent development. Recommendations for further improvement of Chinese LS are provided and implications of Chinese LS on the practice of LS internationally are discussed.

Keywords: Lesson study; Chinese Lesson study; Teaching research system; Teacher professional learning; Improvement science; LS in Education 4.0.

1. Background

1.1. Japanese lesson study and its adaption internationally

Due to Japanese students' high performance in math in the 1995 Trends in International Mathematics and Science Study (TIMSS), Japanese mathematics teaching has drawn international attention (Jacobs et al., 2006). The 1999 TIMSS video study examining nationally representative eighth-grade mathematics classrooms (81 in the US, 100 in Germany, and 50 in Japan) (Stigler and Hiebert, 1999) revealed high-quality mathematics teaching in Japan (90% of classrooms studied were rated as medium and high) in comparison with the classrooms studied in Germany (66%) and in the US (11%). The reason for the Japanese success was uncovered in Stigler and Hiebert's (1999) seminal book, *The Teaching Gap,* in which the authors detailed a Japanese "structured problem solving" mathematics teaching model which includes four major phases. The phases are: (1) teacher poses the problem, (2) students work out the problem individually, (3) the whole class discusses students' solutions, carefully orchestrated by the teacher, and (4) the teacher and students jointly summarize big ideas learned. Stigler and Hiebert further described a unique way of teacher professional development (PD), which aims to ensure teachers can teach mathematics through "structured problem solving" nationwide. This Japanese PD approach typically includes collaborative study of teaching materials, joint design of a lesson, and teaching of the lesson observed by colleagues with a post-lesson debriefing followed by a revision of the lesson plan. This Japanese PD approach has been coined

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as lesson study (jugyou kenkyuu, 授业研究). LS is a nationwide practice with multiple models and with a long history (Lewis, 2016; Makinae, 2019). This approach has been adopted in the US in 1990s (Lewis and Tsuchida, 1998) and then has spreaded globally (Huang et al., 2019a). Japanese LS has demonstrated its effects of promoting teacher professional learning and improving student learning outcomes (Lewis, 2016; Lewis and Perry, 2017), as well as developing a teacher professional learning community (Huang and Shimizu, 2016).

1.2. Chinese LS and PISA studies

The outstanding performance of Chinese Mainland students in mathematics and science on Programme for Student Assessment (PISA) (OECD, 2013, 2019) has prompted international scholars to study mathematics education and mathematics teacher education in China (Fan et al., 2015; Li and Huang, 2018). The Shanghai teaching method has been characterized as mastering teaching and learning through variation (Huang Huang et al., 2021d), while Shanghai lessons have been recommended as exemplary lessons in a popular book, Mathematics Mindset (Boaler, 2018). Naturally, how to prepare and ensure teachers can teach mathematics in such a way has become an interesting question. Prevalent in China is a job-embedded, hierarchical PD system of teaching research activity includes studying of teaching materials, jointly planning a lesson, teaching the lesson, observing the lesson, and having a post-lesson discussion as a core component (Huang et al., 2016; Yang, 2009). The Chinese approach to PD which focuses on examining and polishing a lesson aligned with reform-oriented teaching is coined as Keli (exemplary lesson) study (Huang and Bao, 2006). Further, it has been theorized as Chinese lesson study (Chinese LS) from cultural, institutional, and instructional expertise perspectives in a journal special issue on Chinese LS and its adaptation internationally (see Huang et al., 2017). Li (2019) further tracked and interpreted Chinese LS from cultural and historical perspectives. The recent development of Chinese LS within the context of 21st century competency-oriented curriculum reform is discussed in a follow-up special issue on Chinese LS (Fang et al., 2022). In the following sections, I will provide more details about the features of Chinese LS and interpretation of why and how Chinese LS works in China.

1.3. Similarities and differences between Japanese LS and Chinese LS

Embedded in a nationwide, hierarchical teaching research system (school-based, district-based, city-based, province-based, nation-based), Chinese LS includes multiple modes with different purposes at different levels as well. In general, various types of Chinese LS focus on polishing the research lesson based on classroom observation and collective reflection and emphasize the LS product as “public lessons” or “exemplary lessons” (Huang et al., 2017; Yang, 2019). There are “report lessons” for novice teachers to demonstrate their professional growth, “exemplar lessons” for expert teachers to demonstrate reform-oriented good practice, and “contest lessons” for winning awards being excellence in teaching (Huang et al., 2017a).

With regard to the Chinese LS process, Gu and his team have theorized a mode of LS within the context of curriculum reform in Shanghai (Wang and Gu, 2007). It is called "three foci (teacher belief, gaps identification, and adaptive change) in two rounds of reflections between the iterative research lesson planning for improvement (三关注， 两反思)" (Wang and Gu, 2007, p. 37). More specifically, a teacher starts with planning a lesson aiming at making visible his or her own existing teaching beliefs and behaviors by reflecting on feedback from colleagues and identifying the gaps from what the reform requires. Then the teacher redesigns and teaches the lesson aiming at gaining lived experience of the new standards, ending with reflecting, redesigning, teaching it again based on colleagues' observation feedback and evidence of student learning in order to arrive at a new behavior phase. Gu and colleagues' work also marked the first time that researchers were called upon to work with schools to provide experts guidance (zhuanjia yinglin, 专家引领) (Huang and Bao, 2006).

There are similarities between Japanse and Chinese LS regarding focus on examining and reflecting upon classroom pratice and the nature of job-embedded and nationwide PD activity, with each having a long history and a cultural and institutional support system (Lewis, 2016; Li, 2019; Yang, 2019). Yet, some essential differences between Chinese and Japanse LS are identfied. Specially, the essential components of Chinese LS are: (1) repeated teaching of the same topic; (2) focusing on both content and pedogogy; (3) the involvement of knowledgeable others throughout the LS process; and (4) examplary lessons as products of LS (Huang et al., 2017a; Li, 2019).

2. Why Chinese Lesson Study Works

Chinese LS has played important roles in implementing curriculum reform and improving mathematics instruction over decades (Wang and Gu, 2007; Huang et al., 2019b). In this section, the reasons for why Chinese LS works are explored from multiple perspectives.

2.1. An analysis from a historical and cultural perspective

From a cultural perspective, Chen (2017) argues that the following three core cultural orientations are features of Chinese LS. First, unity of knowing and doing（知行合一） rather than conceptual explication is behind teacher knowing and understanding through embodied actions and practical discourse. Ontologically, in Chinese culture, knowing and doing are integrated. Second, practical reasoning（实践推理）drives the deliberate practice of repeated teaching through group inquiry and reflection. Epistemologically, knowledge of good teaching is not so much talked about in verbal concepts as enacted in teachers' actions in deliberate practice through critical inquiry and reflection. As a Chinese saying states, "Proficiency comes from familiarity"（熟能生巧）. Third, a tendency of emulating those better than oneself（见贤思齐） motivates teachers to learn from "good" exemplars of expert teachers. Methodologically, it is believed that watching model teaching, practicing for making prefect, and learning from making errors are valuable opportunities for teachers. This corresponds to the statements: “doing things” cannot be separated from “being humane,” and “respecting virtues” should go hand in hand with “learning knowledge.” These cultural values about teacher professional learning could help explain why in Chinese LS, repeated teaching of the same topic, developing an exemplary lesson, and knowledgeable others’ involvement throughout the LS process are emphasized.

From a historical and cultural perspective, Li (2019) further argues that the following three principles and practices are crucial for understanding the nature of Chinese LS: (1) respecting and learning from masters and experts; (2) teaching and learning by integrating profound theory and deliberate practice, and (3) learning taking place among learner peers through mutual observation and discussion. They also suggest that these cultural roots could help better understand the nature and features of Chinese LS. Yet, some unintended consequences should be noticed. For example, some teachers have taken a utilitarian or opportunist approach, participating in LS activities mainly for the sake of winning a contest or promotion, social status, and financial incentive (Li, 2019). Furthermore, it is often difficult to agree upon the criteria for “good” lessons during LS amid ongoing curricular reforms (Chen, 2017).

2.2. An analysis from an institutional perspective

From an institutional perspective, both a teacher professional promotion system and an associated teaching research system are fundamentally important for ensuring Chinese LS is implemented at scale. *The professional ranking and promotion system*, established in 1993, has evolved for supporting teachers’ professional development. There are three levels of professional titles: senior（高级）, intermediate (中级, Level 1), and primary (初级, Level 2 and 3). For each level, political, moral, and academic qualifications are specified. In addition, there are some specific titles for honoring teachers with excellence in teaching, research and leadership such as “exceptional teacher” which is equivalent to university professor status (Huang et al., 2016), or “master teacher” and “subject leader” (Cravens and Drake, 2017). This system not only specifies components of teacher professional expertise, but also provides incentives and a culturally supported mechanism for teacher professional development (Li et al., 2011).

Associated with the teacher promotion system, there is a *teaching research system* supporting teacher professional development (Chen, 2020; Ricks and Yang, 2013). Teaching research (Jiaoyan) is a special term that refers to various activities of professional development at different levels (school, district, city, or national), and is organized by teaching research groups (school-based) and institutes (Jiaoyan Jigou). The teaching research system, initially established in 1956 (Wang, 2009), has evolved into a hierarchical system with school, district, county, city, province, and national levels (Yang, 2019). There are different departments including educational bureaus, educational science research academies, and curriculum development centers at both national and local levels. These are responsible for guiding teaching research, overseeing teaching administration in schools on behalf of educational bureaus, providing consultation for educational authorities, mentoring the implementation and revision of new curricula, building the bridge between modern educational theories and teaching experiences, and promoting high-quality classroom instruction (Huang et al., 2016). There are more than 100,000 teaching researchers (inclusive of other disciplines) working in teaching research institutes (Wang, 2009). The teaching researchers play multiple roles, including: (1) interpreting opinions regarding the implementation of teaching plans, syllabi, and materials based on local contexts; (2) providing evidence and suggestions on decision making for local education authorities; (3) organizing a variety of teaching research activities at different levels; and (4) helping teachers study teaching materials, implement teaching schedules, and improve their teaching efficiency. Specific requirements for recruiting teaching researchers have been set by the Ministry of Education and are further specified by local education authorities (Huang et al., 2012). In general, a teaching research specialist must be an excellent teacher with good teaching research ability and leadership.

Within the teaching research system, many teaching research specialists and educational researchers who have excellence in teaching and doing educational research and with needed skills in facilitating teaching research activity could serve as knowledgeable others for facilitating LS. Some advanced teachers are selected to serve as subject leaders at district or city levels to lead in carrying out school-based teaching research including Chinese LS. These subject leaders help teachers interpret the curriculum standards, demonstrate their own teaching, mentor other teachers, and decode instructional expertise through comparing teaching conducted by experts and regular teachers. Chinese LS and district research projects, with the support of subject leaders (or/and knowledgeable others from universities), have made curriculum reform transparent for teachers to ensure their learning to teach reform-oriented lessons (Cravens and Wang, 2017; Fang, 2017).

2.3. Studies on Chinese lesson study

Similar to Japanese LS, Chinese LS has played roles in improving mathematics teaching (Huang et al., 2011), promoting students' outcomes of learning (Huang et al., 2016), developing both teachers' and specialists' professional knowledge and skills (Huang and Han, 2015; Huang et al., 2017b), implementing reform/innovative ideas (Huang et al., 2019b; Zhao et al., 2022), and building connections between research and practice (Huang et al., 2016). In Huang and Li's (2009) study, with the aim of developing exemplary lessons to supplement the textbook, LS groups from a school, a district and a city supported teachers in developing lessons which demonstrated new curriculum-oriented instruction. Huang et al. (2011) further documented how teachers could develop their instructional expertise through developing exemplary lessons and collaboration within the LS mechanism. Huang et al. (2016) they explored how a LS infused by learning trajectory and variation pedagogy could promote students' conceptual understanding of the mathematical algorithm of division of fraction. Similarly, Huang et al., (2019c) revealed that theory-infused LS could develop students' ability to solve word problems. Regarding the effect of LS on curriculum reform, both

Huang et al. (2019b) and Zhao et al. (2022) documented how innovative ideas introduced in curriculum standards could be implemented in the classroom effectively through iterations of LS. Concerning the learning of knowledgeable others (e.g., mathematics teaching research specialists in China), Huang and Han (2015) documented how mathematics specialists and teachers co-learned through boundary crossing during LS. Huang et al. (2017b) detailed what knowledge and skills are needed for being specialists and how specialists develop their professional knowledge. With regard to the roles in linking theories to practice through Chinese LS, Huang et al. (2016), Han et al. (2019), and Zhao et al. (2022) showed how a certain theory (e.g., learning trajectory, variation pedagogy) could inform the LS process and promote student learning outcomes. Recently, Huang et al. (2022) portrayed teachers' expansive learning process through Chinese LS. In the journal special issue on Chinese LS and its adaptation in other countries (Huang et al., 2017), it was argued that Chinese LS is a deliberate practice for developing instructional expertise, a research methodology for linking research and practice, and an improvement science for instruction and school improvement system wide.

To understand recent developments regarding LS in China, a new journal special issue revisits LS's roles within the context of competency-based curricula (Fang et al., 2022). This special issue argues that LS in China continues to serve as a powerful platform to support change in teaching and reveals a new feature of Chinese LS, namely, research-practice partnerships (RPPs) in LS (Farrell et al., 2022) where researchers, who are university faculty members support teachers to implement competency-based (*hexing suyang* 核心素养) curriculum reform through boundary crossings (Engeström and Sannino, 2010). From the lens of learning at the boundary of research-practice partnerships (RPPs), the features of Chinese LS are highlighted in three major themes: (1) the role of university-school partnerships in meeting the new demands of key competency reform; (2) resourceful tools, strategies and structures to support boundary crossing for teachers; and (3) roles and relationships for mutual learning in university-school partnerships. Thus, it urges the need to redefine Chinese LS to engender versatility and hybridity and to enlist mutual learning relationships in future university-school partnerships.

3. Further Development of Chinese LS for Education 4.0

3.1. Features of teaching and learning in Education 4.0

Research shows the positive effects of Chinese LS on improving teaching, developing teachers and implementing new curricula in China. Yet, the world has entered a new era: Industrial Revolution (IR) 4.0 in which the advancement of new technologies blurs the lines between the physical, digital and biological worlds. These advancements are led by the emergence of artificial intelligence, robotics, the internet, autonomous vehicles, bio and nanotechnology, 3-D printing, material science, quantum computing and energy storage (Diwan, 2017; Shwab, 2016). The Industrial Revolution 4.0 affects not only businesses, governments and people, but also education; thus the name

Education 4.0 came into existence. Fisk (2017) identified the following nine trends related to teaching and learning in Education 4. 0. First, learning can be anytime, anywhere. E-learning tools offer opportunities for remote, self-paced learning. The flipped classroom approach allows interactive learning in class, while the theoretical parts can be learned outside the class time.

Second, learning is personalized to individual students. Harder tasks are introduced only after a certain mastery level is achieved. Positive reinforcement promotes positive learning experiences, boosting students' confidence about their academic abilities.

Third, students have a choice in determining how they want to learn. Although the learning outcomes of a course are presented by the institutions, students are free to choose the learning tools or techniques they prefer. Options may include blended learning, flipped classroom, and Bring Your Own Device (BYOD).

Fourth, students will be exposed to more project-based learning. Students apply their knowledge and skills in completing short-term projects which allows them to practice their organizational, collaborative, and time management skills, all of which are useful in their academic careers.

Fifth, students will be exposed to more hands-on learning through field experiences including internships, mentoring projects, and collaborative projects. Technological advancement enables the learning of certain domains, effectively making more room for acquiring skills that involve human knowledge and face-to-face interaction.

Sixth, students will be exposed to data interpretation in which they are required to apply their technological or ethical knowledge to numbers and use their reasoning skills to make inferences based on logic and trends from given sets of data. The manual part of mathematical literacy will become irrelevant as computers and artificial intelligence (AI) will perform the statistical analysis and predict the future trends.

Seventh, students will be assessed differently. The conventional platform to assess students may become irrelevant or insufficient. Factual knowledge can be assessed during the learning process, while the application of knowledge can be tested when students are working on their projects in the field

Eighth, students' opinions will be considered in designing and updating the curriculum. Students' input helps curriculum designers maintain up-to-date, and usefulness.

Lastly, students will become more independent in their own learning. This will force teachers to assume a new role as facilitators who guide the students through their learning processes.

These nine trends of Education 4.0 shift the major learning responsibilities from teachers to students. Hence, teachers should support the transition (Hussin, 2018).

To align with the new responsibility of learning shifted to learners in Education 4.0., a specific set of core skills is needed, which is recommended by the World Economic Forum (2016a). These top 10 skills are: (1) complex problem solving, (2) critical thinking, (3) creativity, (4) people management, (5) coordinating with others, (6) emotional intelligence, (7) judgement and decision making, (8) service orientation, (9) negotiation, and (10) cognitive flexibility. To promote learners' development of these skills, teachers should create conducive learnin environments.

The following top 14 strategies for developing these core skills are recommended (World Economic Forum, 2016b):

1. Encourage play-based learning.
2. Break down learning into smaller, coordinated pieces.
3. Create a safe environment for learning.
4. Develop a growth mindset.
5. Foster nurturing relationships.
6. Allow time to focus.
7. Foster reflective reasoning and analysis.
8. Offer appropriate praise.
9. Guide a child's discovery to topics.
10. Help children take advantage of their personality and strengths.
11. Provide appropriate challenges.
12. Offer engaged caregiving.
13. Provide clear learning objectives targeting explicit skills.
14. Use hands-on approach.

3.2. Recommendation for developing Chinese LS for Education 4.0

As a traditional and powerful teacher professional development approach, Chinese LS should be improved to meet the needs of Education 4.0. By recognizing weaknesses of Chinese LS such as focusing on teacher performance rather than student thinking and focusing on reflection based on experience rather than analytical analysis, several strategies could be adopted to improve the LS process. First, theoretical notions such as learning trajectory (Simon, 1995) and variation pedagogy (Gu et al., 2004; Huang and Li, 2017) could be used as guiding principles during the LS process. Second, the LS process could be carried out as disciplined inquiry (Bryk et al., 2015) by adopting the ideas (pre, post-tests; intended, enacted and achieved goals of learning) from learning study (Marton and Pang, 2006) and investigating a focus-group of students during LS (Dudley, 2012). Thus, the LS process could be enriched as displayed in Fig. 1 (on the next page).

Within the LS cycle, it is crucial to identify important problems to address, and how to measure the outcomes of solving the problems. Before planning a lesson, it is important to understand student learning readiness through a pre-test or interview with focused students. During the teaching and observation, it is necessary to use certain instruments to capture critical teaching moments and student learning evidence. Immediately after the research lesson, a post-test and/or interview are needed to collect student learning outcomes. During debriefing, based on the collected data, analysis of results should be incorporated for revising the lesson plan for the next cycle of LS.

With regard to promoting the Chinese LS systemwide, some ideas from improvement science (Bryk et al., 2015; Lewis, 2015) and networked improvement

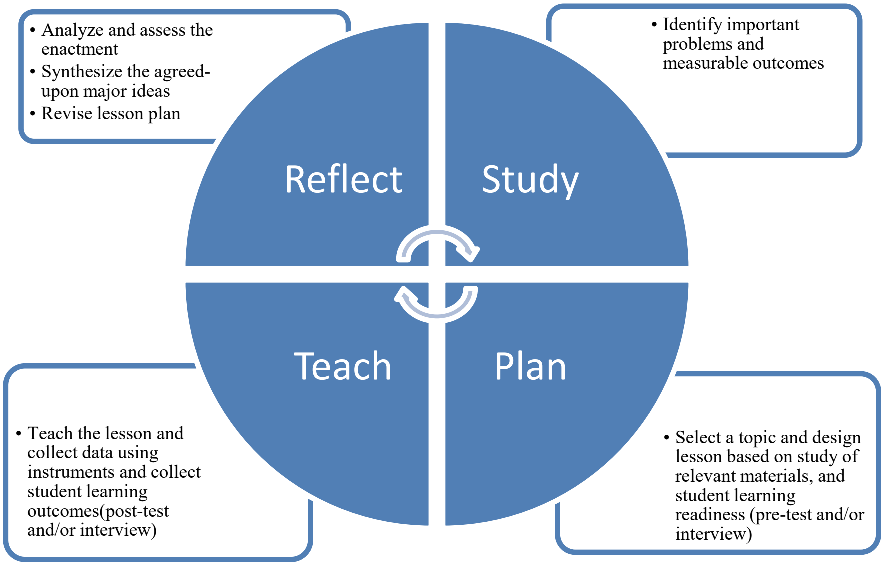

Fig. 1. Enriched Chinese LS process

community (Russell et al., 2017) could be adopted. There are six core principles of improvement. First is to make the work problem-specific and user-centered. It starts with a single question: "What specifically is the problem we are trying to solve?" Second is that variation in performance is the core problem to address. The critical issue is not what works, but rather what works, for whom and under what set of conditions. The third core principle is seeing the system that produces the current outcomes. It is hard to improve what you do not fully understand. It is important to understand how local conditions shape work processes and make hypotheses for change public and clear. Fourth, we cannot improve at scale what we cannot measure. It is important to embed measures of key outcomes and processes to track if change is an improvement. Fifth is to anchor improvement in disciplined inquiry. Engaging rapid cycles of *Plan, Do, Study, Act (PDSA)* make learning faster and improvement quicker. It is not a problem that failures may occur, but it is a problem if we fail to learn from failures. The last core principle is to accelerate improvements through networked communities. We can accomplish more together than even the best of us can accomplish alone.

An examination of PDSA cycles (core principle 5) (Fig. 2) shows that the PDSA and LS cycles are nicely matched. At each phase of PDSA, there is a detailed description about what needs to be done. For example, Plan: analyzing the cause of problem within the system. Act: we have to make explicit the measurable outcome and hypothesis and have a theory in action (protocol). In the context of LS, it is crucial to measure what students learn, and how certain types of intervention link to learning

Fig. 2. Plan-Do-Study-Act (PDSA) circle

outcomes. The PDSA cycle could be repeated to continuously hypothesize and test the improvement. Regarding LS context, building on the product of a cycle of LS (lesson plan and video lessons, measurement and learning evidence), further cycles of LS could continue to address the identified problem. Thus, this type of LS could be conducted across schools in the same district or across districts. In the last principle, the networked improvement communities (NICs) could accelerate the improvement.

There are six principles for building NICs (See Fig. 3), including: understanding the problem, iteratively refining the theory of practice improvement; learning and using improvement methods; utilizing a measurement and analytics infrastructure;

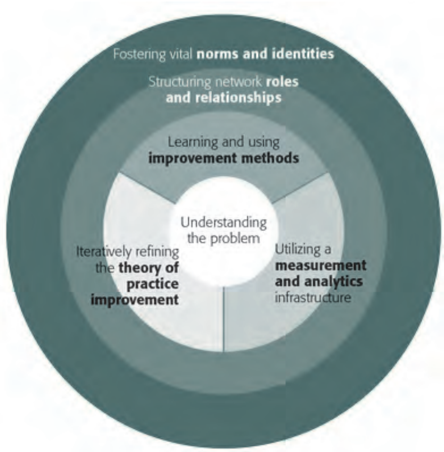

Fig. 3. NICs development framework (adapted fro Russel et al., 2017)

structuring network roles and relationships, fostering vital cultural (norms and identifies (Russel et al., 2017).

Considering the situation of LS in China, there is a culture and a structure (teaching research system and collaborative learning culture), but there is lack of the methodology and theory of practice improvement. The hierarchical and networked teaching research system in China lays a foundation for building networked LS-based improvement communities. If we adopted some principles such as theory of practice improvement and disciplined inquiry, the existing teaching research system could be developed into networked improvement communities (local and nationwide). For example, a school-based LS improvement community could be networked with other school-based LS improvement community within the same district, and the networked improvement communities in a district could be networked across the district, even across cities or regions. Correspondingly, teachers' learning could expand beyond their schools.

In addition, various technologies could be used to strengthen the LS process and develop a networked improvement community. Huang et al. (2021a) proposed a model of technology-assisted LS (Fig. 4). In the special issue on technology and LS (Huang et al., 2021a), studies document the strengths of using technology-assisted LS including: resolving geographical distance; building a productive professional learning community; capturing rich student learning evidence; and promoting deep teachers' reflection. Certainly, technology can assist Chinese LS as well (Huang et al., 2021c). Moreover, an AI-assisted LS model: TEAM model and a Sokrates platform (https://www.habook.com/en/product.php?act=view&id=37) demonstrates its potential to develop a networked improvement community (Huang et al., 2021b). The TEAM model focus on developing smart classroom teaching through four phases: Teaching, assEssing, diAgnosing and reMediation (TEAM) in an online environment based on the TPACK framework (Mishra and Koehler, 2006). The key feature is data-driven decision-making during the teaching. The TEAM model platform includes lesson observation, AI Sokrates analysis, expert

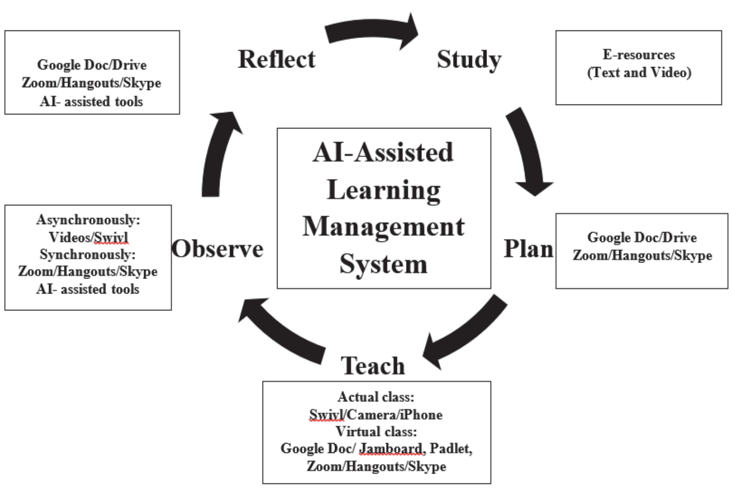

Fig. 4. Technology-assisted hybrid LS (adapted from Huang et al., 2021a)

annotation, and Sokrates cloud. The platform can support LS in several different ways significantly.

First, during the process of observing a research lesson, all observing teachers can enter their observations and comments with regard to use of technology, pedagogy and textbooks through their devices. The platform can collect and analyze all teachers' input automatically. Meanwhile, the research lesson is recorded and analyzed automatically (regarding different types of classroom activities). Immediately after the research lesson, the system generates the analytical information for post-lesson discussion, with the facilitator having the ability to promote the discussions based on analytical data (Fig. 5).

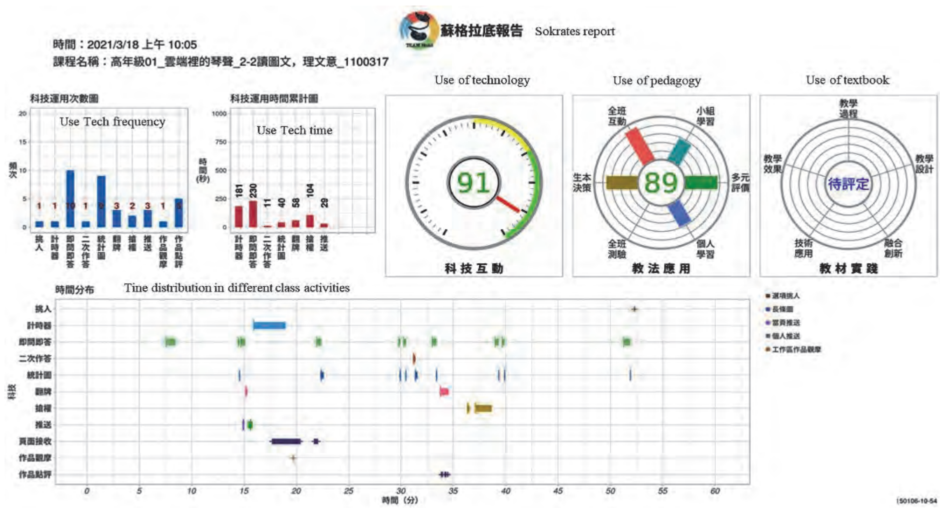

Fig. 5. Dashboad of the Sokrates platform

More importantly, the platform stored relevant data for future sharing.

On the platform, all shared data could be sorted and searched based on theme, subject, grade level, all those lesson study groups are networked. AI-assisted technology may contribute to building true NICS.

Second, both research lessons and participants' comments and analysis are stored in the cloud. Teachers can watch and add comments to the research lessons. On the platform, users can search research lessons based on subject, grade level and teaching research activities (expert comments, self-practicing, professional development analysis report and exemplar lessons). These features of the platform show the possibility for developing a networked LS improvement community substantially at scale although some more adjustments are needed. Examples are how to consider the features of mathematics subject to provide a framework for teachers to enter their subject-specific comments in depth; when emphasizing data-driven, analytical analysis, how to use expertise of facilitators holistically; and when storing the documents, how to make the search more precise and flexible regarding learning goals, student learning difficulties.

4. Implications of Chinese LS for LS Globally

The key features of Chinese LS such as iteration, the involvement of knowledgeable others, focusing on both process and product, and linking theory and practice may provide insight into enrichment of LS around the world. For example, the repeated teaching of the same content (similar to design-based implementation research) (Fishman et al., 2013) has been adopted by UK-research LS which focuses on using multiple cycles of LS with a deep investigation of a group of focus-students. Involvement of the knowledgeable others (or facilitators) in the LS process has been recognized as one of the important factors impacting the success of LS (Takahashi and McDougal, 2016; Seleznyov, 2019). Focusing on both process and products of LS is critical for scaling up LS and building a networked improvement community (Hiebert and Morris, 2011). However, when adopting lessons originated in Asia to other countries, cultural transposition (cultural beliefs, institutional intentions) should be considered (Bartolini Bussi et al., 2017; Ramploud et al., 2022) and necessary modifications made by incorporating local culture and traditions.

5. Concluding Marks

Rooted in Chinese cultural values and supported by the teaching research system and teacher promotion system, Chinese LS has contributed to the improvement of mathematics and science education nationwide over the decades. Meanwhile Chinese LS itself has evolved and developed into new forms and connotations to meet teacher professional development needs in changing contexts. The continuity and change keep the Chinese LS a dynamic and vital professional development vehicle for teachers to meet changing challenges. Chinese LS is local tradition with a long history, but it has evolved to meet the challenges of teacher professional development by taking innovative ideas from the West. At the same, the practice and development of Chinese LS may provide insights into teacher professional development in other countries.

Acknowledgments

I thank Dr. James Calleja from University of Malta, Malta and Dr. Dovie Kimmis from Middle Tennessee State University, USA for their valuable comments and suggestions on an earlier version.

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Introducing the Ladder and Slide Framework: A New Visual Framework of Mathematics Teacher Levels of Integrating GeoGebra in Their Teaching

Kasti Houssam[1]

ABSTARCT The diffusion of technology in the teaching and learning is more complex than other fields. In order to understand the complexity of the factors and processes affecting teachers' integration of technology, in mathematics education in particular, we need to use many complementary lenses. For that end, in this study, we have used three theories: the technological pedagogical content knowledge (TPACK), innovation diffusion theory (IDT), and the zone theory. TPACK describes the types of *knowledge* that teachers need to integrate technology effectively in their teaching practices. IDT describes the developmental *processes* that individuals go through as they adopt/reject a technological innovation. While the zone theory identifies the limiting and assisting *factors* teachers face when they decide to integrate technology in their teaching. The result of this study was a new framework named the *Ladder and Slider* framework to introduce the three theories together using the networking theory. The purpose of the Ladder and Slide framework is to visualize easily the complexity of technology integration, consequently that will influence a better design of professional development. A pilot phase done with four in-service secondary mathematics teachers using GeoGebra in their teaching is presented with the new framework followed by some conclusions and recommendations.

Keywords: GeoGebra; DBR, Zone theory; PD; TPACK; Ladder and Slide.

1. Introduction

The integration of technology by teachers in their lessons depends on many factors. A teacher must have the required knowledge, the availability of some assisting factors and the ability to overcome limiting or hindering factors. In addition, there are many levels for that integration such as recognizing the importance of the technology, accepting it, adapting the lessons using technology, exploring further the role of technology in lessons, and accelerating in the use of technology in teaching. The required knowledge is best reflected by the Technological Pedagogical Content Knowledge (TPACK) (Koehler and Mishra, 2008). The factors, limiting or assisting, are categorized by the zone theory suggested by Goos (2005), and the different levels of technology integration are explained by the Innovation Diffusion Theory (IDT) (Rogers, 1995). As a result, no one theory or framework can capture the complexity of

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integrating technology in lessons. Coordinating and combining many theories will result in better analysis of the issue. For that end, Niess et al. (2009) combined the two theories TPACK and IDT and constructed the TPACK development model.

This study suggests an updated framework under the name of the Ladder and the Slide (LS). This was a result of a PhD thesis that was done over seven years using design-based research over many iterations (Kasti, 2018). The new framework used the coordinating and combining method of networking theories by Prediger, Bikner-Ahsbahs, and Arzarello (2008) to add to the TPACK development method the zone theory as an attempt to fill in the gaps of the "how" and "why" teachers' technology integration level changes.

In what follows a summary about each of the theories.

2. Literature Review

2.1. TPACK

Building on Shulman's (1986) work of PCK, Mishra and Koehler (2006) developed technological pedagogical content knowledge (TPACK) which is the knowledge of how to integrate appropriately certain technological tools to constructively teach a particular domain. As indicated in Fig. 1, to be an effective teacher he/she does not only need to know the content of his subject matter, some pedagogical knowledge, and some technological knowledge but rather the teacher should know how to use the triple knowledge as way to facilitate learning for students to make it more effective, efficient and engaging.

One of the false expectations of change in teachers' professional practice is only increasing their pedagogical knowledge (e.g., Ottenbreit-Leftwich and Ertmer 2010;

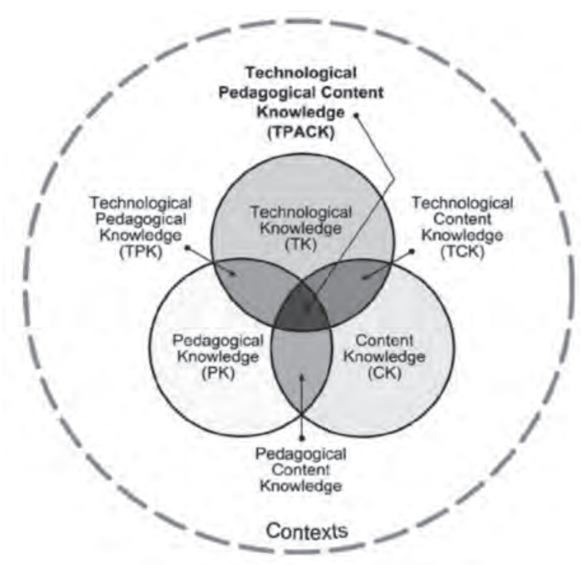

Fig. 1. The TPACK framework (Koehler et al., 2014)

Heitink et al., 2017). One of the reasons could be that, the focus is more on the positive impact of TPACK professional development courses on teacher' perceived TPACK confidence (Chai and Koh, 2017; Doering et al., 2014) and less on the issue of pedagogical change. As Niess (2009) mentioned one of the main gaps in TPACK is being focusing more on knowledge and less on the process, in more details:

> Although the Mathematics Teacher TPACK Standards and Indicators set goals for technology integration, the standards themselves do not provide information on how teachers progressively gain this integrated knowledge for appropriately teaching mathematics with suitable technologies. This recognition raises important questions. How does TPACK develop? Is there a process in which teachers gain mathematics TPACK knowledge? Do teachers suddenly display this knowledge in their professional practice? What is needed is a model that captures the progression of mathematics TPACK as teachers integrate technology into the teaching and learning of mathematics. (Niess, 2009, p. 9)

This gap led Niess et al. (2009) to propose a developmental model for TPACK emanating from Rogers' (1995) model of the innovation-decision process (first introduced in 1962 concerning societal diffusion of innovations). Rogers described a five-stage, sequential process by which a person makes a decision to adopt or reject a new innovation.

In what follows, we will introduce the diffusion of innovation theory then present Niess et al. TPACK development model.

2.2. Innovation Diffusion Theory (IDT)

Roger's (1995; 2003; 2011) Innovation Diffusion Theory (IDT) is the most prominent and widely-used theory to explain the stages an individual, or group of individuals, decides to adopt an innovation (Sträub, 2009). It has been used across disciplines in more than 6000 research studies and field tests to comprehend and predict change, making it the most reliable in the social sciences (Robinson, 2009). An innovation as defined by Rogers (1995) is "an idea, practice or object that is perceived as new by an individual or other unit of adoption" (p. 11). The innovation does not necessarily mean better or objectively new. The decision to adopt an innovation occurs in five phases (Fig. 2).

First is the *knowledge* phase when an individual becomes aware of the innovation through personal experience, mass media or social interactions. This *knowledge* phase can be initiated either through being exposed to the innovation or through necessity. The second stage is *persuasion* whereby an individual has acquired enough knowledge about the innovation to make a judgment about their preference towards it. Third, the individual makes a *decision* to accept or reject the innovation. Fourth, the *implementation* stage is when the individual actually utilizes the innovation. The fifth and last stage is *confirmation* during which the individual reflects on the implementation of the innovation and determines whether they want to continue implementing it or not. (Rogers, 1995, 2003).

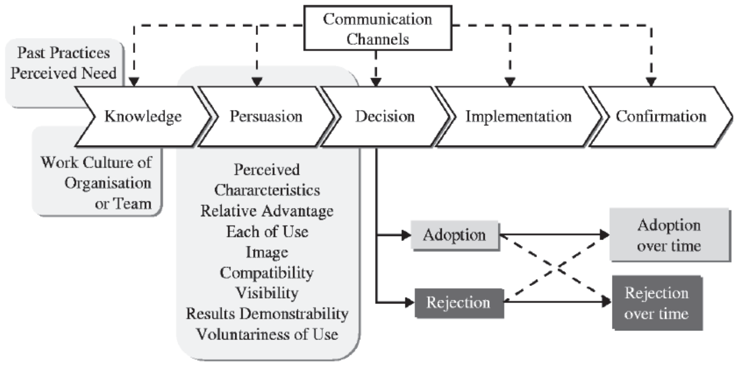

Fig. 2. The IDT framework (Rogers, 2003)

IDT posits that there are four primary components that impact the five stages of adoption just described, including: 1) the innovation, 2) communication channels, 3) the social system, and 4) time. These elements interact with the five stages in a process of diffusion. Niess et al. (2009) networked the TPACK and IDT in one model and called it "TPACK development model" briefed in what follows.

2.3. *TPACK development model*

Niess et al. (2009) combined the four categories of TPACK (Mishra and Koehler, 2006) with the five levels of IDT (Rogers, 2003). They observed many teachers, over a 4-year period, learning about spreadsheets and how to integrate spreadsheets as learning tools in their mathematics classrooms. Analysis of these observations found that teachers progressed through five-stage developmental process when learning to integrate a particular technology in teaching and learning mathematics. They called their new framework the TPACK development model (TDM). The five scale-levels of TDM are (Fig. 3):

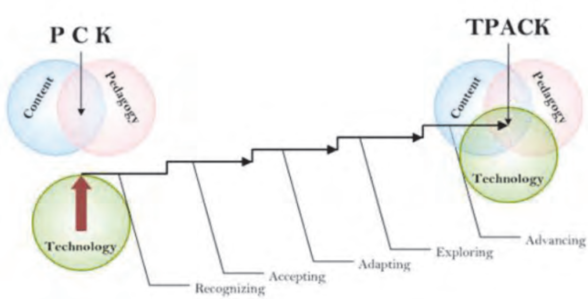

Fig. 3. Visual description of teacher levels in TPACK development model

1. *Recognizing* (knowledge), in which teachers are able to use the technology and recognize the alignment of the technology with mathematics content, yet, do not integrate the technology in the teaching and learning of mathematics.
2. *Accepting* (persuasion), in which teachers form a favorable or unfavorable attitude toward the teaching and learning of mathematics with an appropriate technology.
3. *Adapting* (decision), in which teachers engage in activities that lead to a choice to adopt or reject the teaching and learning of mathematics with an appropriate technology.
4. *Exploring* (implementation), in which teachers actively integrate the teaching and learning of mathematics with an appropriate technology.
5. *Advancing* (confirmation), in which teachers evaluate the results of the decision to integrate the teaching and learning of mathematics with an appropriate technology. (Niess et al., 2009, p. 5)

Niess et al. (2009) described the stages mathematics teachers go through since they decide to adopt certain technology until when they become professionals in using that technology in their teaching. Still the natural question that arises is what happens from one stage to another? What are the assisting and/or hindering factors that come in the way of the teacher when climbing the ladder of technology integration? To answer those questions, we need a third theory namely the zone theory briefed below.

2.4. Zone theory

Goos and Bennison (2008) mentioned that Zone Theory is based on a sociocultural perspective in which learning is viewed as a result of the complex and dynamic interaction among individuals and their environments. Zone theory was initially developed by Valsiner (1997) within the context of child development. This theory is an extension of Vygotsky's (1978) Zone of Proximal Development (ZPD) which is defined as the gap between children's independent ability and the potential performance they can reach with adult guidance or peer collaboration (Vygotsky, 1978). In addition to ZPD, Valsiner's (1997) theory includes two other components, namely: The Zone of Free Movement (ZFM) and the Zone of Promoted Action (ZPA). ZFM refers to environmental restrictions that limit freedom of thought, expression or behaviour; while ZPA refers to the efforts of more experienced individuals in promoting learning (Vasliner, 1997). Within the school context, the interactions between teachers, students, technology, and the teaching-learning environment can be clearly categorised by the zone theory. The ZPD is characterized as the gap between teachers' current technology ability and their ability that can potentially be reached with the help of more experienced individuals. It includes teachers' disciplinary and pedagogical content knowledge and beliefs. ZFM refers to external constraints that limit teachers' use and integration of technology such as student characteristics, curricular and assessment requirements, availability of technological resources and materials. ZFM also includes teachers' own interpretations of the environment which can serve as personal constraints or affordances. Finally, ZPA refers to opportunities that teachers were exposed to through pre-service teacher education or in-service professional development relating to the integration of technology. ZPA can be thought of as professional development strategies. (Goos, 2005; Goos et al., 2007) (Fig. 4). In order for teachers to successfully integrate technology in their classrooms, their ZPA should be within their ZFM and consistent with their ZPD (Goos, 2009). In other words, "…professional development strategies must engage with teachers' knowledge and beliefs and promote teaching approaches that the individual believes to be feasible within their professional context" (Goos et al., 2010, p. 26).

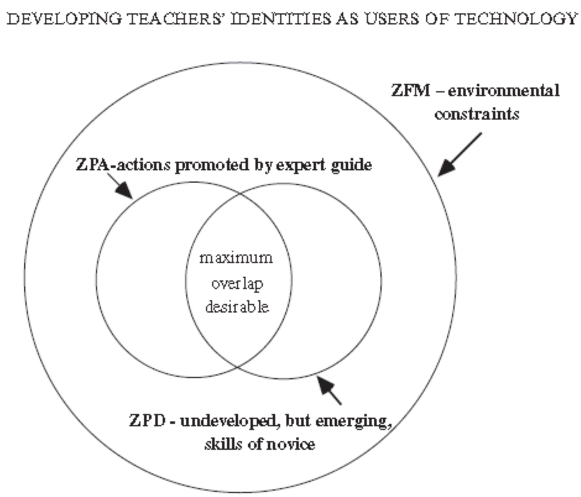

Fig. 4. Relationship between ZPD, ZFM, and ZPA for teachers

In what follows, after we have introduced briefly the three theories and the TPACK development model, we will introduce our new framework that combines the three perspectives (theories) into one visual framework. Then to bring it to life the results of a pilot study with four in-service secondary mathematics teachers will be presented (Kasti, 2018).

3. The Ladder and Slide Framework

Linking TPACK to IDT by Niess et al. (2009) adds to the necessary knowledge teachers must have in order to integrate technology in their classes the integration scale-level. This model still misses how teachers climb the ladder of technology adoption and what are the mediating factors (limiting or assisting) teachers face when they integrate technology in their teaching (Fig. 5). The zone theory suggested by Goos (2005) captures all the factors that mediate technology integration by teachers. Therefore, using the networking theories approach by Prediger, et al. (2008), the well-fitting elements from the aforementioned theories were coordinated and combined. The result is the new framework, namely, the Ladder and Slide (LS) framework.

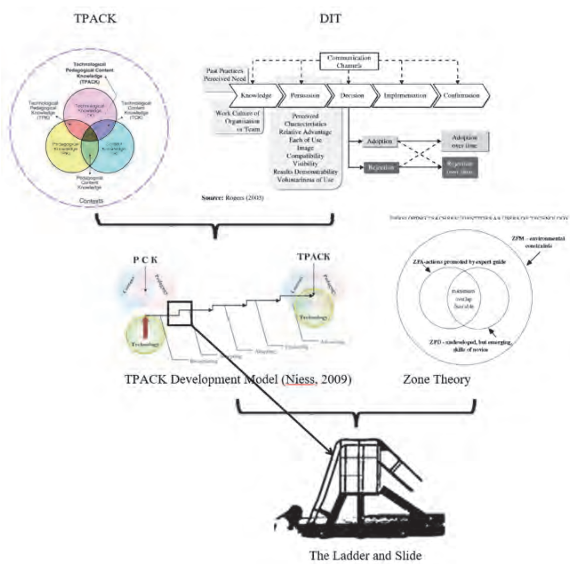

Fig. 5. TPACK development model and the zone theory combined in the new Ladder and Slide framework

Each stage represents one of the five levels described by the TPACK development model (Niess et al., 2009) To go from one integration level to another teachers face a "ladder" that has the following characteristics:

The ladder, as shown in Fig. 6, is made of three main components:

1. the main core represents the ZFM factors teachers face as limiting or assisting factors;
2. the steps represent teachers' backgrounds and beliefs that also can assist them or limit their technology integration. In addition, these stairs represent the ZPD and ZPA of teachers, and
3. the handrail represents collaboration and iteration teachers might get in their transfer from one phase to another.

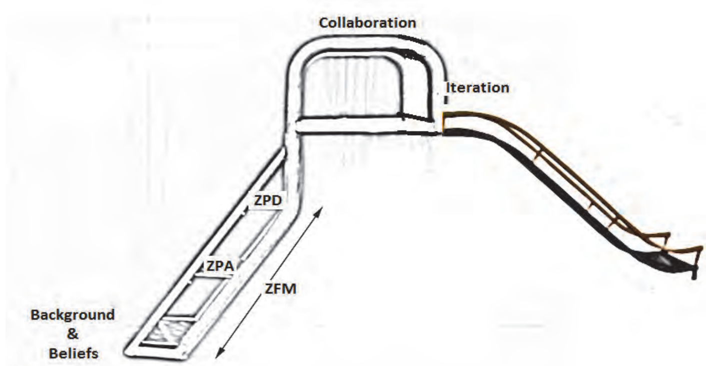

Fig. 6. The Ladder and Slide framework

All the zones play dual roles: limiting and assisting. Hence, that was represented by the ladder in the following way. The core of the ladder represents the assisting role of the ZFM factors whereas the limiting role is represented by the degree of inclination of the ladder; the higher the inclination (slope), the stronger the impact of the limitation. Slope less than one is weak barrier; slope equals to one is moderate barrier and slope more than one is high barrier.

Whereas, the steps represent the assisting role of the background, ZPD and ZPA. They assist teachers in moving up the integration levels. On the other hand, the limiting roles of ZPD and ZPA are represented by the position of the steps; the higher the step, the stronger the impact of the limitation factors. That is, step one is weak barrier, step two is a moderate barrier, and step three is a high barrier.

The handrails represent collaborations and iterations teachers get when moving up the technology integration levels and prevent them from sliding down again. After each level, teachers might slide down (using the slider) to a lesser integration level and that could be due to one or more factors. As the stage increases, the teachers TPACK levels and the extent of using technology in their teaching, increases. In what follows, the results of the pilot phase are represented using the framework.

4. Methodology

The methodology used in the pilot phase study was design-based research with three iterations. Many instruments were used to collect data. The main instrument was an adapted version of Niess et al. (2009). Fig. 7 displays the order of stages that were followed in conducting this research. The study consisted of a pre-intervention stage and an intervention stage.

Stages

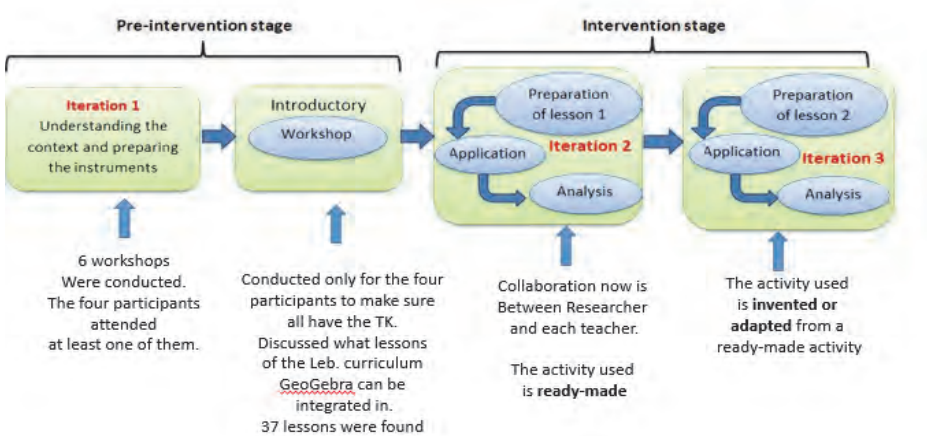

Fig. 7. The research stages as pre-intervention and intervention over three iterations

The collaboration was between the researcher and the four participants as a group during the introductory workshop, afterwards the collaboration was done between the researcher and each of the participants at his/her own school, according to each participating teacher's free time. Each teacher was free to choose the lesson in which GeoGebra will be integrated in. There were two visits for each iteration, one before and another after the intervention (implementing the GeoGebra lesson). Before and after each lesson teachers were interviewed to measure the effect of the intervention on their practices, TPACK and barriers faced. The activity of the first lesson was totally prepared by the researcher but for the second lesson the teachers adapted a ready-made activity or prepared their activity. In what follows, design-based research methodology will be presented.

4.1. Design based research methodology

Wang and Hannafin (2005) define DBR as:

> … a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually-sensitive design principles and theories. (p. 6)

The six basic characteristics of DBR are that it is: 1) pragmatic; 2) grounded; 3) iterative, and flexible; 4) interactive; 5) integrative; and 6) contextual (Wang and Hannafin, 2005).

DBR differs from traditional experimental designs in that it does not occur in controlled settings, but rather in naturalistic settings where myriad systemic variables are taken into account (Collins, 1992). The focus on the evolution of design principles differentiates DBR from action research and formative evaluation designs in that "the design is conceived not just to meet local needs, but *to advance a theoretical agenda*, to uncover, explore, and confirm theoretical relationships" (Barab and Squire, 2004, p. 5). Based on that the design-based research methodology was found perfectly fit for the study. The selection of participants in this pilot phase will be explained next.

4.2. Participants

The pilot study was with four participants, three females and one male. All four teachers were in-service secondary mathematics teachers. The participants were among many secondary teachers who have attended some previous workshops on the use of GeoGebra in teaching and accepted the invitation to participate in the study. But the four were particularly chosen so they differ in one or more of the zones levels. The demographics of the four participants and their starting data are summarized in Tab. 1. Teachers were interviewed and their use of GeoGebra software in their teaching was recorded; in addition, factors that limited their technology integration was recorded and grouped in terms of zones.

Tab. 1. Participant demographics, practice and zones starting level

Name	Age	Highest degree	Teaching experience	Practice level	ZFM	ZPA	ZPD
Amani	50−55	BS	25 years	Low	Moderate	Moderate	Low
Tima	23−26	Masters +TD	2 years	Moderate	Low	Moderate	N/A*
Sara	33−40	BS	7 years	Moderate	Moderate	Low	N/A
Hazem	41−50	Masters	31 years	High	Moderate	N/A	N/A

**Note.* N/A = The zone was not considered a barrier to *GeoGebra* integration.

All names are psudonames; practice level means how often did the participants use GeoGebra in their teaching; the zones were taken in the limiting sense. That is, a "Moderate ZFM" means that the teacher faced some limiting factors related to the zone of free movement. Those levels were set by the study instruments that will be listed and explain next.

4.3. Instruments

This study was an intervention study that followed a DBR methodology. It was implemented in two phases: pre-intervention and intervention, with each phase having its own set of instruments. All of the instruments were adapted by the researcher from previously tested instruments and based on multiple well-known theories. In addition, they were all administered on an individual basis with each participant in his/her school and were tape-recorded after the participant's approval. For further details it is advised to check the complete work (Kasti, 2018).

5. Results

The results of the pilot study using the Ladder and Slide framework is reported in what follows on individual bases.

5.1. Amani and the LS framework

Amani started the intervention considering herself as having the necessary skills (assisting ZPD) to integrate technology in her classroom, but not enough knowledge and confidence to do so (limiting ZPD). She did not receive any related preparation to integrate technology in her teaching as part of her university degree, school preparation or professional development (limiting ZPA) and had a high limiting ZFM. After implementing her first lesson, (Fig. 8), she changed her teaching methodology to be more student-centred than teacher-directed. Amani got encouraged by her students' motivation and her colleagues and administration (assisting ZFM) after conducting the activities. She still had some ZPD problems related to knowledge of the software which she overcame after her second implementation.

Furthermore, after her second implementation, Amani progressed to the highest level (*advancing* level), overcoming most of her ZFM limiting factors and assisted by her ZPA and ZPD factors in collaboration with the researcher. The major impact of the

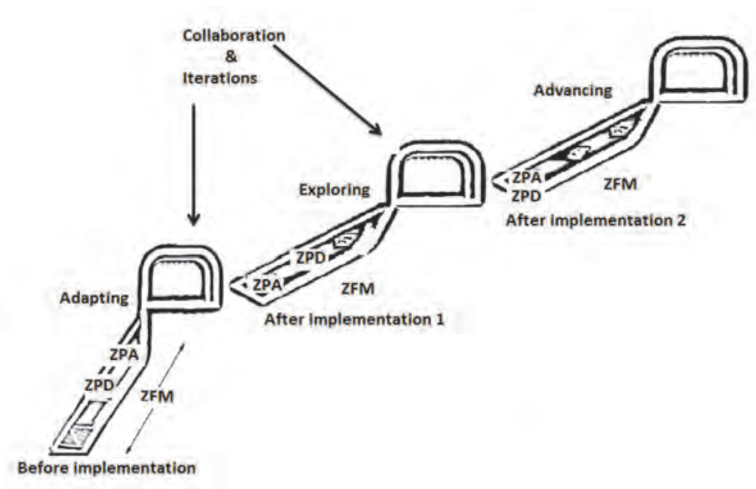

Fig. 8. Amani's story according to the LS framework

intervention on Amani is the increase of self-confidence. In addition, the follow-up drove her to be more committed and apply what she has learned.

When asked why she did not apply what she had learnt from the workshops she attended, she replied:

> I didn't apply due to many reasons (personal and availability of hardware) but now I am happy exploring things and learning a lot. Your presence made me work because I felt more confident, secure, and somehow ethically obliged. (Interview 1, October 17, 2015)

Amani learned a lot from the workshops she attended with the researcher, but that did not make her feel ready enough for *GeoGebra* integration. While immediately after collaborating with the researcher in preparing, implementing and assessing her first lesson, her assistive ZPA increased and continued to increase after the second lesson.

In asking about Amani's evaluation of the whole experience, she said:

> I am very happy, really happy, I enjoyed and enjoying a lot the new way [use of GeoGebra to introduce lessons], even that it is taking time and effort I am seeing videos and trying to learn more. I am enjoying (it) a lot; (I'm) really happy. (Interview 3, December 5, 2015)

5.2. Tima and the LS framework

Tima was a newly graduated teacher with only two years of experience in teaching mathematics for the intermediate and secondary classes. She found herself ready in terms of skills, knowledge and confidence to integrate technology in her classroom (assisting ZPD) despite the fact that she did not receive enough preparation on technology integration in teaching as part of her university degree, school development, or professional development (limiting ZPA; Fig. 9).

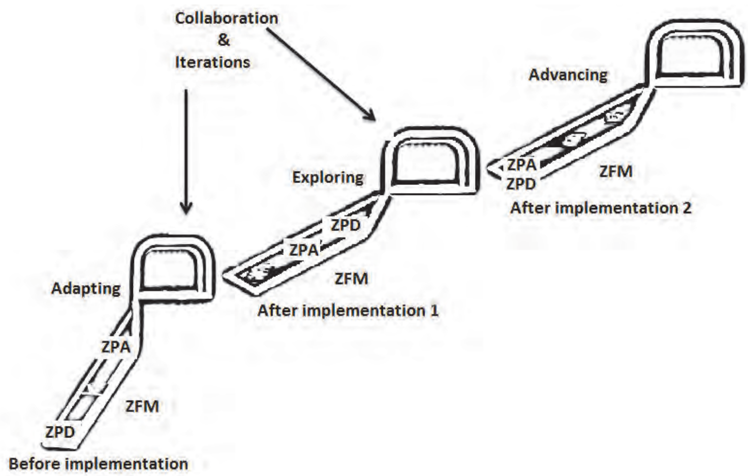

Fig. 9. Tima's story according to the LS framework

The intervention lessons were implemented in her school computer lab, but with old hardware and software and without receiving any encouragement from the administration (limiting ZFM). In the process of the intervention, Tima learned a lot and she overcame most of the barriers she faced. What hindered Tima's integration of *GeoGebra* in her teaching was her lack of knowledge of the proper methods to do that and the knowledge of the lessons to which she could apply it. Regularly, a workshop will solve this knowledge problem, but a new teacher like Tima needs scaffolding to gain more confidence. Hence, iterations and collaborations solved this big gap and allowed Tima to reach the *exploring-advancing* level in which she adapted *GeoGebra* for continuous use in her teaching.

After the intervention, Tima continued to use *GeoGebra* in her teaching. She stated:

> I sensed really it is powerful and now I am really a strong believer of the importance of using GeoGebra in my teaching (and I will) in all the grades I teach (intermediate and secondary). I want to teach every lesson that can be taught using GeoGebra. I am intending to do it. I was honored to be part of your team and learn from you. Now anything I am teaching exercises and problems I am using GeoGebra, everything is clear and easy. I even taught a colleague of mine how to draw using GeoGebra. (Interview 3, February 22, 2016)

5.3. Sara and the LS framework

Sara (Fig. 10) had seven years of experience in teaching mathematics for the intermediate and secondary classes. She found herself ready in terms of skills, knowledge and confidence to integrate technology in her classroom (assisting ZPD) despite the fact that she did not receive enough preparation on technology integration in teaching as part of her university degree (limiting ZPA). In her school, she had no computer lab, no hardware to be used in class, and no applicable software. She got no encouragement from her colleagues or administration and no technical support. The first implementation in this study was her first time she tried a discovery activity done by students in the computer lab. Students' motivation played an assistive role (assisting ZFM) and encouraged Sara a lot. She still had some ZPD problems related to knowledge about the software that she overcame after her second implementation. In the second implementation, students were not as motivated, so her expectations decreased (limiting ZFM). Her ZPA and ZPD were overcome as limiting factors and her ZFM became less steep as a limiting factor (Fig. 10) and she progressed to the highest level (*advancing* level).

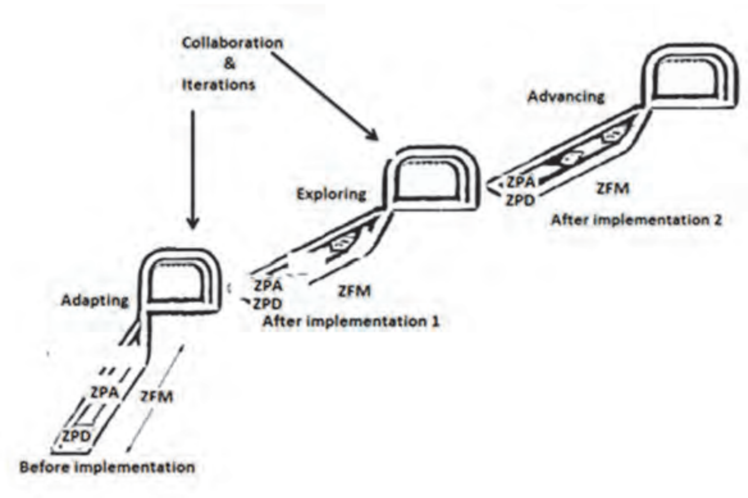

Fig. 10. Sara's story according to the LS

Sara could sense the difference between presenting her lesson using *GeoGebra* and letting her students explore the lesson working with *GeoGebra*, she said:

> There is a difference between when things are done by students is totally different from seeing things. The instructions were very well organized. I was impressed I felt the importance but without the availability things would be impossible or more difficult, to discover and to experience to get IT skills and math thinking skills is impossible without working in the lab. (Interview 2, November 7, 2015)

5.4. *Hazem and the LS framework*

Hazem (Fig. 11) has 31 years of experience in teaching mathematics for the intermediate and secondary classes. He found himself ready in terms of skills, knowledge and confidence to integrate technology in his classroom (assisting ZPD) since he has a degree in computer science (assisting ZPA). He sees students' motivation, curriculum requirements and assessment policies as the most common barriers to technology integration in his class (limiting ZFM). He overcame the availability by asking students to bring their pads or tablets with them at all times. It is

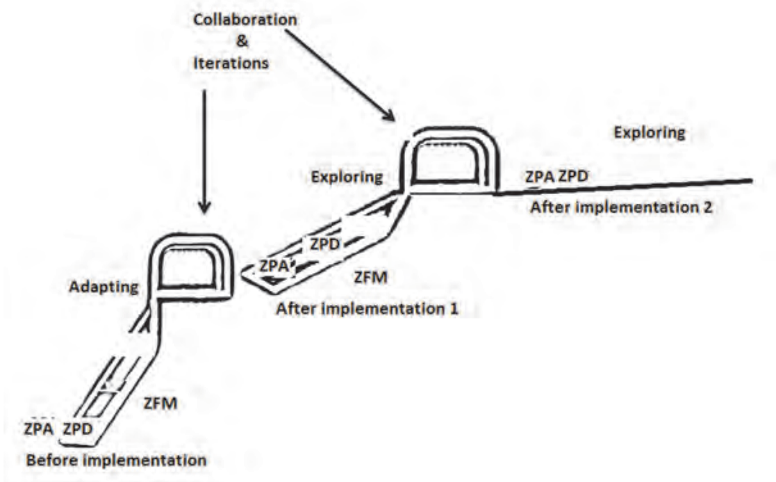

Fig.11. Hazem's story according to the LS framework

evident from Fig. 11 that Hazem smoothly started his *adapting* level with just some ZFM barriers, but after his first implementation, his ZPD appeared as a limiting factor. He identified his need to know more about GeoGebra. Where as the ZFM barriers were being surpassed. After his second implementation, he reported that he was not at the *advancing* level for many reasons some of which were the ZFM barriers he mentioned before.

Hazem tried to overcome *availability and accessibility to hardware* by asking students to bring their own tablets or laptops. He said:

> Class is more interesting with technology but accessibility to computers was a factor (barrier) because GeoGebra is easier to use on computers than on tablets … What helped me was some students were motivated and they shared it, in general it was fine (the activity). (Interview 2, February 11, 2016)

6. Discussion

From the results shown by the Ladder and Slide framework reported for the four in-service secondary mathematics teachers we can deduce the following:

First, the fact that all the teachers in this study reported less ZFM barriers after the intervention (by seeing the slope of the slide getting less in all the four cases) means that the intervention played a role in decreasing the limiting effect of ZFM and that they were able to overcome some barriers. It may also be indicative of the fact that teachers may have anticipated encountering certain inhibiting factors (before the intervention); however, when they had hands-on experience with technology integration, they may have found that those were not the real barriers, but rather some other unanticipated ones. In fact, all of these have been mentioned as common barriers to technology integration in numerous previous research (e.g., Bingimlas, 2009; Chen, 2008; Forgasz, 2006; Lim and Khine, 2006; Oncu et al., 2008). After the intervention, the number of ZFM barriers mentioned decreased to only three which were accessibility and availability of hardware and student motivation. Accessibility and availability of hardware has been consistently reported by in-service teachers as a significant barrier to technology integration (Earle, 2002; Forgasz, 2006; Hew and Brush, 2007; Oncu et al., 2008).

Second, with regard to teachers' ZPD, the teachers were least likely to report that a gap in their knowledge, skills or confidence was an inhibiting factor. This we can see that all four participants one had their first step in the ladder a ZPD (Only one in the second step) and then we can see with iterations the step have changed. This is in contrast with previous research which reveals that teachers' lack of confidence and skills in using technology and their lack of technology knowledge are important factors in whether or not they choose to integrate technology (Boris et al., 2013; Forgasz, 2006; Lim and Khine, 2006). However, after the first implementation, the teachers of this study realized that they needed additional knowledge and skills in *GeoGebra*. This was remedied after the second implementation whereby teachers were somewhat able to fill those knowledge and skill gaps as well as boost their confidence through collaborations with the researcher.

Finally, in terms of ZPA only one participant started with low step in his ladder whereas the others started either high or medium height. That is due to a lack of adequate or sufficient training in technology integration from the findings of this study which indicated that three of the teachers reported not to have had adequate training or experience with technology integration, in general, and *GeoGebra*, in specific. This is consistent with other studies that mentioned this factor as the top most cited issue hindering teachers' use of technology (Bingimlas, 2009). Interestingly, despite the fact that the teachers reported not having adequate training, they still indicated that their knowledge and skills were the least inhibiting factors in their technology integration. Teachers' lack of training and competence in technology integration is in fact directly related to their knowledge, skills and confidence (Becta, 2004; Bingimlas, 2009).

Analyzing these results clarifies the interconnectivity among the zones and how it is best to work on all of them simultaneously. This study also highlights the importance of working in teachers' own environments or contexts. In line with this, Goos (2013) conducted a study where the researcher's role was that of a facilitator of change within teachers' zone systems. Goos (2013) maintains that:

> Focusing on the person-in-practice allows for interpretation of knowledge and beliefs within teachers' professional contexts, while refocusing the lens on the practice-in-person shifts attention to identity formation as practice changes the person. (p. 532)

7. Recommendations

The visualization of data on teachers' zones and TPACK stages as it relates to integrating GeoGebra in teachers' practices, in particular, and any other technology in general, can be a good ground for determining the real reasons behind adopting or rejecting technology in teaching. Moreover, this study highlights the importance of considering various individual and contextual factors that impact mathematics teachers' technology integration. It is insufficient to address teachers' needs by tackling only one type of barrier since all the barriers are interconnected. Consequently, better decide how to prepare differentiated professional development. Those professional development modules should cater to teachers' instructional, curricular and pedagogical needs and beliefs, as well as to be coherent with their classroom and school context. In addition, it could point out what type of follow up is needed to insure the change in teachers' TPACK and practices regarding teachers' technology integration.

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https://doi.org/10.1142/9789811287183_0018

18

What Can History Do for the Teaching of Mathematical Modeling in Scientific Contexts?

Tinne Hoff Kjeldsen[1]

ABSTRACT We will explore the role of history as a resource through which students can gain experience with authentic mathematical modeling in scientific contexts i.e., when mathematical modeling is used as a research tool, a practice, to gain knowledge in other areas. Three modeling episodes from the 20th century will be presented and analyzed with respect to modeling strategies, practices, items used in the modeling construction and cross-disciplinary epistemic issues — and an analytical framework for analyzing modeling episodes in scientific context will be presented. The framework will be discussed with respect to the modeling cycle in mathematics education, highlighting issues in the framework which are not featured explicitly in the modeling cycle. It will be illustrated how and in what sense history makes it possible to invite students into the work place of scientists that used and experimented with mathematical modeling as a research practice, i.e. its significance in creating such teaching and learning environments. Finally, the value of developing students' historical awareness for preparing them for tertiary studies where mathematical modeling might play a role will be discussed.

Keywords: History of mathematics; Mathematical modeling; Mathematics education.

1. Introduction

History can serve a variety of purposes in mathematics education[2] — one of them, which is the focus of the present talk, is to provide a window for students into "mathematics in the making" so to speak (Kjeldsen, 2018). In the following, this role of history will be explored with respect to the teaching and learning of mathematical modeling in scientific contexts — and with this I mean, when mathematical modeling is used as a research tool, as a research practice, to gain knowledge in other areas.

On the one hand, mathematical modeling has come to play an important role in scientific practices during the 20th century, and modeling has also by now been included in mathematics curricula in many countries (Blum et al., 2007). Being aware of the role of mathematical modeling in scientific context, both in mathematics

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[2] For a recent overview, see Clark et al. (2020).

education and in other science educations, might open students' eyes for the possibilities, mathematical modeling has to offer in other disciplines as a research tool.

On the other hand, this is neither easy nor unproblematic. Scientists from different disciplines have different views on what counts as a "good" model, as a "good" explanation, as useful knowledge and what is relevant knowledge. However, to become acquainted with such differences in high school and undergraduate science education, might encourage and prepare students for interdisciplinary studies and collaboration.

In the following, I will argue that and illustrate how history can contribute to making students aware of issues of how mathematical modeling can function as a research practice in science. First I will present and analyze three modeling examples from the history of mathematics from the 20th century: John von Neumann's model of general equilibrium in economics, Vito Volterra's predatory-prey model, and Nicolas Rashevsky's model of cell division. In each of these cases there were reactions and discussions from scientists from the target domain. The work of these three authors together with the reactions and discussions with scientists from the target domain, provide a possibility to bring the actors' "voices" into the classroom. Or, if we look at it from the teaching side, a possibility to invite students into the work place of scientists that used and experimented with mathematical modeling as a research practice. Secondly, I present a framework for analyzing and comparing modeling episodes in scientific contexts to understand modeling strategies, practices and cross-disciplinary epistemic issues.[3] I will discuss the analyses with respect to the modeling cycle in mathematics education, pointing out shortcomings in the sense of elements in the framework, which are not featured explicitly in the modeling cycle. A student project work will be presented to illustrate how students, working with the Rashevsky case, were invited into an authentic modeling workshop. Finally, the value of developing students' historical awareness for preparing them for tertiary studies where mathematical modeling might play a role will be discussed.

2. Case 1: John Von Neumann's Model of General Economic Equilibrium

John von Neumann was born in Budapest in 1903. He immigrated to the USA where he became a professor at the Institute for Advanced Study in Princeton. In 1932 he gave a talk in Princeton where he presented a mathematical model in economics. It was published five years later in Karl Menger's *Ergebnisse eines mathematischen Kolloquiums*. In 1945 it was translated into English with the title *A Model of General Economic Equilibrium*. Von Neumann considered a general economy where there are n goods $G_1, \dots, G_n$ which can be produced by m processes $P_1, \dots, P_m$. He asked the question "Which processes will be used (as "profitable") and what prices of the goods will obtain?" (von Neumann, 1937, p. 75).

He mathematized this economy by setting up a system of six linear inequalities that express relationships between the intensities of the processes, which are

[3] This framework was first presented in Jessen and Kjeldsen, 2021.

represented by $x_1, \dots, x_m$ and the prices of the goods, which are represented by $y_1, \dots, y_n$ (von Neumann, 1937, p. 75−76). The two parameters α and β represent the expansion factor and the interest factor, respectively. The coefficients a_{ij} and b_{ij} represent the amount of the good G_j used in the process P_i, and the quantity of the good G_j produced by the process P_i, respectively.

$$x_i \geq 0$$
$$y_j \geq 0$$
$$\sum_{i=1}^{m} x_i > 0$$
$$\sum_{j=1}^{n} y_j > 0$$
$$\alpha \sum_{i=1}^{m} a_{ij} x_i \leq \sum_{i=1}^{m} b_{ij} x_i \text{ for all } j$$
$$\beta \sum_{j=1}^{n} a_{ij} y_j \geq \sum_{j=1}^{n} b_{ij} y_j \text{ for all } i$$

The first four inequalities are self-explanatory, the fifth one makes sure that we do not consume more of the good G_j than is produced in the economy. And the last one means that there is no profit in the system — everything gets re-invested.

In order to solve this model von Neumann wanted to investigate whether a solution exists, and he was able to prove the existence of a solution to the inequality system. In order to do so, he first transformed the problem into a problem of a saddle point for a certain function. He then proved a new mathematical result, a generalization of Brouwer's fixed point theorem, which he then used to prove the existence of a saddle point and thereby the existence of a solution to the inequality system (see Kjeldsen 2001). With this result, von Neumann had proved that such a general economy has an equilibrium.

It was not a constructive proof it was an existence proof, so von Neumann did not construct a solution. In 1945 when the paper was translated into English by the British economist David Champernowne, Champernowne wrote a note where he raised several critical aspect to the use of von Neumann's model in economics:

"Approaching these questions as a mathematician, Dr. Neumann places emphasis on rather different aspects of the problem than would an economist. […] The paper is logically complete […]. But at the same time this process of abstraction inevitable made many of his conclusions inapplicable to the real world […] the reader may begin to wonder in what way the model has interesting relevance to conditions in the real world. […] utmost caution is needed in drawing from them any conclusions about the determination of prices, production or the rate of interest in the real world." (Champernowne, 1945, pp. 10–15)

As Chapernowne's warning here indicates there is not necessarily an agreement about what constitutes a "solution", when mathematics is used in scientific practices in other domains — it is context dependent. The disciplinary lens that is used, especially when new modes of inquiry are under development, plays a significant role in the acceptance or non-acceptance of modeling results. Despite Champernowne's critic, and other critical voices in economics, von Neumann's paper is considered a rather important paper, and it has played a significant role in the development of theory in economics, see e.g. (Dore et al., 1989). It is an example of models as elements of economic theories. The case also illustrates that how modeling and models are perceived and judged depend on the recipient's conception of the purpose: for von Neumann, the purpose was to prove consistency; does an equilibrium, a solution to the linear inequality system, exist or not? That was the interesting question. Champernowne though, wanted to solve concrete (economic) problems in practice and here von Neumann's model didn't really help.

3. Case 2: Vito Volterra's Predatory-Prey Model

The second case is Vito Volterra's now well-known predator-prey model. Volterra was born in Italy in 1860. He became a professor of mechanics at the University of Turin, and of mathematical physics at University of Rome in the year 1900. He was asked by the biologist, Umberto D'Ancona, if he could explain the observation that the reduced fishing in the Upper Adriatic during WWI, in contrast to what one might think, apparently was more favorable for the predator fish than for the prey (Volterra, 1926). Volterra approached the phenomenon as if it was a problem in mechanics by e.g. neglecting friction from the environment. He explained his approach in a paper published in 1927 where he wrote that:

> To facilitate the analysis it is convenient to present the phenomenon schematically, by isolating those factors one wishes to examine, assuming they act alone, and by neglecting the others. [...] I have started by studying only the intrinsic phenomena due to the voracity and fertility of the coexisting species. (Volterra, 1927 [1978, p. 68])

Volterra only took the predatory and fertility into account, and constructed a hypothetical system based on these two kinds of events. He further assumed that the two populations of fish developed continuously, because he wanted to use the theory of differential equations in his modeling. He further assumed that the birth rate of the prey (ε_1) is constant, so they grow exponentially if they live alone, and he assumed that the number of predators will decrease exponentially in the absent of prey, with ε_2 denoting the death rate. To model the predation, he used a mechanical analogy, which he called the "method of encounters". He envisioned that encounters between two competing species, N_1 and N_2, occur at random as with particles in a perfect gas in a closed container, so the predation is proportional to the product of the numbers of species, that is their densities, so to speak. Based on this analysis, he derived the differential equations below and we can see that in the second set of equations, the encounters between the species are implemented. These are now known as the “Lotka-Volterra” equations (Volterra1927 [1978, p. 80, 95]):

$$\frac{dN_1}{dt} = \varepsilon_1 N_1, \quad \frac{dN_2}{dt} = -\varepsilon_2 N_2,$$

$$\frac{dN_1}{dt} = (\varepsilon_1 - \gamma_1 N_2)N_1, \quad \frac{dN_2}{dt} = (-\varepsilon_2 + \gamma_2 N_1)N_2,$$

Volterra represented the solutions in a graph, showing the now well-known periodic behavior of the two populations, which his model was able to capture. His model was also able to account for the observation of D’Ancona, that the reduced fishery during World War I was more favorable for the predatory fish than for the prey fish.

Volterra was concerned about the validation of his model through empirical data. But here D’Ancona had a different point of view, which he expressed very clearly in a letter he wrote to Volterra in 1935, where he stated that:

> My observations [of the fisheries in the Upper Adriatic] could be interpreted in the sense of your theory, but this fact is not absolutely unquestionable: it is only an interpretation. …You should not think that my intention is to undervalue the experimental research supporting your theories, but I think that it is necessary to be very cautious in accepting as demonstrations these experimental researches. If we accept these results without caution we run the risk of seeing them disproved by facts. Your theory is completely untouched by this question. It lay on purely logical foundations and agrees with many well-known facts. Therefore, it is a well-founded working hypothesis from which one could develop interesting researches and which stands up even if it is not supported by empirical proofs (D’Ancona to Volterra 1935, quoted from Israel, 1993, p. 504).

There is a discussion of epistemic value in this letter. D’Ancona was, which he expressed very clearly, of the opinion that the exploration of a mathematical model that has been derived from a concrete phenomenon though based on crude, simplifying hypotheses and idealizations, can lead to new (valuable) insights even if the model cannot be confirmed by data. D’Ancona’s view in this matter has been interpreted by the Italian historian of science Georgio Israel (Israel, 1993) as indicating a shift towards a more modern abstract conception of modeling.

4. Case 3: Nicolas Rashevsky’s Early Model on Cell Division

The third and last case is Nicolas Rashevsky’s early work on cell division. He was born in 1899 in Chernigov in Ukraine. He held a doctorate in theoretical physics from the University of Kiev. He immigrated to the USA, first to Pittsburg in 1924 where he came to work at the Research Department of the Westinghouse Electric and

Manufacturing Company. In 1934 he moved to University of Chicago, to academia, through a fellowship from the Rockefeller foundation (Abraham, 2004).

Rashevsky's ambition was to build a mathematical biology on a physico-chemical basis — he wanted to mimic the development of physics based on mathematics. He saw the role of mathematics as a 'gateway', he said, to the "hidden fundamental properties of nature" (Rashevsky, 1935, p. 528). Rashevsky was very much outspoken and he wrote many papers where we can follow his modeling and also his thoughts and philosophy of mathematical modeling. Here is how he expressed it in Nature in 1935:

> … very little attempt has been made to gain an insight into the physico-chemical basis of life, similar to the fundamental insight of the physicist into the intimate details of atomic phenomena. Such an insight is possible only by mathematical analysis; for our experiments do not and cannot reveal those hidden fundamental properties of Nature. It is through mathematical analysis that we must infer, from the wealth of known, relatively coarse facts, to the much finer, not directly accessible fundamentals. (Rashevsky, 1935, p. 528).

Here he was questioning the experimental practice in biology. He wanted to have a more theoretical practice, and he promoted the use of what he called 'paper and pencil models', which he had explained in the journal *Physics* in 1931:

> … a physicist has enough confidence in the results of his calculations, that he does no need actually to build a model, and may satisfy himself by investigating mathematically, whether such a model is possible or not. The value of such "paper and pencil" models is not only as great as that of actual "experimental" models, but in certain respects it is even greater. The mathematical method has a greater range of possibilities, than the experimental one, the latter being often limited by purely technical difficulties. (Rashevsky, 1931, p.143–153)

Rashevsky presented his model of cell division to the biologists at a symposium that was held on Long Island, New York in 1934. Here he confronted the biologists, asking the question: "Do we need to assume some special independent mechanisms to explain cell division?" And he gave the answer: No. "Cell division can be explained as a direct consequence of the forces arising from cell-metabolism" (Rashevsky, 1934, p. 188). He also gave the 'recipe' for how that can be done, namely logically and mathematical from a set of well-defined general principles. … and he claimed the superiority of mathematics:

> … it is only natural to assume that the lack of our knowledge of the fundamental causes of biological phenomena, in spite of the tremendous amount of valuable facts, is due to the lack of use of deductive mathematical methods in biology. (Rashevsky, 1934, 188−198).

In his modeling, he drew an analogy to work, he had done while he was at Westinghous. There he had worked on dynamics of colloid particles and division of droplets. He linked this to cell division by conceptualizing a cell as a physical system which is liquid, and from this he derived that, due to metabolism:

"… there will be a difference in concentration outside and inside the system, the concentration outside being greater. […] We have to do with a phenomenon of diffusion governed for a quasi-stationary state by the equation

$$D\Delta^2 c = q(x, y, z)$$

where *D* denotes the coefficient of diffusion, *c* the concentration, and $q(x, y, z)$ the rate of consumption of the substance …" (Rashevsky, 1934, p. 189).

This was the first step in his modeling. The next step was to investigate at the level of molecules, so he derived expressions for the forces produced by a gradient of concentration. He calculated the force exerted on a molecule (A) of the solvent by all molecules (B) of the solute. By integration, he derived an expression f_s for the force exerted on each element of volume of the solvent by the solute, and he also calculated the force f_o acting on each volume as a result of osmotic pressure, and the force f_r of repulsion between molecules. He summarized, saying that:

> …we see that a gradient of concentrations produces a force per unit volume which is the sum of the above three forces $[f_s + f_o + f_r]$ (Rashevsky, 1934, p. 191).

The third step in his modeling was to make further idealizations. He assumed that the cells were homogenous and spherical. He emphasized that he was aware that this is not how cells look like, but that this idealization would give a general qualitative picture. He calculated that under these assumptions, when a cell divides, the volume energy will decrease and the surface energy will increase, and for large radius of the cell, the increase will be less than the decrease. Then he invoked the principle of free energy from physics, and argued that:

> As any system tends to assume such a configuration, for which its free energy has the smallest value possible, one is tempted to infer that … division of a cell will occur spontaneously as soon as, … the cell will exceed the critical size. (Rashevsky, 1934, p. 192).

Well, he was well aware, as he said, that "unfortunately […] things are not so simple", so he only concluded that:

> every cell, by virtue of the processes of metabolism … contains in itself the necessary conditions for spontaneous division above a certain size. (Rashevsky, 1934, p. 192).

The talks from the symposium are published in a proceedings together with the discussions after the talks, so we can follow the discussion between the biologists and Rashevsky after Rashevsky's talk. It is quite clear from the discussion that the biologists didn't approve of Rashevsky's method. They wanted to know what "example in nature would be nearest to this theoretical case?" (Rashevsky, 1934, p. 195). If we look at Rashevsky's assumptions, we can see that they range from speculation to adjustments for the sake of mathematics and the biologists questioned all these assumptions.

From Rashevsky's point of view, his modeling fulfilled his purpose of investigating possible explanations for cell division by deducing consequences and compare them with empirical results. But it did not fulfill the biologists' purpose. They wanted to know the mechanism of cell division not all kinds of imaginative possibilities. According to the biologists, Rashevsky's approach lacked reality and experiments. So here, we see very clearly this clash of practices across disciplinary boundaries with biology on the one side and the mathematical physical domain on the other side (Kjeldsen, 2019)[4].

5. Analysis and Comparison of The Cases: Developing A Framework

Tab. 1 (on the next page) represents a (preliminary) analytical framework for analyzing and comparing modeling episodes in scientific contexts to understand modeling strategies, practices and cross-disciplinary issues. The framework is constructed with inspiration from Boumans' (2015) "model" for modeling, Gelfert's (2018) ideas of explorative modeling supplemented by the notions of modeling strategy and epistemic value, issues which were illuminated by the analyses of the historical actors' actions and disputes in the three episodes. To be more specific: the cases are analyzed and compared with respect to "meta aspects", "items used in the modeling construction" and "explorative model function". The meta-aspects are divided into "motivation", "strategy" and "discussion of epistemic value". The items used in the modeling construction that have been identified in the analyses are "analogies", "mathematical concepts and theories", "theoretical notions from other areas" and "empirical data". Finally, the explorative model functions are the three functions "starting point", "proof of principle", and "possible explanation" as identified by Gelfert. They will be explained below.

Regarding meta-aspects, the analyses of the three cases with respect to the motivation, strategy and the discussions of epistemic values for the modeling, we find that there are various differences. In von Neumann's case, the motivation was to develop theory, in Volterra's case, it was to explain an observed phenomenon, a pattern, and in Rashevsky's case, the purpose of the modeling was to search for causality, to explain cell division in terms of physics and chemistry. If we look at the strategies, we see that von Neumann created and analyzed an abstract mathematical structure, which was his model. Volterra, he developed a hypothetical system from the kinematics of gases, and Rashevsky conceptualized a cell as a liquid system that transforms substances. Comparing the discussions of epistemic value of the model results, we found that in von Neumann's case, there was a discussion between the epistemic value of internal consistency versus the lack of reality; in Volterra's case there was a discussion between Volterra and D'Ancona about whether it was necessary to have the model verified by data or whether it had epistemic value in itself that the model was founded on logical foundation. In Rashevsky's case, there was the discussion of the

[4] For a scientific biography of Rashevsky, see Shmailov, 2016. For issues of interdisciplinary collaboration, see also Keller, 2002.

epistemic value of a possible explanation, as Rashevsky cherished — while the biologists considered this as just pure imagination, which they deemed irrelevant.

If we look at the items that went into the modeling cases there are at least four items that can be identified: analogies played a role that had an effect on the modeling. In Volterra's case it was the collision of molecules, and Rashevsky made the analogy to droplets of liquid in physics. These analogies effected the way the models were constructed. It was through these analogies that Volterra and Rashevsky set up their models.

Tab. 1. Analytical framework for analyzing mathematical modeling episodes in scientific contexts (Jessen and Kjeldsen, 2021)

		Von Neumann	**Volterra**	**Rashevsky**
Meta aspects	**Motivation**	Develop economic theory.	Explain a concrete phenomenon due to reduced fishing.	Explain cell division in terms of physics and chemistry.
	Strategy	Abstract mathematical structure of a general economy.	Simple hypotheses, simplifications and idealizations.	Conceptualized a cell as a liquid system that transforms substances.
	Discussion epistemic value	Existence of solution/ internal consistency vs, lack of reality/not useful.	Verification through data vs. purely logical foundation.	Possible explanation, promising vs. imaginary causes, irrelevant.
Items used in the modeling construction	**Analogies**		Collisions of molecules.	Physical phenomenon of droplets.
	Mathematical concepts/theories/ techniques	Linear inequalities, fix-point techniques.	Calculus, systems of differential equations.	Differential equation (diffusion equation), integration.
	Theoretical notions from other areas		Method of encounters.	Physical forces, surface/volume tension, free energy, principle of free energy.
	Empirical data		Served as motivation not as verification.	Served as control not as verification.
Explorative model function	**Explorative function 1., 2., 3.**	2. Proof of principle.	2. Proof of principle. 3. Possible explanation.	1. Starting point. 3. Possible explanation.

Also the mathematics they chose had an influence: for von Neumann it was systems of linear inequalities, and he used fixpoint techniques to solve the model, Volterra used differential equations, and Rashevsky used the diffusion equation and integration techniques. Notions from other areas were implemented into the target domain and influenced the model constructions. In Volterra's case it was the 'method of encounter', and Rashevsky used the notion of force, surface tension and the principle of free energy from physics. Data played different roles in the three episodes: in von Neumann's case data wasn't really present, in Volterra's case it served as a motivation. He wanted it also to be necessary for verification, but here D'Ancona's view was that he didn't think that was necessary. In Rashevsky's case the data didn't play a role for verification it only played a role as control. He compared his theoretical results of the model with experimental results to check the order of magnitude of the radius for when a cell divides.

The notion of explorative modeling was introduced by Axel Gelfert (Gelfert, 2018). By explorative modeling he means:

> Models …that allow us to extrapolate beyond the actual, thereby allowing us to also explore possible […] scenarios. (Gelfert, 2018, p. 253).

He continues, explaining that

> The use of models, then, is not restricted to the initial goal of representing actual target systems. ... [some] models only aim to provide potential explanations of general patterns, […] without thereby claiming to be able to identify the actually operative causal or explanatory factors. (Gelfert, 2018, p. 253).

Gelfert sees explorative function as one of the key functions of mathematical modeling as a research practice to gain new insights.

He distinguishes between three functions of explorative models. One function is that it can aim at a "starting point". This is exploration in the hope of finding fruitful ways to proceed — in the absence of a well-formed underlying theory (Gelfert, 2018, p. 254). Rashevsky's case very much fits this description. He didn't have a starting point, and he explored the model in terms of that. The second function, Gelfert identified is what he calls "proofs of principles". Here Gelfert uses Volterra as a case, and he says that in Volterra's case we see that "the methodology of differential equations is suitable for generating insights into the dynamics of (discrete) populations" (Gelfert, 2018, p. 257). So, Voterra's model and exploration constituted a proof of principles. And I think that also in von Neumann's case, the modeling can be identified as having this explorative function. It constitutes a proof of the existence of equilibrium of such a general economy. The third explorative modeling function is to come up with "potential explanations". We saw that function very clearly in Rashevsky's case, but also in Volterra's case.

6. Significance of History in The Teaching of Mathematical Modeling

In mathematics education there are various "models" of modeling and the modeling cycle features prominently in this literature. The following figure is one example of a modeling cycle (Fig. 1). It is adapted from Blomhøj and Jensen (2007). It is an analytic model of a modeling process. Our analyses of the modeling constructions in the three cases brought out elements such as the motivation, the underlying agenda, of the modeler, the modeling strategy, the use of analogies, the effect of the chosen mathematics on the construction of the model, the possible import of theoretical notions from other disciplines and their effects on the model construction, the

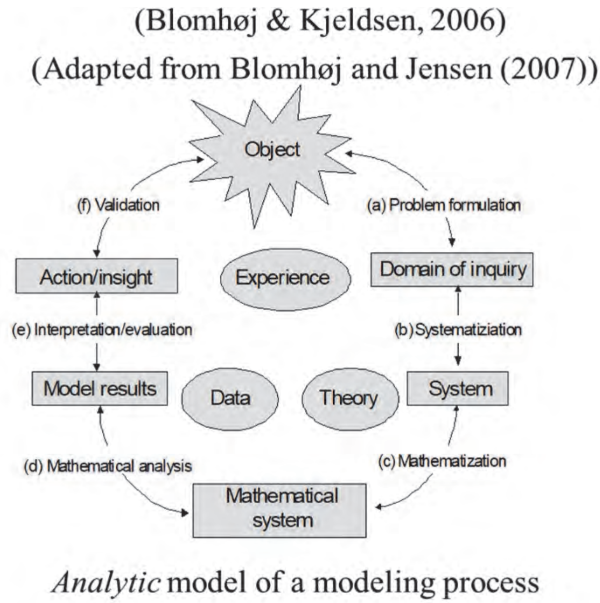

Fig. 1. An analytic model of a modeling process adapted from Blomhøj and Jensen, 2007

explorative function of the model, the issues of epistemic value of model results across disciplinary boundaries. These elements are not featured explicitly in the modeling cycle.

Another account of modeling is Boumans' work in history and philosophy of economics from 2005, which has been used in the construction of the framework in Tab. 1. Based on historical case studies from history of economics, he developed a conception of modeling as "baking a cake without a recipe" (Boumans, 2005, p. 16). He conceives models as being built by fitting together various elements from different sources. The elements, he brings up, are theoretical notions, mathematical concepts, mathematical techniques, stylized facts, empirical data, policy views, analogies, metaphors. In his "model" of modeling, Boumans try to capture what goes into modeling in practice, like mixing the pieces together. So there is a different focus in Boumans' conception of modeling than in the modeling cycle, and it has these various ingredients and tools, which are made explicit. Boumans' model might be a valuable supplement to the modeling cycle in mathematics education.

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modeling in practice, like mixing the pieces together. So there is a different focus in Boumans' conception of modeling than in the modeling cycle, and it has these various ingredients and tools, which are made explicit. Boumans' model might be a valuable supplement to the modeling cycle in mathematics education.

However, the epistemic disciplinary issues — the different practices, the epistemic values and explorative functions that were displayed in the analyses of the historical cases are not captured explicitly in this model either. This is where the historical cases can serve a function in the teaching and learning of mathematical modeling in scientific practices. Historical episodes represent cases where the actors get a voice and they can function in that way in the teaching of mathematical modeling in scientific practice to elicit such issues and to make them explicit objects of students' reflections. They can serve as "invitations" of students into "modeling in the making". Below, I will present an example where this happened in teaching, where a group of students worked with Rashevsky's modeling of cell division that he presented at the Cold Spring Harbor Symposium.

The case of Rashevsky was used in a project work with third semester undergraduate students in an interdisciplinary Bachelor of Science program at Roskilde University in Denmark in 2003. At Roskilde University, problem-oriented teaching where students work together on a project in groups is a pedagogical corner stone in all study programs. In the interdisciplinary Bachelor of Science program, the students work half of the study time throughout a semester in groups of 3–8 students with a bigger project supervised by a professor.

In the spring semester of 2003, there was a group of four students that worked with the Rashevsky case. The problem that guided the students' work in the project was the question, why Rashevsky was unable to get through to the biologists of his time with his ideas. (Andersen et al., 2003, p.2).

So what did they do? The students read and discussed these early works by Rashevsky on cell division including the paper of the talk he gave at the Cold Spring Harbor Symposium. They also read articles and literature from the history and philosophy of science.

In the learning environment that was created through the problem-oriented project work, the students were invited into Rashevsky's "workshop". They obtained access to a modeling process at the frontier of (former) research. The students constructed illustrations of Rashevsky's modeling to support their understanding of how he found expressions for the various forces, that he calculated. In this process, through their work with Rashevsky's paper, the students gained hands on experiences with the mathematization process and techniques of adding up by integration, see (Kjeldsen and Blomhøj, 2009).

The disagreement between Rashevsky and the biologists, that the students studied in connection to Rashevsky's talk, supported their competencies regarding interpretation and validity of the results produced by a model. In this sense, their work with the historical sources enhanced and developed the students' modeling competency.

What the students also gained was that they obtained insights into aspects of the emergence (and struggles) of interdisciplinary fields of research. The historical case gave the students concrete examples of assumptions and beliefs underneath research processes about how the world function, and illustrated explicitly for them that such assumptions guide the questions one asks and the kind of answers one can acquire (Kjeldsen, 2017).

7. Conclusion: What Can History Do for The Teaching of Modeling?

The analyses of the three historical episodes of mathematical modeling in scientific contexts and the reception of the models as research practice in the target domain identified essential factors in modeling constructions in scientific contexts, which are displayed in the analytical framework in Tab. 1. As we have shown in (Jessen and Kjeldsen, 2021), the relation between mathematical modeling in scientific context and upper secondary education is, at least in Denmark, very vague. Our analyses of curriculum in high school showed that “the knowledge to be taught reflects to a minor degree the nature of modeling and practices […] found in the historic cases” (Jessen and Kjeldsen, 2021, p. 54).

Through the historical cases, students can be invited into the workshop of scientists and follow mathematical modeling “in the making”, and thereby gain experience with various modeling strategies, the explorative nature of modeling and the role of e.g. analogies, the chosen mathematical theory and techniques, the implementation of theoretical notions from other areas on the modeling process, the model and its reception — all issues that are essential for modeling as a research practice.

Investigation of such debates from history of science in mathematics education brings the voices of authentic actors into the classroom, and can raise students’ awareness and understanding of methodological issues and debates in interdisciplinary scientific research of today. It illustrates the uncertainty inherent in research at the frontier, where new areas are explored and/or new methods are employed. It also promotes students to reflect about the uses of mathematics to obtain knowledge in other areas, and make students see how, what seems to be a valid scientific approach in one field, can be rejected by experts from a different field — or researchers with a different perspective, see also (Green and Andersen, 2019).

More generally, history can bring authenticity into the teaching and learning of mathematics — it is a source of authentic mathematical (modeling) activities. Chinn and Hmelo-Silver interpret what they call ‘authentic inquiry’ as “activities that scientists engage in while conducting their research” (Chinn and Hmelo-Silver, 2002, p. 171). Such activities are not so easy to implement neither in the teaching of mathematics as a scientific subject in itself nor in mathematical modeling in scientific contexts (or as professional task), as Frejd and Bergsten (2016) have discussed in a recent paper. Here history of mathematics serves a role qua being history. Episodes from history of mathematics, like the three that has been presented here, can provide a window into mathematics and mathematical modeling “in the making”, so to speak. By using historical sources and episodes to invite students into the workshop of past scientists, a learning environment can be created where students can gain insights into and be challenged to reflect explicitly about how scientists get ideas for using modeling to explore research agendas in other areas, which strategies they use, the significance of various choices they make, how they argue, and how they learn. Students may also, as we have seen, come to reflect upon discussions and opinions about what counts as valid arguments, as useful knowledge among the various groups of actors, and realize that there might be differences here. In this sense, they also come to reflect upon the epistemology and the nature of mathematical modeling in scientific contexts. A learning environment that structures and promotes such kinds of reflections is an example of what we have called an ‘Inquiry-reflective learning environment’ (Kjeldsen, 2018), (Johansen and Kjeldsen, 2018), and by integrating historical cases in the teaching and learning of mathematical modeling it is possible to set up such inquiry-reflective learning environments.

I will finish by drawing attention to the notion and significance of developing students’ historical awareness more broadly in mathematics education. The notion of historical awareness is based on the circumstance that both the past and the future are present in the present. The past as recollection and interpretations (of the past) and the future as a set of expectations. To develop students’ historical awareness means to motivate their interest in and ability to ask questions about the past in order for them to gain an understanding of the complex world they live in.

As we have seen, working with historical episodes in an inquiry-reflective learning environment can provide access into people’s/mathematicians’/scientists’ creation of mathematical knowledge and modeling and/or their thoughts about it and their work-life opportunities — illustrating that it is a process that is constrained by the past and that it sets the possibilities for the creation of future mathematical knowledge and uses of mathematics (Kjeldsen, Clark and Jankvist, 2022). To come back to the teaching example presented above. Through their work with the Rashevsky-project, the students developed historical awareness with respect to the role of mathematical modeling in the sciences, and they obtained insights into aspects of the emergence (and struggles) of interdisciplinary fields of research. This helped them to orient themselves with respect to mathematics in their further education. So more generally, the purpose of bringing historical awareness into the mathematics classroom is also to enlighten students, and to give them tools to reflect on their own abilities and possibilities in and with mathematics in their future lives.

Acknowledgments

The framework displayed in Tab. 1 was first presented in (Jessen and Kjeldsen, 2021). I am grateful to the editor of the journal *Quadrante* for accepting the overlap.

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https://doi.org/10.1142/9789811287183_0019

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Teaching Maths in Secondary (Middle and High) Schools: Complex Strategy and its Successful Implementation

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ABSTRACT This article deals with maths education in the middle and high school in Kharkiv City (Ukraine) and in Academic Gymnasium No. 45 in particular. It shows the whole structure of education and the ways of motivation for learning maths at the high level by students. It also shows the obvious success of the strategy of complex maths teaching and analyzes its positive results for the last 25 years.

Keywords: Maths education; Maths competitions; Maths Olympiads; Development of critical thinking; Students' scientific research.

1. Introduction

This article deals with maths education in secondary (middle and high) schools in Kharkiv City (Ukraine) and in the specialized school — Academic Gymnasium No. 45 in particular. It shows the whole structure of education in the chain country-city-school-class and ways to motivate students to learn higher level maths. It also shows the obvious success of the strategy of complex maths teaching and analyses its positive results for the last 25 years.

There are many countries with famous scientific schools and traditions in mathematics. However, our world is changing rapidly. It takes teachers a long time to motivate their students to study science and its applications at universities. So, maths teachers should try to make efforts to encourage and motivate young people to get knowledge, the sooner the better. With the development of our society this is not an easy task, because of a lot of temptations far from science and learning.

The following information concerns the unique experience in Kharkiv City and at our school for a regular creation of student's motivation for deep maths learning.

2. The Unique Experience in Kharkiv City

2.1. Maths education in Kharkiv City

Kharkiv is one of the largest scientific centers of Ukraine (East Europe). There were 3 Nobel Prize winners in Kharkiv. It is the place where an atomic nucleus was split one of the first times in history. The scientists of Kharkiv University cooperated with

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students of city schools. But only at the beginning of 1980s we managed to create extra-curricular maths courses in the city. So, a great number of students were involved and they started learning additional maths topics. It was wide spread in the 1990s and hundreds of students 5–17 years joined these courses. Today there are 3 big extra maths educational centers with about 5 000 students (there are about 114 000 students in Kharkiv schools).

As a result, a lot of students became the winners in Maths Olympiads. It is also important, that learning maths is becoming more and more popular, prestigious and trendy. Furthermore, our city council provides talented students with different grants and scholarships and reflects their success in mass media. Thus, we have the basis for children learning maths in elementary, middle and high schools. Unfortunately, there is a lack of experienced maths teachers in Kharkiv to support students' interest in maths and to develop their abilities.

2.2. Maths education in Academic Gymnasium No. 45 and general principles

Speaking about Academic Gymnasium No. 45 we have our own unique system of such complex maths development of students. We have been working on this task for about 25 years. The following information is about the special features of our teaching concept and how to apply this system successfully in other schools. The main idea is that successful maths teaching at school should be complex (Kryzhanovskiy, 2015) and includes the following:

1) Teaching basic knowledge at lessons via heuristic methods,
2) Preparing for maths competitions at lessons and extra-curricular lessons as well,
3) Preparing science projects with students, under the leadership of Mathematicians in particular,
4) Development of critical thinking skills and a scientific mindset, realization of the importance of maths and its connection to modern computer science.

Maths teachers should not only give basic knowledge but also inspire students to solve problems, including playing and small competitions among themselves. So, I believe it is very important to recognize mathematically gifted children, pay them special attention, involve them into creative maths discoveries at lessons, additional lessons, tutorials, and organize their attendance in maths development centers. Due to participation in different competitions, scientific contests and conferences a lot of students consider maths not only as a strict and boring school subject. It is also important to influence not only advanced students, but the whole class.

Unfortunately, the majority of schools are oriented on only one direction, such as: strict following the course program, preparing for final and entrance exams, or work only with gifted students. But following this strategy we have a tendency of students' losing interest for maths, and maths teaching is becoming less effective. Let children do what they like, but under supervision. Most of them like playing so they can play maths games. If they like to compete, give them a chance to do it at lessons. If they are fond of gadgets, they can use them for maths modeling!

It is important for teachers to avoid putting certain boundaries on students' maths development. For example, we have to publish popular maths books as much as possible oriented not only for the top 5% students, but with a style such that it is available for most of students and their parents. Unfortunately, we have a gap in these types of maths literature. On the one hand, some of these books are for very low students, and on the other hand there are books for advanced students only. It looks like maths books are written in absolutely different styles, end even of different subjects. As a result, the majority of students can't find the appropriate books. It's important for children to know that lessons at school, popular science books, and additional maths literature for Olympiad participants are all aspects of the same science — MATHS!

2.3. Selective exams

My work with the students of Academic Gymnasium No. 45 starts at their entrance exams after the 4th grade. These selective exams give us a chance to find out students with good mathematical abilities. But it does not mean that all gifted children can pass these exams, and all of the selected students will connect their life with maths in the future. Though it is very important to develop the students' personalities and to give them an opportunity for creative research in an appropriate surrounding. There are two steps in our entrance exams:

1) A competition for students of Grades 3−6, which is called "The World of Maths",
2) Entrance tests "Student of Gymnasium".

The first step is oriented on finding out most of the gifted students with a strong and special mindset. As usual, these students have been attending different city maths courses for a while, but some of them are real prodigies. The second step is based on testing the learned basic knowledge of the elementary school and students' abilities of applying it in unusual situations. The winners of these competitions form a special class. In fact, most of these children are really good at maths, so my goal as a teacher is to develop their abilities based on certain topics in maths.

2.4. Three parts of maths education

2.4.1. Usual lessons and the development of students' critical thinking

The further organizational work is conducted in three directions. The first one is making lessons in which compulsory topics are combined with solving Olympiad problems and tasks for the development of their thinking skills. It takes the same time as usual, because gifted students are very quick in standard methods of solving problems and need a challenge. It is very effective to organize a group work at lessons while solving multi-case problems especially in geometry. It influences the development of students' critical thinking and teamwork skills.

For example, let us consider the following problem (Kryzhanovskiy, 2016).

Problem 2.1 In *the given parallelogram there is a height from the vertex of the obtuse angle. This height divides the opposite side by two segments with the ratio 1:7. Find the ratio of the two segments of the diagonal obtained by the intersection with the given height.*

Consider two triangles in Fig. 1: ΔAFE and ΔCFB: $\angle AFE = \angle BFC$ as vertical; $\angle FAE = \angle BCF$ as interior alternate angles for $BC \,//\, AD$, AC transversal. So $\Delta AFE \sim \Delta CFB$. Hence, $\frac{AF}{FC} = \frac{AE}{BC} = \frac{1}{1+7} = \frac{1}{8}$.

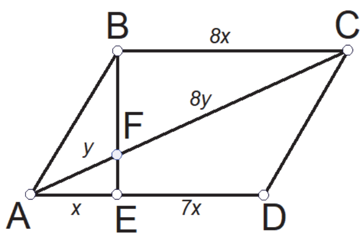

Fig. 1. The first case of the problem 2.1

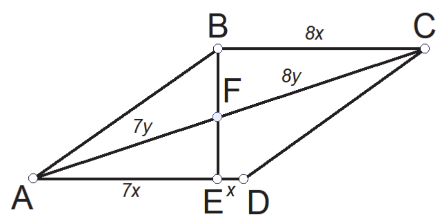

Fig. 2. The second case of the problem 2.1

Let us analyze how our figure corresponds to the given conditions. In the given conditions we have the ratio of two parts of the side of the parallelogram, but nothing about the order of the two parts. So, we have another case in this problem. Let us look at Fig. 2.

In the same way, we have a similarity of 2 triangles *AFE* and *CFB*, and a corresponding proportion: $\frac{AF}{FC} = \frac{AE}{BC} = \frac{7}{1+7} = \frac{7}{8}$.

Hence, in this case we have another answer.

It seems now, that we considered all possible cases. But no! We have two more cases for the location of the given height. Let us consider two new situations on the Fig. 3 and Fig. 4 (on the next page):

These cases are interesting due to the fact, that the height is drawn to one side, but divides proportionally another side.

Let us consider these two situations. In both we have two similar triangles: $\Delta BIC \sim \Delta EID$. Indeed, $\angle BIC = \angle EID$ as vertical, $\angle BCI = \angle EDI$ as interior alternate angles for $BC \,//\, AD$, CD transversal. So, $\frac{CI}{DI} = \frac{BC}{DE}$. With the same way we can obtain that $\Delta AFE \sim \Delta CFB$. Hence, $\frac{AF}{FC} = \frac{AE}{BC}$.

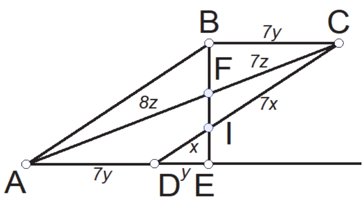

Fig. 3. The third case of the problem 2.1

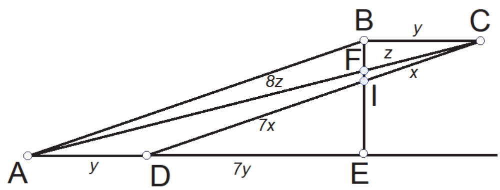

Fig. 4. The fourth case of the problem 2.1

These cases are interesting due to the fact, that the height is drawn to one side, but divides proportionally another side.

Let us consider these two situations. In both we have two similar triangles: $\Delta BIC \sim \Delta EID$. Indeed, $\angle BIC = \angle EID$ as vertical, $\angle BCI = \angle EDI$ as interior alternate angles for $BC // AD$, CD transversal. So, $\frac{CI}{DI} = \frac{BC}{DE}$. Similarly, we have $\Delta AFE \sim \Delta CFB$. Hence, $\frac{AF}{FC} = \frac{AE}{BC}$. So, in the 3rd case $\frac{AF}{FC} = \frac{8}{7}$, and in the 4th case $\frac{AF}{FC} = \frac{8}{1}$.

As a result, we have 4 cases in this problem, and two answers only — 8:7 and 8:1.

Now let us look at an example of another interesting side of complex maths teaching — connections with other subjects. The following shows a commonality between the AM-GM inequality and electric circuits.

Given two electric circuits, where both of them consist of one battery and two resistors (Fig. 5 and Fig. 6).

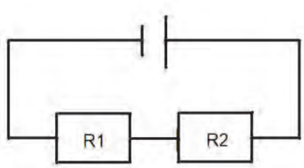

Fig. 5. Two resistors are connected in series with the battery.

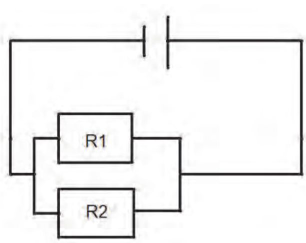

Fig. 6. Two resistors are connected in parallel with the battery.

In the first circuit two resistors are connected in series with the battery. In the second circuit the same resistors are connected in parallel with the battery. Find the minimum ratio between the total resistances in these two circuits.

Due to physical laws for total resistance, we have that in the first circuit $R' = R_1 + R_2$, and in the second one $R'' = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$. But how can we compare these results?

Let us use now the AM-GM inequality. With that we obtain the following:

$$\frac{R'}{R''} = (R_1 + R_2)\cdot\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{(R_1 + R_2)^2}{R_1 R_2} = 4\cdot\frac{\left(\frac{R_1 + R_2}{2}\right)^2}{R_1 R_2} \geq 4\cdot\frac{R_1 R_2}{R_1 R_2} = 4.$$

With the smallest ratio of 4 the resistance for the series circuit is at least 4 times the equivalent of the resistance of the parallel circuit.

2.4.2. Extra maths classes

The second direction is based on working with the children of the special class at our additional maths classes "Solving of Maths Olympiad problems" systematically. At these lessons we study applying certain mathematical methods on appropriate examples adapted for children. As usual, the students enjoy this kind of activity, which content is related to serious mathematics, but it is unusual and often resembles as a game. These extra lessons exist in all grades from 5 to 11 (12), and are held once a week. The most important thing in this approach is adapting famous methods to develop the mathematical skills of children to school requirements.

2.4.3. Individual and group lessons

The third step is individual and group lessons prepare the most gifted students for Olympiads, competitions and scientific conferences. Very often I'm not just a teacher, but also, I partner up with my students while solving actual problems. As usual, a good competition spirit and students' maths ambitions give a quite positive dynamic in the learning process. It is also important that all of the children are involved in creative work, in spite of changing forms and methods of learning.

It is very difficult to divide the topics which are learned at usual lessons and which are used only at Olympiads. Thus, I try to include Olympiad and research problems at our lessons as much as possible. Preparing for maths competitions we try to discover how usual methods of solving problems can be applied at Maths Olympiads.

So, it is obvious that teachers, their students and their parents should have close contact to each other. In fact, teachers and students communicate at lessons, preparing for Olympiads, visiting maths Festivals, at conferences, scientific competitions and at summer maths schools. Thus, at Academic Gymnasium No. 45 in Kharkiv we are going to have the 15th year of our traditional summer school with the profile "maths and computer science". One of the main goals of this school is to stimulate the students' motivation in both maths and computer science studies, in the form of creative communication between students and their teachers.

2.4.4. Three types of maths competitions

To involve students in creative maths learning I try to organize their participation in competitions of 3 types. The first type consists of competitions, available for all students, such as "The Kangaroo", "The championship of maths logical solving problems" (organized by France). The next one consists of competitions for mathematically gifted students, ready for intellectual and mental fighting. For instance, there is a system of national Olympiads in Ukraine, which includes 4 steps, with the final national Ukrainian Olympiad. Our students take also part in the "International Mathematical Tournament of Towns", "maths fest" and Olympiad named after Euler (organized by Russia). The third type is the most difficult and consists of individual and group competitions for the most advanced and gifted students, such as IMO, EGMO, and Romanian Masters. Besides, there are some unique Ukrainian competitions, such as: "Young Maths Tournament" (team research), "Champions Tournament" (including maths, physics, and computer science), and Kiev international Maths and Physics Fest (with the participation of scientists from the Maths Institute of the National Academy of Science in Ukraine).

2.4.5. Students' scientific research

It is very important to mention that the problems of these competitions are often the beginning of scientific research. For instance, the problem, given by prof. Valentine Leyfura, was the starting point for our cooperative research with my student Julia Fil. Consider the triangle ABC, with the points D, E, F belonging to the sides AB, BC, AC respectively. Investigate the perimeter of the given triangle ABC, DEF, using p, r, R — half of the perimeter, the inradius and the circumradius.

This project was presented at the National competition of students' scientific research. We found non-trivial lower and upper bounds for different cases, represented below.

Problem 2.2. *Consider the triangle ABC, with the points D, E, F belonging to the sides AB, BC, AC respectively; p, S, r, R — half of the perimeter, the area, the inradius and the circumradius. Then:*

(1) $\frac{2pr}{R} \le DE + EF + DF \le p$, *given D, E, F — points of tangency of the circle, inscribed in the triangle ABC;*

(2) $3\sqrt{3}r \le DE + EF + DF \le \frac{3\sqrt{3}R}{2}$, *given D, E, F — foots of the angular bisectors of the triangle ABC;*

(3) $DE + EF + DF = \frac{2S}{R}$ *(for right or acute triangles);*

$\frac{2S}{R} < ED + DF + EF \le \frac{3\sqrt{3}}{2}R$ *(for obtuse triangles),*

given D, E, F – foots of the altitudes of the triangle ABC;

(4) $\frac{2S}{R} \le DE + EF + DF < 3\sqrt{3}R$, *given* D, E, F *— points of tangency of the excircles, points* D, E, F *belong to the sides* AB, BC, AC *respectively.*

For example, let us find the upper bound in the inequality (4). First, find the sides and perimeter of the triangle DEF (Fig. 7)

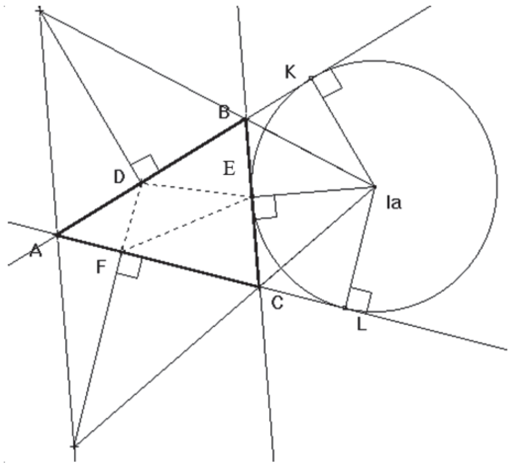

Fig. 7. The fourth case of the problem 2.2

BI_a is the angular bisector of $\angle KBC$ and CI_a is the angular bisector of $\angle LCB$, since I_a is the excenter relative to the vertex A.

So, $\angle ECI_a = \frac{1}{2}\angle ECL = 90° - \frac{\gamma}{2}$, which implies that $\angle EI_aC = \frac{\gamma}{2}$. In the same way, $\angle EI_aB = \frac{\beta}{2}$. Hence, from right triangles ΔBI_aE and ΔCI_aE , we have

$$EC = r_a \tan\frac{\gamma}{2}, \quad BE = r_a \tan\frac{\beta}{2}.$$

Notice that $r_a \tan\frac{\beta}{2}\tan\frac{\gamma}{2} = r$. Indeed, $BC = r_a\left(\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right)$. But, obviously,

$$BC = r\left(\cot\frac{\beta}{2} + \cot\frac{\gamma}{2}\right) = r\cdot\frac{\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}}{\tan\frac{\beta}{2}\tan\frac{\gamma}{2}},$$

so $r_a \tan\frac{\beta}{2}\tan\frac{\gamma}{2} = r$. Thus,

$$EC = \frac{r}{\tan\frac{\beta}{2}}, \quad BE = \frac{r}{\tan\frac{\gamma}{2}}.$$

In the same way,

$$BD = \frac{r}{\tan\frac{\alpha}{2}}, \quad AD = \frac{r}{\tan\frac{\beta}{2}}, \quad CF = \frac{r}{\tan\frac{\alpha}{2}}, \quad \text{and} \quad AF = \frac{r}{\tan\frac{\gamma}{2}}.$$

According to the Cosine Theorem:

$$\begin{aligned}
&DE^2 = DB^2 + BE^2 - 2BD \cdot BE \cdot \cos\beta \\
&= \frac{r^2}{\tan^2\frac{\alpha}{2}} + \frac{r^2}{\tan^2\frac{\gamma}{2}} - 2 \cdot \frac{r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}} \cdot \cos\beta \\
&= \frac{r^2}{\tan^2\frac{\alpha}{2}} + \frac{r^2}{\tan^2\frac{\gamma}{2}} + \frac{2r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}} - \frac{2r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}} - 2 \cdot \frac{r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}} \cdot \cos\beta \\
&= \left(\frac{r}{\tan\frac{\alpha}{2}} + \frac{r}{\tan\frac{\gamma}{2}}\right)^2 - 2 \cdot \frac{r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}} \cdot 2\cos^2\frac{\beta}{2} \\
&= b^2 - \frac{4r^2\cos^2\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}} = b^2 - \frac{2r^2\sin\beta\cos\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} \\
&= b^2 - \frac{2r^2\sin\beta \cdot \frac{p}{4R}}{\frac{r}{4R}} = b^2 - 2pr\sin\beta.
\end{aligned}$$

Here we used the formulas (Prasolov, 2001)

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{r}{4R} \quad \text{and} \quad \frac{p}{4R} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

Note that if we multiply these two formulas, we obtain $\frac{1}{8}\sin\alpha\sin\beta\sin\gamma = \frac{pr}{16R^2}$. Then a useful corollary follows:

$$\sin\alpha\sin\beta\sin\gamma = \frac{S}{2R^2}.$$

Using the Sine Formula for the area of a triangle, we have

$$\begin{aligned}
&b^2 - 2pr\sin\beta = b^2 - 2S\sin\beta = b^2 - ac\sin^2\beta \\
&= 4R^2\sin^2\beta - ac\sin^2\beta = 4R^2\sin^2\beta - 4R^2\sin\alpha\sin\gamma\sin^2\beta \\
&= 4R^2\sin^2\beta(1 - \sin\alpha\sin\gamma).
\end{aligned}$$

Therefore, $DE = 2R\sin\beta\sqrt{1 - \sin\alpha\sin\gamma}$.

With the same way we get,

$$EF = 2R\sin\gamma\sqrt{1-\sin\alpha\sin\beta} \quad \text{and} \quad DF = 2R\sin\alpha\sqrt{1-\sin\beta\sin\gamma}.$$

To get the formula for the perimeter of the given triangle DEF we make an estimation using the Cauchy-Schwarz inequality:

$$\begin{aligned} &DE+EF+DF \\ &= 2R(\sin\beta\sqrt{1-\sin\alpha\sin\gamma}+\sin\gamma\sqrt{1-\sin\alpha\sin\beta})+\sin\alpha\sqrt{1-\sin\beta\sin\gamma}) \\ &\le 2R\sqrt{\sin^2\alpha+\sin^2\beta+\sin^2\gamma}\cdot\sqrt{3-(\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\alpha\sin\gamma)}. \end{aligned}$$

Notice, that from the famous Leibniz formula $MO^2 = R^2 - \frac{a^2+b^2+c^2}{9}$ we obtain the following Leibniz inequality: $a^2+b^2+c^2 \le 9R^2$, and therefore

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma = \frac{a^2}{4R^2}+\frac{b^2}{4R^2}+\frac{c^2}{4R^2} \le \frac{9R^2}{4R^2} = \frac{9}{4}.$$

Now, using the AM-GM inequality we get:

$$\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\alpha\sin\gamma \ge 3\cdot\sqrt[3]{(\sin\alpha\sin\beta\sin\gamma)^2} = 3\cdot\sqrt[3]{\left(\frac{S}{2R^2}\right)^2}.$$

Therefore, $DF+DE+EF \le 2R\cdot\frac{3}{2}\cdot\sqrt{3-\sqrt[3]{\left(\frac{S}{2R^2}\right)^2}} < 3\sqrt{3}R.$ □

Notice that the first expression for the upper bound is exact, but really long and complicated. The last expression $3\sqrt{3}R$ is a good estimation too — not really exact, but it is shorter.

2.4.6. Maths Olympiads and scientific ideas

It is necessary to say that Olympiads and other maths competitions are important in some aspects. As it was already said, competitive and game forms of learning encourage students' motivation at lessons. Besides that, scientific tournaments are a perfect starting point for the first scientific research. But there are also deep problems that show the connection of solving maths problems very quick and serious scientific ideas. It is obvious that the difficulties of these problems depend on the difficulty level of the Olympiad. But involving these scientific ideas is one of the most essential parts of maths competitions. Let us have a look at applying these ideas of work with space basis and with functional equations. This unusual problem (author: Oleg F. Kryzhanovskiy, NYC) was given at Kharkiv Region Maths Olympiad (Kryzhanovskiy, 2012).

Problem 2.3. *Let us name the sum of triangles with sides* $a_1 \le b_1 \le c_1$ *and* $a_2 \le b_2 \le c_2$ *the triangle with sides* a_1+a_2, b_1+b_2, and c_1+c_2. *Name the product of the* *real number $x > 0$ and the triangle with sides a, b, c the triangle with sides xa, xb, xc. Find all functions with a set of triangles as the domain, and a set of real numbers as the range, with following properties:*

(1) *for any triangles T_1, T_2 : $f(T_1 + T_2) = f(T_1) + f(T_2)$ ("additive property");*

(2) *for any triangle T and any real number $x > 0$: $f(xT) = xf(T)$ ("homogeneous property").*

Justify your answer.

Each triangle is defined by ordered triple of positive numbers (a,b,c), where $a \le b \le c$, which represent the triangle's sides. Try to guess the answer, using an analogy of vector components notation: $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$.

Thereby,

$$\begin{aligned} & f(a,b,c) \\ &= f(a(1,0,0) + b(0,1,0) + c(0,0,1)) \\ &= f(a(1,0,0)) + f(b(0,1,0)) + f(c(0,0,1)) \\ &= af(1,0,0) + bf(0,1,0) + cf(0,0,1) \\ &= xa + yb + zc. \end{aligned}$$

However, the "basis" consists of triangles "degenerated" into segments.

It would be easily improved by an operation, inverse to triangle addition — "subtraction of triangles". Indeed, if we have the equalities

$$(1,0,0) = (3,3,3) - (2,3,3), \quad (0,1,0) = (2,3,3) - (2,2,3),$$

and

$$(0,0,1) = (2,2,3) - (2,2,2),$$

then

$$(a,b,c) = a((3,3,3) - (2,3,3)) + b((2,3,3) - (2,2,3)) + c((2,2,3) - (2,2,2)).$$

Since "subtraction of triangles" is not defined, amend last equality by shifting "subtraction" with addition:

$$(a,b,c) + a(2,3,3) + b(2,2,3) + c(2,2,2) = a(3,3,3) + b(2,3,3) + c(2,2,3),$$

or

$$(a,b,c) + a(2,3,3) + b(2,2,3) + 2c(1,1,1) = 3a(1,1,1) + b(2,3,3) + c(2,2,3).$$

Use the function *f* to both parts of the last equality and apply its "homogeneous" and "additive" properties

$$f(a,b,c) + af(2,3,3) + bf(2,2,3) + 2cf(1,1,1) = 3af(1,1,1) + bf(2,3,3) + cf(2,2,3).$$

So we get

$$f(a,b,c) = a(3f(1,1,1) - f(2,3,3)) + b(f(2,3,3) - f(2,2,3)) + c(f(2,2,3) - 2f(1,1,1)).$$

Thus,

$$f(a,b,c) = xa + yb + zc,$$

where x, y, z are any real numbers.

Checking by substitution of the given type function shows, that they satisfy the given.

Finally, we obtained the answer: $f(a,b,c) = xa + yb + zc,$ where x, y, z are any real numbers.

3. Summary

So, the main idea of the given experience is a selection of gifted students, complex development of their mathematical abilities and encouraging students' motivation to study maths science at school and university. The following results show the obvious success of the given method for 25 years in Academic Gymnasium No. 45, Kharkiv, Ukraine:

- More than 300 winners of Kharkiv Region Maths Olympiad;
- More than 50 winners of the final level of the National Ukrainian Math Olympiad;
- 3 winners of IMO (2003, Tokyo (Japan); 2011, Amsterdam (the Netherlands); 2018, Kluzh-Napoka (Romania)).

For instance, the silver medal winner of IMO - 2011 Olexii Kislinskij also won the International Mathematics Competition for University Students with Gold medal in 2012. Recently he graduated from Yale University (the USA) with a PhD in maths in 2021.

- 1 winner (Gold Medal) of EGMO (2017, Zurich (Switzerland))
- More than 80% of school graduates enter Ukrainian and foreign universities on specialties connected with maths and computer science.

The students of Kharkiv schools — the population of the city is 1 500 000 — have even more impressive results. For instance, every year some students from Kharkiv become winners of the IMO.

But the main result of my work is the creation of the gifted student's mindset, involvement of them in the world of scientific research, computer science and IT, and forming them as integrated personalities.

Certainly, it is possible to use my experience at Academic Gymnasium No. 45, Kharkiv, Ukraine in other schools. What we need is a cooperation between students and their parents, teachers and school's senior management, city authorities and different additional mathematical educational centers. Following this scheme of complex maths education this experience might be useful for teachers of other schools. But it doesn't mean that this scheme should be followed absolutely accurate.

Of course, it should be adapted to the actual teachers' approaches. Thus, it helps them to succeed in their work and get great results from their students in maths education.

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https://doi.org/10.1142/9789811287183_0020

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A Constructivist Approach towards Teaching and Learning Mathematics in Singapore: Rationale, Issues, and Challenges

Ngan Hoe Lee[1]

ABSTRACT. Enabling students to achieve a deep and connected understanding of mathematical concepts is an important aim in Singapore mathematics education. While current forms of instruction in the mathematics classroom can engender detailed expositions of a concept and links between targeted concepts and earlier concepts, much of this information is structured by the teacher and neglects the role of students' perspectives of the information that is transmitted to them. With the demonstrated efficacy of constructivist learning designs that build upon students' prior knowledge structures, one of such designs was implemented in Singapore's mathematics classrooms to not only afford deeper learning but also transform mathematics teaching and practice. In this paper, this constructivist learning design that was introduced to Singapore's secondary mathematics classroom is described and its rationale, efficacy, and the measures that were taken to ensure its sustainability discussed. The paper concludes with reflections of how to sustain such constructivist designs beyond research, and suggestions on proliferating their use among the Singapore teaching fraternity.

Keywords: Constructivist learning design; Deeper learning; Mathematics teaching and learning; Sustaining learning designs.

1. A Deeper Understanding of Mathematics: Potential of Constructivist Approaches

Getting students to develop deep and robust understanding of mathematics is a desired outcome of mathematics education, and this objective is emphasized in the recent updates of the Singapore's secondary mathematics curriculum (Grades 7 to 10; Ministry of Education, Singapore [MOE]: Curriculum Planning and Development Division [CPDD], 2019). To achieve this goal, it is essential that students are given the opportunities to explore the interconnected nature of mathematical concepts, which are "products of insight, logical reasoning and creative thinking" (MOE: CPDD, 2019, p. 5), and to participate in processes that afford the *active construction* of these

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interconnections. Unfortunately, pedagogical practices in the current Singapore mathematics classroom are largely didactic in nature (Kaur, 2009), and they may not be adequate in supporting students to engage in deeper learning. A paradigm shift is required, and one possible direction for this shift is for teachers to adopt pedagogies that are informed by constructivist learning perspectives.

Constructivism asserts that knowledge is *actively constructed* by the learner, and that knowledge is the product of one's cognitive acts via a meaning-making process (Applefield et al., 2001; Confrey, 1990a; Ertmer and Newby, 2013; Karagiorgi and Symeou, 2005). It posits that individual construction of knowledge can be influenced by the process of interaction and negotiation (Jaworski, 1994) with teachers (Green and Gedler, 2002; Vygotsky, 1978) and the learning context (Amineh and Asl, 2015). Such interaction and negotiation would enable teachers to clue into students' prior knowledge structures and knowledge construction processes during learning, and into the kinds of knowledge that they could build upon during instruction (Smith et al., 1993). Several learning designs like the "Open Ended Approach" (Becker and Shimada, 1997) and "Productive Failure" (Kapur, 2008, 2010), which involve the use of students' constructions in instruction, have been found to be efficacious in promoting student learning. This suggests that constructivist learning designs that build on students' constructions may have potentials to enhance students' connected understanding, which is one of the core objectives of the updated curriculum.

In this paper, a similar learning design — coined the *Constructivist Learning Design* (CLD) — that was developed by a team of mathematics researchers and educators in 2018, and later implemented in the Singapore secondary mathematics classrooms is described. The CLD's rationale, justification, and efficacy will be first examined, and this will be followed by an identification of the possible issues of sustaining such a design in the classrooms and a description of measures that were taken by the research team to address these issues. The paper closes with a reflection on the challenges of sustaining the use of CLD beyond the research project, with suggestions on propagating the pedagogical innovation among Singapore mathematics teachers.

2. The Constructivist Learning Design (CLD)

2.1. Engineering a constructivist learning environment in learning a new concept

The CLD is based on three propositions about learning from both cognitive constructivist (e.g., Confrey, 1990a; Noddings, 1990) and social constructivist positions (e.g., Brown et al., 1989; Savery and Duffy, 1995). First, constructivists posit that understanding is brought about through an interaction between learners' prior conceptions and the context of learning. This is supported by research on students' misconceptions and alternative conceptions (e.g., Confrey, 1990b), which showed that learners' prior conceptions, whether formal or informal, are activated and used as "resources" in the knowledge construction process. Acknowledging the importance of prior knowledge, constructivists suggest that students learn best in learning environment that allow for the compatibility of their prior constructions to be tested (Savery and Duffy, 1995), and this could be achieved through reflections and comparisons (Billing, 2007).

Second, learning is stimulated via *cognitive conflict or disequilibrium*, which determines the organisation and nature of what is learnt. What is "problematic" leads to and is the organiser for learning (Dewey, 1938; Roschelle, 1992), and this notion is also echoed in Piaget's (1970, 1977) theory of cognitive development, which maintains that knowledge construction is stimulated by internal cognitive conflict as learners strive to resolve mental disequilibrium (Applefield et al., 2001). Getting learners to realise gaps between their current knowledge and that of the targeted one also harmonises with Vygotsky's (1978) "zone of proximal development" (ZPD), where this "zone" illustrates the difference between what a student could achieve independently and what he or she could achieve with the guidance of knowledgeable others (e.g., teachers, peers).

Third, e*valuating the viability of individual understandings and social negotiation are important in the evolution of knowledge*. The catalyst for knowledge acquisition is via dialogue, and understanding is facilitated by exchanges that occur through social interaction, questioning and explaining, challenging, and offering timely support and feedback (Applefield et al., 2001). Meaning can be socially negotiated and understood based on viability (Savery and Duffy, 1995). Transfer is promoted when learning takes place through active engagement in social practices and is facilitated when learners are encouraged to talk about the similarity of representations for both the initial and targeted tasks (Billing, 2007).

From the propositions outlined above, four instructional principles that undergird the proposed CLD were outlined. These include (i) affording the elicitation and building upon of students' pre-existing informal or formal understanding of a concept, (ii) aiding the development of an organised and interconnected knowledge that facilitates retrieval and application, (iii) engaging students' thinking about their thinking and learning through conflict inducing processes, and (iv) building a social surround that allows for interpersonal and social nature of learning and this could be done via collaborative learning (see Lee et al., 2021 for more details). A viable learning approach that could fulfil the above principles also needs to be aligned to the school or national curriculum, and in the Singapore context, the CLD should be aligned with the updated secondary mathematical syllabus which emphasizes on *problem solving* that is central to the Singapore mathematics curriculum framework (MOE: CPDD, 2019). Given that, a possible way to support the generation of students' conceptions in the learning of new mathematical concepts could be through the introduction of a complex problem that targets a new concept that students had not been formally taught. This echoes the "teaching via problem solving" approach coined by Schroeder and Lester (1989) and the "problem-solving first, instruction later" approach from the learning sciences (Loibl et al., 2017), both of which were pointed out by scholars as being suitable in introducing new mathematical ideas (e.g., Kapur and Bielaczyc, 2012; Nunokawa, 2000).

2.2. The CLD: description and justification

The considerations behind a viable constructivist learning design culminated in a two-phased CLD. The CLD comprised (i) a collaborative *problem-solving phase*, where students work collaboratively on a problem targeting a concept that students have yet to learn and attempt to generate innovative solutions to solve the problem, followed by (ii) an *instructional phase* where the teacher builds upon the solutions, creating linkages between the solutions to the targeted concepts.

2.1.1. CLD's problem solving phase

The problem in the problem-solving phase was designed such that it helps in the elicitation of students' prior knowledge structures. In line with other similar "problem-solving first, instruction later" learning designs (e.g., Becker and Shimada, 1997; Kapur and Bielaczyc, 2012), a complex problem targeting a concept or strategy is given to students to solve before the formal introduction of the concept or strategy. The problem, which contains different parameters, provides students' opportunities to tap on their intuitive or formal prior knowledge, and encourages the generation of multiple solutions. Past research suggests that while students were typically unable to generate or discover the correct solutions by themselves, they are able to generate a diverse set of solutions (e.g., Kapur, 2008; 2010; 2012; Kapur and Bielaczyc, 2012). The problem is also solved collaboratively, and such peer collaboration is necessary to allow for the negotiation of meaning of concepts (Lee et al., 2021). Beside keeping the groups on task and providing affective support to ensure that students persevere in solving the problem, the teacher also ensures that students experience conceptual conflict and disequilibrium. While refraining from telling students the solution to the problem, teacher facilitates students problem-solving efforts by pointing out their solutions' potential strengths and limitations and suggest ways to refine their strategies. The students' responses provide teacher with an insight into the gaps that are to be bridged between students' current conceptions and the targeted concept.

2.1.2. CLD's instruction phase

After the problem-solving phase, the teacher organizes students' solutions to the problem, and builds upon these to teach the targeted concept or strategy. The solutions are organised according to their relationship with the critical features of the targeted concept. The teacher then implements CLD's instruction phase, which aims to resolve the conceptual conflict and gaps that were induced during the problem-solving phase, effect the process of assimilation and accommodation (Piaget, 1977), to help students understand why the targeted concept or strategy is the most adaptable one given the problem. In line with past recommendations on how multiple solutions to problems could be consolidated (Kapur and Bielaczyc, 2012; Richland et al., 2017), the teacher discusses the affordances and constraints of each solution type, and compares and contrasts each solution's features with the critical features of the targeted concept, via counterexamples as much as possible. By getting students to consider the viability of their solutions vis-à-vis the targeted concept, the instruction phase acts as a platform for the negotiation and reflection of the concept's meaning, and therefore aids a deeper understanding. In addition, the CLD recommends the use of practice tasks that not only reinforce the procedural knowledge of the concept, but also further develop students' conceptual understanding. This is supported by Chinese Post Teahouse approach (Tan, 2013), a learning design which embodies constructivist principles, which advocates the use of appropriate practice tasks to complement the pertinent ideas that were brought up during instruction.

2.1.3. Justifying CLD: efficacy of similar learning designs

Past studies had shown that problem-centered learning designs that are similar to CLD were effective in helping students acquire better content and conceptual knowledge in K-12 classrooms and also in other settings like tertiary medical education and professional training development (e.g., Hung et al., 2008; Merritt et. al., 2017; Thomas, 2000). A review by Loibl et al. (2017) also demonstrated that a similar two-phased "problem-solving first, instruction later" instructional designs could potentially help to improve students' ability to transfer. For example, it was found that students who experienced the two-phased problem-first "productive failure" learning design significantly outperformed their counterparts in the traditional direct instruction condition on conceptual understanding and transfer problems without compromising on procedural fluency (e.g., Kapur, 2008, 2010, 2012; Kapur and Bielaczyc, 2012). Evidently, these positive findings provided justifications for the implementation of CLD in the actual ecologies of the classroom.

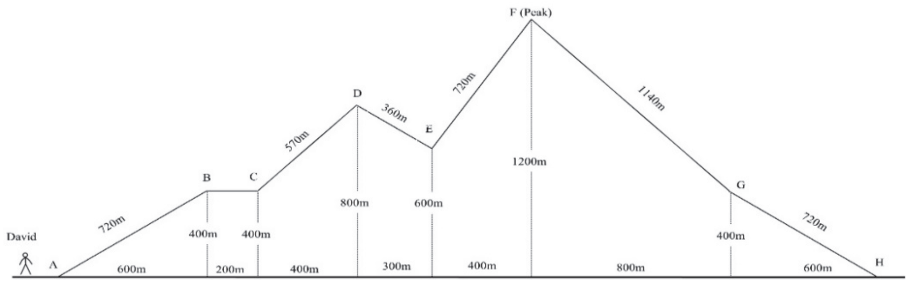
2.2. Implementing CLD and results

One of the CLD units that was designed targets the concept of gradient of linear graphs, which is introduced at secondary 1 (grade 7) level of the Singapore mathematics secondary level curriculum. The canonical gradient concept, which is a measure of steepness and direction of a straight line, is formulated as

$$\frac{\text{change in magnitude and direction of variable 1}}{\text{change in magnitude and direction of variable 2}} \text{ or } \frac{\text{Vertical change}}{\text{Horizontal change}} \text{ or } \frac{\text{Rise}}{\text{Run}}.$$

Together with a team of experienced Singapore mathematics educators, an analysis of the concept was conducted, and 4 critical features that underlie the gradient concept were identified: the (a) quantification/magnitude of steepness; (b) the quantification of direction; (c) the consideration of 2 dimensions/variables; and (d) the consideration of the ratio of 2 variables. Variations of these critical features were crafted within a plausible context, culminating in the complex problem task that we see in Fig. 1. In the problem task, students are asked to develop as many mathematical measures as they can to characterize the steepness and direction of 7 mountain trail sections, using given variables such as horizontal distances, absolute heights, and slope lengths.

The Mountain Trail



David enjoys hiking on a small mountain with a peak of 1200m. The figure above shows a sketch of David's trail, which has 7 sections. He starts at point A, hikes through points B to G, and ends at point H. The vertical heights, the horizontal distances, and the lengths of the slopes (rounded to the nearest 10m) are also indicated in the figure.

Although he is a seasoned hiker, David notices that some sections are steeper compared to others**.** He seeks your group's help to describe **both the steepness and direction of the mountain's slopes mathematically**. Here is what you must do:

(1) Assuming that all other things are equal, please use the information provided in the figure above and come up with as many ways as possible to rank the various sections of the trail both in terms of their steepness and the direction.
(2) For each way of ranking in (1), justify your ranking **mathematically** to describe both the steepness and direction of different sections.

All the best, and remember, don't give up until you have come up with as many methods as possible!

Fig. 1. "The Mountain Trail" problem of a Constructivist Learning Design (CLD) unit targeting the concept of gradient of linear graph

The CLD unit on gradient of linear graphs was implemented to secondary 1 (Grade 7) students from various secondary schools. In a study that was conducted with students from a Singapore mainstream school (Lee et al., 2021), students in the CLD class spent a period (roughly 50 minutes) on the problem-solving phase and generated as many possible solutions as possible, while the instruction phase, which included teachers' consolidation of students' solutions and follow-up practice, took about four 45-minute periods. The CLD groups were able to produce an average of 4 solutions per group (*SD* = 1.95 solutions), and these solutions were categorized into 4 categories that were related to the critical features of the gradient concept, including those that considered (i) one dimension/variable, (ii) a combination of two dimensions/variables, (iii) a ratio of two dimensions/variables; and (iv) solutions that employed angles to determine the steepness and direction of the slopes (see more details of in Lee et al. 2021). During the consolidation of these solutions, the teacher invoked critical features of the gradient concept by actively comparing and contrasting each solution type relative to the others. For example, when analysing solutions that consider only the horizontal distance to measure steepness, the teacher noted that while such one-dimension measures are quantitative, they are insufficient, since slopes with the same horizontal distance have different steepness (e.g., comparing sections CD and EF in Fig. 1). In contrast, solutions with two dimensions might have better affordance.

To ascertain the efficacy and tractability of the CLD, the learning effects of the CLD was compared to its transmissionist, direct instruction (DI) counterpart on the learning of the concept of gradient of linear graphs (Lee et al., 2021). Students in the DI condition differed from the CLD condition in terms of the sequence of the problem solving and instruction phases. They first experienced the teacher-led instruction of the gradient of linear graphs concept guided by the course textbook, and after the instruction, worked on practice problems targeting the necessary procedural and conceptual understanding of the concept, from the same textbook that was used by their CLD counterparts. The teacher also went through the students' solutions, directing attention to the critical features of the targeted concept, and highlighted common errors and misconception. The learning outcomes of both conditions were compared via a 12-item post-test, comprising items assessing students' procedural knowledge, conceptual understanding, and ability to transfer knowledge of gradient to similar contexts (near transfer) and to more advanced concepts, such as gradient of curves (far transfer). Controlling for the effects of students' pre-requisite knowledge using a 4-item pre-test, a multivariate analysis of covariances revealed that the two learning conditions were significantly different in the learning outcomes, with subsequent tests of between-subject effects further indicating that the CLD class had significantly higher scores for conceptual understanding, near transfer, and far transfer (see Lee et al., 2021 for more details).

Like past "problem solving first" approaches, the CLD demonstrated the potential to develop more connected understanding of a concept. These findings provide a positive indication that the CLD has engendered deep learning processes to afford the cultivation of transferrable skills and knowledge (Lee et al., 2021). The demonstrated efficacy of the CLD unit paved the way for the development of more units that cover the major strands of the secondary mathematics syllabus, and these topics include angle properties of circles, standard deviation, and quadratic inequalities (see Ng et al., 2021).

3. CLD in Singapore Mathematics Classroom: Sustaining its Use

The demonstrated efficacy of the CLD and other similar constructivist pedagogical designs attests to their tractability in the mathematics classroom, and their potential in bringing transformative change in learning and teaching. Despite that, a major challenge is to ensure the sustainability of such learning designs in teacher practice, i.e., after the research, teachers are able to continue to employ these instructional innovations in the manner intended by its designers and make valid moves to own the designs, such that they become a part of their instructional repertoire (Coburn, 2003; Fishman, 2005). Teachers play critical roles in determining the degree of success in implementing instructional innovations (Doyle and Ponder, 1977; Ghaith and Yaghi, 1997; Guskey, 1987, 1988; Kennedy and Kennedy, 1996; Stein and Wang, 1988; Zhao et al., 2002). Given this, extant literature on possible factors that hinder teachers from sustaining instructional designs in their practice were surveyed and from which, programme and structures were developed to support teachers embracing a new instructional approach, i.e., CLD, in the classroom.

3.1. Teachers sustaining instructional innovations: knowledge, beliefs, and perceptions

A major reason why teachers do not actively use innovations in their instruction is a lack of teacher capacity (Ball et al., 2008, Shulman, 1986). Employing innovation places demands on teachers' content knowledge (CK) and pedagogical content knowledge (PCK). CK refers to teachers' subject-specific knowledge, while PCK is the subject matter knowledge unique to teaching, such as the knowledge of what makes certain topics easy or hard for students to grasp, of possible students' understandings and preconceptions, and of how to work with students' conceptions (Shulman, 1986). To design other topics using the learning design, teachers would also have to develop their design knowledge (DK) as well. Given these, efforts aiming to support teachers in sustaining the use of instructional innovations like CLD would do well to develop teachers' DK, CK, and PCK.

Teachers' expectations about learning and their perceptions of the utility of innovations also present challenges to continued use of an instructional innovation (Cohen and Ball, 1999; Fishman et al., 2011). Teachers' perceptions of their own students' abilities shape the kinds of instruction employed (Fishman et al. 2011), and their beliefs that certain strategies are more suited for their high achieving students than for the low-achieving ones, and vice versa, were documented in past research (e.g., Desimone et al., 2005; Stanovich, 1986; Young-Loveridge, 2005). With regard to teachers' perceptions of the usefulness of instructional innovations, past studies on various professional development (PD) programs showed that teachers' perceived coherence of the educational innovation with their personal goals, for learning and for their students, predicted their change in classroom practices (Garet et al., 2001; Penuel et al., 2007), and that having the capacity to make adaptations to the innovation is one of the key elements for long-term sustainability to occur (Shaharabani and Tal, 2017).

Taken together, for the CLD to be usable and sustainable for mathematics teachers, its alignment to teachers' knowledge, beliefs about learning, and their perceptions of its utility need to be considered. Given these, PD programmes and support structures were put in place to ensure the sustainability and relevance of the CLD in mathematics classrooms.

3.2. Professional development

To support teachers in implementing and sustaining constructivist learning designs in their practice, quality PD programmes are necessary to allow teachers to be conversant with implementing the necessary tasks and activity structures, and then progressing to eventually independently implementing, and possibly designing new units on their own. The PD principles that were adopted in the CLD research project were drawn from the continuous professional development model advocated by Fallik et al. (2008) and the PD model put forth by Markowitz and colleagues (2008). The *continuous professional development* model advocates the need for collaboration among teachers, partnership between teachers and facilitators, and support for teachers when they embark on any teaching method with their students. The need for teachers to be led through phases of being a learner, an instructor, and an innovator for any new pedagogical approach as they learn how to implement a new instructional method is recommended in Markowitz et al.'s (2008) model.

The recommendations made from these models were infused in the PD workshops and sessions designed for teachers prior to them implementing CLD in the classroom. These PD sessions were designed to not only enhance teachers' CK and PCK in implementing the unit, but also provide them with an embodied sense of what the CLD was from the standpoints of both a learner and teacher. After being introduced to the background, aims, and design principles of CLD, teachers then experienced a "problem-solving phase". Working in small groups, teachers examined a complex problem that were designed and evaluated students' representative solutions that were produced for each task. At the end of the evaluation, they were to provide a lesson plan of how they would build upon the students' solutions and instruct the targeted concept. In the 'instruction phase", a representative teacher from each group presented the group's solution. The trainer of the session, a research team member who is an academic faculty, Master Teacher, or Curriculum Specialist with experience in teacher training, consolidated teachers' responses and discussed ways to effectively compare and contrast student-generated solutions with the targeted concept. Teachers implementing the CLD units were also provided with detailed teacher's guides and with in-situ support by the research team during implementations.

3.3. Networked learning community

To support teachers', use of the CLD during and beyond the research, a Networked Learning Community (NLC) was also set up. Facilitated by both Master Teachers from the MOE's Academy of Singapore Teachers (AST) and a curriculum specialist, the NLC was set up with the aims to help build interested teachers' capacity to implement CLD, and champion its use in Singapore mathematics classrooms. Upon formation, the NLC organized meetings and workshops to equip a core group of interested teachers to deepen their CK and PCK in implementing existing CLD units, with sessions devoted to getting them to work with three chosen targeted concepts, consider their features, how they were being instructed in the curriculum, and students' prior knowledge and conceptions to these concepts. The NLC also enhanced teachers' DK in designing new CLD units. With support from Master Teachers and the curriculum specialist, the teachers worked collaboratively to decide on potential concepts that could be taught using the CLD approach, and then crafted the complex problems for the chosen concepts.

At present, the NLC has an active membership of 11 teachers from 7 schools. The NLC developed 2 units, and these units had at least one iteration in actual classrooms settings. In addition, the NLC also put in place structures to develop teachers who are relatively accomplished in implementing CLD to help seed the design through collegiate networks in their schools.

4. The CLD Beyond the Research: Challenges

The teacher capacity building efforts via PD programmes and resources, and the presence of the NLC platform to build a community of practitioners were measures that were put in place to ensure that a tractable alternative pedagogical approach like the CLD could be sustained during and beyond the research. The CLD's underlying principles are in line with curriculum's emphasis on deeper understanding of mathematics and problem-solving as focus. Such a research endeavor was made possible via a tripartite partnership among representatives from policy (MOE curriculum specialists), practitioner (Master Teachers from AST), and researchers from NIE. Nonetheless, ensuring a wider uptake of pedagogical innovations such as CLD remains a challenge in transforming Singapore education practice. As observed by Hung et al. (2022), the demands of a centralised Singapore education system that propelled Singapore's stellar performance in mathematics international assessments might explain the general inertia among mathematics practitioners in embracing innovations. Apart from identifying similar teacher capacity and beliefs issues brought up in the previous section, Hung et al. (2022) also noted institutional, policy, and cultural level issues behind the inertia. At the institutional level, teachers' educational background, and the lack of exposure to constructivist learning designs in both pre-service and in-service teacher programmes could explain teachers' lack of efficacy and unwillingness to implement such designs. At a policy level, while there is a push for such innovations in the classrooms, the high-stake assessments might disincentivise teachers in taking up instructional innovations that are perceived to be less efficient in getting students to master the necessary content knowledge. At a macro, cultural level, a (i) fear of failure that inhibits teachers' openness to unfamiliar instructional methods with unknown outcomes, and (ii) high power distance (Hofstede, 1991) that propagates the belief that knowledge provided by the teacher is absolute and final are possible inhibitors to pedagogical innovations that require some loosening of teachers' control in instruction.

Evidently, the factors identified by Hung et al. (2022) demonstrate the inextricable influence that the Singapore education system has on teachers in advancing and sustaining pedagogical innovations. To motivate change in teachers, there is a need of concerted actions from policymakers, researchers, and the teaching fraternity to further enhance the mathematics education for teachers to have the space and courage to implement pedagogical innovations independently. Drawing from the implications made by Hung et al. (2002) on how pedagogical innovations in Singapore could be advanced and the research team's experience with implementing and sustaining CLD in the Singapore mathematics classes, it seems that this tripartite synergy among policy makers, practitioners, and researchers is an important mechanism at addressing this gap, motivating a change of classroom culture and transforming practice. At the *policy* level, there could be a stipulation of the use of such innovations nationwide, a development of taxonomy that defines and operationalizes features of effective mathematics lessons, a provision of directives on the use of various assessment methods to assess mathematical competencies, and free up more space and time for teachers to implement these new pedagogies in the classroom. At the *practice* level, the presence of Professional Learning Communities and NLCs could not only help build teacher capacity, but also seed the innovation through its networks. These networks will be instrumental in developing teachers who champion and lead the innovation in their schools, effecting ecological leadership (e.g., Toh et al., 2016) and start micro-cultures that could shift socio-mathematical norms in the mathematics classrooms. As for the role of *research*, the research fraternity could work with policymakers and practitioners to develop the necessary resources in advancing these pedagogies; develop effective PD models that could equip Singapore teachers with the necessary capacities; adopt brokerage roles to understanding the needs of the ground and suggesting the necessary ideas and avenues for teachers to implement these strategies; embrace the essence of action research and teacher inquiry as measures of success of adaptation on the ground, and; continue their roles in helping policymakers and practitioners effect deeper understanding of mathematics.

5. Conclusion

In conclusion, the CLD was developed and put forth as a potential learning design that would promote deep and connected understanding of mathematics in students. As opposed to didactic forms of instruction, the CLD emphasizes more on the processes of problem solving to afford deep and meaningful learning and the development mathematical habits and dispositions in students. Moving beyond the research, it was postulated that cultural factors and teacher capacity were the reasons behind sluggish uptake of pedagogical innovation, and that these factors could be addressed by the concerted efforts to invest in teacher development, and push for a change in school and classroom culture. Furthermore, a tripartite synergy among policy, research, and practice could be important in sustaining new pedagogical approaches that have potential for deeper learning.

Acknowledgement

Research reported in this paper is funded by a grant from the Singapore Ministry of Education (MOE) under the Education Research Funding Programme (ERFP) to the author. The grant was awarded to the research project "Constructivist Learning Design for Singapore Secondary Mathematics Curriculum" (DEV04/17LNH; NTU-IRB reference number: IRB-2018-03-009) and was administered by the National Institute of Education (NIE), Nanyang Technological University, Singapore. The claims and opinions presented herein are mine alone and do not necessarily represent those of the funding agency.

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Fostering Student Agency in learning Mathematics: Perspectives from Expert Teachers in Shanghai

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ABSTRACT Fostering student agency means developing students' willingness and ability to engage in their own learning. This chapter presents the views of .. expert elementary school mathematics teachers in Shanghai on fostering student agency. Interview data show that all of the expert teachers value the importance of getting students take ownership of learning and they believe that teachers can contribute significantly in developing student agency. When described the essential features of a classroom where students exercise agency, in addition to focusing on creating an environment that supports student to take up space and actively engage in learning, the expert teachers in Shanghai placed special emphasis on achieving satisfactory learning outcomes. When sharing their strategies for fostering student agency, they commonly mentioned the importance of teachers as role models for their students, which has been less addressed in the literature.

Keywords: Student agency; Expert teachers; Teacher's role; Primary school mathematics teaching.

1. Introduction

In China from very early times, we have a favorable attitude towards tradition, authority, official rank and self-cultivation, which are still exerting great influence upon people's thinking and behavior (Li and Chen, 1996). A reflection of the favorable attitude towards authority in education is the high respect for teachers' authority. Chinese teachers control a lesson primarily through prepared instructional tasks, lectures, and frequent exchanges of teacher questions and student responses, which have been reported in the research literature (e.g., Leung, 1995; Mok, 2006). At the same time, Chinese are well aware that learners must take responsibility for their learning. It is a common belief in China that all children can learn and succeed, but slower students must devote more time and effort than their peers (Li, 1995). In a study

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of examining Chinese adolescents' goals and sense of agency, Jin Li, found that her 259 participants from grades 7–12 of middle and high schools in China showed many more effort-based personal agency than social agency in their responses, and the learning virtues such as diligence, concentration, and self-generated activities such as do homework and read books outside classroom were the most often expressed. She called for an emphasis on developing student social agency, such as working with peers or seeking help from teachers (Li, 2006).

Fostering student agency in learning mathematics has been repeatedly emphasized in curriculum reforms in China since the 1990s. The Shanghai Municipal Education Commission issued the Action Plan for School Mathematics Education into the 21st Century in 1997 and it clearly indicated that "students are the masters of their own learning. They are the internal factors that affect the learning, while teachers and textbooks are the external factors. Teachers teach so that in the end teachers do not have to teach. Students' development depends to a large extent on their stronger willingness and ability to take responsibility and participate in the learning process" (The Action Plan for School Mathematics Education into the 21st Century project group, 1997, p. 8). In 2001, the Ministry of Education of China initiated a national curriculum reform at school level, writing in the Standards of Mathematics Curriculum for compulsory education that "students themselves are the masters of their own mathematics learning, while teachers are the organizers, facilitators and collaborators of students' mathematics learning" (Ministry of Education of China, 2001, p. 2). This statement appeared again in its 2011 Curriculum Standards, but with some further suggestions and clarifications: "teachers should play a leading role, handle the relationship properly between teacher direct instruction and student learning agency, and guide students to think independently, explore actively, and cooperate and communicate with their peers" (Ministry of Education of China, 2011, p. 3).

2. Student Learning Agency

Fostering student learning agency is about developing students' willingness and capacity to engage in mathematical learning (Schoenfeld, 2018). It is both a learning goal and a learning process (OECD, 2019). As we conducted the literature review, we found that there are various ways to connect research to the theme of learner agency. For example, from the perspective of educational purpose, the OECD (2019) put forward the Student Agency for 2030 Project, which emphasized that "learn how to learn" is an invaluable skill for every active, responsible and engaged citizen. Identifying culturally appropriate approaches to foster student agency is a challenge for educators in every country.

The emphasis on student learning agency is a reflection of the shift from teacher-centered to learner-centered teaching in curriculum reform. The metaphor of the teacher as a facilitator of student learning is sometimes falsely taken to mean that the teacher's role is passive, but Boaler (2003) showed us how a school teacher enabled students to work productively with open problems by establishing an environment in which they could engage in talking about their own thoughts and responding to the ideas of their classmates. Sherin (2002) referred to such a classroom environment as a discourse community and gave us more details of how a middle school teacher worked through the school year tried to find his own way to maintain a balance between a student-centered mathematical discourse process and focusing the discussions on significant mathematical content. Based on the social constructivist stance that learning is essentially a social phenomenon, Walshaw and Anthony (2008) considered mathematical discourse involving explanation, argumentation and defense of mathematical ideas as a defining feature of a quality classroom experience.

There are some essential features of high quality teaching, but creating a productive and engaging environment for learning mathematics has become one of them (Middleton et al., 2017; Schoenfeld, 2018). Schoenfeld and his colleagues have carefully constructed a framework for Teaching for Robust Understanding in Mathematics (TRU) to characterize the kinds of teaching that result in students being knowledgeable, flexible, and resourceful thinkers and problem solvers, and "agency, ownership, and identity" is one of the five dimensions of this framework (Schoenfeld, 2018). This particular dimension is used to examine the extent to which students have the opportunity to present their own mathematical ideas and develop their collective understandings through classroom discourse.

Establishing and maintaining a caring and trusting relations with students so that they can learn in a comfortable climate is a favorable condition for high quality teaching. Noddings (2012) agreed that the teacher-student relation is not an equal relation by nature, but she suggested that listening to students' ideas, understanding their needs, responding with care, and integrating moral education with academic learning all contribute to the establishment and maintenance of caring. In an ethnographic study, Noblit (1993) described the story of Pam, an African American elementary school teacher who used her power to construct a caring, safe and engaging learning environment for her students. Pam's class had many routines and rituals to build collective responsibility, and the strong collectivity made each child stronger as a consequence.

Based on our decades of experience working with mathematics teachers in Shanghai, we believe that after more than 20 years of practice and exploration, expert teachers have gained personal insight into developing student agency. Their wisdom needs to be presented in a holistic manner with research that addresses the practices of groups of teachers. To this end, this study targeted 21 expert elementary school mathematics teachers in Shanghai to understand their views and practices on fostering student learning agency. Specifically, we have the following two research questions:

RQ1: What does a classroom where students exercise agency look like?

RQ2: How can teachers effectively foster student learning agency? In particular, what are the specific roles that teachers play? What teaching strategies do they use?

3. Methods

3.1. Data collection

This study is primarily based on interview data that we collected in 2019. In addition to interviewing these 21 teachers, we also observed and recorded or collected teaching videos from eight of them to help us better understand their views and practices that they mentioned in the interviews. On the other side, we interviewed a total of 18 students from 3 elementary schools. We also asked 38 fifth-grade students to anonymously write down one strength and one possible weakness that impressed them most after an open lesson given by an expert teacher who participated in this study. However, only data collected from teachers were used in this report, and our conclusions were based primarily on the interviews with the 21 expert teachers. In our interviews, we asked the expert teachers to share their perspectives **and** stories on the following questions:

1. How to interpret that "teachers should play a leading role in the teaching process and students are the masters of their own mathematics learning"?
2. How teachers can use questions to assist students exercise their learning agency?
3. Give a picture of a classroom where students exercise agency in learning; and
4. for the subtraction question $50 - 26 =$ ___ , what would you do next if you see that the students have answers 24 and 34?

The teachers were asked to share stories or concrete examples because we wanted to understand their own interpretations of the theoretical views advocated by the official curriculum standards and to learn more about their unique teaching strategies from their stories.

We interviewed with 17 expert mathematics teachers from 15 primary schools in Shanghai and 4 expert teacher educators who used to be primary school teachers in Shanghai but mainly do in-service mathematics teacher training now. They are named as "expert" because 4 of them are outstanding teachers in Shanghai with the rank of professor level (the top professional title to school teachers in China) and the others are senior level teachers (the second high). Seven of them are also called Master (TEJI) Teacher (an honour to recognize outstanding school teachers and principals, but it is not a professional rank. In this study, we coded these 7 Master Teachers as A1−A7 and the other 14 expert teachers as B1−B14). Ten of them were males and 11 were females. Sixteen of them had been teaching for 20 years or more, with the least being 11 years and the most being 41 years. Four of them had not taught Grades 1 and 2, the remaining 17 teachers had taught all grades of elementary school.

Teachers were interviewed by telephone, email, or in person. The main reason for the inconsistent interview format is that very few mathematics teachers in each school have the professional ranks we set, so our participants came from 18 workplaces, which are located in various regions of Shanghai. Also, they were all busy, so it was more convenient for them to be interviewed by phone or email. If we felt that something was not clear in the written responses, we asked the interviewees for further clarification by phone or in person. It seems that the telephone and in-person interviews may have been more detailed and in-depth in some areas due to the interaction, whereas the teachers interviewed by email may have been more comfortable in providing us with details of the relevant learning tasks and implementation processes. All interviews were conducted focusing on the 4 interview questions. Both the in-person and telephone interviews were audio recorded.

3.2. Data analysis

After completing the interviews, we transcribed all audio-recorded interview data into text to facilitate subsequent coding. Data coding and analysis for this study was conducted in the original language of Chinese. Selected data was translated into English to provide evidence for the results of this study. Based on the purpose and the research questions of this study, we conducted a qualitative content analysis to describe and identify themes in the participants' responses. Considering that fostering student agency is culturally influenced, we decided to use an inductive approach, which means that codes, categories, or themes were extracted directly from the data (Cho and Lee, 2014). Firstly, the three authors recorded then summarized independently the interview texts of seven teachers they were responsible for, sentence by sentence and example by example, on worksheets. The worksheets were sent to a second coder for cross-checking. The second coder marked the descriptive labels with which he/she disagreed, added a brief comment, and sent the worksheets back to the first coder. If not all comments were accepted, they would discuss the labels by phone to reach an agreement. Then, each of us took the lead in coding the data for 1−2 interview questions. By repeatedly reading the worksheets and sometimes looking back at the interview transcripts or watching the teaching videos, some key words and ideas emerged and were noted in a list of key terms or short sentences. We categorized all teachers' responses to each interview question with a set of key terms or short sentences. Data coding was cross-checked again, however, this time all three authors checked the coding for each interview question. We organized several video conferences to resolve the coding inconsistencies and fix the problems we encountered in categorization. We then drafted a summary report outlining the preliminary findings for each interview question. In the final stage of data coding, all interview data were categorized according to the research questions of this study and the themes underlying the teachers' responses to each research question were identified. The first author went through all the interview transcripts again, fine-tuned and combined a few repetitive codes under the structure of the two research questions and finalize the codes. We did not code the collected teaching videos, but we selected some teaching episodes from the videos to support the findings of this study.

4. Results

4.1. Essential features of a classroom where students exercise agency

Through recursive analysis of the 21 expert teachers' descriptions of a classroom where students exercise agency, four themes were identified and each feature is represented by 2–3 indicators. In general, all of the four features were valued by the expert teachers.

4.1.1. Full and active participation

The first feature shows the degree of learners' behavioral engagement and cognitive engagement (Fredricks et al., 2004) in their learning. It includes learners' external actions, such as manipulations, as well as internal actions, such as thinking and working quietly and independently, using their hands, brains and hearts. The first indicator "intellectually engage in hands-on or other activities" highlights that manipulative activities must be accompanied by mathematical thinking. Another indicator, "collaborate and communicate among community members", reflects a general recognition among the expert teachers of the importance of social interaction and communication in mathematics learning. Ms. A2 shared with us a story about her teaching of "Recognizing milliliters and liters".

> After the class, a teacher saw me bring back some disposable cups with marks left by students and asked me what they were for. I said: to measure 100ml. The teacher said, "No need, we have 100 ml measuring cups in the school lab, just show them to the children." I told her that I designed an activity called "Play with 100 ml of water" and made 2 requirements: 1). estimate the water level of 100 ml based on 10 ml of water; 2). take the 100 ml of water estimated by your group (use disposable cups only). During the sharing time, one group of students said: "We did it by counting, 10 ml is just a layer on the bottom of the cup, 100 ml is 10 × 10 ml, so I made my mark here." Another group of students immediately raised their hands and said: "This mark is not correct, the water must be more than 100 ml. The cup is smaller at the bottom and it gradually gets bigger from the bottom to the top, so our group thought......"

She ended the story by saying that mathematics is not a subject that depends on experiments, it depends on thinking. Langer-Osuna and Esmonde (2017) also agreed that the integration of thinking and manipulation is crucial for mathematical learning, noting that "students' agency is framed in terms of the degrees of freedom they experience in being able to intellectually engage with mathematics" (p. 645).

4.1.2. Taking up space

This dimension of engaging classrooms also includes two indicators: "discuss questions from students and build on each others' idea" and "involve students in the evaluation process". It implies that instruction focuses on learners' learning, students become part of the classroom community, engage in meaningful, critical, and respectful dialogue, and use their capacity to contribute to the development of the collective agency (Bandura, 2000; Hand, 2012). The following keywords were used more often by the expert teachers in their interviews: “have a voice, agency, ownership, student ideas, reflective listening, questions, questioning, add, contribute, build on, solve problems, agree, disagree, justify, persuade, whole class discussion, authority, facilitator, openness, support, self-study, community, self-assessment, peer assessment”. This is a key feature of a classroom where students take ownership of their learning, and as Cobb et al. described, “authority being distributed across the students and the teacher. The students were expected to determine the reasonableness and adequacy of solutions as they presented, listened, and asked questions” (p. 54).

In general, Chinese students rarely ask questions in class (Jin and Cortazzi, 1998), developing their willingness and capacity to listen attentively, think critically, and communicate friendly takes years, and responding appropriately to students’ ideas on the spot can be challenging for any teacher; however, the expert teachers who participated in this study generally agreed that it made a great deal of sense to use their authority to support students taking up space and exercising agency in their learning. According to Mr. A6, “whether students have the courage to present their own ideas separates good teachers from average teachers”. Mr. A5 said, “having students ask questions is the key to a good lesson, the key to personalized learning and deep learning”. Explaining why she believes teachers should not be the only source of mathematical ideas and assessments, Ms. B4 said:

> If only teachers evaluate the students, their comments are not diverse enough. Involving students in the evaluation process can push them listen to their peers and think reflectively. And, at the end of a lesson, I often invite students to evaluate their performance today. Successful peer assessment make students learn from each other, and make progress together; Self-assessment helps students to see new things they have learned and to ask questions they have, and it promotes their own development.

In contrast to “full and active participation”, the feature of “taking up space” concerns collective agency rather than individual agency in learning.

4.1.3. A pleasant and supportive learning environment

It is enjoyable for students to try and explore on their own and eventually discover a pattern or solve a problem, but limited by their age, experience, and knowledge, they usually need the support of their teachers and peers to be successful. The first indicator is “engage in interesting tasks with a certain degree of openness”. The reason why open-ended but interesting problems were recommended is to allow students of all levels to participate. The second indicator is “have adequate time and resources for inquiry”. Ms B7 said in her interview: “If you really want students to explore, you must give them time. Some students won’t succeed if they haven’t reached a certain number of attempts. Maybe after a few more tries, they will find the pattern”. The last indicator is “listen each other carefully and allow sharing of uncomplicated, flawed replies”. Ms. B5 shared with us a story that happened in her class.

> I remember we talked about a multiplication equation, there are five rows of flowers, each row has six flowers, so 6×5, a total of thirty flowers. One child said I have another way, $5+5+5+5+5+5$, and the other children snickered. I said seriously, "I think he is very good. ...... What's so funny?" Then I turned to him: "So why do you think the other students reacted that way?" "Because my way is tedious". "It's okay, you deserve more praise. First, you know why it needs improvement, and second, you have the courage to voice your idea."

The importance of attentive listening and mutual respect has been stressed in the literature (Cobb et al., 2009; Middleton et al., 2017; Ministry of Education of China, 2011; Noddings, 2012). Middleton et al. (2017) stated that "engagement varies depending on the level of social risk students feel comfortable taking" (p. 687).

4.1.4. Satisfied learning achievement

The expert teachers in Shanghai placed special emphasis on achieving satisfactory learning outcomes. Its indicators are "learn with great interest" and "develop conceptual understanding, procedural fluency, proficiencies and good habits". They involve students' learning achievement in cognitive, affective, and meta-cognitive aspects. The first indicator is related to emotional engagement (Fredricks et al., 2004) in learning. When describing a classroom where students exercise agency, Mr. A4 said, "the lesson must be engaging. Not only are students happy to learn, but teachers are also enthusiastic to teach. It looks like all of us are enjoying our time in the classroom." In an evaluation report of the UK-China Mathematics Teacher Exchange Programme, Boylan and his research team described their views as outsiders to Shanghai mathematics: "Shanghai mathematics education is a mastery approach and so is premised on the belief that all pupils can succeed as mathematical learners. Classroom practices and organization of mathematics teaching follow from this belief. Shanghai whole class interactive teaching aims to develop conceptual understanding and procedural fluency." (Boylan et al., 2016, p. 15). The expert teachers we interviewed believe that it is their responsibility to help students, especially the late-bloomers, grow, even though it may take years to see them succeed.

4.2. Strategies for fostering student agency

All 21 expert teachers agreed that teacher can contribute significantly in developing student agency and shared with us some of their own teaching strategies. Mathematics teachers take on four distinct roles as diagnosticians of students' thinking, conductors of classroom discourse, architects of curriculum, and river guides who are flexible in the moments of teaching, which are the "images" of expertise in mathematics teaching identified by Russ et al. (2011), and this can serve as a framework to organize our findings about how teachers foster students' learning agency, but it is important to add one more "image", namely *Ren Shi* (人师, role models for students, in Chinese) presented by Hsieh et al. (2018).

4.2.1. Strategies used by teacher as architect of curriculum

These expert mathematics teachers in Shanghai value and excel at teaching design. They carefully design teaching goals, tasks, learning processes, and formative assessments prior to class so that students can actively and successfully engage in deep learning. They shared with us a lot of instructional tasks or questions they designed and explained their intentions as well. We found three strategies they use associated with careful teaching design. Due to space limitations, a brief description of each strategy is provided.

Use appropriate tasks/problems to help students succeed. Tasks play a critical role in students' learning outcome and learning opportunities (Hiebert and Wearne, 1993). Carefully designed situational task sequences can turn a learners' attention to abstract similarities and develop their conceptual understanding (Margolinas, 2013). Similarly, the Shanghai teachers suggested to use a coherent sequence of tasks/problems to guide students in constructing knowledge on their own, to design tasks with a degree of openness to allow all students to have a voice, and ask questions within "zone of proximal development".

Use interesting tasks to engage students. They recommended presenting tasks in real-life or interesting contexts, inviting a group of students to work as a team to teach their classmates like a teacher, inviting students to ask relevant questions based on the given conditions, and solving problems from students.

Have students try the tasks before class. They proposed assigning tasks before class so that students come prepared; ensuring enough time for exploration, personal reflection, and discussion of common errors and misconceptions; and having students to learn new content on their own that is similar to a topic they have already learned.

"Knowing millimeters and decimeters" was a lesson that Mr. A4 gave to Grade 2 students. In the first part of this lesson, Mr. A4 organized eight activities to help children visualize 1 mm. They are: recognizing it on a ruler, drawing it on paper, telling the class how they feel about it, finding it in the classroom, making it with two fingers, estimating and measuring the thickness of a book in millimeters, talking about examples of using millimeters as a unit in their lives, and discovering the relationship between millimeters and centimeters with the help of a ruler. He applied the techniques such as "use a coherent sequence of tasks to guide student learning", "design tasks with a degree of openness to allow all students to have a voice", "ask questions within ZPD" and "present tasks in real-life or interesting contexts". In teaching of converting between units of length, he presented the staircase model (the children had seen this model before when they studied numbers) on the blackboard, put a millimeter card on the step, and ask: "1mm up, 1mm up, grow up to 10 mm, it is ___?" The whole class answered in chorus "1 centimeter". He continued, "In fact, when we count by ones and reach 10 ones, we go up a step and create a new unit a ten; and when we count by tens and reach 10 tens, we go up another step, a new unit, a hundred. How do you think what we learned before helps us learn today?" With his help, the class successfully mastered the relationship between units of length and discovered a connection between the learning of numbers and the learning of units of length, thus deepened their understanding of base-10 system. He also arranged two engaging exercises at the end for formative assessment, “Who’s Missing?” and “Uncovering the Truth,” which asks children to identify and correct units used incorrectly in a short story. This example illustrates how teachers can effectively involve students in learning and develop their sense of measurement by designing engaging tasks before class.

4.2.2. Strategies used by teacher as conductor of classroom discourse

The image of “mathematics teacher as conductor of classroom discourse” refers to teachers develop trusting classroom communities, direct and shape the classroom discourse, and motivate students to build on each other’s’ idea.

Arrange communication of different sizes. The Shanghai expert teachers flexibly organize discussions of different sizes for two-person, four-person and the whole class, with particular emphasis on whole-class discussions. The use of two-person exchange allows for greater efficiency. Using whole-class dialogue facilitates the resolution of disputes, demonstrates different ways to solving the same problem, and also allows the teacher to enact a mentoring role, such as intervening early when students are observed to be having learning difficulties.

Form a pleasant and supportive learning environment. The expert teachers mentioned techniques such as seeing several students raise their hands to wait a few extra seconds to get more responses from the class; assisting individual students to clarify their thoughts and expressions, and respecting and praising the effort and courage of students who answered inappropriately. Walshaw and Anthony (2008) also highlighted the importance of scaffolding students’ ideas.

Engage students in productive class dialogue. In China, due to the large population, developing economy, and collective culture, large classes are common (Jin and Cortazzi, 1998). The expert teachers suggested some techniques to improve student communication, such as forming a rule in mathematics classroom that agree requires a reason and disagree requires a counterexample; inviting different students to explain or comment on a peer’s method to show different ideas, inspire each other, and build on each others’ ideas; and having students guess what questions the teacher could ask to motivate students to think and learn how to ask questions. Primary school teachers in China often lead the class in applauding a student’s wonderful ideas, which has a powerful encouraging effect on students.

4.2.3. Strategies used by teacher as diagnostician of students’ thinking

Teachers’ expertise in subject matter knowledge and pedagogical content knowledge (Ball et al., 2008) enables them to provide children with timely and helpful diagnosis-based guidance and feedback (Fraivillig et al., 1999). These Shanghai mathematics teachers talked about strategies they used both in designing instruction before class and implementing their lesson plans during class.

Analyze student learning to inform teaching. Specific techniques that the expert teachers shared with us were: clarifying what students know, what they don’t know, what they want to know, and what they can know with the help of others; diagnosing student learning based on their work and the questions they ask; and using students' work, especially their errors and misconceptions, as teaching resources.

Listen to students and quickly discern their real thoughts. They suggested that teachers give timely and helpful feedback, especially on what was done well; guide the class to discover where and why they went wrong and how to avoid repeating it; categorize students' various methods mathematically to reveal the nature of these methods that may not look alike; and decide quickly which ideas should be discussed further.

Assess learning outcomes after intervention. The Shanghai teachers preferred to arrange an exercise after their intervention. Using the lesson "subtraction with trading" as an example, Mr A5 explained that his "specialized exercise" means that the teacher prepares six to eight subtraction equations and has the class verbally classify them into two groups: with trading or without trading, no further calculations are needed. This is because once students have determined that a subtraction equation requires a trading, it then becomes a familiar task that students can easily complete. Another technique the teachers advocated is inviting students to participate in assessment, including both self-assessment and peer assessment. They agreed with the idea of assessing *with* students, rather than just *to* students. They suggested to have students who answered incorrectly share their new understanding and give their reminders to the class at the end of the class.

4.2.4. Strategies used by teacher as river guide for learning journey

Any unexpected responses or contingencies may arise during the learning journey, and teachers are expected to respond quickly and appropriately to them and modify their original lesson plans accordingly. The recommended strategies are:

- *Elicit student responses.* These expert teachers suggested two approaches to break the ice. One is to ask a different question to elicit responses or break down difficulties. The second is to provide manipulatives or draw a diagram to facilitate thinking.
- *Help students in a smart way.* Teachers should not overlook mistakes, but some errors may be caused by carelessness, and students can find and correct them in explanations to the class. In this case, for example, for the equation $50 - 26 = ?$ teacher could ask the student who answered 34 how to change the equation, then the answer 34 would be correct. Another teaching tip is to postpone teacher comments for those students who are particularly active or too far ahead, and instead listen to other students' ideas and come back to comment later when appropriate.
- *Plan ahead, but be flexible.* Shanghai has unified mathematics textbooks, but designing the teaching process and preparing learning tasks is a daily task for every teacher. The expert teachers advised young teachers not only to be flexible in adjusting their original lesson plans and not to be forced to complete all the prepared tasks, but also to record post-lesson teaching reflections on their lesson plans in order to improve their future teaching.

4.2.5. Strategies used by teacher as role model for students

There is an old Chinese saying, "A teacher for a day is a father for a lifetime", referring to the deep feelings between teachers and students. Teachers need to be strict with their students, but also love them, care about their needs, interests, thoughts and emotions, assist them to succeed, and always behave as a role model for students. The expert teachers believed the following two points are important.

Value the moral development of students. Moral education should be integrated into the intellectual education of every subject in school. The expert teachers placed special emphasis on: praising good behavior, expressing your expectations and love for your students; establishing teacher authority through excellence in personality and scholarship; and modeling the virtues of seeking the truth, convincing others with reason, and being willing to correct any mistakes.

Guide students on how to learn. They emphasized that teachers should develop students' enthusiasm for inquiry, knowing that inquiry begins with observation and a reasonable guess, followed by verification and finally expression. They should assist students in asking questions, stating and explaining their ideas confidently, and being willing to learn from others. They should proactively use new technology platforms or other resources to facilitate students' self-learning. Attention also should be given to developing good learning habits, such as "listening and comparing others' ideas with one's own," "saying full sentences and using stem sentences (Boylan et al., 2016, p. 71)," "reading textbook and questions carefully and circling key points," "writing neatly and formally," and "using estimation and other ways to check an answer".

5. Conclusion

For teachers who intend to foster student agency in learning mathematics, it is crucial to identify the essential characteristics of a classroom in which learners exercise agency. From our interviews with 21 expert elementary school mathematics teachers in Shanghai, we learned that their perspectives on this are as follows: students collaborate with their teacher, participate in learning activities behaviorally, cognitively and affectively. As members of a learning community, they take the initiative to ask questions or contribute ideas, build on others' thinking, and deepen their own understanding through comparison and reflection. Of course, the active engagement cannot be achieved without having a pleasant and supportive learning environment. At the same time, being a good lesson must also result in satisfied learning outcomes, not just be fun for students. Thus, in their view, it is more like a teacher-directed and learner-engaged classroom, where both teacher guidance and student engagement are essential.

Consistent with what is stated in the curriculum standards (Chinese Ministry of Education, 2011), all the expert teachers involved in this study agreed that students themselves are the masters of their own learning and students' learning should be placed at the center of education, but teacher's leading role must be maintained. Eleven teachers explicitly pointed out that student exercise agency is actually a reflection of the teacher's leading role, the more fully the teacher's leading role is enacted, the more motivated and proactive the students will be in their learning. Since achieving satisfactory learning outcomes is a common goal for both teachers and students, teacher guidance and student agency can coexist in harmony. Their teaching stories also illustrate some strategies on how to achieve such coexistence. For example, in the lesson “Knowing millimeters and decimeters” we mentioned earlier, Mr. A4 prepared a series of focused, coherent and engaging learning tasks. He introduced parallel learning activities to visualize 1 millimeter and 1 decimeter, but students took greater ownership in the latter set of learning activities because he believed that the experience of learning 1 millimeter would assist them in learning 1 decimeter on their own. Conversions between units are often difficult for children, and they try to remember them by rote. His solution was to draw a staircase model on the board to make converting length units simple and meaningful. The expert teachers know where to focus their instruction, which learning barriers to break through are critical to success, and what learning tasks could scaffold students’ thinking and make their learning easier and productive. Teacher-planned activities and the various mathematical discussions that surround them are critical to learning (Ben, 2016; Walshaw and Anthony, 2008).

If we divide teachers’ work in a lesson into four phases: preparation, initiation, development, and closure, then we find that the teachers could play different main roles in each phase. According to the data we collected, teachers seemed to have more control in the preparation and initiation phases. Once the lesson started, they encouraged their students take an active role and were prepared to intervene as conductors and river guides when appropriate. In all phases, teachers always disciplined themselves with the expectation that they were role models for their students. Teacher’s role is multifaceted, professional, and irreplaceable (Russ et al., 2011).

The call to foster student agency in learning requires that student voice be heard, but actually doing so in teaching remains very complex due to the time constraints of a lesson. As a result, the expert teachers participated in this study placed great emphasis on designing and structuring learning tasks, quickly diagnosing student thinking, organizing whole-class dialogue, using new technology platforms to facilitate students’ self-learning, and involving students in assessment. It is worth noting that in China, primary school math teachers typically teach only math and will continue to teach the same class for several years. The members of each class are essentially fixed, and they work together as a learning community for all subjects. These are conducive to the development of good learning habits, classroom norms and interpersonal relationships that take years to develop.

Enabling students to take ownership of their learning requires, on the one hand, that teachers have excellent and varied expertise and the courage to explore, change and frequently reflect on their teaching practices; on the other hand, it requires that students have the willingness and ability to engage in active learning. Finally, and most importantly, it requires teachers to have a deep love and commitment to education and to their students. In the interviews, two teachers talked about the same lesson they taught a few years apart and told us how and why their teaching design had changed. It can be said that they have strong teacher agency in their teaching. Building teacher agency is the key to addressing student agency.

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https://doi.org/10.1142/9789811287183_0022

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Characteristics of Teacher-Student Interaction in Mathematics Classroom of Chinese Senior High Schools in the Information Technology Environment

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ABSTRACT Based on the Flanders Interaction Analysis System (FIAS) and the Information Technology-Based Interactive Analysis Coding System (ITIAS), nine high school math lessons from the National and Local Public Service Platform for Educational Resources were selected as the research objects and were analyzed to investigate the characteristics of teacher-student interaction in mathematics classroom of Chinese senior high schools in the information technology environment.

Keywords: Teacher-student interaction; Mathematics classroom; Information technology environment.

1. Research Background

At present, China's senior high school mathematics curriculum reform advocates that teachers should build a good teacher-student cooperative relationship, engage students into classroom teacher-student interaction activities, stimulate students' interest in learning, so as to realize students' independent learning and improve the classroom teaching effect.

Some people think that the "quality lesson" should be the model of teaching. But are these quality lessons really positive in terms of teacher-student interaction? There is a paucity of evidence to support this. To investigate the teacher-student interaction in "quality lesson", this study chooses the lessons from the campaign launched by the Ministry of Education to promote teacher professor development — "Every teacher should have one high-level lesson; In the practice of each lesson, a famous teacher emerges" (referred to as "one good lesson from one teacher, one good teacher from one lesson").

2. Literature Review

2.1. Previous studies of "Quality Lessons"

Generally speaking, high quality teaching means outstanding level of teaching. For high-quality classroom, it means to provide high-quality classroom teaching, achieve

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the purpose of cultivating outstanding talents, and meet the educational needs of parents and the society.

However, the classification standards of classroom teaching quality have always been different. Quality classroom is a concept of development, which is restricted by political, economic and cultural factors in different time and environment, and is the result of comparing with certain objects.

German scholar Meyer (2006) put forward ten characteristics of quality classroom teaching: (1) clear classroom teaching structure, (2) a high proportion of effective teaching time, (3) A classroom atmosphere conducive to learning, (4) Clear teaching content, (5) Creating meaningful teacher-student exchanges, (6) Diversified teaching methods, (7) To promote the individual development of students, (8) cleverly set exercises, (9) Having clear learning expectations, and (10) A complete classroom teaching environment.

Chinese scholar Lan Ye (2005) believes that a good lesson should have the following points: (1) valuable (meaningful), (2) efficient, (3) generative, (4) normality, and (5) rooms to be improved.

Through review of mathematical quality lessons at home and abroad studies, a certain commonality has been found between Chinese and western lesson for high-quality research. For instance, the mathematics classroom of high quality should have good classroom atmosphere; the students can actively participate in teaching; the teacher as a facilitator of classroom teaching can guide the student to study independently.

However, there are many differences. The research on the quality of mathematics class in China is mainly speculative, which is often the summary of the experience of the quality class. Foreign research is mainly empirical, through the development of some scales to observe and evaluate the classroom.

2.2. Studies on classroom interaction between teachers and students

The study of teacher-student interaction originated in the 1970s, when the American educator Brickley first introduced the theory of interaction into the field of education. Subsequently, researchers constantly shifted the focus of interaction research to teacher-student interaction.

The research on teacher-student interaction mainly focuses on: (1) the essential characteristics of teacher-student interaction; (2) the mode of teacher-student interaction； (3) the influencing factors of teacher-student interaction; (4) observation tools for teacher-student interaction.

2.3. Relevant research on teacher-student interaction observation tools

Flanders Interaction Analysis System (FIAS) is a kind of classroom Interaction Analysis System proposed by American scholar N. A. Flanders in the 1960s. This system innovatively uses quantitative analysis to observe classroom behavior and plays a very important role in classroom observation research.

In order to study the quality of Classroom interaction among students in different learning periods, a team led by Piata from the University of Virginia in the United States developed Classroom Assessment Scoring System (CLASS). However, the system requires professional observers to observe and evaluate, and the operating conditions are relatively strict.

The TIMSS Video Study is part of The Trend of International Mathematics and Science Study (TIMSS). This project carried out a large-scale video research, and developed a coding framework for mathematics classroom video, which mainly analyzed six dimensions including the content, mode, organization form, language, teaching fragments and overall quality of mathematics classroom teaching.

Mathematical Quality of Instruction (MQI) is a widely used Mathematical classroom assessment tool. The MQI tool was developed by academic Heidhill and his colleagues. The evaluation system of MQI reflects the interactive relationship among teachers, students and content in the process of mathematics teaching, and evaluates the quality of classroom teaching from five important dimensions. The MQI tool does not evaluate the actual classroom teaching process but evaluates the recorded classroom teaching videos. The idea is to divide each video into roughly the same length segments of 5 or 7.5 minutes. Based on the encoding of these segments, the coder gives each segment a score on five dimensions, and then calculates the score for a lesson.

Because the Flanders interactive analysis system was produced in the last century, the analysis of the current classroom information digitization is not comprehensive enough, Chinese scholars have carried out corresponding research and improvement on the Flanders interactive analysis system.

Gu and Wang (2014) put forward the Information Technology Based Interaction Analysis System (ITIAS) supported by Information Technology.

Fang et al. (2012) proposed Improved Flanders Interaction Analysis System (IFIAS).

Through the analysis of existing teacher-student interaction research tools, it is found that all kinds of classroom evaluation tools have different foci and advantages and disadvantages in the study of teacher-student interaction in the classroom. In general, most of the tools pay more attention to the evaluation of classroom teaching effectiveness. Although they are applicable to a wide range, there are few evaluation tools with mathematical subject attributes.

3. Methodology

3.1. Research questions

This study aims to answer the following questions:

1) What is the status quo of the teacher-student interaction in the “quality lesson”?
2) What are the main characteristics of high school mathematics class in the aspect of teacher-student interaction?

3.2. Selection of research samples

According to the habits of Chinese teachers, this research divides the main mathematics classes of high school mathematics into: lesson of math concepts, lessons of math principle and lessons of math exercises.

It is determined that the representatives of the mathematical concept class are "3.1.2 Meaning of Probability", the representatives of the mathematical principle class are "4 Projection Theorem of Right Triangles", and the representatives of the mathematical exercises class are "3.3 Coordinate and Distance Formula of the Intersections of Straight Lines".

Tab. 1. Research samples

Teacher code	Teaching type	Topic	level	gender
G1	Main goal of learning mathematical concepts	Meaning of probability	A	female
G2	Main goal of learning mathematical concepts	Meaning of probability	B	female
G3	Main goal of learning mathematical concepts	Meaning of probability	C	male
Y1	Main goal of learning mathematical propositions	Projection theorem of right triangles	A	female
Y2	Main goal of learning mathematical propositions	Projection theorem of right triangles	B	male
Y3	Main goal of learning mathematical propositions	Projection theorem of right triangles	C	male
X1	Main goal of solving the math-topic	About the "line intersection coordinates and distance formula" exercise	A	male
X2	Main goal of solving the math-topic	About the "line intersection coordinates and distance formula" exercise	B	female
X3	Main goal of solving the math-topic	About the "line intersection coordinates and distance formula" exercise	C	male

3.3. Research methods

The following three methods were used in the study:

(1) Lesson study/Video study;
(2) Classroom Observation;
(3) Quantitative analysis.

Nine lessons from the above three topic were chosen as the objects of Video study.

3.4. Analysis framework

Based on the existing research, this study proposes an improved mathematics Classroom Interaction Analysis System (MCIAS), and mainly makes the following adjustments to the FIAS System:

(1) due to the information technology is widely used in classroom teaching, interaction between teachers and students gradually from words to information technology as the medium of multi-dimensional interaction, so words are no longer the sole cause of the interaction, and the use of information technology and classroom

activities, such as the classroom blackboard writing without verbal interaction influence the interaction between teachers and students.

China's mathematics curriculum standards emphasize that teachers should implement open teaching in the teaching process, create a teaching situation conducive to the development of students, stimulate the learning autonomy of students, and promote the all-round development of students. The raising of open questions is beneficial to the creation of teaching situation, so teachers' questioning is divided into two categories: "raising open questions" and "raising closed questions".

Students are the main body of learning, and stimulating students' initiative in learning is the key to achieve good teacher-student interaction. Students' active talking is divided into "active response" and "active questioning". Since student discussion has become an important part of classroom speech interaction, the category of "discussion with peers" is added under the dimension of "student language".

The original coding system was very rough in dealing with "silence", and many important classroom information was ignored, which could not reflect the characteristics of mathematics. In this study, the nonverbal behavior of both teacher and student are included in the teacher-student interaction, and the original "invalid speech" is detailed, and the codes "13 for silent thinking", "14 for students' practice", "15 for students' use of technology", "16 for teachers' demonstration", "17 for teachers' use of technology" and "18 for ineffective silence or confusion" are added. More detail please refer to Tab. 2 (on the next page).

3.5. Research process

3.5.1. Coding

Follow the Flanders Analysis method, according to the time sampling method, every 3 seconds is a sample of the classroom teaching. Record the encoding in a table chronologically. A record point in the table represents an action recorded every 3 seconds, each row represents 20 actions recorded in 1 minute, and the column represents the number of minutes of the lesson. There are about 700 to 1000 codes in one lesson.

3.5.2. Construct analysis matrix

After encoding the interaction between teachers and students in mathematics class, the observation record table is organized into a data matrix of order 18×18. Among them, the number of rows and columns of the matrix represent the 18 kinds of teacher-student behaviors stipulated by the coding system (Tab. 2).

The specific methods are as follows:

Each time, two adjacent data are taken from the encoded data sequence as an "order pair". The former data represents the number of rows of the matrix, and the latter data represents the number of columns of the matrix, which are accumulated in the corresponding matrix cells.

Assumes that the encoding of speech act between teachers and students, in turn, record of 4, 8, 9, 7, 10, 13, 17, 16, 5, 6, coding and link into a “sequence”, get nine corresponding “sequence”, (4, 8), (8, 9), (9, 7), (7, 10), (10, 13), 13 (2) and (17, 16), (16, 5), (5,6).

Where, (4, 8) means counting once in the cells of the fourth row and eighth column of the matrix, and (8, 9) means counting once in the cells of the eighth row and ninth column of the matrix.

The analysis matrix can be obtained by filling all the data into the cells of the matrix in turn.

Tab. 2. Coding system

Category	Coding	Item	Description
Teacher Talk	1	Teacher’s manner of acceptance emotion	Teachers do not accept, clarify, or express students’ feelings in a threatening manner.
	2	Teacher encouragement and approval	Approve or encourage student behaviour
	3	Accepting students’ ideas	Accepting students ideas; clarifying or developing their opinions or ideas.
	4	Raising open questions	Asking questions based on teachers’ opinions or ideas and expecting students’ answers
	5	Raising close questions	
	6	Teaching	The teacher provides facts or opinions on the content or steps of the procedure, expresses his or her own opinion, or quotes from authoritative scholars.
	7	Instruction	Give instructions or orders that the student can comply with with a view to the student being able to carry out.
	8	Criticizing or defending teacher's authority	Change the behavior of students with harsh language and make it into acceptable behavior; scold and blame the students.
Student Talk	9	Passive response	(Response to code 4) Students respond to teacher questions. The teacher assigns the students to answer the questions, or triggers them to speak. Students are restricted in freely expressing their ideas.
	10	Active response	Students take the initiative to express their emotions and attitudes towards teacher behavior; students can express their opinions or ideas freely.
	11	Ask questions actively	Ask questions voluntarily and express your opinions freely.
	12	Discuss with peers	Students will discuss and exchange views with their peers.
	13	Static and silent learning	According to the teacher’s questions or instructions, students think independently, read silently, take notes, watch the teacher play videos, courseware, demonstration experiments, etc.
Student Behavior	14	Students practice	Students perform written exercises on the blackboard; students participate in games and demonstrate experiments; students participate in experimental operations independently or in groups.
	15	Students use technology	Students participate in teaching activities through information technology. Such as personal tablet, graphics calculator, answering machine and other equipment to learn
	16	Teacher demonstration	Teachers write on the blackboard, use traditional teaching AIDS or physical teaching AIDS for teaching, and operate equipment for experiments.
Teacher Behavior	17	Teachers use technology	Teachers use computers, tablet computers, slides, projectors, geometric drawing boards and other information technologies to conduct teaching activities.
	18	Invalid silence or confusion	The classroom is in a state of helpless teaching silence or chaos

3.5.3. Ratio analysis

Matrix analysis can be used to explain the deeper meaning of the interaction between teachers and students in the classroom. These indicators are called variables. The variable here is mainly the ratio value of the interaction between teachers and students calculated through the analysis matrix.

For example, the proportion of teachers' discourse, the proportion of students' discourse, the proportion of teachers' operation, the proportion of students' operation, the proportion of silence, the proportion of teachers' questioning and the proportion of teachers' response.

Tab. 3. Ratio analysis

Variant	Formula	Description
Teacher talk ratio	$\left[\frac{\sum_{i=1}^{8} R(i)}{\text{Total}}\right] \times 100$	The ratio of teacher talk time in all the teaching time. The higher ratio indicates that the teacher talks more (Norm = 68)
Student talk ratio	$\left[\frac{\sum_{i=9}^{12} R(i)}{\text{Total}}\right] \times 100$	The ratio of student talk time in all the teaching time. The higher ratio indicates that the student talks more (Norm = 20)
Teacher operation ratio	$\left[\frac{\sum_{i=16}^{17} R(i)}{\text{Total}}\right] \times 100$	The ratio of teacher demonstration time or using information technology time in all the Teaching time. The higher ratio indicates that the teacher operates more.
Student operation ratio	$\left[\frac{\sum_{i=14}^{15} R(i)}{\text{Total}}\right] \times 100$	The ratio of student practice time or using information technology time in all the teaching time. The higher ratio indicates that the student operates more.
Silence ratio	$\left[\frac{\sum_{i=13}^{18} R(i)}{\text{Total}}\right] \times 100$	The proportion of the non-verbal teacher-student interaction time in the total teaching time. The higher the proportion, the less students have verbal interaction behavior and the more non-verbal interaction behavior.
Teacher response ratio	$\left[\frac{\sum_{i=1}^{3} R(i)}{\sum_{i=7}^{8} R(i) + \sum_{i=1}^{3} R(i)}\right] \times 100$	The ratio of discourse time that teachers respond to students' views and emotions in the discourse time that teachers are not directly related to teaching (except the teachers' questions and talking). The higher the ratio, the more teacher responds to the students. (Norm = 42)
Teacher question ratio	$\left[\frac{\sum_{i=4}^{5} R(i)}{\sum_{i=7}^{8} R(i)}\right] \times 100$	The ratio of the time when teachers ask questions in the time directly related to teaching (the time when they ask questions to teaching). The higher the rate, the better the teacher is at teaching through asking questions. (Norm = 26)

3.5.4. Validity of coding

In order to verify the scientific nature of the teacher-student interaction analysis system in mathematics classroom, the researcher invited three partners to code the classroom video of G1 teacher in the case study according to the coding framework and coding rules, and analyze the consistency with the researcher's own coding.

The analysis results showed that under the premise of 99% confidence, the Pearson correlation coefficient of the researcher and the three fellow observers was all greater than 0.993**, and the observed data were all highly correlated. Thus, the coding system has the characteristics of less subjectivity and strong objectivity.

The other six examples fall into two categories: lesson of math principle and lesson of math exercises.

4. Results and Discussion

In this section, the results of lesson of math concept (Teacher G1, G2 and G3) are presented though three types of lesson are studied due to the limitation of the layout (Fig. 1 − Fig. 3).

4.1. The status quo of teacher-student interaction

The analysis of lesson of math concepts is arranged by their classroom language structure, question-and-answer of teachers and students, teachers' teaching style, teachers and students' emotion, teachers and students' behavior interaction.

Coding	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	20	6	3	7	8	0	0	1	0	0	0	1	0	2	4	0
4	0	0	4	10	1	1	0	0	10	6	0	0	6	1	0	0	2	0
5	0	0	0	2	10	1	1	0	15	0	0	0	3	0	0	0	0	0
6	0	0	0	5	8	129	4	0	0	0	0	0	1	0	0	5	18	0
7	0	0	1	6	1	1	21	0	3	3	0	0	1	10	2	6	4	0
8	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
9	1	0	18	4	1	2	2	0	3	0	0	0	0	0	2	0	0	0
10	0	0	5	1	0	3	3	0	0	2	0	0	0	0	0	0	0	0
11	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	1	0	0	0	0	0	0	11	0	0	0	0	0	0
13	0	0	2	1	2	7	3	0	2	0	0	0	47	0	1	0	0	0
14	0	0	2	0	0	0	7	1	0	0	0	1	0	33	4	0	0	0
15	0	0	0	1	1	0	2	0	0	0	0	0	0	3	29	0	2	1
16	1	0	0	2	0	7	4	0	0	1	0	0	0	0	0	55	1	0
17	0	0	0	3	3	11	2	0	0	1	0	0	7	0	1	3	57	2
18	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	2	7
Total	2	0	52	41	32	170	59	1	33	15	1	12	65	48	39	71	90	10

Fig. 1. Classroom interaction behavior analysis matrix of teacher G1

oding	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	1	0	1	0	1	1	0	0	0	0	0	0	0	0	0	2	0
2	1	0	0	0	0	3	0	0	2	0	0	0	0	0	0	0	3	0
3	0	8	24	9	4	4	0	0	6	1	0	0	1	0	0	0	2	1
4	0	0	1	50	0	5	5	0	11	7	0	2	2	1	0	0	0	2
5	1	0	0	0	33	1	5	0	12	0	0	0	4	1	0	0	0	0
6	0	0	0	9	8	157	2	0	0	0	0	0	0	0	0	3	11	0
7	1	0	0	2	2	1	5	0	15	2	0	3	0	1	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	0	30	8	3	0	3	0	78	0	0	0	1	0	0	0	0	1
10	0	0	5	2	2	0	1	0	0	10	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	1	0	0	3	0	0	0	0	60	0	0	0	0	0	0
13	0	0	0	2	0	1	3	0	1	0	0	0	3	0	0	0	0	1
14	0	0	0	1	1	0	0	0	0	1	0	0	0	16	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	1	0	4	0	0	0	0	0	0	0	0	0	19	2	0
17	0	0	0	0	4	12	0	0	0	0	0	0	0	0	0	4	0	0
18	1	0	0	0	0	1	4	0	0	0	0	0	0	0	0	0	0	2
Total	6	9	60	86	57	190	32	0	125	20	0	65	11	19	0	26	20	8

Fig. 2. Classroom interaction behavior analysis matrix of teacher G2

Coding	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
2	0	0	1	2	0	1	1	0	0	0	0	0	0	0	0	0	0	0
3	0	1	44	5	18	10	3	0	1	0	0	0	1	0	0	1	1	3
4	0	0	2	14	0	0	3	0	29	0	0	0	2	0	0	0	0	1
5	0	0	2	3	14	0	2	0	56	0	0	0	0	0	0	0	2	0
6	0	0	0	8	11	98	3	0	0	0	0	0	0	0	0	0	2	1
7	0	0	1	1	1	3	17	0	8	1	0	1	3	6	0	0	1	1
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	3	37	13	25	6	8	0	16	1	0	0	1	0	0	1	1	3
10	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	1	0	0	0	0	0	0	8	0	0	0	0	0	0
13	0	0	0	0	2	1	3	0	2	0	0	0	92	0	0	1	0	0
14	0	0	0	1	2	0	3	0	0	0	0	0	0	97	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	24	1	1
17	0	0	0	3	3	1	0	0	0	0	0	0	0	0	0	0	35	1
18	0	0	1	1	2	3	1	0	2	1	0	0	1	0	0	0	0	10
Total	1	5	88	51	79	123	44	0	115	2	0	9	101	103	0	27	43	22

Fig. 3. Classroom interaction behavior analysis matrix of teacher G3

4.1.1. Analysis of classroom speech structure

Results show that G3 and G1 give students more time to think and promote their autonomous learning (Tab. 4).

Tab. 4. Language structure of lesson of math concept

Code	Teacher talk ratio （%）	Student talk ratio （%）	Silence ratio （%）	Ratio of teacher talk to student talk
G1	48.31	8.1	43.59	5.96
G2	59.95	28.61	11.44	2.10
G3	48.09	15.5	36.41	3.10
Norm	68	20	11/12	3.4

On the whole, the ratio between teachers' speech and students' speech fluctuates alternately, and the verbal interaction between teachers and students is more frequent (Fig. 4–Fig. 6). Students have more time to communicate with teachers and students, and students can express their views more freely.

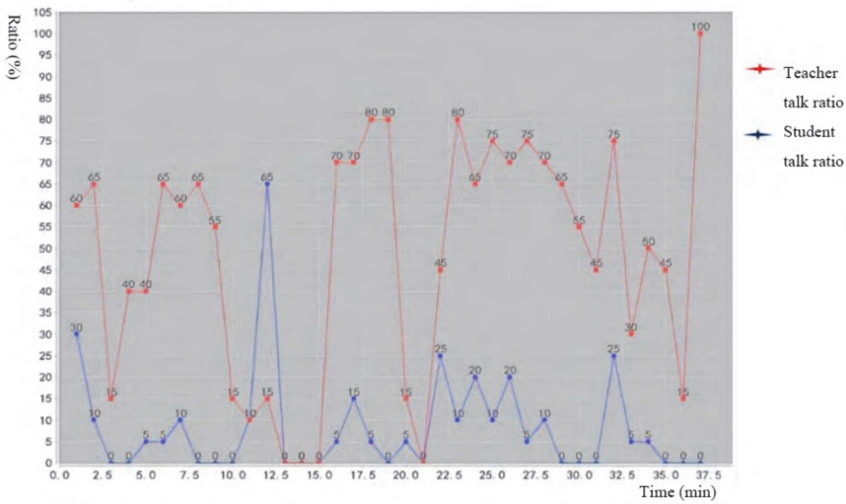

Fig. 4. Dynamic feature of ratio of teacher talk to student talk (for G1)

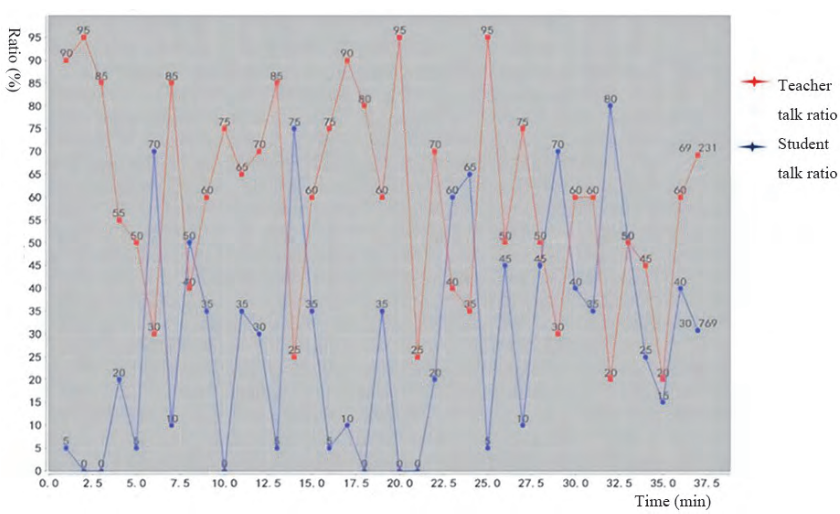

Fig. 5. Dynamic feature of ratio of teacher talk to student talk (for G2)

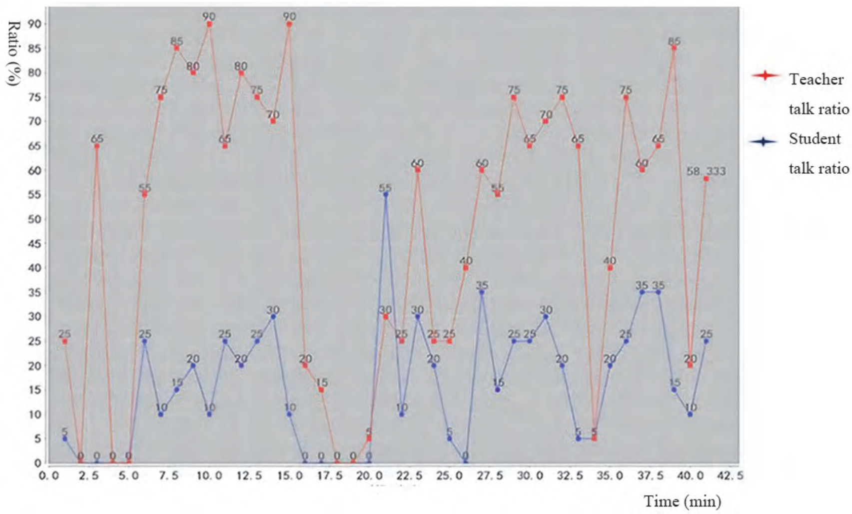

Fig. 6. Dynamic feature of ratio of teacher talk to student talk (for G3)

4.1.2. Teacher asking and student answering

As can be seen from the Tab. 5, the questioning ratio of teachers G1, G2 and G3 is greater than 26%, indicating that the three teachers are good at using questions to conduct teaching in class, and teachers G1 and G2 are good at using open questions to cause students to think. Among them, the number of open questions of G2 teachers reached 86, accounting for 60.14% of the total question ratio, while G3 teachers had more closed questions, and the rate of open questions was 39.23%, but the question ratio of G3 teachers was the highest, indicating that G3 classroom is driven by problems and promote the interaction.

Tab. 5. Statistics of teacher questions

Code	Teacher question frequency	Teacher question ratio%	Open question frequency	Close question frequency	Open question ratio%
G1	73	30.04	41	32	56.16
G2	143	42.94	86	57	60.14
G3	130	51.38	51	79	39.23
Norm	-	26	-	-	-

Tab. 6 shows that the student ratio of G1 teachers alone is close to 34%, while the spontaneous student ratio of G2 to G3 teachers is much lower than that of the norm, which is just in line with the G3 that drives classroom teaching through the closed problem and is proposed in the teaching process.

Tab. 6. Statistics of student response

Code	Passive response frequency	Active response	Active asking questions	Spontaneous student ratio %
G1	33	14	1	31.25
G2	125	20	0	13.79
G3	115	2	0	1.71
Norm	-	-	-	34

4.1.3. Teacher teaching style

From the perspective of teaching effect, Flanders divides teachers' speech into positive reinforcement and negative reinforcement, and from the teaching methods, it can also be divided into direct influence and indirect influence, the ratio of indirect influence and direct influence, positive reinforcement and negative reinforcement can be used to analyze teachers' teaching style inclination. Tab. 7 illustrates the different teaching style of the three teachers.

Tab. 7. Comparison of teaching style

Teacher code / Description	G1(%)	G2(%)	G3(%)
Percentage of numbers of indirect influence	17.27	29.84	27.55
Percentage of numbers of direct influence	31.04	30.25	20.54
Ratio of indirect influence to direct influence	0.577	0.987	1.34
Percentage of numbers of positive reinforcement	7.42	10.35	11.56
Percentage of numbers of negative reinforcement	8.1	4.36	5.41
Ratio of positive reinforcement to negative reinforcement	0.917	2.38	2.13

It can be seen from the above table that the indirect influence of teacher G1 is much less than the direct influence, and less than 1, indicating that teacher G1 prefers the teaching style and adopts less questions and opinions for students, and the positive and negative reinforcement ratio of teacher G1 is less than 1, indicating that there is less positive reinforcement, less encouragement and praise in class, and more teaching and instruction in teaching.

The ratio of indirect and direct effects of teacher G2 is close to 1, indicating that teacher G2 has complementary indirect and direct effects in the classroom; G2 has the highest ratio of positive and negative reinforcement, which G2 like to give positive feedback, and accepting students' feelings, encouragement and opinions in the classroom are higher than G1 and G3, and the evaluation language is rich.

The ratio of indirect and direct influence of teachers G3 is the highest, indicating that the indirect teaching style adopted by G3, and the ratio of positive and negative reinforcement is greater than 2, indicating that they like to accept and encourage students' emotions through positive reinforcement speech, which makes the classroom atmosphere more harmonious.

4.1.4. Teacher emotion

In classroom teaching, good classroom atmosphere can promote the emotional communication between teachers and students, so as to help to form a relaxed and harmonious psychological atmosphere, which is easier to stimulate students' learning motivation. According to the Flanders interactive analysis system, the classroom teaching atmosphere of the three teachers was analyzed by using both the positive integration lattice and the negative defect lattice in the analysis matrix.

Tab. 8. Statistics of teacher emotion

Description \ Teacher code	G1	G2	G3
Numbers of active integration of areas	20	33	46
Percentage of actively integrated regions in total	2.70%	4.50%	5.66&
Numbers of times in the defect area	3	3	8
Percentage of defect areas in total	0.4%	0.4%	0.98%

From the number of defect area, three teachers have up to 8 times, and the percentage of defect area is not more than 1%, this shows that the classroom rarely change students' behavior through instruction or critical words. According to the analysis of video, three teachers use more peaceful words to communicate with students, let students feel the relaxed classroom atmosphere, and promote their active interaction with teachers.

4.1.5. Behavior of teacher-student interaction

In the classroom interaction between teachers and students, in addition to the effective teacher-student language interaction, there is also a large number of teacher-student nonverbal behavior interaction, which contains the ineffective speech interaction in silence. If these teacher-student behavior interaction conducive to teaching is ignored, it is obviously unable to obtain the real situation in the classroom. These effective teacher-student interaction behaviors can be divided into teacher behavior and student behavior; now information technology has been widely used in classroom teaching, including teachers using multimedia, computer, interactive whiteboard and network technology, and students using touch-screen smart desks, electronic bags, tablet computers. Of course, the classroom also contains many non-information technology of traditional classroom behavior interactions, such as teachers use traditional teaching tools, teachers use blackboard writing, students use equipment for experiments and students conduct written exercises. By statistical analysis of teacher behavior and student behavior in the system, the result is shown in Tab. 9.

Among them, G1 teachers' operational teaching behavior accounted for 21.73% of the total time, For 12.15% of the time, These include using an interactive touch-screen whiteboard and classroom management software; Students' learning through operational behavior accounted for 6.48% of the total time, The use of information technology time accounted for 81.2% of the operation behavior, Students use tablet computers for classroom exercises, and conduct a large number of throwing experiments with coins in class, and use the classroom software to upload the experiment results, and cooperate with teachers for learning, realizing the classroom dual-screen interactive learning.

Tab. 9. Behaviors of teacher-student interaction

Description	G1 (%)	G2 (%)	G3 (%)
Percentage of silent thinking and learning	8.77	1.5	12.42
Percentage of student practice	6.48	2.59	12.67
Percentage of student using technology	5.26	0	0
Percentage of student operation	11.74	2.59	12.67
Percentage of teacher demonstration	9.58	3.54	3.32
Percentage of teacher using technology	12.15	2.72	5.29
Percentage of teacher operation	21.73	6.26	8.61
Percentage of invalid silence or chaos	1.35	1.09	2.71

4.2. Characteristics of teacher-student interaction

It is found that the selected high-quality lessons have excellent performance in teacher-student interaction on the whole, but they have their own characteristics in specific analysis dimensions, as follows:

a) Teachers' speech structure is good and speech act is moderate, and there is not too much control over the classroom. Teachers gives students more time for verbal interaction.
b) A large amount of time is left for non-verbal interaction in high-quality classrooms, and the types of teacher-student interactions in the classrooms are diverse.
c) The teacher-student conversation time and students' speaking time are long, and the teacher-student verbal interaction is frequent and lasting for a long time.
d) Teachers are good at interacting by asking questions and stimulating students' enthusiasm for learning by asking open questions, so students can express their opinions freely and have a better verbal interaction with teachers and peers.
e) Teachers are good at responding to students' words through the indirect influence of praise, encouragement and adoption, and are also good at interacting with students in the classroom through detailed and close questioning.
f) Teachers can organize the classroom in an orderly manner.
g) Teachers mostly use direct control methods such as lectures and instructions to mobilize students, but students are less proactive.
h) Teachers are good at adopting teaching techniques of positive reinforcement, giving positive feedback to students' opinions or emotions, and accepting, encouraging and praising students' opinions and emotions.
i) In terms of interaction, female teachers prefer to teach in the classroom through lectures, while male teachers are more inclined to interact by requiring students to conduct written exercises, classroom board performances, or participate in activities and experiments and so on.
j) Among the selected high-quality lessons, compared with provincial and municipal high-quality lesson teachers, national high-quality lesson teachers are more adept at using the blank in the classroom to give students more time to think and express.

5. Conclusions

Though the high-quality lessons have excellent performance in teacher-student interaction, there are still some room to be improved. The teachers should update the educational concept, optimize the classroom speech structure; should pay attention to the classroom to ask questions, cultivate students' awareness of asking questions; create a good classroom atmosphere and increase the depth of interaction and attach importance to the use of information technology and promote the diversification of interactive forms.

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https://doi.org/10.1142/9789811287183_0023

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The Effects of Interventions with Mathematics Manipulatives on Generalization and Maintenance for Children with Autism Spectrum Disorder: A Meta-Analysis of Single-Case Studies

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ABSTRACT There have been reviews or meta-analysis showing that using manipulatives is an effective intervention for learning mathematics for students with disabilities, including autism spectrum disorders (ASD), without concentrating on the effects on generalization and maintenance. We conducted a meta-analysis to evaluate the effect of manipulatives on generalizing and/or maintaining mathematical skills for individuals with ASD and whether the effect varies with different participant characteristics, study design, intervention characteristics and mathematical content, focusing on the single-case studies. After application of the What Works Clearinghouse design standards, a total of 11 studies were included in the review: three studies collected data points during generalization phases, five studies collected data points during maintenance phases, the other three studies collected both generalization and maintenance data. Aggregate Tau-U and non-overlap of all pairs effect sizes (NAP) were calculated for each study and conducted moderator analyses. Overall, effect size scores ranged from small to significant effects across all comparisons. On average, most comparisons from the baseline to generalization and maintenance produced medium to large effects. Whereas, minor effects were found in most of the intervention of generalization and maintenance comparisons. Further moderator analysis regard to generalization and maintenance revealed that out of seven variables analyzed, only manipulatives types served as a moderator for maintenance. The findings suggest that manipulatives interventions were likely to result in mixed effects on mathematical skill generalization and maintenance within children with ASD, especially virtual manipulatives. Limitations and implications for future research and practice are discussed.

Keywords: Manipulatives; Mathematics; Autism spectrum disorder; Generalization and maintenance; Single-case research; Meta-analysis.

1. Introduction

Manipulatives, one instructional approach, are widely used in mathematics classes (Carbonneau et al., 2013), defined as objects designed to represent explicitly and

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concretely mathematical ideas that are abstract (Moyer, 2001). They have both visual and tactile appeal and can be manipulated by learners through hands-on experience which supports students' conceptual understanding of mathematical content (Moyer, 2001). The National Council of Teachers of Mathematics (NCTM, 2000) has recommended to use manipulatives for students to develop mathematical understanding. Furthermore, systematic reviews on the use of manipulatives to improve mathematical outcomes for students indicated that using manipulatives was effective at improving mathematical outcomes (Carbonneau et al., 2013; Sowell, 1989).

Manipulatives come in two types: the physically represented form of concrete manipulatives, and the computer-generated form of virtual manipulatives (Bouck and Flanagan, 2010; Moyer et al., 2002). Concrete manipulatives are physical objects that are used to engage students in hands-on learning of mathematics to introduce math concepts. Examples of concrete manipulatives include Base 10 Blocks, Pattern Blocks, algebra tiles and fraction pieces. They allow students have numerous materials to manipulate and opportunity to sort, classify, weigh, stack and explore. Therefore, when students use concrete manipulatives to explore and master concepts, they were more engaged and motivated (Moyer, 2001). However, researchers pointed that there are two challenges associated with concrete manipulatives. One is the case that dealing with multiple physical pieces may distract students' thought process, and the other is multiple pieces may increase cognitive load, leading to a lack of mathematical concepts (Suh and Moyer, 2008). The increased presence of technology in today's classrooms supports an exploration of virtual manipulatives (Bouck et al., 2018) which may represent an appropriate substitute for concrete manipulatives (Bassette et al., 2020). Virtual manipulatives are defined as an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that enable it to be manipulated, that presents opportunities for constructing mathematical knowledge (Moyer-Packenham and Bolyard, 2016). They are viewed as three-dimensional objects that appear on a computer screen and can be transformed in multiple ways by the user and similar to concrete, except the images of manipulatives available through websites (i.e., internet-based) or tablets (e.g., app-based) (Bassette et al., 2019). Compared with concrete manipulatives, virtual ones can be altered randomly according to need, including changing the shape of the objects and they are more accessible because of the online environment. Anyhow, existing reviews showed both concrete and virtual manipulatives were effective for students in math learning (Bouck and Park, 2018; Peltier et al., 2020).

Furthermore, the use of manipulatives was an evidence-based effective instructional approach to teaching math to students with disabilities, including autism spectrum disorder (ASD) (Bassette et al., 2020; Bouck and Park, 2018; Peltier et al., 2020). ASD, one neurodevelopmental disorder, is characterized by persistent deficits in social communication and social interaction across multiple contexts and restricted, repetitive patterns of behavior, interests, or activities (DSM-5, 2013) and currently affects one in every 54 eight-year-olds (Maenner et al., 2020). Research showed that students with ASD often have trouble developing problem solving and critical thinking skills in math (Hua et al., 2012), which require high level thinking and comprehension of abstract concepts. Also, Whitby (2013) concluded that students with ASD demonstrated difficulties with abstract concepts. Considering the challenges which students with ASD face in the processing of abstract concepts, learning with manipulatives was a good way for them.

There is some evidence to suggest that manipulatives interventions improved the mathematical performance for ASD. For instance, Shurr et al. (2021) examined effects of manipulatives in the acquisition of double-digit addition and word problem-solving abilities of three elementary students with ASD using a single-case experimental design and the results presented that these interventions produce better outcomes than baseline. Jimenez and Besaw (2020) investigated the impact of virtual manipulatives, paired with graphic organizers and systematic instruction, for two elementary students with ASD and moderate intellectual disability to gain early numeracy skills and indicated a functional relationship between the use of virtual manipulatives and student math skills, supported by statistical analysis with a large effect. Additionally, students were able to generalize and maintain the skills across new math contexts. In study of Bassette et al. (2020) which compared the use of concrete and virtual manipulatives in teaching subtraction skills to elementary students with ASD by an alternative treatment design, all participants improved the ability after the treatment while only two of three participants demonstrated improved maintenance scores. Therefore, in general, manipulatives are effective instructional practice for ASD in math learning while the effects of manipulatives interventions in generalization and maintenance are yet to be discussed.

Earlier, Sowell (1989) carried out meta-analysis of the use of manipulatives applied to mathematics learning. In the study, each achievement and attitude effect size were estimated using a formula from Glass et al. (1981) and the results of 60 studies showed that mathematics achievement was increased through the long-term use of concrete manipulatives. One limitation of Sowell's synthesis was that it did not examine whether instructional characteristics or other factors moderate the effectiveness of manipulatives in math learning. Therefore, Carbonnean et al. (2013) performed a meta-analysis of 55 studies, including 30 studies designed of quasi-experiment, 13 experiments and 12 within subjects, to examine the efficacy of teaching mathematics with concrete manipulatives when compared to instruction with no concrete materials and to identify potential moderators including instructional and methodological characteristics. Aggregated mean effect size calculated by Cohen's d of 0.37 was statistically significant, in favor of the use of concrete manipulatives. Further, the effect of interventions using concrete manipulatives was moderated by both instructional (e.g., developmental status, math topic, instructional time) and methodological (e.g., peer-review status, research design, test type) characteristics of the studies.

The above two reviews all focused on typically developing children. For students with disabilities, Bouck and Park (2018) reviewed 36 articles involving mathematics manipulatives, both concrete and virtual, and summarized each study of participant characteristics, study design, mathematical content and manipulatives, effect of manipulatives and quality indicators. Of the 36, 21 were single-case design studies and

15 were group design studies. Of the 21single case studies, seven met the Horner et al. (2005) quality indicators and two of the seven studies which could be evaluated relative to the Gersten et al. (2005) experimental/quasi-experimental quality indicators were met the quality indicators, which could conclude that most of the researches that exist were low in scientific credibility or couldn't be evaluated by quality indicators. What's more, Bouck et al. (2018) collated the results of each study to analyze the impacts of manipulatives while the effect size was not calculated. Based on existing reviews and meta-analysis, Peltier et al. (2020) conducted further investigation on the effectiveness of mathematics manipulatives on students at risk or identified with a disability and explored whether the effects vary based on systematic differences related to intervention design or population characteristics. They focused on the studies with single-case experimental design, evaluating the methodological quality of studies based What Works Clearinghouse (WWC) (Kratochwill et al., 2013), measuring the effect of mathematics interventions on child outcomes using manipulatives via visual analysis, Tau-U and between-case standardized mean difference (BC-SMD), and identifying the different effects do participant characteristics, manipulative characteristics and interventions have by moderator analysis. Overall, 53 studies were met inclusion criteria and 48 studies were included in the omnibus effect size. Omnibus Tau-U effect size was 0.91 and the BC-SMD for individual studies ranged from 0.03 to 18.58, suggesting manipulatives were effective at improving the mathematical performance of students at risk or identified with a disability. Thirty-three of the 48 studies met indicators with or without reservations. Moderator analyses revealed that, of all the variables, only disability category served as a moderator.

As discussed above, they both concentrated on students with disabilities, not exclusively on ASD. What's more, the existing reviews of literature for mathematics manipulatives tend to highlight the effect of intervention, that is whether students acquire mathematical skills after intervention. However, learning for students with disabilities, including ASD, occurs in four stages: acquisition, fluency, generalization and maintenance (Alberto and Troutman, 2009; Collins et al., 2012; Shurr et al., 2019). Acquisition is the initial learning of a new behavior or response. Fluency is how well a learner can perform a specific behavior. Acquisition and fluency were most of the researchers focusing on. While, if a learner acquires a skill that does not maintain or generalize, instruction has little meaning. Generalization, perhaps the most important phase of learning, is the ability to perform a behavior across different conditions, including people, settings, activities, materials, and times of day. If learners cannot generalize or apply behaviors that have been acquired, then learning has no purpose. Maintenance refers to the ability of a learner to perform a behavior over time. In general, while students need to first acquire a skill before they can become fluent, the ultimate goal is for students to maintain the skill over time and generalize across settings, context, people, and materials (Collins et al., 2012). Each of the stages is significant for learning, while few researchers focused on skill generalization and maintenance for students with ASD. Lafay et al. (2019) conducted a systematic review to examine the immediate effects as well as maintenance and transfer of interventions with manipulatives on mathematics learning disabilities (MLD) by assessing the methodological quality. A total of 38 studies were listed, with 16 group studies and 22 single-case studies. To determine the level of methodological quality of each study, they utilized the quality indicators outlined by Gersten et al. (2005) to each group study and Horner et al. (2005) to each single-case study. The results suggested that mathematics interventions overall with manipulatives were effective for MLD. Yet, because few articles that assessed maintenance and transfer and meet the quality standards, it was unable for Lafay et al. (2019) to conclude that interventions in these studies are evidence-based practice. Lafay et al. (2019) gave a systematic review of the effects of intervention with manipulatives on immediate learning, maintenance and transfer in individuals with MLD and did not calculate the effect size or take instructional variables, student characteristics and other possible confounding and moderating variables into consideration.

Consequently, the purpose of the present study was to examine the effects of generalization and maintenance in mathematics for ASD using manipulatives. In addition, considering the necessity of individualized interventions for children with ASD, single-case experimental design can be more appropriate. Single-case experimental designs identified the influence of variables on specific behavior of a specific individual by monitoring their performance in manipulating independent variables. The performance of the monitored individual over a period of time is recorded. Individual's performance can be compared under different experimental conditions or manipulations of independent variables. As such, each individual is considered as a unit of analysis and acts as his or her own comparison (Odom et al., 2005). Meta-analysis techniques are then used to synthesize and analyze data from many single-case experiments, identifying the effect of the use of manipulatives in maintaining and/or generalizing mathematical performance of individuals with ASD by using a single metric applicable to all studies.

To sum up, we focused on the single-case studies and aimed to extend the literature by evaluating the methodological quality by WWC Design standards, reporting the effects of generalization and maintenance by calculating effect sizes; and identifying whether effects vary based on different variables related to participants characteristics, intervention design or mathematical content. This meta-analysis will help researchers to find out the current state of the extant literature and make decisions on manipulative selection and intervention design to maximize individuals' performances in generalization and maintenance. The following research questions will be solved: (a) What is the status of the extant literature regarding the measures on maintenance and/or generalization of effects of manipulatives interventions on mathematical content for participants with ASD; (b) What is the magnitude of effect (i.e., Tau-U effect sizes, non-overlap of all pairs (NAP)) of manipulatives interventions for maintaining and/or generalizing mathematical performance of individuals with ASD and (c) What effects do participant characteristics, study design, intervention characteristics and mathematical content have on maintenance and/or generalization of the effects of mathematics interventions using manipulatives?

2. Method

A comprehensive search was conducted for all studies investigating the effects of manipulatives for ASD in math study during generalization and/or maintenance phases. Search methods were consistent with the Preferred Reporting Items for Systematic Reviews and Meta-analysis (PRISMA, Moher et al., 2009), including four steps. Fig. 1 (on the next page) contains a detailed description of the search to identify eligible studies. First, an electronic search was conducted within seven electronic databases (i.e., ProQuest, ProQuest Dissertations & Theses Global, Academic Search Premier, Education Resources Information Center, PsycARTICLES, PsycINFO, and Teacher Reference Center) unlimited to the year of publication. The selection was restricted to peer-reviewed articles or dissertations published in English. For each search, the following search terms were used: (Field 1) *manipulative* with app**, OR *computer**, OR *virtual**, OR *digital**, OR *technolog**, OR *math**, OR *concrete**, OR *physical**; (Field 2) *autis**, OR *Asperger*, OR *autism spectrum disorder*, OR *ASD*, OR *PDD*, OR *pervasive developmental dis**, OR *developmental dis**, OR *DD*; (Field 3) *math**, OR *problem solving*, OR *numeracy*, OR *computation*, OR *geometry*, OR *statistic*, OR *concept*, OR *algebra*, OR *calculation*, OR *fraction*, OR *arithmetic*. The search resulted in 1668 articles and 1456 after excluding the duplications.

2.1. Inclusion criteria

To be included in the review, the following criteria for inclusion were used for eligibility. The studies (a) used a single-case design, (b) included at least one participant with ASD, (c) used a manipulative (i.e., a concrete or virtual/digital object a student would manipulate or move to aid in understanding or solving mathematics problems) as primary intervention component, (d) had at least one dependent variable relative to mathematical learning or skill acquisition, (e) collected maintenance and/or generalization data for the dependent variable relative to mathematics.

2.2. Abstract search and full text review

Applying the aforementioned inclusion criteria, a review of titles and abstracts excluded 1395 articles. If a decision could not be made upon the title and abstract alone, the article was retained for full-text screening. The full texts of each of the remaining 61 studies were screened against inclusion criteria. Two unavailable studies were excluded. Finally, a total of 17 articles were included for further analysis.

2.3. Hand search

Once all electronic files were audited and studies were chosen for inclusion in the review, the second step was to conduct a hand search within the following journals: *Exceptional Children, Journal of Special Education, Remedial and Special Education, Journal of Positive Behavior Interventions, Research in Development Disabilities,* *AJIDD-American Journal on Intellectual and Developmental Disabilities* and *Education and Treatment of Children*. All articles published from January 2019 to September 2021were screening for eligibility, while no more articles were included.

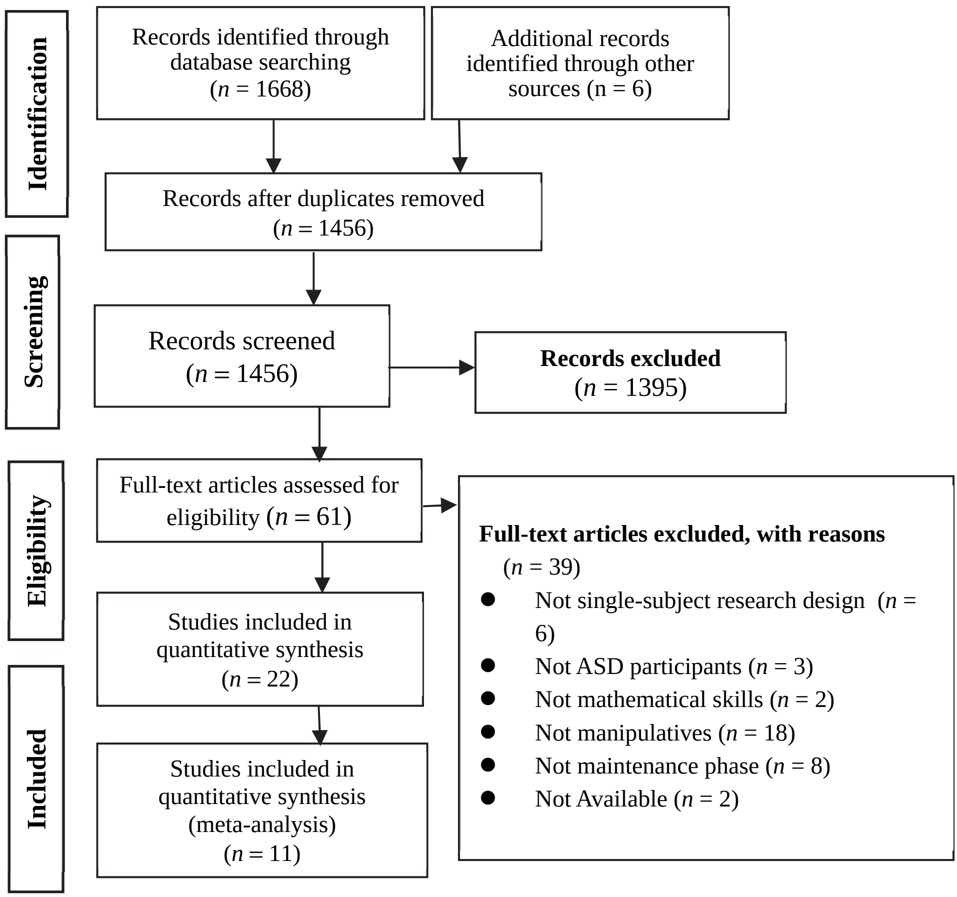

Fig. 1. Literature searches and results

2.4. Reference and citation search

The third step was to conduct a reference search. Five relevant meta-analysis and review articles' references were screened and yielded one article which met the inclusion criteria. Finally, to increase the likelihood that all the potentially relevant studied were identified, a citation search was conducted by reviewing all articles which had cited the included studies. Five additional studies were identified in the reviews. In total, 22 articles were included in this review for further analysis.

2.5. Quality assessment of studies

The methodological quality of the 22 included studies was evaluated using the WWC Pilot Single-Case Design Standards (Kratochwill et al., 2010). Design Standard 1 evaluated whether the data in the article were presented in graphical and/or tabular format. Design Standard 2 measured if the independent variable was systematically manipulated. Design Standard 3A to 3C assessed for inter-assessor agreement (IAA). Design Standard 3A evaluated whether the study reported IAA. Design Standard 3B measured if the study collected IAA in each phase and at least 20 percent of data points in each phase. Design Standard 3C assessed whether the values of IAA were at least 0.8 measured by percentage agreement or 0.6, if measured by Cohen's kappa. Design Standard 4A to 4B were about the intervention. Design Standard 4A evaluated whether the study demonstrated at least three attempts to treatment effects at least three different points in time. Design Standard 4B measured whether the study met criteria involving the number of data points depending on the design type. Design Standard 5A to 5C were additional criteria specially for multiple probe designs, as following: (a) initial pre-intervention data collection sessions must overlap vertically, (b) probe points must be available just prior to introducing the independent variable, and (c) each case not receiving the intervention must have a probe point in a session where another case either first receives the intervention or reaches the pre-specified intervention criterion. Following the application of the design standards, each article was assigned to a score for overall design classification to indicate whether the study's design "Met Design Standards", "Met Design Standards with Reservations", or "Did not Meet Standards".

2.6. Coding of studies

Referring to the work conducted by previous researchers (Bouck and Park, 2018; Carbonneau et al., 2013; Peltier et al., 2020; Spooner, 2019), the included articles that met WWC standards with or without reservations were summarized on the following categories: (1) participant characteristics, including gender, age and co-occurring diagnosis; (2) study design, (3) intervention characteristics, containing interventionist, the type of the manipulatives used (e.g., concrete, virtual) and the instructional sequences (e.g., concrete-representational-abstract (CRA), virtual-representational-abstract (VRA), virtual-abstract (VA) , and virtual-representational (VR) sequences) (Bouck et al., 2021) (4) mathematical content including number and operation, algebra, geometry, measurement, data analysis and probability of the NCTM (2000).

Furthermore, according to Schlosser and Lee (2000) and Neely et al. (2016), generalization and maintenance variables could be summarized according to the following categories: (a) generalization dimension, (b) generalization assessment design, (c) maintenance assessment design, and (d) latency to maintenance probes. The generalization dimension included three categories: (a) setting (i.e., the data was collected across setting), (b) material or behavior, (i.e., the data was collected across material or behaviors) (c) person (i.e., the data was collected across persons). The generalization assessment design contained three categories: (a) single probe (i.e., one data was collected in a generalization session), (b) multiple probes (more than one probe were collected in the duration of the study), and (c) continuous probes (i.e., the data were collected during the baseline, intervention and post-intervention session).

The maintenance assessment design variable included three categories: (a) single probe (i.e., only one maintenance data point was collected in a post-intervention phase), (b) multiple probes (i.e., more than one probes were collected during the maintenance phase), and (c) sequential withdrawn (i.e., the intervention components were sequentially withdrawn in consecutive experimental phases). In addition, the description of the latency to maintenance probes was coded (i.e., 2-week follow-up). We also coded the number of the sessions during the maintenance phases.

2.7. Data extraction

Numerical values for each graphed data point in each study were extracted to format graphed data into comma separated files by a web-based tool WebPlotDigitizer (https://automeris.io/WebPlotDigitizer/). In the event that data measured skills of participants without ASD, these data were excluded from the analysis. The extracted data were categorized by baseline, intervention, generalization and maintenance phases.

2.8. Data analysis

In the review, Tau-U and NAP, measures of effect size, were calculated for each study with the web-based Tau-U and NAP calculator, available on singlecaseresearch.org (Vannest et al., 2016). Due to no consensus on which effect size measure is best for addressing the complexity of single-case studies currently, it is better to compute more than one measure when synthesizing the literature. Tau-U is interpreted as the percent of nonoverlapping data minus the percent of overlapping data (Parker et al., 2011). NAP is defined as the percentage of all pairwise comparisons across Phases A and B, which show the percentage of data which improve across phases (Parker and Vannest, 2009). In this meta-analysis, Tau-U and NAP were selected as the effect size measure because they both have greater statistical power and precision, simple calculation and the ability to calculate confidence intervals. In addition, they tend to be less susceptible to outliers (Parker et al., 2011). In the study, data in the maintenance or generalization phases were contrasted with both baseline and intervention phases within a participant/condition. For example, for one condition in a multiple-baseline design with ABC design, in which C collected maintenance or generalization data, one contrast would be A-C and a second B-C. Different resulting Tau-U or NAP scores indicate different effects. Tau-U scores less than or equal to 0.62 indicate a small effect, 0.63-0.92 a medium effect, and 0.93 and above a large effect (Parker et al., 2011). And for NAP scores, values less than or equal to 0.65 indicated a small effect, 0.66–0.92 a medium effect, and 0.93 and above a large effect (Parker and Vannest, 2009).

After Tau-U effect sizes were calculated, we enter these data into the Comprehensive Meta-Analysis software program (Version 3; Borenstein et al., 2005), along with each associated standard error (SD_{Tau}) to conduct moderator analyses. A random effects model was preferred in this case because the studies included in this meta-analysis vary in the participants, outcome measures, procedures, and settings, and it was hypothesized that the variance between studies was on account of systematic differences instead of sampling error alone (Borenstein et al., 2009; Lipsey and Wilson, 2001). Moderator analyses generating an effect size for each potential moderator and its associated subgroups and statistically significant were detected by analyzing associated p value for the between study variance (i.e., Q_b).

2.9. Inter-observer agreement

Inter-Observer Agreement (IOA) was conducted on all aspects of the searches, including the initial screening of inclusion criteria, descriptive study characteristics and data extraction, to ensure all the appropriate studies had been included and correct information had been recorded. Agreement between raters defined as both raters have determined whether to include or exclude the same study, and if both raters agreed that the information represented in the data extraction table was an accurate representation of the study. IOA scores were calculated by dividing the agreements between the two raters by agreements plus disagreements and multiplying by 100.

In the study, the first and second authors, who have got professional training, screened the articles independently. For each stage of the literature review, 50% of articles were screened by the second author. The IOA results of the two raters were as follows: 97.4% for abstract screening, 92.3% for the full text review, 100% for hand search, 100% for reference search, 97.8% for citation search. Also, IOA was conducted on all the articles which meet the inclusion criteria of the WWC Design Standard coding and the result was 99.5%. Additionally, the IOA was 97.2% for study coding and 99.3% for data extraction of all the articles which were determined to have “Met Design Standards” or “Met Design Standards with Reservations”. All disagreements between two raters were discussed and resolved by consensus.

3. Results

3.1. Quality of studies

Of the 22 articles that met the pre-set inclusion criteria, 11 of 22 articles (50%) were determined to have “Met Design Standards” or “Met Design Standards with Reservations” which included two dissertations. Specially, three of the 11 articles gathered data points during generalization phases, five of them gathered data points during maintenance phases, only three of them gathered both generalization and maintenance data points.

3.2. Characteristics of studies

Tab. 1 shows the general characteristics of each of the 11 articles in which the participant characteristics, study design characteristics, intervention characteristics and mathematical content of individual studies are diverse. Tab. 2 and Tab. 3 present the tau-U and NAP scores of generalization and maintenance respectively.

Tab. 1. Descriptive of individual studies

Studies	Gender	Age	Co-occurring Diagnosis	Study Design	Interventionist	Manipulative Type	Instructional Sequences	Mathematical Content	Generalization Dimension	Generalization Assessment Design	Number of Generalization Sessions	Maintenance Assessment Design	Latency to Maintenance Probes	Number of Maintenance Sessions
Agrawal (2013)	M5 F1	E	ADHD (2) PDD-NOS (1)	MBD-P	Researcher	Concrete	CRA	Number and Operation (fraction)	Material or behavior	Multiple probes	3	Multiple probes	4 weeks in experiment 1; 2 weeks in experiment 2	2
Bassette et al. (2019)	M3	E	--	ATD	Researcher	Concrete and virtual	--	Number and Operation (subtraction)	--	--	--	Multiple probes	1-2 weeks	3
Bassette et al. (2020)	M3	E	--	ATD	Researcher	Concrete and virtual	--	Number and Operation (fraction)	--	--	--	Multiple probes	1-2 weeks	3
Bouck et al. (2019)	F1	M	--	MPD-P	Researcher	Virtual	VR	Number and Operation (fraction)	--	--	--	Multiple probes	2 weeks	2
Bouck, Park, Levy et al. (2020)	F1	M	--	MPD-P	Researcher	Virtual	--	Number and Operation (division)	Material or behavior	Multiple probes	3	--	--	--
Bouck and Park (2020)	F1	M	--	MPD-P	Researcher	Virtual	--	Number and Operation (addition)	Material or behavior	Multiple probes	5	Multiple probes	2 and 4 weeks	4
Bouck et al. (2020)	F1	M	--	MPD-P	Researcher	Virtual	VR	Number and Operation (fraction)	--	--	--	Multiple probes	2 weeks	2
Cihak and Grim (2008)	--	H (4)	ID (4)	MPD-B	Teacher	Concrete	--	Number and Operation (money)	Setting	Multiple probes	3	Single probe	6 weeks	1
Weng (2019)	M5	M (3) H (2)	ID (1)	ATD	Researcher	Virtual	--	Number and Operation (money)	setting	Multiple probes	4	--	--	--
Yakubova et al. (2016)	M3 F1	P (1) E (3)	--	MBD-B	Researcher	Concrete	CRA	Number and Operation (mixed)	--	--	--	Multiple probes	3weeks	3
Yakubova et al. (2020)	M2 F1	M (3)	ADHD/SLD/ EP (1) APD/SLD (1)	MPD-P	Researcher	Concrete	--	Number and Operation (fraction)	Material or behavior	Multiple probes/ single probe	3,3,1	--	--	--

NOTE. P = preschool-aged; E = elementary-school-aged; M = middle-school-aged; H = high-school-aged; ADHD = attention deficit and hyperactivity disorder; PDD-NOS = pervasive developmental disorder not otherwise specified; ID = intellectual disabilities; SLD = specific learning disorder; EP = epilepsy; APD = auditory processing disorder; MBD-P = multiple-baseline design across participants; ATD = alternative treatment design; MPD-P = multiple-probe design across participants; MPD-B = multiple-probe design across behaviors; MBD-B = multiple-baseline design across behaviors; CRA = concrete-representational-abstract; VR = virtual-representational

Tab. 2. Tau-U and NAP effect sizes per study: Generalization

Study	Baseline vs Generalization		Intervention vs Generalization		Mean Effect Size Per Study	
	Tau-U	NAP	Tau-U	NAP	Tau-U	NAP
Agrawal (2013) Experiment 1	1.00 CI_{95} [0.77, 1.00]	1.00 CI_{95} [0.77, 1.00]	0.51 CI_{95} [0.27, 0.74]	0.75 CI_{95} [0.52, 0.99]	0.76 CI_{95} [0.59, 0.92]	0.88 CI_{95} [0.71, 1.00]
Agrawal (2013) Experiment 2	1.00 CI_{95} [0.77, 1.00]	1.00 CI_{95} [0.77, 1.00]	0.61 CI_{95} [0.37, 0.85]	0.81 CI_{95} [0.57, 1.00]	0.81 CI_{95} [0.64, 0.97]	0.90 CI_{95} [0.74, 1.00]
Bouck, Park, Levy et al. (2020)	1.00 CI_{95} [0.40, 1.00]	1.00 CI_{95} [0.40, 1.00]	-0.47 CI_{95} [-1, 0.15]	0.27 CI_{95} [-0.35, 0.89]	0.28 CI_{95} [-0.33, 0.89]	0.64 CI_{95} [0.03, 1.00]
Bouck and Park (2020)	0.80 CI_{95} [0.33, 1]	0.90 CI_{95} [0.43, 1]	-0.40 CI_{95} [-0.93, 0.13]	0.30 CI_{95} [-0.23, 0.83]	0.23 CI_{95} [-0.27, 0.74]	0.62 CI_{95} [0.11, 1]
Cihak and Grim (2008)	1.00 CI_{95} [0.77, 1.00]	1.00 CI_{95} [0.77, 1.00]	0.48 CI_{95} [0.27, 0.70]	0.74 CI_{95} [0.53, 0.95]	0.73 CI_{95} [0.57, 0.89]	0.87 CI_{95} [0.71, 1.00]
Weng (2019)	0.92 CI_{95} [0.56, 1.00]	0.96 CI_{95} [0.60, 1.00]	-0.07 CI_{95} [-0.29, 0.15]	0.46 CI_{95} [0.24, 0.68]	0.30 CI_{95} [0.09, 0.51]	0.65 CI_{95} [0.44, 0.86]
Yakubova et al. (2020)	0.50 CI_{95} [-0.10, 1]	0.75 CI_{95} [0.15, 1.00]	-0.30 CI_{95} [-0.91, 0.31]	0.35 CI_{95} [-0.26,0.96]	0.20 CI_{95} [-0.20, 0.60]	0.60 CI_{95} [0.20, 1.00]
Mean Effect Size	0.94 CI_{95} [0.75, 1.00]	0.97 CI_{95} [0.78, 1.00]	0.23 CI_{95} [0.04, 0.43]	0.62 CI_{95} [0.42, 0.81]	0.63 CI_{95} [0.55, 0.71]	0.81 CI_{95} [0.73, 0.89]

Tab. 3. Tau-U and NAP effect sizes per study: Maintenance

Study	Baseline vs Maintenance		Intervention vs Maintenance		Mean Effect Size Per Study	
	Tau-U	NAP	Tau-U	NAP	Tau-U	NAP
Agrawal (2013) Experiment 1	1.00 CI_{95} [0.73, 1.00]	1.00 CI_{95} [0.73, 1.00]	0.41 CI_{95} [0.13, 0.68]	0.70 CI_{95} [0.43, 0.98]	0.71 CI_{95} [0.51, 0.90]	0.85 CI_{95} [0.66, 1.00]
Agrawal (2013) Experiment 2	1.00 CI_{95} [0.73, 1.00]	1.00 CI_{95} [0.73, 1.00]	0.59 CI_{95} [0.32, 0.87]	0.80 CI_{95} [0.52, 1.00]	0.80 CI_{95} [0.60, 0.99]	0.90 CI_{95} [0.71, 1.00]
Bassette et al. (2019)	0.67 CI_{95} [0.16, 1.00]	0.83 CI_{95} [0.33, 1.00]	0.36 CI_{95} [-0.15, 0.86]	0.68 CI_{95} [0.17, 1.00]	0.51 CI_{95} [0.15, 0.87]	0.76 CI_{95} [0.40, 1.00]
Bassette et al. (2020)	0.00 CI_{95} [-0.49, 0.49]	0.50 CI_{95} [0.01, 0.99]	-0.20 CI_{95} [-0.71, 0.31]	0.40 CI_{95} [-0.11, 0.91]	-0.10 CI_{95} [-0.45, 0.25]	0.45 CI_{95} [0.10, 0.80]
Bouck et al. (2019)	1.00 CI_{95} [0.28, 1.00]	1.00 CI_{95} [0.28, 1.00]	0.00 CI_{95} [-0.69, 0.69]	0.50 CI_{95} [-0.19, 1.00]	0.49 CI_{95} [-0.21, 1.00]	0.75 CI_{95} [0.04, 1.00]
Bouck et al. (2020)	1.00 CI_{95} [0.25, 1.00]	1.00 CI_{95} [0.25, 1.00]	-0.38 CI_{95} [-1.00, 0.29]	0.31 CI_{95} [-0.35, 0.98]	0.27 CI_{95} [-0.44, 0.98]	0.64 CI_{95} [-0.07, 1.00]
Bouck and Park (2020)	1.00 CI_{95} [0.49, 1.00]	1.00 CI_{95} [0.49, 1.00]	-0.75 CI_{95} [-1.00, -0.18]	0.13 CI_{95} [-0.44, 0.69]	0.17 CI_{95} [-0.37, 0.71]	0.59 CI_{95} [0.05, 1.00]
Cihak and Grim (2008)	1.00 CI_{95} [0.63, 1.00]	1.00 CI_{95} [0.63, 1.00]	0.48 CI_{95} [0.13, 0.83]	0.74 CI_{95} [0.39, 1.00]	0.73 CI_{95} [0.48, 0.99]	0.87 CI_{95} [0.61, 1.00]
Yakubova et al. (2016)	0.78 CI_{95} [0.55, 1.00]	0.89 CI_{95} [0.65, 1.00]	-0.08 CI_{95} [-0.30, 0.14]	0.46 CI_{95} [0.24, 0.68]	0.34 CI_{95} [0.18, 0.50]	0.67 CI_{95} [0.51, 0.83]
Mean Effect Size	0.83 CI_{95} [0.62, 1.00]	0.92 CI_{95} [0.71, 1.00]	0.16 CI_{95} [-0.04, 0.37]	0.58 CI_{95} [0.38, 0.78]	0.53 CI_{95} [0.45, 0.62]	0.77 CI_{95} [0.68, 0.85]

3.2.1. Participant characteristics

Of all studies, there were 32 participants. Excluding four subjects whose gender was not specified in the study, 21 of the 28 participants were male (75%). For the age, 15 participants (46.88%) were elementary-aged students, 10 (31.25%) were middle school students, six (18.75%) were high school students and only one (3.13%) was a preschool student. Besides ASD, five participants (15.63%) had comorbid disabilities of intellectual disabilities, two (6.25%) with co-occurring attention deficit and hyperactivity disorder and one (3.13%) with pervasive developmental disorder not otherwise specified. What's more, one ASD case was diagnosed as specific learning disorder (SLD), and the other one was diagnosed as comorbid auditory processing disorder and SLD.

3.2.2. Study design characteristics

Among all the studies, most of the articles (n = 6, 54.55%) were designed of multiple probe design, in which, a multiple probe across participants design of single-case research design was used in five studies and the remained used a multiple probe across behavior design. Besides, two (18.18%) articles were designed of multiple-baseline design including multiple-baseline across participants and multiple-baseline across behavior. The remaining three articles (27.27%) were designed of alternative treatment design.

3.2.3. Intervention characteristics

Of the eleven studies, ten studies' interventionists were researcher (90.91%), and only one study's interventionist was teacher (9.91%). For the type of the manipulative used for intervention, four of the eleven studies (36.36%) employed concrete manipulative (e.g., base 10 blocks, colored chips, flashcards), five (45.45%) used virtual manipulatives (e.g., Fraction Tiles app, Number Lines app), and two (18.18%) conducted both concrete and virtual manipulatives. Also, two studies (18.18%) used the CRA framework and two (18.18%) studies used the virtual-representation (VR) framework.

3.2.4. Mathematical content

All of the studies focused on the number and operation. Furthermore, five studies (45.45%) addressed fraction problems; four (36.36%) focused on the basic operations, such as the subtraction; and two (18.18%) were about money.

3.2.5. Maintenance characteristics

Across the eight articles that collected maintenance of intervention effects, seven collected multiple maintenance data points (87.5%) and only one collected single maintenance follow-up data point (12.5%). None collected maintenance data using a sequential withdrawal design.

For all articles, data was collected anywhere from one week following completion of the intervention phase to six weeks following the intervention. Six articles (75%) collected maintenance data within four weeks after the conclusion of the intervention phase. One article (12.5%) collected data up to six months following the intervention phase. Moreover, one article conducted two experiments, whose follow-up probes were harvested two weeks in experiment one and four weeks in experiment two. Finally, half of the eight studies extracted at least three data points within each maintenance phase.

3.2.6. Generalization characteristics

Of the six studies which collected data point during generalization, five of them collected multiple probes in the duration of the study. The remaining one study collected only one probe for one of the subjects and multiple probes for the other subjects. Additionally, two of the six studies assessed generalization of effects across settings, and four evaluated across behavior and materials.

3.3. Overall effect size

Overall, raw data for a total of 211 separate contracts (i.e., baseline/intervention vs. maintenance/generalization) from 11 articles with 32 participants were extracted to calculate effect sizes. Tab. 2 and 3 present the results of Tau-U and NAP scores across articles. Results from Baseline vs. Maintenance comparisons were medium (mean NAP = 0.92 and mean Tau-U = 0.83) with a variable range of effect sizes (NAP = 0.50~1.00 and Tau-U = 0.00~1.00). Intervention vs. Maintenance comparisons produced small findings with a mean NAP of 0.58 (0.13~0.80) and Tau-U of 0.16 (−0.75~0.59). Results from Baseline vs. Generalization comparison were significant with a mean NAP of 0.97 (1.00~0.75) and Tau-U = 0.94 (0.50~1.00). Intervention vs. Generalization comparisons were small with a mean NAP of 0.62 (0.27~0.81) and Tau-U = 0.23 (−0.47~0.61).

3.4. Moderator analysis

To identify whether effects with respect to generalization and maintenance varied across participant characteristics, study design, intervention characteristics and mathematical content, seven variables were examined: age, gender, co-occurring diagnosis, study design, interventionist, the type of manipulatives, mathematical content. Tab. 4 and 5 summarize the results from the analysis of the moderator analysis.

Firstly, for baseline to maintenance in comparison, of all the variables analyzed, the type of manipulative variable was the only variable that had statistically significant differences between the categories analyzed ($Q = 6.64$, $p = 0.04$). And the mean effect size for studies with virtual manipulative was statistically greater than that of concrete or virtual/concrete manipulative. The participant characteristics variables, including gender, age and co-occurring diagnosis, did not function as moderators. Meanwhile, study design, interventionist variable and mathematical content did not find statistically significant.

Whereas, for baseline to generalization comparison, all of the variables, including participant characteristics, study design, intervention characteristics and mathematical content, did not show statistical differences.

4. Discussion

Through a complete screening process, 22 studies examining mathematics manipulatives interventions for individuals with the diagnosis of ASD meet the inclusion criteria, among which 11 studies met WWC standards. This meta-analytic review examined 11 studies aiming to analyze varying study characteristics, to evaluate the extent to which intervention using manipulatives for individuals with ASD contributed to generalization and maintenance in mathematics, whether the effect vary with different participant characteristics, study design, intervention characteristics and mathematical content, and to provide suggestions for practice and future research.

4.1. Major findings and implications

4.1.1. Quality of evidence

In analyzing the 22 studies, we first evaluated the quality of evidence of the studies. We found 11 of them have met the WWC design standards with or without reservations. Since most of the studies were designed of multiple probe design, failing to meet the additional criteria specially for multiple probe designs was the primary reason that studies did not meet the WWC design standards. The same failure of additional criteria specially for multiple probe designs has also been notes in the study on using mathematics manipulatives with students at risk or identified with a disability (Peltier, 2020), indicating that the studies using multiple probe design needs to design the experimental process more carefully, and the experimental data should be collected and recorded reasonably in future studies. Additionally, failing to meet the design standards for IAA and insufficiency of data points in each phase (i.e., fewer than three data points in a phase) were also one of the reasons why the study did not meet WWC design standards, suggesting that the data integrity will need to be monitored more carefully in future studies. The poor experimental design may affect the credibility of the results. The effect size of these studies which did not meet the WWC design standards requires more careful interpretation. Therefore, the calculation of effect sizes and a moderator analysis in this study focuses only on studies which have met the WWC design standards with or without reservations.

4.1.2. Participant and intervention characteristics

We examined the study characteristics before analyzing the magnitude of effects of the mathematics manipulatives interventions. The results indicated that school-age children have been the main focus of manipulative studies on individuals with ASD. This may show a lack of knowledge about the effectiveness of manipulatives interventions for young children with ASD. In order to address the disparity between study populations, more research is needed on manipulatives interventions maintaining and/or generalizing relevant mathematics skills.

In examining the types of manipulatives interventions, we found that more studies paid attention to virtual manipulatives, which was similar to the finding in previous research (Bouck and Park, 2018). Additionally, two studies used the CRA instructional sequence and two adopted the VR instructional sequence. Based on the application of the indicators and standards, Bouck, Satsangi, et al. (2018) confirmed that the CRA instructional sequence was an evidence-based practice for students with learning disabilities. Students with ASD may benefit from this instructional sequence as well, and this assumption could to be verified in future studies. Meanwhile, with the development of virtual manipulatives, exploring the effect of VR framework is also a direction for further studies.

4.1.3. Mathematics content

We noted that target mathematical topics of studies included all focused on the number and operation, especially fraction problems and basic operations. On one hand, fractions and operations are important basic knowledge and skills of mathematics. Fraction problems are generally considered as the foundation of learning algebra and more advanced mathematics (Fuchs et al., 2014). Meanwhile, students often have difficulty learning fraction knowledge. Many middle and high school students are still unable to master the ideas and procedures taught about fractions in the elementary grades (Ni, 2001). Besides, operation is an important component of solving mathematical word problems (Fuchs and Fuchs 2002). Without computational accuracy and fluency, students would not be able to engage in higher level problem-solving skills, let alone actively participate in inclusive general education classrooms (Butler et al., 2001). NCTM (2000) even listed fluent computation as a goal for mathematics instruction. On the other hand, using manipulatives is beneficial to the instruction of number and operation. First of all, the number and operation are considered procedural skills or procedural understanding (Rittle-Johnson, 2017), which is the ability to both know which procedure to follow and complete the appropriate steps to arrive at the correct answer. Besides, there were very mature manipulatives to teach number and operation, such as Base 10 Blocks for teaching computation, fraction pieces of the fraction and flashcards for money-related skills. Interactions with manipulatives may help them better understand the knowledge. Moreover, several studies have identified manipulatives as an effective strategy for students with ASD (Bassette et al., 2020) and usually designed instruction with steps by steps (Shin et al., 2017). Therefore, for ASD, using manipulatives appears to be an effective way to teach number and operation.

However, for other mathematical content such as measurement, algebra, geometry, statistics and probability, more study is needed to evaluate the effectiveness of manipulatives. We point out the omission as a suggestion for further research.

4.1.4. Generalization and maintenance characteristics

As showed in the literature review, lack of articles involved the effects of manipulatives interventions for participants with ASD on maintaining and/or generalizing the related math skills which showed the neglect of generalization and maintenance. Notably, three of the 11 studies collected data points during generalization phases, five of them collected data points during maintenance phases, only three of them gathered both generalization and maintenance data. This is due to the fact that generalization and maintenance measures are somewhat more difficult to implement. As noted, one way to ensure that learners are maintaining what they have been taught is to conduct periodic probes over time where learners are required to perform targeted skills, requiring long-term follow-up of participants. However, in the real experiment, it may be impossible to track the subjects due to the school holidays or personal factors of participants, and therefore not enough maintenance probes can be collected (e.g., Saunders, 2014). Let alone generalization which needs to be measured across individuals, materials or settings, putting forward higher requirements for researchers. In spite of this, it suggested that practitioners and researchers should pay more attention to the generalization and maintenance of skills, which are the core segments in learning, despite the difficulties in measuring or collecting the performance of students during generalization and maintenance.

Additionally, inclusion of maintenance and/or generalization phases in quality indicators has not been identified as a requirement for methodological soundness, resulting in a lack of focus on maintenance and/or generalization phases in single-case study designs (Kratochwill et al., 2013). As Collins (2012) pointed, however, learning to do a skill in one context with one instructor did not necessarily mean individuals with significant disabilities would apply that skill (i.e., maintain and generalize it) whenever and wherever it was needed or would be useful. Thus, we suggest that future studies pay more attention to implementing the generalization and maintenance phases.

Moreover, included articles collected mean 2.5 probes during maintenance phases and 3.5 probes during generalization phases. Interpretation of single-case research data depends on the trend and slope of a data path (Kratochwill et al., 2013). A minimum of three data points is necessary to meet basic design standards, with more data points leading to stronger conclusions regarding the data set. Although half of the studies collected less than three probes during maintenance or generalization, most collected at least two probes which were a strength of the literature base. On the other hand, however, all of the studies collected maintenance data less than six weeks following completion of the intervention. Given the latency of the maintenance probes and these short follow-up time, whether the effects of these interventions can be sustained in a long run after the training period is questionable.

4.1.5. Magnitude of effects on generalization and maintenance

We especially focused on the magnitude of manipulatives intervention effects on maintaining and/or generalizing the mathematical performance of individuals with ASD. In general, it is gratifying that the use of manipulatives really improved mathematical performance of students with ASD during generalization and maintenance probes as compared to baseline probes as both Tau-U and NAP scores were positive in most of these studies.

Data from the present study provided information on the change from baseline to generalization and maintenance phases and the change from intervention to generalization and maintenance sessions. As mentioned above, the purpose of mathematics manipulative intervention included grasping the mathematical content across time and conditions as compared to baseline. Omnibus effect size for change from baseline to generalization and maintenance indicated the desired change in mathematical performance. For intervention to generalization and maintenance, the omnibus effect size indicated the slight effect. While the effect size for the change from intervention to maintenance was slightly above zero. Through the intervention, the participants did gain the targeted content and could generalize them across the conditions. However, as time went on, they may have forgotten some of these.

Meanwhile, we also concerned whether the effects on generalization and maintenance of manipulatives in math vary among participant characteristics, study design, intervention characteristics and mathematical content. Through the moderator analysis, for both generalization and maintenance, the effects regarding the effectiveness of manipulatives for supporting mathematics instruction were consistent across the participants included. This was positive because the results presented manipulatives were effective instructional methods for all students with ASD regardless of the age, gender and co-occurring diagnosis. Of note, effects were also consistent across implementer (i.e., teacher vs. researcher), which is consistent with findings from Peltier et al. (2020). This is promising because the findings suggest, with training, teachers can implement the intervention and yield comparable effects as researchers with expertise in the intervention. We thus call for the relevant training of teachers in the use of manipulatives.

The effectiveness of manipulatives for supporting mathematics instruction was consistent across a variety of mathematical content (e.g., addition, subtraction, division, mixed operation, fraction and money). However, as pointed out before, the target mathematical topics of the included studies all focus on numbers and operations and thus lack topics such as measurement, data analysis and probability, algebra and geometry. Consequently, this conclusion may not be representative and should be interpreted prudently.

Effects regarding the effectiveness of manipulatives for maintaining mathematical performance were significantly different among various types of manipulatives. The virtual manipulatives yielded larger effects than the concrete ones, and even better than the combination of virtual and concrete manipulatives. Many researchers investigated the potential of using virtual manipulatives in math learning for individuals with ASD recently and suggested that, comparing to the concrete manipulatives, students preferred virtual manipulatives (Bassette et al., 2019; Bassette et al., 2020; Bouck et al., 2014). Additionally, systematic reviews and meta-analysis which suggested that virtual manipulatives were more effective for students with disabilities in skill acquisition, comparing with concrete manipulatives (Bouck et al., 2018; Peltier et al., 2020). The participants with better skill acquisition may perform better over time.

Since it is difficult for children with ASD to understand abstract concepts, teachers should try to use manipulatives, especially virtual manipulatives, in classroom teaching to help students learn, maintain and generalize mathematical concepts.

The other explanation for this result may be that children with ASD are individuals with high visual abilities (Fossett, 2004) and it is difficult for them to stay focus in one lesson (Bai et al., 2015). Virtual manipulative is a digital interactive experience that depict mathematical concepts and is one form of visual stimulation. For children with ASD, they may be an effective source of help to be applied in their learning style. Meanwhile, one of the features of virtual manipulatives was that they could focus students' attention on particular aspects of mathematical objects — aspects that they otherwise may not have paid attention to (Anderson-Pence, 2017). In addition, Suh and Moyer (2008) pointed that the concrete manipulatives may distract students' thought process and increase cognitive load. While virtual manipulatives can not only compensate for the deficiencies, but also provide additional visual information which was not available with the concrete manipulatives. Students also can be provided with individualized scaffolds through virtual manipulatives. What's more, according to Reimer and Moyer (2005), one advantage of virtual manipulatives is the capability of connecting dynamic visual images with abstract symbols, a limitation of regular manipulatives. Thus, teachers could try to make more use of virtual manipulatives in the classroom for better learning effect. Furthermore, as an example of assistive technology, virtual manipulatives can support the mathematics learning of school-age students in online and blended learning environments.

However, researchers should seek to systematically compare the implementation of concrete and virtual manipulatives for students with ASD. What's more, key details about the teacher's (or researcher's) practice, such as what he or she says to children and shows them at key moments in teaching, are often omitted. Thus, additional research on the specific process of applying virtual manipulatives to improve mathematics performance of students with ASD is still needed.

4.2. Limitations and future studies

While these findings suggested that manipulatives were effective in maintaining and/or generalizing mathematical skills for students who were identified as ASD, there were some limiting factors to consider when evaluating the results.

First, the review may be impacted by potential publication bias. Only studies published in peer-reviewed journals were included. Second, another limitation of this meta-analysis is related to the number of studies ($n = 11$) that were included. Only eight studies collected data points during maintenance phases and six collected generalization data. Due to the small number of studies, the analysis was limited to examining a few moderating variables. Third, effect sizes for individual studies were based on the data presented in the articles, which were extremely limited in some cases. Many of the articles presented limited generalization and maintenance data, with as few as one data point in some cases. Due to the limited number of data points used to calculate individual effect sizes, the effect sizes may have been influenced by typical variability rather than changes in the independent variable. A fourth limitation of the current study is about the outcome. The results are of limited generality because this analysis only included studies with single-case design which meet the WWC design standards with or without reservations and all included participants were identified with ASD.

As a consequence, firstly, it was necessary for future researchers to pay more attention to implementing the generalization and maintenance phases, and conduct more high-quality studies to examine the effectiveness of manipulatives in maintaining and/or generalizing mathematical skills for students identified with ASD. Meanwhile, researchers might collect more probes during generalization and maintenance phases so that present stronger evaluations of generalization and maintenance. Secondly, through the literature review of the 11 studies, this analysis has identified that the target mathematical content was limited to the number and operation. Future work can further investigate the effectiveness of manipulatives in the instruction of measurement, algebra, geometry, statistics and probability. At the same time, based rich literature, the applicability of different kinds of manipulatives to different mathematical contents may be further explored, namely determining what types of content can be effectively taught using concrete or virtual manipulatives and what type of content is difficult. Thirdly, as mentioned above, the study has found that manipulatives were effective instructional approach for maintaining and generalizing mathematical content. Future work can investigate how to maximize the potential learning benefits of concrete and virtual manipulatives and try to design effective instruction. Furthermore, as mentioned earlier, children with ASD may benefit from the CRA instructional sequence, but this finding needs to be validated by more studies. In addition, with the development of virtual manipulatives, how to apply it to improve the mathematics performance of students with ASD has become an urgent problem to be solved. At the same time, additional research on exploring the effects of VR framework is also needed and still emerging. Last but not least, a related direction for future work is identifying other factors which may be influential in the instructional effect of manipulatives but which have not been accounted in the above analysis such as factors that were not identified due to the limited literature available. Related to the virtual manipulatives, exploring the effect of VR framework is also a direction for further study. Besides, more specific information should be sought in future reviews.

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24

Are You Really Teaching Mathematics? What Education Can Learn from History

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ABSTRACT It is believed that a knowledge of the history of mathematics could improve or expand an individual's understanding of the nature of mathematics, and hence may challenge teachers' epistemological beliefs of mathematics and, as a result, cause teachers to reconstruct their beliefs. Wilder reminds us that mathematics is a part of, and is influenced by, the culture in which it is found. As such, the culture dominates its elements, and in particular its mathematics. For instance, a Chinese mathematician living about the year 1200 C.E. would have mainly focused on computing with numbers and solving equations without paying attention to geometry as the ancient Greeks understood it. In contrast, a Greek mathematician of 200 B.C.E. would have focused more on geometrical proofs than on algebra and numerical computation as the Chinese practiced it. This paper aims to question the conventional view that treats mathematics as a significant instrument for developing one's personal career, instead advocating that we should regard mathematics as a cultural discipline of human endeavor in our teaching. I will interpret the history of mathematics in terms of a sociological macro-view and investigate the rise and fall of mathematics in the European and Chinese cultures to shed more light on the intellectual value of mathematics in education.

Keywords: History of mathematics; Mathematical culture; Teaching of mathematics.

1. Introduction

Mathematical knowledge is one of the oldest wisdoms of human beings. Both the Six Arts (rites, music, archery, charioteering, calligraphy, and mathematics) of ancient Chinese culture and the Quadrivium (arithmetic, geometry, music, and astronomy) of ancient Greek philosophy regard mathematics as a significant liberal art for educating a scholar. However, it was not unusual for distinct civilizations to have been devoted to scrutinizing an identical mathematical problem through their varied approaches. For instance, Archimedes asserts that the area of any circle is equal to a right-angled triangle in which one of the sides of the right angle is equal to the radius and the other to the circumference of the circle. Yet an ancient Chinese mathematical text, *Nine Chapters on the Mathematical Art* (九章算術), claims that multiplying half the

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circumference by half the diameter of the circle yields the area. With its useful applications in agriculture, natural sciences, engineering, and business, mathematics has become a fundamental subject in school (American Association for the Advancement of Science, AAAS, 1989). This glimpse into its history shows that mathematics can be seen as a locally-developed global language (Liu, 2017).

On the other hand, with the coming of the era of globalization, mathematics education has been obliged to respond to international trends and domestic needs. Therefore, it is treated as a globally-exchanged local practice. Few would deny that teachers are the most significant key persons contributing to the success or failure of the teaching mathematics in school. In what way and to what extent they interpret and transmit mathematical knowledge thus deserves attention. Not surprisingly, not all mathematics teachers hold an identical belief about what mathematics is and how it should be taught. As Thom (1973) proposed, "[A]ll mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (p. 204). Following this position, Hersh (1979) asserted, "[T]he issue then is not: what is the best way to teach? but, what is mathematics really all about?" (p. 34). In this paper, I will discuss the developmental nature of mathematics in terms of its micro history to investigate the effects of the history of mathematics on the teaching and learning of mathematics and, as well, to investigate the rise and fall of mathematics in the European and Chinese cultures from a sociological perspective. It is hoped that this approach may shed light on the intellectual value of mathematics and propose a potential philosophy for teachers' practices in teaching.

2. A Brief Review of the Development of Mathematics

The Greeks' logically deductive approach to mathematics dominates contemporary mathematical research. However, this was not the case thousands of years ago. Several other ancient civilizations, such as Babylonia, Egypt, Arabia, India and China, had been demonstrating highly developed mathematical knowledge in different ways. Prior to conducting a macro analysis of the development of mathematics, a brief review of ancient mathematics will be helpful.

2.1. Ancient Babylonian mathematics

Though there is a debate over the earliest appearance of the ancient Babylonian mathematics, its origin can be dated back to at least the third millennium B.C.E. Thanks to the hundreds of unearthed clay tablets in the Assyrian areas, we know that there had been a high-level investigation into the geometrical ratio in ancient Babylon. The clay tablets can be categorized into two kinds: problem texts and table texts. For instance, the clay tablet, YBC 7289 (Fig. 1), contains a diagram and numbers. A sexagesimal 30 is inscribed along one edge of the square and sexagesimal sets of 1; 24, 51, 10 and 42; 25, 35 are written along the diagonal and in the lower segment of the square, respectively. The diagram and numbers have been decoded as follows:

$$1;\ 24,\ 51,\ 10 = 1 + \frac{24}{60} + \frac{51}{3600} + \frac{10}{216000} = 1.414212963 \approx \sqrt{2}.$$

Fig. 1. Babylonian clay tablet YBC 7289

If we multiply "1; 24, 51, 10" by 30, we get

$$30 \times 1.414212963 = 42.42638889 = 42 + \frac{25}{60} + \frac{35}{3600} = 42; 25, 35.$$

The clay is believed to be used for measuring land during 1900−1600 B.C.E. Namely, if you own a square plot of land with 30 as its edge length, then its diagonal would be "42; 25, 35" long. It can be seen that the ancient Babylonians had a good understanding of the irrationals. Another clay tablet, Plimpton 322 (Fig. 2), has drawn much attention and debate about its use.

Fig. 2. Babylonian clay tablet Plimpton 322

Comprising four columns and 15 rows of numbers containing Pythagorean triples, the tablet is dated much earlier than any other civilization's recorded insight into the triples. There is no consistent agreement about what the clay tablet was for (Britton, Proust and Shnider, 2011; Resnikoff and Wells, 1973). Recently, Mansfield and Wildberger (2017) claimed that Plimpton 322 is a table of Babylonian exact sexagesimal trigonometry, but were soon challenged by Lamb (2017). Regardless of the controversy it has caused, Plimpton 322 represents a significant achievement of ancient Babylonian mathematics.

2.2. Ancient Egyptian mathematics

Our current knowledge of Egyptian mathematics mostly relies on the *Rhind Papyrus* and *Moscow Papyrus*. The particular features of Egyptian mathematics are one, unit fractions wherein each fraction in the expression has a numerator equal to 1 and a denominator that is a positive integer, and two, its method of false assumption. For finding *aha,* the Egyptian word for the unknown quantity in an equation, hypothetical numbers were initially used to fit a simpler equation, followed by a revision of the hypothetical numbers to fit the original equation. For instance, the 24th problem in the Rhind Papyrus states, "*aha* and its one-seventh is 19. Find *aha.*" The method of false assumption starts by assuming that 7 is the hypothetical *aha*:

$$7 + 7\left(\frac{1}{7}\right) = 8.$$

Multiply $2 + \frac{1}{4} + \frac{1}{8}$ on both sides,

$$\left(2 + \frac{1}{4} + \frac{1}{8}\right)\left(7 + 7\left(\frac{1}{7}\right)\right) = \left(2 + \frac{1}{4} + \frac{1}{8}\right)8 = 19.$$

Then *aha* is

$$16 + \frac{1}{2} + \frac{1}{8}.$$

The 14th problem of the Moscow Mathematical Papyrus, another ancient Egyptian papyrus containing mathematics, calculates the volume of a truncated pyramid: $V = \frac{1}{3}h(a^2 + ab + b^2)$, where *a* and *b* are the base and top side lengths and *h* is the height. In particular, the ancient Egyptians knew how to apply the Pythagorean Theorem and method of false assumption to solve simultaneous equations. One of the two problems appeared in the *Berlin Papyrus*:

> You are told the area of a square of 100 square cubits is equal to that of two smaller squares, the side of one square is 1/2 + 1/4 of the other. What are the sides of the two unknown squares?

In modern terminology, it can be represented as follows:

$$\begin{cases} \dfrac{x}{y} = \dfrac{4}{3} \\ x^2 + y^2 = 100, \end{cases}$$

where x and y are the lengths of the sides of the two squares.

We start by assuming that 3 and 4 are the two hypothetical side lengths. Since $3^2 + 4^2 = 5^2$ and $100 = 10^2 = (2 \cdot 5)^2$, we can multiply the length of both sides by 2^2, $6^2 + 8^2 = 10^2$. We can then assert that 6 and 8 are the actual lengths of sides x and y.

To determine the volume of a pyramid, ancient Egypt developed the method of false assumption, a unique technique in the history of mathematics, to solve quadratic simultaneous equations.

2.3. Ancient Greek mathematics

Though ancient Greek mathematics was grateful to the heritage of Babylonia and Egypt, the Greeks created a totally different culture and changed the nature of mathematics (Kline, 1962). Ancient Greek thinkers found the ways to apply mathematics to the fields of commerce and engineering, yet what impressed them most was the power of mathematical reasoning, the power of revealing the structure and nature of the physical world. According to Plato, mathematical objects are immaterial, just like God, goodness, courage and the human soul. Doing mathematics is the best way to understand the immaterial world. Plato asserted that the study of numbers facilitates the conversion of the soul itself from the world of generation to essence and truth, and an officer who had studied geometry would be a very different person from what he would be if he had not. Furthermore, based on the firm belief that the physical world is rationally designed, mathematics is the key to reveal the secret under the veil and, hence, astronomy became the chief scientific interest of the ancient Greeks.

Upon entering the Alexandrian period, a mixed interest in theoretical reasoning and practical investigation had risen. Euclid of Alexandria (born c. 325 B.C.E.), Aristarchus of Samos (310−230 B.C.E.), Eratosthenes of Cyrene (276−198 B.C.E.), Archimedes of Syracuse (287−212 B.C.E), and Claudius Ptolemy of Alexandria (100−170 C.E.) were the representative figures. Euclid's *Elements* synthesized previous known mathematical propositions and demonstrated them in a deductive fashion. On the basis of conventional astronomical phenomena and the basic calculation of geometrical objects, Aristarchus estimated the sizes of and the distance between the sun, Earth and moon, and Eratosthenes made a remarkably accurate estimation of the circumference of the earth. Archimedes expertly employed intuitive thinking and theoretical reasoning, and as well, skillfully combined physical principles and mathematical propositions to derive and rigorously prove a range of mathematical theorems. Some of his works can be found in the so-called Archimedes' palimpsest.

Ptolemy's *Almagest*, a construction of a geocentric model of the universe, was the most influential mathematical and astronomical treaties until the appearance of Copernicus' *On the Revolutions of the Celestial Spheres* in 1543. Though ancient Greeks made little contribution to the study of numbers and the solving of equations, their mathematics not only reached the peak of the world at that time but also established the modern paradigm of mathematics.

2.4. Ancient Indian mathematics

Our knowledge about the mathematics of ancient India is mostly based on the scripts written in Sanskrit, a language used before the middle of the first millennium. However, unlike the aforementioned clay tablets in Babylon and papyrus in Egypt, very few original sources can be traced to reconstruct their mathematical knowledge with certainty. The first known texts written in Sanskrit are the "Vedas" (literally "knowledge"), which is a canon of hymns for religious ritual. The *Śulbasūtras* (literally "*rope-rules*") Vedic texts are the only sources of Indian mathematics during the Vedic period. Because altar construction usually requires doing area-preserving transformation, the geometric procedures were connected with sacrificial ritual in this manner. In one of the *Śulbasūtras*, called *Baudhāyana Śulbasūtras* (the *Śulbasūtras* texts are associated with the author's name), the process of transformation between circle and square were addressed as follows (cited in Katz, 2007):

- If it is desired to transform a square into a circle, [a cord of length] half the diagonal [of the square] is stretched from the centre to the east [a part of it lying outside the eastern side of the square]; with one-third [of the part lying outside] added to the remainder [of the half diagonal], the [required] circle is drawn.
- To transform a circle into a square, the diameter is divided into eight parts; one [such] part after being divided into twenty-nine parts is reduced by twenty-eight of them and further by the sixth [of the part left] less than the eighth [of the sixth part].

The above techniques pertaining to the circulature of a square imply the value of π to be 3.088. But this value is not consistent throughout. It can be found that π is approximated by other values elsewhere. The value was, therefore, obtained empirically, without a systematic approach.

In 327 B.C.E., Alexander the Great conquered some small kingdoms of northeastern India and started to spread Greek influence into this ancient civilization. In spite of the constant ups and downs among the different kingdoms in this land and despite the fact that Alexander's ambition ended with his premature death, the study of astronomy was always encouraged, triggering the development of trigonometry (Katz, 1998). The earliest known Indian text involving trigonometry is *Paitāmahasiddhānta*, written in about the early fifth century and containing a table of half-chords (*jyā-ardha*, Fig. 3). Note that the half-chord (sin *a*) in ancient India was different from the contemporary concept. We define sin *a* as the ratio of the line segment to the radius, but ancient Indians thought of sin *a* as the line segment itself. Following the study of trigonometry, the techniques of approximation and solving indeterminate equations were developed. India went on to achieve its mathematical peak in the 12th century and maintained it until the mid-16th century.

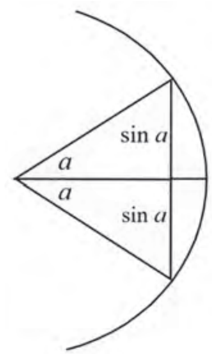


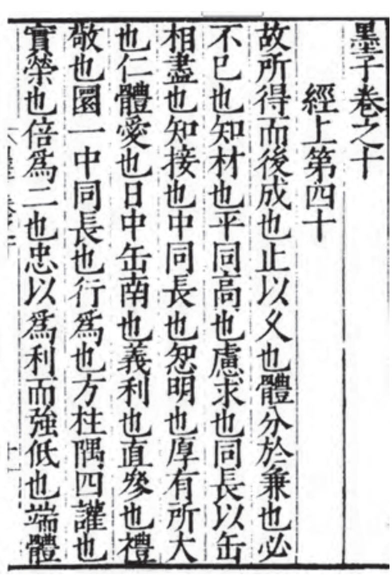
Fig. 3. The half-chords in Paitāmahasiddhānta

2.5. *Ancient Chinese mathematics*

Zhoubi Suanjing (周髀算經), *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*) is one of the oldest mathematical books dedicated to astronomical observations and calculations in ancient China (ca. 100 B.C.E.). It addresses a special case of the Shang Gao Theorem (商高定理), the Chinese version of the Pythagorean Theorem) and implicitly shows a general proof. The book begins with a conversation between the Duke of Zhou (周公) and the mathematician Shang Gao about the method for using *Bi* (髀 gnomon) to measure the width of the land and the height of the sky. Though the methodology was moderate and Shang Gao's mathematical reasoning was appropriate, the results were not all correct due to being based on a canopy heaven cosmological model, an umbrella-like heaven that rotates about a vertical axis rooted on a flat earth, which was adopted at the time. More than just the Shang Gao Theorem, the *Zhoubi Suanjing* is a collection of various ancient astronomical texts. However, the compilers of this book could have modified the original data (Li and Sun, 2009). Despite its flaws, this book deserves the title of 'the principal surviving document of early Chinese science' (Cullen, 1996), the earliest paradigm for demonstrating China's use of mathematical methods in astronomy.

In addition to astronomy, acoustics and optics were other branches of physics well studied in ancient China. *Guanzi* (管子), an ancient text traditionally attributed to the philosopher Guan Zhong (管仲, ca. 7th century B.C.E.), proposed the Method of Subtracting and Adding Thirds (三分損益法) to create musical scales, similar to the Pythagorean tuning system. *Mojing* (墨經, the *Mohist Canon* written by Mozi (墨子)

and his followers, ca. 4th century B.C.E., Fig. 4) sequentially claimed eight propositions of optics for describing the phenomena of light, shadow, and pinhole imaging. It should be noted that, among all schools of philosophical thought in ancient China, Mohism (墨家) is unique in its inclusion of a discourse on mathematics and mechanics. According to Mozi, the reason for something is what must be before it will come about. There are two kinds of reasons: minor reasons and major reasons. The definition of a minor reason is "*having this, it will not necessarily be so; lacking this, necessarily it will not be so*" (小故，有之不必然，無之必不然). Obviously, the minor reason is the necessary condition in terms of modern logic. Major reason, on the other hand, is "*having this, it will necessarily be so; lacking this, necessarily it will not be so* (大故，有之必然，無之必不然), which are the sufficient and necessary conditions. The Mojing also demonstrates a Euclidean style of thought in defining dimension as "having something which it is bigger" (厚有所大也), circular as "having the same lengths from one center" (圜，一中同長也), and point as "the unit without dimension which precedes all others" (端，體之無序而最前者也). The principle of leverage was also discussed in the *Mojing* for interpreting the function of moving heavy objects in the steelyard, about 200 years earlier than Archimedes. However, the lack of quantitative analysis makes it impossible to carry out a mathematical discourse. Mohism was a very influential school of thought during the Warring States period (戰國時期) and was the largest rival to Confucianism (非儒即墨). However, Mohism was almost forgotten due to the Qin Dynasty's promotion of Legalism (法家) and the Han Dynasty's promotion of Confucianism.


墨子卷之十
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故所得而後成也止以久也體分於兼也必
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敬也圜一中同長也行爲也方柱隅四讙也
實榮也倍爲二也忠以爲利而强低也端體

Fig. 4. Mohist Canon

The most famous and influential ancient Chinese mathematical book is the *Jiuzhang Suanshu* (九章算術, *Nine Chapters on the Mathematical Art*). The author and original date of the book are unknown but it is estimated to have been written shortly after 200 B.C.E. Actually, in some sense, the *Jiuzhang Suanshu* is less like a mathematical treatise and more like a how-to reference manual. It presented 246 problems in life, business, and measurement, followed by answers and algorithms but without formal proof or derivation. In ancient China, Liu Hui (劉徽, 225−295) and Zu Chongzhi (祖沖之, 429−500) were the two most significant mathematicians prior to the Tang and Song dynasties. Liu Hui's major contribution is his commentary on the *Jiuzhang Suanshu*, demonstrating a typical Chinese style of inductive argumentation. Zu Chongzhi is well-known by his world-leading approximation of π. The Tang dynasty (618−907) government recruited and trained officers to do practical mathematics including measuring, taxation, and calendar making. Though the Tang government compiled and corrected the *Ten Mathematical Canons* (算經十書) as the official mathematical texts for imperial examinations, the mathematical texts studied by these imperial officers included problems and skills for solving problems without dealing with any new methods. "Thus there was no particular incentive for mathematical creativity" (Katz, 1998, p. 193).

3. A Cultural Survey

Fig. 5 is a chart graphing the ratio of GDP of all major powers from the year one A.D. to 2017 (Visual Capitalist, 2017). The wider the color band, the stronger the economy. It appears that, prior to the rise of the European Renaissance in the 14th century, China and India were the two greatest economic powers and, coincidentally, the

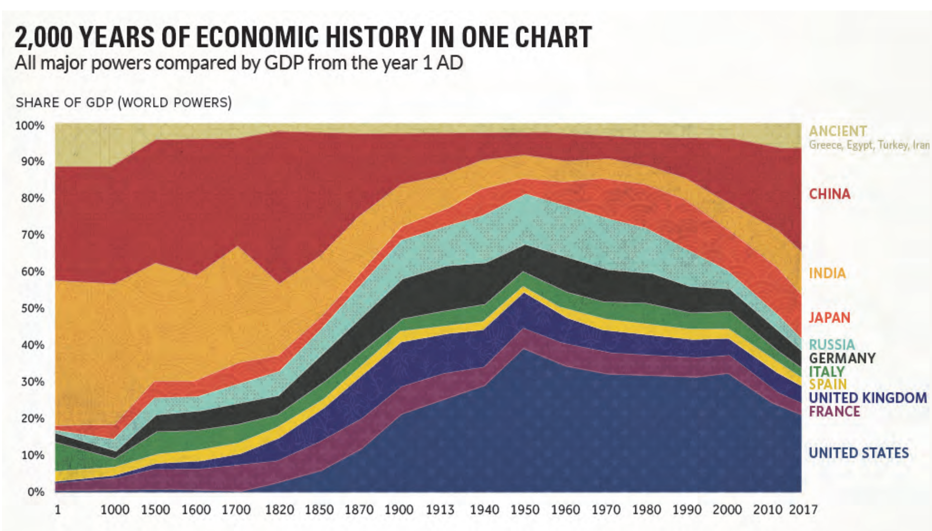

Fig. 5. 2,000 years of economy history (Visual Capitalist, 2017)
https://www.visualcapitalist.com/2000-years-economic-history-one-chart/

achievements in mathematics of both civilizations reached their peaks during this period, respectively. Following the Tang dynasty, the Song dynasty (960−1279) is regarded as the golden or the greatest age of China (Fairbank, 1992; Stavrianos, 1971) for its high cultural achievement, and has even been named “the Eastern Renaissance” (Miyazaki, 1950). The contribution made by several mathematicians during the Song marks the peak of ancient Chinese mathematics, such as Jia Xian’s (賈憲) method for extracting the square and cubic roots, Qin Jiushao’s (秦九韶) Chinese Remainder Theorem, Li Ye’s (李冶) techniques for solving polynomial equations, and Zhu Shijie’s (朱士傑) method for solving high-order simultaneous equations of several variables. Despite its high achievement in computational arithmetic and instrumental techniques, the failure to use mathematics to reveal physical laws is one reason why ancient Chinese mathematics and science stagnated. This could be attributed to traditional Chinese philosophy which regarded the whole universe as an organic and self-sufficient system “in which there was no room for Laws of Nature, and hence, no fixed regularities to which it would be profitable to apply mathematics in the mundane sphere” (Needham, 1956, p. 325).

Besides, Brahmagupta proposed a general form of Heron’s formula for the area of cyclic quadrilaterals. Following Āryabhaṭa and Brahmagupta’s mathematical tradition, Bhāskara II studied the solution of quadratic, cubic and quartic indeterminate equations. He also proposed preliminary concepts of infinitesimal calculus and integral calculus as early as the 12th century.

Fig. 5 also shows that the economy of China was getting stronger again during 1700~1820, which was the period under the reign of the emperors Kangxi (康熙), Yongzheng (雍正), Qianlong (乾隆), and Jiaqing (嘉慶). During that time, China eagerly welcomed Western mathematics and sciences, and developed its own mathematical community. The aforementioned phenomena suggest a link between the development of mathematics and the degree of economic growth or cultural openness.

Raymond Wilder put forward the concept of mathematical culture specifically in the early days. He gave a speech on the cultural basis of mathematics at the International Congress of Mathematicians in 1950, expounding on the connotation and importance of mathematical culture. He claimed that he believed that only by recognition of the cultural basis of mathematics would a better understanding of its nature be achieved (Wilder, 1950, p. 259). Because diverse mathematical practices developed and evolved in different civilizations in response to common problems that were encountered within a cultural context, Hersh (1997) expressed a cultural view that “mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context” (p. xi).

To explore mathematics in culture is to understand the macroscopic development of mathematical knowledge. The first dawn of mathematical development came from the investigation of nature. Afterwards, due to the uniqueness of the evolutionary methods and processes of various human civilizations and societies, different rational paths and thinking cultures were created, and a “homogeneity and heterogeneity” of mathematical knowledge was born. Mathematics has been influenced by agriculture, commerce, industry, warfare, engineering, philosophy, psychology, and astronomy, and an understanding of mathematics requires one to take these key factors into account (Struik, 1948). The invention of typography in Europe contributed to the dissemination of mathematical knowledge during the Renaissance, which not only contributed to the mathematization of science in the 16th and 17th centuries, but even triggered the scientific revolution. However, this relationship between mathematical knowledge and social culture in the development of mathematics was not an inevitable trend. China, where typography originated earlier, did not have a similar developmental path. Rather, its development is related to the mathematical traditions of various cultures. Wilder (1950) reminds us of the interactive relationship between internal and external tensions between mathematics and other disciplines. In addition to the influence of the host culture, cultural infiltration among different ethnic groups may also lead the direction of mathematics in new directions. For instance, the ancient Greek mathematics was influenced by ancient Egypt and Babylonia, which then influenced the mathematics of Arabia and India. After that, because of the advantage of its symbolic system in operation and abstraction, the style of Western mathematics became mainstream, resulting in the gradual disappearance of cultural differences in contemporary mathematics.

4. Conclusion

As asked in the title of this paper, are you really teaching mathematics? What can education learn from history? In the book "Mathematics in Western Culture", Kline (1954) demonstrated that mathematics is a subculture of the entire human culture. Actually, his larger attempt was to reveal that mathematics is "a major cultural driving force in Western civilization" (p. ix). This seemingly weird claim will of course attract criticism, but Kline believes this is due to a long-standing public misunderstanding of the nature of mathematics. He maintains that mathematics, although a body of knowledge, does not contain truths. Science is indeed pursuing the truth of the physical world, and mathematics just acts as a beacon, guiding science to its purpose. In addition to society's need for a response to its problems, "over and above all other drives to create is the search for beauty" (p. 5).

The purpose of this study is to maintain that mathematics is not only an educational product but also a cultural creation. Mathematics problems arose from culture; the culture of mathematics then developed to become a part of mathematical knowledge, a knowledge which further influences other fields and forms other cultures. The interaction between mathematics and culture is not only related to mathematical knowledge itself but invisibly affects the reality of mathematics education. If the general public see mathematics only as a tool for solving problems, this view will mislead the public's understanding of the nature of mathematics. As Burton (2009) pointed out, the orientation of mathematics in a regional culture may constitute a barrier for certain ethnic groups to enter the domain of mathematics, and also shape the public's understanding of the culture of mathematics. Therefore, Burton believes that it is necessary to pay attention to the socio-political attitudes, values and behaviors towards mathematics in a society. We must see how to teach mathematics in a new light, namely, that the teaching of mathematics is a cultural inquiry as well as an educational issue.

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25

Effects of Instructional Videos on Students Learning

Rachel Ka Wai Lui[1]

ABSTRACT E-learning has become popular these years, and the advantages of flipping the classroom are also widely depicted in literature. However, the widely used element in video instruction, overlaying of a small video of the instructor over lecture slides, is understudied. A new technology called Learning Glass, which can be used for recording lectures and allowing instructors to write lecture notes while maintaining face-to-face contact with students, was used to record instructional videos. The effect of the presence of instructors in instructional videos for university students in two metropolitan universities (Los Angeles and Hong Kong) was studied. Participants were randomly assigned to watch a video with and without the presence of the instructor. The extent to which the participants have grasped the video materials was assessed via pre- and post-tests. Participants' satisfaction towards the video was also evaluated via a survey near the end of the experiment. The effect of the instructor's presence and where the participants come from was studied. It was found that the instructor's presence did not impose a statistically significant difference towards participants' acquisition of the video contents. One possible reason is that individual learning preference is more important than instructing all learners with one approach. It was, on the other hand, found that participants from Los Angeles were more willing to recommend videos to the others and to watch more for learning. This may be related to the fact that e-learning is more popular in Los Angeles. Results of this study may help us recognize the implication of the presence of the instructor in videos as well as providing a better learning environment in the future.

Keywords: Instructional videos; Video design; Gesture; Instructor's presence; Learning Glass; E-learning.

1. Introduction

E-Learning has played a more and more significant role in education over the past few years. Many institutions, for instance, offer online courses for distance learning for a wide variety of audiences. For non-distance-learning courses, the approach of flipping the classroom has also been popular due to various advantages (e.g., Bergmann and Sams, 2012). To deliver the contents to the course participants in both cases, various learning resources such as online quizzes and presentation slides have to be provided. Often instructional videos form the central part of these learning materials as a way to

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provide course participants with a direct narration of the course materials.

Instructional videos can come in a wide spectrum of styles (Chen and Wu, 2015). They can be a recording of a lecture for sharing with students; they can also be of the voice-over style, which mainly displays presentation slides accompanied with synchronous narration by the instructors on the contents. The picture-in-picture approach, on the other hand, allows presentation slides and voice narration to be shown simultaneously together with the instructor's appearance.

The effects of different types or different features of videos on the learning effectiveness are rarely studied. Chen and Wu (2015) compared the learning effectiveness of different types of video lectures and reached a conclusion that the learning performance with videos showing the lecturer is significantly better than voice-only videos. Some researchers proposed that the presence of a lecturer in videos could enhance the learning process, probably by non-verbal communications such as gestures (Valenzeno et al., 2003).

Gestures are regarded as an important tool to present abstract ideas and enhance students' comprehension of course materials in classroom (Alibali and Nathan, 2007). Kizilcec, et al. (2014) found that students strongly preferred instruction with the face and perceived it as more educational. Some studies demonstrated that children exposed to gestures of the lecturer in instructional video had better understanding of taught concepts about linguistics and symmetry compared to those who were exposed to videos without gestures (Valenzeno et al., 2003; Church et al., 2004). However, those experimental settings mostly stimulated the classroom environment, aiming to examine the effect of gestures in face-to-face learning mode. The role of gestures in E-Learning at the university level, especially in the instructional videos for self-learning purpose, is seldom investigated and receives little attention currently. If a link can be established between gestures and learning performance in E-Learning, it might be useful in designing the teaching materials and enriching learning experience for college students.

2. Objective

In this study, we investigated the effect of instructor's gesturing in instructional videos for university students and test if the learning performance would be significantly different for videos with or without the presence of the instructor. Results of this study may help us recognize the implication of instructor's presence in instructional videos and provide a better learning environment in the future.

3. Hypothesis

We hypothesized that instructional videos with the presence of instructors would not affect the student performance. If students are given identical tests before and after watching an instructional video, the performance of the students who watched the videos with or without the presence of instructor will not have statistically significant difference.

4. Methodology

4.1. Experimental video

Two recordings from the same 10-minute presentation in English were created. The presentation was about statistical concepts of mean and median, emphasizing the mean as the balancing point of data and how the mean and the median related to the shape of data distribution. The instructor was a psychology professor who has 8 years of teaching experience in statistics at a university in Los Angeles, US. One recording (Recording 1) included the video, audio and presentation slides, such that the gestures and facial expressions of the instructor were clearly visible in the recording (see Fig. 1). The recording was prepared with Learning Glass, which was a transparent screen that the instructor could write on. Using the Learning Glass, the instructor was recorded as forward facing in the video which allows for natural eye contact with the camera, gesturing and demonstrating what has been written on the glass. The instructor mainly used gestures to indicate the position of data on the number line and underscore the concept of balancing point. Another recording (Recording 2) was identical to the first one except the absence of the instructor, so it contained the same audio track and presentation slides only. Both videos are 9 minutes long.

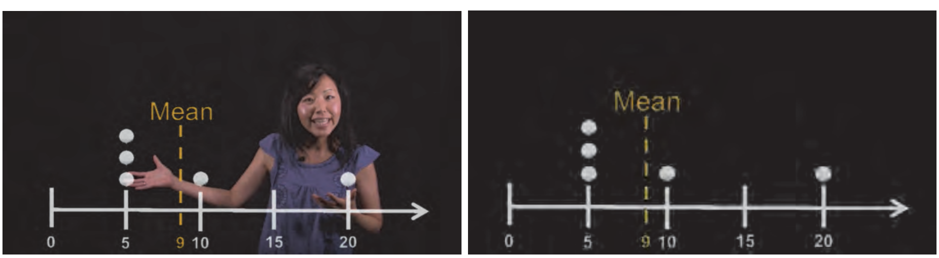

Fig. 1. An illustration of Recording 1 (Left) and Recording 2 (Right)

4.2. Pre-test and Post-test

The questions of pre-test and post-test were identical. The pre-test was administered to assess participants' prior knowledge about mean and median. The post-test evaluated the potential enhancement of their knowledge after watching the video. They consisted of ten questions each. In the first three questions, students had to figure out the relationship between mean and median in the given graphs. The remaining questions focused on mean as a balancing point and the concept of mean and median. All questions carried equal scores. Participants were, furthermore, asked to rate their learning satisfaction at the end of the post-test, i.e. "How likely are you to recommend this video to a friend who's interested?" and "If you were taking a statistics class, how interested would you be in another short video with the same professor?" in a seven-level Likert scale with 7 representing the highest level of satisfaction.

4.3. Procedures

Undergraduate students from a university in Los Angeles ($n = 211$) and from a university in Hong Kong ($n = 129$) were invited to participate in the study. Participating students from both universities all use English as the medium of instruction but come from a variety of study backgrounds.

4.4. Participants

Participants were invited by emails which included a link to access the experiment. At the beginning of the experiment, the participants completed the online pre-test. Then they were randomly assigned to one of Recording 1 and Recording 2. After watching the recording, the participants took the post-test. The improvement in the understanding of participants towards the topic could be measured by the increase in the score of the two tests. Statistical method was applied to test if there is a significant difference between the two groups. The whole process was launched in online format. Participants could take part in their leisure. This simulated the actual online learning process that caters for diversified learning habits. Participants were expected not to spend more than 30 minutes to complete the experiment.

5. Results

Tab. 1 shows the average gained scores of the participants from both universities. The average scores of the pre-test and the post-test in the two groups (Recording 1 and Recording 2) were calculated. The gain scores were used to understand how the videos enhanced the knowledge of the participants. A two-way ANOVA was conducted using gained scores as the dependent variable, while location and the presence of the instructor were used as independent variables with $\alpha = 0.05$.

Tab. 1. The average gained scores of the participants from both universities

Average gained scores	Recording 1	Recording 2	All participants
University in Los Angeles	2.23 ($n = 98$)	2.19 ($n = 113$)	2.21 ($n = 211$)
University in Hong Kong	2.05 ($n = 61$)	2.37 ($n = 68$)	2.22 ($n = 129$)
All participants	2.16 ($n = 159$)	2.26 ($n = 181$)	2.21 ($n = 340$)

Tab. 2 shows the results of the two-way ANOVA. Since 0.923 and 0.871 are both greater than 0.05, the two factors, location and the presence of the instructor, are statistically insignificant to the performance of the students which were measured by the gained scores.

Tab. 2. The results of the two-way ANOVA

Source	Value	Standard error	t	Pr > |t|	Lower bound (95%)	Upper bound (95%)
Intercept	2.195	0.168	13.037	< 0.0001	1.864	2.526
University in Hong Kong	0.022	0.231	0.097	**0.923**	−0.431	0.476
University in Los Angeles	0.000	0.000				
Instructors-with	0.040	0.247	0.162	**0.871**	−0.446	0.526
Instructors-without	0.000	0.000				

In addition, the participants were asked to rate their learning satisfaction at the end of the post-test. Tab. 3 summarises the results to the question "How likely are you to recommend this video to a friend who's interested in learning more about mean?", while Tab. 4 shows the results of the two-way ANOVA with location and the presence of the instructor as independent variables with $\alpha = 0.05$.

Tab. 3. The results to the question "How likely are you to recommend this video to a friend who's interested in learning more about mean?"

Average scores (0−7)	With instructor	Without instructor	All participants
Participants from Hong Kong	5.16 (n = 61)	4.72 (n = 68)	4.93 (n = 129)
Participants from Los Angeles	5.55 (n = 98)	5.24 (n = 113)	5.38 (n = 211)
All participants	5.40 (n = 159)	5.05 (n = 181)	5.21 (n = 340)

From Tab. 4, both p-values are less than 0.05. This means that both factors are statistically significant.

Tab. 4. The results of the two-way ANOVA

Source	Value	Standard error	t	Pr > |t|	Lower bound (95%)	Upper bound (95%)
Intercept	5.219	0.124	42.085	< 0.0001	4.975	5.463
University in Hong Kong	−0.457	0.163	−2.799	**0.005**	−0.777	−0.136
University in Los Angeles	0.000	0.000				
Instructors-with	0.355	0.158	2.243	**0.026**	0.044	0.667
Instructors-without	0.000	0.000				

The participants in Los Angeles are more likely to recommend this video to a friend who's interested in learning about mean in comparison with the participants in Hong Kong. In addition, the presence of an instructor also has a positive effect on whether they would recommend the video to a friend.

Tab. 5 summarises results to the question "If you were taking a statistics class, how interested would you be in another short video with the same professor?", while Tab. 6 shows the results of two-way ANOVA with location and the presence of the instructor as independent variables with $\alpha = 0.05$.

The two-way ANOVA showed that the factors of location and the presence of an instructor are both significant. The participants in Los Angeles are more likely to watch another short video with the same professor in comparison with the participants in

Hong Kong. In addition, the presence of an instructor also has a positive effect on whether they would be interested in another short video with the same professor.

Tab. 5. The results to the question "If you were taking a statistics class, how interested would you be in another short video with the same professor?"

Average scores (0–7)	With instructor	Without instructor	All participants
Participants from Hong Kong	5.02 ($n = 61$)	4.60 ($n = 68$)	4.80 ($n = 129$)
Participants from Los Angeles	5.77 ($n = 98$)	5.34 ($n = 113$)	5.54 ($n = 211$)
All participants	5.48 ($n = 159$)	5.07 ($n = 181$)	5.26 ($n = 340$)

Tab. 6. The results of the two-way ANOVA

Source	Value	Standard error	t	Pr > |t|	Lower bound (95%)	Upper bound (95%)
Intercept	5.342	0.121	44.102	< 0.0001	5.104	5.580
University in Hong Kong	−0.743	0.159	−4.665	< 0.0001	−1.057	−0.430
University in Los Angeles	0.000	0.000				
Instructors-with	0.423	0.155	2.733	0.007	0.119	0.727
Instructors-without	0.000	0.000				

6. Discussion

Based on our results, the presence of the instructor in videos does not impose a statistically significant difference in students' performance between the pre-test and the post-test. Previous educational studies have suggested that adopting one teaching approach for all students may not be the most effective; taking care of learning preferences of different students would be more important. To investigate the effect of instructional video design on learning effectiveness, the narrative style of the instructors in the video, the design of the presentation slides and the relation with the subject contents can be our future directions.

In terms of perception of the video, our results showed that participants who had watched a video with the instructor in it (Recording 1) were more likely to recommend it to the others as well as watching more videos of the same instructor. The presence of instructors can capture viewers' attention through the use of various body languages. Such presence can also help showcasing the involvement of the instructors in the course, thus fostering more satisfaction in the students.

Participants from Los Angeles, furthermore, are more likely to recommend our videos (both Recording 1 and Recording 2) to their friends as well as watching more, compared to their peers in Hong Kong. Such a difference can be related to the environment of the two locations. Although both cities are metropolitan, they show differences: Hong Kong, with a population of around 7.5 million, is about ten times smaller than Los Angeles County, which has a population of 10 million. With relative ease of public transportation, remote learning in Hong Kong is not very common. In contrast, the population in Los Angeles is more spread out with more reliance on private transportation. Because of this, participants from Los Angeles would be more used to e-learning and watching instructional videos for learning.

The current study has its own limitations. The results, for example, may be related to the subject contents of our video, and it is still up to further research to assess if similar findings can be obtained for other subject areas. Our results, furthermore, can be affected by the use of Likert scale in our surveys, as participants with different cultural backgrounds can interpret the scale differently: some participants may be more willing to use the whole Likert scale, while some may tend to restrict themselves more to the middle part.

7. Conclusion

The effect of instructors' presence on learning effectiveness is studied among participants from Los Angeles and Hong Kong. While the presence of the instructor in the video does not demonstrate statistically significant difference in participants' acquisition of the video contents, it does help to encourage participants to watch more similar videos and to recommend it to their peers. Compared to their counterparts in Hong Kong, participants from Los Angeles are also more willing to recommend the video to their friends and to watch more of the same style. This may be related to the fact that e-learning and the use of instructional videos for learning are more widely practised in Los Angeles. These findings can be of value for instructional video design so as to improve teaching and learning effectiveness.

Acknowledgments

The author would like to express my gratitude to my friends and colleagues Professor Karen Givven from UCLA, Professor James Stigler from UCLA and Professor Ji Son from California State University LA who supported me and offered deep insight into this study.

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https://doi.org/10.1142/9789811287183_0026

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On the Notion of Mathematical Competence

Mirko Maracci[1]

ABSTRACT This contribution analyzes the origin of the competence construct, its evolution and how it is conceptualized by different authors in different fields. The objective is to reveal the complexity of the idea that the construct is meant to capture; in fact, only by bringing out this complexity can we hope to make the construct truly operational and useful for practice and educational research. In particular, I discuss the multidimensional artefact-like character of the construct of competence trying to reveal the several distinct related dimensions which contribute to form this single theoretical concept.

Keywords: Competence; Mathematical competence; Multidimensional construct.

1. Introduction

In this contribution I discuss the complexity of the idea of competence and of mathematical competence at two distinct levels:

- At a general level, the discussion focuses on how the idea of competence is conceptualized in different domains and from different subjects; the interest is on the variety of ways in which the idea is conceptualized within and outside mathematics education, and on the different elements that these conceptualizations bring to light.
- And at a more particular level, the discussion focuses on a specific elaboration of the idea of competence in education, on its relevance for mathematics education, on the features these conceptualization helps to grasp and its intrinsic complexity.

The discussion also concerns the issue of the objectives, for which one needs or wants to define the idea of competence. This leads to reflect on the *artefact-like nature* of the notion of competence. These aspects are intertwined with each other, and emerge together throughout the discussion.

The structure of the contribution is the following. First of all, I will try to make clear the reasons for my interest in the idea of competence (Section 2), and introduce the issue of its elusiveness (Section 3) which will be further developed in the following sections. Then, I will outline the origin and development of the idea of competence in the last decades (Section 4) and examine the idea as it emerges from literature in social and

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behavioral sciences (Section 5). To this end, I will refer to the surveys by Weinert (2001) and Mulder (2017). I will present some constructs in mathematics education, which can be related to the idea of competence (Section 6). These constructs resonate at different levels with a specific definition of the idea of competence elaborated in the field of education. Drawing on this definition and I will discuss the various critical issues that characterize in an intrinsic, unavoidable manner the idea of competence (Section 7). I will then try to trace some conclusive reflections on this theme (Section 8).

2. The Interest for the Idea of Competence

The idea of competence has acquired great importance in recent years in the context of the debate on education and training promoted in Europe (and consequently in Italy, where I come from) following the so-called Lisbon strategy. This debate has been prompted in by diverse Political institutions (e.g. in Europe by the European Parliament and Council, in the context of the so-called Lisbon strategy) but also economical organizations (such as OECD), which defend the adoption of the term, and the perspective it conveys, in order to highlight the importance of an approach to teaching and learning not bound to specific discipline content knowledge.

However, the interest in the idea of competence is not restricted to the European context or to the debate on the Lisbon strategy in Europe. In fact, the discourse on the idea of competence involves various fields of social sciences, psychology, and educational sciences, with particular reference to the world of work, to incoming and ongoing vocational training, to professional qualification, to mobility ... But the term is also used outside the purely professional context.

All that does not remain at a declarative level, but has practical consequences, in that curriculum and assessment are now expected to be framed by this perspective. As a consequence, teachers have to cope with new professional challenges concerning the design of educational activities and assessment of students' learning by focusing on competence development.

3. An Elusive Idea

As mentioned above, the idea of competence is at the intersection of different fields and research domains and it is used by different actors — researchers, teachers, policy-makers, various stakeholders — within different communities and with different aims. But contributions from different fields are not always clearly connected to each other.

In such a situation, it does not help that the idea of competence is rarely defined — its meaning is assumed to be clear or intuitive — or it is defined, maybe inevitably, by quasi-synonyms: capacity, ability, proficiency… (Kilpatrick, 2020). The idea of competence, then, appears to one of the most elusive in education (Kilpatrick, 2020). This same view has been expressed over the years by several authors. For instance, Dolz and Ollagnier in the early 2000s edited a book titled "the enigma of competence in education" (2002, my translation); Gilbert and Parlier (1992) used the metaphor of the *sponge-word* to refer to the term competence: as a sponge, it gradually absorbs all the meanings attributed to it by those who employ it, but, when pressed, it empties and does not reinstitute any meaning. Although Gilbert and Parlier wrote in 1992, the danger they point to is still present.

In next sections, Section 4 and Section 5, I analyze the origin of the idea of competence, its evolution and how it is conceptualized by different authors in different fields. This analysis is important, on the one hand because it contribute to detect the complexity of the idea that the construct of competence is meant to capture. On the other hand, because only by bringing this complexity to the light can we hope to make the construct truly operational and useful for educational practice and research. Furthermore, such reflection is needed to make more fruitful the discussion between the different communities that revolve around the world of education. The extensive use of the term does not suffice in itself to assure that there is a common shared perspective amongst the various stakeholders.

4. The Evolution of the Idea of Competence in Education

The notion of competence has distant roots: the term derives from Latin *competens* and *competentia*, the meaning of which are close to the current common-sense meanings, but the origin of the idea can be traced back to Greek philosophy (Mulder et al., 2007; Pellerey, 2013). Beyond these ancient roots, it is difficult to clearly trace the evolution of the idea: different authors have identified various, apparently independent origins of the emergence and diffusion of the idea of competence in different fields in the last decades.

In the educational field, according to Pellerey (2004), the idea of competence begins to be explored around the 1960s, as a means to describe the expected outcomes of teaching interventions. At first, outcomes are described in terms of "final observable and, in some way, measurable behaviours" (ibidem, p. 35, my translation). This initial approach reveals the dominant behavioural approach of those years, characterized by the identification of the idea of competence with that of performance, which pervades the school context for a long time. Later on, the notion of competence is progressively enriched thanks to the reflection developed in the world of work since the end of the seventies. From those years we witness the so-called "de-taylorization" of work (Terraneo and Avvanzino, 2006), which leads to a radical change of paradigm in the conception of the relationship between work and production, and of the organization of work itself. In a Taylor-like work organization, the activity, that the worker must perform, is broken down into simple units, prescribed operations, predefined in a complete and detailed manner. When the organization of work must appeal to the initiative and versatility of the workers, it is necessary for them to develop and be able to mobilize competences related to facing the unexpected, innovating or deciding in uncertain situations (ibidem, p. 17). This new paradigm leads to rethinking professional training and, with it, education in general. In this context, McClelland's contribution appears particularly relevant (1973, cited in Pellerey, 2004; in Mulder, 2007; and in

Mulder et al., 2007), in that launches a series of studies aimed at promoting competences as a tool for personnel selection.

In previous years, Chomsky (1968) introduces, in linguistics, the distinction between performance and competence. Even if the meaning that Chomsky attributes to the term "competence" differs from those generally assumed in the educational field, and even if his influence on educational research will be felt only later (Pellerey, 2004), nevertheless this distinction is of crucial importance. Competence becomes therefore conceived as the abstract capacity possessed by an individual, while performance is considered as the possible manifestation of a competence. Likewise, it is recognized that the quality of performance does not depend only on the set of knowledge and skills that the individual may or may not possess, but it is based on a number of factors that are not directly observable.

This brief overview gives an idea of the rich elaboration that took place around this idea, which over the years has been influenced by different theoretical approaches and paradigms. Despite this heterogeneity, general trends can be traced in the development of the idea of competence. Based on the analysis of Mulder, Weigel and Collins (2007), Marzano and Iannotta (2015) identify three main directions along which the development of the notion of competence took place.

- *From simple to complex*. Competence is an improvement of the knowledge already owned by subject that involves the activation of knowledge, skills and dispositions. The process engages the cognitive, the motivational and the emotional dimension.
- *From the outside to the inside.* According to this process, knowledge draws attention to all those subjective dimensions that are not directly observable outside, but that form the basis of individual behavior.
- *From theoretical to pragmatic.* Competence is specifically assumed and it is related to a given context, losing its general sense. Competence is identified with the subject's ability to use operational strategies for the solution of the problem related to specific culture and contextual dimension" (ibidem, p. 10).

5. The Idea of Competence in Social and Behavioural Sciences

From the reconstruction of the historical development of the idea of competence presented in the previous section, a common general understanding of the idea emerges. In fact, there emerges a shared attempt to characterize competence as a system of "prerequisites" or "conditions" necessary to undertake effective actions in the context of certain activities, "[A]a set of capabilities […] which are necessary conditions for effective performance" (Mulder, 2017, p. 1079). However, as we have already mentioned, there are significant differences in the landscape of social and behavioral sciences. To get an idea of this variety we can refer to the analyses of Weinert (2001), and Mulder (2017).

The former investigates the ways in which competence has been defined, described, or interpreted theoretically — in social and behavioral sciences and identifies nine different theoretical approaches to the notion of competence.

The latter tries to extrapolate from the definitions used in the literature (in particular, with reference to vocational training) the main components that can be involved in the definition of the competence construct, and clarify them.

In both, the focus is ultimately on competence as an attribute of the individual.

5.1. A survey on the idea of competence

In his survey Weinert (2001) identified 9 different approaches to the idea of competence, I mention only some of them to give the flavour of the great variety of conceptualizations Weinert recognized:

- System of *general cognitive resources* independent of the content and context of the activity — e.g. working memory, processing speed
- System of *highly specialized knowledge, skills, routines* ... which depend on the content or context of the activity — e.g. chess playing, piano playing, automobile driving, mathematical problem solving, trouble-shooting in complex systems
- System of *cognitive resources and motivational action tendencies*, including factors such as motivation, sense of self, sense of self-efficacy, belief systems.
- *Metacognitive resource system*, either declarative or pragmatic, concerning the management and regulation of one's own cognitive resources; the ability to use knowledge about own knowledge.
- System of *key-competencies*.

With respect to these multiple meanings Weinert (2001) observes that "Unless one argues that the individual prerequisites for the array of cognitive performances and goal directed actions must include all primary mental abilities, all learned skills, knowledge and strategies, the entire complex of learning and achievement motives, and all important vocational skills, the various definitions of competence listed […] are mutually exclusive on a phenomenological, conceptual or theoretical level".

5.2. The several dimensions implied in the idea of competence

Mulder analysed the definitions used in the literature, in vocational training, and extrapolated the various dimensions involved in those definitions. Several different dimensions emerge from this survey: *contextuality*, *developability*, *measurability*, *definability*, *centrality*, *knowledge inclusion*, *dynamic nature*, *mastery level*, *performativity*, and *transferability*.

In order to illustrate these dimensions, I invite readers to consider the following issues, and ask themselves whether or to what extent they think of competence as:

- general capacity independent on the context or specific to a given context;
- a modifiable or immutable psychological trait;

- something directly measurable or inferable;
- something transferable.

Or ask themselves whether and to what extent:

- knowledge domain is taken into account in the conceptualization of competence
- particular traits of competence are central;
- possible factors that trigger the mobilization of a competence are taken into account;
- different levels of possession of a competence are considered;
- competence is related to (high-level) performance;
- competence is extent is it definable.

The way we answer these questions reveal our understanding of the idea of competence, but it may reveal also the possible objectives for defining the idea as precisely as possible.

5.3. A provisional synthesis

Insofar we consider the competence as the prerequisite system that an individual must possess in order to perform an activity effectively, it is clear that each of the approaches described by Weinert and each of the dimensions detected by Mulder highlight relevant elements. Hence, one can raise the question about which dimension or component should be given especial value.

In my opinion, the interest in a definition of an educational construct, is that of providing a tool to organize, frame, clarify, addressing certain phenomena, situations, or problems. The question therefore arises: for what purposes is it necessary / appropriate to define the notion of competence?

Now, depending on the purposes, different conceptualizations of the idea might be appropriate or useful, different components might need to be emphasized. For instance, if one needs or wants to select people for some objectives, s/he might not need to wonder whether the components are modifiable or not, or whether they are modifiable through purposefully designed activities. But one needs to consider this issue if the idea of competence is meant for designing teaching interventions. Moreover, if the aim is to re-design curriculum or to organize teaching-learning activities, one might consider in different ways the dimension of affect. So, to me, how we conceptualize the idea of competence depends on and should be put in relation to the reasons why we think we need to do that; which phenomena, problems or situation is our elaboration of the idea competence expected to frame.

6. The Idea of Competence in Mathematics Education

In mathematics education, many constructs have been introduced that to some extent can be connected to the idea of competence: mathematical competence, mathematical literacy, mathematical proficiency, numeracy, ... Some of them were introduced before the term competence gained the current diffusion. The differences among these constructs are not only lexical ones of course; the authors who developed these constructs and coined the respective terms, did so because they felt that the existing ones did not grasp what they actually meant to.

My objective is not to make a complete overview, I will just mention some of them to illustrate some common aspects and some differences. In the next sections I will briefly outline the following constructs: mathematical habits of mind (Cuoco et al., 1996, Levasseur and Cuoco, 2003), mathematical proficiency (Kilpatrick et al., 2001), and mathematical competence (Niss, 2003, Niss and Højgaard, 2019).

6.1. Mathematical habits of mind

Cuoco et al. (1996) denounce that the mathematics, which students' study and have studied for generations in high school, "has very little to do with the way mathematics is created or applied outside of school" (1996, p. 375), and contend that "[M]much more important than specific mathematical results are ***the habits of mind*** used by the people who create those results" (ibidem, p. 375, my emphasis).

Habits of mind are particular ways of thinking, of facing situations, and disposing to act in the different situations. Without pretending to be exhaustive, the authors propose a repertoire of habits of mind which should be pursued in mathematics teaching. Students should become *pattern sniffers, experimenters, describers, tinkerers, inventors, visualizers, conjecturers, guessers*… Habits of mind are described and illustrates through several examples. Not all these habits are always appropriate or useful, so students should develop also an *awareness* of when to act one way or another.

To state that mathematics education should explicitly aims at fostering the development of these habits does not mean that they should be explicitly taught the same way in which content knowledge is. According to Levasseur and Cuoco (2003), these habits can and should be developed by students as they do mathematics; "the crucial element is that students be given the opportunity to develop mathematical understanding through problem solving" (p. 27). According to them, habits of mind can, therefore, be used to frame and organize mathematics curricula.

6.2. Mathematical proficiency

The construct of mathematical proficiency is elaborated in the context of a project sponsored by the US National Research Council (Kilpatrick et al., 2001). The goal of the project was to develop recommendations for teaching mathematics, teacher training and curriculum training, based on research, in order to improve the quality of learning of all students in the years from pre-kindergarten to grade 8. The problem of characterizing what can be defined as effective or successful learning of mathematics was therefore posed within the project. Within this context, the expression *mathematical proficiency* is introduced and defined through its components (strands):

- *conceptual understanding*, which refers to the student's understanding of mathematical concepts, operations and relationships;

- *procedural fluency*, the student's ability to perform mathematical procedures in a flexible, accurate, efficient and appropriate way;
- *strategic competence*, the student's ability to formulate, represent and solve mathematical problems;
- *adaptive reasoning*, the ability to think logically and to elaborate reflections, explanations and justifications of mathematical arguments; And
- *productive disposition*, which includes the inclination to see mathematics as a sensible, useful and useful subject to learn, combined with the belief in the value of work and in one's own self-efficacy.

Among these components, both the dimension of disciplinary knowledge and the metacognitive dimension (strategic competence) are explicitly taken into account, as well as the affective dimension with reference to the personal disposition (productive disposition). All components are intertwined and interdependent: *mathematical proficiency* is not a one-dimensional trait, it cannot be achieved by focusing on just one or two of these (Kilpatrick et al., 2001, p. 116). This means that the development of the conceptual understanding also feeds on the development of the other components, and vice versa.

The components of *mathematical proficiency* are identified on the basis of studies in mathematics education and cognitive psychology. In particular, with respect to the literature in mathematics education, different consonances can be recognized between the construct of mathematical proficiency and research on mathematical problem-solving (e.g. Schoenfeld, 1985, 2007).

6.3. Mathematical competence

The KOM project (Niss, 2003; Niss and Højgaard, 2011) was promoted by the Danish Ministry of Education between 2000 and 2002 with the aim of identifying on the one hand any critical elements of the Danish education system and on the other adequate tools to address these critical issues. Among the latter, the heterogeneity with which mathematics is considered and treated at different school levels is highlighted.

> Mathematics is perceived and treated so differently at the different levels that one can hardly speak of the same subject, even if it carries the same name throughout the system […]. In other words, there are problems with the identity and coherence of mathematics as a subject across the levels. (Niss, 2003, p. 3).

In this context, the notion of mathematical competence is assumed as a unifying element to be able to define what it means to "master mathematics" at all school levels, and therefore as a tool to be able to articulate the description of the curricula and the learning outcomes expected at end of each cycle of education, and to describe students" learning progress in mathematics through different school levels. At the origin of this approach, there is therefore the common concern of defining the dimensions along which to build and organize the curriculum. To this end, part of the work of the KOM project was directed to the elaboration of the notion of mathematical competence.

> Mathematical competence then means the ability to understand, judge, do and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. (Niss, 2003, p. 7).

The entire framework of the KOM project has been recently revisited with the aim of updating the terminology and clarifying some definitions while keeping the overall system unchanged (Niss and Højgaard, 2019). In particular, mathematical competence is now defined as "*someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations*" (ibidem, p. 12). In this new formulation, the challenging nature of the situations emerges with greater clarity, at the same time it is not explicitly stated whether these situations should concern intra- or extra-mathematical contexts, or both. The term *readiness* in the definition refers only to the cognitive aspects and not to the volitional and affective ones (ibidem, p. 12). In fact, the affective and volitional dimension is intentionally, deliberately excluded from the definition of mathematical competence. This decision clearly distinguishes the approach to mathematical competence taken in the KOM project from the approach to mathematical proficiency seen previously.

Mathematical competence is structured into eight components called *mathematical competencies*, which define, in a certain sense, the nature and characteristics of the mathematical actions to be undertaken to face the different challenges that may arise in different situations: *mathematical thinking competency*, *problem tackling competency, modeling competency*, *reasoning competency, representing competency, symbol and formalism competency*, *communicating competency* and *aids and tools competency*. Each of them has a dual nature: *analytical*, which consists in the ability to understand and examine aspects of mathematical activity conducted by others, and *productive*, which consists in the ability to carry out mathematical activities in the first person.

Unlike the five strands that define *mathematical proficiency*, these eight competencies, although linked to each other, are seen as independent components of *mathematical competence*. As well as *habits of mind*, these competencies are identified starting from the analysis of the characteristics of the potential action of an expert, characteristics that therefore constitute the reference term for the development of students" competence in mathematics.

6.4. A second provisional synthesis

These perspectives share a common starting point. In fact, they start, with some differences, from a feeling of unsatisfaction and an explicit critique towards the objectives of mathematics teaching as described in the curricula. Moreover, they also share common objectives, at some extent at least. To say it in Kilpatrick's words "*Competency frameworks are designed to demonstrate to the user that learning* *mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures*" (2020, p. 112).

The ability to use mathematics and mathematical skills to cope with given situations is conceptualized and described as a multi-dimensional construct constituted by different components specifically related to mathematics and mathematical activity. But the approaches differ with respect to the specific components highlighted, their nature, the relation between these components and between them and the general idea.

7. A Triadic Structure of Competence in Educational Domain

The constructs described in Section 6 can be considered (at least partially) resonant with definitions coming from the field of education. Here, I report two definitions from Pellerey and Perrenoud respectiveley:

> Competence is the ability to cope with a task or a series of tasks, to be able to initiate and orchestrate one's internal resources — cognitive, affective and volitional ones — and to use the available external ones in a coherent and fruitful way. (Pellerey, 2004, p. 12, my translation)

> The ability of a subject to mobilize all or part of her/his cognitive and emotional resources to deal with a family of complex situations. (Perrenoud, 1996, p. 15, my translation)

These definitions share a common triadic structure: both refer explicitly to ***resources*** (including non-cognitive ones) which need to be mobilised to address some ***kinds of tasks*** or face given situations in ***effective ways***. Beyond the apparent simplicity, each of these features — resources, tasks, effectiveness — constitute a critical issue inherent to this type of conceptualization of competence, and I am tempted to state to any kind of conceptualization which could be interesting and helpful for educational purposes.

7.1. Resources

Both the definitions refer to the mobilization of internal and external resources. Though it is not explicitly stated, external resources include not only available tools, but also other human agents. With that respect, more explicit is the definition of Zarifian (1999, quoted in Terraneo and Avvanzino, 2006): "Competence is the ability to mobilize networks of actors around the same situations, to share issues, to assume areas of joint responsibility" (Zarifian, 1999, p. 77, my translation).

Wittorski (1998) pushes the discourse further and suggests the possibility of attributing competence to a collective as such. That raises the issue of the relationship between individual and collective competences. This issue cannot be solved by introducing the reference to generic relational skills (also important): individuals need to develop a common image of the activity to be carried out as a whole, of the different phases and of the various individual contributions, and that it develops a specific language to manage the interaction. Considered from this point of view, we can say that relational competence specializes in relation to a specific domain, mathematics in our case. The fact that carrying out a task requires the participation of complementary competences does not mean that these competences should or could be less developed: "Paradoxically, the stronger the collective competence, the more the individual competences become indispensable." (Terraneo and Avvanzino, 2006, p. 19, my translation)

In education, shifting the focus — from the individual"s competence to the competence of a collective — raises sensitive questions both respect to individual assessment and respect to how to promote the development of individual and collective competences in a balanced way.

Beyond the individual/collective duality, Pellerey's and Perrenoud's definitions draw the attention to internal resources needed to face a given situation. Internal resources include not only cognitive resources but also metacognitive and affective ones. From point of view of mathematics education, this choice, on the one hand, recognizes and values the importance of the role of metacognitive and affective factors in shaping mathematical activity, especially in an educational context. On the other hand, it makes the whole picture even more complex, or, better, it reveals more clearly the complexity of the whole picture.

Internal resources, be they cognitive, volitional or affective, depend largely on the specific domain evoked by the situation. Thus, there remain the issues to identify the resources needed to carry out mathematics activities efficiently, to suitably characterize them; there remain the issues whether it is possible to set the development of these resources be explicit educational goals, and in case how to intentionally promote them. The research studies discussed in Section 6 address and frame these issues in different ways.

7.2. Tasks

Competence is defined as the ability to cope with a task or a family of tasks. But, what is exactly meant by task?

Duncker (1935) defines a problem as something that arises when a living being has a goal but does not know how to reach it. Drawing on this definition, Zan (2007) proposes to distinguish between problem and task — depending on the existence or not of a goal and of its sense for the individual — and between problem and exercise — depending on whether the solver has or does not have a procedure available to reach the goal. In this same line of thought, Terraneo and Avvanzino (2006) propose to distinguish, in the context of work psychology, between prescribed and actual task, between explicit prescription, implicit prescription and perceived prescription. It is a type of distinction that is absolutely pertinent to the educational field, too.

Another important issue concerns the distinction between real-life or in-context tasks and simulated or contextualized tasks; with the recognized potential but also the limits of proposing simulated tasks in the classroom (Mulder et al., 2007; Palm, 2002).

Many conceptualizations of the idea of competence stress the challenging nature of tasks. This necessarily leads to relativize the idea of competence, to problematize the performance as an indicator of the presence or absence of a competence, to recognize the importance of devoting a specific attention to the affective dimension.

Finally, one speaks of family of tasks, tasks sharing analogies, but what does it mean? Who is expected to see these analogies, whose point of view is assumed? For instance, one could consider a given *field of experience*, that is as a field of human cultural experience recognized as homogeneous and unitary (Boero et al., 1995). Or, one could consider a family of tasks, as a set of situations whose mastery requires a certain system of concepts, procedures and symbolic representations strictly connected between them, that is as given *conceptual field* (Vergnaud, 1995). The organizing principles assumed in the two evoked perspectives are quite different, and this difference is not without implications form an educational point of view.

Finally, let's note that the descriptions of tasks and resources are strictly connected, in fact the more clearly the "family of tasks" at stake, or the "domain of competence", can be defined, the easier is the analysis of the resources to be developed to operate effectively in that domain. At the same time, we can say that a task can belong to the "domains" of different competences and require the mobilization of different competences.

7.3. Efficiency

Finally in the definitions of competence we are examining, there is an indirect reference to the theme of assessment, as the question arises of what "fruitful and effective mobilization of resources" means, and how it is assessed. This leads to the necessity of considering who assesses, why, for what purposes, what is assessed and how.

When an individual faces a certain situation, the individual her/himself, an expert, a collective (of possible non-experts) can, each, be involved in the evaluation of the action of the individual, of the outcome of the mobilization and orchestration of her/his resources, of its eventual fecundity and efficacy. Each of these agents can have different criteria for the assessment, focuses on different possible indicators. The situation itself can provide, or not, feedback, which can used for the assessment.

There might be several reasons why it can be necessary to establish whether the individual succeeds to cope with the situation in an effective way. For instance, the task can be relevant in itself and needed to be solved. The assessment of how the task has been accomplished can be needed for devise future initiatives depending on the accomplishment of the task itself. In some contexts, assessment is made for selecting people, for certifying competences, for promoting learning.

With that respect it is worthwhile noticing that if one considers an educational context, two dimensions are simultaneously present:

- The assessment of the accomplishment of the task per se (which can be made by different agents)
- The assessment of the accomplishment of the task as an indicator of competence, which is made by an evaluator (basically the teacher).

With respect to the issue of the assessment in schools, it seems to me that the discourse is backlog: one pretends to evaluate competences after a teaching practice that is not aimed at developing them. It seems to me that Mulder and colleague's criticism of some vocational training programs should sound like a warning to school: "the emphasis on competence assessment is unbalanced, and [that] it frustrates learning and development more than it supports it" (Mulder et al., 2007).

8. Conclusions

The discussion developed does not solve the problem of the definition of mathematical competence and of its elusiveness. This was not the objective. Rather, I meant to discuss the complexity of the idea of competence and to put the issue of its possible characterization it in a different perspective: the choice of an approach to the definition of competence brings into play objectives and systems of values; therefore, the first step in being able to start a fruitful discussion on this issue, at any level, is to explain them.

Competence, as we have seen, is a multi-faceted complex idea. It emerges, in any conceptualization, as a ***multidimensional construct***. By stating that competence is a *construct*, I mean to highlight its artefact-like nature: it is a conceptual construction which can be useful to frame, organize clarify, complex phenomena. In this sense it is not an attribute of an individual but express the point of view of someone on something. To me the key-issue is what idea we mean to capture through this notion and for what purposes we need/want to capture this idea.

It is *multidimensional*: as we have seen, any conceptualization refers to "several distinct but related dimensions or components treated as a single theoretical concept" (Edwards, 2001, p. 144). It needs to be rich to capture complex phenomena but needs no to be too complex for being useful. Diverse components can be considered with different emphasis; the relation amongst these components and between them and the more general idea need to be clarified. Since we need the breadth and comprehensiveness of a multidimensional construct and the precision and clarity of its single dimensions.

Personally, some of the reasons for which I began to investigate this notion are still there, for instance: how can one foster the development of students" mathematical competence? What does it mean? How can one attest the development of competences?

We need to make the notion of mathematical competence more operational, to define the unit of analysis in educational research on mathematical competence and problematize how we can capitalize on the existing research results in mathematics education concerning many aspects which can be related to the idea of competence.

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https://doi.org/10.1142/9789811287183_0027

Invited Lecture 27

The Power of Mathematical Task for Teacher Training: The Case of Suma y Sigue

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ABSTRACT The need to improve teachers' preparation to teach mathematics is shared by many countries. E-learning professional development (PD) programs appear as an attractive option due to their flexibility and availability. Suma y Sigue is an e-learning PD program for Chilean teachers that focuses on the development of Mathematical Knowledge for Teaching (MKT). The program is characterized based on a constructivist perspective of learning by using a contextualized problem-based approach. This article describes the instructional design of the program learning activities that demonstrate how mathematical tasks centered on the construction of MKT are articulated and implemented. The learning performance of the participants in a specific course within the program is analyzed. The findings show empirical evidence of improvement in teachers' knowledge. The detailed description of the course and participants' performance can aid PD developers to design principles and the use of different instructional strategies, especially when the course focuses on MKT development.

Keywords: E-learning; Instructional design; Professional development; Mathematical knowledge for teaching.

1. Introduction

Improving teachers' knowledge and skill to teach mathematics is a need in many places and contexts. Ball and Bass (2000) believe that teachers' mathematical knowledge should be strong enough to allow them to deal flexibly with the complexity of teaching mathematics to different students. Ball (2003) emphasizes the importance of "designing courses in mathematical knowledge for teaching, helping instructors and professional developers teach them well, and doing so at scale" (p. 38). There are

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recommendations for Professional Development (PD) programs for teachers, which acknowledge the need for a clear focus on Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008) especially to impact teachers' work with students in the classroom (Campbell and Lee, 2017). Many efforts have been made to understand the role of strategies used to deliver PDs activities and bring learning opportunities for teachers to improve their MKT (Copur-Gencturk et al., 2019). Borba and Llinares (2012) identify knowledge-building practices in technology-mediated workgroup interactions among several key topics that require further research in online mathematics teacher education. Yet, it is not evident the types of learning activities that might help teachers to develop such knowledge.

Suma y Sigue is a PD program designed based on e-learning modalities (fully online or through blended learning) for Chilean teachers, which is focused on the development of teachers' MKT. The program is based on a constructivist perspective of learning by using a contextualized problem-based approach. Martínez et al. (2020) studied teachers' satisfaction by participating in the blended model of the Suma y Sigue PD program. Starting in 2020, the program changed its format by adopting an e-learning approach. This was planned due to the necessity to reach teachers in remote locations and was also precipitated by the COVID-19 pandemic. In this study, we describe the instructional design of a virtual learning environment that allows teachers to develop their MKT through an instructional model with a high autonomous learning component. We also emphasize how the learning sequencing of this model promotes teachers' engagement in mathematical tasks, and how it guides a user in both the reinvention of elementary mathematics and the development of professional mathematical knowledge. In this paper, we aim to explain the Suma y Sigue design structure in online model by focusing on the activity design and the learning sequence of a particular course called "Working with multiplication and division". Moreover, we analyze the learning results of participating teachers in this course, discussing the relationship between course activities and change on teachers' MKT.

2. The Importance of Professional Development Programs and Teachers' Knowledge

For the past three decades, efforts to improve the competencies of math teachers and thus the quality of math education have been a constant feature of educational policies around the world. To this end, policymakers and educational organizations have implemented various plans and programs to reform education. In the field of reform, teacher preparation plays a key role in enhancing teachers' professional competencies (Barber and Mourshed, 2008). By focusing on in-service teachers' education, PD programs have been known as an important area of research in promoting mathematics education and also as a goal of various governments and research communities during the last decade (Koellner et al., 2011; Martin and Mulvihill, 2020).

Chile has begun extensive reforms to improve the quality of education (Santiago et al., 2017; Toledo and Wittenberg, 2014), particularly in mathematics (Saadati et al., 2023; Martínez et al., 2020). For example, through the new national Teacher Professional Development System in 2016, teachers have been encouraged and granted the right to participate in PD programs. Thus, it is expected that teachers will have access to free and relevant education in order to further develop their professional careers and improve their knowledge and professional competencies. That system brings opportunities to develop different types of PDs in Chile to support educational reform.

Considering the reforms in mathematics education, efforts have been focused on improving the knowledge of teachers so that they are able to carry out teaching in a way that allows them to help students understand mathematics conceptually. In fact, teaching mathematics is a serious and demanding arena of work and teachers need to be prepared to handle it (Ball, 2003; Ball et al., 2005). The quality of teaching depends on the knowledge and teaching skills of the teachers, so focusing on improving MKT is a key strategy to address this need (Campbell and Lee, 2017).

Copur-Gencturk and colleagues (2019) examined the successful characteristics of a PD in improving teachers' MKT and learning, such as PD tasks, materials, and agendas. They showed that a focus on curricular content knowledge and reviewing student work seems to be important to improve teachers' content knowledge for teaching. Garet et al. (2001) proposed two groups of features for effective teacher PD programs. The first one is structural features such as the form of activity, its duration and collective participation, and the second group refers to the core features which include content knowledge, active learning, and coherence. Considering these two groups of features is a must to design an effective PD program to improve teachers' MKT.

In general, providing opportunities for in-service teachers to develop MKT requires constructing and implementing a specific type of task that should: (a) create opportunities to unpack, make explicit, and develop a flexible understanding of mathematical ideas; (b) provoke a stumble due to a superficial understanding of an idea; (c) help to make connections among mathematical ideas; (d) lend themselves to constructing multiple representations and solutions methods; and (e) provide opportunities to engage in different mathematical practices (Suzuka et al., 2009).

2.1. Contextualized problem-based learning

According to the Realistic Mathematics Education (RME) theory (Freudenthal, 2012), contextualized problems are inextricably linked to mathematics learning, with "realistic" contexts serving both as a source for initiating the development of mathematical ideas and as settings to later apply mathematical knowledge. Context problems function as anchoring points for a guided reinvention of mathematics by the students, helping to bridge the gap between informal and formal mathematical knowledge (Gravemeijer and Doorman, 1999).

Even though mathematical literacy is defined in terms of an individual's capacity to solve problems in a variety of real-world contexts (OECD, 2018), most problems which students face in mathematics classrooms can be solved by a simple and straightforward application of one or a combination of the four basic arithmetic operations and are not closely related to students' experiential worlds (Depaepe et al., 2010). Moreover, teachers and students, mostly focus on the mathematical structure of a problem, ignoring the realistic aspects that could help them to make sense of its structure (Depaepe et al., 2010; Peled and Balacheff, 2011). Thus, it is necessary that teacher education addresses the need to improve teachers' understanding of the value of real-life problems to mathematics learning (Peled and Balacheff, 2011). Considering contextualized problem-based learning in professional development programs may aid teachers to use more realistic situations as a starting point for mathematical activities in the classroom.

Using teaching situations as the context for problem solving activities may also impact teacher competencies and pedagogical knowledge. Zaslavsky and Sullivan (2011) indicate that worthwhile teacher education tasks are those that are motivated by the desire to foster the orientation in prospective teachers to the study of practice. Case-based teaching can be used for creating meaningful settings for teacher learning (Putnam and Borko, 2000). This approach allows enacting tasks which are idiosyncratic to teacher education and explores the richness and complexity of genuine pedagogical problems (Putnam and Borko, 2000).

2.2. Virtual learning and mathematics teacher education

The idea of virtual learning has been underlined in the mathematics learning literature and recently has been translated into mathematics teacher training (Borba and Llinares, 2012; Goos et al., 2020; Martínez et al., 2020). In this shift, constructivism helps educators design virtual courses and learning environments, in which learners build their own knowledge. Constructivists believe that learners construct knowledge (rather than acquiring it) individually through their interactions with the environment (including other learners) from their authentic experience, mental structures, and beliefs, which are themselves mediated by the prior knowledge (Ernest, 1996; Simon, 1995; Thompson, 2014).

Before the widespread use of the Internet, mathematics knowledge belonged to teachers and textbooks, and mathematics teaching happened in formal classrooms or teacher-centered settings with a mandated curriculum (Borba et al., 2012). After the availability of the Internet and the use of new technologies, the perspective on learning has changed. Learning can be happening face to face or virtually, synchronous or asynchronous, in a classroom or through the Internet (as e-learning), especially in a large and distributed community.

With the use of the Internet or a shift to e-learning in mathematics education, three fundamental foci within mathematics education can undergo a radical change; mathematics knowledge, teaching and the context of classrooms (Borba et al., 2012). In fact, thoughtful replacement of face-to-face education with online learning includes three main features: a fundamental change in course design to optimize learners' interaction with the learning environment as well as with other learners; the ability to restructure and replace traditional class contact hours with the flexibility to choose the learning time; the flexibility to choose different learning activities according to the needs of learners, including the content of educational materials in various forms of documents, videos, animations, simulations.

Research showed that teachers often resist participating in intensive long-term PD programs, especially when the commitment involves traveling from their school to another location (McConnel et al., 2013). Moreover, a lesson we learned from the Covid-19 pandemic and the closure of schools, is the importance of online learning. Therefore, given the key features of e-learning, a PD program in a virtual modality can be a solution for continuing teacher education. It brings several benefits for teachers such as: reducing the attendance time of in-service teachers in face-to-face or synchronous workshops, which is an inconvenient factor for participants due to conflict with their work schedule (Eroğlu and Kaya, 2021), arranging courses for a longer period which is recommended to increase the efficiency of a PD program (Garet et al., 2001), and offering interactive online learning plans, which help teachers to have rich and flexible knowledge about the subject they teach (Borko, 2004).

In line with the importance of online learning in teacher education, Borba and Llinares (2012) suggest that online activities can transform teacher collaboration and cause individual development. However, designing a virtual PD course needs specific attention.

Goos et al. (2020) described the effectiveness of a blended learning PD program to address the lack of mathematics content knowledge and mathematics pedagogical competencies among in-service mathematics teachers. In Martínez et al. (2020), the design of the PD program aimed at improving teachers' MKT is discussed in detail. This work presents how to use different instructional strategies in e-learning modalities to develop teachers' MKT. Although there are significant benefits for teachers, constructing virtual learning environments to improve teachers' MKT is a difficult task for PD developers. To help developers, it is vital to present samples of PDs with a full description of the principles of design and its materialization.

3. The Case of Suma y Sigue: An E-learning PD Program

The Suma y Sigue program was developed at the Center for Mathematical Modeling (CMM), a research institution of the University of Chile. Its development included the joint work of several teams, including content development, graphic design, and programming. The content team was made up of teachers, mathematicians, and experts in mathematics education. This composition was essential to focus the contents of the course on the professional knowledge involved in teaching mathematics and the design of learning activities that were relevant to that effort. The creation of a single course took around 8 months. The development and implementation of the program was supported by collaboration agreements between the CMM and the Chilean Ministry of Education (MINEDUC).

3.1. Design principles

The instructional model of "Suma y Sigue" aims to improve teachers' MKT, through contextualized learning activities focused on the deep analysis of elementary mathematics. For this, three fundamental principles are established, which are discussed in more detail in Martínez et al. (2020). The first principle is based on the constructivist perspective of learning, that is, it is considered that knowledge is not passively received by learners but actively constructed by them using their previous knowledge (Thompson, 2014; Ernest, 1996). For this reason, in the program, the contents emerge as the teachers solve mathematical and didactic problems that force them to use their previous knowledge and restructure it to find a solution. A second principle considers the importance of using contextualized problems in realistic situations. This materializes through the use of mathematical problems placed in contextualized situations that allow solvers to make sense of formal mathematics (Freudenthal, 2012; Gravemeijer and Doorman, 1999), and also, with the use of didactic problems that put teachers in plausible classroom situations (Putnam and Borko, 2000; Zaslavsky and Sullivan, 2011). Finally, the third principle is articulated around the MKT model (Ball et al. 2005, 2008), which proposes that the teaching of mathematics requires specific knowledge, which can be distinguished, characterized and developed. Some of the courses focus heavily on the subject knowledge components of the MKT model, that is, on common and specialized knowledge, while other courses incorporate the pedagogical knowledge components to a greater extent.

3.2. Materializing the principles in design

The program's online activities are built around a mathematical story which allows articulating the different types of tasks necessary to develop MKT. As the story unfolds, new conflicts arise, which makes it possible to modify the didactic variables of a task or change the type of task addressed. For instance, a discussion among the characters of the story can help to produce a conflict around a mathematical idea, triggering a questioning process that leads users to unpack mathematical concepts. Technology allows the design of dynamic learning scenarios in which tasks and content are progressively displayed, facilitating not only addressing different types of knowledge in an integrated way but also a scaffolded learning process.

3.3. Structure of the program

The "Suma y Sigue" program offers thirteen courses dedicated to teachers who teach mathematics at different primary and secondary school levels. Each course addresses topics specific to a curricular domain (Numbers and Operations, Geometry and Measurement, Algebra and Patterns, Probability, and Data). The courses are organized in two modules, each module is made up of two or three asynchronous virtual workshops followed by a synchronous workshop (Fig. 1). At the end of each module, participants answer an online test whose items address the mathematical and pedagogical content of the course. The courses last approximately 34 hours distributed over 10 weeks. About 25 hours are of asynchronous work and 9 synchronous work.

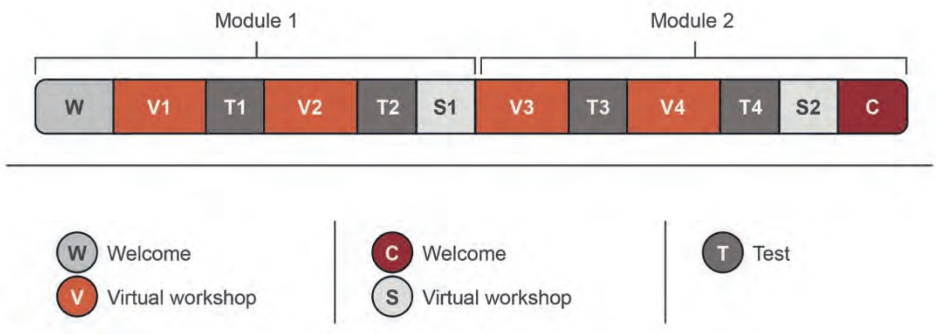


Fig. 1. Course structure

3.3.1. Virtual workshops

The learning unit of a workshop is known as a virtual activity. Each activity has a specific objective defined according to the mathematical content addressed. A contextual situation is proposed for each activity, acting as a frame for arranging different tasks. The resolution of these tasks requires building specific mathematical knowledge to address the situations. Both the context and the learning tasks are dynamically integrated, in the sense that they become more complex throughout the activity, allowing participants to consider a variety of aspects of MKT. Each workshop is made up of three to six virtual activities and ends with a systematization section that summarizes all the contents seen in the activities.

The virtual activities are designed to trigger autonomous learning which requires involving and keeping the teacher focused on the construction of knowledge. Each activity has three phases: engagement, construction, and systematization. In the engagement phase, a contextualized situation is introduced, usually in daily life or a classroom setting, using a variety of learning resources such as cartoon stories, dialogues between characters, and animations. Depending on the complexity of the knowledge involved, the construction phase is structured in an activation-institutionalization-practice cycle, which is repeated as new MKT elements are incorporated. In the systematization phase, the contents addressed through the activity are summarized. We refer to Martínez et al. (2020) for a more detailed description. In Section 4 we provide an example, describing the different phases of activity of the first virtual workshop of the course called “Working with multiplication and division”.

3.3.2. Synchronous workshops

At the end of each module, an asynchronous workshop is held. The objective of these workshops is to discuss pedagogical issues related to the mathematical contents of the course. These workshops are organized in three stages: the first one called “Activation” occurs before the synchronous session. Teachers must reflect individually on a mathematical or classroom situation. In the second stage called “Synchronous discussion,” the participants meet by zoom to discuss the situation, which is connected with a reflection on the teaching of the content. Collaboration is promoted here through small group discussions, the conclusions of which are then brought back to the full group. The last stage is called “Discussion projections” in which participants reflect on the learnings achieved in the discussion and consider new deepening questions. The participants must analyze a document that systematizes the main ideas of the synchronous virtual discussion and introduces new questions. The discussion happens in a virtual forum moderated by the course tutor.

3.4. Assessment

The evaluation process of the “Suma y Sigue” courses fulfill two purposes: on the one hand, the learning of the participants is qualified to certify the approval of the course (summative assessment) and, on the other hand, feedback is given to the participants on their performance (formative assessment). As for the summative evaluation, teachers take 4 tests throughout the course. These tests are applied at the end of each workshop on the fixed dates scheduled before. Each test has a total of 7 items in multiple-choice, true and false, and open response format. These items assess the mathematical and didactic content reviewed in each workshop. Regarding formative evaluation, the platform offers constant feedback as teachers progress through the course. Depending on the complexity of a question, after the participant submits their answer, a feedback narrative capsule (Narciss, 2008), named “Exploring a possible solution”, is displayed to her/him containing feedback and justification of correct and incorrect answers. This allows participants to reflect about the knowledge required to answer the question. These feedback capsules have different levels of complexity involving written explanations, pictorial representations and animations.

The course approval criteria include achieving at least 60% correct answers in the tests, having completed at least 80% of the platform activities and attending the two synchronous workshops.

4. A Course Description

The “Working with Multiplication and Division” course is aimed at teachers who teach mathematics from 2nd to 4th grade. This course has four asynchronous virtual workshops. The first one, “Multiplicative situations”, is devoted to analyzing different types of problems and representations associated with multiplication and division, providing different interpretations of these operations with whole numbers, addressing the transit between them, and establishing some basic properties. In the second virtual workshop, “Multiplication”, the justification of properties of multiplication will be addressed using generic examples. Also, the construction of the multiplication tables and different calculation strategies, including the standard algorithm, will be justified in this workshop. For the third virtual workshop, “Division”, various contextualized problems are proposed to address the properties of division in connection with calculation strategies. In the last virtual workshop, called “problem solving”, a series of stories showing students engaging in problem solving activity are proposed to analyze the relevant aspect of this type of activity in the classroom, such as the use of representations, understanding different solutions strategies, recognizing errors as an opportunity to enrich learning, and correctly interpreting mathematical results.

The two content-focused synchronous workshops are taught after teachers have finished the virtual workshops 2 and 4 respectively. In the first one, which will be described in more depth below, the teaching of the multiplicative situations of grouping and combination type, their models and representations are discussed. The second synchronous workshop is focused on the teaching of different division strategies. For that, a video clip showing a classroom situation in which students are using different strategies for solving a multiplication task.

4.1. The “multiplicative situations” virtual workshop

In this workshop, three types of multiplicative situations are introduced to provide meaning and connections between multiplication and division. For that, a story involving the discussions of two characters that work at a restaurant is constructed. This context is useful to motivate teachers to think about different types of situations, such as the distribution of pastries among diners, the combinations of dishes for a dinner, and the assignment of waiters to rooms. This story allows teachers to build connections between possible interpretations and representations of multiplication/ division, understanding the role of the numbers involved accordingly. This workshop aims to help teachers develop specialized mathematical knowledge that they need to teach these operations with students from 2nd to 4th grade, according to the Chilean curriculum.

Four activities comprise the “Multiplicative situations” virtual workshop: “Grouping and arranging pastries” which introduces multiplications as a solution to grouping and array-type problems; “Choosing the menu” in which multiplication is the solution to combination problems; “Sharing in the restaurant” where division appears as a solution to grouping problems associated with the question “How many are there in each group?”; “Setting groups for events”, that connects division with grouping problems associated to the question “How many groups are there?”. We describe the first activity in more detail showing its different construction cycles.

4.1.1. Grouping and arranging pastries

This activity starts by introducing Anaís, a cook, who drew the following diagram to represent the number of pastries that she should bake for each table, as shown in Fig. 2 below:

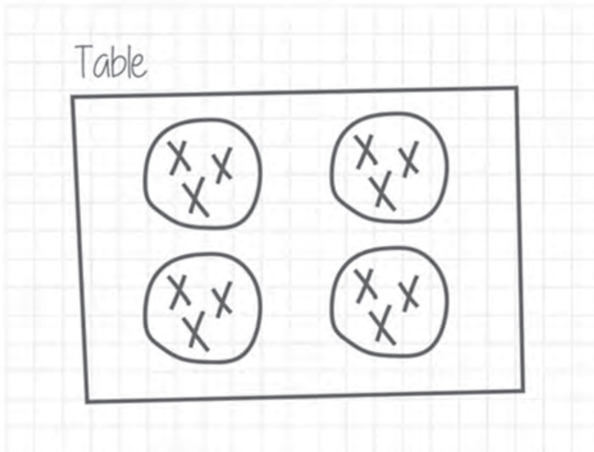


Fig. 2. Pastries' arrangement

In the first question, teachers are asked to recognize how Anaís is organizing the pastries, distinguishing the number of groups and the number of elements of each group, as well as the addition that corresponds to this organization (Fig. 3).

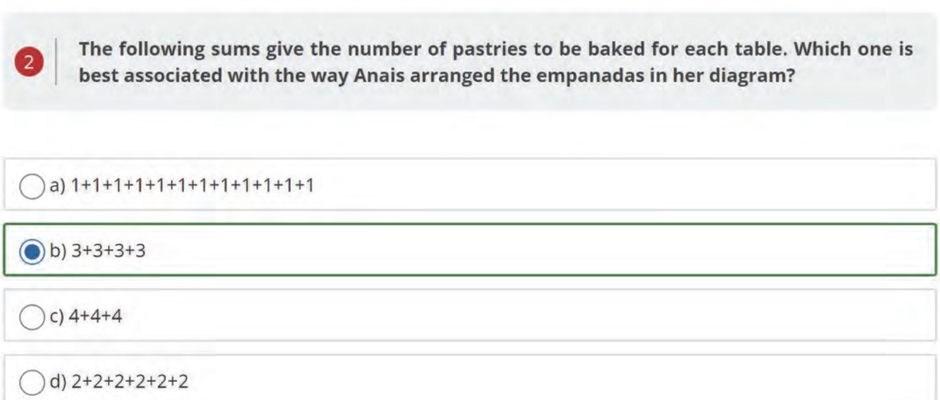


Fig. 3. Question example

After a teacher submits her/his answer a narrative feedback capsule is displayed showing the transit between both representations (pictorial and symbolic), and justification to understand the reason behind representations and the transit (Fig. 4).

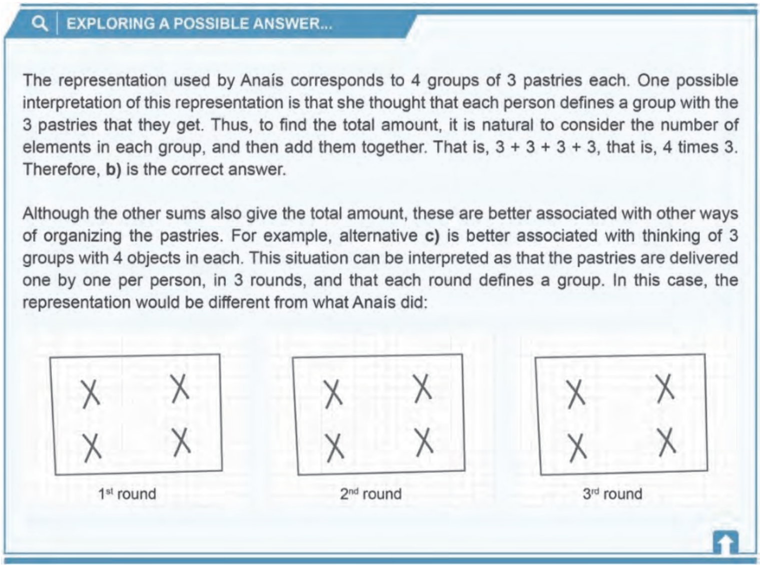


Fig. 4. An example of a feedback capsule

The connection between this situation and the multiplication of 4 × 3 is established in a content capsule that defines the multiplication of a × b as an iterated addition (Fig. 5), ending the "engagement phase" of the activity.

Multiplication is an operation between two numbers a and b, called factors, with which another number is obtained, called the product, and denoted a • b.

When a and b are whole numbers, multiplying a by b corresponds to adding a times the number b.

This is:

$$a \cdot b = \underbrace{b + b \cdots + b}_{a \text{ times}}$$

We will call a sum like this, where the addends are all the same, a repeated sum.

Fig. 5. A multiplication definition

The first "construction phase" starts with a question, in which teachers must decide whether a situation is described by 2 × 5 or 5 × 2 according to the definition given above. Then, a question regarding plates of pastries with different types of elements is introduced, to show that even though the number of pastries is 12, it is not connected to a multiplication. After deducting that having groups with an equal number of elements is necessary to relate grouping problems with multiplication, the connection is established in a content capsule (Fig. 6).

The product $a \times b$ between two whole numbers a and b can be interpreted as the number of objects in a groups, each with b elements.

Problems involving groups with the same number of objects each will be called **grouping problems**. As we saw, multiplication allows us to obtain the total number of objects when the number of groups and the number of elements in each of them are known.

Fig. 6. A content capsule that addresses grouping problems and multiplication

The next construction phase starts with Juan Pablo, a cook, who has placed some pastries in a tray as follows (Fig.7). A conversation between Anaís and Juan Pablo unfolds (Fig. 8).

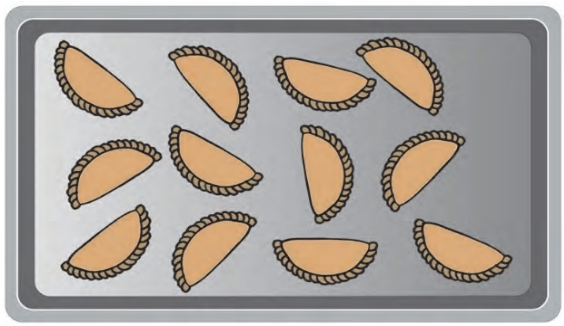


Fig. 7. Pastries in a tray

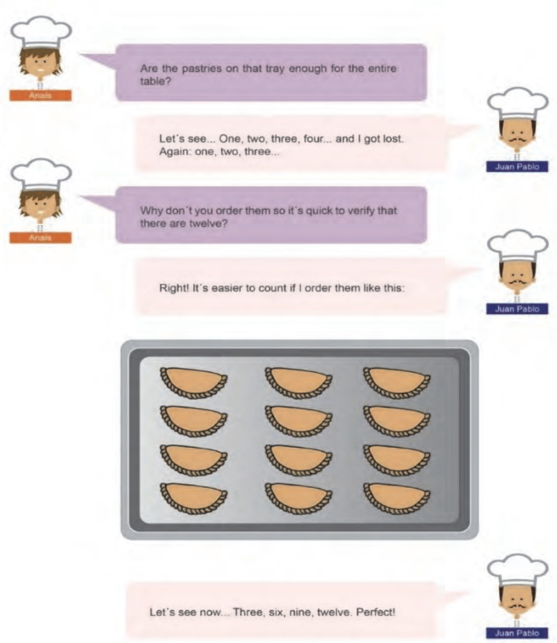


Fig. 8. A dialogue leading to an array of pastries

Teachers are asked to recognize the Juan Pablo's counting of pastries with the multiplication, and then to connect the grouping problem with the array as shown in the dropdown question below (Fig. 9):

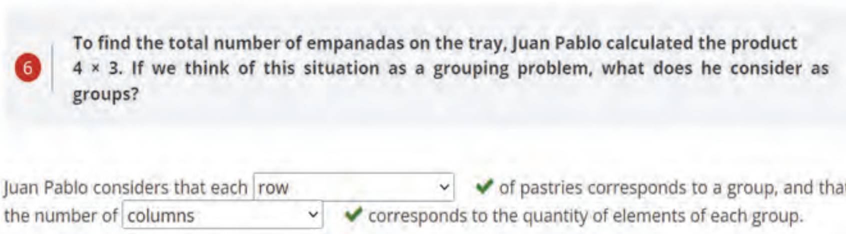


Fig. 9. A question connecting a grouping problem with an array representation

After this, some characteristics of array problems are addressed through questioning, so that teachers deduce that, given groups with the same number of elements, it is always possible to organize their elements in an array, but that not all array diagrams correspond to grouping problems. Thus, it is established that under certain conditions, both types of problems, grouping and array provide two possible representations for one scenario. This conclusion ends the second construction cycle.

The last construction cycle of this activity is focused on the deduction of the commutative property using an array. It starts with Juan Pablo assembling the pastries and placing them on a table, as shown in Fig. 10:

Fig. 10. An array of pastries

Then, a conversation between Juan Pablo and Anaís follows:

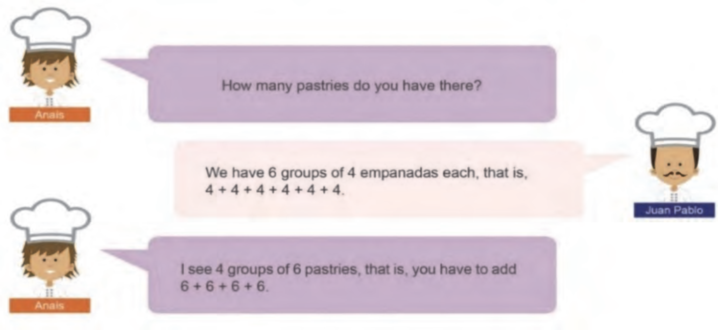


Fig. 11. Two iterative addition expressions for the number of pastries

This dialogue presents two different expressions to compute the number of pastries. Teachers are asked to recognize them and connect them with the array representation, showing that the two different iterative additions are indeed equal. This cycle ends with a discussion regarding the use of generic examples to justify properties, in this case, the commutative property of multiplication.

4.2. Description of a synchronous workshop

The first synchronous workshop of this course delves into didactical aspects relevant to teaching of multiplicative situations, and it is held at the end of the second virtual workshop. In the first stage "activation", an activity involving the analysis of a classroom video is proposed, in which 4th grade students solve a problem. Teachers are asked to recognize the type of problem proposed, the role of the quantities involved, and then to describe and interpret the different answers given by the students to the problem proposed. In the main activity of the "synchronous discussion" stage the problem shown in Fig. 12 is introduced for discussion among participants:

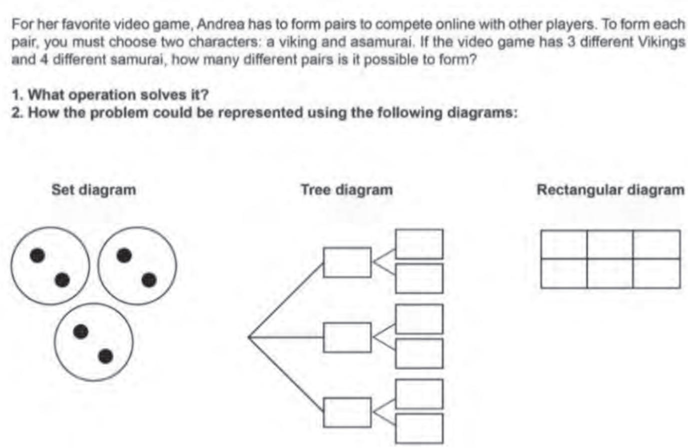


Fig. 12. A problem for discussion

In the group discussion that ensues, teachers are expected to recognize that there are representations that are more suitable than others to model a particular situation, and that the representations themselves are helpful to understand why the operation that models the situation is a multiplication. Later, teachers are asked if the video game problem above can be modeled through with the phrase in Fig. 13 below, and to recognize differences between the grouping and combination situations.

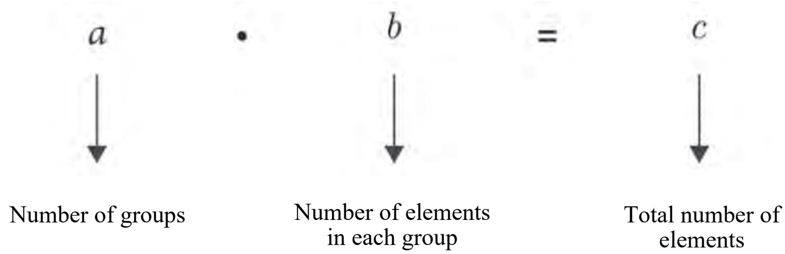


Fig. 13. Symbolic representation

In the last stage "discussion projections" participants are presented with a document in which a classification of multiplicative situations is proposed. Then, starting with a set of problems, they are asked to propose a teaching sequence for them, justifying the reasons that support such progression.

5. Methodology

A quasi-experimental design with pre- and post-tests was adopted, applying two analysis techniques to describe how the teachers participating in the course can be distinguished or classified based on their MKT before and after participating in the course.

5.1. Participants

The study involved 124 primary school teachers from different regions of Chile, who enrolled in the program voluntarily. However, only 91 who answered all the items of the instrument in the pre and post-test were considered in the analysis section. This is because of the limitation of the statistical analysis technique (the latent class analysis or LCA) that applies here. It is suggested to eliminate those teachers who do not participate or answer all items in order to reduce the classification bias (Rose et al., 2017). The LCA as a multivariate technique classifies teachers according to the latent construct arranged in the observed variables (teachers' responses as their selection of all distractors or correct responses) obtained through the multiple-choice items in the instrument (Vermunt and Magidson, 2002). This assumes the evocation of the intended knowledge when interpreting the stimuli arranged in the items.

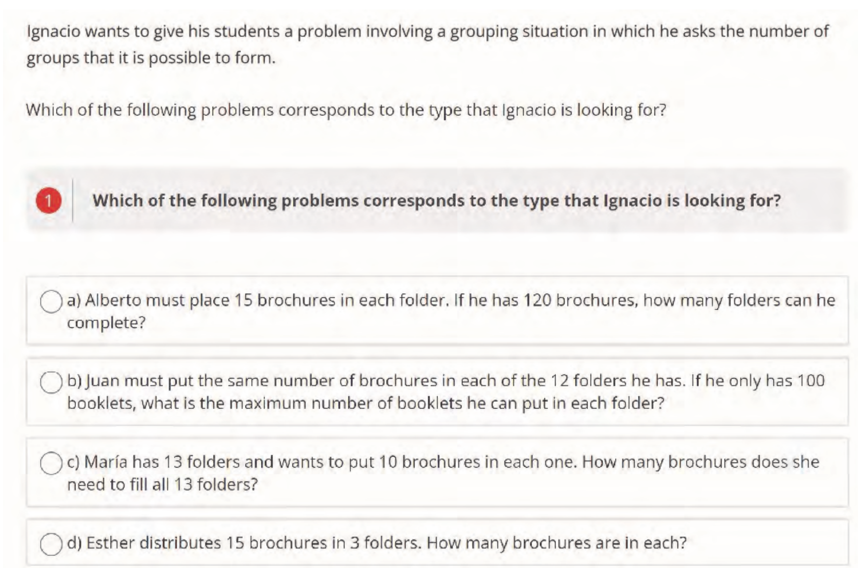
5.2. The instrument

The instrument used in this study included 10 items of multiple-choice, true and false and open-ended responses. It was requested to be answered by participating teachers before and after the course at their convenient time. The analysis was done only considering the 5 multiple-choice items, due to their psychometric properties, which presented good indices of discrimination, difficulty, and reliability (Mean_discr = 0.2; Mean_dif = 0.5, α = .68).

Tab. 1. Domains of mathematical knowledge to teach that evaluates each item

Item number	Assessed domain
1	Specialized mathematical knowledge (posing multiplicative problems).
2	Common mathematical knowledge (solving multiplicative problems).
3	Specialized mathematical knowledge (identifying properties that justify a calculation procedure).
4	Specialized mathematical knowledge (identifying properties that justify a calculation procedure).
5	Common mathematical knowledge (estimation strategies).

The instrument was designed to evaluate the teachers' knowledge of teaching multiplication and division, including common knowledge of the content and specialized knowledge of the content. Tab. 1 presents the details of each specific domain of knowledge measured by the instrument. Fig. 14 shows two items (1 and 5) used to explain the learning results of participating teachers in this course.


Ignacio wants to give his students a problem involving a grouping situation in which he asks the number of groups that it is possible to form.

Which of the following problems corresponds to the type that Ignacio is looking for?

1 **Which of the following problems corresponds to the type that Ignacio is looking for?**

a) Alberto must place 15 brochures in each folder. If he has 120 brochures, how many folders can he complete?

b) Juan must put the same number of brochures in each of the 12 folders he has. If he only has 100 booklets, what is the maximum number of booklets he can put in each folder?

c) María has 13 folders and wants to put 10 brochures in each one. How many brochures does she need to fill all 13 folders?

d) Esther distributes 15 brochures in 3 folders. How many brochures are in each?

Fig. 14. Items 1 and 5

5.3. *Data analysis method*

As we mentioned earlier, two methods of analysis are conducted. First, due to the non-normal distribution of the sample, McNemar's proportion comparison test (McCrum-

Gardner, 2008) was used to assess the progress of the participants' MKT in each item of the instrument. Then, the LCA is used to describe how the teachers participating in the course are classified based on the probabilities of answering the items which describe their MKT. Thus, the MKT of the participants is modeled and characterized by using variable indicators distributed in the items of the instrument through this technique. Thus, latent classes were determined separately for the pre-test and then for the post-test. Consequently, the LCA is used to explain the movement of the teachers or change the class membership between these latent classes associated with the pre-test and post-test. This change describes the modification of MKT among participants.

The fit of the classification or psychometric properties was examined using the Akaike Information Criterion (AIC) (Akaike, 1998) and Bayesian Information Criterion (BIC) (Schwarz, 1978) as suggested by the literature. As a reference, the smaller the AIC and BIC values, the better the model fit (Vrieze, 2012).

6. Results

6.1. Difference between pre-and post-test

As shown in Tab. 2, teachers show significant progress in their achievement in solving items 1 and 4 from the pre- to the post-test. Item 1 involves specialized knowledge related to the identification of multiplicative grouping problems and Item 4 requires the identification of properties underlying a multiplicative calculus procedure (see Tab. 1). Significant progress in teachers' performance is also observed in Item 5, which involves some common mathematical knowledge for quantity estimation.

Tab. 2. Percentage of correct answers per item

Item number	Pre-test (%)	Post-test (%)	Δ_{diff} (%)	χ^2 (p-value)
1	45 (49%)	74 (80%)	29 (32%)	27.034 (0.000)
2	42 (46%)	47 (51%)	5 (5%)	0.761 (0.382)
3	27 (29%)	32 (35%)	5 (5%)	0.516 (0.472)
4	38 (41%)	54 (59%)	16 (17%)	5.113 (0.023)
5	46 (50%)	63 (68%)	17 (18%)	6.918 (0.008)

Note: pre= number of correct answers in the pre-test; post= number of correct answers in the post-test; the absolute difference between the correct answers of the post and the pretest; = McNemar's chi-square.

6.2. The LCA: classification of the participants based on their MKT

6.2.1. Before the course

From the LCA applied to the pretest responses, teachers are classified into two classes according to their performance in the MKT pre-test (AIC = 1138.48; BIC = 1216.32; $G^2 = 298.14$; $\chi^2 = 1168.59$).

Class 1 includes about 22% of participating teachers (20 teachers), while the rest of teachers (71 teachers) belong to Class 2. As can be seen in Tab. 3, what distinguishes the classes is the performance on Items 2, 3 and 5. Items 2 and 5 require common knowledge of multiplication and division for solving problems. Item 3 requires identifying the properties that underlie a computational procedure.

Tab. 3. Probabilities of answering correctly an item

Item number	Pre-test		Post-test	
	Class 1 (.22)	Class 2 (.78)	Class 1 (.11)	Class 2 (.89)
1	.52	.49	.27	.88
2	.30	.50	.13	.57
3	.16	.33	.00	.40
4	.45	.41	.58	.60
5	.00	.63	.27	.74

6.2.2. After the course

The classification of participants based on their performance after terminating the course, confirms that the teachers are still classified into two classes or groups according to their MKT post-test. The data shows a goodness fit index for the classification (AIC=928.26; BIC=1006.09; G^2=174.91; χ^2=499.17).

Tab. 3 shows that the teachers of Class 2 have a significantly better performance than those of Class 1 in Items 1, 2, 3 and 5. Moreover, 89% of the teachers (81 teachers) belong to this class (Class 2) with the best performance. In Item 4, which involves the identification of properties that underlie a mental calculation strategy, both groups show a similar performance, having probabilities of around 60% of answering the item correctly.

6.3. Teacher movement between classes according to their performance

Tab. 4 illustrates the movement of teachers according to their performance in the evaluations carried out before and after taking the course (pre-test and post-test). As we discussed earlier, Class 1 included those teachers who show a lower performance compared to teachers in Class 2 during the pre-test. The analysis shows that 90% (18 out of 20) of the teachers in this class (Class 1), move to Class 2 according to the post-test. Class 2 in the post-test included the teachers with better achievement in post-test. Only 11% (8 out of 71) of the teachers in Class 2 from the pre-test, move to Class 1 from the post-test.

Tab. 4. Number of teachers in each class

		Post-test		Total
		Class 1	Class 2	
Pre-test	Class 1	2	18	20
	Class 2	8	63	71
Total		10	81	91

When comparing class assignments by the LCA before and after the course using the non-parametric Mann-Whitney test, a significant difference between both moments is observed ($W = 3685.5$, $p = .046$).

7. Discussion and Conclusion

This article describes the instructional design of an e-learning PD program with a high degree of autonomous asynchronous work that aims to improve teachers' MKT. The instructional design of the learning activities is shown for a particular course, highlighting the instructional strategies and program's features that promote active learning focused in the different domains of MKT. In addition, for this course, a pre- and post-test design was used to see how the course brings change on teachers' MKT.

Developing an e-learning program that includes a high level of autonomous work has benefits for teachers, such as providing flexibility in scheduling. Also, the program's virtual asynchronous activities are less demanding of high-speed broadband internet than synchronous activities that rely on software like Zoom or Meet. Indeed, accessibility to a high-speed internet connection is a limitation for most people in Chile (Sepulveda-Escobar and Morrison, 2020), particularly for teachers living in rural areas. So, this type of instructional design that privileges autonomous work on a platform that facilitates interaction between the participant and the content, can be a successful alternative to provide access to PD with territorial equity. Even though designing the asynchronous virtual activities of the program was costly and time consuming, the dissemination of the program was more robust and less dependent on highly qualified instructors, which is critical to maintaining the quality of a large-scale PD program (Carney et al., 2019; Roesken-Winter et al., 2015).

We highlight the characteristics of the course activities from two different perspectives. By focusing on the relationship between activities and the construction of MKT we can point out how activities design materializes the principles declared by Suzuka et al. (2009). The course activities are built on the basis of construction cycles helping to progressively unpack elementary mathematical knowledge required to analyze a situation. For example, in the activity of "Grouping and arranging pastries" the concept of multiplication from grouping situations is gradually built and developed. In the first construction cycle, a type of pictorial representation is connected with a symbolic representation (the numerical phrase) emphasizing in both the meaning of numbers involved. In the second cycle, the situation is extended by presenting a second pictorial representation, which allows reinterpretation of the factors of a multiplication, thus extending the meaning of this operation. In the third cycle, the commutative property is argued through a generic example and the use of an adequate representation. On the other hand, the use of dialogues allows proposing questions that provoke a stumble due to a superficial understanding of an idea, in this case, they can point to considering groups with different numbers of elements in a grouping problem, or representing combination situations with set diagrams.

Moreover, we can characterize the nature of the course activities by considering the strategies that encourage inquiry among learners (Lim, 2001). Several of these strategies are evidenced in the course description: (1) Designing problems from a simplest version (required less cognition) to the most complicated version of the problem which mainly required a high level of cognition; (2) Providing a different representation of one scenario, that can help learners to have a better understanding of the problem; (3) Using different technological tools with colors, animations, illustrations for providing a better visualization for learners that can also capture the attention; (4) Making time for reflection by providing feedback constantly during an activity, which leads learners to reach a correct reasoning and answer; (5) Designing a task with a problematic scenario, which starts with what learners already know and continues to a situation where they become curious about knowing a new concept.

The empirical finding regarding learning outcomes of the course "Working with multiplication and division" showed progress in teachers' common and specialized knowledge for teaching according to the MKT model (Ball et al., 2008). Knowing about the movement of teachers between classes from the beginning to the end of the course, by using the LCA technique, provides evidence about the change on their MKT. In the case of our study, a majority of teachers (89%) classified in a class characterized by having a good performance in most of the items at the end of the course. This is in contrast to the initial situation of the participants, in which the majority of teachers had at most 0.55 probabilities of correctly answering just 2 of the 5 items. This result is, in fact, evidence of the change that the course promises to bring to participants' MKT (Martínez et al., 2020).

Regarding the relationship between the characteristics of the course and the progress observed in the items, we consider that the instructional design and characteristics mentioned above contributed to the teachers' progress in their knowledge. For instance, the activity described in Section 4.1 and the participants' achievements for item 1 shown in Tab. 2. Although this study did not aim to find a causal relationship between instructional design and learning outcomes, research shows that a curriculum-based PD leads to greater effectiveness in teachers' MKT (Copur-Gencturk et al., 2019).

In conclusion, we would like to emphasize the importance of discussing the design of e-learning-based activities in the field of math teacher programs. Indeed, the type of activities and the instructional design that are beneficial to develop MKT in teachers have not been studied well. In addition, it is rare to find works that detail the type of activities carried out so that other PD developers can learn from these experiences. In this sense, the present work proposes to advance in this subject, showing principles of design, its materialization by using different instructional strategies that allow focusing the course on the development of MKT. Although in this article we report teacher learning according to a pre- and post-test, we do not have a good understanding of how teachers interact with the content or how different elements may affect the learning process. Future studies are needed to establish relationships between different features of course instructional design and teachers' learning on specific MKT domains.

Acknowledgments

This research was supported by FB210005 and FB0003 Basal funds for Centers of Excellence ANID-Chile, UNESCO Chair "Preparing teachers to teach math in the 21st century", collaboration agreements CMM-MINEDUC, Projects FONDECYT 3201094 and FONDECYT 3220465.

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https://doi.org/10.1142/9789811287183_0028

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Attitudes in Mathematics Education

Pietro Di Martino[1]

ABSTRACT Attitudes towards mathematics has a long history in mathematics education research. Over the time, research on attitudes and, more in general, on affective aspects developed a wide range of methodologies and perspectives in mathematics education, playing a growing role in the field. In this chapter, I will describe the development of the research about attitude in mathematics education, discussing the main issues emerged in this field. In particular, I will discuss the definition problem, that is the emergence of the need for a clear definition of the construct, and the ground for the development of *our* (TMA) three-dimensional model of attitude (Di Martino and Zan, 2010). In the last part of the chapter, some fields of application of the TMA model will also be discussed.

Keywords: Attitude towards mathematics; Affect in mathematics education; Qualitative research.

1. Introduction

The awareness that the learning process of mathematics is strongly affected by affective factors was born and developed in the field of Mathematics Education during the Eighties.

Mason, Burton and Stacey (1982) published the book "Thinking Mathematically". The main aim of the book was to unfold the processes which lie at the heart of mathematics, and, within this scope, authors underlined as their "*experience with students of all ages has convinced us that mathematical thinking can be improved by (…) linking feelings with action*" (ibidem, p. ix).

The role of affect in the specific development of mathematical thinking was stated so clearly for the first time and the emotion management was described as a fundamental part for the thinking development: "*Probably the single most important lesson to be learned is that being stuck is an honourable state and an essential part of improving thinking*" (ibidem, p. ix).

This is a significant breakthrough in the field of mathematics education: affect was no longer considered "auxiliaries" in the learning process of mathematics, rather a crucial part for the development of mathematical thinking and, therefore, an important key for the interpretation of the widespread students' difficulties in mathematics.

In the same period, several scholars arrived at the same conclusion because of their research about problem solving. At the end of a long research into human problem-

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solving process, Schoenfeld (1983, p. 330) stated: “*The point here is simply that “purely cognitive” behavior — the kind of intellectual performance characterized by discussion of resources, heuristics, and control alone — is rare. The performance of most intellectual tasks takes place within the context established by one's perspective regarding the nature of those tasks. Belief systems shape cognition, even when one is not consciously aware of holding those beliefs*”.

This stance marks the end of a naïve era, the overcoming of the assumption that the mathematical thinking and its quality is determined only by cognitive elements.

This awareness finds its definitive consecration in 1989, when the book “Affect and Mathematical problem solving” was published (Adams and McLeod, 1989). The book was developed in a particular period: on the one hand, most research on problem solving were developed in the wake of Polya’s seminal work, giving no attention to affective issues, on the other hand, many results of the research about problem-solving showed the limits of a purely cognitive approach.

Based on these results, McLeod (1989, p. 23) clearly underlined how: “*Limiting one’s research perspective to the purely cognitive seems acceptable for those interested mainly in the performance of machines; however, researchers who are interested in human performance need to go beyond the purely cognitive if their theories and investigations are to be important for problem solving in classrooms* […] *Affective issues play a central role in mathematics learning and instruction* [...] *If research on learning and instruction wants to maximize its impact on students and teachers, affective issues need to occupy a more central position in the minds of researchers*”.

In the first chapter of the book, Mandler (1989, p. 3) was clear about the need to seriously develop research investigating the role of affect in problem-solving and, more in general, in the teaching and learning of the mathematics: “*The* problem-solving *and* teaching-and-learning *literature is full of remarks that have a single message: <Someday soon — maybe tomorrow — we must get around to doing something about affect and emotion>. I am delighted to see that* tomorrow *has come”.*

Few years later, McLeod (1992) depicted the state of the art of the research on affect in mathematics education, trying to systemize the field.

McLeod recognized three main constructs emerging in the field of affect: emotions, beliefs and attitudes. These constructs differ in the stability of the affective responses and in the level of intensity of the affects that they represent (Fig. 1).

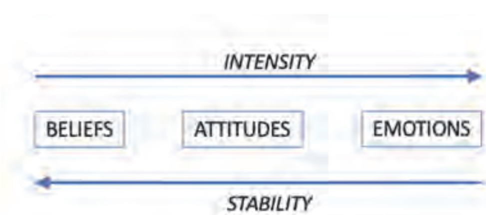

Fig. 1. The degree of stability and intensity of beliefs, attitudes and emotions

In McLeod's view, emotions, beliefs and attitudes also differ for their "cognitive component". However, the basic assumption was that all of them have a strong relationship with cognition and research in the field of mathematics education should have been investigate the dynamics of this relationship.

To do that, the first needed step was to ground affect and its main constructs in a strongest theoretical foundation.

Some year later, Goldin (2004) underlined how this need did not appear to be fully satisfied, since a precise and shared language for describing the affective domain was still missing.

However, if some progress had been made in describing emotions (Zan et al., 2006) and beliefs (Törner, 2002), this was not the case for the concept of attitudes: "*probably the most problematic concept in McLeod's framework*" (Hannula et al., 2011, p. 38).

In this context and in an interpretivist perspective, Rosetta Zan and I began to get interested in the definition of the construct "attitude towards mathematics", imagining the potential of the construct for a better and more complete interpretation of the students' difficulties *in* mathematics (Fig. 2).

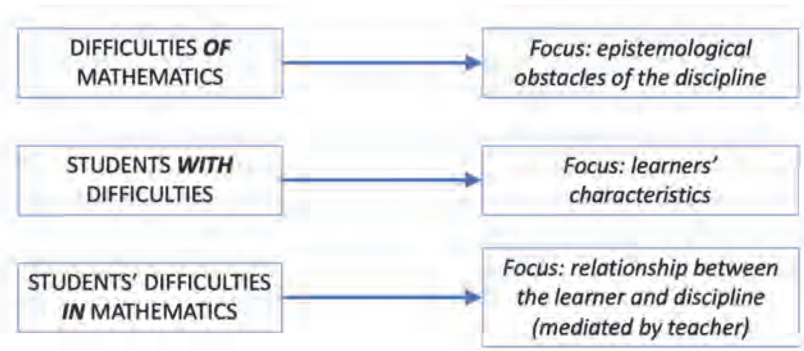

Fig. 2. The different kinds of difficulties and their focus

2. Toward a (Working) Definition of Attitude towards Mathematics

According to Boero and Szendrei (1998, p. 199): "If we claim that research in mathematics education must be similar to research in any "normal" science, "cumulation" and "universality" of research results are needed, and the existence of the progress must be evaluated comparing new results with previous ones".

In line with this characterization of research in mathematics education, in 2001 we developed a critical overview of the existing literature about attitude (Di Martino and Zan, 2001), trying to reconstruct if and how the several studies about attitude in mathematics education answered to the following question: what is attitude towards mathematics? The answer to this question is also crucial for characterizing what are a positive and a negative attitude towards mathematics.

It emerged that a large portion of studies about attitude did not provide a clear definition of the construct itself. Attitude tended rather to be defined implicitly and a posteriori through the instruments used to measure it. However, in the studies where an explicit definition of attitude was given, we recognized three main typologies of definition (Fig. 3).

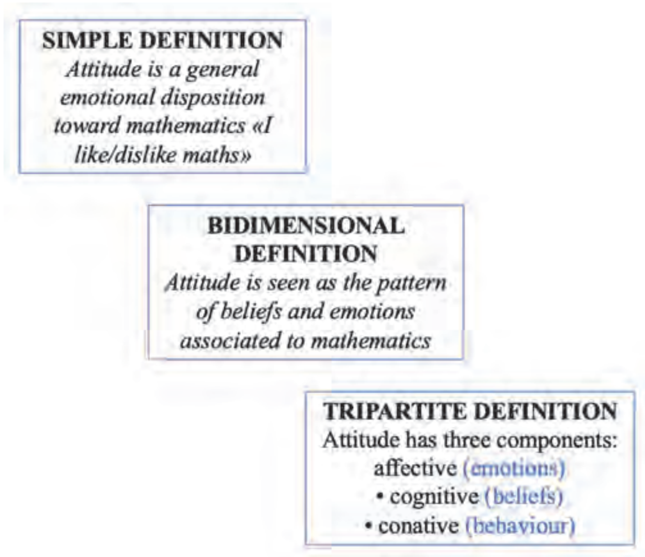

Fig. 3. The different characterizations of attitude in 2001

Considering this variety of approach, the debate about which is the correct definition was particularly intense. On the other hand, according to Kulm's position (1980, p. 358) "*It is probably not possible to offer a definition of attitude toward mathematics that would be suitable for all situations, and even if one were agreed on, it would probably be too general to be useful*", Daskalogianni and Simpson (2000) suggested to consider the definition of attitude as a working definition. Attitude is considered as "*a construct of the observer's desire to formulate a story to account for observation*" rather than "*a quality of an individual*" (Ruffel et al., 1998, p. 1).

In this perspective, the development of the (working) definition of attitude must be related and functional to the research problem and, therefore, the real question we needed to reply was the following: which is the most adequate definition of attitude for our research interests?

Being particularly interested in the third kind of difficulties in Fig. 2 — students' difficulties in mathematics — we needed to develop a definition of "attitude towards mathematics" strongly related to students' relationship with mathematics, as well as, to teachers' practice. We had the ambition to characterize in an operative way what "positive" and "negative" attitude toward mathematics are, developing a theoretical tool for the interpretation of students' difficulties in mathematics, capable to suggest didactical strategies to overcome these difficulties.

According to this aim, a first investigation about teachers' use of the diagnosis 'negative attitude' in their school practice was developed by Polo and Zan (2005) within an Italian research project.

It emerged that the diagnosis "*This student has a negative attitude toward mathematics*" was frequently (85.6% of the sample) used by teachers regardless of their school level. On the other hand, several different meanings of the diagnosis "negative attitude" emerged (Fig. 4).

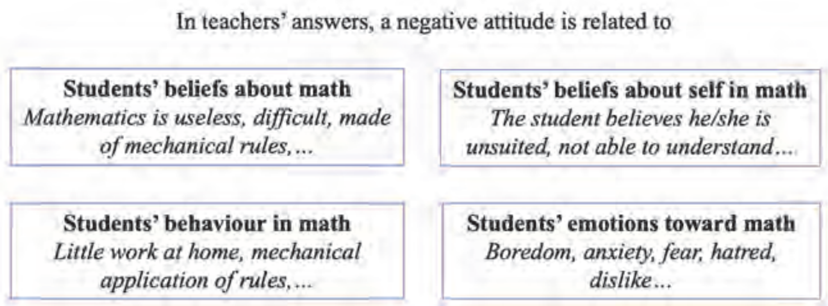

Fig. 4. The different meanings of "negative attitude" in teachers' view

Another result of the research conducted by Polo and Zan was particularly interesting: it emerged how the diagnosis "that student has a negative attitude towards mathematics" usually represents a sort of claim of surrender of the teacher in the face of students' difficulties in mathematics rather than an interpretation for steering didactical intervention. The "negative attitude" diagnosis was a black box in a nutshell.

In our view, clarify the meaning of (positive/negative) attitude from a theoretical viewpoint was the key to open the black box, turning the "negative attitude" diagnosis into a useful instrument for teachers and researchers.

To do that, we developed our main research about attitude (Di Martino and Zan, 2010, 2011), studying students' relationship with mathematics at school through the collection of several autobiographical essays: *Me and mathematics: my relationship with maths up to now* (Fig. 5).

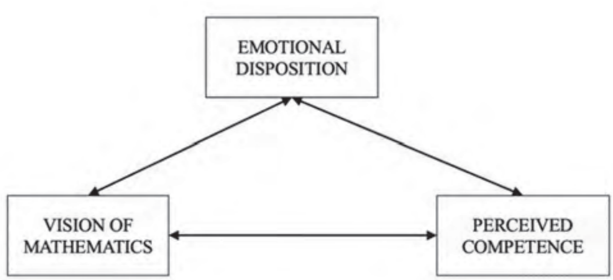

Fig. 5. The description of the data collected

Lieblich et al. (1998) describe different approaches for the analysis of narrative materials. They identify two main independent dimensions: holistic vs categorical (this dimension refers to the chosen unit of analysis: the complete narrative or a specific part of the story), and content vs form. Combining these dimension results in four modes of analyzing narrative data, each of which responds to a specific research interest.

Through a categorical-content approach, we found three recurrent expressions in students' narrative: "I like/dislike mathematics", "I am able/ unable to do mathematics", "mathematics is". These three expressions identified three core themes: the emotional disposition towards mathematics, the vision of mathematics, the perceived competence in mathematics. At the end of the analysis, we obtained only 32 essays (less than 2 % of the entire sample) that did not refer to at least one of these three themes.

Coherently with our initial assumptions, we developed our Three-dimensional Model for Attitude (TMA) based on the themes students used to describe their relationship with mathematics (Fig. 6).

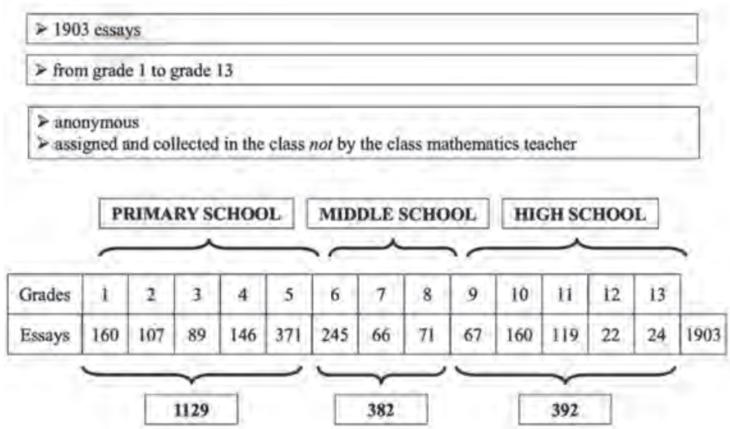

Fig. 6. The three-dimensional model for attitude (Di Martino and Zan, 2010)

The multidimensionality of the model suggests the development of different profiles of attitude towards mathematics. The multidimensionality therefore underlines the inadequacy of the positive/negative dichotomy for attitude referred only to the emotional dimension, suggesting to consider an attitude as *negative*, when at least one the dimensions is *negative*. In this way we can outline different *profiles* of negative attitude, depending on the dimension that appears to be *negative*.

On the other hand, the developed definition of attitude could be a valuable tool for didactical diagnosis and intervention if and only if its complexity is limited and reasonable. In this perspective, we decided to reduce the complexity of each of the three dimensions in the model to the following dichotomies:

- Emotional disposition: positive/negative
- Vision of mathematics: relational/instrumental (Skemp, 1976)
- Perceived competence: high/low

In this way, TMA model identifies eight different profiles of attitude towards mathematics (Fig. 7): a unique profile of positive attitude towards mathematics and seven different profiles of negative attitude, depending on the dimension that appears to be negative.

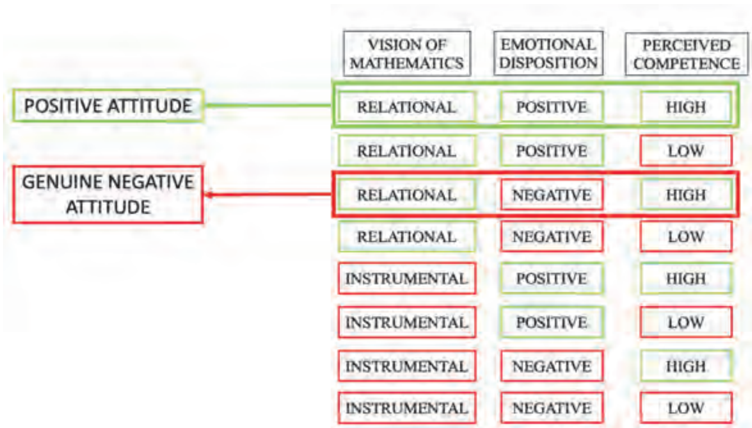

Fig. 7. The profiles of attitude

The so called *genuine* negative attitude towards mathematics — the profile characterized by a relational view of math, a high perceived competence but a negative emotional disposition — is particularly interesting, It is based on an epistemologically correct vision of mathematics and it is not related to a story of difficulties with mathematics, therefore, in our view, the negative emotional disposition is genuine, a sort of personal taste that we should accept despite our passion for mathematics (this means that I believe a didactical intervention is not needed).

From a quantitative point of view, it is important to underline that this *genuine* profile was not represented in our data, and it is not represented now either collecting other essays (up to now we collected almost 2000 essays).

3. The Possible Uses of the TMA Framework

The developed TMA framework has the potential to be used for multiple research interests, appropriately modifing the involved variables (Fig. 8).

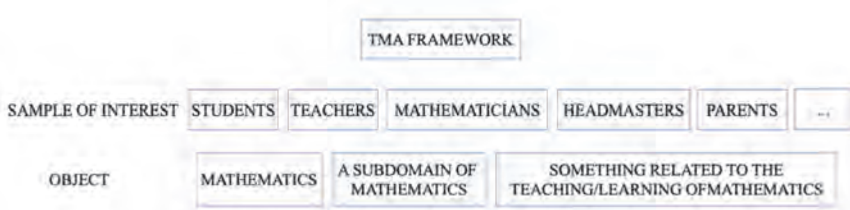

Fig. 8. The possible research variables

The first variable is the sample of interest, that is the group of people we are interested in. Several possibilities exist: students, teachers, mathematicians, students' parents, headmasters, politicians, etcetera.

The second variable is the *object*. It can be mathematics as well as a *subdomain* of mathematics (geometry, algebra, problem solving, etcetera) or something related to mathematics, for example the attitude towards the teaching of mathematics. However, the TMA framework can be also used for objects different from mathematics: for example, Bocchialini and Ronchini used TMA framework for assessing business students' attitudes toward finance (Bocchialini and Ronchini, 2019).

The third variable is represented by the dichotomy between static vs developmental research on attitude. In the first case, the focus is on the current attitude of a certain sample, in the latter case, the focus is on the evolution over a period of attitude (typically this period is characterized by a didactical intervention or by a discontinuity in the educational path, for example the transition between two different school levels).

I want to conclude this chapter with a synthesis of the research I developed (with different colleagues) using the TMA theoretical framework, in addition to the study of students' attitude towards mathematics in the Italian context.

First, the TMA theoretical framework has been used for studying the attitude towards having to teach mathematics of pre-service primary teachers. These studies confirmed how negative attitudes towards mathematics and towards the idea having to teach mathematics are very common among the future teachers in primary school. On the other hand, we discovered and described an interesting phenomenon that we called the *desire for math-redemption*: it happens when there is a positive attitude towards the *challenge* of having to teach mathematics, motivated by the desire of a personal reconstruction of the (negative) attitude towards mathematics developed during the school experience as student (Coppola et al. 2013).

The TMA theoretical framework has been also used for studying the in-service teachers' attitude towards the national and international standardized assessments (Di Martino and Signorini, 2019). The results of the study — that involved of all school levels — confirmed a generalized What emerges is a complex picture that includes positions of principle against the standardized assessments and their uses, but also more specific criticism towards the design of the test. Understanding the teachers' attitude towards this kind of assessment appears to be crucial also to exploit the informational and developmental potential of the standardized assessment (Di Martino and Baccaglini-Frank, 2017).

In the last period, I have applied the TMA framework to the evolution of attitude in two different cases. The first one concerns the evolution of pupils' attitude towards problems in the period from kindergarten to the end of primary school (Di Martino, 2019). The results of the study showed a worrisome evolution of the pupils' attitude towards problems, in terms of all the three components of TMA. Kindergarten pupils reported a very promising view of problems, not fixed to a stereotypical model: this view is held until the end of grade 1 in primary school, then a deterioration of this idea begins, and this deterioration appears to be linked to precise didactic choices.

Currently, I am using the TMA framework for analyzing the mathematical crisis in secondary-tertiary transition (Di Martino and Gregorio, 2019). The secondary-tertiary transition in mathematics is described by Clark and Lovric (2008; 2009) as a rite of passage. This rite of passage consists of three stages: the separation stage, the liminal stage, the incorporation stage. The liminal stage is characterized by several individual's crisis and it is very interesting to analysis the evolution of attitude from the separation stage to the liminal stage.

To conclude, I strongly believe that research on attitude still can and must say much about several significant phenomenon for mathematics education, and the TMA framework has the potential to give a contribute.

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https://doi.org/10.1142/9789811287183_0029

29

Mathematical Instruction and Textbook Use in Post-Secondary and Tertiary Contexts: A Discussion of Methods

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ABSTRACT In my work, I seek to understand how interactions between instructors, students, and resources — both inside and outside of the classroom, create opportunities for mathematics learning in post-secondary settings. Various methodological decisions have advanced this work. I showcase the evolution of two inter-dependent research strands that together have helped me understand the centrality of resource use by instructors and students and its implications for student learning.

Keywords: Instruction; Resources; Post-secondary education; Tertiary education.

I investigate how and why resources, instructors, and students interact to create rich opportunities for mathematics learning in post-secondary and tertiary settings. In this paper, aggregating findings from various studies, I reflect on the types of measures I have used to describe teachers' practice inside their classrooms and using textbooks. The presentation is organized in three sections. I present studies characterizing instruction first, followed by studies of textbook use. I conclude with a reflection on the methods used.

1. Characterizing Instruction

Following Cohen, Raudenbush, and Ball (2003) characterization of instruction as the interactions between instructor students and content allowed me to observe classroom activity occurring in real time. In earlier studies, using observations and audio-recordings of lessons, I relied on low inference codes, such as counts of audible speaking turns (i.e., speech that is given by a speaker before being interrupted or giving the floor to another speaker) or their length (in number of words) (Mesa, 2010a, 2010b). The counts of audible speaking turns can be identified by speaker, and a very simple ratio of number of student-turns to number of teacher- turns, when done at a large scale, provide important information about patterns of interaction between teachers and

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students. When the ratio is 1 then number of student- and teacher-turns are the same; ratios over 1 indicate more student-turns than teacher-turns. Using close 150 lessons from five different studies, ranging from developmental to graduate level courses in about 40 different institutions; in some studies, with maximum variation sampling (e.g., developmental to graduate courses) and homogeneous (e.g., community colleges, successful calculus institutions, inquiry-based learning) it is possible to find averages of these ratios, and they are quite revealing (Mesa, 2011). In Fig. 1, I present these ratios for various types of courses at the university and at community colleges[2] upper division and first year courses at university; pre-college and developmental level courses at community colleges, and inquiry-based learning courses at university. In university courses it is typical for teachers to dominate the talk in the class; in courses taught at community colleges, that is not the case, and in courses that use inquiry, the difference is more remarkable, at least relative to other courses taught at university (Mesa, 2009, 2011).

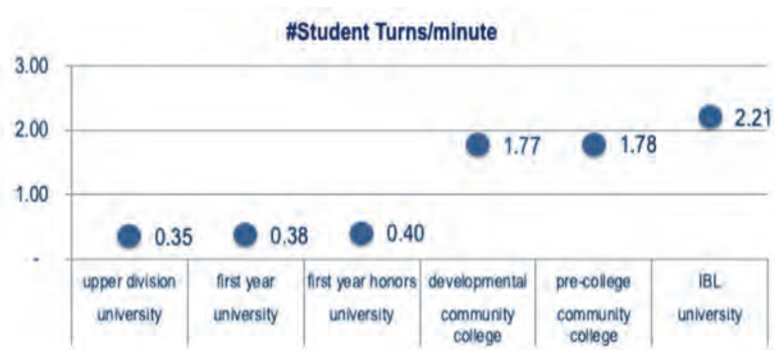


Fig. 1. Ratios of student turns to teacher turns by type of course and setting

Counts of words, is also revealing; a teacher-turn on average is 40 words, whereas student turns are on average between four and five words. Student turns that are between one and three words can be about 51% in non-inquiry classes, but less than 10% in inquiry classes. This suggests that even when students participate in classroom, their contributions are limited, except when the classes use inquiry. These results seem to make sense when we think that during mathematics classrooms, the prevalent mode of instruction is lecturing (Mesa and White, 2022), whereas in inquiry-based classrooms, students then to either speak among themselves and ask each other questions, or present information at the board without intervention from the teachers. Thus, these low inference codes corroborate classroom participation patterns that we know exist in post-secondary classrooms.

These codes though are insufficient to further characterize the quality of the interactions. To look at these, I have relied on analyses of examples and questions that

[2] Community colleges are post-secondary institutions that prepare students for professional vocational work and that provide courses equivalent to the first two years of a college degree.

teachers use in classroom under the assumption that the content and ways in which these are phrased have the potential of triggering a particular cognitive process in students. To analyze examples and tasks, I initially used the revised Bloom's taxonomy (Anderson et al., 2001), which classifies knowledge into four distinct dimensions — factual, procedural, conceptual, and metacognitive — and identifies six different types of cognitive processes — each increasing in complexity, as they are assumed to require more cognitive resources: Remember, Understand, Apply, Analyze, Evaluate, and Create Fig. 2. Using this taxonomy with small modifications, it has been possible to classify tasks used in classrooms (Mesa, 2010b) and in homework and in exams (White and Mesa, 2014). Classifying questions into non-mathematical and mathematical (Novel or Routine) has also been useful to gauge the cognitive demand of mathematical work done in classrooms (Mali et al., 2019, Meta et al., 2014). Naturally, inferring the cognitive demands of examples used and questions asked is more difficult, as it requires rigorous training and understanding of the context in which the questions and problems are asked: A question or a problem asked in a calculus class might be of low cognitive demand because the students might already be familiar with the content, whereas the same question or problem asked in college algebra class might be novel. Thus, all these analyses are paired up with the context in which the courses take place, including the place in the sequence of the course, the course objectives, the profiles of the students taking the class, and the goal of the course in a student's major.

Knowledge Dimension	*Cognitive Processes Dimension*
1. **Factual Knowledge:** Basic elements students must know to be acquainted with a discipline or solve problems in it, including knowledge of terminology and of specific details. 2. **Conceptual Knowledge:** Interrelationships among the basic elements within a larger structure that enable them to function together. It involves knowledge of classifications and categories, of principles and generalizations, and of theories, models, and structures. 3. **Procedural Knowledge:** How to do something, method of inquiry, and criteria for using skills, algorithms, techniques, and methods. It includes knowledge of subject-specific skills and algorithms, of specific techniques and methods, and of criteria for determining when to use appropriate procedures. 4. **Metacognitive Knowledge:** Knowledge of cognition in general as well as awareness of one's own cognition. It includes strategic knowledge, knowledge about cognitive tasks (including appropriate contextual and conditional knowledge), and self-knowledge.	1. **Remember:** Retrieve relevant knowledge from long-term memory, including recognizing and recalling. 2. **Understand:** Construct meaning from instructional messages, including oral, written, and graphic communication. It involves interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining. 3. **Apply:** Use a procedure in a given situation. It involves executing and implementing. 4. **Analyze:** Break material into its constituent parts and determine how the parts relate to one another and to an overall structure or purpose. It involves differentiating, organizing, and attributing. 5. **Evaluate:** Make judgments based on criteria and standards. It involves checking and critiquing. 6. **Create:** Put elements together to form a coherent or functional whole and reorganize elements into a new pattern or structure. It involves hypothesizing, designing, and producing

Fig. 2. Revised Bloom's Taxonomy (Anderson et al., 2001)

These analyses, using a similar corpus of data, have revealed different proportions of routine tasks and questions asked in classes in which the mode of instruction is lecturing, versus classes in which the mode of instruction is inquiry as seen in Fig. 3.

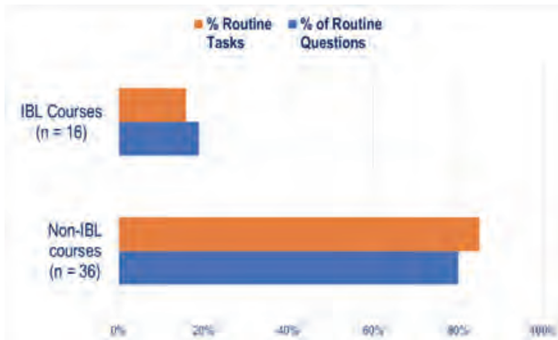


Fig. 3. Proportion of routine tasks and questions by mode of instruction

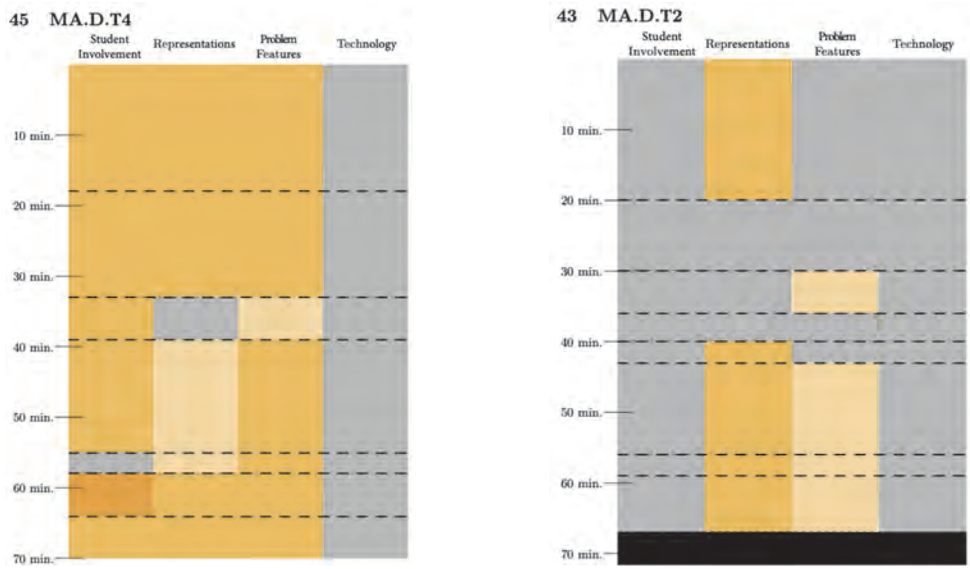
We see a stark contrast in the cognitive demand for the cognitive demand of questions and problems in courses that are inquiry versus those that are not inquiry. In 16 inquiry courses, the proportion of routine questions or tasks is under 20 percent, whereas in other courses, it is above 80 percent. One can infer that it is likely that the cognitive demand expectations of work assigned to students in the two types of courses is different. These differences might lead us to think twice about the advantages of continuing using non-inquiry lessons in our mathematics courses.

To further investigate this phenomenon, and as part of a national study of calculus in the United States (Bressoud, 2013; Bressoud et al., 2013; Bressoud et al., 2015; Bressoud and Rasmussen, 2015), we analyzed the problems solved during calculus lessons attending to four aspects of the reform: student involvement, use of representations, features of the problems, and technology. Coding problems as having more or less of each of these features, allowed us to create a sort of heatmap for each lesson observed, that identified for each of the four aspects the extent to which reform was enacted (see Fig. 4).

Student Involvement (0-3)	Representation Use (0-2)	Feature Use (0-2)	Technology Use (0-1)
0: Instructor only	0: Symbolic only	0: Only practicing skills or known methods	0: No technology
1: Individual work and Instructor only	1: One representation that is not symbolic	1: Context and Diagram, but no Proof/Justification, Multiple methods, Open Ended	1: Some technology
2: Group, pair, class discussion, no student presentations	2: Any two representations	2: Proof/Justification, Multiple methods, Open Ended	
3: Student presentations			

Fig. 4. Categorization of reform features in calculus lessons (Mesa and White, 2022; White and Mesa, 2018)

The gray color represents "business as usual" whereas darker shades of yellow more enactment of reform features. We can compare now, full lessons in terms of the level of reform enacted. These two lessons from the same institution, taught by two different instructors show important differences, with the lesson on the left (Fig. 5a) enacting more elements of reform than the lesson on the right (Fig. 5b) — which appears mostly gray. These representations do suggest that students in these two courses, taught at the same institution, are experiencing mathematics differently.



(a) (b)

Fig. 5. Maps of two calculus lessons taught by two different instructors at the same institution.(Mesa and White, 2022)

Now, while analyzing tasks and problems is useful, and as seen here, instruction is more complex than that. Understanding what hoes into high quality instruction involves not only the types of problems and questions that teachers select; these decisions attend merely to the teacher content interaction of the definition of instruction and only superficially address the student content interaction. To understand how instruction operates, a more complex system of observation is needed.

Building up on work done with elementary school practitioners (Heather et al., 2008; Heather C. Hill, Blunk, et al., 2008; Heather et al., 2005; Heather et al., 2004), we have developed an instrument that documents the quality of instruction in algebra courses taught at community colleges; the instrument, *Evaluating the Quality of Instruction in Postsecondary Mathematics,* EQIPM, addresses the interactions between students, teacher, and content with 14 codes, grouped into three distinct hypothesized dimensions, as seen in Fig. 6[3]. The 14^{th} code is hypothesized to relate to

[3] The definition of the dimensions and the codes is provided in the Appendix.

all the dimensions. A categorical confirmatory analysis revealed that these are distinct dimensions (Lamm et al., 2022). The final instrument includes 12 codes (Tab. 1)

Evaluating the Quality of Instruction in Postsecondary Mathematics (EQIPM)		
Quality of Student-Content Interaction	Quality of Instructor-Content Interaction	Quality of Instructor-Student Interaction
1. Student Mathematical Reasoning and Sense Making 2. Connecting Across Representations-Student 3. Situating the Mathematics-Student	4. Instructors Making Sense of Procedures 5. Connecting Across Representations-Instructor 6. Situating the Mathematics-Instructor 7. Supporting Procedural Flexibility 8. Organization in the Presentation of Procedures 9. Mathematical Explanations	10. Instructor-Student Continuum of Instruction 11. Classroom Environment 12. Inquiry/Exploration 13. Remediation of Student Errors and Imprecisions
14. Mathematical Errors and Imprecisions in Content and Language		

Fig. 6. Codes for the EQIPM instrument (Mesa, Duranczyk, Watkins, & AI@CC Research Group, 2019, February; Mesa et al.,

Tab. 1. Three-factor ordinal confirmatory factor analysis results with *Remediation of Student Errors and Difficulties* removed

Factor	**Item**	**Std. Loading**	**SE**
Student-Content Interaction	Student Mathematical Reasoning and Sense Making	0.778	0.058
	Student Connecting across Representations	0.722	0.061
	Student Situating the Mathematics	0.531	0.127
Instructor-Content Interaction	Instructor Making Sense of Mathematics	0.629	0.078
	Instructor Connecting across Representations	0.463	0.083
	Instructor Situating the Mathematics	0.324	0.091
	Mathematical Explanations	0.456	0.101
	Supporting Procedural Flexibility	0.403	0.104
	Organization in the Presentation	0.666	0.097
Instructor-Student Interaction	Instructor-Student Continuum of Instruction	0.890	0.081
	Classroom Environment	0.788	0.110
	Inquiry / Exploration	0.599	0.095

Chi-Square = 58.004, p value 0.177; CFI = 0.967; RMSEA = 0.039; SRMR = 0.074

These findings across multiple studies using different types of analytical tools that demand low and high inference coding, strongly suggest that typical mathematical instruction in post-secondary and tertiary levels is constituted by interactions primarily led by the instructor, with limited student participation, on tasks that are mostly routine and with low reform activities (such as collaborative work, student presentations, multiple representations used and significant use of technology), and that mathematics lessons in which inquiry is used do not follow these patterns.

With an instrument the available instrument, it would be possible to seek to identify cases in which instruction is of high quality, possibly in inquiry-based contexts, or professional development programs based on the items in the instrument; the instrument could be complemented with other instruments that assess inclusion, diversity, and equitable practices.

2. Studying Textbook Use

Research on mathematics textbook use by faculty and students is in its infancy. There are over a dozen of studies about undergraduate students' use of mathematics textbooks, and a handful of studies about how mathematics faculty use their textbooks for teaching. A main reason for the scarcity of research is methodological, as it is difficult to conduct such work in naturalistic settings, that is, in real classrooms with real faculty and their students. Most of the existing work has been conducted in laboratory like settings, with a handful of participants, and have focused on how people understand or make sense of what they read (Shepherd et al., 2010; Sierpinska, 1997; Weinberg and Wiesner, 2011; Wiesner et al., 2020). Some studies have investigated student use via surveys (Gueudet and Pepin, 2018; Weinberg et al., 2012). How can textbook use be studied in actual classrooms? The problem becomes more complex when studying online textbooks. What methodological tools are there to study textbook use?

Textbooks are an integral part of teaching and learning activities in post-secondary mathematics; not only are they a source for inspiration for organizing presentations and activities for classroom work, but they are also fundamental for designing homework or other student assessments. As part of a large scale study digital textbook use (Beezer et al., 2018), we selected three textbooks, Active Calculus[4] (herafter AC, Boelkins, 2021), A First Course in Linear Algebra[5] (hereafter FCLA, Beezer, 2021), and Abstract Algebra Theory and Applications[6] (hereafter AATA, Judson, 2021), chosen because they target different profiles of students in a mathematics program. These textbooks are written in PreTeXt[7] a markup language that provides a structure to the content of the textbooks, and a unique identifier for each textbook element. The textbooks can include computation cells (in Sage[8]) and are highly linked. They can also be rendered in any device without losing the quality of the mathematics or the graphs and can be printed in any format, including Braille.

The textbooks used in the project, have been modified to facilitate interactivity: in response to specific questions, students can type answers, and when they submit them, teachers can read their answers in real time. If students had submitted answers to the questions as preparation for the class, teachers could in a quick glance, make decisions about what to address in the lesson, perhaps change and example, or perhaps skip one that might not be needed. The blank spaces appear after each question in the Preview Activities in the calculus textbook, and the Reading Questions, in the other two textbooks. The Reading Questions for AATA appear at the end of each chapter, whereas in the other textbooks, the feature appears within each section of the chapter (see Fig. 7).

[4] https://books.aimath.org/ac/frontmatter.html
[5] https://books.aimath.org/fcla/front-matter.html
[6] https://books.aimath.org/aata/frontmatter.html
[7] https://utmost.aimath.org/pretext/
[8] https://www.sagemath.org/

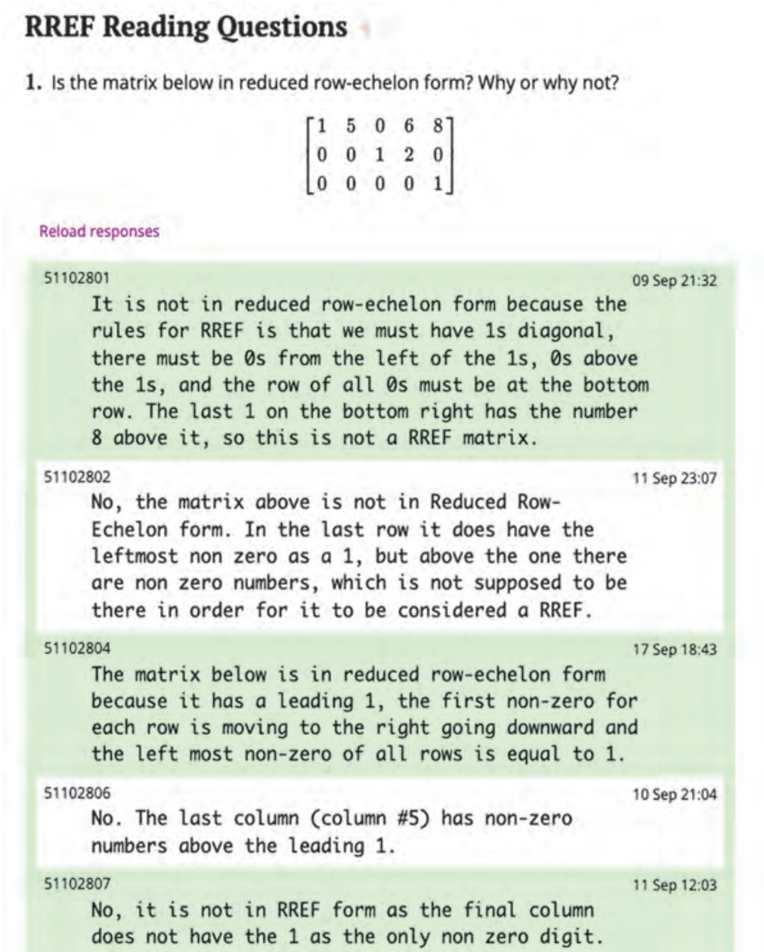
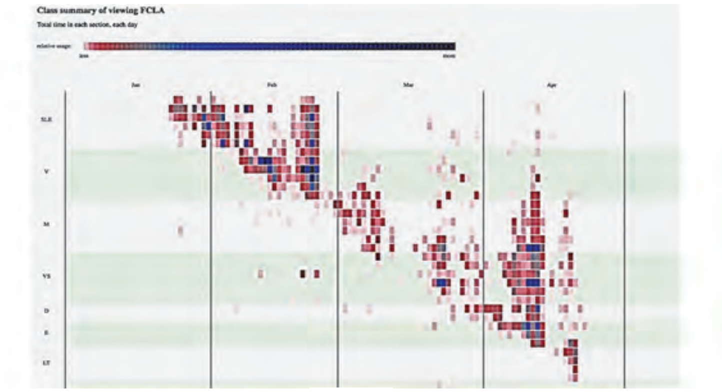


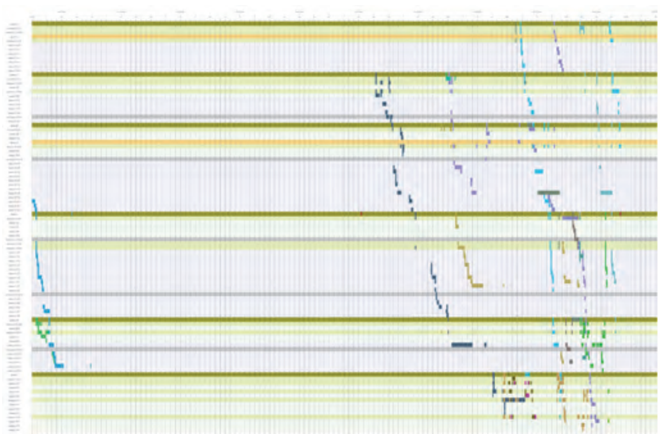
Fig. 7. Teacher view of responses provided by students to one of the reading questions in the FCLA course, in the section Reduced Row Echelon Form (https://books.aimath.org/fcla/section-RREF.html)

To interact with the feature, the users need a username and a password, which facilitates the identification of how much time a user spent viewing a particular section of the textbooks in any given day. This information is used to create heatmaps of use, heatmaps that can be created at various levels of detail: course, chapter, section, and user (Fig. 8).

For this project we collected data from over 55 instructors and their students (close to 900), over ten semesters; we collected a wide array of data: surveys, tests of knowledge, lecture notes, periodic surveys with open and close ended questions, and for a few sections, an intensive data collection process including observations, interviews, and student focus groups. Fig. 9 illustrates the process of data collection within any given semester.



(a)



(b)

Fig. 8. Heatmap viewing data (a) at the course level; (b) at the user level. different colors reflect different users

	Pre term	Week in the term 3	6	9	12	15	Post term	Summer
Teacher surveys	X							
Teacher logs		X	X	X	X	X		
Course syllabi		X						
Lecture notes[a]				X				
Workshop								X
Computer-generated student and teacher viewing data[b]								
Campus visits[c]				X				
Student logs		X	X	X	X	X		
Student surveys				X				
Students tests		X				X		
Student grades							X	

Notes: [a]. Varies by textbook. [b]: Collected continuously throughout the term. [c]: Includes classroom observations, teacher interviews, and a student focus group.

Fig. 9. Data collection for the UTMOST project

We found that students report extensive use of their textbooks, in particular features such as definitions, examples, and theorems, as they prepare their homework or study for exams; they also make extensive use of the narrative text and of the preview activities or reading questions as they prepare for class. We identified expected differences by textbook. AC students read the narrative text, focus on activities, and complete the WeBWorK practice problems; FCLA and AATA users, attend to definitions, theorems, and proofs. Sage use varies substantially by instructor. It is remarkably difficult to establish any connection between time spent viewing the textbook ad how well instructors think their students are doing in the course. Many faculty reported being surprised that students who appeared to be viewing the textbook a lot may do well, not well, or just OK in the course. The same occurred for students who did not view the textbook much. Part of the reason has to do with the number and variety of resources students report using, including other textbooks, online forums or video sites, peers, family, tutoring, and their instructors. Some students preferred using a printed copy of the textbook and thus their viewing did not get recorded. An internal quantitative analysis confirmed prior research that indicates that students who are more motivated perform better in their courses (earn higher grades in the course). Having a textbook available online or in printed form does not seem to be of any advantage measurable by grades or gains in knowledge.

The multiple data sets collected as part of this project and the sheer number of participants has led us to data reduction and analysis techniques that use natural language processing (Kumar et al., 2016) and networks of knowledge graphs (Hamilton, 2017). Using both manual coding (Mesa and Mali, 2020) and natural language processing, we have identified multiple uses students give to their textbooks, as anticipated by the instrumentational approach (Rabardel, 2002), our theoretical framework, which defines instruments as a combination of artifacts and schemes of use; the schemes of use encompass goals, rules of actions, operational invariants, and possibilities for inferences (Vergnaud, 1998). We have been able to identify that students read the textbooks to reverse engineer processes shown in examples, to check their work, to learn definitions, to understand how proofs work, to anticipate what will happen in class, to propose questions that might be useful for their learning, and for self-directed study. Faculty tend to think that students in lower division courses only read the textbook to find out examples that can help with their homework (Mesa & Griffiths, 2012). This is not the case. Students will read the textbook is asked by their instructors and will use it for their own learning. Instructors too, use textbook features, such as the reading questions to identify whether they have understood the material that they are about to teach (Mesa et al., 2021). Moreover, the data collected from viewing the textbooks can be reliably mapped to the uses students describe for particular features (Kanwar and Mesa, accepted). Instructors, in using their textbooks for planning, also use a wide range of resources, from graduate school notes, or their own notes from prior terms, or the internet, or repositories available in their campus. Instructors may attend to the course objectives or to student thinking as they plan or to a combination of both (Liakos et al., 2021; Mesa, 2023). We have confirmed that instructors will integrate the textbook into their usual ways of planning and enacting instruction — even though the textbooks are open source, making changes to those textbooks is very difficult at this time. We have also learned that students use their textbooks in more ways than other studies report, in part thanks to the continued follow-up and because of the use of the heatmaps that assist them in recalling how they are viewing their textbooks; we have also made their teachers aware of the many ways the students use their textbooks, thus amplifying their knowledge of student work with resources. Finally, we have demonstrated that making connections between amount of textbook viewing and student performance is problematic because the reasons why users view and use textbooks and resources vary and because in general, they take advantage of more resources beyond their own textbooks. There is simply no way to connect time spent on a single resource and performance, as measured as a grade in a course.

3. Concluding Thoughts

My work to understand instruction and the use of textbooks in teaching and learning has required a wide range of methodological approaches that allow making coarse descriptions of practice (when using low inference codes) to descriptions that are more nuanced but that require much interpretation. The accumulation of data across terms and studies have increased the potential for pattern identification that has led to significant generalizations. The current developments in data science and artificial intelligence can be seen as great tools that can harness the potential of analyzing large bodies of data with higher levels of accuracy and reliability. The availability of these tools in online environments make it possible the data collection of participants who do not need to be closely located to a researcher. If we have learned something about the pandemic of the COVID-19 is that we are more connected than ever, and that proximity is not a prerequisite for having meaningful interchanges about work.

There are also ethical questions about the handling of the vast information that is being gathered. While we still rely on institutional review boards to ensure that the use of personal information is done correctly in ways that protect the users, we still rely on mechanisms put in place before the advent of these new technologies. Ensuring that such information be managed in ways that will not harm participants is a key point to attend to. An important question remains open: What is the potential of the interactions between resources, teachers, the students, and the content to create rich opportunities for mathematics learning in post-secondary and tertiary settings? Much work needs to be done, but my sense is that most work is methodological is we are interested in learning about resource use in naturalistic settings.

Acknowledgement

This material is based upon work supported by the National Science Foundation (IUSE 1624634, 1821509). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The University of Michigan is located on the traditional territory of the Anishinaabe people. In 1817, the Ojibwe, Odawa, and Bodewadami Nations made the largest single land transfer to the University of Michigan. This was offered ceremonially as a gift through the Treaty at the Foot of the

Rapids so that their children could be educated. Through these words of acknowledgment, their contemporary and ancestral ties to the land and their contributions to the University are renewed and reaffirmed. The Research on Teaching Mathematics in Undergraduate Settings (RTMUS) research group at the University of Michigan has provided support for this work over the years.

Appendix: EQIPM Dimensions (in gray boxes) and Items (in white boxes)

Dimensions of Quality
Student-Content Interaction: Attends to actions that demonstrate students' thinking about the mathematics by either describing their own thinking and reasoning, making connections among representations of mathematical ideas, or stating relationships with other mathematical topics and ideas they had seen before
Student Mathematical Reasoning and Sense-Making: Assesses student utterances that showcase reasoning and sense-making about mathematical ideas.
Connecting Across Representations-Student: Assesses connections that students express within, between, and across representations of the same mathematical problems, ideas, and concepts.
Situating the Mathematics-Student: Assesses connections students express to other aspects of the algebra curriculum, related topics, or the broader domain of mathematics, situating and motivating the current area under study within a broader context.
Instructor-Content Interaction: Attends to actions that reflect instructors' engagement with the content by making sense of the mathematics, making connections among representations of mathematical ideas explicit, situating the content in the larger structure of mathematics, describing the procedures not just as steps to follow but attending to why the steps are required or what other procedures can do the same job, presenting the ideas in a coherent organization, and providing mathematically sound explanations
Instructors Making Sense of Mathematics: Assesses how instructors leverage known and new mathematical ideas or students' personal knowledge or experiences, in order to make meaning of the mathematics presented.
Connecting Across Representations-Instructor: Assesses connections that instructors express within, between, and across representations of the same mathematical problems, ideas, and concepts.
Situating the Mathematics-Instructor: Assesses connections instructors express to other aspects of the algebra curriculum, related topics, or the broader domain of mathematics, situating and motivating the current area under study within a broader context.
Mathematical Explanations: Assesses how instructors provide mathematical reasons and justification for why something is done.
Supporting Procedural Flexibility: Assesses how instructors support the development of procedure use by identifying what procedure to apply and when and where to apply it.
Organization in the Presentation: Assesses how complete, detailed, and organized the instructor's presentation of content is when outlining or describing the mathematics or describing the steps used in a procedure.
Instructor-Student Interaction: Attends to actions describing how the instructor and the students relate to each other, specifically how the instructor shares the instructional space with the student, how the class environment supports students' participation and learning, how opportunities are created for students to engage in mathematical exploration, and how errors and difficulties are managed.
Instructor-Student Continuum of Instruction: Assesses the degree to which either the instructor or the students contribute to the development of the mathematical ideas (abstract concepts, formulas, notation, definitions, concrete examples, pictorial examples, and rules/properties). It captures who is responsible for the development of those ideas.
Inquiry / Exploration: Assesses the degree to which mathematics exploration and inquiry occurs.
Classroom Environment: Assesses how instructor and students create a respectful and open environment in their classroom in which expectations for high quality mathematical work is the norm.
Remediation of Student Errors and Difficulties: Assesses remediation (either for the whole class or with individuals/small groups) in which student misconceptions and difficulties with the content are addressed by attending to their reasoning.
Cross-Cutting Item: Mathematical Errors and Imprecisions in Content or Language: Assesses mathematically incorrect or problematic use of mathematical ideas, language, or notation by students and instructors. This item is scored differently than the rest of the items in the instrument, with a rating of 0 indicating no errors or imprecisions present, and a number between 1 and 4 to indicate low to high severity of the errors and imprecisions.

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https://doi.org/10.1142/9789811287183_0030

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Proposed Pedagogical Content Knowledge Tool for Assessing Teachers' Proficiency in Mathematical Knowledge for Teaching

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ABSTRACT Studies have indicated that the development of Mathematical Knowledge for Teaching (MKT), is rooted in teaching experience occasioned in teachers' daily work. To determine the role of teaching experience in the development of MKT, a special tool was required to capture all the MKT tenets and their combinations for analysis of mathematics teacher's proficiency. In this article the effectiveness of a tool developed purposely to examine the relationship between years of teaching experience and the development of Mathematical Knowledge for Teaching (MKT) is shared. This article has been drawn from a larger study on MKT proficiency status carried out in Kenya involving 117 trained secondary school mathematics teachers with varying years of teaching experience and academic backgrounds. Both descriptive and inferential statistics were found to be interpreted accurately using this tool. Using this tool, this study found a very weak positive relationship ($\beta = 0.171$) between teaching experience and MKT proficiency. The study established that MKT proficiency is not progressive, it is non directional and can regress in spite of teaching experience. From this finding it is my proposition that this pedagogical tool can sufficiently be used to discuss exhaustively teachers' MKT proficiency.

Keywords: MKT; PCK tool; Teaching experience; PCK proficiency.

1. Background and Context of Study

Mathematical knowledge for teaching (MKT), the mathematical knowledge that is specifically useful in teaching mathematics, is claimed to have its development deeply rooted in the experience afforded to teachers in their daily work (Ball, 1993). Proficiency in MKT is described as the deep connected understanding of mathematics as a subject and the flexible knowledge about effective strategies of presenting mathematical content to learners (Ball, 1993). The extent to which MKT is refined by years of experience has not been well documented. This article shares a Proficiency Status Tool (PST) (Miheso and Margot, 2016) that can quantify qualitative data for exploration and determination of teachers' MKT proficiency. The The process of using

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the tool includes examining each component of the framework (see Fig. 1) adapted from Ball et al. (2008). In this paper, the efficacy of this proficiency status tool is applied to measure the relationship between teachers' MKT proficiency and teaching experience.

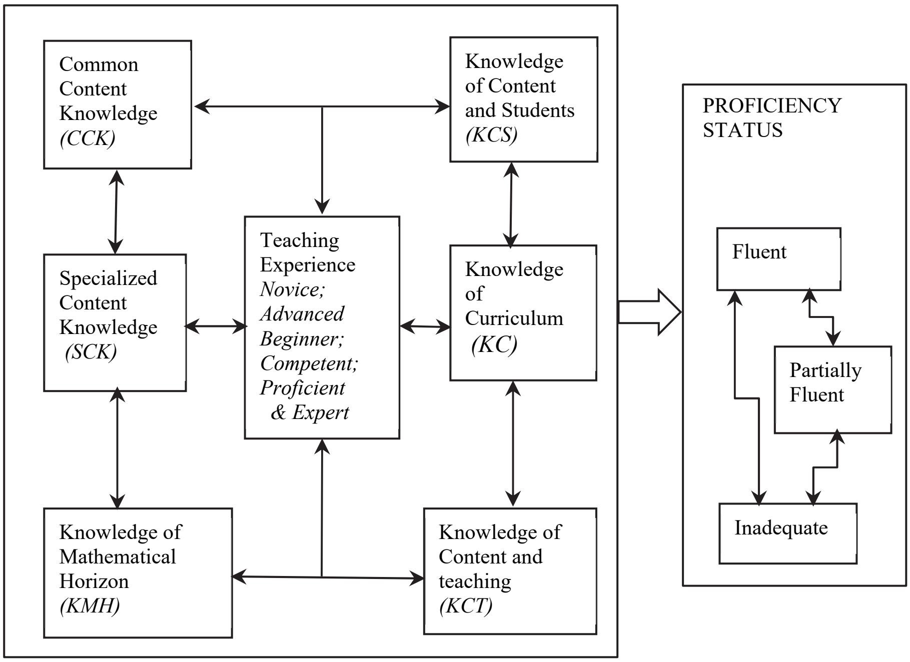

Fig. 1. Conceptual framework: MKT proficiency development process by experience

According to Ball (1993), the development of MKT, an accumulation of a defined body of knowledge, is deeply rooted in the practise of teaching. Accordingly, the proficiency tool was applied on the practice of teaching to investigate proficiency status of teachers' MKT in the Kenyan context. The study combined the teaching experience categories using the novice–expert (Livingstone and Borko, 1989) model, the Mathematical Knowledge for Teaching (MKT) framework developed by Ball et.al (2008) and the study's generated MKT rubrics to develop the proficiency status tool.

The proficiency status tool is a descriptive model that quantitatively characterizes teacher's proficiency on a continuum of fluent, partially fluent and inadequate. For this study, (i) "*Proficiency Status is Fluent*" represents a status in which teachers' transformation of content into comprehensible concepts for learners is powerful and easily discernible; (ii) "*Proficiency Status is partially fluent*" describes the status where the teacher displays good content mastery (CCK) and acceptable basic pedagogical strategies and procedural text guided knowledge of student as a learner. Teachers at this level drift within the "novice–expert" stages of advanced beginner, competent and proficient professional intermittently at various stages of curriculum implementation. The status category (iii) "*Proficiency is inadequate*" *is* characterized by observable teacher's insufficient MKT competency. The teacher is textbook dependent, uses textbook worked examples. Teachers within this category facilitate a learning environment that is passive (KCS).

Based on these descriptors, this tool was used to determine the relationship between years of teaching experience and teacher's proficiency status in Mathematical knowledge for teaching, A mixed methods approach was adopted to profile teacher's proficiency status of MKT and then map these status categories to their teaching experience. The main assumption of the study was that, Mathematics teachers' proficiency status in MKT is dependent on the cumulative proficiency of its components as described in MKT model (Ball, 1993). Experience by number of years was regressed against these variables to determine the significance of this relationship.

2. Developing the Pedagogical Status Tool (PST) Using MKT Rubrics

A generic rubric guideline for the written tasks was first developed using a numeric value (0 to 4) assigned to progressive competency descriptions based on MKT problem solving typical memo as displayed in Tab. 1 below.

Tab. 1. Generic MKT rubric

Score	Description
0	No response, incoherent explanation, wrong response
1	Partial incomplete interpretation, explanations that fault correctly identified prerequisite knowledge without justification
2	Correct interpretation of student action without explanation
3	Correct interpretation of student action with clear correct explanations
4	Correct interpretation, clear explanation, evidence of powerful pedagogy through use of analogies, correct remedial response

Based on this generic rubric guide, a specific MKT 5- point rating scale ranging from; correct response, partially correct (2 levels), incorrect and no responses was adapted for each task item in the teacher's pedagogical questionnaire. In addition, for this study, three proficiency categories (Tab. 2), fluent, partially fluent and inadequate were bench-marked using mean scores and categories to interpret the scores generated using the rubrics. The lower and upper limits for each category was generated based on the general performance, teaching culture of study location, what provides for a proficient teacher based on literature and the authors professional experience with teacher education.

Tab. 2. Bench-marking proficiency categories

Proficiency Categories	Fluent	Partially fluent	inadequate
Competency benchmarks by mean score	$4 \geq M \geq 2.5$	$2.5 \geq M \geq 1$	$M \geq 1$

2.1. Analyzing data using PST

To measure MKT proficiency, each task was broken into MKT components and strand types. (Tab. 3) Mean scores were then determined for each task from teacher scores.

Findings indicate that tasks with a combination of higher cognitive MKT components such as knowledge of Mathematical Horizon (KMH), Specialized Content Knowledge (SCK) returned poorer scores than those that demanded Common Content knowledge (CCK) and Knowledge of Curriculum (KC). These findings reveal that experience as a stand-alone determinant does not sufficiently explain teachers MKT fluency.

Tab. 3. Task MKT components mean score tool descriptors

Item no.	MKT Knowledge assessed	Task strand	Mean (max=4): SD	Proficiency descriptions
1	CCK, KCS, KCT, SCK, KMH, KC (6)	Data handling/Number /Operations	1.1(1.09)	Partially fluent
2	CCK, KCT, SCK, KMH (4)	Algebra, Geometry	1.6 (1.04)	Partially fluent
3	CCK, SCK (2)	Number and Operations	0.92 (1.4)	Inadequate
4	KCS KCT, CCK (3)	Algebra	0.56 (0.94)	inadequate
5	SCK, KCS, KMH, CCK (4)	Measurement/Geometry and Algebra	1.3 (1.2)	Partially fluent
6	CCK KCS, SCK, KMH (4)	Algebra/Geometry	1.18 (1.08)	Partially fluent

Teacher's performance on specific MKT components to determine respective fluency for each knowledge type indicated that fluency level of proficiency was determined in the KCT component of MKT (Tab. 4).

Tab. 4. Status of performance levels by knowledge type

Pedagogical knowledge type	**Mean score (Max. score 4)**	**Rating**
Common Content knowledge	2.5	Partially fluent
Specialized content knowledge	0.98	Inadequate
Knowledge of Content and Student	0.97	Inadequate
Knowledge of Mathematical Horizon	0.63	Inadequate
Knowledge of Content and Teaching	3.5	Fluent
Knowledge of Curriculum	2.0	**Partially fluent**
OVERALL SCORE FOR MKT	1.76	PARTIALLY FLUENT

Fluency in KCT which is teachers knowledge of content and teaching is a strong indicator of the type of knowledge that is the most impacted by years' experience

The tool afforded returns of a mean score in KCS (0.97), SCK ($M = 0.72$), This finding which is described in the MKT proficiency status tool as i*nadequate* is a strong indicator of a content-based teaching approach with minimum student support. The results also reveals partial fluency were attained in some items. Based on these results several interpretations and recommendations were made possible from this analysis

2.2. *Using PST to measure MKT performance by years of experience*

When teachers performance on tasks was compared by years of experience on each of the tasks, an average mean score of 1.11 against a maximum score of 4 was found. The study showed consistently higher performance for the competent and proficient categories in comparison to the novice and expert categories completely flipping the experience by years stage model by Dreyfus and Dreyfuss (1989) and Novice expert model by Livingstone and Borko (1986).

When a regression of performance on the cumulative mean scores for each category with categorized years of experience was carried out, a result ($\beta = .171$, $F = 1.857$, $p = 0.66$) was found. This indicated a weak positive relationship and in addition the relationship was found to be not statistically significant.

3. Conclusions

The MKT tool can be used to provide a more detailed evaluation of teacher's MKT proficiency across all the mathematical strands. Findings from this study which revealed an overall partially fluent proficiency status ($M = 1.76$), suggest that teachers' schemata require focused efforts on teachers' knowledge to support their systematic progress towards fluency in MKT. Using PST, it was possible to determine the non-statically significant relationship ($\beta = 0.171$) between proficiency status and teaching years of experience. This finding reveals that mathematics teachers, irrespective of experience, have inadequate proficiency status in the KMH component of the MKT framework.

Teaching mathematics at secondary school requires deep understanding in all the strands offered at this level. Variation in proficiency status across strands was revealed indicating skewed teaching experience. It was possible through use of the PST to determine similarities in both the less experienced teachers and the more experienced teachers in their proficiency displays of MKT. This relates to the characterization of the KCS and SCK components ($M = 0.98, 0.97$) as a need effort of all the MKT components. Inadequate display of proficiency levels in three categories from both most experienced and less experienced teachers refutes a direct relationship between experience and proficiency levels in MKT. This study revealed that the ability to refine and support student understanding as they engage in problem solving require purposeful targeted experiences.

Consequently, this study concludes that the role of experience by number of years should not be used to explain the proficiency status of teachers, but experience should be used to support the development of proficiency among mathematics teachers.

Teaching Mathematics at secondary school level is a complex process, but teachers can be supported to develop their fluency levels in MKT. Through the findings, Using PST, it is possible that MKT components that display low proficiency among secondary school teachers can be targeted for professional developing. This would help develop focused and relevant programs that address the MKT that teachers need to develop their teaching knowledge. This would make experience of the practise of teaching a resource for proficiency development of MKT.

This article posits that it is possible to quantify MKT for a deeper analysis of the deeply interwoven web of MKT components present in the daily tasks that teachers encounter in their work of teacher using Proficiency Status Tool (PST). Accordingly, teachers' needs can be addressed through determining their general and specific MKT proficiency using this tool.

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https://doi.org/10.1142/9789811287183_0031

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Language and Learning Mathematics: A Sociocultural Approach to Academic Literacy in Mathematics

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ABSTRACT This invited lecture summarized my work on language and learning mathematics. I described a theoretical framework for academic literacy in mathematics (Moschkovich, ... 5a, 2015b) that can be used to analyze student contributions and design lessons. The presentation included a classroom example and recommendations for instruction that integrates attention to language. Although the example is from a bilingual classroom, the theoretical framing and the recommendations are relevant to all mathematics learners, including monolingual students learning to communicate mathematically.

Keywords: Language; Learning; Sociocultural.

1. Introduction

This talk summarized a sociocultural framework for academic literacy in mathematics (Moschkovich, 2015a, 2015b) that uses a complex view of both mathematics and language, focuses on understanding (not computation), and emphasizes mathematical practices (Moschkovich, 2013a). To support all students in learning mathematics we need to shift from simplified views of mathematical language as single words to a broader definition of academic literacy — not just learning words but learning to communicate mathematically. Mathematics instruction must shift from focusing on low-level language skills (i.e., vocabulary or single words) to using an expanded definition of academic literacy in mathematics that includes mathematical practices and discourse. This sociocultural framework can be used to uncover how students participate in mathematical practices, hear how language provides hybrid resources for mathematical activity, and design lessons that include attention to language.

1.1. Why integrate language into research on learning mathematics?

In the talk, I described how I integrated language into research on mathematics learning and teaching. I first summarized a theoretical framework that is a socio-cultural approach to "academic literacy in mathematics." I then used a classroom example to illustrate that theoretical framework and make recommendations for instruction.

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This integration was motivated by theorical and practical goals. Integrating language into research on mathematics learning provides a fuller theoretical account of mathematical thinking that includes language(s). Integrating research on learners who use more than one language into research on mathematics learning/teaching is important not only to include this student population, but also because language is central to a full theorical account. Research with bilingual learners provides a window on language: while language can be invisible in monolingual situations, a setting with bilingual learners makes language visible. In this way, bilingual learners are a gift.

There are also practical reasons for this integration. Clearly, this integration is crucial to improve instruction for students who are learning the language of instruction (LOI) and are bilingual/multilingual. Nevertheless, integrating language is important not just for those student populations imagined to have issues with language, it is important for all students. Mathematics is a discipline associated with power and authority; power and authority are enacted through and mediated by language. Integrating language is particularly important for students from communities with a history of lack of opportunities to learn mathematics due to imagined issues with language. However, this integration is important for all students because all students experience power and authority through language practices in mathematics classrooms.

1.2. A little history: themes in my research

I started out looking closely at student conceptions of linear functions. I examined how students understand and use the connections between equations and their graphs. I was especially interested in how a discussion with a peer supported learning about equations and lines (Moschkovich, 1996). This led me to thinking about the mathematics register and mathematical discourse (Moschkovich, 2002, 2007a). I worked with bilingual students and this led me to exploring the role of language in learning mathematics, documenting how bilingual students communicate mathematically (Moschkovich, 1999, 2007b). Moschkovich (2002) was my first attempt to understand and use the concept of register. I also aimed to shift from a view of language as an obstacle to a resource, what I would now call from a deficit to an asset view of bilingual learners.

I used Vygotskian and neo-Vygotskian theories of learning (i.e., Forman, 1996; Vygotsky, 1978, 1979, 1987) to frame my research. One goal was to reconcile my theoretical commitments to Vygotskian perspectives with more cognitive views of mathematical thinking. I initially struggled to answer several (fundamentally Vygotskian) questions, especially where to see mediation by social-cultural artifacts. I came to see that, for me, the answer to this question lay in clarifying the category of mathematical practices, connecting those to mathematical discourse, and using appropriation to describe how learners learn to use those practices.

Using Vygotskian theories and work in sociolinguistics (e.g., Gee, 1990), I analyzed discussions of mathematical problems among students or between a learner and an adult. My analyses have focused on identifying and describing central aspects of mathematical practices. In one article, “Appropriating mathematical practices: A case study of learning to use and explore functions through interaction with a tutor," I returned to the theme of mathematical discourse and added an analysis of mathematical practices (Moschkovich, 2004). As I realized that the construct "mathematical practices" was central to my work, I wrote a chapter titled "Issues regarding the concept of mathematical practices (Moschkovich, 2013).

More recently, starting in 2015, I integrated these themes — student thinking, language, discourse, and practices — into a theoretical framework that I call "academic literacy in mathematics (Moschkovich, 2015a, 2015b)." Here I summarize that framework and briefly define the three components of academic literacy in mathematics: mathematical proficiency, mathematical practices, and mathematical discourse. I have used that framework to analyze classroom discussions (Moschkovich, 2015a) and describe how a teacher provided scaffolding for mathematical practices (Moschkovich, 2015c).

1.3. Theoretical framing and assumptions

The framework draws on sociocultural and situated perspectives of learning mathematics (Brown et al., 1989; Greeno, 1998) as a discursive activity (Forman, 1996) that involves participating in a community of practice (Lave and Wenger, 1991), and using multiple material, linguistic, and social resources (Greeno, 1998). Mathematical activity is assumed to involve not only individual mathematical knowledge but also collective mathematical practices and discourses.

A sociocultural perspective brings several assumptions to defining academic literacy in mathematics. The first assumption is that mathematical activity is not merely cognitive or individual; instead, it is simultaneously cognitive, social, and cultural. Second, the focus is on the potential for progress in what learners say and do, not on learner deficiencies. The third assumption is that participants bring multiple perspectives to any situation and that meaning is not static but situated: representations and utterances have multiple meanings; meanings for words (or inscriptions) are situated, constructed while participating in practices, and negotiated through interaction (Moschkovich, 2008).

A sociocultural perspective of academic literacy in mathematics[2] provides a complex view of mathematical proficiency as participation in discipline-based practices that involve reasoning, understanding, and communicating. A situated and sociocultural perspective on bilingual mathematics learners (Moschkovich, 2002) shifted the focus from looking for deficits to identifying the mathematical discourse practices evident in student contributions (e.g., Moschkovich, 1999). The sociocultural

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[2] This sociocultural perspective builds on previous work where I described a sociocultural view of mathematics learners who are bilingual and/or learning English (Moschkovich, 2002), of mathematical discourse (Moschkovich, 2007a), and of mathematical practices (Moschkovich, 2013). In other publications (e.g., Moschkovich, 2008), I described how mathematical discourse is situated, involves coordinated utterances and focus of attention, and combines everyday and academic registers. The definition of academic literacy in mathematics used here brought together and built on different aspects of those analyses.

perspective in Moschkovich (2002, 2004, 2007a) also provided a theoretical framework for recognizing the mathematical practices in student contributions.

2. Academic Literacy in Mathematics

Academic literacy in mathematics includes three integrated components: mathematical proficiency, mathematical practices, and mathematical discourse. These three components are intertwined, should not be separated during instruction, and cannot be separated when analyzing student mathematical activity or designing mathematics lessons.

The view of academic literacy in mathematics presented here is different than previous approaches to academic language in several ways. First, the definition includes not only cognitive aspects of mathematical activity — what happens in one's mind, such as mathematical reasoning, thinking, concepts, and metacognition — but also social and cultural aspects — what happens with other people, such as participation in mathematical practices — and discourse aspects — what happens when using language (reading, writing, listening, or talking about mathematics). Most importantly, the components of academic literacy in mathematics work together, not separating mathematical language from proficiency or practices.

This definition goes beyond narrow views of mathematical language as vocabulary, definitions, or formal language because these views limit learners' access to high-quality instruction. A focus on single words or vocabulary limits access to complex texts and high-level mathematical ideas and to opportunities for students to understand and make sense of those texts. The assumption that meanings are static and given by definitions limits students' opportunities to make sense of mathematics texts for themselves. The assumption that mathematical ideas should always and only be communicated using formal language limits the resources that students can use to communicate mathematically, excluding or dismissing resources such as informal, everyday, or home language(s) that have been documented as important for communicating mathematically.

In contrast, the view of mathematical language used here assumes that meanings for academic language are situated and grounded in the mathematical activity that students are actively engaged in. For example, meanings for the words in a word problem do not come from the definition in a word list provided by the teacher. Instead, students negotiate meanings as they work on a problem, communicate with peers, and develop their solutions. A complex view of mathematical language also means that lessons must include multiple modes (not only reading and talking but also listening and writing), multiple representations (gestures, objects, drawings, tables, graphs, symbols, etc.), and multiple ways of using language (formal school mathematical language, home languages, and everyday language). In addition, this definition ensures that academic literacy in mathematics goes beyond simplified views of mathematics as computation. It includes the full spectrum of mathematical proficiency, balancing procedural fluency with conceptual understanding; it also includes mathematical practices and emphasizes student participation in discourse practices.

2.1. Defining academic literacy in mathematics

Academic literacy in mathematics is more complex than simply combining alphabetic literacy with proficiency in mathematics. Reading and solving a word problem entails not only proficiency in mathematics but also competencies in using mathematical practices and discourses. Typically, "literacy" is interpreted as referring to words and "mathematics" as referring to numbers. For example, we could imagine that solving the word problem below involves "literacy" in the activity of reading and understanding the words, and "mathematics" in the activity extracting the numbers and relating them through arithmetic operations:

> Jane, Maria, and Ben each have a collection of marbles. Jane has 15 more marbles than Ben, and Maria has 2 times as many marbles as Ben. All together they have 95 marbles. Find how many marbles Maria has.

However, reading and solving this word problem entails not only mathematical proficiency, proficiency in the content of mathematics, but also competencies in using mathematical practices such as making sense of the problem. If students are asked to communicate their solutions to a peer or to the whole class, then solving this word problem also involves mathematical discourse, communicating one's thinking and describing one's solution. These three components cannot be separated when considering mathematical tasks, analyzing student mathematical activity, or designing mathematics instruction.

The complexity of academic literacy in mathematics is evident in another word problem:

> A boat in a river with a current of 3 mph can travel 16 miles downstream in the same amount of time it can go 10 miles upstream. Find the speed of the boat in still water.

Solving this word problem requires much more than reading and understanding the text and then deciding what arithmetic operations to use or what equation to write. One surely cannot solve this word problem by using key mathematics vocabulary words to extract the correct numbers (or quantities or variables), relate them using the correct arithmetic operation, or write the correct equation. Instead, what is required are all three components simultaneously: mathematical proficiency, mathematical practices, and mathematical discourse. Possible mathematical practices needed for this problem include modeling a situation (as one carefully imagines what is going on in the situation) and using or connecting representations (if one draws a picture). This word problem also illustrates how academic literacy in mathematics is not principally about technical vocabulary. The crucial vocabulary for understanding this problem situation is not mathematical or technical vocabulary. Instead, the challenging vocabulary might be upstream, downstream, and "in still water."

Simplified views of academic language in mathematics focus on words, assume that meanings are static and given by definitions, separate language from mathematical knowledge and practices, and limit mathematical discourse to formal language. In contrast, academic literacy in mathematics as defined here includes three integrated components: mathematical proficiency, mathematical practices, and mathematical discourse. This view of academic literacy in mathematics is different than previous approaches to academic language in several ways. First, the definition includes not only cognitive aspects of mathematical activity — such as mathematical reasoning, thinking, concepts, and metacognition — but also sociocultural aspects — participation in mathematical practices — and discursive aspects — participation in mathematical discourse.

This is an integrated view of three components working in unison, rather than isolating academic language from mathematical proficiency or mathematical practices. Second, this integrated view, rather than separating academic language from mathematical proficiency or practices, views the three components as working in unison. Separating language from mathematical thinking and practices can have dire consequences for students. This separation can make students seem more deficient than they are, since they may express their mathematical ideas through imperfect language, but may still be engaged in correct mathematical thinking, and they may participate in mathematical practices through other modes, for example using objects, drawings, or gestures to show a result, describe regularity in data, or illustrate a mathematical concept. Lastly, this definition includes the full spectrum of mathematical proficiency, balancing fluency in computing with an emphasis on conceptual understanding, reasoning, and communicating.

2.1.1. Defining mathematical proficiency

A description of mathematical proficiency (Kilpatrick et al., 2001) shows five intertwined strands: Conceptual understanding, Procedural fluency, Strategic competence, Adaptive reasoning; and Productive disposition. Procedural fluency is knowing how to compute. Conceptual understanding is fundamentally about the meanings that learners construct for mathematical solutions: knowing the meaning of a result (what the number, solution, or result represents), knowing why a procedure works, and explaining why a particular result is the right answer. Reasoning, logical thought, explanation, and justification are closely related to conceptual understanding. Student reasoning is evidence of conceptual understanding when a student explains why a particular result is the right answer or justifies a conclusion. The five strands of mathematical proficiency provide a cognitive account of mathematical activity focused on knowledge, metacognition, and beliefs. However, from a sociocultural perspective, mathematics students are not only acquiring mathematical knowledge, they are also learning to participate in valued mathematical practices (Moschkovich, 2004, 2013a).

2.1.2. Defining mathematical practices

The term practice shifts from purely cognitive accounts of mathematical activity to assuming the social, cultural, and discursive nature of doing mathematics. I use the terms practices drawing on Scribner's (1984, p. 13) practice account of literacy to "highlight the culturally organized nature of significant literacy activities and their conceptual kinship to other culturally organized activities involving different technologies and symbol systems". This definition implies that mathematical practices are culturally organized, involve symbol systems, and are related conceptually to other mathematical practices. From this perspective, mathematical practices are not only cognitive — i.e., involving mathematical thinking and reasoning — but also social and cultural — arising from communities and marking membership in communities — and semiotic — involving semiotic systems (signs, tools, and their meanings).

Academic mathematical practices can be understood in general as using language and other symbols systems to think, talk, and participate in the practices that are the objective of school learning. There is no single set of mathematical practices or one mathematical community; practices vary across communities of research mathematicians, traditional classrooms, and reformed classrooms. However, across these various communities, there are common practices that can be labeled as academic mathematical practices. Examples of mathematical practices include problem solving, sense-making, reasoning, modeling, abstracting, generalizing, using or connecting mathematical representations, imagining, and looking for patterns, structure, or regularity.

2.1.3. Defining mathematical discourse

A sociocultural framing of mathematical practices connects practices to discourse . In particular, discourse is central to participation in many mathematical practices, and meanings for words are situated and constructed while participating in mathematical practices. Academic mathematical discourse has been described as having some general characteristics. In general, particular modes of argument, such as precision, brevity, and logical coherence, are valued (Forman, 1996). Abstracting, generalizing, and searching for certainty are also highly valued. Generalizing is reflected in common mathematical statements, such as "The angles of any triangle add up to 180 degrees", "Parallel lines never meet", or "a + b (always) equals b + a". What makes a claim mathematical is, in part, the detail in describing when the claim applies and when it does not. Mathematical claims apply only to a precisely and explicitly defined set of situations and are often tied to mathematical representations (symbols, graphs, tables, or diagrams). Many valued academic mathematical practices involve mathematical discourse.

The academic literacy in mathematics framework goes beyond low-level language skills, using a view of mathematical discourse that includes multiple modes, symbol systems, registers, and languages. Mathematical discourse is assumed to be multi-modal and multi-semiotic (using multiple sign systems). Meanings are not provided by static dictionary definitions, but situated in local history, practices, and socio-cultural context. Mathematical discourse draws on hybrid resources; during classroom mathematical discussions students use both everyday and formal mathematics registers.

I use the phrase mathematical discourse because there are multiple meanings for "language" and to emphasize that discourse is much more than language. I do not use the phrase "academic language" because it can be reduced to single words, vocabulary, or grammar. In contrast, I use a view of mathematical discourse not as a list of words with precise meanings but the communicative competence (Hymes, 1972/2009) necessary and sufficient for competent participation in mathematical practices. Work on the language of disciplines (e.g., Pimm, 1987) provides a complex view of mathematical language as not only specialized vocabulary — new words and new meanings for familiar words — but also as extended discourse that includes syntax and organization, the mathematics register (Halliday, 1978), and discourse practices (Moschkovich, 2007a). The mathematics register includes styles of meaning, modes of argument, and mathematical practices and has several dimensions such as the concepts involved, how mathematical discourse positions students, and how mathematics texts are organized.

3. A Classroom Example of Mathematical Practices

If students are participating in academic literacy in mathematics, we can see and hear them actively participating in mathematical practices, many of which are discursive. This classroom example illustrates using the framework to uncover how students use mathematical practices and hybrid language practices (Gutierrez et al, 1999) to participate in a mathematical discussion.

The lesson excerpt comes from a third-grade bilingual Spanish-English classroom in an urban California school. In this classroom, there were thirty-three students. In general, this teacher introduced students to topics in Spanish and then later conducted lessons in English. The students had been working on a unit on two-dimensional geometric figures. For several weeks, instruction had included vocabulary such as the names and properties of different quadrilaterals in both Spanish and English. Students had been talking about shapes and the teacher had asked them to point, touch, and identify different quadrilaterals. The teacher identified this lesson as a lesson where students would be using English to discuss different shapes.

Below is an excerpt from the transcript for this lesson involving descriptions of a rectangle. (Brackets indicate transcript annotations.)

1. Teacher: Let's see how much we remembered from Monday. Hold up your rectangles. ... high as you can. [students hold up rectangles] Good, now. Who can describe a rectangle (for me)? Eric, can you describe it? [a rectangle] Can you tell me about it?
2. Eric: A rectangle has ... two ... short sides, and two ... long sides.
3. Teacher: Two short sides and two long sides. Can somebody tell me something else about this rectangle? If somebody didn't know what it looked like, what, what ... how would you say it?
4. Julian: Parallel(a). [holding up a rectangle]
5. Teacher: It's parallel. Very interesting word. Parallel, wow! Pretty interesting word, isn't it? Parallel. Can you describe what that is?

6. Julian: Never get together. They never get together [runs his finger over the top length of the rectangle].
7. Teacher: OK, what never gets together?
8. Julian: The parallela ... they ... when they, they get, they go, they go higher [runs two fingers parallel to each other first along the top and base of the rectangle and then continues along those lines] they never get together.
9. Antonio: Yeah!
10. Teacher: Very interesting. The rectangle then has sides that will never meet [runs fingers along top and base of an invisible rectangle] those sides will be parallel [motions fingers vertically in parallel lines]. Good work. Excellent work.

3.1. Uncovering mathematical practices

One recommendation for instruction is to use this framework to focus on mathematical practices, not "language" as words, vocabulary, or formal definitions. An overemphasis on correct vocabulary and formal language limits the linguistic resources for learning/teaching math with conceptual understanding and precludes students from participating in valued mathematical practices.

What mathematical practices did Julian use? There were several mathematical practices evident in Julian's original utterance in line 8. Julian was abstracting and generalizing. He was describing an abstract property of parallel lines and making a generalization saying that parallel lines will never meet. He was also imagining what happens when the parallel sides of a rectangle are extended. If we focused only on whether he did or did not use mathematical vocabulary, we would miss Julian's use of these important mathematical practices.

Emerging language and ideas are imperfect and may be difficult to understand. In this example, uncovering the mathematical practices in Julian's contributions is challenging. Julian's utterances in turns 4, 6, and 8 are difficult to hear and interpret. He said the word "parallela" with hesitation. His voice trailed off, so it is difficult to know whether he said "parallelo" or "parallela." His pronunciation could be interpreted as a mixture of English and Spanish; the "ll" sound pronounced in English and the addition of the "o" or "a" pronounced in Spanish. Was this hesitation due to issues with the pronunciation or the mathematical idea? It is impossible to answer this question! Instead, the framework can be used to focus on mathematical practices and to notice that Julian accurately described a property of parallel lines. If we focus only on formal vocabulary, we miss the mathematical practices. Instead of focusing on single words or formal vocabulary, it is more important to listen for the meaning of the whole utterance, since that is the way to uncover mathematical practices.

3.2. Uncovering hybrid language practices

The second recommendation is to use the academic literacy in mathematics framework to treat everyday discourses as resources. Everyday and home registers have been documented as providing resources for communicating mathematically. Students are likely to use hybrid language practices that combine everyday and formal registers.

What language resources did Julian use to communicate his mathematical ideas? Julian's pronunciation in turns 4 and 8 is an example of a hybrid language practice. His utterances can be interpreted as a mixture of English and Spanish, the word "parallel" pronounced in English, and the added "a" pronounced in Spanish. In Spanish, the word parallel would agree with the noun (line or lines), in both number (plural or singular) and gender (masculine or feminine; "parallel lines" translates to "líneas paralelas," "parallel sides" translates to "lados paralelos"). The grammatical structure in turn 8 can also be interpreted as a mixture of Spanish and English. The apparently singular "parallela" in turn 8 was preceded by the word "the" (which can be either plural or singular) and then followed by a plural "when they go higher."

Julian also used colloquial expressions such as "go higher" and "get together" rather than the formal terms "extended" or "meet." These everyday expressions were not obstacles but resources to communicate a mathematical idea. These phrases are instances of everyday phrases used with mathematical meaning. Julian used hybrid language resources that drew on both everyday and academic registers. He did not use technical phrases but an everyday phrase with mathematical meaning. The discussion was mathematical not because it involved technical mathematical terms, but because it involved mathematical concepts and practices. This example illustrates how the everyday and academic registers are not in opposition and how both can provide resources to communicate mathematical ideas.

3.3. Teacher moves

The excerpt also illustrates how this teacher, rather than requiring students to use an idealized version of perfect language, accepted and built on students' hybrid use of language to support student participation in a mathematical discussion. The teacher used several teachers' moves such as asking for clarification, probing what students mean, and revoicing student statements.

Revoicing is an important way teachers can build on students' own use of mathematical practices or add new mathematical practices to a discussion. In turn 5, the teacher accepted Julian's response, revoicing it as "It's parallel," and probed what Julian meant by "parallela." In turn 10, the teacher revoiced Julian's contribution in turn 8: "the parallela, they" became "sides," and "they never get together" became "will never meet, will be parallel."

In this case, the teacher's revoicing made Julian's claim more precise, introducing a new mathematical practice, attending to the precision of a claim. In line 10, the teacher's claim is more precise than Julian's claim because the second claim refers to the sides of a quadrilateral, rather than any two parallel lines. Revoicing also provided opportunities for students to hear more formal mathematical language. The teacher revoiced Julian's everyday phrase "get together" as "meet" and "will be parallel." Revoicing can be used to scaffold mathematical practices and use formal language (Moschkovich, 2015c).

4. How Instruction Can Focus on Academic Literacy in Mathematics

The view of academic literacy in mathematics described here integrates mathematical proficiency with mathematical practices and discourse. Separating language from mathematical proficiency limits learners' access to conceptual understanding. Separating language from mathematical practices curtails students' opportunities to participate in mathematical practices. Not allowing students to use informal language, typically acquired before more formal ways of talking, also limits the resources to communicate mathematically. Lastly, focusing on correct vocabulary curtails opportunities for students to express themselves mathematically in what are likely to be imperfect ways, especially as they are learning new ideas.

In contrast, the view of academic literacy in mathematics described here focuses on mathematical practices and uses and expanded view of language that includes informal language as a resource. Mathematics lessons that integrate language provide students opportunities to participate in mathematical practices, negotiate meanings, and use multiple discourses and registers. Teachers can support students as they negotiate meanings for mathematical language; this negotiation is best when it is grounded in students' own mathematical work, instead of giving students definitions separate from their mathematical activity (Moschkovich, 2015a, 2015b).

For students learning mathematics, informal language is important, especially when students are exploring a new mathematical concept or discussing a problem in small groups. By learning to recognize how learners actively use hybrid language practices to engage in understanding, reasoning, and communicating, teachers can provide opportunities for students to participate in all three components of academic literacy in mathematics. Students can use informal language during exploratory talk (Barnes, 2008) or when working in a small group (Herbel-Eisenmann et al., 2013). Such informal language reflects important mathematical thinking (for examples, see Moschkovich, 1996, 2008). In other situations, for example, when presenting a solution or writing an account of a solution, using more formal academic mathematical language becomes more important.

Mathematics instruction needs to shift from simplified views of language as vocabulary and carefully consider when and how to emphasize correct vocabulary and formal language. Such views severely limit the linguistic resources teachers and students can use to teach and learn mathematics, and separate language from mathematical practices. Focusing instruction on vocabulary limits students' access to the five strands of mathematical proficiency and curtails students' opportunities to participate in mathematical practices. We must leave behind simplified views of language as vocabulary, embrace the multimodal and multi-semiotic nature of mathematical activity, and shift from monolithic views of mathematical talk or dichotomized views of everyday and mathematics registers (Moschkovich, 2010). An overemphasis on correct vocabulary and formal language limits the linguistic resources teachers and students can use in the classroom to learn mathematics with understanding.

The question is not whether students should learn vocabulary but rather when and how instruction can best support students as they learn not only the meanings of words and phrases but also how to participate in mathematical practices. Vocabulary drill, practice, definitions, or lists are not the most effective way to learn to communicate mathematically. Instead, vocabulary acquisition (whether it is in a first or second language) occurs most successfully in instructional contexts that are language-rich, actively involve students in using language, require both receptive and expressive understanding, and require students to use words in multiple ways over extended periods of time (Blachowicz and Fisher, 2000; Pressley, 2000). To develop oral and written communication, students need to participate in negotiating meanings (Savignon, 1991) and in tasks that require student output (Swain, 2001). Instruction should provide opportunities to actively use mathematical language to communicate about and negotiate meaning for mathematical situations.

Overall, dichotomies such as everyday/academic or formal/informal are not useful for research or practice (Moschkovich, 2010). Classroom discussions draw on hybrid resources from both academic and everyday contexts, and multiple registers co-exist in math classrooms. Everyday ways of talking should not be seen as obstacles to participation in academic mathematical discussions but as resources teachers can build on to support students in learning more formal mathematical ways of talking. Teachers need to hear the mathematical content in students' everyday language, build on that everyday language, and support or scaffold (Moschkovich, 2015c) more formal language. Everyday language is not only a starting place for learners; it supports reasoning, facilitates communication, and grounds meanings.

With a complex definition of academic literacy in mathematics, teachers can choose (or design) tasks that support academic literacy in mathematics, provide opportunities for learners to participate in academic literacy in mathematics, and recognize academic literacy in mathematics in student activity. When designing instruction, teachers can consider how each component of academic literacy in mathematics might appear and how to provide students opportunities to participate in each of the three components. If students are participating in academic literacy in mathematics as defined here, then we see or hear them engaged in the full spectrum of mathematical proficiency as they participate in mathematical practices, many of which are discursive.

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https://doi.org/10.1142/9789811287183_0032

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Research on Discussion in Mathematics Teaching: A Review of Literature From 2000 to 2020

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ABSTRACT For decades, reformers have emphasized discussion over recitation and lecture. Yet, traditional communication patterns are still dominant in mathematics classrooms internationally. In an effort to better understand this challenge, the present study investigates patterns and contributions of research on discussion in mathematics teaching. Based on systematic search in the Eric database, and in selected journals of mathematics education, 72 studies were reviewed. Based on analysis and discussion of the reviewed studies, it is suggested to develop conceptual clarity and include definitions of core terms like discussion, to consider alternative methods for studying discussion in teaching, and to consider shifting the focus from teacher actions to the entailments of the work of leading mathematical discussions.

Keywords: Discussion; Teaching; Mathematics; Literature review.

1. Introduction

The idea of discussion in teaching is not new. It is associated with the dialectic principles of the Socratic dialogues (cf. Sattler, 1943), Dewey's (1916) thinking about participation in a democratic society, and more. In an early textbook on social psychology, Ross (1908) stated that, "It is coming to be recognized that there is nothing of concern to human beings which may not profitably be discussed in the right spirit, by the right persons, at the right time" (p. 309). Later, Schwab (1954) described discussion as "indispensable to a good liberal education" (p. 51), and Cockroft (1982) listed discussion as a core element of mathematics teaching. Where traditional teaching involves communication formats like recitation, lecture, and teacher explanation, reform pedagogies often involve exploration and discussion (Smith, 1996).

A simplistic view of discussion in teaching is that the teacher should avoid telling the students, and instead step aside and let students discuss. In this sense, teaching by discussion would seem to involve less effort and a less prominent role of the mathematics teacher. Chazan and Ball (1999) were among the early critics of such a view. From analysis of two episodes of discussion, they unpacked the complexities and challenges in the role of the teacher in discussion and examined moves that teachers

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could make to moderate discussions. In another study, Lampert (2001) unpacked how the work of leading discussions consists of numerous problems that teachers must solve. Many of these problems involve navigation of seemingly unsolvable dilemmas, like "simultaneously teaching individual students and engaging the group as a whole in worthwhile activity," and "keeping the discussion on track while also allowing students to make spontaneous contributions that they considered to be relevant" (p. 174). More recently, leading discussions has been described as a core practice of mathematics teaching (Jacobs and Spangler, 2017), and volumes have been written to help teachers carry out this practice (e.g., Chapin et al., 2009; Kazemi and Hintz, 2014). Although books like these, along with frameworks like the "five practices" (Stein et al., 2008), have provided teachers with useful tools to support their planning, initiation, and orchestration of mathematical discussions, research still indicates that change in practice is slow, and that traditional teaching is still dominant. Everyone seems to agree about the need for change, and about the direction for such change, but the field has still not managed to change practice in this direction. This is a challenge. Being faced with a challenge like this, it seems logical to carefully consider previous efforts to approach it. I was therefore surprised to find that there were few reviews of research on discussion in mathematics teaching. There have been some reviews of research on discourse in mathematics education (e.g., Ryve, 2011), and Jacobs and Spangler (2017) provided a useful review of research on core practices, where leading discussions was one of two core practices they considered in detail, but I have not been able to find any comprehensive reviews of research on discussion in mathematics teaching. To mitigate this, the present study investigates what characterizes recent research on discussion in mathematics teaching. The aim of the study is to uncover trends in research, consider what has been emphasized and not, and thereby initiate professional deliberation about limitations and potential shifts in research on discussion in mathematics teaching. The following research questions are considered:

1. What are the core problems in studies of discussion in mathematics teaching?
2. What aspects of discussion are focused on in studies?
3. What methods are used for studying discussion in mathematics teaching?
4. What common reference literature can be identified in studies of discussion in mathematics teaching?

The study has been organized as a review of research on discussion in mathematics teaching, focusing on the last twenty years (2000–2020). Before presenting methods and results from the literature review, I elaborate on key terms and conceptual underpinnings.

2. Conceptual Background

The first key term to elaborate on is that of *discussion*. What do we mean by discussion? On the one hand, discussion is common term that is used frequently in everyday speech, and we "discuss" the weather, the news, last night's TV show, or a recent sports event.

On the other hand, discussion can refer to a more specific form of communication where we investigate or examine a complex issue to reach a solution.

The word "discuss" originates from Latin and is composed by two parts. The first part, "dis," means apart, and the last part, "quatere", means to shake. Etymologically, then, discussion means to shake something apart. If we confer with a contemporary dictionary, like the Oxford English Dictionary, discussion is defined like this:

> Treatment of a subject, in speech or writing, in which the various facts, opinions, and issues relating to it are considered; the action or process of talking about something in order to reach a decision or to exchange ideas (Discussion, n.d.)

A few things are worth noticing about this definition. First, discussion is always about *something*. There must be some subject or issue to be discussed. Discussions can be verbal or written, and written discussions can be synchronous or asynchronous, like in an online discussion forum. This study focuses on verbal discussions that take place synchronously in the context of the mathematics classroom. Second, discussion is "the action or process of talking about something in order to reach a decision or to exchange ideas." This indicates that discussion always has a *purpose*. Dillon (1994, p. 8) brings all these aspects together in the following definition:

> Discussion is a particular form of group interaction where members join together in addressing a question of common concern, exchanging and examining different views to form their answer, enhancing their knowledge or understanding, their appreciation or judgement, their decision, resolution or action over the matter at issue.

At least four perspectives are worth highlighting from this definition. First, Dillon describes discussion as a *particular* form of group interaction, and he makes a clear distinction between discussion and recitation. In recitation, the teacher typically asks a question, a student responds, and the teacher then evaluates their response; this is often referred to as an IRE pattern of communication. Discussions typically follow a different pattern of communication, where students often ask questions — not only to the teacher, but also to other students — and teachers do not always provide an evaluation of students input, but they might instead prompt students to comment on each other's thinking. Second, the emphasis on the question of common concern is useful to keep in mind. In a discussion, there must be a particular question or problem that a group wants to solve. In the classroom, the group typically consists of a teacher and their students. Third, there must be an exchange of ideas or views in a discussion; it is not sufficient to have a contribution from only one person. Different views must be exchanged and examined to constitute a discussion. Fourth, there might be different purposes of a discussion. Some discussions aim at enhancing knowledge or understanding, whereas other discussions aim at reaching a decision that might lead to some action.

The next key term in this study is "teaching". Studies of discussion in teaching necessarily involve a conception of teaching. Like discussion, the words "teach", or "teaching" can be used in different ways in everyday speech. The Oxford English Dictionary defines teaching as "the imparting of instruction or knowledge; the occupation or function of a teacher" (Teaching, n.d.). Two things are worth noticing about this definition. First, teaching is always about something; there must be some content or knowledge at play. Second, teaching can also refer to the occupation or function of a teacher. Teaching is thus not only about actions that teachers perform, but it also refers to an occupation or professional practice.

The research literature applies different definitions of teaching, and these definitions relate to the dictionary definition referred to above. Many define teaching as the activities that are carried out by teachers. As an example, Gage (1978, p. 14) defined teaching like this: "By teaching I mean any activity on the part of one person intended to facilitate learning on the part of another". One interesting aspect of this definition is the focus on activity. Researchers, like Gage, who apply a process-product paradigm for studying teaching, often consider teaching as something teachers do, or activities performed by teachers. Another interesting aspect of the definition is the implied relationship between teaching and learning. Although teaching is the activity of one person, it has the intention of facilitating learning in another person. The definition seems to imply that teaching is something that teachers do, whereas learning is something students do. Although there is an intention of facilitating learning, teachers are dependent on someone else (the students) to be successful in their profession (Cohen, 2011).

Other researchers seem to attend more to the second aspect of the dictionary definition, when they define teaching as professional practice, or as work that needs to be done. For instance, Lampert (2010) describes teaching as a practice, and she frequently refers to the "work of teaching". Ball and Forzani (2009, p. 497) are in the same tradition when they define the work of teaching as "the core tasks that teachers must execute to help pupils learn". Their focus is more on identifying and understanding the tasks than on evaluating how particular teachers execute these tasks. This involves a shift in focus from considering teaching as something teachers do toward the tasks or core components of the work that teachers are faced with. The research literature describes these core components of the work in different ways. For instance, Lampert (2001) describes them as *problems* of teaching, indicating a metaphor of teaching as problem solving. Cohen (2011) describes the work of teaching by considering its *predicaments*; teachers are faced with numerous predicaments that they must deal with. Again, Ball and colleagues (2008) describe the core components of the work of teaching as *tasks of teaching* in their practice-based theory of mathematical knowledge for teaching (cf. Hoover et al., 2014). These are (mathematical) tasks that teachers are routinely faced with and must carry out in their work.

Considering teaching as work differs from the more conventional way of thinking about teaching as actions teachers perform. Studies that conceptualize teaching as something teachers do often focus on identifying patterns in teachers' actions or communication, or they attempt to evaluate the effectiveness of teachers' actions when compared with some outcome variable. In contrast to this, consideration of teaching as work often leads to studies that aim at understanding what is involved in teaching (e.g., Ball, 2017), or developing a language to describe the core components of this work and pedagogies for learning or improving it (e.g., Boerst et al., 2011; Ghousseini, 2015).

3. The Literature Review

Selection of studies was carried out in two phases. The first phase involved manual searches in a selection of research journals in mathematics education. The following journals were included: *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *The Journal of Mathematical Behavior*, *Mathematical Thinking and Learning*, *Mathematics Education Research Journal*, *ZDM*, *International Journal of Science and Mathematics Education*, and *Journal of Mathematics Teacher Education*. The first six were the same journals that Ryve (2011) considered in his review of research on discourse in mathematics education, but I decided to add the last two since they have become prominent in recent years.

Searching the archives of these journals for studies on discussion in mathematics teaching that were published between 2000 and 2020, 35 articles were included after initial screening and coding. Based on what was learned from the coding of these articles, a second phase was initiated that included more systematic searches in the Eric database. From the first phase of review, I observed that relevant articles tended to have key words like *teaching*, *mathematics*, and *discussion* in the title or abstract. A search for peer-reviewed journal articles in English, with these keywords — teaching, mathematics, and discussion — as search terms in the title or abstract, gave 146 articles. After initial screening and coding, and after deleting duplicates from the first phase, 37 additional articles were included in the review. Altogether, a total of 72 studies were included from the two phases of the literature review.

In both review phases, studies were excluded from the review if they were 1) not empirical (e.g., theoretical articles or review articles), 2) not about discussion, or 3) not about mathematics (e.g., some articles in the second phase focused on discussion in other subjects).

To answer the first research question, studies were coded in terms of:

- focus of the study
- how (much) the study emphasized discussion
- problem of the study (generic problem that was approached in the study)

To answer the second research question, concerning what aspects of discussion were focused on, studies were coded according to the following perspectives:

- definition (if the study provided explicit definition of discussion)
- phase (what phase in the work of leading discussions that was emphasized)
- talk moves (if the study included emphasis on talk moves or similar)

- norms (if the study included emphasis on establishing norms for discussion)
- demands (if the study focused on knowledge demands of leading discussions)

The third research question concerned methods to study discussion, and the following aspects were considered to answer this question:

- participants (number of participants)
- level (e.g., primary, or middle school)
- teachers (e.g., future, beginning, or experienced teachers)
- setting (e.g., professional development or teacher education context)

To investigate the fourth research question, reference lists of all articles were scanned, and references that relate to discussion were identified. These references were counted and compared across the total set of articles included in the review.

To illustrate the coding of studies, I briefly describe the study by Langer-Osuna and Avalos (2015). This study initially came up in both searches, since it was published in one of the journals that was targeted in the first phase (ZDM), and since it had a clear focus on discussion in mathematics teaching in the abstract. The study focused on implementation of progressive classroom practices. The overall problem of the study was about "how teachers facilitate discussion." The primary focus in the analysis was on the orchestration of discussion, and talk moves were discussed, although the authors focused more on students' use of talk moves than on the teacher's use of talk moves as a tool to orchestrate the discussion. The authors mentioned that norms have been established in the classroom, but the study as such did not have an explicit focus on the establishment of norms, and the study did not focus on knowledge demands of the work of leading discussions. It was a small-scale study that analyzed data from the grade 4 classroom of one practicing mathematics teacher in the United States. The setting was professional development of in-service teachers. Although the authors defined various kinds of talk that might take place during discussions, they did not define the concept of discussion as such. When considering the reference literature used, this study frequently referred to literature on dialogic education, like Littleton and Mercer (2013).

4. Results

Below is a presentation and discussion of results from the analysis of the studies in response to the four research questions.

4.1. Problems of the studies

Specific research questions are likely to differ across studies, and they are thus difficult to compare directly. Instead of comparing the specific research questions of the studies, I tried to identify the more general or overarching problems that the studies seem to address (Tab. 1). This corresponds with the way Hoover et al. (2016) identified problems in their review of studies of mathematical knowledge for teaching.

Tab. 1. Problems of the studies

Problem	No. of studies
What contributes to/supports discussion	21
How teachers facilitate discussion	11
What contributes to student learning in discussion	11
How teachers attend to students in discussion	8
What contributes to development of discussion	7
What contributes to participation in discussion	5
What students experience or learn from discussion	3
What demands teachers are faced with in discussion	2

By considering the problem statements in the articles — this includes the specific research questions, but also the overall framing of the problem in the studies — inductive codes were developed to describe more generic types of problem statements. As an example, Hintz and Tyson (2015) presented two research questions in their article: “1. How do an elementary teacher and his or her students listen to each other during a mathematical discussion? 2. How does the teacher support students to listen as mathematical sense-makers?” (pp. 301–302). I considered the overall problem in their study to be “How teachers attend to students in discussion”.

We notice here that 21 studies approached a generic problem of what contributes to or supports discussions. One subgroup of studies in this category investigated use of diverse types of technology to support discussion. For instance, Hensberry and colleagues (2015) investigated how simulations can provide a context that supports whole-class discussions, whereas Slavit (2002) explored how an electronic forum can support classroom discussions. Another subgroup of studies that focused on what contributes to or supports discussion investigated various kinds of tools or frameworks. For instance, Casa (2013) investigated how a “talk frame” can be used as a tool to support discussion in mathematics classrooms. In another study, Wu and colleagues (2009) explored use of graph organizers and the “mathematician’s chair” as tools to support problem solving discussions in mathematics. A third subgroup of studies focused on how teachers’ knowledge or beliefs might support discussions. For instance, Bray (2011) investigated how teachers’ knowledge and beliefs influenced the way they handled student errors in classroom discussions, whereas Cengiz and colleagues (2011) studied how mathematical knowledge for teaching influenced teachers’ instructional actions in discussions.

A second and related group of studies emphasized how teachers facilitate discussion. Many of these studies were small scale studies that investigated how one or a few teachers approach facilitation of classroom discussions in mathematics. Some studies involved attempts to try out unusual ways of organizing discussions, for instance by introducing random grouping of students (Carter, 2019). Other studies unpacked different components of the work of facilitating discussion. For instance, Selling (2016) explored what teacher moves that were used to make mathematical practices explicit for students, and what was made explicit about these practices. In another study, Zolkower and Shreyar (2007) described the moves a teacher made to press students to express their mathematical thinking verbally in "thinking-aloud discussions".

A third group of studies focused on what contributes to student learning in discussion. Many studies analyzed what teacher actions that support student learning in discussions. An example is the study by Vale and colleagues (2019), who explored the leading of problem-solving discussions through lesson study. They found that teachers attend more productively to student responses — and can select appropriate student responses — when they have solved the problems for themselves first and engage in the practice of anticipating children's responses. In another study, da Ponte and Quaresma (2016) found that an appropriate level of challenge in problems was necessary to foster productive learning situations. In yet another study, Lim and colleagues (2020) found that teachers' use of follow-up questions can stimulate student learning and participation in discussions.

A fourth group of studies considered how teachers attend to students in discussion, which can be an aspect of the facilitation of discussion. Attending to students is closely related to teacher noticing, and Scherrer and Stein (2013) explored how an intervention influenced what teachers notice during classroom discussions. In another study, O'Connor (2001) investigated how a teacher's use of questions in discussion can stimulate students' thinking. In yet another study, Hintz and Tyson (2015) investigated the listening of teacher and students in classroom discussions. They highlighted "complex listening", which involves listening evaluatively, interpretively, as well as hermeneutically, and they argued that this way of listening is necessary to facilitate mathematical sense-making. For the teacher, this involves, among other things, to take a "listening stance" and be curious about students' thinking.

Other groups of studies focused on what contributes to development of (e.g., Aguirre and del Rosario Zavala, 2013), or participation in (e.g., Ing et al., 2015), mathematical discussions. A few studies investigated what students experience or learn from discussion (e.g., Gellert and Steinbring, 2014), and two studies focused on the demands that teachers are faced with in discussion (e.g., Leikin and Dinur, 2007). Finally, there were four categories that only had one study each.

4.2. *Aspects of discussion in focus*

Tab. 2. Aspects of discussion that are in focus

Focus in studies	No. of studies
Orchestration	37
Talk moves	18
Norms	10
Demands	10
Definition	7

More than half of the studies focused on the orchestration of discussion, and many of them also involved some focus on talk moves. Studies applied different notions of "moves" teachers can use to support discussions. Some referred to them as "teacher moves," or "didactic moves," whereas others referred to a commonly known list of "talk moves" (Kazemi and Hintz, 2014). Whereas much emphasis has been placed on moves teachers can make as they talk in discussions, less emphasis has been made on the complexity of listening in discussions. This was the primary focus in the study by Hintz and Tyson (2015), who considered different forms of listening in conjunction with other kinds of moves teachers can make while leading mathematical discussions. Despite different definitions and terms, all these studies had some emphasis on moves teachers can or should make during discussions.

Ten studies focused on norms. Again, I was generous in my interpretations and included studies that only briefly mentioned norms although the study was not primarily about norms. Only a few studies had an explicit focus on norms or development of norms for discussion. For instance, Kline (2008) had a primary focus on establishing a classroom environment for discussion. Based on her experience from long-term professional development with teachers, she unpacked what needs to be considered "when establishing a tone that encourages children to think during whole-group discussions, including addressing children's diverse thinking approaches and using their incorrect solutions" (p. 145).

Another group of ten studies had a focus on knowledge demands that might be entailed in teaching with discussion. An example is the study by McCrone (2005), where a fifth-grade classroom was observed over a full year, focusing on how student contributions to discussions develop over time, and the challenges the teacher was faced with in this work. The analysis in this study also involved emphasis on negotiation of sociomathematical norms, like the common belief that the teacher is the one who has authority to decide whether proposed solutions are valid.

Finally, only seven out of 72 studies defined what they meant by discussion — again with a generous interpretation of what constitutes a definition. For instance, McCrone (2005) defined discussion as "one aspect of discourse, namely, to describe the nature of small group and whole group discussions centered on making sense of mathematics problems" (p. 112). A more concise definition was given by Tyminski et al. (2014) who referenced Pirie and Schwarzenberger (1988, p. 461), who defined a mathematical discussion to be "purposeful talk on a mathematical subject in which there are genuine pupil contributions and interaction." Tyminski et al. (2014, p. 465) also clarified that a "discussion can be called mathematical to the extent it contributes to students' mathematical understanding and reasoning." These were exceptions, however, as most studies did not define discussion, and many applied different interpretations of discussion in their studies.

4.3. Methods for studying discussion

Several aspects were considered in the analysis of methods for studying discussion, but the most striking difference was found when comparing the sample size of studies (Tab. 3).

Tab. 3. Sample size of studies

Sample size	No. of studies
Small scale (<10)	55
Medium 1 scale (10–29)	5
Medium 2 scale (30–70)	2
Large scale (>70)	6

We notice that 55 of the 72 studies were small scale studies with less than ten participants; many studies only focused on one teacher[2]. Three of the large-scale studies focused on students rather than on teachers and thereby had a larger sample size than if they had focused on the teachers. The emphasis on how students experience discussions was shared across these studies, and some also focused on how students learn from discussions.

This tendency of mostly small-scale studies with a qualitative design is not surprising, since most studies focused on various aspects of the interactions between teachers and students in discussion, and few, if any, instruments have been developed to measure aspects of discussions quantitatively.

The study by Bragg (2012) was one of the few large-scale studies, and the focus was on investigating how game playing might contribute to mathematical learning. The study measured the impact of an intervention by use of achievement tests. The participants ($n = 112$) were thus students. In another study, Ing et al. (2015) investigated students' participation in discussions, and again the sample consisted of students ($n = 71$). These students were from six classrooms, and the team spent six months observing the classrooms before initiating the formal phase of data collection. Lesson videos were coded with a particular emphasis on student participation — primarily in terms of student explanations and engagement with the ideas of other students — and teachers' support. The study also included a written post-test of students' thinking. In yet another large-scale study, Lemonidis and Kaiafa (2019) measured the effect of including storytelling strategies on students' learning of fractions, and they compared results from an experimental group and a control group (each with $n = 38$).

The study by Jackson et al. (2013) was among the few large-scale studies that focused on the teachers ($n = 165$), and this study explored the relationship between use of cognitively demanding tasks and students' opportunities to learn in discussions. These researchers video recorded mathematics lessons over two days with each of the participating teachers, which constituted a total of 460 lessons that were analyzed using

[2] Four studies did not provide clear information about sample size, so therefore the total in table 3 only adds up to 68.

"an expanded version of the Instructional Quality Assessment" (p. 658). This instrument was developed from the Mathematical Tasks Framework, and it targets the interactions between teacher and students in discussions. The instrument identifies opportunities to learn, but it does not measure student learning. This can be considered among the most significant studies in the review, and it provides an interesting example of a study that involved the use of frameworks to measure aspects of mathematical discussions.

Although some of the large-scale studies were impressive in size and scope, many of the small-scale studies also provided important contributions. For instance, O'Connor's (2001) study unpacked important aspects of leading mathematical discussion, with a particular emphasis on how teachers' use of questions in a position-driven discussion might support the development of students' mathematical thinking. This was one among several studies that illustrated how case studies of one teacher can also provide important insights into the work of leading mathematical discussions. Another example was the study by McCrone (2005). Whereas Ing et al. (2015) studied students' participation in discussion in a large-scale study, McCrone (2005) investigated the development of student participation in discussions by following a teacher and her grade 5 students over a year. Through a longitudinal qualitative study, she unpacked how the role of the teacher and her use of questions might stimulate student participation by supporting their development from non-active to active listening, and to draw upon other students' thinking.

4.4. Reference literature

In his review of research on discourse in mathematics education, Ryve (2011) identified several core theoretical and epistemological traditions that were referenced. When reviewing research on discussion in mathematics teaching, few theoretical frameworks or traditions emerged. Surprisingly, it was difficult to identify any core body of literature, and there was significant variation in the literature that was referenced across studies. Candidates for a core body of literature from before 2000 were:

- Ball's (1993) unpacking dilemmas of teaching elementary mathematics
- Lampert's (1990) study of altering roles in mathematics classroom discourse
- Stein et al. (1996) with their analysis of cognitive demands in mathematical tasks
- Yackel and Cobb (1996) on sociomathematical norms in discussions

Even though these were among the most frequently used references, each of them was only referenced in a few studies (5–10 studies). More recent candidates for a body of core literature on discussion (published after the year 2000) were:

- Chapin et al.'s (2009) sourcebook on classroom discussions in mathematics
- Kazemi and Hintz (2014) with their book on structuring and leading discussions

- Lampert's (2001) seminal work on teaching with problems
- Stein et al. (2008) and their five practices for orchestrating discussions

Again, the number of references to these more recent candidates for core literature was relatively small. It was also surprising to notice that many core references from the general literature on discussion in education (e.g., Dillon, 1994) were mostly absent from the list of references in studies that were included in the review.

5. Concluding Discussion

Based on the present review of research on discussion in mathematics teaching, I will highlight three issues that are worth attending to in research on discussion in mathematics teaching. With each of these issues, I will point at limitations of research and suggest efforts to progress.

The first issue revolves around conceptual clarity. Dillon (1994) noticed that there was confusion of terms in studies of discussion in education, and he stressed the importance of distinguishing discussion from other types of interactions and providing clear definitions. Similarly, in his review of research on discourse in mathematics education, Ryve (2011) found a lack of conceptual clarity, and he argued that this might threaten the cumulative nature of research. The present review shows that few studies of discussion in mathematics teaching define what they mean by discussion, and studies tend to use the term in diverse ways. This is a significant challenge for our field. Coherence of terms might not be a requirement in research, but clarity is. If studies fail to clarify what they mean by the core constructs they investigate — like discussion — successive studies will be hard pressed to build on them. Ryve (2011) argued that this was critical for research on discourse, and I argue that this is equally important in research on discussion in mathematics teaching. An everyday concept like discussion might be particularly elusive in this respect, since everyone uses it, and everyone thinks they know what it means.

A second issue relates to methods for studying discussion in mathematics teaching. It is interesting to notice that most studies of discussion in teaching are small-scale, qualitative case studies. These studies provide illustrations of what discussions might look like, and they explore various aspects of discussions — often providing existence proofs. This tendency might be related to a general lack of instruments to measure important aspects of discussion in mathematics teaching. In one of the few quantitative studies, Jackson et al. (2013) applied an adapted version of the Instructional Quality Assessment, and this is one candidate measure for use in research on discussion in mathematics teaching. In their review of research on the core practice of leading classroom discussions, Jacobs and Spangler (2017) also noted that most studies were inductive case studies, and they suggested that development and use of observation instruments can be a promising method for studying discussion in mathematics teaching. Similarly, in their review of research on mathematical knowledge for teaching, Hoover et al. (2016) emphasized measurement work — not simply use of measures in correlational studies and assessment of practice, but development of measures as tools that may contribute to conceptualization of core constructs that are studied.

Finally, a third issue in research on discussion in mathematics teaching relates to the underlying conception of teaching. Research on mathematics teaching builds on a long history of research on teaching, where a process-product paradigm has been prevalent. Within this paradigm, teaching is defined as something teachers do to help students learn, and studies of teaching have often considered process variables concerning teachers' actions or performance in relation to outcome variables, like some measure of student learning. Similarly, research on discussion tends to focus on the actions or moves teachers make and the underlying goals of these moves (Jacobs and Spangler, 2017). Focusing on teacher moves makes sense, in particular within a context of teacher education where the emphasis is on learning to lead discussions. Yet, it might be productive to shift toward a conception of teaching as professional work, where studies focus more on entailments of this work than on how teachers carry out the work. Ball (2017) and others have initiated a similar shift in research on mathematical knowledge for teaching, where the emphasis is on investigating problems, dilemmas, demands, or tasks that are entailed in the work of teaching, and this has laid the foundation for productive developments in this area of research (cf. Hoover et al., 2016). Research on discussion in mathematics teaching, however, still tends to emphasize actions by teachers or students in discussion, and the effectiveness of such actions. I suggest that a shift toward unpacking entailments of the work of leading mathematical discussions might stimulate further progress. Identifying demands of the complex work of leading discussions might lay the foundation for developing a professional practice that acknowledges the skills and knowing that are involved in leading mathematical discussions. One aspect of the work of leading discussion that might benefit from further research is the complex work of developing a classroom climate for discussion. Too many studies of discussion in mathematics teaching tend to investigate the orchestration of discussions in a context where such norms have already been established.

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https://doi.org/10.1142/9789811287183_0033

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Arguments or Findings Regarding Language as Resource for Mathematics Learning and Teaching

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ABSTRACT Language as resource is a challenging research approach in mathematics education because it examines how language can function to support mathematics learning and teaching. The approach originally started to develop in response to discourses of non-mainstream languages and cultures as problems or obstacles to mathematics teaching and learning. In this text, I revisit and bring together four empirical studies in order to discuss four major findings that are arguments to explain the complexity and importance of the language as resource approach. These four arguments are: 1) the huge potential of all languages to make mathematical meaning; 2) the critical realization of some languages in the mathematics classroom; 3) the critical communication of mathematical meaning in classroom teaching talk; and 4) the huge potential of teaching talk to support mathematics learning for understanding.

Keywords: Language as resource; Sociocultural language-based stances; Mathematics learners and learning; Mathematics teachers and teaching.

1. Language for/as Communication and Language as Resource

I remember very well my years as a secondary school mathematics teacher. In 1997, I met Sergev, the fifteen-year-old son and brother of a Bosnian migrant family who had arrived in Barcelona less than one year earlier. He spoke a little English and Spanish, and three days per week attended the special lessons in the school for students who were in the process of learning Catalan, the language of instruction. For the rest of the days, I was his mathematics teacher in the ordinary classroom where he seemed to enjoy doing mathematics. One day, another student solved a quadratic equation on the board and wrote down the numerical solution “3”, without any sign. She had explained in Catalan that the negative solution made no sense in the context of the word problem given. When the bell rang, Sergev remained in his seat and I approached him. He then went to the board, pointed at the number symbol 3, and used Catalan, Spanish and English, none of which were his home languages, to share the following observation:

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A l'escola meva este número es siempre positive. Aquí es positive i negative.
[In my school this number is always positive. Here this number is positive and negative.]

Sergev was trying to adapt what he knew about mathematical symbols and quadratic equations to figure out why there was only one number written as a solution. I thought that he had possibly not understood or made sense of the explanation in Catalan regarding why only the positive solution had been written on the board. Since he knew there should be two solutions, he reinterpreted the number 3, with no sign accompanying it, as a way of representing both the positive and negative solutions of the equation. We engaged in Catalan-Spanish-English mathematical talk about the negative solution and in reasoned conversation regarding what counts as a solution of an equation that is modeling an everyday context. Over the course of the conversation, Sergev insisted on the distinction between the solutions of the equation and the solution of the problem. Multilingual talk, rather than linguistically accurate talk in just one language, created opportunities of mathematics teaching and learning other than those possibly created by the written symbols and algebraic expressions. Together, we managed to make talk work as a resource at the service of communication and mathematical understanding. Our flexible use of language allowed us to move from how to proceed to solve a quadratic equation towards what counts as a solution.

The episode with Sergev illustrates in practice the basic idea that mathematics teaching and learning involves the successful use of *language for/as communication* (for the communicative approach to mathematics teaching and learning, see, e.g., Morgan et al., 2021; Pimm 1987/2021; Planas et al., 2018). We developed spoken communication that helped us to overcome our respective lack of linguistic knowledge of the first languages of each other, and to engage in reasoned mathematical discussion. Sergev and I experienced both language *for* mathematical communication and language *as* communication of mathematical understanding, even if at times this could be limited communication. In close connection to these notions of language for/as communication, *language as resource* (Planas, 2018, 2019) plays an important role in classroom research on mathematics and language, with the meaning for "resource" assumed to establish a discourse of language as supportive and respectful of school learners, their learning needs, and the values of their cultures, communities and funds of knowledge. There is, however, a diversity of interpretations regarding what is meant by this notion or approach and what it adds to the communicative approach, influenced by the diversity of interpretations regarding what language is. Many variables affect the choice of how to define "language".

1.1. A sociocultural understanding of language

In the context of multilingual classrooms, there is a tendency to start at the level of entire "national" language systems, or the set of linguistic resources that children first develop in their homes. This is a problematic perspective that leads to a paradox or tension that is difficult to solve. Decisions about what counts as a distinct language are political: do we count varieties of South American Spanish, of Castilian, of Indigenous Creole Spanish, of …? In order to further define language and to work out aspects of specific varieties of language, research in mathematics and language tends to focus on special cases of linguistic peculiarities, such as specific groups of people who supposedly live under similar linguistic-cultural conditions, for example people who migrated from South American regions to Catalonia. This form of analysis keeps producing normative attributions of equality within groups and categories of difference are created, despite the attempt to overcome them. This paradox does not seem to be solvable and can only be countered by permanent awareness of the inevitably reductionist access to any group of people according to linguistic-cultural backgrounds. The problem of distinguishing between varieties of languages extends to distinctions between "everyday" registers or languages (whose "everyday"?) and mathematical languages (whose "mathematics"?). Once the scope of the definition of language is determined, often with reasons for limiting the extent of multimodality, the question still arises as to what units of language are of interest. Linguists conventionally distinguish nested ranks of grammatical units such as morpheme, word, phrase, clause and sentence, in combination with levels of analysis such as utterance, interaction, text, genre, register and discourse. Depending on which units are privileged, mathematics education research on language addresses the processes of mathematics teaching and learning very differently.

In my work, I adopt sociocultural stances in the understanding of language not just as a tool for communication, but as one of the most important means of experiencing, interpreting and shaping the worlds around us through multiple processes of meaning making (Gee, 2004; Halliday, 1985; Wells, 2009). Language is thus part of what constitutes mathematical learning, and is made visible as learners talk among themselves or with others, and in how they communicate meanings and make sense of them. In line with this definition of language, Planas (2018) conceptualized the language as resource approach in classroom mathematics research by building on two arguments: 1) the huge potential of all languages to make mathematical meaning; and 2) the critical realization of some languages in the mathematics classroom. In that paper, I also claimed that enhanced talk amongst diverse languages and meanings in the construction of the language of the classroom would yield further opportunities to learn school mathematics. Since then, broadening the research to include the focus on the languages of teachers in mathematics content teaching has provided substantial insights by bringing up two more arguments: 3) the critical communication of mathematical meaning in classroom teaching talk; and 4) the huge potential of teaching talk to support mathematics learning with understanding. In the rest of this paper, I elaborate on these four arguments by presenting data and findings from four classroom studies conducted in schools of Barcelona over the last two decades. Collaborations with colleagues in other countries and continents with similar research interests suggest that the findings seen in this particular context are also distinctive of mathematics teaching and learning in a larger quantity of classrooms, this being typical of the arguments that support the importance of the language as resource approach.

The two majority languages spoken in Barcelona are of the Romance family: Catalan and Castilian, as Spanish is known in this part of Europe. Like in many other parts of the world, the language policy in education has long been an issue, with a monolingual orientation that values one language over others. The Catalan-only language policy is used as a means to address social inclusion since it is expected to help integration and ease historical tensions coming from various decades of Catalan being forbidden (for recent discussion of the political and linguistic intricacies of the case of Catalonia, see, e.g., Aramburu, 2020). As would occur with a monolingual Castilian-only policy, several challenges are posed to school students who are not expected to learn mathematics through their home languages, as well as to teachers who are not expected to promote languages other than Catalan in classroom interaction. A consequence is that while the language policy in education was created to address social inclusion, it has unintended effects on different language groups by increasing the gap in their access to classroom participation and mathematics learning opportunities. This is the context in which I conduct research and developmental work, in the belief that all students, regardless of their languages, cultures and socioeconomic backgrounds, have the right to learn mathematics with understanding.

2. The Huge Potential of All Languages to Make Mathematical Meaning

I start with the argument first reported in the field literature, regarding the recognition that all languages have the potential to support mathematics teaching, learning, and content meaning making. Mathematics education research guided by this argument and built on views of language as social, challenges monolingualism as the acceptable norm (Planas, 2021a), and invites research and practice that consider the students' home languages and how they are present in and mediate the generation of mathematical meaning across discourse practices. Two decades of findings in classroom research on mathematics and language (Planas et al., 2021; Essien and Msimanga, 2021) reveal that multilingual students engage in multilingual practices that enable them to complete mathematical tasks given or presented in the language of instruction. The successful use of all the languages at their disposal, through flexible language switching and translanguaging (Sapire and Essien, 2021) has been proved to be common in situations of working on meaning in order to clear up misunderstandings, follow up more complicated mathematical reasoning, and ask one another questions.

Drawing on the argument that all languages are potential resources for mathematics learning, Planas and Setati (2009) reported strategies of bilingual migrant students in Barcelona in order to successfully deal with mathematical tasks introduced in the language of instruction. With minimal pedagogical intervention from the classroom teacher, bilingual students born in South America switched languages to provide continuity to small group work by means of selecting one language and then shifting to the other depending on the conceptual complexity of the task. In that study, the collection and analysis of lesson data was aimed at investigating shifts between Catalan [C] and Spanish [S] in mathematical communication. The research questions were: Do Spanish-dominant bilingual students in Catalan classrooms switch languages during mathematical activity? If so, what are some of the factors that seem to promote language switching with a group of these students in specific lessons? An assumption framing these questions was that the contrast between mathematical participation in a small and the whole group was related to the language mostly spoken in the interaction.

All the lesson transcripts were analyzed using a version of sociocultural discourse analysis (Gee, 2004; Halliday, 1978, 1985), that is, an analysis being concerned with the content, form and function of spoken language and the processes through which shared meaning and understanding are developed in social context. This included coding themes and quantifying and comparing code frequencies to arrive at patterns which were then exemplified by lesson extracts that showed language forms and functions across participation in mathematical activity. The focus of the analysis was therefore on both the activity around mathematical content and the language forms in use. The flexible switching between Catalan and Spanish in student interaction, and hence the permeability of mathematical participation through these two languages, was one evident pattern. The analysis allowed us to uncover a second pattern which served to corroborate our initial assumption. We detailed the systematic change to the home language when there was also a change in the nature of discourse from descriptions of observations and procedures to mathematical explanations. We thus concluded that the students switched to their shared home languages as soon as the conceptual demands in mathematical talk increased. Finally, these were the two patterns specified:

- Bilingual students flexibly use their languages when engaged in describing observations and procedures in small group work.
- These students tend to use their common home languages when engaged in conceptual building and mathematical explanations.

As an insight into bilingual mathematics learning and creative translanguaging practices in small group work, Tab. 1 presents a transcript of data from Planas and Setati (2009). It belongs to the part of a secondary school lesson unit called "Our dynamic planet," which included mathematical activities that encouraged students to pose questions and solve problems in real contexts. This unit had been designed the year before by a group of teachers in the school as part of the development of teaching materials in support of mathematics learning with understanding. In the third lesson, the teacher wanted the students to think about "How can you mathematically represent a tornado?" From the transcript below, it is clear that language switching between Catalan and Spanish occurs for communication and discussion on the possibilities of graphically representing, through combinations of plane isometries, the spatial movements that make up a tornado. Máximo and Eliseo use their home language to discuss the arrow representing the perpendicular movement around the tornado axis and its orthogonal axis (horizontal and vertical) on the task sheet. The group discussion, of which the transcript in Tab. 1 is a fragment, follows a sequence of descriptions of characteristics of a tornado and explanations of mathematical representations through reasoning and concept building around composition of isometries.

Tab. 1. Transcript of lesson interaction (Planas and Setati, 2009, pp. 43, 48)

Máximo:	[C] *Hem de decidir les fletxes que dibuixem i ja està.* [We need to decide the arrows that we draw and that's all.]
Eliseo:	[C] *Primer pensem les fletxes, després les dibuixem i després en parlem.* [First we think about the arrows, then we draw them and then we talk about it.]
Máximo:	[S] *Esta idea de las flechas no es fácil. Tenemos que imaginar los diferentes movimientos que existen dentro del tornado.* [This idea of the arrows is not easy. We have to imagine the different movements that exist within the tornado.]
Eliseo:	[S] *Una flecha tiene que ser una línea recta para que el tornado baje. Tenemos la t para la translación.* (…) [An arrow needs to be a straight line for the tornado to go down. We have the t for the translation.] (…) [C] *El que hem de fer és entendre què és un tornado i després li busquem un nom.* [What we need to do is to understand what a tornado is and then we find a name for it.]
Luna:	[C] *Hem de fer les fletxes com ahir?* [Do we need to make the arrows like yesterday?]
Eliseo:	[C] *El que hem de fer és entendre què és un tornado i després li busquem un nom*. [What we need to do is to understand what a tornado is and then we find a name for it.]

The nature of the mathematical discourses and how bilingual or multilingual students engage in them through bilingual practices has been researched in various other contexts in which English is the language of instruction. González, Andrade, Civil and Moll (2009) with Latino bilinguals in Arizona, and Setati and Adler (2000) with multilingual students in South African townships, characterized language switching as a successful strategy for convergence towards school mathematics in the English language of instruction. Language switching was claimed to be a "natural" unproblematic effect of what multilinguals do with language in mathematics classrooms, even when the educational policies and classroom norms are differently oriented and refrain students from flexibly using their languages to communicate their thinking. It was additionally claimed that multilinguals within mathematics classrooms behave as people who speak more than one language generally do, and that language switching does not necessarily stand for lexical gaps, linguistic difficulties or deficient language abilities. Like in Planas and Setati (2009), these studies show classroom discourse to be the site in which everyday languages, the school language in general and the school mathematics language in particular become connected.

3. The Critical Realization of Some Languages in the Mathematics Classroom

The episode with Sergev in the introduction suggests the role of the teacher in establishing a favorable climate for using all the languages available at the service of communication and mathematics teaching and learning. Establishing such a favorable climate is, however, not straightforward. In line with this, the second argument refers to the different valorization of the languages in the mathematics classroom, and hence the different distribution of mathematics learning opportunities amongst their speakers. In my context, for example, influential ideologies underlying language policy situate Castilian as a language that is more valuable than other forms of Spanish spoken by South American migrants, who tend to be seen as powerless groups of people (Planas, 2021a). Far from viewing language as a neutral object in the classroom, it is therefore necessary to address questions concerning the several visible and invisible messages that are sent to students (who are in particular language users) through the differing valorizations of languages (and language uses). It is complex to know whether language uses provoke valorizations or valorizations provoke language uses. Both directions are at the heart of debates concerning multilingual mathematics classrooms: Do students facilitate particular positions in the classroom by the fact of using a language in their discussions with others at certain moments? Is it that talking with some of the other participants in the classroom leads to the use of a certain language alongside the creation of particular positions? The debate about what comes first, however, is not directly related to the educational and pedagogic debate.

As said by one of the teachers in Barcelona, the situation is more complex than just letting South American migrant students use their home languages in the classroom. In Planas and Setati-Phakeng (2014), valorizations between the languages and their users, on the one hand, and the mathematical discussions, on the other, were documented in a variety of secondary and primary school lessons and in interviews with mathematics teachers in the urban areas of Barcelona, Johannesburg and Pretoria. Like in Planas and Setati (2009), in that study, the data were transcribed and analyzed using a version of sociocultural discourse analysis to examine how teachers used talk (in the classroom lessons and the interviews) to represent their students as learners and knowers of mathematics. A combination of qualitative and quantitative methods enabled the study of how the realization of the languages of students to do mathematics was mediated by specific valorizations of the teachers. Again, this implied coding themes and quantifying and comparing code frequencies to find commonalities which were then exemplified by lesson and interview extracts that showed language forms and functions across or about participation in mathematical activity. On this occasion, the analysis allowed us to uncover the following major themes or findings:

- [From the interviews] Teachers referred to students whose home languages and knowledge were not helping them to speak mathematics accurately.
- [From the lessons] Teachers focused on the mathematical content but also on the linguistic accuracy in the talk of the students in the interaction.

The transcript below illustrates a classroom conversation between the teacher and Luis, a migrant student from South America whose knowledge of the language of instruction was good (Tab. 2). Luis is provided the opportunity to learn both that *senar* is the Catalan name for the notion of odd number, and that his statement of a mathematical property is relevant to the understanding of the notion. During group work aimed at talking $2x + 1$ with words, the use of specialized vocabulary was required. Odd [number] is not a priori a difficult word but the Catalan name for it, *senar*, is quite different from the Spanish name, *impar*. Instead of elaborating on the arithmetic meaning that can be given to an algebraic expression, the teacher interrupted the mathematical discussion started by Luis, who had found a way of referring to the category of odd numbers, "even plus one", to put the focus on the language of instruction. This teacher acted differently in other turns: ''So you don't spend time on these things", or "Even plus one is a good property to remember". The focus on the names in the language of instruction somehow unfocused Luis' thinking of odd numbers versus even numbers, both as relative to the grouping process with 2 as the number to group by, and 1 or 0 as the remainders.

Tab. 2. Transcript of lesson interaction (Planas and Setati-Phakeng, 2014, pp. 888−889)

Luis:	[C] *He estat molt temps perquè no podia recordar la paraula en català de* [S] *l'impar.* [I spent a lot of time because I couldn't remember the Catalan word for odd.]
Teacher:	[C] *Doncs no perdis temps en aquestes coses, busca una altra manera de dir-ho.* [So you don't spend time on these things, you find another way to say it.]
Luis:	[C] *Umm… Puc dir* [S] *par más uno?* [Umm… Shall I say even plus one?]
Teacher:	[C] *Ara que saps la paraula, eh?... senar, digues-la.* [Now that you know the word, eh?… odd, you say it.]
Luis:	[C] *I ara també sé una altra manera, per si em torno a oblidar de la paraula.* [And now I also know a different way, just in case I forget the word again.]
Teacher:	[C] *Millor que no la oblidis. És una paraula important en matemàtiques. I sí! parell més u és una bona propietat per recordar, ser consecutiu d'un nombre parell.* [You'd better not forget it. It's an important word in mathematics. And yeah, even plus one is a good property to remember, that of being consecutive to an even number.]
Luis:	[C] *I millor que no oblidi la propietat!* [And I'd better not forget the property!]
Teacher:	[C] *I tant! És més important arribar a conèixer la propietat.* [Yeah! It's more important to get to know the property.]

Although the home language of Luis was heard in the small group discussions and in social talk in his school, it was not the language of whole class communication. Students like Luis were able to participate in many aspects of the mathematics classroom but for him to participate in the whole class discussions he had to be singled out by the teacher explicitly switching to Spanish or inviting him to participate by accurately using the language of instruction. In our observations we did not witness any instances of Spanish dominant students in classrooms where they were in a minority speaking up in Spanish in front of the whole class, unless the teacher had directed the question to them. Although this singling out was probably well received in the schools in Barcelona due to the affiliation to Spanish and the overall bilingual environment, still we wonder about how students in this age group perceive being treated differently from their peers through a language switch or through mentions to their learning of the language of instruction, e.g., "Now that you know the word…"

There is no doubt that language ideologies have an impact on the students' lives at multilingual schools and on their mathematical learning. Learning is often judged from the way it is communicated, and communication has a great deal to do with languages. Such ideologies are instilled so deep inside a society that students sometimes anticipate what will be the effects of certain uses of their languages, and thus rearrange their communication opportunities. Research on this topic has stated the sociopolitical dimension of learning mathematics in multilingual classrooms, but also in classrooms with students who are linguistically disadvantaged for a variety of reasons, including impoverished socioeconomic status. In the example with Luis both constraints and opportunities can be uncovered in how this student goes on with his mathematical learning in a classroom in which the language of instruction is not a home language.

4. The Critical Communication of Mathematical Meaning in Classroom Teaching Talk

The third argument or set of findings around the complexity and importance of the language as resource approach refers to the fact that the communication of mathematical meaning in classroom teaching talk is not always sufficiently explicit or precise. Communication in the mathematics classroom, as in all other language contexts, is made up of communicative intent and intended meaning on the one hand, and communicative effective function and interpreted meaning on the other. Mathematics teaching and teachers need to support students in their communication of the intended mathematical meaning, but also need themselves to successfully resolve their communicative intents of mathematical meaning. One of the challenges with mathematics teaching talk is that the intended content meaning is not always communicated as clearly as expected.

In Planas (2019, 2021b), this discussion was addressed for the case of algebraic contents through the analysis of a number of instances of two teachers' talk in their school lessons. The analysis was guided by the identification and interpretation of classroom teaching talk with meaning potential to support the communication and learning of specific algebraic contents. In the teaching of equations, for example, teaching talk with potential to support the communication of specialized meaning for the concepts of algebraic equivalence and equal sign was revealed as both crucial and critical. Once more, a version of a sociocultural discourse analysis particularly framed within Functional Linguistics (Halliday, 1978, 1985), allowed the construction of two main themes:

- Teachers give names and explanations with potential to support the communication of meaning for specific mathematical contents.
- The use of some other names and explanations in teaching talk could have increased the opportunities of communicating important content.

These findings were corroborated in a recent investigation (Planas, Alfonso and Rave-Agudelo, submitted) with data from seven transcripts of lessons of four secondary school teachers and two transcripts of one primary school teacher. These were all representations of content lessons from design-based studies in which I had collaborated with teachers and doctoral students in seminars to create the tasks. These data were now imported to identify quality aspects of teaching talk at the word and sentence levels of language. The first column of Tab. 3 lists the five mathematical contents that were the object of learning in the one or two lessons with each of the five teachers. In the classroom studies, the teaching had not been prepared or planned to meet specific learning challenges. To assess teaching talk, we took decisions as to which content learning challenges might need support in talk oriented to teach the curricular content in play. We selected two learning challenges per content from the specialized literature; for this, we were further advised by five colleagues with expertise in those areas of research and who were knowers of the local mathematics curricula and school system. We realized that names and explanations in teaching talk can be interpreted by considering how they support the overcoming of content learning challenges which are highly predictable regardless of whether or not manifested in school students' talk. Since the lesson transcripts included students' talk, we explored both cases. This distinction is not trivial given the need in teaching talk to support content learning challenges even when their manifestations cannot be traced in the available products of the students. Some or even many of the learners might be experiencing important content learning challenges selected from the specialized literature that were not made public in talk or written products.

Tab. 3. Elements and examples of teaching talk (Planas, Alfonso, and Rave-Agudelo, submitted)

Content	Challenge	Names and Explanations
Probability	Equiprobability bias (Green, 1982)	It is possible or impossible, not the most possible, but it can be *the most likely.* // These eleven outcomes are *not equally likely or equally probable.* // *From impossible to certain,* we can find *very unlikely, not likely, likely…*
	Representativeness bias (Tversky and Kahneman, 1973)	This sequence may be *random-looking* but it is *not more probable.* // You can follow a random procedure and obtain an *ordered-looking* result. // Do not judge *more frequent* and *more probable* as if they were more familiar.
Areas	Area-perimeter confusion (Stavy and Tirosh, 2000)	Two-dimensional figures have *enclosing lines or perimeters* and also have *areas or enclosed regions.* // Let us think of *the smallest area for this perimeter.* // *Numbers related to areas* can be smaller than *numbers related to perimeters*.
Decimals	Whole number bias (Resnick et al., 1989)	By adding more *decimal places*, we do not make a number larger. // 0.61 is smaller than for example 0.62 because 0.61 is *one hundredth less*. // The smallest number has the greatest *number of digits after the decimal point.*
Algebraic equation	Equal sign misconceptions (Kieran, 1981)	The *equal sign* in the equation is a *relational symbol for equivalence*. // *Equal* indicates *balance* between the expressions on each side. // On both sides of the *equal sign* you write *equivalent expressions.*
Triangles	Prototypical thinking	*A triangle height* is *an either internal or external segment.* // The three *height feet* are not any point of each of the three

	(Clements and Battista, 1992)	sides. // It is not any angle either; it is *the perpendicular angle.*

The second column of Tab. 3 is for the content learning challenges whose consideration was traced in the transcribed talk of the teacher. For example, names and explanations with meaning potential for reducing equiprobability biased thinking were traced in teaching talk throughout the secondary school lesson where the experiment of throwing two dice and summing up the numbers was presented. In the lesson transcript, we found an explanation by the teacher in response to some evidence of biased thinking:

> Student: It is like throwing one die, but now you have more to choose, from two to twelve, and so the probability to guess is one over eleven.
> Teacher: These eleven outcomes are not equally likely or equally probable.

Prior to this conversation, the teacher had given explanations with important names whose meaning potential supported debiased probability reasoning, such as: "It is possible or impossible, not the most possible, but it can be the most likely." For the representativeness bias, we also traced some support in the form of content names and explanations in the teacher talk over the two lessons. The other four contents and five lessons each had two associated learning challenges and, for all of these lessons, clear support in the form of content names or explanations in teacher talk could not always be traced. The third column of Tab. 3 reproduces examples of content names (in bold) and explanations of importance for supporting all learners, especially those who face the associated learning challenge.

5. The Huge Potential of Teaching Talk to Support Mathematics Learning for Understanding

The fourth argument or set of findings regarding the complexity and importance of the language as resource approach refers to the enormous potential of teaching talk to support mathematics learning for understanding (or mathematics learning that implies leveraging multiple mathematical meanings and connections amongst them). Given the critical communication of specific mathematical meaning in classroom teaching talk, however, the realization of this potential cannot be taken for granted. Teachers need to develop professional expertise about how students use language to learn mathematics, and about how to use language to teach mathematics. For this, they need time and space to practice and work collaboratively with others towards productive mathematical talk in teaching. In order to improve the impact of mathematics professional development in classroom practice, increasing attention has been paid to work with teachers guided by teaching needs (Kazima et al., 2016), which may vary across cultures and groups of teachers (Essien, Chitera and Planas, 2016).

In this section I present the study conducted with two secondary school teachers, Jana and Maia, in the first round of the research and developmental project introduced in Planas (2019, 2021b). These teachers had expressed various concerns with the teaching of equations, which was an icon of mathematical knowledge in their schools with some of the families especially interested in test results on this content. They had several years of mathematics teaching experience, and worked in two different schools of Barcelona at the time of the study. The results of their students in the annual tests for the past years had shown poor conceptual understanding of equations, in contrast to the good performance in the resolution of equations and in the translation of word problems into algebraic expressions. Poor conceptual understanding was revealed in the beliefs that: two different equations can have the same numerical solutions; and an equation can be simplified into numerical solutions without an operation sign. One of the students had written in a recent test, "… and so the equation is $+2$ and -2", which reminds us of Sergev and his clear understanding of equations, solutions of equations and solutions of problems modelled through equations.

The response to the demand of the teachers was to interrogate their talk when teaching equations. In most of my collaborative experiences of work with teachers, they did not normally feel that the mathematical richness of the classroom practices can be hampered by under specificity in talk, nor did they tend to feel that language was a content in mathematics teaching (Planas and Civil, 2009). Hence, the response was in a sense a surprise for Jana and Maia who were, as they said, expecting to engage in developmental work oriented to learn and practice mathematical tasks of explanation and modelling around the qualities and types of equations in the local curriculum. We finally agreed on exploring possibilities of improving content teaching of equations through improving teacher talk. For this, five 90-minute sessions were held. Even though the two teachers graduated in mathematics, there was initial time for revising mathematical knowledge on the equation concept and preceding the work driven by language-based tasks. There was a session organized around the task in Tab. 4, whose English version, with only some of the underlined examples of lexical elaborations produced during the session, does not intend to reproduce these sentences as if they were exactly equal in meaning to those discussed in Catalan. The sentences selected show choices in language that can inform mathematics teachers in the use of sentences for teaching equations in the secondary school. Although the original sentences from the lessons of Maia or Jana made good sense and could be said to work in teacher talk, they were not followed or preceded by complementary sentences adding content meaning, and were not placed into pedagogic general talk or application of routines. Even so, by presenting the sentences separated from the lesson context in which they were said, and whereas this was done intentionally in the design of the developmental task, the potential regarding newer meaning increased.

For each given sentence (left column, Tab. 4), the written practice was organized into individual writing, group discussion of the two individual proposals, and final shared writing on the worksheet. Jana and Maia decided on the mathematical meanings whose communication they wanted to prioritize in the re-elaboration of the original sentences (right column, Tab. 4). I pushed them to think of the individual writing as an opportunity for referring to the meanings that they missed most in test results and conversations with students, and that remained unfocused in the original sentences. We also reflected on the fact that, during their period of elaboration, mathematical procedures are conceptual in nature (Kieran, 2013). Some of the examples in Tab. 4 were published in Planas (2021b, pp. 282–283). The alternative sentences are not solutions in the sense of being totally adequate; they are just more appropriate in the sense of being closer to the idea of communicating mathematical meaning within languages of equations beyond the representation of operational routines. They are also more appropriate in the sense of taking the opportunity to name relevant equation-related terms such as the names for the variable and the known and the unknown coefficient. In the final discussion, we talked about what we could possibly learn from the further elaborations of the original sentences. Jana said that even in the school lessons that are planned to practice the manipulation and resolution of equations, teacher talk can and must provide opportunities for students to step back and reflect upon what they are doing and why, in different lessons and teaching moments. Maia gave value to sentences with the names for quadratic equation, resolution and formula alongside descriptions such as "the formula for the resolution of a quadratic equation can be expressed in different ways using different letters for the variables." We discussed the questions of what is meant by the coefficients usually expressed with the letters a, b and c, and how this is told in relation to the concept of variable expressed with letters such as x, y and z. Regarding $ax^2 + bx + c = 0$, both teachers explained cases of students for whom a, b, c represented letters from which to generate numerical values. We concluded that the use of "letter" instead of coefficient and variable could be hindering conceptual learning. Two findings about knowledge gained by the teachers were:

Tab. 4. Examples of responses of the teachers in a workshop

What does the teacher say?	*What could the teacher say?*
We can solve a quadratic equation with a formula.	We can solve a quadratic equation with a formula. <u>That is, we can obtain the numerical values for x that solve the equation.</u>
We will modify the initial written equation.	We will modify the initial written equation. <u>In other words, we will look for ways of writing the same equation for the final application of the formula.</u>
Get a sequence.	Get a sequence, <u>which is to say, get a sequence of equivalent equations, or equations with the same solutions.</u>
Every equation, you change it a bit.	Every equation, you change it a bit. <u>By changing it a bit, I mean adding, subtracting, multiplying or dividing both sides with the same numbers so that the solutions do not change.</u>
You have to use the transposition rules.	You have to use the transposition rules. <u>That is, the rules for the generation of equivalent equations.</u>
You go mapping one written form to another up to the general formula on the board.	You go mapping one written form to another up to the general formula on the board. <u>All the equations will be the same because the same numerical values solve them all.</u>

- Direct naming of concepts through the use of specialized names in teaching offers opportunities to listen to important content meaning.
- Teaching talk that communicates content meaning in clear and precise forms increases opportunities of mathematical learning for understanding.

6. Conclusions and Ways Forward

In this paper, I have summarized sociocultural language-based research in mathematics education that has provided arguments for the importance of a notion of language as resource. This notion highlights the possibility that the access to and creation of opportunities for mathematics teaching and learning may be explained in part by the mathematical quality and clarity of the languages of teaching, rather than simply in terms of the capability of individual students or the skill of their teachers. The language as resource approach, as shaped by the arguments presented, encourages further investigation of the relationship between teaching languages and mathematical meaning communication in classroom encounters where participants use their languages for a diversity of purposes, activities and discourses. Few researchers have tried to relate the critical realization of some languages in school mathematics classrooms and the critical communication of mathematical meaning in teaching talk to access to and development of mathematics learning opportunities. Yet these relationships are of crucial interest and their understanding will possibly give rise to practical implications for all students, classrooms and mathematical contents.

The current state of sociocultural research on mathematics education and language suggests that strengthening and scaling up collaborations with teachers in schools and teacher educators in developmental settings may be part of the way forward. The last set of findings presented in the previous section, regarding developmental work with teachers to identify and enhance the potential of teaching talk to support mathematics learning for understanding, offers a promising path to follow in the direction of impacting positively on classroom practices and processes. Developmental work on the clarity and quality of content teaching talk over the course of classroom discourse practices, will enable mathematics teachers to focus on what they can do and say in teaching to increase their support for all students, regardless of their (everyday) languages, for mathematics content learning with understanding.

Acknowledgments

My thanks are for the students, families, teachers and schools who supported the studies, for my research group, GIPEAM, and for the Spanish and Catalan funding of Grants PID2019-104964GB-100 and EIN2019-103213, and SGR-2017-101.

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https://doi.org/10.1142/9789811287183_0034

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Specifying Mathematical Language Demands: Theoretical Framework of the Language Specification Grid

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ABSTRACT The didactic perspective on mathematics and language focuses on topic-specific instructional approaches for integrating language learning opportunities into mathematics instruction. From a didactic perspective, a sound and research-based specification of language demands is crucial for providing well-focused learning opportunities. For this, the paper (1) presents the topic-specific specification grid as a useful practical tool for specifying mathematically relevant language demands and (2) explains its underlying theoretical framework by making explicit the four incorporated lenses: epistemic, conceptual, functional, and discursive. The theoretical framework for specifying topic-specific language demands combines various linguistic theory elements and is empirically grounded in findings on typical language demands while mathematics learning.

Keywords: Meaning-related language; Form and function; Conceptual understanding; Discourse practices.

1. Introducing the Didactic Perspective on Language-Responsive Mathematics Teaching and the Need for a Theoretical Framework

Various scientific disciplines have identified students' language proficiency as an important factor for successful mathematics learning: From a psychometric perspective, strong correlations between students' language proficiency and mathematics achievement have been found in assessment studies (e.g., Abedi and Lord, 2001). From a linguistic perspective, these correlations have been explained by exploring the epistemic role of language as a tool for mathematical thinking and knowledge construction (Schleppegrell, 2007). From a sociolinguistic perspective, the language proficiency construct was extended from national languages to social language varieties, with emphasis on the school academic language to which many socially under-privileged students have only limited access (Snow and Uccelli, 2009). From an educational classroom research perspective, the language gap in students' mathematics

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achievement has been explained as an opportunity gap, as students with limited academic language can often only participate peripherally in classroom discourse practices (Herbel-Eisenmann et al., 2011). These findings led to educational calls for enhancing academic language in all subject-matter classrooms (Thürmann et al., 2010).

> The realization of language-responsive instructional approaches for mathematics classrooms still raises many questions, mainly from a didactic perspective: Didactics is the discipline that generates theoretically founded and empirically based knowledge about teaching and learning of a subject matter. It is typical of European didactic traditions that HOW-questions are enriched by deeper topic-specific WHAT-questions (van den Heuvel Panhuizen, 2005). For language-responsive mathematics instruction, these questions are:
>
> - WHAT language demands are most relevant to being treated so that students can benefit most in their mathematics learning?
> - HOW can language learning opportunities be integrated into mathematics classrooms so that students can learn effectively?

The HOW-question has been treated by adopting principles from second-language education (e.g., Gibbons, 2010) and adapting them for mathematics teaching. Subject-specific design research studies, classroom observation studies, and controlled trials have contributed to an empirically grounded set of design principles for language-responsive mathematics instruction (see research overviews in Erath et al., 2021): (1) enhance rich discourse practices, (2) establish a variety of language routines, (3) connect multiple representations and language varieties, (4) include students' multilingual resources, (5) use macro-scaffolding to sequence and combine learning opportunities, and (6) vary and compare language aspects (form, function, etc.) for raising students' language awareness.

However, a targeted realization of these design principles for a particular mathematical topic depends heavily on unpacking what exactly is meant by the overall language proficiency construct. The WHAT-question points to decisions about which language demands can be circumvented and WHAT needs to be explicitly treated as learning content in mathematics classrooms (Bailey, 2007; Moschkovich, 2010; Prediger and Zindel, 2017). Hence, this specification of language demands relevant for a specific mathematical topic is a core didactic challenge for mathematics teachers and for designers of language-responsive instructional approaches and curriculum materials. A recent analysis of US algebra textbooks revealed that in particular for students with low language proficiency, the suggested mathematics and language learning content is poor, often restricted to procedural fluency as a mathematics learning goal and to isolated technical vocabulary as a language learning goal (de Araujo and Smith, 2021). It is the task for didactics as a scientific discipline to develop an empirical and theoretical foundation, based on which designers of language-responsive curriculum materials or teachers can specify language demands that are really relevant for understanding a specific mathematical topic.

This paper presents a practical tool, the specification grid for specifying topic-specific language demands (Section 2), and the underlying theoretical framework that justifies and explains the connections between its elements (Section 3). This theoretical framework combines classical sources in linguistics and language-education research and draws upon subject-specific empirical and design-based mathematics education research focused on language learning.

2. Specification Grid: A Practical Tool for Specifying Language Demands

2.1. Typical pitfalls in specifying language demands for mathematics classrooms

When mathematics teachers' practices start to include language in their mathematics instruction, three typical pitfalls occur in their specifying practices:

- Many mathematics teachers (Turner et al., 2019; Prediger et al., 2019) and curriculum designers (de Araujo and Smith, 2021) start by training isolated vocabulary without connecting it to understanding the mathematics in view.
- Most mathematics teachers exclusively focus on technical language, whereas they falsely assume that important school academic-language demands are already part of students' everyday language (Prediger, 2019; Prediger et al., 2019).
- Some mathematics teachers aim at enhancing discourse practices (e.g., discussing multiple solutions for a calculation task), but not those that are most critical for developing students' conceptual understanding (especially explaining meanings and describing general patterns; Setati, 2005; Erath et al., 2018; Prediger et al., 2019)

In all of these cases, teachers can invest a lot of energy and classroom time to teach students to master a specified language demand, but when the language demands themselves are peripheral to mathematical understanding, they distract rather than support students' mathematics learning. That is why Moschkovich (2015) pleaded for a focus on discourse practices rather than vocabulary, and Setati (2005) emphasized the need for conceptual rather than procedural talk.

2.2. The specification grid for language demands as a practical tool

To overcome these often-documented pitfalls, we developed a practical tool named "the specification grid" that can support mathematics teachers, curriculum designers, and researchers to specify mathematically relevant language demands (Fig. 1).

The specification grid incorporates four lenses that foreground different components of the grid and their interplay that are to be explained, connected to the grid, and then theoretically founded (Section 3):

- Epistemic lens: Language as a thinking and learning tool;
- Conceptual lens: Focus on conceptual understanding of mathematics;

Fig. 1. Specification grid for identifying mathematical language demands (Prediger, 2019)

- Functional lens: Double use of form-function relationship, where language is viewed as serving particular purposes and not viewed only as forms;
- Discursive lens: Discourse practices as the essential language learning content.

The epistemic lens entails that the specification question is posed as "What language demands are most relevant to being treated so that students can benefit most in their mathematics learning?" This means that we do not only focus on the communica-tive function of language as a medium of communication, but also on the epistemic function of language as a thinking and learning tool (Pimm, 1987; Snow and Uccelli, 2009) for mathematics learning, not a generic academic-language proficiency. In the specification grid, the epistemic lens is incorporated by determining the relevance of particular language demands from their role for learning particular mathematical content goals. Practically, this is realized by starting the specification process in the first column of Fig. 1, by setting the mathematical content goals of a particular teaching unit; in this example it is the equivalence of fractions.

The conceptual lens is incorporated in the specification grid by the rows that distinguish between conceptual understanding and procedural skills. Both kinds of knowledge are relevant in mathematics education, but procedural skills still tend to be prioritized in classroom practices (Hiebert and Grouws, 2007). In our practical specification example in Fig. 1, Question 1 leads to distinguishing understanding the meaning of the equivalence of fractions from the procedure of expanding fractions.

The discursive lens treats the discourse practices as the key language unit in view. The functional lens is incorporated into the specification grid by two form-function relationships: For the epistemic function of expressing the procedural skills and conceptual understanding, the discourse practices provide the key language units. In the practical example, Question 2 leads to specifying the discourse practice of "reporting procedures" for articulating the procedural knowledge and the very distinct discourse practice of "explaining meanings" for articulating the conceptual understanding. Question 3 focuses the second form-function relationship of lexical and syntactical means needed for realizing the discourse practices in phrases.

In an empirical study on teachers' specification practices for the case of equivalence of fractions (Prediger, 2019), most teachers immediately identified the typical formal vocabulary such as numerator, denominator, expand, and multiply as relevant for expressing the equivalence of fractions, but without being aware of their functional connections, in other words, that these phrases can only be used in the discourse practice of reporting procedures that can underpin the procedural skill. Much less often, the teachers specified the discourse practice of explaining meanings, although it is epistemically relevant for developing conceptual understanding. Teachers were not aware that explaining meanings requires other phrases to express mathematical structures and relationships such as "describes the same share, but more coarsely structured," that we have termed meaning-related phrases (Pöhler and Prediger, 2015). Meaning-related phrases can require new academic vocabulary, but also syntactical means for expressing complex relationships, for example, binary relations for comparisons ("this number is larger than this number" rather than "this number is large and this number is small") or sophisticated syntactical constructions such as "increases more slowly" with adverbs fine tuning the meaning of verbs (Prediger and ŞahinGür, 2020).

Summing up, the specification grid supports specifications of mathematical and language learning content in an overall epistemic lens. The vertical arrows in the specification grid in Fig. 1 make explicit the relevant distinction related to procedural and conceptual aspects in a conceptual lens, and the horizontal arrows incorporate the functional lens entailing important functional connections between content sub-goals, the discourse practices for each sub-goal, and the lexical or syntactical means to realize the discourse practices. Positioning the discourse practices in the middle column reflects their relevance in a discursive lens.

In a PD research project, we showed that with the support of the specification grid, mathematics teachers learned to identify mathematically relevant language demands with a higher accuracy (ŞahinGür and Prediger, 2019), so it turned out to be practically useful. In the next section, the theoretical backgrounds of the lenses are described, together with the empirical findings strengthening the claims underlying the postulated connections and distinctions of vertical and horizontal arrows.

3. Four Lenses Underlying the Specification Grid and Their Background

Without aiming at a comprehensive account of all perspectives on language and mathematics learning (as provided by Morgan et al., 2014, or the early book by Pimm, 1987), this section explains the theoretical framework for the practical purposes sketched in Section 2, led by four lenses, each intertwined in pairs.

3.1. Backgrounds for the functional and epistemic lenses: studying language in its epistemic function for students' knowledge construction processes

Whereas linguistic research sometimes studies language as a form with relevance in itself (e.g., different lexical or syntactical phenomena), mathematics education research on language has adopted a functional lens from the beginning, that is, the function of language for and in mathematics teaching and learning was considered. Early on, Austin and Howson (1979) and Pimm (1987) described the two major functions of language for mathematics classrooms: the communicative function as a tool for classroom interaction and the epistemic function of language as a tool for thinking and learning. Epistemic, in this context, means related to students' individual or collective knowledge construction processes. Vygotsky (1934) explains the epistemic role of language by the relevance of interiorizing external operations (learned in social interaction) by inner language. The epistemic lens can be considered as a first-level realization of the functional lens on language. The sketched Vygotskyan theoretical background, however, does not sufficiently help to disentangle what aspect of language is relevant. Accordingly, Morgan et al. (2014) promoted the specification question "What are the linguistic competencies…required for participation in mathematical practices?" (p. 851) as crucial for future research. Its systematic treatment requires further lenses.

3.2. *Epistemic and conceptual lenses: language as a thinking and learning tool for developing conceptual understanding*

As Vygotsky (1934) and Cummins (1979) had already pointed out, the epistemic function of academic language is particularly relevant for higher order thinking skills and for understanding abstract scientific concepts, whereas more elementary ideas and concepts can be learned with less elaborate language. More recent linguistic and language-education theories confirm this connection of elaborateness of language and thought, and even define academic language by its epistemic function: Academic language is "the language that is used by teachers and students for the purpose of acquiring new knowledge and skills…, imparting new information, describing abstract ideas, and developing students' conceptual understanding" (Chamot and O'Malley, 1994, p. 40).

These theoretical backgrounds underpin our MuM research group's decision to focus the epistemic lens on language mainly on a conceptual lens, in other words, for the development of conceptual understanding. This decision was fueled by empirical findings that language proficiency has more of an impact on the conceptual understanding than procedural skills, as shown by studies in Grade 3 (Ufer et al., 2013) and Grade 10 (Prediger et al., 2018). Even if concep-tual understanding and procedural skills must always be developed in mutual dependence (Kilpatrick et al., 2001), specifying language demands with a conceptual lens thereby focuses on those types of knowledge that pose more language challenges for teachers and students, namely, conceptual understanding.

In the theoretical framework, the adopted conceptual lens draws upon defining conceptual understanding as grasping the meaning of concepts. Meanings are not Platonist ideas, but socially constructed networks of mental representations. Hiebert and Carpenter's (1992) definition of the meaning of a mathematical concept was that it is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. […] [It] is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (Hiebert and Carpenter, 1992, p. 67)

This characterization of conceptual understanding as a network with strong connections has been further elaborated in at least three ways: (a) with respect to multiple external representations, (b) with respect to concept elements, and (c) with respect to the epistemic and semiotic processes required to build the networks.

(a) In many instructional approaches, multiple external representations count as crucial tools for developing conceptual understanding. Indeed, the translation from manipulatives to graphical representations to symbolic representations has proven effective for developing conceptual understanding (Lesh, 1979), in particular when the representations are not only juxtaposed, but explicitly connected (Renkl et al., 2013). The focus on connecting rather than only implicitly translating is in line with Hiebert and Carpenter's (1992) emphasis on connections as characteristic in understanding. With respect to language in mathematics learning, the emphasis on manipulatives and graphical representations must not be mistaken as a substitute for explicit verbalizations. Instead, various empirical studies have shown that explicit negotiations of mean-ings are required before students "see" the relevant structures in graphical representations (Meira, 1998; Steinbring, 2005).

(b) Beyond external representations, Hiebert and Carpenter (1992) pointed to knowledge elements that are to be connected in the mental representation of a mathematical concept or theorem. These knowledge elements in the connected networks can comprise concept elements (e.g., a particular basic mental model or a subconstruct of a concept) and procedural elements (e.g., the procedure of expanding fractions by multiplying numerator and denominator by equal numbers is connected to graphically structuring a fraction bar into a finer structure; Korntreff and Prediger, 2022).

(c) Based on this characterization of conceptual understanding as a network of mental representations of knowledge elements in multiple external representations, we distinguish four epistemic processes that are needed to develop understanding (Prediger and Zindel, 2017; Korntreff and Prediger, 2022): Students

- mentally construct knowledge elements (e.g., by explicating them in the semiotic processes of translating or connecting multiple external representations),
- connect several knowledge elements into a network,
- compact the knowledge elements or external representations into new conceptual entities which can then serve as new elements of higher order, and
- unfold the compacted concepts into their constituent knowledge elements when necessary (e.g., for justification or explaining connections).

To unpack the inverse epistemic processes compacting and unfolding, our theoretical framework draws upon Drollinger-Vetter's (2011) interpretation of Aebli's

(1981) conceptualization, who describes compacting as the highly relevant epistemic process in which networks of (internal or external representations) and knowledge elements are encapsulated into new conceptual entities, which can then be the elements for networks of higher complexity. Successful compacting processes can be reversed, that is, the encapsulated concept or procedure can be unfolded back into the constituent knowledge elements. In our research, we showed that for many students, this reversibility is only fragile or not achieved, and requires elaborated discourse practices with concise phrases (Prediger and Şahin-Gür, 2020).

3.3. Discursive and functional lens: discourse practices and means to enact them

In sociolinguistics, the differences between everyday language and academic language have been characterized by different dimensions: in the lexical dimension (e.g., by specialized vocabulary, composite or unfamiliar words, and specific connectors), in the syntactical dimension (e.g., long and syntactically complex sentences, passive voice constructions, and long noun phrases and prepositional phrases), and in the discursive dimension (e.g., turn-taking organization, situative language use, and subject-specific text types; Snow and Uccelli, 2009; Heller and Morek, 2015).

Most mathematics education researchers have emphasized that the atomistic lexical and syntactical dimensions must be subordinated to the discursive dimension, as the mathematical discourse is the major language unit relevant for mathematics learning (Adler, 2001; Moschkovich, 2010; Setati, 2005; Herbel-Eisenmann et al., 2011). This widely agreed discursive focus has been adopted with many different theoretical backgrounds: Ryve's (2011) research overview identified a huge variety of conceptualizations of discourse, spanning from complex culturalistic or conversational perspectives to commognition perspectives. Here, we follow Moschkovich (2010, 2015) in her emphasis on the discursive lens in the epistemic function for meaning-making and her conceptualization of mathematical discourse that "draws on hybrid resources and involves not only oral and written text, but also multiple modes, representations (gestures, objects, drawings, tables, graphs, symbols, etc.), and registers (school mathematical language, home languages and the everyday register)" (Moschkovich, 2015, p. 2). To further narrow the discursive lens down, we base our discursive lens on interactional discourse analysis (Quasthoff et al., 2017), in which patterns of discourse are considered as socio-culturally evolved and then interactively co-constructed in classroom discourses (on the micro-sociological level).

The key units of language in interactional discourse analysis are discourse practices, which are

> defined as multi-unit turns … interactively co-constructed, contextualized and functionally oriented towards particular genres such as narration, explanation or argumentation. By making use of conventionalized genres, discourse units in their joint achievement in interaction rely on patterns available in speech communities' knowledge. (Erath et al., 2018, p. 4)

Typical discourse practices are narrating, explaining, arguing, and reporting. As Morek and Heller (2015) explained, academic discourse practices are those optimized by school purposes, in particular explaining what and explaining why to convey or construct knowledge and arguing to negotiate divergent validity claims in classrooms.

The theoretical construct of discourse practices resonates with the constructs "language actions within text genres" (explaining, describing, and evaluating; Roll et al., 2019) and "discourse functions" (classifying, defining, describing, evaluating, reporting, explaining, and exploring; Thürmann et al., 2010; Dalton-Puffer et al., 2018). The three constructs overlap in (a) the adopted epistemic lens that conceives cognitive processes and language practices as tightly connected; (b) the characterization as pattern of language in discursive dimension, serving for certain typical communicative and epistemic purposes; and (c) the educational emphasis on their dual nature as being learning medium and learning goal. The two constructs differ substantially in that discourse practices and discourse functions are explicitly also related to oral language (whereas the language actions in text genres are restricted to written language) and in that discourse practices are characterized, not in psycholinguistic ways as individual mental practices, but in sociolinguistic ways by their interactive co-constructed nature.

Whereas Roll et al. (2019), Bailey (2007), and Thürmann et al. (2010) identified their lists of discourse functions/language actions by analyzing textbooks and written curricula, Dalton-Puffer et al. (2018) and our studies (Prediger and Zindel, 2017; Erath et al., 2018; and others) investigated transcribed mathematical teaching learning processes to identify the relevant oral and written discourse practices.

Tab. 1. Empirically identified list of the most important mathematical discourse practices

Mathematical discourse practice	**Explanation**
Naming	• Stating numbers/results, naming words • Assigning individual words/information/elements to something without an explanation
Narrating	• Non-condensed narratives of everyday experiences, mostly organized sequentially without extracting any mathematical structure
Reporting procedures	• Reporting individual procedures in sequential but concrete ways (e.g., previous solution paths) • Elucidating general procedures in sequential but generalized ways
Explaining meanings	• Interpreting a concept/formal element in graphical representations, contexts, etc. • Articulating how two external representations are connected
Arguing	Justifying a connection by reducing to aspects established as true, e.g., • justifying choices of representations by referencing structural elements • justifying choices of operations by referencing their meanings • refuting a conjecture by providing counterarguments
Describing mathematical structures	• Articulating the structure of a context situation, e.g., a functional relationship or part-whole relationship
Describing general relationships	• Example-oriented (generic) verbalization of relationships • General verbalization of relationships (e.g., with word variables)
Evaluating	• Formulating/justifying an independent evaluation judgment about facts by drawing upon mathematical knowledge/reasoning • Forming an opinion

For the didactic perspectives adopted in the MuM research group (with its focus on what-questions; see Section 1), the discourse practices as characterized by interactional discourse analysis (Erath et al., 2018; Quasthoff et al., 2017) needed to be refined by what is treated. Tab. 1 lists the repeatedly iden-tified relevant discourse practices.

Some researchers of discursive dimensions have pleaded for focusing the discursive dimension instead of the lexical and syntactical dimension (e.g., Barwell, 2012). But this separation of forms and functions has already been problematized by Solano-Flores (2010). He sketches four research traditions: two focusing on language in its function for mathematics learning (language in its epistemic function as a process in investigating development and cognition, e.g., for meaning-making, and language more in its communicative function as a system in investigating social interaction) and two traditions focusing on language forms (language as a structure, studied regarding lexical or syntactical difficulties in tests, and language proficiency as a factor investigated with respect to the achievement of different student groups). Solano-Flores (2010) pleaded for combining the four traditions to capture more complex language issues.

This combination is realized in functional linguistic perspectives when consid-ering discursive, syntactical, and lexical dimensions in their functional connections:

> Lexical and morpho-syntactical forms prevalent in academic texts… are … made for…presenting information in highly structured ways… that enable the author/speaker to take an assertive, expert stance toward the information presented….The high frequency of nominalizations and expanded noun phrases… can be explained by their functions…for knowledge transfer:…avoid ambiguity…condensing previously given information. (Heller and Morek, 2015, p. 176; see also Schleppegrell, 2007)

In our functional lens, we follow Moschkovich (2015) and interactional discourse analysis (Heller and Morek, 2015) to subordinate the dimensions, not by priority but functionally. This means that lexical and syntactical dimensions are considered for identifying the necessary means to textualize and mark the discourse practices. For example, topic-independent lexical means for reporting procedures comprise temporal connectors for marking the sequential structure (e.g., “at first,” “then,” “later,” “finally”), whereas explaining meanings or arguing requires integrating connectors (e.g., “for this,” “because of,” “this means”). Realizing discourse practices with sequential structure is easier for most students than realizing discourse practices with integrative structure and global coherence, because the integration of structures also requires a mental condensation of ideas and more condensed phrases (Schleppegrell, 2007; Erath et al., 2018). These necessities explain why reporting procedures is enacted more successfully by many students than explaining meanings (Erath et al., 2018).

Summing up, specifying the relevant discourse practices needed to articulate a particular mathematical learning goal allows researchers and designers to specify relevant language demands. Discourse practices are well-defined units in the discursive dimension and carry with them lexical and syntactical features as means for engaging in them.

3.4. Discursive and conceptual lens: discourse practices of explaining meanings and meaning-related phrases as the essential language learning content

Combining a discursive and conceptual lens, Setati (2005) already hinted at the problem that most discourses in her observed classrooms were shaped mainly by procedural talk and rarely by conceptual talk. Many other researchers have similarly problematized that in classrooms with mainly procedural talk, students find too few learning opportunities for conceptual understanding. Hence, the discourse practices involved in collective meaning-making were analyzed in depth (Moschkovich, 2010; Barwell, 2018), yet so far mainly without using these analyses for identifying the topic-specific language demands involved in realizing conceptually strong discourses.

From our particular perspective on discourse practices in a functional and discursive lens, the distinction between procedural and conceptual talk is reflected by the distinction between two rows in the specification grid (Fig. 1). The functional connection makes visible that the discourse practices of reporting procedures can mainly support the learning of procedural skills, whereas the development of conceptual understanding of mathematical concepts requires the discourse practice of explaining meanings (Pöhler and Prediger, 2015; Prediger and Zindel, 2017). For practical purposes, the distinction into these two discourse practices is sufficient and insightful for professional development of teachers (e.g., Prediger, 2019).

For research purpose and subtler design decisions, however, the distinction can be further refined by more in-depth analyses of the semiotic and epistemic processes introduced in Section 3.2 and their verbalization in discourse practices, in particular for collectively unfolding compacted concepts and external representations. Three epistemological and ontological characteristics of mathematical concepts shape the necessity of elaborate discourse practices and elaborate lexical means for the epistemic processes of developing their understanding:

a) Mathematical concepts are abstract. As the meaning of mathematical concepts cannot simply be grasped by pointing to external objects ("This is a table."), language is required for negotiating meanings of abstract entities. Therefore, the relevance of multiple representations in the interactive processes of mean-ing constructions have been outlined (e.g., Lesh, 1979).
b) Mathematical concepts are relational in nature: Steinbring describes "The particular epistemological difficulty of mathematical knowledge — contained in the specific role of … signs and symbols — consists in the fact that mathematical knowledge does not simply relate to given objects, but also that relations, structures and patterns are expressed in it" (Steinbring, 2005, p. 4; similar Barwell, 2018).

c) Mathematical knowledge elements must be connected in dynamic networks. Whereas students quickly acquire temporal connectors (e.g., “first,” “after that,” “then”), the connections that have to be made explicit when building and connecting knowledge elements into networks of understanding require more elaborate connectors.

Whereas many authors have expressed the hope that processes of meaning-making can and should completely rely on the individual resources that students bring into the mathematics classroom, these particular epistemological and ontological characteristics of mathematical concepts pose particular challenges when articulating ideas in the four epistemic processes of constructing, connecting, compacting and unfolding mathematical concepts. In particular, the process of unfolding requires not only vague, ambiguous, and perhaps deictic everyday resources (e.g., “this, here,” “there, you know”), but also academic phrases for realizing concise and explicit explanations of meanings.

With meaning-related phrases, our research group established a new construct that encompasses all topic-specific lexical (and sometimes also syntactical) means required to express the abstract and relational nature of a particular concept in an explicit and concise yet informal way. Although some students with high academic-language proficiency might have sufficient individual meaning-related phrases in their individual resources, students with lower academic-language proficiency have been shown to lack exactly these meaning-related phrases. For example, even many seventh graders fail to crack complex percent information and connect the part and the whole only with “and,” without explicitly expressing the part-whole relationship, using, for example, “out of” (Pöhler and Prediger, 2015), and many 10th graders struggle with coordinating two quantities in functional relationships as they cannot articulate “the price depends on the weight” or “the price grows with the weight” (Prediger and Zindel, 2017). The topic-specific meaning-related phrases have thereby turned out to be a key area of language-learning content to enable all students to engage in demanding discourse practices such as explaining meanings and arguing. In classroom interaction, establishing shared meaning-related phrases also strengthens the possibilities for joint knowledge construction processes (Prediger and Pöhler, 2015).

3.5. Outlook: epistemic and discursive lenses in depth and for designs

Although our investigations of processes of developing conceptual understanding have already substantially contributed to specifying the mathematically relevant language demands, still further research is necessary to deepen the empirical exploration of the epistemic and discursive lenses. In our current research, we investigate how the four epistemic processes (mentally construct knowledge elements, connect knowledge elements, compact knowledge elements, and unfold compacted knowledge elements) are articulated in the discourses and what students’ major language challenges are when engaging in these epistemic processes.

We can provide quantitative evidence that classrooms in which teachers engage students in rich discourse practices and constantly connect different knowledge elements have significantly higher learning gains than others without these rich discourse practices and important epistemic processes (Neugebauer and Prediger, 2023). Further qualitative research can reveal deeper insights into the underlying mechanisms.

This, in turn, can also inform the design of language-responsive mathematical learning opportunities, which should be the overall goal of research.

Acknowledgments

The core of this paper has grown since 2009 in my MuM research group in Dortmund, in research projects funded by the National Ministry of Education and Research (BMBF-grants no. 01JC1112, 01JM1703A, 03VP02270) and Deutsche Forschungsgemeinschaft (DFG-grant PR 662/14–1 and/14–2). I thank my team involved in the lovely and intense collaboration that pushes our ideas further and deeper every day.

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https://doi.org/10.1142/9789811287183_0035

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Digital Technologies, Cultures and Mathematics Education

Ana Isabel Sacristán[1]

ABSTRACT I focus here on several aspects of cultures related to digital technologies and mathematics education. One first aspect is that any integration of digital technologies for mathematical (or other) teaching and learning creates and transforms the classroom culture. On the other hand, in order for learning to be meaningful through the use of digital technologies, these may need to be embedded in a certain "culture" that empowers students to engage pro-actively with those technologies. I present different types of teaching and classroom "cultures" that have been found when using digital technologies, how these impact mathematical learning, as well as different conditions and teacher-training opportunities for the use of digital technologies found in different countries, illustrating all of these with examples from my own experience and from literature. I discuss how the different conditions and access opportunities in different regions and cultures create digital gaps. Finally, I discuss what could be done to support teachers to create meaningful contexts and classroom cultures when integrating digital technologies within established school systems (but at the same time transforming these), so that these can empower learners (e.g., to "do mathematics") and promote the construction of knowledge.

Keywords: Digital technologies; Classroom cultures; Mathematics education; Constructionism.

1. Introduction

I discuss here ideas and issues related to how different cultural aspects affect the integration of digital technologies in mathematics education.

Digital technologies permeate our lives, but I ask: How are they used for (mathematical) teaching and learning? Are they integrated in ways that promote significant learning and enhanced mathematical practices? I argue that they generally do not, except in exceptional cases. My aim here is to reflect, on the one hand, on how existing cultures affect how technologies are integrated in mathematical classrooms and for mathematical learning. On the other hand, when digital technologies are available and/or integrated into school practices, I reflect also on what kind of cultures

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are, or can be, created for students and teachers — in educational institutions and classrooms — to promote mathematical learning.

Firstly, I believe that existing cultures affect how digital technologies are integrated for mathematics education and in its teaching and learning; secondly, that when digital technologies are integrated they create new cultures; and thirdly, that new cultures need to be created in order to have a significant and effective mathematics education impact.

That is, while it is important to create different school cultures for integrating digital technologies, at the same time, and conversely, digital technologies create new cultures in the classroom, as is illustrated in Fig. 1. This is part of what I will reflect upon.

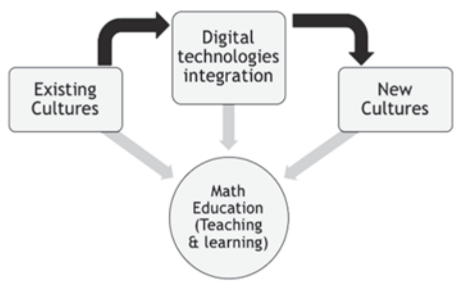

Fig. 1. Cultures and digital technologies integration (a two-fold process); and their effect on mathematics education

But I will also reflect (i) on the constraints and what I call the *inertia of classroom cultures*; (ii) on the barriers to meaningful and transformative technology integration in mathematics classrooms, which include digital gaps and issues of access and equity; as well as (iii) on teachers' cultures. In relation to the latter, I believe that teachers are key players for integrating digital technologies in the classroom and that there needs to be more teacher involvement in both the design of technological implementations as well as of related resources.

2. Towards a Significant Integration of Digital Technologies for Mathematical Learning: Revisiting the Constructionist Paradigm

My views related to the integration of digital technologies for mathematics teaching, learning and thinking, are informed in great part by the constructionist school of thought.

2.1. The core ideas of constructionism

Constructionism is a term coined by Seymour Papert to describe a paradigm of learning based on the tenet that building knowledge structures in the mind "happens especially felicitously in a context where the learner is consciously engaged in constructing" (Papert, 1991, p. 1), and not only in just constructing but in constructing something *shareable*. It is a shift in education, from learners as receivers of information, to learners as creators, constructors and main actors; where students engage in activities where they learn *to do mathematics* rather than learn about mathematics (Papert, 1971) — activities where they have *objects-to-think-with* and can develop powerful ideas (Papert, 1980). These are some of the core ideas of constructionism.

An interesting aspect is that, from its inception, constructionism has been related to computer programming, a highly expressive activity. Papert explained, in his seminal work *Mindstorms* in 1980, that in constructing with, or programming the computer, the learner "establishes an intimate contact with some of the deepest ideas from science, from mathematics, and from the art of intellectual model building" (Papert, 1980, p. 5). He also said that "in teaching the computer how to think, [students] embark on an exploration about how they themselves think" (Papert, 1980, p.19). But the emphasis is not only on computer programming, but also on the entire learning culture: A culture or learning environment where there is collaboration, exploration and sharing — again, the sharing idea is very important in constructionism– and where students engage in personally meaningful projects. This "personally meaningful" aspect is central to constructionism.

2.2. Digital technologies for personal expressiveness and the joy of learning

Digital technologies, with their possibility of customization and adaptability are platforms where learners can find and create something personally meaningful, which can bring joy to the learning activity, and thus help them learn best.

Sinclair, Healy and Noss (2015, p. 2) wrote: "A more learnable mathematics should also be one that is worth learning" and using Bruner's expression, they point to the 'sense of delight' (Bruner, 1969, cited in Sinclair et al., 2015) that is involved in many aspects of using digital technologies. But they also warn that

> the technologies offered to students have become increasingly easy to use … and the tasks much more goal-oriented toward the learning of particular concepts. While no one would advocate over-complicated technologies, it is evident that those which offer more expressiveness, though at times they might additionally require a steeper learning curve, also offer more potential — and expressiveness and potential are essential ingredients of both delight and intellectual travel. (Sinclair et al., 2015, p. 2)

There are several important points in the above quote, that I would like to reiterate and comment upon.

First, the observation that the technologies have become increasingly easy to use, while the tasks have become much more goal-oriented. I agree with this, and would add that in doing so, technology-based tasks also lack open-endedness — something which would give learners the opportunity for more creativity and the engagement in more personally meaningful activities.

Their mention of not advocating over-complicated technologies reminded me of another aspect: I argue that, fairly often, something simple, but deep, allows for rich exploration, much more than overly complicated tasks or technologies.

A central point which they then discuss, is that technologies that offer expressiveness have more potential, even if they can be more difficult, and they end with an important idea worth repeating: "expressiveness and potential are essential ingredients of both delight and intellectual travel" (Sinclair et al., 2015, p. 2). In fact, there is a joy in having mathematical insights, but this doesn't mean it is easy. The satisfaction of solving a challenge or problem is much more satisfying than just getting through an activity without facing any challenges.

For example, for decades I was involved in promoting Logo programming in schools; teachers found it difficult, and students also said that it was more difficult than doing things in, for instance, dynamic geometry. Nevertheless, both teachers and students also felt more rewarded when they were able to achieve their goals. In particular, there is the story of a thirteen-year-old boy who told me that he liked Logo because it required thinking, whereas software that just required pointing and clicking, like office suites, made "human beings obsolete" (personal communication, 2009).

However, challenging, creative constructionist implementations are not easy to achieve, particularly in classrooms with established cultures and curricula. As Agalianos et al. (2001 and 2006) pointed out, the use of technologies in schools is shaped by social, economic and political forces and constraints (see also Ruthven, 2008).

One influence on what happens in schools, is the general technology-use culture in today's society. So, let us dwell briefly on what the general technology-use culture and tendencies are, to then analyse what happens in classrooms — that is, what are the prevalent school cultures and practices in terms of technology-integration.

3. Technology-use Cultures in Society and in Schools

3.1. The prevalent culture of technology-use in today's society

Most people today mainly use technology, first, for information and communication: that is, for looking up information (e.g., through Google or Wikipedia); for email and social media (e.g., WhatsApp, Facebook, Instagram); for video conferencing (e.g., Zoom); and for entertainment purposes (e.g., downloading streaming videos from YouTube or Netflix). Those are probably the main uses of technology (and the mentioned apps are among the most popular ones — Most popular apps, 2022).

We can summarize some of the main uses in society as information, communication and entertainment, through *connectivity, social media and mobile media*. Are these three modes present in mathematical educational practices?

In general, even though curricula in many countries around the world emphasize the use of digital technology, there are discrepancies with the reality in classrooms. But before discussing that, let us reflect on the trends, over the last decades, of digital technology use in schools and, in particular, in terms of its use for mathematics.

3.2. School cultures and practices for technology-use in mathematics classrooms in the last decades

In the 1980s, there was, of course, logo — possibly the first educational software, dynamic geometry, computer algebra systems, and spreadsheets — all of which can facilitate, to certain degree, some expressiveness —; although, at the same time, we also had computer assisted instruction, more directed towards practice, tutorials or simulations (Aydin, 2005).

But from the 1990s to the early 21st century, the use of much of the open expressive digital tools and software declined due, for instance, to a lack of adaptability and alignment to conventional classroom practices and curriculum (Ruthven, 2008; see also Agalianos et al., 2001 and 2006); and then, easier-to-use technologies began to dominate (as mentioned by Sinclair et al., 2015). Then, if technology was used in mathematics classrooms, what was prevalent were, either general non-educational and non-mathematical tools (e.g., presentation tools, such as PowerPoint, Word and LaTeX, as I reported in Julie et al., 2009 for the case of Latin-American classrooms), or specific interactive apps for a particular mathematical content or topic. Another use has been the projection of videos (e.g., as reported in Miranda and Sacristán, 2012 and 2013).

Such uses continue until today, although there are new trends towards coding and computational thinking, as discussed further.

We can summarize the observed uses of technology in mathematics classrooms in the past two decades as those:

- *for demonstration/presentation;*
- *for faster computation;*
- *for easier visualization; and*
- *for information*

All of the above are information and communication technologies (ICT). That is not harnessing the full potential of digital technologies, that is not using expressiveness, that is not letting students create and be constructors. It is mostly a teacher-centred use and the tasks are usually the same or similar to paper-and-pencil ones, just with the add-on of technology. In fact, Litke (2020) points to how teacher-centred instructional formats tend to be the norm in the United States, as well as in other nations. Thus, even in developed countries like the U.S.A., teacher-centred instruction is still prevalent.

3.3. Current trends towards coding, computational thinking and different learning approaches

Nevertheless, in the last decade, there are some trends that are more in tune with the precepts of constructionism.

Computational thinking and coding have become a general trend. We have already mentioned at the beginning of the paper how Seymour Papert (1980) considered that computer programming is a way to develop mathematical thinking. The relationship between computational thinking and mathematical thinking is worth dwelling on. One could consider that the computational thinking is part of mathematical thinking, although there are differences. A more profound discussion is beyond of the scope of this paper, although it has been examined by other authors (e.g., Weintrop et al., 2016; Kallia et al., 2021). Carolyn Kieran also provided a very interesting discussion on computational thinking versus mathematical thinking in a plenary panel at the 42nd PME-NA conference (Hoyles et al., 2020), which is worth looking at in full. A main point that she made is that "digital technologies afford multiple varieties of mathematical activity that can offer experiences that involve coding but also those that do not" (Kieran in Hoyles et al., 2020, p. 76). So, there is also a distinction between computational thinking and coding.

Another important trend in the past decade, has been more emphasis on the learning approaches. Previous to that, when promoting computer use, there was little acknowledgement of "the epistemological and cognitive dimensions associated with such change or the complexity associated with the appropriation of tools into mathematical and teaching practices" (Healy, 2006, p. 213) — although Healy was referring to the case of Brazil, this has been the case in many places.

More recently, however, the USA's 2017 *NMC/CoSN Horizons Report* (Freeman et al., 2017), which aims to identify trends in teaching and learning, pointed to: (i) an increasing use of collaborative learning approaches and of blended learning; shifts from students as consumers to creators and a recent push for coding literacy; (ii) the rise of STEAM learning which seeks to engage students in interdisciplinary learning breaking down traditional barriers between different classes and subjects – this would address one of the criticisms raised by Papert (2006) in his last plenary, as I will discuss further below —; (iii) a rethinking of how schools work, shifting to deeper learning approaches (e.g. project-based learning, etc.) and a redesigning of learning spaces (e.g., ideas like flipped classrooms).

Nevertheless, changes are slow and more so for changing the way mathematics is taught. In 2014, Clark-Wilson and her colleagues pointed out that "[i]nnovative research projects and proposals, and curriculum development don't seem to have had much impact on students' learning of mathematics" in the transformative way that was initially anticipated" (Clark-Wilson et al., 2014, p. 1). Furthermore, as we will see through the examples given in section 5.1, not only have such innovative projects and development not been able to transform students' learning and teaching practices, but they are rarely sustained in the long term.

An exception to that is Brock University's (in Canada) Mathematics Integrated with Computer and Applications (MICA) programme, which has been sustained for over two decades. In this programme, students (mathematics majors and future teachers) learn to design and program interactive and dynamic computer environments (microworlds) to investigate mathematical concepts, conjectures and real-world applications (see Buteau et al., 2016). It is an educational program that has been deemed constructionist; it is also a real world, authentic and sustained implementation.

4. Barriers and Obstacles for Changing Classroom Cultures

However, in general, as we already mentioned, there are social, economic and political forces that are barriers to the uptake and integration of digital technologies and to change of classroom cultures so that these technologies can promote more meaningful teaching and learning. Let us look at that more closely.

4.1. The constraints and inertia of school cultures and systems

As I said at the beginning of the paper, in order to have a meaningful integration of digital technologies in schools, there is a need to change classroom practices and cultures. That is, there is a need to overcome what I call the inertia of current classroom practices, adapt and change them.

In this regard, Seymour Papert, in his plenary talk at the ICMI 17th Study Conference in Vietnam mentioned the followed, which is worth reproducing here:

> Are we going to continue using the new technology to implement what was only there because there wasn't the technology? […] We would never have had airplanes, […] if we had constrained new transportation to follow the schedules of the sailboats and the horse-drawn carriages. That's what we are doing in our schools. […] we have an education system that is rooted in every aspect in the very idea of grades 1, 2, 3, 4, 5, the very idea of cutting up knowledge into the subjects, the order in which to do them, what we do – all this should be put in question. […] Our schools are dictated […] by a technology that's now obsolete, the pencil and paper. Digital technology is the liberator, of allowing completely new things — but, paradoxically, we are caught in a trap of using it to do the same stuff. (Papert, 2006)

What I discussed in section 3.2, and some of main observed uses of technology in mathematics classrooms that I listed there, show that Papert was right: digital technologies are generally used to teach and serve the old; that is, they serve to cater the existing curricula, with much of their potential overlooked.

4.2. Difficulties and determining factors for implementing digital technologies in (mathematics) classrooms

Thus, changing classroom cultures, and overcoming their inertia, is hard. The NMC Horizon Reports consider that the changing role of teachers and educators is difficult, even a "wicked challenge" (Becker et al., 2017). Let us consider what are some of the difficulties, constraints and determining factors that make those changes difficult — in particular for promoting mathematical teaching and learning —, from observations of classroom practices and teachers' experiences. I will begin by drawing some of these from a programme that I was involved with.

Between 1997 to 2007, the Mexican Ministry of Education sponsored what was called the Mexican Teaching Mathematics with Technology (EMAT) programme (see Ursini and Sacristán, 2006; Sacristán and Rojano, 2009), which was carefully designed as a research-based programme; it was also research-linked because much associated studies were carried out. Though the programme was well designed and there were areas of success, we also identified some challenges and difficulties at different levels (see Trigueros and Sacristán, 2008): some at the *teacher, student and classroom level*; some at the *school level*; some at the *local authorities' level*; and some at the *government level* — in fact, it was the federal government that cancelled this programme in 2007 (as described in Trouche et al., 2012; and also, in Sacristán et al., 2021).

So, at the latter level, there can be various policy, administrative and bureaucratic factors that impede successful integration. Some examples are administrators not allowing the use of computers by students; or a lack of support personnel for making sure that computers work or to assist teachers with their use.

At the teacher-student level, some of the issues that have been found (see Sacristán, 2017), are, on the one hand, issues of *time*, where teachers simply lack the time to prepare tasks that integrate technology, or cannot find time within the pre-set curricula to incorporate such tasks.

Moreover, when mathematical digital tools *are* used, sometimes teachers are unable to take advantage of their affordances (maybe due to a deficient Pedagogical Technology Knowledge — PTK —, Thomas and Palmer, 2014). For example, there was a case where we visited a school to see how dynamic geometry was being used (see Sacristán, 2017): The students had to construct bisectors of a triangle and they did it fine, but then they erased it and started all over again. We asked them why. They explained they wanted to make the triangle bigger; that is, they had no idea that they could drag the vertices of the triangle and make the triangle bigger. because the teacher had never conveyed that. So dynamic geometry was not used dynamically; it was just used as a drawing tool. Thus, sometimes the tools' main affordances are not known.

Other observed issues are similar to those identified by Thomas and Palmer (2014), some of which relate to teachers' Pedagogical Technology Knowledge (PTK):

- the teachers' *content knowledge* of mathematics; as well as
- their *attitudes, beliefs and confidence* in and for technology use (also that of their students);
- knowledge of *how to use the technology* both in *technical terms* as well as in *pedagogical* ones (the instrumental genesis of the digital tools for mathematics teaching);
- the *integration of technology tasks with the established curriculum*; and
- *asses sment demands* — how do you assess what takes place with technology use?

This is why there is a need for continuous professional development and support in the use of digital technology, as discussed in section 6 below.

All of the above issues, found to impede effective technology integration, are very similar to what a 2011 report listed as the challenges and barriers that UK mathematics teachers' encounter and their concerns for why to use, or not use, technology in their practices:

- a lack of confidence with digital technologies;
- fears about resolving problems with the technology;
- fears about knowing less than their learners;
- access to digital technologies;
- inappropriate training;
- lack of time for preparation;
- a lack of awareness of how technology might support learning;
  not having technology use clearly embedded into schemes of work (Clark-Wilson et al., 2011, p. 20)

That report also includes the following factors among the barriers to a more creative student-focussed use of digital technologies:

- an inadequate guidance concerning the use of technological tools in curriculum documentation;
- assessment practices;
- and “a perception that digital technologies are an add-on to doing and learning mathematics” (Clark-Wilson et al., 2011, p. 6).

A more recent issue is that there is an overload of information and of resources of varying quality. The internet is full of resources and teachers may not know what to choose. In relation to this, Santacruz-Rodríguez identified different types of resource selection criteria by teachers (see Santacruz-Rodríguez and Sacristán, 2019). A first criteria is the technical (also called ergonomic) one, which relates to the ease of use of a resource; others are curricular, mathematical and didactical. Many teachers choose technologies because they fit with something that they want to teach, that is, can be used to cover a specific content. Or they may simply choose a technology because it is easy to use, without necessarily taking much into account mathematical or didactical criteria. In the latter case, the selection criteria could be more for instructional, teacher-centred technology that serves existing (paper-and-pencil) curriculum (which, again, is due to the inertia of classroom cultures).

4.3. School cultures in developed countries vs. those in developing countries

I believe it is also necessary to discuss what happens in developing countries in contrast to developed ones, because additional challenges arise in the first, for technology integration.

In Sacristán et al. (2021), we analysed and compared the situations in India and Mexico. In that chapter, we presented there how there is a lack of equipment and lack of connectivity; and, in rural schools, even a lack of electricity. Computer labs are still common, so students must go to a separate room to use computers; this means that computer technologies are not integrated with and within disciplinary topics, such as mathematics. There is also a lack of professional development for teachers.

For example, in a survey partly reported in that chapter (Sacristán et al, 2021), in rural schools in the region of Oaxaca, Mexico, half of the schools did not have access to internet, and some schools had only two computers for the entire school.

Thus, in most developing countries, there are issues of restricted access. Prevalence of old hardware seems to be the norm, so teachers have to cope with what they have. Or, when there is only a computer lab, they often cannot use it for mathematics classes, because it is used exclusively for computer science.

We have even documented the case where computer labs were used as storage rooms (Herrera-Salgado, 2011). Or there are extreme cases, such as one in Ghana, where computer science is taught using chalk blackboards because the minimum hardware is not available (see Sacristán et al., 2018).

Thus, there are issues that affect digital integration such as: digital gaps and fears; access (particularly in rural schools); and issues of professional development. These examples show the discrepancies and the digital gap between developed and developing countries. Thus, that which is readily available and taken for granted in some countries, is scarce in others.

At the same time, however, there are a few commonalities in all countries. In particular, it seems that the transition to meaningful integration of digital technologies for mathematical teaching and learning has been much slower than anticipated.

4.4. Political and social forces (e.g., the COVID-19 pandemic)

Another important discussion point is that of policy. Top-down policies can generate some changes, but so do changes in society. For instance, the changes that have happened due to the COVID-19 pandemic have pushed the use of technology, although perhaps not in the most significant or favourable ways.

In Mexico, in the late 1990s, there was a push to insert digital technologies in schools by the government, with the EMAT (mentioned above) and other parallel or related programmes (e.g., Enciclomedia — see Trouche et al., 2012, and Sacristán et al, 2021). Then, in 2007, the government stopped those important programmes, and other smaller initiatives were not very successful.

Thus, when the COVID-19 pandemic hit, schools were unprepared to deal with the necessary distance education. Elementary and middle-schools relied on television programs and YouTube videos. And teachers used WhatsApp to send homework to students and vice versa (students sent homework back to the teachers). That is, technology was not being used to change the way mathematics is taught, nor did it encourage mathematical learning. On the other hand, it is a social situation that pushed for more technology use, in particular, for online teaching and learning. It would be interesting if we could harness that momentum to veer educational practices and cultures towards more innovative uses of digital technologies in schools.

5. Harnessing the Potentials of Digital Technologies for Creating Innovative School Cultures

Something that was called for by Seymour Papert (2006) in his keynote at the ICMI 17th Study Conference in Vietnam, was that we should devote 10% of our time to reflect on how technology can create new mathematical ideas and practices –something which we have dubbed Papert's 10%.

Thus, one aspect that we need to consider is how take advantage of prevalent technologies and trends, if we are to break away from what he criticized, when he said that what we do in schools is like operating airplanes with schedules of horse-drawn carriages (Papert, 2006). How can we take advantage, for instance, of the trend towards coding and computational thinking? Or, how can we take advantage of connectivity, mobile technologies and social networks (mentioned in section 3.1) for mathematical activities, rather than just for communicating (such as the non-mathematical use during the pandemic in Mexico)?

In the following section, I present some selected examples of constructionist implementations that have taken advantage of connectivity.

5.1. Selected examples of "connected" constructionist projects

The first example is not exactly a project or implementation, but an example worth discussing: that of the Scratch programming environment (https://scratch.mit.edu/). Scratch is a beautiful community where people share their projects online, programmed in Scratch; children are creating in Scratch and they are remixing them. In https://scratch.mit.edu/statistics/one can see the trends of how Scratch has been used; its use has been increasing over the years and during the COVID-19 pandemic, it's increase was huge. Yet, if we ask how mathematical is its use? The answer is probably not that much. In the UK, however, the ScratchMaths curricula (http://www.ucl.ac.uk/scratchmaths) was developed in an attempt to bridge computer programming in Scratch with mathematical thinking and learning. The emphasis of ScratchMaths may have been more on computer programming for mathematics, rather than on the social collaboration through the connectivity of Scratch, but it is still worth mentioning. Some results of ScratchMaths are reported in Benton et al. (2017). One interesting point that arose from ScratchMaths, is the issue of *fidelity* (see Hoyles et al., 2020): No matter how well designed a programme is, there is a question of how it is then implemented by the teachers; that is, there is a gap between the intent of the developers and the understanding the teacher.

Thus, technologies are often appropriated in ways unanticipated by their developers and may not yield the expected results (they may even go against some of their fundamental principles; for instance, when in constructionist designs, students are not allowed to create).

Other examples of "connected" constructionist projects are older, but they are worth presenting. One is the Weblabs European project (WebLabs, 2011; Noss and Hoyles, 2005) that took place from 2002 and 2005, and in which I was fortunate to participate. Students, 10 to 14 years old, from several countries across Europe, engaged in scientific and mathematical investigations. The project investigated new ways of representing expressive mathematical and scientific knowledge. Students programmed models of their ideas using the non-text-based computer programming environment ToonTalk (http://www.toontalk.com). One very interesting thing was that the participants collaborated online, sharing and discussing their investigations and constructs using a type of blogs called WebReports, which was something that was very innovative at the time. Unfortunately, it didn't continue after the project ended.

Inspired by WebLabs, we developed a project in Mexico, which we called the iMat online virtual mathematics laboratory, which was a distance education learning environment (see Olivera and Sacristán, 2012). In this project, university students engaged in collaborative mathematical explorations through model-eliciting (Lesh et al., 2000) activities, to discuss and reflect upon various types of real-life mathematical problems. One of the difficulties that our team encountered was in designing the activities. It was very difficult to break the inertia of how established curricular tasks are commonly structured and come up with innovative designs that really engaged learners. After months of struggle, we came up with activities related to selected themes (linear motion; free-fall and gravity; population growth; cryptography), where students were involved in tasks such as analysing videos and other data, and constructing models (e.g., of the gravity on the Moon). Digital technologies (e.g., video software for frame-by-frame analysis; a virtual ruler for measurements; spreadsheets or CurveExpert (http://www.curveexpert.net) for finding mathematical equations to fit the data; and modelling software, such as Modellus — https://modellusfq.blogspot.com/) were used in such tasks; as well as for collaborating, sharing, discussing online (as was done in WebLabs) and proposing new explorations, through a web-based discussion forum. The experiences were very rich and rewarding, but after a couple of years iMat was discontinued because it required too much effort and did not fit with the established curriculum.

This is what tends to happen. There are many wonderful innovative projects and programmes across the world, but there is a lack of continuity. More often than not, the projects' funding ends, they are abandoned, and there is rarely any uptake by others. For instance, the WebLabs files can now only be found in archive.org's Wayback Machine (WebLabs, 2011).

In fact, many interesting projects remain unknown and do not get much projection beyond some publications. Related to that latter point, I must also mention the very interesting work carried out for many years by Chronis Kynigos and his colleagues at the Educational Technology Lab of the University of Athens, Greece (http://en.etl.eds.uoa.gr/educational-technology-lab-etl.html). They have developed wonderful constructionist materials (software and projects) over several decades, but they lack resources — as does MIT to disseminate Scratch — so their work has not expanded much beyond their group.

So, even when there are wonderfully designed innovations, we see other social forces and trends (lack of continuity, of support, or issues of fidelity) that restrict their possible successful implementations in schools.

5.2. Can digital technologies serve as catalysts to change school cultures?

Hoyles pointed out that her interest in digital technologies has been to "help learners open windows to mathematical knowledge by using digital technologies in innovative, future-oriented and intellectually rigorous ways" (in Hoyles et al., 2020, p. 70). I believe that digital technologies can act and help to create new cultures in the classroom. But the questions remain: How can we promote the necessary changes? How do we break with the inertia of school cultures to harness the potentials of digital technologies for mathematical thinking and learning and so they can serve as catalysts for creating new school cultures?

One aspect is that it is important to start with what is already being done. That is, to work with and within the school systems and curricula. Curricula and classroom cultures are not going to going to change soon, so we need to work within them, rather than against them. But there has to be an openness to change.

If there is some openness, one approach to change, is to gradually adapt new pedagogies and designs to integrate digital technologies in innovative (e.g., constructionist) ways. For that, I particularly like structured environments that follow the extended microworlds idea proposed by Hoyles and Noss (1987), where a microworld takes into account several components: the student, the context in which the learning takes place, the pedagogy, which includes the teacher and how they orchestrate all the didactical materials, as well as the technology itself, that is the technical aspect (see also Sacristán et al, 2009).

But none of this can be achieved without support from the authorities and, most importantly, without considering the teachers (promoting their involvement in generating change, resources and decision-making), as well as supporting and training them (professional development). Thus, I continue by focusing on the role and importance of the teachers for generating technology integration in math classrooms.

6. The Role and Importance of the Teacher

The teacher is the key player for successful implementations of technology-centred educational innovations. However, as Paul Goldenberg said, it is necessary to "provide instruction and time for teachers to become creative users of the technology" (Goldenberg, 2000, p. 8). We saw that, in the EMAT programme, where even the most motivated and supported teachers — who were directly and continuously supported by us — that it took them three years to actually grasp and change their ways of teaching, and to change their classroom cultures (Trigueros and Sacristán, 2008).

Thus, changes need to take place in gradual steps, and fit in with what is already being done. We have also discussed, in section 5.1, the issue of fidelity (see Hoyles et al., 2020), where teachers need to develop an understanding of innovative designs in order to implement them with more fidelity to the main principles, in order to ensure a higher probability of success. Wright (in Aldon et al., 2017, 54:02) gave the following points to be taken into account to help teachers adopt innovative technologies: *ease of use; small steps* — that fit well into teachers' existing practice —; have a *perceived immediate gain* to students' learning; and support (technical support, support from a professional learning community, and *support* from someone who will give initial boost to the innovation and sustain its promotion).

This is why continuous professional development (PD) and support in the use of digital technology is of upmost importance. As Celia Hoyles mentioned at the ICTMT 2017 in Lyon, France: "You will never achieve things in the classroom without proper professional development" (C. Hoyles, personal communication, July 2017). As I will discuss further below, that is also related to the need to build communities of teachers. In Faggiano et al. (2021), we identified some theories that inform the design of professional development programs for integrating digital technologies in mathematics education, and which provide insights for future PD implementations. We should also consider the role of connectivity and distance education, and changing the role of the teachers and their professional development

Thus, we need to provide teachers with professional development but, at the same time, also involve them as active collaborators in generating the changes for meaningful technology integration; that is, involve them in designing resources and taking decisions.

When teachers are involved as co-creators (working *with* teachers), they can appropriate themselves better of new resources: if they feel they are participants themselves, they have more ownership, and the appropriation is easier to achieve. For that, there needs to be collaboration with researchers, so that teachers feel they are also decision-makers; this is crucial in terms of motivation, in affecting their beliefs and overcoming their fears and apprehension. Also, involving teachers in the design process, may help them improve their Pedagogical Technology Knowledge (PTK).

Pepin et al. (2017) explain that teachers' design capacity of resources depends and can be refined by (i) how they understand and transform existing resources ("re-mixing") to (re-) design instruction; (ii) the organization of collectives of teachers and designers, not only for design, but to share and observe other teachers' experiences; and (iii) multi-national efforts to create quality digital resources.

The communities, collectives or networks, in which teachers and designers can participate, can also be online ones (e.g., France's Sésamath — http://www/sesamath.net/ — see Trouche et al, 2012).

In terms of the multi-national efforts to create quality digital resources, a notable mention is the European Mathematical Creativity Squared (MC2) Project (http://www/mc2-project.eu), which had amongst its aims, to rethink the nature of open educational resources, create Communities of Interest in four European countries, and collectively design and produce digital content for creative mathematical thinking (M C Squared Project, n.d.).

Thus, teachers' collectives or communities of practice, where they can share, reflect and be supported by peers, experts and researchers, are important. I personally have been involved in a couple of experimental programmes where we have had in-service teachers reflecting and collaborating in small communities (e.g., Parada et al., 2013; another is mentioned in Sacristán et al., 2011 and Trouche et al., 2012). Participation in communities where teachers can share and reflect on their practice, seems to help and enrich the integration of technologies in their practices and create the needed changes in the classroom cultures.

To work with teachers is also a way to involve them in Papert's 10%, that is, for them to reflect on what new knowledge can emerge from the use of technology (Papert, 2006); to think about their own learning and their students learning; and to think differently, so that they can innovate and change school cultures.

7. Concluding Remarks

To conclude, I would like to summarize some of the main points discussed in this paper:

- The inertia of the classroom and the paper-and-pencil cultures limit change.
- The teacher is a key player for successful and transformative technology integration.
- However, we need to promote models of collaboration (such as communities and networks) between teachers, researchers and policy-makers: (i) to enhance teachers' professional development, (ii) to empower them, and (iii) to provide means for sharing, discussing and improving resources and their implementation.
- The educational systems also need to change and provide flexibility for teachers to have time to engage in collaboration and innovation.

It is thus that we may be able to "ride the wave" with society's trends in technology-use to harness them and veer them towards meaningful mathematical learning opportunities and new practices. In particular, mobile technologies are more accessible in many regions and social strata, so we need ways to take advantage of them for more mathematical practices, teaching and learning. In general, we should find ways to shift the emphasis in classrooms from technologies for communication, information and presentation, to technologies that promote mathematical thinking.

I end with a call to action, akin to Papert's 10%, to reflect on how else we can use digital technologies to be catalysts for innovation of mathematical practices and learning, and that change school cultures.

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Professional Development of Mathematics Teachers: Perspectives and Experience from East Africa

Veronica Sarungi[1]

ABSTRACT Teacher professional development is important in order for teachers to effectively address changing contextual realities. Effective professional development builds on teachers' experience and relates to their practice. The paper presents guiding ideas and lessons learnt from teacher development component of a research project that aimed at improving numeracy performance of pupils by focusing on teachers' assessment practices. Based on conclusions, recommendations are made for possible approaches to future PD especially in similar contexts.

Keywords: Pedagogical content knowledge; Professional development; Reflection.

1. Background

Villegas-Reimers (2003) views professional development (PD) as development in one's role as a professional. Mathematics teachers through participation in PD can work towards enhancing their professional competences. Professional development of mathematics teachers should be a continuing process in order to provide support in changing educational contexts. Teachers and their PD are part of the educational reform process but not always in central place (Dachi, 2018). Guskey (2002) maintains that PD should result in change in teachers' classroom practices, beliefs, attitudes, and ultimately influence positively students' learning outcomes. PD that results in change can be viewed as effective since it leads to improvement in professional work. At the same time, Korthagen (2016) contends that successful PD is one that takes into account the person of the teacher and what they value in their practice. Meaningful PD takes into account the experience of teachers in their professional learning process.

This paper will showcase a research project that had an emphasis on mathematics teacher professional development. The priority of the research project was to increase numeracy performance among pupils in selected Tanzanian primary schools by focusing on teachers' classroom assessment practices. The guiding ideas for professional development component will be discussed and also the experience of teachers and teacher educators from the project. Final discussion will be on possible

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recommendations for future mathematics professional development especially in similar contexts.

1.1. Country context

Tanzania is located in East Africa and has a population of over 57 million (National Bureau of Statistics, 2021). Current school system is one year pre-primary, seven years of primary followed by four years of lower secondary and three years of upper secondary (Tanzania Institute of Education, 2019). Education up to end of primary school is mandatory. In 2016, fees were removed from government primary schools, which caused a sudden large influx of pupils and teacher-pupil ratios to increase dramatically. Curriculum reform in Tanzania has been gradually shifting from content-based to a competence-based approach. A competence-based curriculum was first implemented in 2005 and introduced concepts of constructivism and learner-centered strategies. Starting from 2015 the primary school curriculum was reformed to address some emerging challenges and changes implemented in gradual phases (Tanzania Institute of Education, 2019).

A challenge for teacher education is that the complementary initial teacher preparation curriculum is implemented several years after the classroom curriculum and so usually in-service PD is expected to bridge the gap between old and new approaches to learning. Primary school teacher preparation takes two years and admission is after lower secondary school. Primary teachers are generalists in that they are expected to teach all subjects offered at primary level. For standards 1 and 2, each class has one teacher taking all periods and the focus is on basic literacy commonly referred to in the context as the 3R's (reading, writing, arithmetic) (Tanzania Institute of Education, 2019). From standards 3 to 7, there is introduction of separate subjects namely Mathematics, English language, Kiswahili language, Social Studies, Science and Technology, Civics and Moral Education, and Vocational Studies (Tanzania Institute of Education, 2019). Teachers from grades 3 take one or more specific subjects across one or more standards. In some schools there is a tendency to allow teachers to specialize informally by teaching the subjects of their preference but mathematics is generally not a preferred choice for most teachers because it is seen as one of the difficult subjects to teach.

1.2. The AFLA research project

The Assessment for Learning Africa (AFLA) research project aimed to increase numeracy performance among pupils in selected primary schools by focusing on teachers' classroom assessment practices. Another goal was to understand how assessment for learning (AfL) could be used and applied in challenging urban contexts. The research project took place in three sites across two countries, Tanzania and South Africa, with the Oxford University Centre for Educational Assessment (OUCEA) being the lead. Six primary schools were selected in Tanzania from a district of the largest urban settlement. All schools had large number of pupils per classroom as this was the greatest challenge in schools encountered after the removal of fees with a sudden influx of students before additional resources such as classrooms and teachers could be deployed. Classroom sizes ranged from 85 to 215 pupils. The research project was from 2016 to 2019 and had two major types of activities namely teacher development and specially designed tests to measure students' numeracy levels. This paper will focus on the teacher development activities.

The teacher development in AFLA had two major components. The first was a series of eight (8) workshops spread across one academic year. The second component was lesson observations and mentoring talks that took place between workshops. Tab. 1 shows the dates and activities for the workshops. The first three workshops took place on consecutive days immediately after the official country launch. The workshops planned to enhance teachers' use of AfL while taking into account contextual realities. For example, workshop seven had not been planned to be a joint session with educational leaders at school and ward level but during the formal launch many participants pointed out the importance of having a session for this group of leaders and this was accommodated by having one workshop structured so that some sessions were separately for teachers and a combined session with these leaders. Apart from AfL, the workshops also focused on building teachers' skills in reflection so as to enable them to learn from own practice as they tried out new strategies.

Tab.1. Summary of activities of Teacher Development Workshop (all done in 2017)

Workshop number	Activities, focus areas, notable features	Month, date
One	Setting the scene for AfL	January, 25
Two	Experience of AfL	January, 26
Three	Model class teaching	January, 27
Four	Critical Incident Analysis	April, 11
Five	Questioning	April, 12
Six	Feedback (part 1)	June, 29
Seven	Feedback (part 2) and combined session with leaders	August, 8
Eight	Peer and self-assessment & Reflecting back	September, 1

Lesson observations were planned with teachers of the six case schools who taught the target classes of standard four. The fourth year of primary had been selected because pupils had completed at least three years of mathematics teaching but were still considered lower primary pupils. There was mutual agreement on the date and time for observation between the teacher and the teacher educators who did the observations. All teachers were familiar beforehand with the observation schedule used. A non-participant observation was made of the lessons and after the lesson the mentoring talk began with a teacher self-evaluation (both written and oral) followed by a discussion.

2. Experience about Professional Development from Research Project

The research project was aimed at understanding how AfL could be used and reflection about the experiences of the teacher development component also give insights that could be applied to teacher professional development in the future. The section will present six such experience and also link to achievements of the research project in terms of change to practice and research objectives.

2.1. Listening rather than telling

The approach in all workshops was to listen to teachers rather than telling them what to do. In case a new concept was to be introduced, a series of activities was designed through which participants would eventually arrive at a similar conclusion as intended by facilitators even if sometimes modified to accommodate contextual realities. For example, in order for participants to appreciate the need for AfL, the results of the baseline test both in summary form and sample work was shared with participants. Teachers were then asked to discuss in groups and respond to different questions such as "what can be said about the pupils' intentions about mathematics?" In their discussions, teachers pointed out that apart from assessment of learning, which is the norm, there needs to be other forms of assessments that can obtain such information since this is crucial. Another example, of listening rather than telling is when teachers were asked during workshop five to state challenges to using questioning in their mathematics lessons and then in groups work on possible solutions. What was achieved was that the emerging techniques were co-constructed between the facilitators and participants and contextually-relevant thus meaningful to the teachers.

2.2. Experiencing novel teaching strategy through new mathematics

The three major areas of teacher knowledge are content knowledge, pedagogical knowledge and pedagogical content knowledge (Shulman, 1986). Through previous experience working on mathematics teachers' PD it was known that apart from pedagogical knowledge (PK) of new strategies, teachers would also need enhancement of their content knowledge (CK). A design choice of the PD was to embed the new mathematics as part of novel teaching strategy so that teachers could experience the sense of being learners and also feel less threatened by the new content. One example was during workshop six when in response to a query raised by teachers in the previous workshops on how to deal with unexpected responses during questioning, the feedback approach was modeled by asking participants to discuss in groups different questions linking perimeter and area. The choice of the topic was in turn based on lesson observations and awareness that teachers were about to introduce the concept of areas to their learners. Through this combined approach to CK and PK that took place in every workshop, it was possible to address matters of teachers' conceptual understanding as well as showcasing AfL strategies.

2.3. Acknowledging emerging issues

As stated earlier, the whole research project had the approach of including participants' views. Apart from the inclusion of leaders, another example of acknowledging emerging issues is the choice of material for critical incident analysis in workshop four. During the opening workshops the issue of harsh disciplining had emerged as a possible conflicting matter. The choice of the material for the critical incident analysis introductory session was therefore an education cartoon strip from a well-known school magazine, which highlighted the negative effects of harsh discipline tactics. In later workshops, teachers expressed how they had shifted to alternate forms of behavior management and their realization that the teaching of mathematics was positively impacted by this change. Emerging issues also included teachers' needs that were presented during mentoring talks. For example, the number tray as a concrete tool for visualizing operations and place value was presented at the fifth workshop after some teachers during mentoring talks mentioned their unfamiliarity with this resource. The acknowledgement of emerging issues ensured that the PD addressed contextual realities appropriately and as needed.

2.4. Learning from and with each other

During the workshop's participants were encouraged to share best practices in teaching mathematics and application of AfL strategies. Apart from providing important information to the facilitators for understanding the application of AfL and generally mathematics teaching, the practices shared were then modified during discussions and also adapted as observed in subsequent classroom observations. Discussions during mentoring talks confirmed that the new practice seen was a result of what had been shared during workshops as was the example of allowing pupils to give the name of an animal of their choice to their group. Thereafter, teachers to motivate pupils during group work used the positive characteristics of the chosen animal. Thus, there was a diffusion of some of the best practices across the case schools and classes.

2.5. Adapted reflection for better practice

Another achievement of the research project and its teacher development component was to make explicit reflective practice that was already part of the recommended teaching and learning guidelines in Tanzania. While reflection was part of the lesson plan template, many teachers were not sure on what to write and how to use the information for improving future lessons. The AFLA project introduced a simple four-part template based on critical incident analysis for group reflection but also a self-evaluation form with guided questions that was completed after lessons observed by a mentor. These observed lessons were less than five for the duration of the project but teachers had the option of using some or all of the questions as self-evaluation for other lessons. While teachers did not use the forms often beyond the required sessions due to large workload, nevertheless, what could be deduced in the final workshop was that ability to reflect had improved since their reflections tended to focus on academic matters and was more critical. In addition, during mentoring talks teachers were able to link proposed future plans to reflections about current observed practices.

2.6. Contextualized change to practice

The teacher development activities were all done in 2017. Two rounds of interviews were conducted, one at the end of 2017 and the second in 2018. Additionally, during the dissemination workshop in 2019, teachers were invited to discuss and validate some of the emerging findings. What could be discerned was that as a result of the PD that there were changes to teachers' practice but these changes depended on what they were doing and what was already in practice. For example, teachers mentioned consciously giving pupils' more opportunity be that in inviting volunteers to the blackboard "teach other pupils" or by telling the intended topic ahead of time so that pupils could then later share about prior or home knowledge. Another example was the improved use of group work. Previously, group work was seen as a means to manage very large class sizes. After PD, teachers mentioned how group work could be used for AfL strategies such as eliciting information about learners' knowledge and feedback through peer assessment. Finally, teachers mentioned that they consulted with other teachers but only in case when they faced specific challenges. The consultation was a positive change since previously the tendency was to see PD and advice as being external only.

3. Conclusions and Recommendations

Reflecting on the experience of the teacher development activities, four conclusions can be drawn. First, involving teachers' experiences and ideas especially with regard to contextual factors and possible solutions ensured that teachers were willing to actively participate in the PD activities. Their willingness was evidenced in opening up their classrooms for observations and new practice observed several months after a given workshop. While it is uncertain how sustained these changes were, these positive shift in practice was probably due to having their experiences respected and included in PD activities. Second, there was win-win situation in combining activities for enhancing teachers' content knowledge and specific tasks related to pedagogical knowledge. The facilitators could demonstrate to teachers how pedagogical content knowledge (PCK) specific to mathematics was applied with the new technique while teachers could to some extent share in their learners' experience ahead of time. The third conclusion is on the importance of adaptation of the PD activities. The process as well as content of the workshops was modified to address emerging issues in order to remain relevant to teachers' needs and realities. While the overall aim was maintained, that is to work teachers' assessment practices so as to improve numeracy outcomes, other factors as mentioned by teachers was also included in subsequent plans and implementations. The last conclusion is about the importance of both personal and community learning. Individual teachers were encouraged to work on self-evaluation and implementing new learning in personal practices but there was also an element of collaboration both with the mentors and during workshops to develop an improved shared understanding of AfL in mathematics classrooms in context of large classrooms.

Based on these conclusions, four recommendations emerge for professional development especially in similar contexts. First, it would be very important especially when bringing seemingly new innovations to teachers, to start the PD from teachers' experience rather than focusing on deficits. By respectfully listening to teachers there would be an opportunity for all stakeholders to develop professionally. Second, it may be useful to have teachers undergo the experience of being learners and thus reflect back on what it would take to facilitate learning. These opportunities for experiential learning could then build teachers appreciation of the complexities of PCK. In the case of the research project, AfL strategies were the basis for creating these learning opportunities. It may be necessary to research on what could provide similar opportunities for such combined approach and perhaps the use of technology in teaching mathematics could be a possibility. The third recommendation is the need for a flexible approach to accommodate emerging issues in PD. The design of PDs may benefit from having the both content and process not firmly defined so that adaptation to contextual realities can occur when and as needed. The final recommendation is on the importance of developing professional learning contexts that would enable reflections and exchange of ideas for professional growth. PD needs to be a continuing process but in order to achieve this in a feasible manner especially in challenging contexts it is important to shift perceptions about PD as being externally facilitated and done as one off workshops towards the view of professional learning communities. Hopefully, there will be a realization that PD is not for blame nor praise but part of being a teacher.

Acknowledgments

The AFLA project was funded through ESRC-DfiD fund (ES/N010515/1). I would like to acknowledge the country lead for Tanzania and Co-Principal Investigator of the whole project, Professor Anjum Halai of the Aga Khan University as well as the Principal Investigator, Professor Therese Hopfenbeck of OUCEA for their outstanding leadership and support.

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Argumentation towards Educational Change in Mathematics

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ABSTRACT We refer to four general theories of argumentation that provide insights on innovative current approaches in mathematics education. Through several examples of tasks, we show the richness of argumentative practices in the learning and teaching of mathematics that have some bonds with these general theories of argumentation. We show, however, that these theories do not capture the specific processes and the complexities of argumentation in the learning and teaching of mathematics according to innovative pedagogies. We pledge for new advances in mathematics education based on design-based research that fosters deliberative, epistemological, rhetorical, and structural aspects of argumentation.

Keywords: Argumentation; Argumentative designs; Model for argumentation.

1. Introduction

Argumentation is as old as the history of civilization. Van Eemeren and Grootendorst (2016) trace its origin in the birth of democracy in Ancient Greece. Cities were ruled by kings that used their power to take advantage of their citizens. Some literate people decided to represent dispossessed citizens in tribunals against them. These literate people, later on called Sophists, developed techniques to convince judges of the rightfulness of the claims of their defendants. They first acted for political reasons — to challenge laws decreed by dictators and Gods and to defend citizens against unjust claims, but gradually found in their techniques sources of income and developed sophisticated techniques to fool judges. Argumentative techniques became tricksy. Socrates, who featured in Platonician philosophy, used argumentative techniques to oppose and defeat Sophists on their own ground and to reach eternal truths. Therefore, from its inception, argumentation was polysemic — it was a rhetorical, deceiving, and epistemological tool at the same time. Interestingly, Plato excluded mathematics from its argumentative epistemology and reserved argumentation to reach the truth in social domains. As exemplified in Meno, for Plato, mathematical proofs are reached through logical moves only.

The Platonician view of mathematics has dominated the scene for more than 2000 years. Although, as Netz (1998) noticed, Greek mathematicians used geometrical

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figures as rhetorical tools for convincing their audience of the correctness of their proofs, the apodictic presentation of mathematical knowledge suggested that any hesitation in quest of a mathematical truth uncovers the limits of the human mind. No room is left for discussion or argumentation when mathematical truths are at stake.

The polemic about the intuitionists and the formalists constitutes one of the first cracks in this epistemological view. Hilbert's formalism contrasted with Brouwer and Poincaré's intuitionism as fundamental for the foundation of mathematics. These giants in mathematics did not articulate additional details on the elaboration of mathematical knowledge. Polya's How to solve it? (1945/2004) and Lakatos' Proofs and Refutations (1976/2015) are landmarks in the epistemology of mathematics. Although there is no unanimity on their contributions among mathematicians, their contribution to the realm of education is enormous. Polya's heuristics and Lakatos' refutations point at crucial moments in mathematical activity during which the epistemological status of statements is at stake. Argumentation is in the air.

Philosophers of mathematics such as Rav (1999) bridged between these two perspectives but stressed insight and meaning of mathematical actions over formal logical structure when describing proof as a sequence of claims, where "the passage from one claim to another is based on drawing consequences on the basis of meanings or through accepted symbol manipulation, not by citing rules of predicate logic" (p. 13).

The formalism-intuitionism controversy is echoed in two pedagogical approaches to proofs in mathematics education. In traditional education, proving activities tend to be substantiated by a formalistic approach. Proving is often disconnected from conjecturing (Aaron and Herbst, 2019), a great emphasis is given to proper proof inscription (e.g., Dimmel and Herbst, 2020), proof comes without conviction and explanation (Hanna, 2000) and is presented as merely devoid of human agency (Morgan, 2016). A formalistic approach to proofs, in the educational context, implies that while the presenter's sole responsibility is to state true statements considered as proofs by expert mathematicians, the responsibility of their readers is "to convince themselves" of their correctness. Proof presenters are not expected to convince their audience; their presentation is monologic and often does not contain informal arguments like diagrams or specific numeric examples (Fukawa-Connelly et al., 2016). This formalist approach to the teaching of proofs is repeatedly criticized by contemporary thinkers in mathematics education as causing students to be excluded from the "mathematician society" and to feel that if they do not understand proofs presented, "there must be something wrong with me."

In contrast, among educators in mathematics that promote novel pedagogies, the practices of mathematicians that stress the non-formalism of mathematics are models for educational practices. These novelties are particularly salient in the domain of mathematical proofs. In pioneering efforts, Mejia-Ramos and Inglis (2009) surveyed argumentative and proving activities in mathematics education in published research journals. They relied on De-Villier's (1990) model of proof functions, which is based on sub-activities whose nature is argumentative: proof construction, proof comprehension, and proof presentation. Construction activities are divided threefold into the exploration of a problem (related to the discovery function), estimation of the truth of a conjecture (referring to the verification function), and the justification of a statement estimated to be true (related to the explanation and systemization functions). The comprehension proof activity includes understanding a given argument and evaluating an argument concerning a given set of criteria. As for proof presentation, Mejia-Ramos and Inglis differentiate between two types of proof presentations: First, to explain the argument as a claim to a given audience and convince them that this claim is true. Second, demonstrate to an expert one's understanding of the given argument. In their review of research on mathematical proofs in education, Mejia-Ramos and Inglis (2009) found that most of them focused on proof construction, a minority involved proof comprehension, and none examined proof presentation. Proof presentation, therefore, is considerably understudied.

Mejia-Ramos and Inglis' observations are not neutral. They convey a deep concern about the teaching of proofs in mathematics classrooms. Indeed, many scholars have reported a superficial preoccupation with technicalities and refraining from giving students an open space to explore through dialogue. A recent example can be found in Dimmel and Herbst's (2020) report on what they call "proof transcription", a prevalent American proof-related activity in which teachers require "mark-for-mark reproductions of written proofs that students would copy to the board from a note sheet" (p. 72). The researchers worry that this activity involves an obsession with notational details that reduces the opportunities for students to develop other mathematical communication skills and does not foster a sense of discovery and the gaining of mathematical insights (de-Villiers, 2020; Dimmel and Herbst, 2020). These concerns partly explain the decline of proof-related activities in mathematics classrooms, which also originates from the typical, non-dialogic educational strategies that do not engage students meaningfully (de Villiers, 2010; Herbst and Brach, 2006). In proving activities that do not emphasize the discovery function of proofs through dialogic processes, students are more inclined to perceive proofs as a tedious chore to satisfy the teacher instead of an exciting task to satisfy their own curiosity (Lavie et al., 2019).

In this worrying context, several researchers have invested efforts in promoting new tasks on mathematical proofs. We do not review these efforts. We refer to our own line of research (Schwarz et al., 2010), which stresses the ubiquity of argumentation in mathematical practices related to the elaboration of proofs that model the practices of mathematicians. Schwarz and colleagues have identified three different argumentative activities: (1) Enquiring — an initial probing stage that concerns conjecturing solutions. It includes preliminary actions for making sense of a problem and setting a tentative plan for the solution process. (2) Proving — activity aims to find logical consequences to turn conjectures into proofs. (3) Inscribing proofs involves translating and rearranging the proof as a chain of logical inferences in a formal way.

In their model of argumentative activities in mathematics, proof plays a central role both as a process and as an artifact that is a product of the argumentative activity.

In their efforts to convey the ubiquity of argumentation in authentic proof activities, Mejia-Ramos and Inglis, as well as Schwarz and colleagues, may aspire to achieve the same pedagogical ideal. However, their use of the term argumentation is not exactly the same. Indeed, many researchers who relate to argumentation in their studies use different definitions, which may lead their results to be misinterpreted. Hence, in the next section, we will discuss four general theoretical models for argumentation. Each model emphasizes different aspects and functions of argumentation. We will show that these models provide a "grammar" for argumentative activities in mathematics.

2. Succinct Considerations about the Theories of Argumentation

Many general theories of argumentation have been developed in the last 70 years. These theories were developed by philosophers and logicians who were not acquainted with the world of education and the world of learning. As noted by Schwarz and Baker (2017), this fact suggests that a general theory of argumentation for learning is necessary, to which they contribute. However, we claim that the general theories of argumentation are relevant to argumentation in mathematics. We review the four leading theories succinctly. Two monologic theories were developed by Perlman (Perelman and Olbrechts-Tyteca, 1958/2012) in his New Rhetoric and Toulmin (1958/2003) in his The Uses of Argument, which are, respectively, discursive and structural. Both Perelman and Toulmin see argumentation as a technique for structuring discourse in order to lead the auditory to accept it; the second perspective sees it as a complex and differentiated structure of interrelated statements. Both theories are monologic. Both are highly relevant to education. For example, Perelman's New Rhetoric merges Aristotelian dialectic and persuasive discursive techniques that may help the audience (the learners) become convinced of the correctness of the argument. Toulmin's argument schemes provide a language for specifying the roles of various types of statements in argumentative discourse.

The two other general theories of argumentation are dialogical. Van Eemeren and Groothendorst (2016) have developed a pragma-dialectic model of argumentation, which is modeled as a critical discussion. This critical discussion is discursive. It is conceived as a multiparty game, with a starting position, allowable and obligatory "moves" (speech acts), and rules for deciding who won or lost. This relates to a constructivist theory of truth, according to which what is true is not correspondence with facts or states of affairs but rather what has emerged as the "winner" from a societal debate. It is also based on dialogical logic (Barth and Krabbe, 1982/2010). The theory is intended to be both descriptive and normative — deciding what a reasonable way to discuss, for which set of rules governs the dialogue game. Argumentative discussions go through several stages: confrontation, opening, argumentation, and concluding. Plantin's (2005) argumentation dialogue arises once the discourse of one person is not accepted (or is called into doubt, questioned) by another person, who then produces a counter-discourse concerning it. Argumentation dialogue is a confrontation of discourses, from which emerges a question to be debated, to which discourse and counter-discourse are justifications for the answers either "Yes" or "No".

We suggest that all four theories of argumentation help understand pedagogical novelties used to promote mathematical ideas through argumentative processes. Interestingly, Toulmin (1958/2003) thought that his model of argumentation was applicable for many contents but excluded mathematics from the realm of application of this theory. Ironically, the Toulmin model is the predominant model used by researchers in mathematics education, probably because research in mathematics education has shown that learners generally rely on informal considerations to elaborate mathematical claims.

3. New Directions in Argumentative Activities in Mathematics Education — Theoretical Examination

This section presents several examples of activities designed to encourage argumentation in mathematics. Most of them have been implemented in Israeli schools. Our focus on this particular context does not point at provinciality but at the importance of knowing the exact circumstances that afford the deployment of argumentation. We show that these examples refer to some extent to the general models of argumentation we just reviewed. We show that this reference sheds light on the argumentative nature of these activities. However, we show that the general theories fall short in capturing some other critical aspects, such as the role of resources and the role of dialogic norms. More generally, we show the decisive role of educational design in affording various aspects of mathematical argumentation. We then stress the importance of theorizing several aspects of mathematical argumentation (epistemological, dialogical, rhetorical, structural) and show that it characterizes novelty in mathematics education.

3.1. Critical discussions in mathematical tasks

We begin our review of innovative tasks in mathematics education with the pragma-dialectic model that van Eemeren and Groothendorst (2004) developed — a model of critical discussion. It requests different reasoned arguments as a starting point of the discussion. The six-cards task (Schwarz et al., 2000) is presented in Fig. 1. Different preconceptions about decimal numbers (also called conceptual bugs) could be detected when the students solved the task alone. Examples of preconceptions are (a) identifying 4.3 and 4.03 as being the same as "0" does not count, or (b) claiming that 4.7 is less than 4.3 because dividing a whole into seven parts leaves less for one part than when dividing the whole into three parts. They were then arranged in dyads. The students were encouraged to discuss their solutions, and in the case of disagreement, to check their hypothesis with a calculator. Schwarz and colleagues identified the 'two "wrongs"

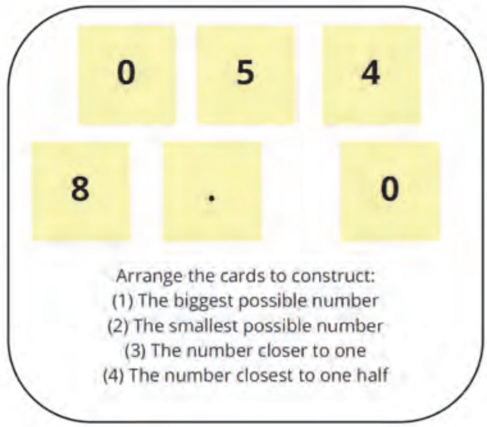


Fig. 1. The six-card task

make a right if they argue together' phenomenon through the six-cards task. The students criticized each other and were able to fix the bugs of their mates mutually. The six cards task led to a conceptual change with respect to the understanding of decimal numbers. This change was shown to be triggered by the deployment of argumentation in interactions among dyads. This argumentation can be referred to as a critical discussion (van Eemeren and Grootendorst, 2004). Each student had a firm preconception that led to a productive interaction among disagreeing peers. However, we doubt that van Eemeren and Groothendorst would envision such a kind of argumentation. Schwarz and colleagues showed that the discussion was nurtured by incessant hypothesis testing undertaken by the students. Explanations were convincing only when they followed the testing of conjectures with the calculator.

The two wrongs that make a right phenomenon is interesting but rare, though. More generally, critical discussions do not easily emerge in mathematical tasks. The argumentation in the case of the six cards task was productive because the different preconceptions did not relate to different levels in mathematics. Students with different preconceptions adopted wrong strategies but, at the same time, had comparable levels. This situation led them to criticize each other in a constructive way, co-elaborate on a right answer, and achieve a conceptual change through argumentation. In another experiment, Schwarz and Linchevski (2007) designed the Blocks task (Fig. 2). Students solved the task alone. They were then arranged in dyads and were provided a balance to test their hypothesis. In this case, too, their interactions led to conceptual change (in proportional reasoning), and some examples of argumentative processes that led to this change could be detected (Schwarz and Linchevski, 2007). However, a fine-grained analysis of the talk of dyads showed that this was generally not the case (Asterhan et al., 2014). Rather, when students whose strategies were additively interacted with students whose strategies were multiplicative, and both failed to solve the Blocks task alone, the students with multiplicative strategies dominated the talk, and conceptual change happened through explanations rather than through argumentation. This experiment suggests that, in contrast with other disciplines (like civic education or history) for which argumentation among students can be easily designed, the emergence of argumentation as a critical discussion in mathematics relies on a meticulous design that ensures some symmetry between the members of the group. The two wrongs-may-make-a-right phenomenon is then correctly labeled through if they argue together since engagement in a critical discussion hardly happens in mathematics when students have different levels. We attribute this specificity of argumentation in mathematics because mathematical levels confer power to stronger students upon weaker students — a fact that avoids the deployment of argumentation.

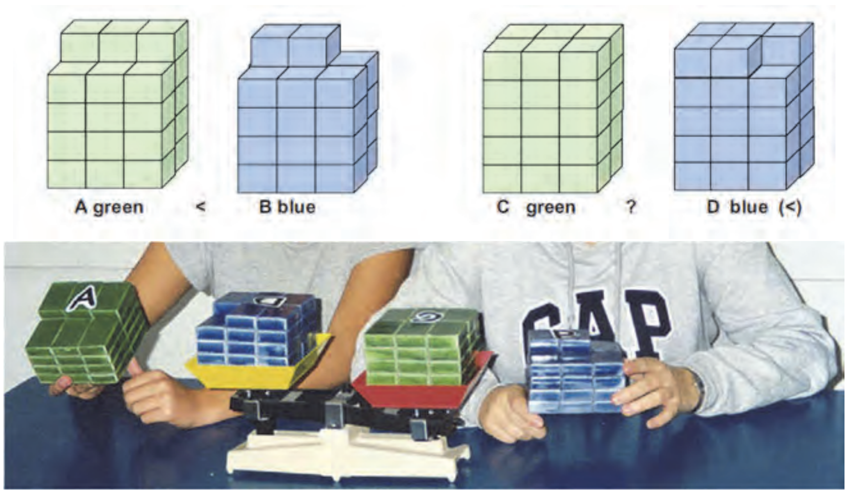


Fig. 2. Top: An example of a blocks task (the correct answer in parenthesis)
Bottom: The balance used as hypotheses checking device

Like the six-cards task, the blocks task relies on a sophisticated learning environment designed to foster a critical discussion towards conceptual change. The design included the provision of a hypothesis testing device and carefully pre-chosen minimal guidance interventions introduced by the experimenter in case students needed help to make progress. Conceptual change indeed occurred (progress was observed in proportional reasoning three weeks after the experiment), but argumentation in a critical discussion was rare.

3.2. Examples of tasks encouraging proofs and refutations

We have stressed the importance of refutations in the structural model proposed by Toulmin. The elaboration of reasoned arguments is not the only part of this model — a fact that is often ignored by educators that refer to it. The realization of the Toulmin model in mathematical tasks is not easy. Hadas et al. (2002) used several tasks that confronted students with contradiction (or uncertainty for the very least) between initial conjectures/predictions and findings/conclusions after an investigation in a Dynamic Geometry software. They form an activity in which students are encouraged to establish an initial argument/conjecture and then to gradually abandon this argument for a more elaborated and informed one that results from their own inquiry. In fact, they refute their initial arguments and feel the necessity to prove their final argument. Fig. 3, left, shows one of the tasks Hadas and colleagues developed — the three angles task, in which the students are asked to determine the relationship between the three angles denoted in the diagram. The right part of Fig. 3 shows the map of the epistemology of the resolution of the task (Hadas et al., 2002). The map shows that junior high schoolers are almost inevitably led to claim first that the three angles are always equal, and Dynamic Geometry manipulations refute this claim. The alternative claim that the three angles are equal in some cases is again refuted through DG manipulations, a fact which invites them to claim that the angles are never equal. As Hadas et al. (2002) showed, the students are not sure that their claim is correct since it is too surprising. They feel the necessity to prove this claim and succeed in this endeavor. Their argumentation at that stage is aimed both at elaborating an argument (a Toulmin-based argumentation) and a self-conviction (a Perelman-based argumentation). The task invites students to bring forward conjectures that are refuted through manipulations of Dynamic Geometry software. The refutations are informal – undertaken by creating displays that constitute counter-examples of a conjecture and lead students arranged in small groups to construct an argument as a mathematical proof. However, we should say that the presence of resources such as Dynamic Geometry software is also crucial. Argumentation accompanies an inquiry process mediated by technologies.

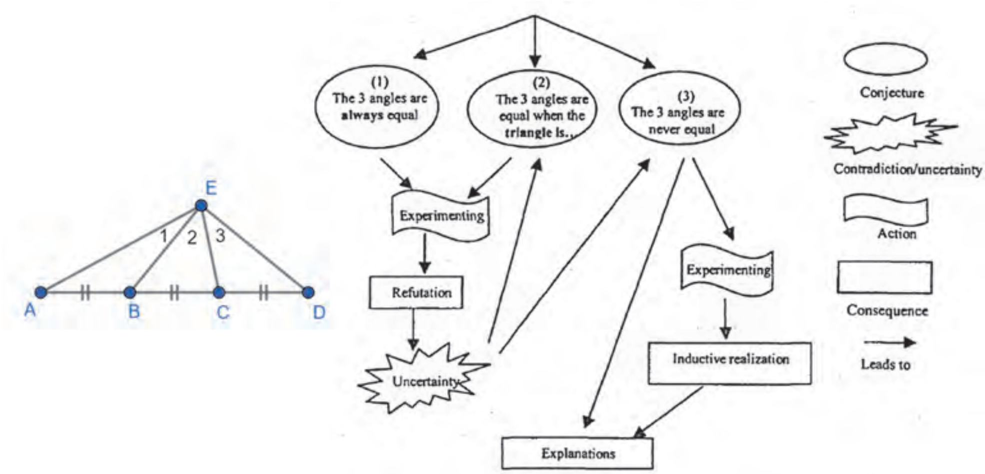
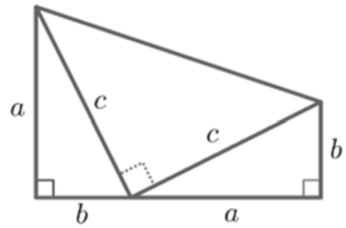


Fig. 3. The three angles task (left) and a map of its epistemological resolution (right)

Although the design of tasks that afford the elaboration of a Toulmin argumentative structure is challenging, this design has been successfully undertaken in several instances in elementary, secondary, and higher education (e.g., Prusak et al., 2013). Toulmin's theory is often used to model the guidance of teachers in the elaboration of mathematical arguments. It is useful to describe how teachers can coordinate students' contributions to co-construct mathematical arguments. This description often reflects traditional teacher-centered guidance, but it sometimes describes a more subtle kind of guidance. For example, Conner (2022) exemplified such a description. This co-construction was also made possible through multiple resources — diagrams, video clips, micro-worlds, and inscriptions on the board, which the teacher used in this co-construction.

3.3. Examples of tasks that encourage the identification of problems

Proof-Without-Words (PWW; Nelsen, 1993) are mathematical texts that allude implicitly to theorems known or unknown. Fig. 4 displays a PWW that alludes to the proof of the Pythagorean Theorem. The reader of such a PWW is expected to fill in the gaps and complete the proof based on the diagram's limited information. In order to fill in the gaps and construct a proof based on the clues given by the diagram, one must identify the proposition to be proved, identify the different components of the diagram and the relations between them and the proposition, realize the dependencies and (in)equalities of different terms in the diagram and justify them based on prior knowledge, determine the order of constructions and phases of the proof and, finally, understand how and to what extent the idea shown in the concrete diagram can be generalized (Marco and Schwarz, 2019). Marco et al. (2021) suggested the gap-filling framework for analyzing students' argumentation when working collaboratively to develop a proof based on a PWW. The theory of gap-filling is a reader-oriented theory taken from literary criticism (Perry and Sternberg, 1986), whose fundamental premise is that any text contains a limited amount of information and that the reader constantly adds information to the text to construct meaning and make sense of it. The fact that students independently identify gaps in a PWW and fill these gaps based on their prior knowledge makes this activity befit the Plantin model, which emphasizes problematization as the departure point for argumentation. Even before the student presents her argumentation to peers or the teacher, she develops her mathematical argumentation in front of a diagrammatical text while interacting with it. This subterranean layer of argumentation does not seem to be better understood by Plantin's model or any other of the models we mentioned. However, it can probably be more productively studied using theories such as Herbs's (2004) conceptual framework of modes of interaction with diagrams. Marco et al. (2022) used the notion of gap-filling to redesign the PWW artifacts striving to enhance students' interactions with them and improve their proof constructions.



$$\frac{(a+b)^2}{2} = \frac{c^2}{2} + 2\frac{ab}{2}$$

$$a^2 + b^2 = c^2$$

Fig. 4. A PWW for the Pythagorean Theorem

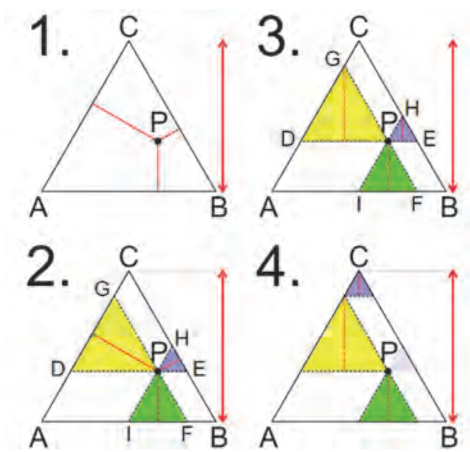


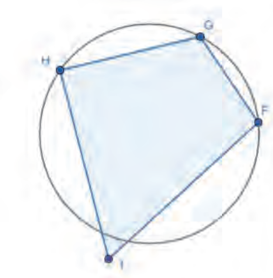
Fig. 5. Viviani's theorem — will students be able to understand both the proposition and its proof?[1]

The same kind of text — a PWW, provided to groups of students, may invite students to discover the problem to be inquired about and proved. This is the case in Fig. 5, which diagrammatically hints at a proof for the Vivianni Theorem. If students are not familiar with the theorem, ask groups of students to look at the picture and conjecture a mathematical claim and then prove it, further point at the Plantin model of argumentation in which the identification of the problem is crucial.

3.4. Examples of tasks that help teachers convince their students and students to convince each other

To show another example of the usefulness of the different theories of argumentation to describe novelty in mathematical practices, let us consider the use of the Viviani PWW in another setting: teachers may use this mathematical text as an artifact in a whole-class collective argumentation to convince the students of the correctness of a theorem. The visualization is a powerful device that teachers can exploit in explanations. In this case, the Perelman model is adequate, as it helps students adhere to what the teacher explains. The use of several PWWs by the teacher may strengthen the students' adherence to the truth of the Pythagorean Theorem (Marco et al., 2022).

Let us consider other activities labeled as "Who-Is-Right" (WIR) tasks. Fig. 6 displays a circle passing through three points and not passing through a fourth one.



Reut: According to the diagram quadrangle GFIH cannot be inscribed in a circle.
Boris: I disagree - it cannot be inscribed in this circle but may be inscribed in another one!

Who is right?

Fig. 6. Who-Is-Right task

Two claims about the (im-)possibility that a circle would (not) pass through four points are suggested. Such mathematical texts were developed by Koichu et al. (2021) and provided to small groups of students. Such a setting may encourage the development of a critical discussion since the opinions suggested in the text reflect common opinions held by junior-high-school students (Koichu et al., 2021). However, the opinions stated in the WIR are not necessarily the opinions held by the discussants. The activity is then more a way for the group to be convinced that one of the opinions is correct. Koichu et al. (2021) have found that enactment of WIR tasks increases students' engagement in looking-back strategies. These are reflective post-solution dialogical moves that include "queries on verification of the obtained solution(s), comparative consideration of alternative solutions, and formulation of implications for future problem-solving." (p. 831). They argue that considering the question "why is the other solution wrong?", is different from addressing the question "why is the chosen solution right?" Answering the former requires the students to use various argumentative practices. Aside from the reported advantages of WIR tasks in promoting looking bake strategies, we see their potential for advancing argumentation skills. In a typical problem, the students encounter a problematic situation and should produce a solution. In the WIR context, the students are confronted with an erroneous solution and a correct one and should decide which one is more persuasive and uphold their decision. The task itself contains a text with a discussion that prompts "discussion on discussion". This simple, but productive, argumentative design which is most suitable to the Perelman rhetorical model, is also suitable for van Eemeren and Groothendorst's model, as the students need to decide which of the two interlocutors is more persuasive.

4. Discussion

The examples of activities designed to trigger argumentative activities and the successes we reported on this design suggest that the design of argumentative activities is at the heart of educational change in mathematics. We confess that such examples do not represent very frequent kinds of activities in mathematics classrooms. Rather, activities in mathematics classes generally consist of the engagement in exercises that lead to the skillful resolution of problems and prepare students for exams in which similar problems are posed. Why is the link between mathematics education and theories of argumentation so weak? We suggest that the weakness of this link is not fortuitous and that it points at weaknesses in mathematics education that innovators aim to palliate. To begin with, students are often requested to solve problems in which the question is given. The curiosity of the students is not aroused. The inadequacy of Plantin's theory of argumentation dialogue points at the lack of care in the progressive identification of problems and questions in mathematics. Secondly, the pragma-dialectical model (van Eemeren and Groothendorst, 2004) fits a situation in which several standpoints have a priori comparable epistemic statuses. The fact that in mathematics, solutions are generally either "right" or "wrong" makes critical discussions difficult to happen. We have stressed the difficulties of designing activities in which critical discussions occur. A promising venue in this direction that we cannot develop here because of length limitations is to promote interdisciplinarity. Thirdly, Perelman's rhetorical model refers to persuasion (rather than conviction), led by the teacher.

Persuasion has not a good press in mathematics. The teacher is expected to present clear and logical statements and not to persuade her students (see an exception in Gabel and Dreyfus, 2022). The only model that seems to be relevant is the Toulmin model. However, this model is monologic. Many innovators in mathematics education pledge for dialogic teaching and situate the Toulmin model in dialogue. Mathematics educators should be cautious, though. Pseudo-dialogues during which the teacher leads students to an inexorable conclusion are frequent in dialogic education (Alexander, 2005). Finding the balance between the attainment of rigorous mathematical ideas and the attentiveness to students' voices in mathematical classrooms is a huge challenge.

Besides the weaknesses of mathematics education that the inadequacy of general theories of argumentation uncovered, the innovative examples we presented show that the scope of these general theories is limited. For example, we have shown the importance of texts in innovative activities in mathematics. The general theories do not clearly relate to such texts (written texts, videos, pictures, or diagrams). We described peer discussions around texts, but the role of the text in the argumentation is not addressed and covered by the models. More generally, a learning environment was presented based on a meticulous design for each of the examples we presented. Abundant literature on design for disciplinary engagement (Engle and Conant, 2002), or argumentative design (Andriessen and Schwarz, 2009) provides design principles for argumentative activities in mathematics, such as the provision of resources (for example, for raising hypotheses and checking them), conferring authority to students (e.g., through collaborative settings), the problematization of tasks, the creation of sociocognitive conflicts, and providing ground rules for high-quality talk (Accountable Talk, Explanatory Talk, etc.). We should stress the surprisingly untapped research direction in the role of texts in mathematics education in general and in particular in argumentative activities.

The specificity of mathematical argumentation is especially salient in the potentiality of tools for checking hypotheses/conjectures toward elaborating proofs. In two of the examples we presented, these tools equipped students with an inquiry channel through which they could feed argumentation and, by such, could enhance certainty in their claims towards conceptual change and other learning gains.

There is always a breach between evolving theories in education and actual classroom practices. One of the research roles is to narrow this gap and enrich relations between theory and practice. However, theories may become too popular and hinder actual classroom practice development. Some of the most popular theories in argumentation used in mathematics education are 70 years old. Their authors did not imagine mathematics as a domain of application of their theories. Toulmin even declared that his theory is not adequate for mathematics. We believe that his image of mathematical activity was flawed — he probably believed that mathematicians' thinking is solely based on logical inferences. The structural but experimental model he suggested is adequate for certain aspects of mathematical activity. The popularity of his model does not reflect these aspects, though. We suggest that this adoption often strengthens conservative models of mathematical education in which the teacher dominates the elaboration of mathematical ideas. The Toulmin model provides more clarity to this kind of teaching but does not revolutionize mathematics education.

We hope that we succeeded in showing that argumentative theories may inspire designers to initiate considerable changes in practice in mathematical education. The theories provide the general grammar of argumentation, but the educational design should be meticulous. While deliberative, rhetorical, epistemological, or structural aspects may inspire designers, argumentation in the mathematical class involves instruments, hypothesis-testing devices, texts, and technologies that theorists of argumentation did not envision. With such resources, identification of problems, critical discussions, elaboration of arguments/proofs, or their presentation is interwoven with inquiry processes. Texts such as "Who Is Right?" tasks or PWWs may help students identify problems before discussing and solving them, as conveyed by Plantin's model of argumentation dialogue. If the theorems conveyed by PWWs are familiar to students, they may help students reconstruct the argument that proves the correctness of these theorems. Alternatively, teachers may use PWWs to convince students of the correctness of theorems they are familiar with. The three examples of tasks that encourage critical discussions show the challenges that their design involves.

In a nutshell, the interactions between different argumentation theories and mathematical practices and advances in educational design are rich grounds for educational changes in mathematics education.

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Influence of University-Based Learning Opportunities on the Professional Development of Future Mathematics Teachers

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ABSTRACT The article focuses two areas related to learning opportunities of future mathematics teachers within their university studies, namely the elementary mathematics from a higher standpoint and practical activities as part of university studies. Thereby the article refers to several comparative studies on mathematics teachers' professional competence which are shortly summarized in the beginning. Afterwards along with conceptual considerations about elementary mathematics from a higher standpoint examples for the integration of the concept in studies on teachers´ professional competence as well as practical experiences from university courses are described. Subsequently empirical results from a study evaluating the professional development of future teachers in longer practical activities are depicted.

Keywords: Future mathematics teachers; Mathematics teachers' professional competence; Mathematics teacher education; Elementary mathematics from a higher standpoint; Practical activities.

1. Introduction

University-based learning opportunities constitute to a large part the professional competence of a teacher and thereby form a central basis for the continuous professional development of teachers. A core issue of university-based teacher education though of course is the imparting of theoretical competences. Alongside yet also other components, for example integrated practical experiences in school, are parts of teacher education at a university. This leads to questions of how to concretely form respective programs and thus for example to the question which contents should be included. Furthermore, with regard to the efficiency of teacher education an arising question is how to conceptualize and measure the future teachers' competences understood as an outcome of teacher education.

In the following against this general background two aspects of university-based teacher education for future mathematics teachers are discussed. The first aspect affects the question which topics should be included with regard to the subject matter knowledge, though especially focusing on the idea of elementary mathematics from a

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higher standpoint. After these considerations related to knowledge-based elements of teacher education the next section addresses the area of practical activities. Therefore, referring to the debate of the efficacy and conceptualization of respective components of university-based teacher education, some results of an empirical study are summarized. As the paragraphs commonly refer to different studies of future teachers' and practicing teachers' competences, in the first section a short summary of some central respective studies precedes.

2. Studies on Mathematics Teachers' Professional Competence

During the last around 15 years the empirical interest on the professional competence of (future) mathematics teachers remarkably raised and several respective studies have been carried out. Often, these studies aimed at both, an empirical analysis of mathematics teachers' professional competence as well as the further development of belonging theoretical frameworks. That is why we can, parallel to a constant growth of empirical results, also identify a constant development and ramification of theoretical approaches towards mathematics teachers' professional competence.

An example for these studies is the TEDS-M-study (Teacher Education and Development Study in Mathematics) (Blömeke et al., 2014; Tatto et al., 2012), an international comparative study with around 23 000 participants from 17 countries. TEDS-M indeed covered two studies, one focusing on future primary mathematics teachers and one focusing on future secondary mathematics teachers. TEDS-M examined and compared the various national policies and institutional learning opportunities and the future mathematics teachers' professional competence understanding the latter as an outcome of the respective mathematics teacher education system.

With regard to the understanding of professional competence the theoretical framework of TEDS-M takes up the understanding of competence by Weinert (2001) with its distinction between cognitive and affective-motivational aspects. Concerning the cognitive aspects, that is the future teachers' professional knowledge, TEDS-M refers to the prominent approach by Shulman (1986) and distinguishes between content knowledge, pedagogical content knowledge and general pedagogical knowledge. Concerning the affective-motivational aspects TEDS-M focuses professional motivation and self-regulation and beliefs both about mathematics as well as about teaching and learning mathematics (Döhrmann et al., 2012).

The TEDS-M study (together with its predecessor study MT21 (Mathematics Teaching in the 21st century), Schmidt et al., 2007) laid the basis for a still ongoing extensive research program covering various aspects of professional competence of mathematics teachers (Kaiser and König, 2019).

Another example of a study is the COACTIV-study (Professional competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy) (Kunter et al., 2013). COACTIV focused on practicing mathematics teachers teaching in lower secondary level. Especially, COACTIV was able to set the teachers' data in relation to their students' achievement data gained within the German extension of PISA 2003 (Baumert et al., 2010).

With regard to its model of teachers' professional competence COACTIV also refers to the concept of competence by Weinert (2001) and distinguishes between professional knowledge, beliefs, values and goals, motivational orientations and self-regulation. Professional knowledge in particular is further divided into content knowledge, pedagogical content knowledge, pedagogical and psychological knowledge, organizational knowledge and counseling knowledge (Baumert and Kunter, 2013).

Two other studies, which are related to each other, are the "Mathematics Teaching and Learning to Teach Project" and the "Learning Mathematics for Teaching Project" (Ball et al. 2008), focusing on the work of elementary teachers. The theoretical core concept is an attempt to further develop and precise the approach by Shulman by introducing the concept of "mathematical knowledge for teaching", which is understood as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p. 395). This concept from a theoretical perspective covers the following domains assigned to the areas of subject matter knowledge and pedagogical content knowledge: common content knowledge, horizon content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching and knowledge of content and curriculum (for details Ball et al., 2008). From an empirical perspective in turn there is a distinction between the domains of knowledge of students and content and knowledge of content, along with a distinction of content areas (Hill et al., 2004). Also, in this project there was the possibility to analyze the relation between teachers' knowledge and students' achievement (Hill et al., 2005).

Along with the ongoing development of theoretical positions about teachers' professional competence, also the theoretical frameworks as well as the methodical approaches of the respective studies further develop. Thus, in more recent studies the understanding of professional competence often changed toward an understanding of competence as continuum (Blömeke et al., 2015). As a consequence, approaches trying to evaluate teachers' professional competence closer to reality in comparison to paper-and-pencil-tests came to the fore. A typical approach for this new kind of instruments are video-based studies. Such a study is for example the TEDS-FU-study, a follow up study to the above-mentioned TEDS-M-study. The participants of TEDS-FU all also participated in TEDS-M at the end of their teacher education and were in the fourth year of their professional practice when participating in TEDS-FU making the study a longitudinal study offering insights into the professional development of mathematics teachers in the first years of their professional practice. The theoretical framework of TEDS-FU extended the framework of TEDS-M with references to the concept of noticing (Van Es and Sherin, 2002) and the expert-novice perspective (Chi, 2011; Berliner, 2001). Departing from these concepts the theoretical framework of TEDS-

FU as mentioned focuses on the idea of competence as a continuum especially referring to the situation-specific skills, that is perception, interpretation and decision-making (PID-model) (Kaiser et al., 2015; Kaiser et al., 2017).

3. Elementary Mathematics from a Higher Standpoint

Internationally as well as often on national levels there is a large variety of different systems of teacher education. And even though the general distinction between mathematics, mathematics pedagogy and general pedagogy can serve as a possibility to classify different components of various teacher education systems, still the concrete topics covered by a certain teacher education program within the respective areas strongly differ amongst different systems (Blömeke and Kaiser, 2012). In particular with regard to content knowledge each system has to answer the question which kind of mathematics it wants the future teachers to gain competencies in. And in turn, studies on the professional competence of mathematics teachers have to answer the question which kind of mathematics they want to include into its theoretical framework and therefore measure. Both questions lead to the same core, that is the question, which mathematics teachers need for successfully teaching mathematics in school.

The last question is not a new one. One famous approach is the idea of "elementary mathematics from a higher standpoint" by Felix Klein (Klein, 2016a, 2016b, 2016c). The ideas were indeed developed more than 100 years ago, but are still often referred to, for example also in prominent positions at ICME-conferences (e.g., the lecture of the recipient of the Felix Klein Medal in 2007, Jeremy Kilpatrick, at ICME-11 (Kilpatrick, n.y.) and "The Legacy of Felix Klein" as one of the themes at the "Thematic afternoon" at ICME-13 (Weigand et al., 2019)).

Klein develops his ideas against the background of overcoming the "double discontinuity" describing the situation, when a future mathematics teacher finds no relation between former school mathematics and university mathematics during his university studies and afterwards when working in school has to teach school mathematics which she or he then cannot relate to her or his university studies. That is why Klein has two aims for his lecture. "On the one hand, there is an effort to impregnate the subject matter, which the schools teach with new ideas derived from modern developments of science and in accord with modern culture. […] On the other hand, the attempt is made to take into account, in university teaching, the needs of the school teacher." (Klein, 2016a, p. 1). His aim is "to show you the *mutual connection between problems in the various disciplines*, these connections use not to be sufficiently considered in the specialized lecture courses, and I want more specifically to emphasize the relation of these problems to those of school mathematics" (ibid., p. 2). It is important to understand that this indeed goes along with the necessity of a sound basis of knowledge with regard to subject matter. Thus, with regard to teacher education this can be summarized to what Klein "look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge taught to you here as vivid stimuli for your teaching" (ibid., p.2).

The idea of elementary mathematics from a higher standpoint was taken up in several theoretical considerations as well as theoretical frameworks of empirical studies. For example, in the TEDS-M study the understanding of the mathematical knowledge of the future teachers prominently integrates aspects offering connectivity to the idea of elementary mathematics from a higher standpoint as in the conceptualization of TEDS-M "a teacher's mathematical knowledge was expected to cover from a higher and reflective level at least the mathematical content of the grades the teacher would teach. In addition, a teacher was considered to need to be able to integrate the educational content into the overall mathematical context as well as to connect the content to higher levels of education" (Döhrmann et al., 2012, p. 327f).

Also, in the COACTIV study the respective conceptualization amongst others prominently refers to the idea of elementary mathematics from a higher standpoint (Krauss et al., 2013). Here against a theoretical background of a distinction between four types of mathematical knowledge ranging from academical to everyday knowledge they "conceptualize the CK needed for teaching as knowledge of the second type: a profound mathematical understanding of the content of the secondary school mathematics curriculum" (Baumert and Kunter, 2013, p.33).

Another example for an attempt further developing corresponding constructs of the mathematical knowledge of teachers both on a theoretical as well as an empirical level is the introduction of the concept of school-related content knowledge (Dreher et al., 2018). This knowledge is described as "conceptual mathematical CK about interrelations between academic and school mathematics" (ibid., p. 329). More concrete the following facets are distinguished: "(1) knowledge about the curricular structure and its legitimation in the sense of (meta-)mathematical reasons as well as knowledge about the interrelations between school mathematics and academic mathematics in (2) top-down and in (3) bottom-up directions" (ibid., p. 330).

Concerning university-based parts of teacher education particularly an integration of elementary mathematics from a higher standpoint into future teachers' university courses seems to foster the future teachers' performance (Buchholtz and Kaiser, 2013). Furthermore, future teachers' professional knowledge about elementary mathematics from a higher standpoint internationally obviously strongly differ (Buchholtz et al., 2013).

The preceding considerations also lead to the question, how future mathematics teachers during their university studies can be supported in overcoming the double discontinuity, besides or subsidiary to the previously summarized idea of focusing on knowledge in the sense of elementary mathematics from a higher standpoint. For this, closing this paragraph in the following two quite practical experiences are described arising from a former project for fostering future mathematics teachers for upper secondary level in the first phase of their university studies (Schwarz et al., 2013, Schwarz et al. 2014).

Thus, a first import issue is the necessity of a sufficient knowledge about school mathematics when entering mathematics teacher education. Indeed, in the project the majority of students had good or very good respective proficiency, however some students also had difficulties in answering belonging questions and for example could not sufficiently calculate with fractions. It is obvious, that the requirements of a mathematical teacher education are difficult to meet for these students.

The second issue is related to the idea, that conceptualizations of competence next to cognitive aspects also contain an affective and motivational component (Weinert, 2001, Blömeke et al., 2015). Thus, as a direct consequence, it is required to not only take knowledge in the sense of cognitive aspects into consideration, when aiming at the development of future teachers' competences, but also consider these affective-motivational aspects. This for example could include offering the students opportunities for accompanied reflection on their studies and their perception of it, especially focusing on the differences between school mathematics and university mathematics, which the students of course realize and often are irritated about. In the project it was already also helpful to create institutionalized opportunities in which the students amongst each other could talk about their studies and particularly realize that they are not "alone" with the challenges arising from the differences between school and academics mathematics.

4. Phases of Practical Activities during Future Teachers' University Studies

In the preceding part, the article was orientated on aspects of professional knowledge possibly acquirable through belonging university courses. In contrast in the following part the focus is laid on practical activities as part of future teachers' university studies. Concerning these practical activities one can state that there is a large variety of different realizations in international comparison but also already on national levels concepts often differ. Furthermore, practical activities in teacher education are broadly discussed from various perspectives and also the empirical results about respective parts of university studies are multilayer and varying (e.g., König et al., 2016; Arnold et al., 2014; Besa and Büdcher, 2014; Zeichner, 2010).

The project shortly described in the following (Schwarz et al., 2020) evaluates practical activities of master students at the University of Vechta, who want to become primary mathematics teachers. During their practical activities the students continuously work in school for 18 weeks, accompanied by university seminars which are hold together by university teachers and teachers from school. The study falls back on instruments of TEDS-FU (Kaiser et al., 2017; Kaiser et al., 2015) and uses them within a pre-post-test-design with two measurement points, the first before and the second at the end of the practical activities. The video-based tasks were open and focused basically on situation-specific skills. The data of 29 students was evaluated according to qualitative content analysis (Mayring, 2014).

The results with regard to the influence of practical activities on the professional competences of the master students mirror the respective differentness of empirical results. Separately analyzing the tasks, in all tasks there was a large proportion of students formulating answers with corresponding adequacy before and after their practical activities, yet not necessarily with corresponding foci in the answers. In some tasks furthermore, the proportion of students who were able to formulate more adequate answers was bigger than the proportion of students formulating less adequate answers after their practical activities or there even were no students of the latter group. In contrary, in other tasks, the proportion of students formulating less adequate answers at the second measurement point was bigger than the proportion of students formulating more adequate answers then. In general, students formulating more adequate answers after their practical activities then were able to more substantially respectively more often refer to relevant aspects of the video and correspondingly in contrast, students formulating less adequate answers at the second measurement point then less substantially respectively less frequently refer to relevant aspects of the video. Instead for example in these cases a stronger focus is laid on superficial aspects of the shown lessons instead of a deeper going analysis of the learning processes.

Sometimes however a less adequate answer after practical activities nevertheless also could hint at a growth of the professional competences of the students. For example, when students at the second measurement point based on their experiences in school also expect and write about quality criteria which were not included in the video and therefore the students' answers cannot meet the requirements connected to noticing aspects from the video.

Keeping in mind that the tasks partially differ concerning their maximum score, with regard to a total sum score adding the varying scores from all tasks as a very first approach to the master students' development more than 60% of the participating students indeed reached a higher score after their practical activities. This result therefore gives hints for the assumption of a usefulness of the practical activities with regard to situation-specific skills. More detailed and quantitatively based results from a bigger sample will soon be available from the project TEDS-Validation-Transfer, a project within the TEDS-research-program (Kaiser and König, 2019), which focuses on the development of professional competences of future mathematics teachers with regard to professional noticing during practical activities as part of their university studies.

5. Summary and Conclusion

Professional competence of (future) mathematics teachers is a widely discussed issue, both under a theoretical as well as an empirical perspective. The article against this general background addresses two aspects of university-based teacher education, namely the integration of elementary mathematics from a higher standpoint into teacher education courses as a more theory-related part and the integration of in-school-activities into teacher education as a more practical-related part of teacher education.

Both areas are suitable as connecting factors for central discussions about future mathematics teachers' university-based studies as both areas exemplarily refer to different perspectives on future mathematics teachers' university-based education. The question of whether and how to integrate elementary mathematics from a higher standpoint into university courses for mathematics teachers, that is integrating an area of knowledge especially orientated on the demands of mathematics teachers, thus exemplarily refers to the perspective that university-based future mathematics teacher education is university-based education *for mathematics teachers*. This perspective emphasizes the necessity to very concretely consider what kind of competences mathematics teachers in particular need in order to offer them an education truly fitting to the demands of these group of students. This of course can or even has to include further subdifferentiations, for example with regard to the teaching level.

The considerations about the efficiency of practical activities as part of university studies in turn also relate to questions of how to integrate these activities into the future teachers' university education. Thereby instead of a simple unconnected integration of practical activities into the future teachers' curriculum it is, as often realized, nearby to connect the practical activities with university-based offers such as seminars for theoretically basing and reflecting the experiences in school. Considerations like that then exemplarily refer to the perspective that university-based future mathematics teacher education is the part of teacher education *taking place at a university.* This perspective can lead to reflections of how to gain benefit for the teachers` education from the particular opportunities and strengths offered by the typical characteristics of a university education.

Hence concludingly the article refers to the idea that bringing together the perspectives of a *university-based* education *for mathematics teachers* offers an approach to considerations about the first phase of teacher education not only with regard to the aspects shortly affected in this article but in general with regard to various facets concerning university-based mathematics teachers' education. A deliberate combination of the two perspectives offers a framework against which respective aspects concerning future mathematics teachers' university education can be contextualized. A discussion of the two perspectives and its combination therefore can contribute to the conceptualization and the measurement of the first phase of teacher education as a basis for the professional competence of future mathematics teachers.

Acknowledgments

The author would like to thank Jessica Hoth for the joint work in evaluating future teachers' practical activities and Philip Herrmann, Birgit Richter and Jens Struckmeier for the joint work in supporting future mathematics teachers in the first phase of their studies. The author would like to especially express his greatest appreciation to

Gabriele Kaiser, also for giving distinction in both projects, but primarily for her wonderful support, great advice and continuous inspiration.

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https://doi.org/10.1142/9789811287183_0039

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Mathematics: A Code for Interdisciplinary Dialogues

Hyunyong Shin[1]

ABSTRACT In this talk, I will introduce mathematics as a code for interdisciplinary dialogues through a story on infinity.

Keywords: Mathematics; Infinity; Interdisciplinary dialogue.

1. Overture

What is the answer to the following question?

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=?$$

How about this solution:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right).$$

$$\text{Let } S=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

$$\text{Then } S=1+\frac{S}{2}, \text{ so } S=2.$$

Seems to be good.

What is the answer to the following question?

$$1+2+4+8+\cdots=?$$

How about this solution:

$$1+2+4+8+\cdots=1+2(1+2+4+\cdots).$$

$$\text{Let } S=1+2+4+8+\cdots$$

$$\text{Then } S=1+2S, \text{ So } S=-1.$$

I have applied similar strategy as before. However, it seems to be no good.

What's the difference between these two infinite situations? What is happening in infinity?

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I have a circle and inscribed regular polygons infinitely many (Fig. 1).

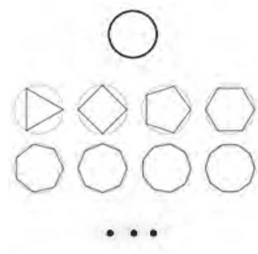


Fig. 1. A series of inscribed regular polygons

The regular polygons approach the circle. And the lengths of regular polygons approach the length of the circle.

Now I have another series of curves (Fig. 2).

$$f_n : [0,2\pi] \to \mathbb{R},$$

$$f_n(x) = \frac{1}{n} \sin nx$$

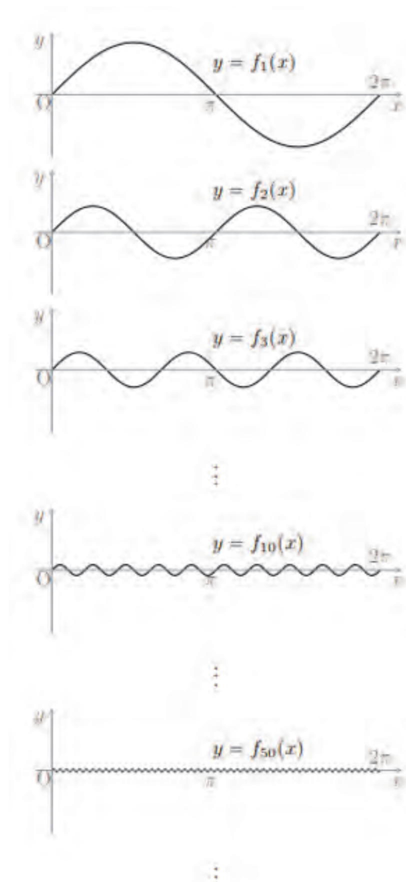


Fig. 2. A series of curves

The curves approach the line segment over $[0,2\pi]$. However, the lengths of the curves do not approach the length of the line segment.

What is the difference between these two infinite situations? What is happening in infinity? We are curious about infinity.

Mathematics seems to be a reasonable language and grammar for infinity. So it is interesting to try an interdisciplinary dialogue on infinity based on mathematics.

At first, we note that a difference between rational and irrational numbers is finiteness and infiniteness. A real number r is rational if and only if r can be presented as a finite simple continued fraction. In other words, a real number r is irrational if and only if r can be presented as an infinite simple continued fraction. For examples, I have some simple continued fractions.

$$1$$

$$1+\frac{1}{1}$$

$$1+\cfrac{1}{1+\cfrac{1}{1}}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}$$

These are finite, so rational numbers.

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$$

This is infinite, so is an irrational number. Infinity is in irrational numbers.

Hyperbolic geometry (Fig. 3) and elliptic geometry are examples of non-Euclidean geometry. It is the parallel axiom which distinguishes Euclidean geometry from the non-Euclidean. Parallel axiom without infinity is not so meaningful in mathematics. Infinity is in non-Euclidean geometry.

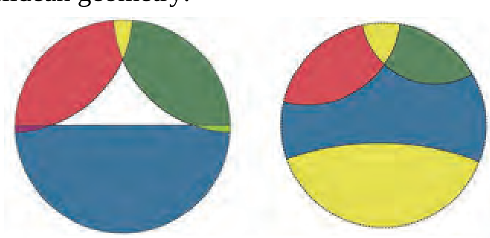


Fig. 3. Properties of hyperbolic geometry

George Pólya has proved that there are 17 wallpaper patterns. Using Pólya's argument, we can easily show that there are 7 frieze patterns. We note that mathematical classification of wallpaper and frieze patterns needs symmetries and infinity. Let us consider some examples.

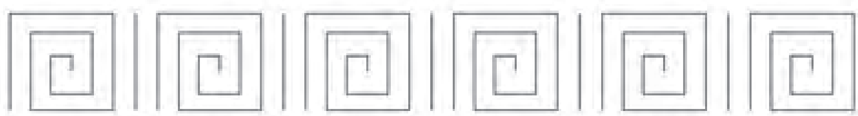


Fig. 4. A frieze pattern with translation symmetry

This frieze has translation symmetry (Fig. 4). It means that we can move the frieze to the right or to the left without any change of the pattern. To say so, we have to assume that the frieze be repeated infinitely in the right and in the left.

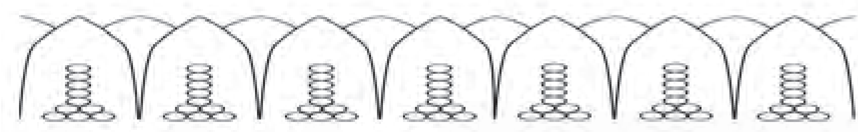


Fig. 5. A frieze pattern with reflection symmetry

This frieze has reflection symmetry (Fig. 5). To say that the following vertical lines are axes of symmetry (Fig. 6), this frieze is assumed to be repeated infinitely in both directions.

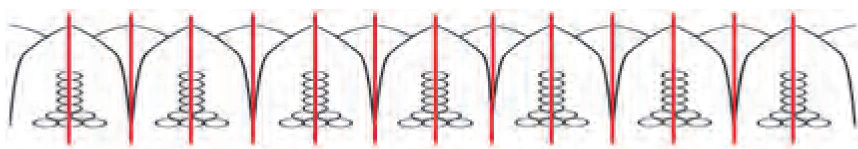


Fig. 6. Axis for reflection symmetry

This frieze has 180°-rotation symmetry (Fig. 7).

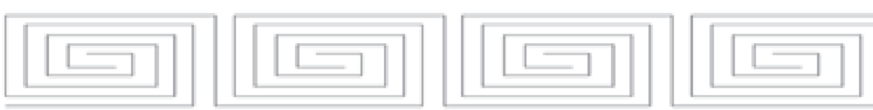


Fig. 7. A frieze pattern with rotation symmetry

To say that the following points are axes of symmetry (Fig. 8), we have to assume that the frieze be repeated infinitely in both directions.

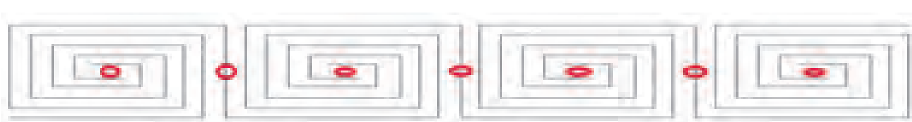


Fig. 8. Axis for rotation symmetry

The same is true in the case of wallpaper. Wallpaper is assumed to be repeated infinitely in four directions. Mathematical approach to patterns requires infinity.

2. Infinity in Music

In *School of Athens* by Raffaello, there are some mathematicians. Pythagoras is one of them. Raffaello presents him with the tetractys. In tetractys, there are some rational numbers: three quarters, two thirds, one half, and eight ninths. These rational numbers are the basis of Pythagorean Scale in music. Those numbers are on the neck of guitar.

About two thousand years after Pythagoras, the tetractys was replaced by the Euler Tonnetz. Euler Tonnetz represents the equal temperament which is based on irrational numbers. Equal temperament uses irrational numbers in substitution for rational numbers. A basic difference between Pythagorean scale and equal temperament is finiteness and infiniteness. Infinity is in music theory.

These are traditional instruments of China, Korea, Vietnam, and Russia as shown in Fig. 9.

Fig. 9. Old traditional music instruments

Today's versions of these instruments are shown in Fig. 10. Compare the positions of frets. They are not same.

Fig. 10. New traditional instruments

How about here? The positions of frets are exactly same as in guitar. In guitar, the positions of frets form a curve (Fig. 11).

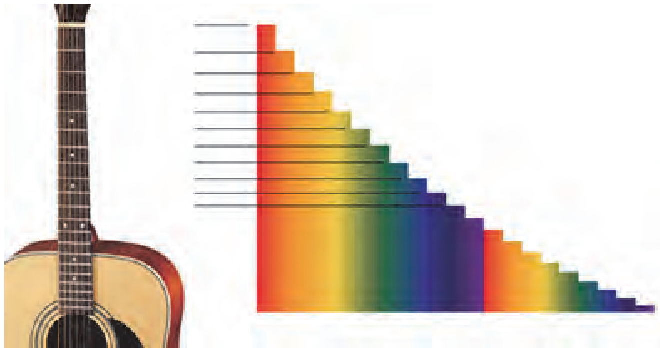
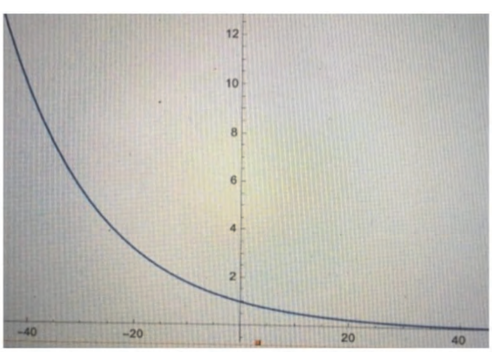


Fig. 11. A curve in guitar

The same curve can be found in piano and in pan flute (Fig. 12).

Fig. 12. Curves in piano and pan flute

The curve is given by this function (Fig. 13). Irrational number is in musical instruments. Infinity is there.



$$y = \left(\frac{1}{\sqrt[12]{2}}\right)^x$$

Fig. 13. Mathematical curve

Geomungo is one of the traditional instruments of Korea. Geomungo can be seen in paintings in tombs of 5th century, and in paintings of 18th century of Korea. Geomungo has a long history and is popular in Korea. This is an old geomungo (Fig. 14). Let us pay attention to positions of frets. There seems to be no mathematics there.

Fig. 14. Old geomungo

However, in today's version seen (Fig. 15), the positions of frets are similar to those in guitar. The irrational numbers are in geomungo. The infinity is even in geomungo.

Fig. 15. Frets in new geomungo and guitar

3. Infinity in Paintings

To understand and to classify the 7 frieze patterns, 17 wallpaper patterns, or Escher's patterns, we need symmetry and infinity.

Max Bill was educated at Bauhaus, and has served as a director of a design school. He has had a dream of new form of art based on mathematics: "I am convinced it is possible to evolve a new form of art in which the artist's work could be founded to quite a substantial degree on a mathematical line of approach to its content." A substantial degree on a mathematical line of approach must require infinity. There is infinity in Max Bill's paintings. Maldonado has also served the same design school as a director. There is infinity in Maldonado's works. Infinity is in art.

Le Corbusier thought that a house is a machine for living in. He wanted the various postures of human being to be considered in architecture. His "Modulor" gave him a solution. Le Corbusier believed that there are many divine proportions in human body.

The divine proportion is an irrational number. Infinity is in “Modulor”. Infinitely is in architecture.

4. Infinity in Literatures

Dostoevsky mentions non-Euclidean geometry. Infinity in his novel. Tolstoy mentions infinitesimal, Newton’s law of gravity, continuity, and discontinuity in *War and Peace*. Infinity is in Tolstoy’s novel.

The Man without Qualities is a novel of Musil. Musil thought that mathematics is the mother of natural science. Mathematics is important in this novel. In fact, the lead character of the novel is Ulrich. He is a mathematician. And one of the themes of the novel is mathematics and mysticism. Infinity should be there.

In Broch’s novel *The Sleepwalkers*, the crisis of foundations of mathematics in the early 20th century is the basic background. The infinity is the key and the essence in the foundation of mathematics. *The Aleph* is a novel of Borges. In mathematics, $\aleph$ denotes the trans-finite cardinality. In the novel, the aleph is a point in space that contains all other points in the world. This is quite similar to the definition of an “infinite set” in mathematics. In mathematics. an infinite set contains infinitely many proper subsets which are equivalent to itself.

Queneau was a member of Oulipo. To him, mathematics was a source of inspiration.

Queneau has proposed *The Foundations of Literature* in the same spirit of *The Foundations of Geometry* by David Hilbert. According to Hilbert, point, line, plane can be replaced by table, chair, and drinking glass. In *The Foundations of Literature*, point, line, and plane are replaced by word, sentence, and paragraph respectively. The parallel axiom in *The Foundations of Literature* is this: “A sentence having been given, and a word not belonging to this sentence, in the paragraph determined by the sentence and this word, there exists at the most one sentence including this word which has no other word in common with the first given sentence.” Queneau eventually claimed that every sentence includes an infinity of words: one perceives only a very few of them, the others being in the infinite or being imaginary. In a book by Queneau, there are 10^{14} sonnets. It is not possible for anyone to read all of those poems. 10^{14}sonnets are finitely many in mathematics, but infinitely many in literature.

In Pynchon’s novel *Against the Day*, some deep mathematics are mentioned quite seriously. Infinity is there. Szymborska was interested in π, the circumference rate. She has written a poem under the title π. π became a poem. The infinity of π became a poem. Infinity is in literature.

5. Infinity in Philosophy

In Plato’s *Meno*, Socrates, Meno, and a servant of Meno have a dialogue. The theme of the dialogue is the length of a side of a square with area 8, which is an irrational number. At first, the servant thought that he knew the length. During dialogue, he became to know that he didn't know the length. At last, the servant became to know himself. Infinity is in Plato's dialogues. Aristotle, in his book *Physics*, discusses Zeno's paradoxes. Without infinity, Zeno's paradoxes do not make any sense.

Newton and Leibniz developed a theory of infinity, called "differential calculus." Their philosophical approaches, however, were quite different. Leibniz thought that infinities and infinitesimals are fictions after all, though well-founded ones. To Leibniz, differential calculus was a logical fiction. Philosopher Berkeley did not accept the theory of Newton and Leibniz. In one of his books, Berkeley refuted differential calculus quite critically. Differential calculus is not only mathematics, but also philosophy. Differential calculus is a theory on infinity.

David Hume, a philosopher of empiricism, did not accept the infinite divisibility. His attitude for infinity was totally different from the conventional mathematics. Infinity is in philosophy.

Karl Marx has written about 850 pages of manuscripts on differential calculus. Differential calculus is a basic theory of motion. It is possible for him to try to have a basic principle of social change from differential calculus. It is also probable that Marx has tried to get a theoretical foundation of communism from differential calculus.

Infinity is even in politics.

Theology discusses existence, love, perfection, greatness, immortality of God. Without infinity, such discussions are not possible. Infinity is in theology.

6. Finale

Infinity is in music, art, literature, philosophy, politics, and theology. How about infinity in mathematics? Euclid, in his book *The Elements*, proposed the parallel axiom. Infinity is there. Euclid also proved that there are infinitely many prime numbers. Infinity is in Euclidean mathematics.

Archimedes has obtained the bound for π from a regular polygon of 96 sides. It is quite clear that Archimedean curiosity on circumference rate was not stopped by his bound. He might have imagined the infinity of π much more. Infinity is in Archimedean mathematics.

As far as infinity is concerned, volume of sphere as well as π is interesting. Archimedes was so glad to have this ratio which exists among the volumes of cone, sphere, and cylinder (Fig. 16).

Liu Hui, a Chinese mathematician, tried to compute the volume of sphere using the sphere inscribed in the intersection of two cylinders of equal radius at right angles (Fig. 17).

Unfortunately, he was not successful. Liu Hui, however, obtained the bound for π, which was as sharp as Archimedean bound.

$$3.141 < \pi < 3.142.$$

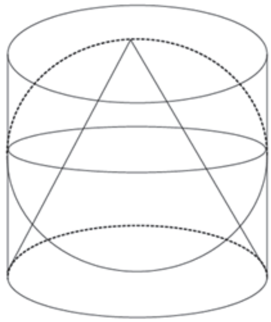


Fig. 16. Cone and sphere in cylinder

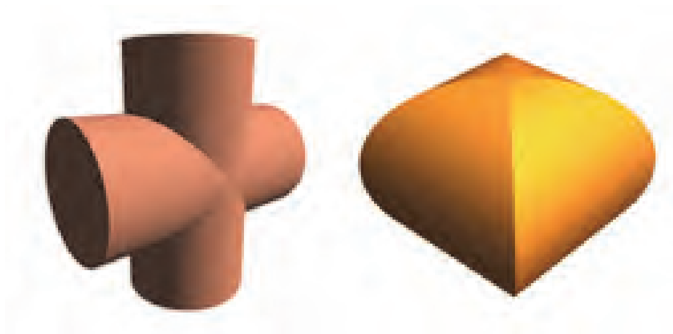


Fig. 17. Intersection of two cylinders

It was Zu Chongzhi who challenged the problem again many years after Liu Hui. Zu Chongzhi was successful in getting $\frac{4}{3}\pi r^3$. He used so called "Cavalieri's principle". However, we know that Zu Chongzhi was more than one thousand years older than Cavalieri.

Zu Chongzhi, on the other hand, obtained the following bound for π.

$$3.1415926 < \pi < 3.1415927.$$

This bound was so good as not to be sharpened more for the next many hundred years. Probably, Zu Chongzhi's curiosity on π was not stopped by this nice bound. He might have imagined the infinity of π much more. There was infinity in Chinese mathematics of many years ago.

Bolzano has struggled with infinity. Bolzano eventually came up with a mathematics of infinity.

Cantor developed a mathematics of infinity. How many points are there in the one-dimensional figure (Fig. 18)?

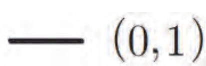


Fig. 18. Points on a line

How many points are there in the two-dimensional figure?

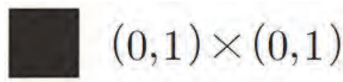


Fig. 19. Points on a square

Cantor proved that these two infinite sets are equivalent. They have the same cardinality. After proving this fact, Cantor shouted:

"I see it, but I don't believe it."

In infinity, Cantor's heart had difficulty in following his own head.

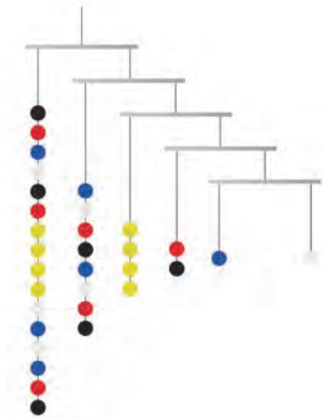


Fig. 20. Mobile for a geometric sequence

This mobile (Fig. 20) can be extended as much as we wish. This mobile says that $2^n < 2^{n+1}$.

What will happen if n goes to infinity? Cantor was surprised with the following equality:

$$2^{\aleph_0} = 2^{\aleph_0+1}.$$

Clearly $n < 2^n$. What will happen if n goes to infinity?

$$\aleph_0 < 2^{\aleph_0}.$$

Cantor was also surprised with the above inequality. Now it is easy to give natural numbers between n and 2^n. What will happen if n goes to infinity? What are there between these two trans-finite cardinalities $\aleph_0$ and $2^{\aleph_0}$?

Unfortunately, Cantor could not answer this his own question until his death. It was Gödel who challenged this problem again some years after Cantor's death. Gödel's solution eventually became the continuum hypothesis. Gödel, furthermore, proved the incompleteness theorems. The incompleteness theorems revealed unexpected properties of axiomatic mathematics involving infinity. For example, a consistent system of axioms involving infinity cannot be complete. In such a system, there is a fact that cannot be proved to be a fact by axiomatic mathematics. There are non-provable facts as well as provable facts. In other words, there are non-provable non-facts as well as provable non-facts.

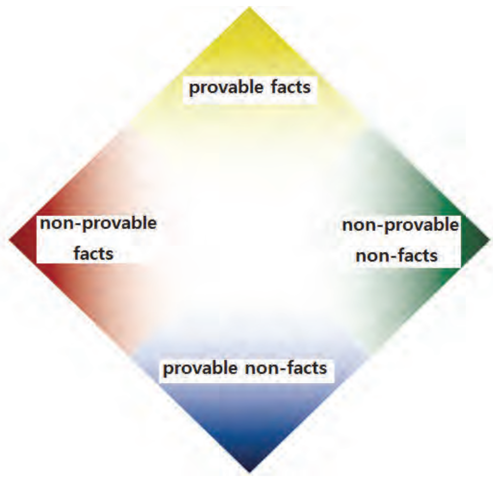


Fig. 21. Provable and non-provable

Furthermore, there is a big area which is beyond the axiomatic mathematics.

Zermelo was interested in axiom of choice which is an axiom of infinity. Based on axiom of choice, Banach and Tarski have proved a theorem called “Banach-Tarski paradox”. Banach-Tarski paradox says that axiom of choice transforms an apple into two apples of same volume (Fig. 22).

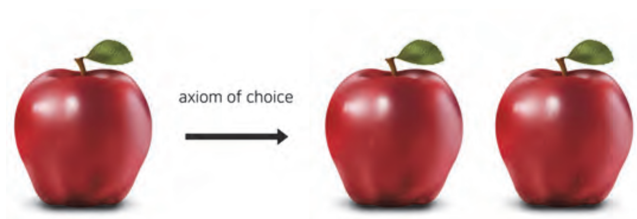


Fig. 22. Axiom of choice

Now let me close my talk. Mathematics says that infinity is mysterious. However, infinity is imaginable through mathematics. Mathematics might be the best language and grammar for infinity. Mathematics could be a code for interdisciplinary dialogues on infinity.

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Duos of Artefacts, a Model to Study the Intertwining of Tangible and Digital Tools in Mathematics

Sophie Soury-Lavergne[1]

ABSTRACT A variety of tangible manipulatives and digital environments are commonly used in mathematics education. Instead of comparing and opposing the two types of artefacts, we propose to study their combination, with a simplified model of just two artefacts. We define duo of artefacts as a specific combination of complementarities, redundancies and antagonisms between a tangible artefact and a digital one in a didactic situation (Soury-Lavergne, 2021). A duo is designed to provoke a joint instrumental genesis regarding both artefacts, and to control some of the schemes and mathematical conceptualizations developed by pupils during its use. Learning is described in terms of evolution of conceptions in the sense of Balacheff (2013). This lecture illustrates the model with two examples of duo of artefacts for primary school, one in arithmetic and one in geometry. We argue that in addition to be a research tool, duos of artefacts are also a way to support the integration of technology into teachers' practices.

This lecture is about the intertwining of digital and tangible artefacts when manipulatives are introduced in the situations with technologies. My work has its origins in a collaboration with my esteemed Italian colleague, Michela Maschietto, who is in charge of the laboratory of mathematic machines in Modena. In 2010, I was working on technologies, especially dynamic geometry, and she introduced me to some mechanical machines for problem solving in geometry and arithmetic. We began to work as a duo of persons before elaborating together the idea of duo of artefacts. My research question is about how to design and to provide students and teachers with digital technologies and didactical situations using these technologies, that would generate meaningful uses regarding the learning of mathematics and also that could be appropriated by the teachers and be integrated into their practices. Since the eighties and the emergence of personal computers in education, the problem has not been solved. The idea of "duo of artefacts" is a proposal to tackle the question.

This paper present first some assumptions that ground my thinking about technology and the learning of mathematics, which explain how I came to the idea of focusing on the articulation of tangible and digital technologies. Then, I will present in detail the combination of digital and tangible artifacts that constitute a duo of artifacts, which is a model to study systems of instruments. I will illustrate it by two examples, one in geometry, and one in arithmetic. The conclusion raises the main characteristics of the model of duo of artefacts that may be used both for

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research purposes as well as for providing teachers and students enhanced learning environments. The whole lecture is available on video at the following url: https://videos.univ-grenoble-alpes.fr/video/20249-icme14-invited-lecture-s-soury-lavergne/

1. Rationale for Studying the Intertwining of Digital Technologies with Tangible Tools

There is currently a growing interest for studying the respective contributions of digital and tangible tools in the learning and teaching of mathematics, like the recent two special issues of the journal Digital Experiences in mathematics Education as shown. In their introduction, Nathalie Sinclair and Richard Nemirowsky raised a lot of questions: “How do the learning affordances of digital and tangible tools differ from each other? Are there optimal combinations of digital and tangible tools? Are there sequences for their alternate use that appear to enhance learning experiences? What theoretical frameworks can help us understand their differences and complementarities?” (Nemirovsky and Sinclair, 2020, p. 108). Duo of artefacts is an attempt to provide first answers to these questions.

1.1. A still lasting concern: integration of technologies in teaching practices

In our community of research about technology, we have a huge concern with is the still low level of integration of technologies in teaching practices, especially in primary school. In 2006, Michele Artigue, who at this time was the vice-president of ICMI, before becoming the president from some years, has concluded the ICMI Study on technologies with a sentence that is still true today: “successful technological integration at large scale level is still a major problem, and this seems to be a general phenomenon” (Artigue, 2010, p. 472).

The work of Ghislaine Gueudet and Luc Trouche (2009) has pointed out the critical role of teaching resources in teachers’ practices. The resources of the teachers have been used as a means to characterize teacher expertise (Pepin et al., 2017; Gitirana et al., 2018). In the case of primary school teachers, there is a significant role of manipulatives in the resources of the teachers. So, a way to perhaps provoke an evolution in the integration of technology in teaching practices for primary school teachers is to build on their existing system of resources. Thus, we have to take into account the presence of manipulatives in those systems, and to understand how technology can be integrated and intertwined to existing systems of resources. It may help to understand and support the integration of technologies into teacher’ practices.

1.2. Questioning the respective role of digital and tangible entities in math learning

Since the first usage of computer in math education, its relationship to tangible objects has been questioned. For Seymour Papert, both computers and tangible objects are a means to carry powerful mathematic ideas into the mind. In the introduction of his famous book "Mindstorms: Children, Computers, and Powerful Ideas" (Papert, 1980), he took the example of the gears of his childhood to explain the double relationship of these gears to the abstract and to the senses. For him, the gears are connected to the formal knowledge of mathematics, and also to what he called the "body knowledge" the sensorimotor schemata of the child. In the nineties, with Sherry Turkle (1992), they claim that "connecting abstract mathematical ideas to the senses" is a characteristic of computers. So, following Turkle and Papert, we may see the computers as a means to replace tangible objects in this idea of making abstract mathematical ideas more concrete.

Later, the definition of virtual manipulatives, that Lauren Resnick and her colleagues (1998) and then Patricia Moyer-Packenham gave (2016), is only centered on digital aspects and digital artefacts. But they still point a relationship with the physical aspects, saying that the important thing is not the fidelity to physical object but the fidelity of the dynamic behavior of the digital object to the mathematical concept. The work of Andrew Manches and his colleagues (2010) compares the use of digital manipulative to the use of tangible manipulatives in problem solving strategies. They have shown that each kind of manipulatives have specific aspects that are important, that guide the mathematical ideas developed by the children and that constraint the strategies pupils can develop in different ways. But there is no evidence for one being better than the other. Their conclusion is that we have to take advantage of the respective contribution of tangible and digital manipulatives.

So, the issue is in the combination of tangible and digital manipulatives to construct knowledge. Which is also the conclusion of Julie Sarama and Douglas Clements (2016), in the book of Patricia Moyer-Packenham about digital manipulatives.

1.3. Theoretical frameworks for mathematical knowledge and tools

Theoretical frameworks for the study of mathematical knowledge and tools have been developed in France. I will first begin with the theory of conceptual fields, from Gerard Vergnaud (2009). I want to precise that Gérard Vergnaud is one of the fathers of the French didactic of mathematics, which passed away a few weeks before the ICME conference, saddening the whole French community of research in math education. For Vergnaud, the root of mathematical conceptualization is in the action. More precisely, the important concept is the concept of schemes, which are the invariant organization of behavior for a class of situations. The epistemic component of schemes, which are the operational invariants, is a key element to understand how knowledge can be built by a user while using tools, either digital of tangible.

The second French theoretical framework is from Nicolas Balacheff (2013), who call it cK¢ for "conception, knowledge and concept". He is developing the work of Vergnaud, drawing on the definition of concept to elaborate a theory to model mathematical knowledge, that enables to understand "student's understanding". For

Nicolas Balacheff, “conception” is modelling every subject way of knowing, be he a student, a teacher or even a mathematician. There is no difference in the nature of their knowledge. The difference lies in the sets of problems they can solve, in the set of representations they manipulate, in the set of operators they are using and also in the kind of control they engage when solving problems.

The instrumental approach of Pierre Rabardel (1995) explains how knowledge develops when using tools. The fundamental distinction made by Rabardel is between artefact and instrument: an instrument is an artefact in situation, associated with usage. When a subject uses an artefact, he develops scheme of use. It is the association of these schemes of use and the part of the artefact that is involved in the schemes that constitutes the instrument. The instrumental genesis synthetizes the conjoint development of schemes and the evolution of the artefact. As Vergnaud explains, there is an epistemic dimension in schemes, with the operational invariants. Indeed, the theoretical distinction between artefact and instrument and the instrumental genesis enable to grasp the construction of knowledge when a subject turns an artefact into an instrument.

1.4. Setting up the problem

The above theoretical frameworks produce two assumptions dealing with the construction of knowledge when using either digital or tangible tools: (i) knowledge (i.e. conception in the sense of Balacheff (2013)) develops through instrumental genesis when using artefacts and (ii) knowledge resides into schemes and their operational invariants.

Therefore, initial questions about the intertwining of digital and tangible artefacts turn into:

- How to design and combine tangible and digital artefacts for the construction of mathematical knowledge by students?
- Does the combination of digital and tangible artefacts help teachers to develop a resources system including technologies?

2. Duo of Artefacts, a Model to Design and Study Systems of Instruments

To answer the previous questions, there is a need for simplification of the learning situation which involves a multiplicity of artefacts in the hand of the students and the teachers. To tackle this complexity, the idea was to reduce it by looking for a simplified model, that is just a pair of artefacts. My purpose is to show that, by carefully choosing the characteristics of the two artefacts we select, we obtain a simple system that is still relevant to study and to understand students’ learning and teachers’ practices with technology. But there are conditions to be fulfilled for two artifacts in order to form a duo.

A first condition is to have some complementarities between the two artefacts. A very pragmatic solution is to choose one tangible artefact and one digital artefact to obtain complementarities between the two.

2.1. Tangible artefact, digital artefact?

The tangible artefact is defined by its physical properties as an object, like mass, color, movement. It is subject to gravity. It is visible and cannot disappear. It can be manipulated under the physical constraints. In opposition to the tangible one, the representations in the digital artefact are not constrained by the same laws of physics within the user interaction. Nevertheless, digital representations have physical properties too. Indeed, they are produced and embedded in hardware. And they appear to be tangible too, as far as one can operate on it like on tangible objects.

Thus, there is a need for some clarification about the choice of words. I choose to distinguish tangible from digital, even if it is subject to discussion. Some colleagues prefer to use physical artefact instead of tangible, although it must be noted that physical is a property of both kind of artefacts. Some others may choose "concrete" versus "abstract". But Sarama and Clements (2016) explained that concrete is not only physical: many students, and us also, manipulate mathematical objects, like numbers, as if they were concrete. Concrete results from the connection of these objects with meaningful experiences, which can surely be the case with digital artefacts. Behind each choice of vocabulary, there is the necessity to distinguishes two kinds of artefacts in a continuity of objects and properties. A first demand is that these two artefacts have complementarities.

Our proposal with Michela Maschietto is a very pragmatic one in order to select two artefacts with complementarity. When choosing a tangible artefact on one side and a digital artefact on the other, that is an artefact embedded in a digital environment, we are sure to get both differences and potential complementarities between the two. With the tangible artefact, we get the gestures, the student's bodily involvement and the eye-hand sensory-motor loop. With the digital artefact, we get extended possibilities of feedback and some specific behavior adapted to the mathematical knowledge at stake.

2.2. Two artefacts are not necessarily a duo

Even with complementarity, there is no reason that any set of two artefacts may turn into a system of instruments during the instrumental genesis. In fact, each of them can be the object of two separated instrumental genesis, resulting into two independent instruments. For instance, if you consider a calculator to produce the result of an operation and a compass to draw a circle, they are complementary because each of them is adequate to a specific problem but they do not necessarily form a system of instruments. Complementarities will give a purpose to use both artefacts. But two artefacts are not necessarily a duo, resulting in a system of instruments (Fig. 1). Our question is about identifying the characteristics of two artefacts in order to form a system of instruments during the instrumental genesis.

Gaetan Bourmaud (2007) has studied systems of instruments in the framework of the instrumental approach. He concludes that there is a need of complementarities but also a need of redundancies between the two artefacts and even some antagonisms.

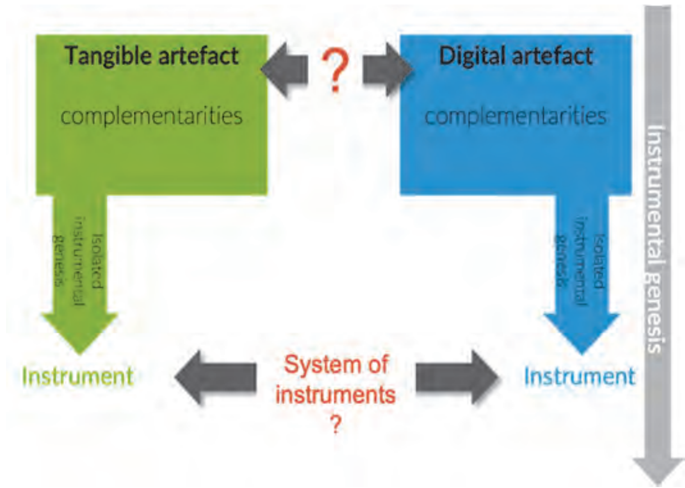

Fig. 1. Two complementary artefacts do not necessarily develop into a system of instruments across an instrumental genesis

Redundancies in a system of instruments ensure some adaptability, flexibility and robustness of the system. And, more surprising, a last characteristic of a system of instruments is antagonism between the elements. Antagonism seems counter-intuitive. It characterizes functionalities that are present in one bartefact but inefficient or even divergent in the other. Bourmaud explains that a system of instruments is both "(2013) more and less than the sum of the functionalities of each artefact" (2007, p. 65). In a learning perspective, it seems to be very important, since learning is also overcoming some obstacles and adapting a way to solve problem to a new situation. When it is about learning, easing the action is not always the goal. Considering that learning is the evolution of schemes by accommodation and assimilation, using one artefact then another, with the necessity to adapt and to change the solving strategy may be powerful.

2.3. Duo of tangible and digital artefacts and genesis of a system of instruments

Our proposal is to take two artefacts (Fig. 2) that present:

- Complementarities between each other, that will justify the interest to use of both of them, and not only one.
- Redundancies that help user to link the two artefacts and that create robustness and adaptability
- And antagonisms that provoke adaptations, evolutions and finally learning

But without a purpose to use the artefact, without a task to achieve, there will be no instrumental genesis. It calls for a last characteristic for two artefacts to become a duo to teach or to learn mathematics: a didactical situation. The didactical situation (Brousseau, 1997) brings the purpose of using both artefacts and characterizes the meaning of the knowledge that will be constructed by using them.

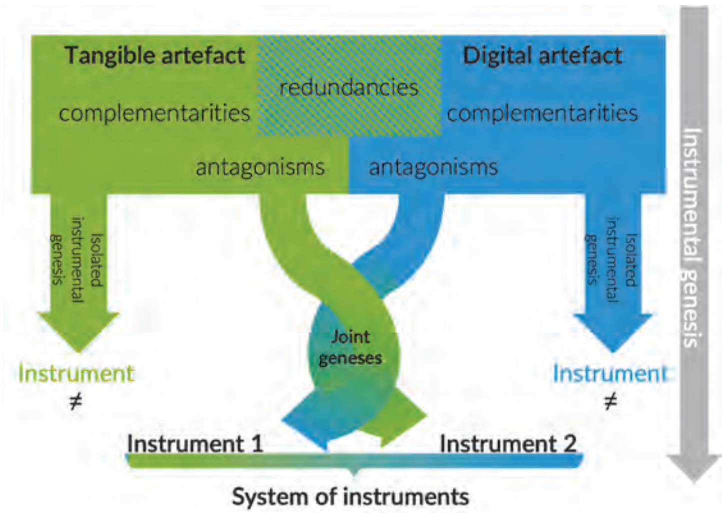

Fig. 2. A duo of artefacts, with complementarities, redundancies and antagonisms may develop into a system of instruments

3. A First Example of Duo of Artefacts in Geometry: A Duo to Learn How to Construct a Triangle

Anne Voltolini (2018) has elaborated a duo of artefacts to address a difficulty of teacher's practices, which is the construction of a triangle given the length of its three sides.

In French primary schools, the construction with compass is introduced as a very procedural way to construct triangles: draw a first side, open the compass according to the length of the second side, pick the compass on one segment endpoint and draw an arc of circle, open the compass according to the length of the third side, pick the compass on the other segment endpoint and draw a second arc of circle, intersecting the first one, this intersection is the third vertices of the triangle. In consequences, students do not relate the procedure to any geometrical properties. Research in didactics have produced several explanations about the causes of students' difficulties, and thus the difficulties encountered by the teachers. Students' difficulties result at least from three aspects.

First, this construction of the triangle is based on the dimensional deconstruction of geometrical figure (Duval, 2005). Most students see the triangle as a two-dimensional surface. But to succeed in the construction, they have to anticipate a point, the third vertices of the triangle, which is a zero-dimensional object. Thus, the construction requires a dimensional deconstruction from a two-dimensional object to a zero-dimensional object, a point which is rarely conceptualized at this level of education. The second explanation is the fact that the two sides that have to be constructed, once the first side is drawn, are not produced by the compass, which is the tool involved in the construction. The association of the artefact compass to the drawing of straight lines is not direct. The last cause is the fact that the artefact compass is at the origin of different kind of instruments. The main instrument developed by students when using compass is a compass to draw circles. It may also be another instrument which is to transfer lengths. None of them is related to the construction of sides of triangle or construction of points.

3.1. Dynamic segments, compass, and a new students' conception of the triangle

The solution Anne Voltolini has designed is a duo of artefacts associated to a possible new students' conception of the triangle. This new conception is a "line conception", which play the role of intermediary conception between the conception of a triangle as a surface and a conception of the triangle as a set of dots (defined by three vertices). A line conception of the triangle refers to the triangle as a set of line, specifically as a closed broken line of three segments. Voltolini assumed that this line conception would be accessible to primary school students and would help students to understand the construction of a triangle with the compass. To support the genesis of this new conception, she has also identified a new kind of instrument that can be elaborated from the artefact compass. The key idea is that the artefact compass can be an instrument to "rotate segments around one of their endpoints". According to these two hypotheses, she has built a didactical situation, in which students have to use dynamic segments to form a broken line at the interface of the computer and then to try to close the broken line in order to obtain a triangle if possible (Fig. 3).

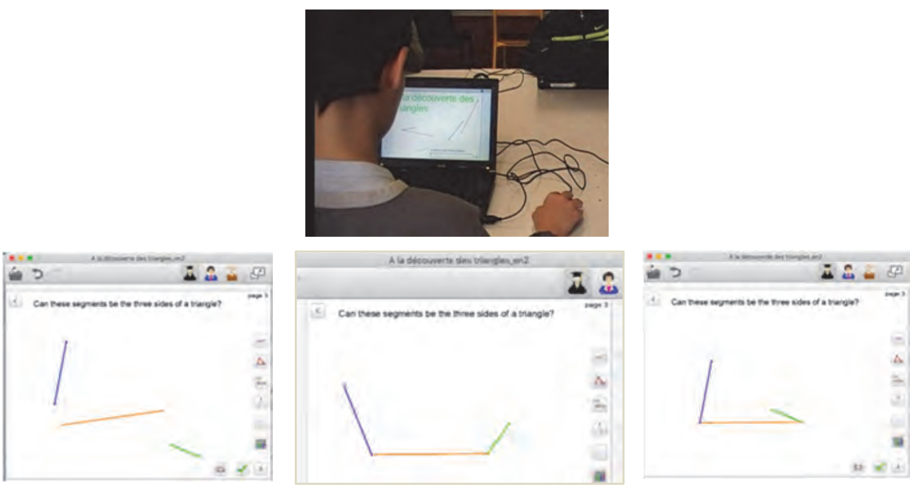

Fig. 3. In the digital environment, a triangle may be formed by translating and rotating dynamic segments

The problem can also be posed within the paper and pencil environment. A broken line of three segments is drawn and the task is to construct the triangle with the same lengths (Fig. 4). In this situation, compass may appear has a means to rotate segments.

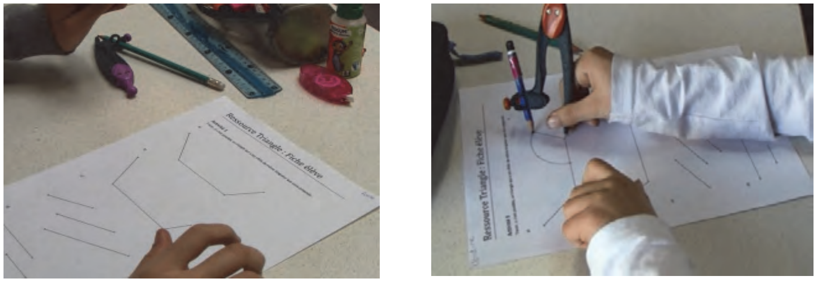

Fig. 4. In the paper and pencil environment, a broken line may be transformed into a triangle with the same segment length by using the compass

The digital environment allows to create constraints and feedback to bring students to the construction of the triangle. A main constraint is created by the behavior of dynamic segments. Segments cannot rotate and translate simultaneously. Thus, the user has to operate the two movements successively. This separation of the two movements is not possible in the tangible world, for instance when forming a triangle with sticks, because movement of sticks is a combination of translation and rotation. From three separated segments, translating the segments to form a broken line, then rotating the two segments at the extremity to close the broken line is an efficient strategy to form a triangle. This constraint brings the rotation of the segments to the front.

Furthermore, the digital environment gives the possibility to block the movements in order to force students to anticipate and to look for other ways to decide if there will be a triangle or not, leading them to the triangle inequality.

3.2. Analysis of a duo formed by dynamic segments and compass

The analysis of this duo consists first to identify the two artefacts of the duo. On one side there is the dynamic segments, on the other side the tangible artefact which is the compass. There may be a joint genesis because:

- Complementarities between the two artefacts make each of them useful: dynamic segments force the dissociation of the two movements and the compass emphasis the trace of the endpoints of the segments.
- Redundancies help user to link the two artefacts, create robustness and adaptability: both environments focus on rotation and on segments endpoints. But this is a rather low level of redundancies, not very explicit at first glance for the users.
- Antagonisms provoke adaptations, evolutions and finally learning: from the digital environment to the paper and pencil environment, the segments are not dynamic anymore. When they encounter the paper and pencil task, students have to develop a new way to close the broken line and to get the triangle. They spontaneously call upon the compass.

3.3. *Evolution of 5-grade students' conception about triangles*

Anne Voltolini has conducted an experimentation with two teachers over three consecutive years, with a methodology of design-based research (Coob et al., 2003). Her aim was to observed the evolution of the students' conceptions about triangles and the emergence of a new instrument related to the compass.

She has analyzed in detail the work of 34 pupils using her duo of artefacts. She has observed students activity within the digital and the paper-pencil environments and analyzed their gestures and behavior (Fig. 5) in order to characterize the components

Fig. 5. Student's hands rotation to show the segments movement to form a triangle, as an indicator of the genesis of the compass as an instrument to rotate segments

of their conceptions about triangle and the characteristics of their instrumental genesis related to the compass. The results show that 30 of the 34 students evolved toward the one-dimensional conception of triangle, which is a very encouraging result. But only five of the students reach the dot conception of a triangle and three students stayed to their initial conception of the triangle like a surface (Fig. 6). About the compass as an instrument to rotate segments, 31 students built this instrument and half of them could identify the circle as a geometrical object involved in the construction of triangles (Fig. 6).

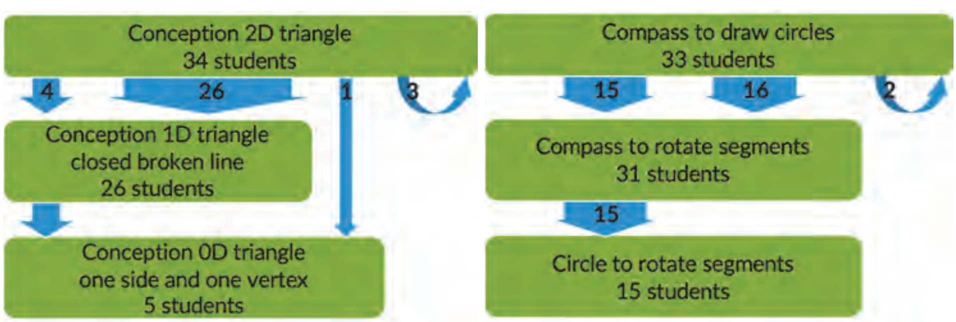

Fig. 6. Evolution of students' conceptions about triangle (on the left) and evolution of the instruments related to the compass (on the right) for 34 students of 5-grade

3.4. Conclusion about duo of artefacts in geometry to provoke conceptual understanding of triangles and their construction

This first example of duo of artefacts in geometry provides an example of a joint genesis of a system of instruments from digital and tangible artefacts and its consequence on learning. A first issue in the analysis of a situation with a duo of artefacts is to select the two artefacts of the duo. In this case, the duo is formed by dynamics segments, on one side, associated to a compass on the other side. The redundancy between the two artefacts is not very strong. Nevertheless, the duo enables the genesis a new instrument associated to the artefact compass, that is to turn segments around one endpoint. Furthermore, this new instrument has its roots in the dynamic segments that have been manipulated in the digital artefact.

The second result of Voltolini's work is the conception of triangle as one-dimensional object. It is a way to enact dimensional deconstruction (Duval, 2005) from two dimensions to one dimension, and maybe a step toward a dot conception of triangles. The line conception of triangle is a reachable learning objective, thanks to the duo and the instrument compass to rotate segment. It is clearly expressed by Luna, a young student who links both the triangle and the compass, the knowledge and the instrument: "The broken line, it helps, because, before, you don't know that you have to use compass to draw a triangle" (our translation).

4. A Second Example of Duo of Artefacts in Arithmetic: The Pascaline and E·Pascaline

The second example concerns arithmetic. It is the duo formed by the pascaline and the e·pascaline that we have designed together with Michela Maschietto (Maschietto and Soury-Lavergne, 2013, 2017; Soury-Lavergne and Maschietto, 2015). This example may be a kind of exemplar in the sense of Kuhn (1977), that is a solution to concrete problems, accepted by the group as paradigmatic.

4.1. Designing the pascaline and e·pascaline duo

The Pascaline is a small mechanical machine (Fig. 7), to write numbers and to do calculations. When using the pascaline, which is a set of wheels, you click on the wheel to write numbers with the digits displayed on the teeth. There are different kinds of strategies to write numbers, to calculate and to solve problems. Among them, two main procedures. One procedure is by iteration. You use only the unit wheel and you pass the numbers one after the other. You follow the number sequence, one click on the wheel for one number. The other procedure is by decomposition. You use each of the three wheels, to write or to calculate, according to the base-10 place value system to write numbers. The actions of the user on the pascaline are different according to these two procedures, which makes the value of the pascaline for teaching number system.

Fig. 7. The pascaline (to the left) and the e·pascaline (to the right) in a duo of artefacts

The e·pascaline has been designed with the Cabri Elem technology (Fig. 7). The idea was to add features and feedback to the functioning of the pascaline: to enrich the functioning of the pascaline; to emphasize the mathematical properties that are relevant for learning base 10 place value system; but also to minimize the prevalence of some of other properties that may distract from learning at stake.

With Michela Maschietto (Maschietto and Soury-Lavergne, 2013), we choose to preserve some visual fidelity to ensure some redundancies and connection between the tangible Pascaline and the digital counterpart (for instance the colors or the purple arrow, we have used drawing of many pupils to determine the one to keep). Also, we used the digital environment to create additional constraints, that would provoke students' adaptation of their procedure. A main constraint is on the possible actions to turn the wheels: you don't act directly on the wheel but you have to click on a small button, which have the shape of an arrow. Sometime, the arrow disappears, preventing you to make the wheel turning. This behavior should provoke an evolution of the user procedure, especially a transition from the iteration procedure to the decomposition procedure. We have also implemented different kinds of feedback, including an evaluation of the student's answer.

4.2. Complementarities, redundancies and antagonisms in the pascaline and e·pascaline duo

The complementarities of the pascaline and the e·pascaline in the duo lie in the following characteristics. The tangible pascaline produces sound and haptic feedback that are not existing in the e·pascaline. The e·pascaline offers also additional functionalities in comparison to the tangible pascaline, like the reset button to display zero on the three wheels, a counter of clicks which displays how many times you have click on a wheel and an evaluation feedback with smileys, expressing success and failures. There are also redundancies. They concern not only the visual aspect (Fig. 8) but also the two main strategies. The iteration and the decomposition strategies are available in both artefacts, even if not always with the e·pascaline. The antagonisms between the two artefacts concern the possible actions on the artefacts. With the e·pascaline, user action on the wheel may be controlled. Depending on the situation, the action on the wheels can be free or can be restricted or even stopped.

Fig. 8. Students' activity with the duo of artefacts pascaline and e·pascaline

Finally, there is a didactical situation framing the students' activity, to make them learn to write numbers, to calculate, to solve arithmetic problems. The didactical situations are partially embedded in the digital environment. It is not possible to do the same with the tangible pascaline.

4.3. 1st-grade students' conceptions about numbers and base-10 place value system

Like in the domain of geometry, learning is modelled by an evolution of conceptions. Thus, different conceptions about numbers and base-10 place value system must be distinguished to understand how the duo can provoke learning (Soury-Lavergne and Maschietto, 2015). According to Balacheff (2013), a conception is defined by the description the problems it enables to solve, the operators, the semiotic system to express the operators and the problems, and the types of control.

Following these elements of description, a first conception about numbers consists in seeing numbers as measuring a set of entities. For instance, number 17 is seen as 17 units. Numbers form a sequence; they follow or precede each other. Digits are placed side by side to form an iconic writing of a number and each number has a proper name, without a necessary relation with the other names of numbers. Adding is "counting on" and the decomposition of numbers (like the decomposition of 8 into 3 and 5) is not seen as a calculation. Another conception is embedding the previous one and a first view on base-10 place-value system. It means that number measures an organized set of entities, grouped in 10 and some units. The number code obeys to principles in order to deduce mathematical properties, like $17 = 10 + 7 = 1\mathrm{t} + 7\mathrm{u}$. Adding is using the information given by the digit in the number code and the sequence of number is seen as generated by adding one unit. The aim of teaching with the duo at primary school will be to make students evolve from one conception to the other.

4.4. Learning with the pascaline and e·pascaline duo, the case of addition

We have conducted several observations of classes, in different contexts in particular in two classes of 6-year-old pupils in France. I will take the example of addition with the pascaline and the e·pascaline to illustrate the change in students' conception.

With the pascaline, iteration and decomposition procedure are possible. We have observed that students do not use the iteration procedure spontaneously. But once they discover it, they use intensively the iteration procedure even with large numbers over one hundred. In fact, they show their expertise in mathematics by counting numbers one by one. The iteration procedure did not take too long and did not generate enough errors to lead pupils to look for another procedure. The e·pascaline works similarly to the pascaline. But, with the e·pascaline, the new constraint on action forces students to drop out the iteration procedure.

If you use the e·pascaline, you will experiment that if you want to add 14+16, you can first display 14 on the wheels, and then begin to add 16 by clicking 16 times on the unit wheel. But after some clicks, you will be stopped because the action button disappears. It creates a problem-solving situation for first grade students. They couldn't find out how to complete the calculation with the e·pascaline. In one of the class, some students have asked to use the pascaline to perform the calculation. One they obtain the result, with the pascaline, they wrote the result on the e·pascaline to get the evaluation feedback. They couldn't mobilize another strategy, which show the difficulty to consider numbers through their numeral writing and to exploit the base-10 place value system.

The problem couldn't be overcome without the teacher intervention. In one class, teacher intervention introduces the decomposition procedure. In the other class, the teacher took in charge the evolution of the procedure in a different way. She first brought the students to a common statement: we are blocked. Then she has formulated the problem: how to overcome the limitation of the use of the unit wheel? Finally, she gave pupils a hint by asking them to look for different additive decompositions of numbers. One of those decompositions is 16 as being 10+6 and with the conversion of 10 units into one ten, the problem could be solved.

In conclusion, the feedback helps to bring the problem to the front, but it is not enough to make pupils change procedure. These first studies have confirmed the great resistance of pupils to abandon iterative strategies in favor of decomposition strategy. It also reveals a conceptualization of number that does not yet incorporate the principles of number decimal writing. Nevertheless, it offers students and teachers a field of experiences that distinguishes two ways to operate with numbers that they can bring to the discussion.

4.5. Teaching with the pascaline and e·pascaline duo

Our aim with duo of artefacts is not only providing students with new environment and new opportunities to learn mathematics, but also to help teachers to integrate technology into their system of resources. So, we have conducted an experiment, with 8 voluntary teachers of first grade which were note involved in the design of the duo, to understand if the use of a duo of artifacts is possible and the condition of success in its integration (Maschietto and Soury-Lavergne, 2017).

The experiment has been conducted in France over a period of 12 weeks. We have made some direct observations, we have received regular reports from the teachers, including students' productions, and we have interviewed the teachers on a regular basis. Our question was: Could teachers integrate the duo in their system of resources? Our criteria to evaluate the integration of the duo by teachers were: (i) the creation of new resources and new situations of use of the duo of artefacts in their class; (ii) the actual organization of a class spatial/temporal configuration that enable to give students access to both artefacts; (iii) the awareness of some didactical aspects of the digital and tangible use of the duo.

One teacher left the experiment for medical reason, but the 7 other teachers have developed a wide range of ways to use the duo. First, they have implemented various didactical configurations, with the use of the pascaline or the e·pascaline, in collective setting, or with pair of students, or with student using individually the pascaline or the e·pascaline. They have also combined simultaneous use of the pascaline with the e·pascaline, for instance, every student has a pascaline and the e·pascaline is displayed on the wall (Fig. 9). There is also successive use, first the pascaline then the e·pascaline or conversely, the e·pascaline is first collectively used in class, then students work individually with their pascaline.

Fig. 9. A 1st grade class using simultaneously the pascaline (on students' desks) and the e·pascaline displayed on the wall

The teachers have produced additional resources and situations to those provided by the research team (Soury-Lavergne, 2014). However, one very important point is that they begin to be aware of the students' strategies and they tried to act and control it by using one or the other artefact. For instance, when students were blocked with the e·pascaline because they wanted to use the iterative strategy, one of the teachers gave access to the tangible pascaline, to help the student. By doing so, she has modified the didactical situation and allowed the pupils to solve the problem without having to change their procedure. It may be a problem for the learning, if the evolution toward the decomposition procedure never occurred.

Nevertheless, it reveals that the teacher played with the didactical characteristics of the duo. Therefore, the duo of artefacts became a system of instruments in the hand of the teachers. They have developed ways to exploit the complementarities and the antagonisms of the two artefacts of the duo. In conclusion, there are evidences of appropriation of the duo by teachers, to teach arithmetic's and to look for conceptual understanding, which means that there is an instrumental genesis of the duo of artefacts among every of the seven teachers.

5. Conclusion, Duos of Tangible and Digital Artefacts, a Means to Study Learning and Teaching with Technology

As a conclusion, I will first address the question of "why designing a duo of artefacts?" before dealing with "how to design it" and "would it help to teach and learn mathematics?"

A first reason to design a duo is because it provides students with a rich learning experience. The two examples of duo, one in geometry and one in arithmetic, have demonstrate that using a duo of artefacts helps students to develop new conceptions about mathematical concepts. Like a one-dimensional conception of the triangle associated to the compass to turn segments or to gain new insight in base-10 place value system for numeral writing, as a tool to solve problem. It is also a way to support dissemination and actual appropriation of digital technology by teachers. In our experiment, we have obtained evidence that it is possible. Teachers became aware of the complementarities of digital technology regarding the already used manipulatives. A duo also provided them with flexible configurations and possibilities for adaptation. A last point in concerning the involvement of teachers into research projects. Involving teachers in the design of a duo of artefacts appears to be efficient to enroll teachers in research. Teachers and researchers can share an initial aim and build collaboration on each other expertise.

The design of a duo of tangible and digital artefacts constitutes the main part of this lecture. The first point is to identify or to create two artefacts, with complementarities, redundancies and antagonisms. A pragmatic choice is to take a tangible and a digital artefact, which is an easy way to ensure complementarities, redundancies and antagonisms between the two. But it is not a necessity. The tangible artefact brings haptic feedback and gesture which are not so easy to obtain with digital artefact. The digital artefact brings feedback about procedures and evaluation which are also not easy to obtain with tangible manipulatives, which have a lot of limitations. Finally, a critical point is the elaboration of a didactical situation, that require the use of both artefacts, and initiate the instrumental genesis. The didactical analysis which is behind the identification of student conception and their evolution thanks the duo of artefacts is also a critical point.

The idea of duo of artefacts raises also new questions for research. Some are related to the model: for instance, how to decide which are the artefacts of the duo to be considered in a given situation to produce the analysis? And a more general question:

which duo of tangible and digital artefact for a given mathematical knowledge? Currently we do not have a stabilized technique to design a duo of artifacts. But it seems to be relevant in a lot of different situations. Several researchers have already considered the combination of digital and tangible artefacts in their own work for different pieces of knowledge. Duos of artefacts are a tool to focus the analysis on the intertwining of tangible and digital tools in mathematics.

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https://doi.org/10.1142/9789811287183_0041

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Developing Caring and Socio-politically Aware Beginning Teachers of Mathematics

Marilyn E. Strutchens[1] and Brea Ratliff[2]

ABSTRACT In this article, we elaborate on the following goals that we have for developing caring and socio-politically aware beginning teachers of mathematics and the strategies that we use to reach them: 1) Understand what it means to achieve equity, access, and empowerment in a mathematics classroom; 2) Develop equitable pedagogical strategies, 3) Examine and overcome barriers related to student engagement and achievement, 4) Confront negative beliefs about students from different race/ethnicity, socio-economic status, gender, ability; and sociolinguistics groups and move forward in a positive manner, 5) Develop an advocacy stance.

Keywords: Equity; Secondary mathematics; Prospective teachers; Introduction.

The following quote from a recent graduate of our program depicts attitudes and beliefs that we hope all the graduates of our program hold:

> You both [Marilyn Strutchens and W. Gary Martin; program faculty] have broadened my perspective on life and mathematics in numerous ways. Mathematics is now so much more exciting and fun for me, and I want my future students to see the joy and beauty in it that I have found in your program. I have a better understanding and perspective on what challenges my students face, and I walk through life more aware of struggles and obstacles that others face. I want to help my students overcome anything that tries to prevent them from reaching their fullest potential. In the world we live in, students need teachers that will create a safe and supportive place where they can thrive and grow to be whatever they want to be. If I can do that for my students, I will have fulfilled my purpose.

As mathematics teacher educators prepare teacher candidates it is important that we help them to think about the social justice issues existing in the world and how these issues impact their lives and their students'. Some of the major social justice issues impacting the United States and other countries in recent years include voter suppression and manipulation, effects of climate change, health care disparities which have been made more evident with COVID-19, refugee crisis and immigration, racial

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injustice, gun violence, and inequitable treatment of the LGBTQ + community (Hamilton, 2020).

Some of the issues mentioned above led several mathematics education organizations to call for change. The Association of Mathematics Teacher Educators (AMTE) issued the following statement:

> The Association of Mathematics Teacher Educators (AMTE) stands in solidarity with Black Americans in the face of racial injustice. We are dismayed by the inhuman and unjust treatment of Black Americans by law enforcement personnel in recent months with the deaths of George Floyd, Breonna Taylor, and Ahmaud Arbery. We acknowledge the inequities that the COVID-19 pandemic has illuminated related to health care, economic standing, and education. As an organization, AMTE believes that racism must be interrogated in this country. We cannot look at what is happening to Black Americans and other oppressed groups as problems that they alone need to solve. (AMTE, 2019)

TODOS: Mathematics for All also stated:

> Our position is to prioritize antiracist mathematics education for all students as we prepare to return to school this fall and the years to come. An antiracist position in mathematics education is a pledge to dismantle systems and structures that maintain racism within teaching and learning mathematics from challenging belief systems that perpetuate microaggressions to disrupting the role mathematics classes play in pushing students out of schooling. We pledge to more thoroughly develop and lead the way with frameworks for antiracist mathematics classrooms. (TODOS: Mathematics for All, 2020)

It is important for mathematics teacher educators to prepare themselves to facilitate the growth of caring and socio-politically aware beginning teachers of mathematics. How do we as mathematics teacher educators prepare ourselves to foster the growth of prospective teachers? As mathematics teacher educators, each of us must become cognizant of the lived experiences of Black, Indigenous, and People of Color (BIPOC) by reading the history of the United States through a social justice lens. We must learn ways to empower and provide access to students who often are judged by the color of their skin and not by their knowledge and abilities. We must develop the knowledge and skills that we want our teacher candidates to develop, such as those listed in AMTE's (2017) *Standard for Preparing Teachers of Mathematics* related to social contexts of mathematics teaching and learning:

Standard C.4. Social Contexts of Mathematics Teaching and Learning

Well-prepared beginning teachers of mathematics realize that the social, historical, and institutional contexts of mathematics affect teaching and learning and know about and are committed to their critical roles as advocates for each and every student. Indicators include:

C.4.1. Provide Access and Advancement;
C.4.2. Cultivate Positive Mathematical Identities;

C.4.3. Draw on Students' Mathematical Strengths;
C.4.4. Understand Power and Privilege in the History of Mathematics Education;
C.4.5. Enact Ethical Practice for Advocacy (AMTE, 2017, p. 21).

In addition to developing the dispositions and skills ascribed by AMTE (2017) for prospective teachers, mathematics teacher educators must take the following actions according to Aguirre et al. (2017):

(1) Stop using deficit-oriented language in mathematics education work and help educate others about how such language perpetuates negative framings of children and communities.
(2) Deepen one's professional knowledge base and mentoring practices with mathematics and social justice as a dual focus.
(3) Acknowledge and learn about the systems from which you benefit from unearned privilege.
(4) Read outside of mathematics education literature to better understand target and non-target identities and how they are related to various systems of privilege and oppression.
(5) Cite mathematics education researchers from around the world who do equity-focused work, especially scholars of color.
(6) Engage colleagues and friends in explicitly talking about race, class, gender, and other systems of privilege and oppression.

These action steps may also be taken by teacher candidates along with the mathematics teacher educators.

1. Goals and Strategies to Enable Prospective Teachers to Become Caring and Socio-politically Aware Beginning Teachers of Mathematics

After mathematics teacher educators have begun taking the preceding action steps, they must set goals for their teacher candidates to enable prospective teachers to become caring and socio-politically aware beginning teachers of mathematics. Below is a list of goals that we have for our secondary mathematics preservice teachers at Auburn University and in the following sections, we elaborate on each of the goals and how we help our prospective teachers to obtain them: 1) Understand what it means to achieve equity, access, and empowerment in a mathematics classroom; 2) Develop equitable pedagogical strategies, 3) Examine and overcome barriers related to student engagement and achievement, 4) Confront negative beliefs about students from different race/ethnicity, socio-economic status, gender, ability; and sociolinguistics groups and move forward in a positive manner, 5) Develop an advocacy stance.

1.1. Prospective teachers need to understand what it means to achieve equity, access, and empowerment in a mathematics classroom

In order for prospective teachers to understand what it means to achieve equity, access, and empowerment in mathematics classrooms they must understand what these constructs mean. We use multiple definitions of equity to help prospective teachers to understand how difficult it is to achieve equity in the mathematics classroom. The Aspen Education and Society Program and the Council of Chief State School Officers (2017, p. 3) states that educational equity means that every student has access to the educational resources and rigor they need at the right moment in their education across race, gender, ethnicity, language, disability, sexual orientation, family background and/or family income. This definition is used to help teacher candidates see the importance of all students having access to meaningful instruction and resources needed for success. Another definition shared with students indicates the bidirectional mutual respect of equity: Equity is extended from a unidirectional exchange — as primarily benefitting growth of students and student groups that have historically been denied equal access, opportunity, and outcomes in mathematics to a reciprocal approach (Civil, 2008). A meme is also shared that has three scenarios to help students to visualize what equity means. One picture has students standing on the same number of crates regardless of their heights as they are staring over a wall to see a game. One student can clearly see the game, another student can barely see the game, and the third student is not tall enough to see the game at all. This picture represents equality, each student gets the same amount of help independent of their needs. In the second picture each student gets what he needs to see the game which represents equity. In the third picture there is not a wall, representing no barriers for any of the children which is liberty.

After discussing the definitions, students are asked to read articles and book chapters that help them to recognize equitable or inequitable situations. We examine the Mathematics Teaching Practices (National Council of Teachers of Mathematics, 2014), the five equity-based teaching practices (Aguirre et al., 2013), and This We Believe: Keys to Educating Young Adolescents (Association for Middle Level Education, 2010). After discussing these practices, students are asked to view videos that feature equitable pedagogy to see how well they identify equitable teaching strategies. One video used is Looking for Squares by Connected Mathematics. The goal of this lesson is for students to develop an early understanding of the concept of square root as the length of a side of a square. Some students drew squares that were "upright" or drawn with sides that were parallel horizontal and vertical lines, while others constructed "tilted" squares that were rotated at 45-degree angles. The teacher facilitated mathematical discourse among the students as they shared their findings with the class. Teacher candidates use checklist versions of the mathematics teaching practices and the five equity-based practices to view the video. They also answer the following questions developed by Rousseau-Anderson (2007) to discuss what they observe: Who has access to the learning that is occurring? Are all students able to participate in the learning process? Who has access to the resources that support learning?

1.2. Prospective teachers need to develop equitable pedagogical strategies

Before prospective teachers view and examine videos for equitable practices, they experience the tasks as learners and then discuss the affordances of the tasks as teachers. These discussions help the prospective teachers to view the videos through learning, equity, and access lenses. Prospective teachers are also observed during their field placements and clinical residency experiences by their university supervisors, mentor teachers, and peers through the same lenses. Below is a note from a peer observation that took place during a paired placement clinical residency experience:

> Today in 7th grade I observed my co-intern teach a lesson on addition and subtractions of integers and rationals! Students practiced addition & subtraction problems and came up with their own rules for adding and subtracting negative and positive numbers. These are the rules the classes came up with: 1) If you add two positive numbers, you will get a positive number. 2) If you add two negative numbers, you will get a negative number. 3) When you add a negative number and a positive number, your answer will have the same sign as the number with the highest absolute value. I was really impressed with student responses as they noticed the patterns and came up with the rules! We all got really excited when students came up with the absolute value rule! I thought my co-intern did a really great job engaging students and making them explain each problem. There were times that students got a little rowdy when they stated their claims on the rules, but the conversation was so great that we didn't get angry with the students. Not all of the classes got a lot of practice with adding decimals and fractions, but the lessons and student engagement today were great.

In addition to discussing the different checklists for equity-based teaching strategies teacher candidates learn about culturally inclusive mathematics lessons which fit into the following categories:

(1) Use students' culture to help students learn mathematics. (Ford, 2005; Ladson-Billings,1995)
(2) Ethnomathematics. (D'Ambrosio,1985; Furuto, 2016)
(3) Identify diverse cultural contributions. (Various Websites)
(4) Explore mathematics using cultural artifacts. (Variety of Resources)
(5) Use mathematics to study social or cultural issues. (Gutstein, 2003, 2006)
(6) General uses of mathematics. (Various websites)
(7) Use multicultural literature as a context for mathematical problem solving. (Strutchens, 2002)

These lessons help teacher candidates to think about teaching mathematics in a variety of cultural contexts. One social justice lesson that is shared with students is called Double Periods (Conway et al., 2018). In this lesson, students developed a statistical question and gathered data, which suggested inequities in course enrollment by race/ethnicity. The question was personally meaningful to many students, as their teacher had provided opportunities for students who were not on the “honors” track but who had good grades in ninth-grade Algebra I to accelerate their course taking so that they could enroll in AP statistics. While some members of the class were initially taken aback by the implication that race/ethnicity may play a role in course taking, the personal experiences of these “non-honors” students helped to frame the class’ discourse about causes for this inequity in course taking as being largely about opportunity rather than ability or interest. Students’ use of mathematics and data empowered then to inform the principal of the school of their findings and start making changes in their school culture (Conway et al., 2018).

Moreover, teacher candidates continue reflecting on equitable practices during their Clinical Residency and Management Seminar. They constantly reflect on implementing the MTPs and other equitable practices (Strutchens et al., 2022). Some prospective teachers are in pairs which enables them and their teachers to create professional learning communities focused on student learning. Below are quotes from prospective teachers related to access and equity in their clinical experiences:

> Overall, my partner’s lesson went really well. I really liked how she noticed students using a variety of strategies so when it came time for the students to present their solutions, she had the students who solved the problems using different methods all present. This allowed all of the students to see there was more than one way to solve this problem and that not one method is "better" than the other. I also thought the problem she selected for the students to explore today was a good example of a multiple entry level problem. There were several entry points to the problem that allowed students who struggle and students who are advanced the productive struggle they needed.”

Second period ran a lot smoother today than it has in the past. The parallel teaching seemed to work out today. The students were given the opportunity to work in small groups and have an instructor with them the whole time. In my group, I feel like my students got to ask all of the questions they were confused on. I also didn't hear any arguing over the groups. I really appreciated the willingness the students had to make this new strategy work.

1.2. Prospective teachers should examine and overcome barriers related to student engagement and achievement

Socio-politically aware teachers need to be aware of the barriers that prevent students from learning mathematics. These barriers can range from physical resources and opportunities to learn to teachers’, students’, parents’, and school administrators’ beliefs and stereotypes. The continuum of caring as discussed by Secada (2003) is something that we discuss with students: “Caring could be used to protect students’ emotional and psychological well-being, where teachers seek to avoid all risk of adding further to their children’s trauma, or caring could be used to motivate proactive interventions, where teachers push students to increase their knowledge to have a variety of options. This continuum is in alignment with the challenge posed by Aguirre et al. (2017):

> There is a longstanding, thoroughly documented, and seemingly intractable problem in mathematics education: inequity. Children of certain racial, ethnic, language, gender, ability, and socio-economic backgrounds experience mathematics education in school differently, and many are disaffected by their mathematics education experience. (p. 125)

We share a video of an African American student, Amari Mitchell, who at the time was a junior at Hoover High School, describing an experience in his mathematics classroom (Dunigan, 2017). Amari described how his mathematics teacher went out of his way to provide the White students in his class with additional attention and support, while he was left struggling to make sense of the teachers' instructions. As a result, Amari learned to depend on his parents to teach him the mathematics he didn't learn in school. Amari conjectured that while some teachers, like the mathematics educator, only care about the students that look like them, others do not. In the second half of the year, a new mathematics teacher was assigned to his class, which led to a completely different, positive educational experience. By challenging this space which previously marginalized his school mathematics experience (Aguirre et al., 2013), the new teacher worked collaboratively with Amari and his parents to ensure his success. This video challenges the teacher candidates to think about what students are learning beyond mathematics in the classroom.

It is important that teacher candidates think about these issues and other social context issues, such as addressing beliefs related to gender abilities in STEM fields, addressing the beliefs about and needs of emergent multilinguals, addressing the beliefs about and needs of Indigenous students, addressing beliefs about and needs of lesbian, gay, bisexual, transgender, queer or questioning (LGBTQ) students, addressing the beliefs about and needs of students in poverty situations, and addressing the needs of students who identify as being in the intersection of multiple groups. Below are prospective teachers' quotes related to access and equity in their clinical experiences:

> Something I noticed my cooperating teacher doing was not calling on a variety of students. She allows the students to sit in "comfortable seating" around the room. One of those options is a round table in the back of the classroom. Throughout the day I noticed several times when a student sitting back there would raise [their] hand to answer a question get overlooked by someone in the middle of the room. If this were my room, I would try and make a mental note of where everyone is sitting to ensure I am calling on students in an equitable way.

During homeroom today, we had a student come to school with a giant hole in his shorts. He happened to have some gym shorts in his backpack, so he chose to change.

A little while later in that same class. I noticed my cooperating teacher hand sewing the [student's] shorts. It reminded me that teachers are more than just teachers.

These quotes show that the teacher candidates are reflecting on their observations beyond the topics they are teaching.

Mathematics identity is another topic that we discuss in class that teachers may impact in both positive and negative ways. Mathematics identity includes beliefs about self as a mathematics learner; one's perceptions of how others perceive him or her as a mathematics learner, beliefs about the nature of mathematics, engagement in mathematics, and perception of self as a potential participant in mathematics (Solomon, 2009). We ask teacher candidates to think about their own mathematics identities through a self-assessment as a mathematics learner: They write down and discuss at least three adjectives which describe themselves as a mathematics learner. They also are asked to think about the factors, which helped to shape their beliefs about themselves as learners and doers of mathematics. We also do an activity where they examine and discuss three cases: Calvin, Caroline and Craig. They examine each of the cases and answer the questions on their own, then share their thoughts with their elbow partners. We then discuss how teachers affirm mathematics identities by providing opportunities for students to make sense of and persevere in challenging mathematics. This form of participation builds a high sense of agency in students.

Students with a high sense of agency make decisions about their participation in mathematics. Below is prospective teacher's quote related mathematics identity from a paired placement experience:

> One student constantly raised her hand for us to validate her solutions. My partner and I both told her to really think about what she is doing and to use her prior knowledge and what she learned the previous days to try on her own. By the end of the period, her mathematical identity was boosted.

Social Identity Petals Activity is an activity that we do with the prospective teachers during one of their pedagogical courses. They are asked to write their identities on the petals of the flower, drawing from the following categories: ethnic background, geographic origin, religious background, gender, talent/ability, hobbies, personal and family influences, race, socio economic class, age, and/or other cultural influences. After they have written their personal identities on the petals, they are asked to think about those that are readily identified in social situations and those that are not. They are also asked to discuss how characteristics that are readily identified by others affect how others interact with them? They then share their petals with their groups. Similarities and differences are also discussed. As a group, they also discuss how they are similar and different from their students. They also discuss beliefs or customs that they have that may conflict with the beliefs and customs of their students. Then they are asked what they can do to make their classroom environment more inclusive? This activity leads to a rich discuss about the different groups of people to which we each belong.

1.3. Prospective teachers must confront negative beliefs about students from different race /ethnicity, socio-economic status, gender, ability; and sociolinguistics groups and move forward in a positive manner

The social identity petals activity leads into a discussion on stereotypes or other beliefs held by people that impede the mathematical empowerment of groups of students. We discuss how they have the power to increase access and success for students in nontraditional programs by interrupting the cycle of negative micro-messages, bolstering student self-efficacy, and challenging cultural stereotypes. (National Alliance for Partnerships in Equity, 2015). Next, prior to viewing a video, we discuss words like privilege that may cause teacher candidates to not hear what is being said by the student. Privilege is a special right or advantage that only one person or group has. Recognizing privilege simply means being aware that some people must work much harder just to experience the things one takes for granted (if they ever can experience them at all) (Gina Crosley Corcoran, 2017, HuffPost). The concept of intersectionality recognizes that people can be privileged in some ways and not privileged in others (Gina Crosley Corcoran, 2017, HuffPost). The video comes from Black Students Talk about the Achievement Gap in Alabama Schools (Dunigan, 2017). The student talks about her experiences as a black student. Kameryn, a 2017 graduate of Clay-Chalkville High School, mentioned the decrease in the number of Black students taking AP classes at her high school and how she believed school personnel held deficit perspectives about their academic abilities. Instead of being encouraged to enroll in AP courses, Kameryn felt Black students were often urged to pursue athletic opportunities. After mentioning that the responsibility for change lies with the teachers and Black students, Kameryn encouraged Black students to resist the temptation of only pursuing athletics. She also suggested teachers should disclose the existence of a wider selection of post-secondary opportunities to all students. Kameryn also believed that due to White privilege, Black students were often forced to double their efforts to achieve the same level of academic access that their White counterparts were afforded. After viewing the video, we discuss micromessages. According to the National Alliance for Partnerships in Equity (2015): Cultural stereotypes exist about people and careers, because of stereotypes, we have implicit biases; micromessages are the manifestation of implicit biases; positive and negative micromessages accumulate which causes high or low self-efficacy and behavior is the result of self-efficacy.

1.4. Prospective teachers need to develop an advocacy stance

To help teacher candidates develop empathy for students and an advocacy stance, teacher candidates follow a student around for a day. They learn much about what students experience on a day-to-day basis. Vignette 7.4. A Student Teacher's

Revelation Related to Emergent Multilingual Students (AMTE, 2017) highlights the importance of teacher candidates following a student around for a day. The vignette also shares the secondary prospective teacher candidate's revelations in learning to meet the needs of her emerging multilingual students. This story illustrates the importance of not only having teacher candidates work with a diversity of learners but also having them reflect on and process those experiences to better understand how they can build on the cultural and linguistic resources that students bring to the classroom to support the learning of each student.

This vignette may be used in a methods course to help teacher candidates consider what they would do if faced with a similar situation. Prospective teachers are asked to think about the following questions: What role did reflection play in the student teacher's ability to advance the mathematics learning of her multilingual students?

What could preservice teachers gain from reading this vignette? What could be done in programs to prepare preservice teachers for this experience? (AMTE, 2017, pp. 131–132). Here is a quote from a prospective teacher's related to access and equity during their clinical experiences:

> Something I have been learning recently is that being a teacher means balancing a lot of responsibilities at once. I really want to improve on making sure I'm taking action to meet my students' needs. I need to consider students with an IEP, advanced students, students who need to makeup work, students in ISS, etc. My goal is to take notes of what I need to do for each student in order to help them all succeed.

In addition to working with other teachers and school personnel, teacher candidates need to involve parents as partners in their students' education. We share strategies for working with parents and resources for families. Below are some resources:

(1) The Algebra Project. (Moses et al., 1989; Moses and Cobb, 2001)
(2) Multicultural Literature as a Context for Mathematical Problem Solving: Children and Parents Learning Together. (Strutchens, 2002)

Another way that we help teacher candidates to develop an advocacy stance is to help them to become aware of the dangers of tracking and other policies that keep students from reaching their full potential (NCTM, 2018). We encourage them to interrogate situations that are not inclusive or look like they are separating students base on their demographics in ways that lessen the students' opportunity to learn. We also encourage teacher candidates to interrogate situations that are inequitable to teachers which in most cases are simultaneously inequitable to students.

2. Conclusion

Throughout the paper, I discussed a list of goals and strategies used during the mathematics education program at Auburn University: 1) Understand what it means to achieve equity, access, and empowerment in a mathematics classroom; 2) Develop equitable pedagogical strategies, 3) Examine and overcome barriers related to student engagement and achievement, 4) Confront negative beliefs about students from different race/ethnicity, socio-economic status, gender, ability; and sociolinguistics groups and move forward in a positive manner, 5) Develop an advocacy stance. These goals have led us to foster the growth of well-prepared beginning teachers who use equity-based strategies and care deeply about their students.

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For Human Flourishing, Build Mathematical Virtues, Not Just Skills

Francis Edward Su[1]

ABSTRACT A great mathematical education should build mathematical virtues, not just mathematical skills. Virtues are what make mathematical experiences enriching and they serve one well no matter what one does in life. They enable human beings to flourish.

Keywords: Mathematical practices; Human flourishing.

1. Introduction

During a job interview at a high-powered company, a student of mine was asked this question:

> How many gallons of water are flushed in New York City during a commercial break of the Super Bowl?

Although the Super Bowl is an annual football game watched by millions of people across the United States, this company was not asking the question because they had any particular interest in American sports or New York City infrastructure. Rather, they were interested in seeing how my student, a mathematics major, could think and reason her way to answer a question of this nature — a question with some inherent uncertainty and ambiguity, with many strategies for determining an answer. Such questions arise often in the work of this company. As she talked out a solution strategy, the interviewer would probe her thinking and suggest ways to circumscribe her answer.

Notice that the company wasn't trying to assess the job candidate's *skills*, such as whether she could apply the quadratic formula or could calculate an integral. Rather they were interested in *virtues*, such as her ability to strategize (in planning a path to a solution), her resourcefulness (in suggesting potential sources of data to draw on), and her creativity in putting that information together. They were also assessing her collaborative abilities — how well she could talk through a mathematical problem with others.

If virtues are what employers are looking for when they hire a math major, why does a mathematics education mostly focus on skills? Of course, skills are important, but to this employer, virtues were even more important than skills.

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2. The Difference Between Skills and Virtues

In this article I'll use the term *skills* to refer to mathematical content knowledge such as facts, algorithms, and formulas and fluency with them. By contrast, *virtues* are human qualities rather than specific items of knowledge. Most people equate virtues with moral virtues such as truthfulness and honesty, but there are many other character qualities that can be considered virtues. For instance, in the Aristotelian philosophical tradition (Aristotle, Ross, and Brown, 2009), courage and wisdom are virtues. However, I want to enlarge our conception of virtue even further.

One way to define virtue is: *an excellence of character that leads to excellence of conduct*. So there's a "being" aspect and a "doing" aspect. The "being" aspect attends to shaping one's character so that the "doing" aspect (good action) naturally follows. In a mathematical context, persistence is one of the many intellectual virtues that can be cultivated by a great mathematics education. Shaping one's character to be persistent in mathematical problem-solving leads to a lifetime of fulfilling choices to grind away at tough problems.

Examples of Mathematical Skills

Fluency with facts, algorithms, formulas
Factoring polynomials
Taking a derivative
Computing things

Examples of Mathematical Virtues

Persistence
Curiosity
Creativity
Thirst for Deep Understanding
Habits of Generalization
Independent Thinking
Capacities to: Define, Quantify, Abstract, Visualize, Strategize, Collaborate

Virtues are important for both practical as well as personal reasons.

On a practical level, virtues are more important to employers than skills. Often that's because the highly technical skills an employer needs are often learned on the job, rather than taught in school. Also, skills are easily replaceable, while human virtues are not. Computers can take derivatives and factor polynomials quicker and more accurately than humans can. (That doesn't necessarily mean that we shouldn't teach those subjects, but it should cause us to consider that the purpose of teaching certain skills is often for the underlying virtues they build, such as developing a deep understanding.)

On the other hand, computers and even artificial intelligence are not (yet) able to be creative and to generalize in the same way that humans can. Moreover, the skills needed in any profession may change over time — just think of all the technological gizmos today that use ideas which didn't exist twenty years ago. But the virtues needed from a math education will not. Employers will always need employees who are persistent problem-solvers, who are collaborative but are also able to independently learn new things.

On a personal level, virtues are more important to one's life than skills. At the end of our lives, are we more likely to appreciate that we knew how to bisect an angle, or to appreciate the ways that being curious and creative enriched our lives?

Virtues provide a better answer to the question "Why do I need to know this stuff?" than the ones teachers often give: "Because you'll need this stuff later." That answer is unsatisfying to students, and it's often not even true. Furthermore, it doesn't help students see that doing mathematics can be relevant to their lives *right now*. A better answer would be: "Because mathematics enables you to marvel at the hidden structures of the world and solve problems you've never seen before, and thinking mathematically builds aspects of character in you that enable you to flourish."

Unlike mathematical skills, which are usually useful only if you end up in a mathematical profession, mathematical virtues will benefit you no matter what you do in life. Think about how much the average person needs to factor a quadratic. Then compare that to how being a creative solver will help you in any profession, or how being able to visualize makes your life richer.

3. Virtues Are More Than Mathematical Practices

Some attention to the role of virtues in math education can be found in the Common Core Standards for Mathematics (NGA Center and CCSSO 2010), an initiative in the U.S. to establish consistent standards across the states for what students should know. These standards include not just standards for mathematical content in each grade level, but also include "Standards for Mathematical Practice" that cut across grade levels. For instance, some of them are:

- Reason abstractly and quantitatively.
- Make sense of problems and persevere in solving them.
- Look for and make use of structure.

These practices describe attitudes built by a great math education that are indeed virtues; they are useful in any profession or life circumstance. Embedded within them, you can see some of the other virtues I've called out: persistence, thirst for deep understanding, habits of generalization. The Mathematical Practices are valuable, but they do not come close to capturing all the virtues that a math education fosters. There are many other intellectual virtues we exhibit when we think well.

And we should not forget the *affective* virtues that are also part of a great mathematical experience. These are what the classroom or workplace a place of wonder, delight, and joy.

Examples of affective mathematical virtues

An Expectation of Enchantment

Disposition towards Beauty
Hospitality in Welcoming Others to Mathematics
Hopefulness
Self-Confidence
Affection for Mathematics

A great math education also builds all these virtues too. Such virtues make one's life richer, and enable one to flourish.

4. Building Virtues by Attending to Basic Human Desires

At ICME-14, I spoke about the purpose of mathematics in promoting human flourishing, a vision that was also described in detail in my recent book (Su 2020a). Virtues are at the heart of what it means to be human and to flourish in this world. Here, I want to build on that presentation by outlining some practical ways to cultivate virtue in the classroom by attending to basic human desires that all our students have. None of these strategies are earthshakingly new, but their framing around human desires may offer some perspective about why they are successful.

For instance, beauty is a basic human desire that all human beings share. We all long to behold beautiful things, and we are curious to understand what others find beautiful. Our mathematics classrooms can and should tap into this desire. Some ways to do this include:

- Making your classroom a place of sensory beauty, such as through artwork or music.
- Highlighting wondrous beauty (the beauty of ideas) and insightful beauty (the beauty of reasoning) and helping students see how they contribute to an appreciation of life.
- Providing space for students to reflect on beauty.

Attending to a desire for beauty cultivates in our students' affective virtues, such as *reflection* and *joyful gratitude*, which contribute to positive identities in mathematics activities. It can provide experiences of *transcendent awe*, when we see the connectedness of mathematical ideas to each other and to explaining the world. It also promotes *habits of generalization*, because we are trained to look for beautiful overarching patterns where we might not expect them.

Another basic human desire is exploration. From our earliest moments as infants, we are eager to explore our environment and learn about the world. How do we nurture that kind of exploration in the classroom? Some strategies include:

- Changing dull computational problems to exploratory problems (one where creative thinking is required and many solutions are possible);
- Praising good questions, not just good answers;
- Showcasing enchanting ideas (and making it a professional goal to learn enchanting ideas).

Cultivating an attitude of exploration can help our students build virtues such as *imagination* and *curiosity*. And as we showcase beautiful and enchanting ideas, students develop the *expectation of enchantment*. In the most rewarding math experiences, budding mathematicians learn to expect this enchantment. It's what keeps them coming back for more. So we should also make it a professional goal as teachers to be continually learning enchanting ideas in mathematics by regularly doing reading about mathematical ideas.

Play is another basic human desire. Humans enjoy fiddling with things, focusing our attention on fun diversions, interacting with others in activities (like games) that intermix structure with freedom, and marveling at the surprises that result. Mathematics teaching should also make mathematical experiences feel playful. Some ways that this can be done:

- Giving students space and time to play, such as open-ended questions that aren't answered right away.
- Lowering the stakes of "right answers."
- Adopting "rough draft thinking" (Jansen, 2020): ideas don't have to be perfect the first time they emerge, and they can always be improved through revision.
- Viewing each mathematical idea from multiple perspectives (just like one does in a game, viewing a strategic situation from the viewpoints of others).

Such practices cultivate in our students' virtues, such as concentration and perseverance, because play can be an intensely pleasurable focus that shuns the other distractions of daily life. They build hopefulness, because when you tinker with a problem long enough, you are exercising hope that you will eventually solve it. And they train in us the ability to change perspectives, a virtue that serves us well in solving math problems but also in understanding the life experiences of others.

These are just a few examples of ways that attending to basic human desires can make our classrooms places of human delight and connection, and cultivate virtues that will enrich our students' lives no matter what they do in the future. I explore several other human desires and their attendant virtues in Su (2020a).

5. Assessing Virtue

This begs the question: if virtues are just as (if not more) important than skills, why do most math educational experiences focus primarily on skills? One reason — in my opinion, a primary reason — is that skills are easy to assess, but virtues are not. It's easy to grade skills — just make a worksheet, have students compute twenty multiplication problems, and check for correct answers. It's harder to assess virtues, since that feels messier — we have to find ways to elicit student thinking, and decide how to assess whether they've been persistent or curious. That feels subjective, doesn't it? And yet, even worksheets — if you consider the choices made about what kinds of problems to include — are subjective assessments. Thus, the fact that assessing virtues is messy and possibly subjective doesn't mean we shouldn't try.

I understand that assessing virtues is hard to do, and I hope math educators will continue to work on methods and rubrics to assess some of the ones I've mentioned in this article. But let me close by suggesting a few ideas about how one might begin.

For several years now, and especially during the pandemic, I've been including reflection questions on my assignments that attempt to get at virtue development. See Su (2020b). For instance, here's a question that I use to evaluate persistence:

> Take one homework problem you have worked on this semester that you struggled to understand and solve, and explain how the struggle itself was valuable. In the context of that problem, describe the struggle and how you overcame that struggle. You might also discuss how struggling built aspects of character in you (endurance, self-confidence, competence to solve new problems) and how these virtues might benefit you in later ventures. Thoughtful answers receive at least 9 out of 10 points.

Having students reflect on what they've gained from struggle is more formative than summative, since I don't try too hard to make find distinctions between answers — I give nearly full credit for any thoughtful answer. If you try it, I encourage you not to sweat fine distinctions. It's much better to make the questions low stakes, and let students know in advance that any thoughtful answer will receive full or nearly full marks. What's crucially important is to notice that by the process of reflection, students are engaging in building the virtue you are hoping to assess. Asking reflective questions sends a strong message to my students that I care about more than just skills. Of course, that message should be reinforced by all that you do and say in the course, not just delivered in homework.

I also like to see how my students' curiosity has grown with questions like this one:

> What mathematical ideas are you curious to know more about as a result of taking this class? Give one example of a question about the material that you'd like to explore further, and describe why that is an interesting question to you. A thoughtful question will receive 9 points, and especially insightful question will receive 10 points.

It's been remarkable to me to see the kinds of questions students ask, and it makes me reflect on how I can encourage more of these questions during our in-person class time.

Finally, here's a question that I use to evaluate a disposition toward beauty.

Consider one mathematical idea from this course that you have found beautiful, and explain why it is beautiful to you. Your answer should: (1) explain the idea in a way that could be understood by a classmate who has taken classes X and Y but has not yet taken this class and (2) address how this beauty is similar to or different from other kinds of beauty that humans encounter. Thoughtful and correct answers will receive at least 8 points, and especially insightful answers will receive 9 or 10 points.

I know some readers will be skeptical that such a question could be meaningful, since it seems far-fetched that one could tell whether a student could have developed such a disposition. And yet I find that it is not hard at all to tell. Moreover, I've learned useful ways of thinking about my own subject matter from students!

For example, here are a couple of responses I received on this question in a course about symmetries of polynomials (Galois Theory).

[In speaking about straightedge-and-compass constructions:]

> Similar to the Mona Lisa or other timeless classic art, these constructions can be appreciated by anyone, regardless of upbringing or ability. In the same way that everyone finds a flower beautiful, these constructions can be appreciated as well… I find the depth of constructions similar to the relationship between traditional and modern art. The construction of the equilateral triangle is easily appreciated, just like a traditional painting. Both are accessible. On the other hand, the construction of the 17-gon is difficult to understand, and even more difficult to attempt to replicate. When constructions are taken to their limit, they are obtuse. I feel like modern art is in a similar position. In my art history class last semester, my classmates and I were often left puzzled after the professor discussed a banana taped to a wall or a toilet made of gold. We often had to ask "what is the meaning of this piece?" Someone seeing the construction of a 17-gon for the first time might ask the same question. Ultimately, I believe the accessibility and universal nature of these constructions make them beautiful.

[In speaking about roots of polynomials that always appear together:]

> This kind of beauty is reminiscent of the idea of a soul mate: someone who in some sense matches you perfectly and who you always want to be together with. Roots of the same minimal polynomial have the same "nature" in some way, and never occur without each other. The idea of two separate things being connected on some deep level also occurs in a lot of other ideas of dualism (you can't have shadow without light, you can't have cold without warmth, etc) which have been present in religions and human beliefs for longer than recorded history, and must appeal to us on some deep level. I guess it is beautiful to see that his idea that is so deeply ingrained in us as has equivalent which can be proved even in the world of mathematics.

Did these students possess a disposition toward beauty? Most certainly yes, and they were encouraged by asking a formative question. One benefit of asking such questions that I did not anticipate was how much it would help me as a teacher — by showing me what they are thinking and informing the examples and analogies I would use when I teach the subject again. And most importantly, I actually *enjoy* reading their reflections, much more than grading any other question.

6. Conclusion

In summary, a great mathematics education should build much more than content knowledge and fluency with skills. A great math education should build intellectual virtues, like curiosity, creativity, and a thirst for deep understanding. It should also build affective virtues, like hopefulness and an expectation of enchantment. It should even build communal virtues, like collaboration and hospitality in welcoming others to mathematics. These are virtues that build healthy math identities, serve us well both practically and personally, and make our lives richer. They also foster more equitable mathematics experiences. Through the development of virtues, mathematics education can enable human flourishing.

Acknowledgments

Many thanks to Matt DeLong for helping me sharpen my thinking about virtue.

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Language in Mathematics Education: Issues and Challenges

Konstantinos Tatsis[1]

ABSTRACT In this chapter I present some theoretical and methodological approaches on language in mathematics education. Language can be mainly viewed as the means to represent mathematical meanings or as constructing mathematical meanings itself. These approaches in turn lead to different methodologies, which in turn lead to different types of results. I argue that researchers should be cautious before adopting a particular theoretical framework, since sometimes frameworks are based on neighbouring concepts. I provide examples of such concepts, namely positioning and norm. The complexity of language and interaction in mathematics classrooms calls for more holistic and less dichotomised approaches. The combination of approaches is also an effective way to conduct research on language in mathematics education; examples of such combinations are provided.

Keywords: Language; Discourse; Norm, Positioning.

1. Introduction — Views on Language in Mathematics Education

The relationship between mathematics and language has been a topic of research for decades. It is worth noting that research in (mathematics) education and in language seem to share some common characteristics: both have initially focused on statistically and laboratory-based studies, but gradually have shifted to more interpretative and less taxonomical approaches, as Austin and Howson (1979) note. These authors were among the first to also note that the field of language in mathematics education deserves our attention, since "In the teaching and learning of mathematics, language plays a vitally important role" (p. 162).

So, a first question that we may ask is what is the role of language in mathematics education? Assuming that we view mathematics as precise and unambiguous (Ambrose, 2017), the role of language is communicating mathematical meanings among human agents. In that case, the effectiveness of communication resides on the agents' ability to code and decode the meanings entailed in language. However, our experience tells us that communication in the mathematics classroom is far from unproblematic and unambiguous. Rowland offers a characteristic example of a student's answer: "The maximum will probably be, er, the least 'll probably be 'bout

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fifteen" (Rowland, 2000, p. 1). The mathematical proposition extracted from this sentence would be the following: "There are fifteen segments" (Tatsis and Rowland, 2006, p. 257). So, why would the student add hedges in his proposition or, in other words, why would he choose to make his contribution vaguer than expected? A short answer to this is because language in the mathematics classroom may be used to communicate more than mathematical concepts; in this case it is used to convey the speaker's uncertainty. Such situations, which are common in mathematics classrooms, are the object of studies which acknowledge that language is more than a "neutral" communication system; it is a constitutive part of the world it describes and, concerning mathematics, there cannot be mathematics without language (cf. Derrida's claim that there is "nothing outside the text" (1976, p. 158)).

The idea of language in mathematics education seen as a social and interactive phenomenon has followed the shift from quantitative linguistics to sociolinguistics, pragmatics, discourse analysis and ethnomethodology (Schiffrin, 1994; see also Ingram (2018) for an overview on mathematics education). In mathematics education, we have witnessed a similar shift towards analyses of interactions (e.g., Krummheuer, 2007), where the focus can be on the establishment of a discursive community (Kieran et al., 2002), on the norms that regulate the verbal exchanges (Cobb and Yackel, 1996) or on the students' roles during the interactions (Tatsis and Koleza, 2006).

A common characteristic of these approaches is that they have adopted and eventually adapted concepts derived from sociology and social psychology. Two characteristic examples are the concept of norm and the concept of positioning, which will be discussed in the next section.

2. How Novel is Your Approach? Neighbouring Concepts in Research

The big development of research in mathematics education — as witnessed by the increase of research papers, scientific journals and conferences — has led to a thrive of theoretical approaches. This has been also made possible by the easy access that researchers have to a multitude of studies worldwide, coming from different disciplines. Interdisciplinary and transdisciplinary approaches have been also made possible by the modern means of communication and exchange of information. Researchers, especially the younger ones, usually find themselves in the position of having to choose between an existing framework to ground their work or create a new one; their most frequent choice is presented in the next quote:

> It has become the norm rather than the exception for researchers to propose their own conceptual framework rather than adopting or refining an existing one in an explicit and disciplined way. This prolific theorizing might be represented as the sign of a young and healthy scientific discipline. But it may also mean that theories are not being sufficiently examined, tested, refined and expanded. (Editors of Educational Studies in Mathematics, 2002, p. 253)

So, we, as researchers or reviewers, find ourselves in the position to examine the actual novelty of an approach, but mostly how well it is scientifically grounded. In order to do so, we need to sometimes study the historical background of the proposed theoretical constructs. I present two examples related to my own research, in order to highlight the similarities among concepts that are met in research with different names.

2.1. Norm and neighbouring constructs

The concept of norm, especially the subconstructs of the social and the sociomathematical norm, is a characteristic example of a concept which was successfully transferred from sociology to mathematics education. This was mostly done by the work of Paul Cobb, Erna Yackel and their colleagues (e.g., Cobb and Yackel, 1996; Yackel and Cobb, 1996; Yackel et al., 2000). The social norms are related to the structure of classroom activity in general and may include

> … explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives in situations in which a conflict in interpretations or solutions had become apparent. (Cobb and Yackel, 1996, p. 178).

The sociomathematical norms are related to mathematical activity and refer to which contribution counts as “a different mathematical solution, an insightful mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation” (Cobb and Yackel, 1996, p. 179). The origins of the concept of norm can be traced in the concept of prescription, as described, e.g., in the work of Biddle and Thomas (1966), who define prescriptions as

> behaviours that indicate that other behaviours should (or ought to) be engaged in. Prescriptions may be specified further as demands or norms, depending upon whether they are overt or covert, respectively. (p. 103)

In mathematics education research, one may find few more concepts, which seem related to the concept of norm. The first one is the concept of obligation, which, according to Voigt (1994), is an interactive construct that connects various routines in the mathematics classroom: teacher’s and students’ actions are constrained by some obligations, and this may especially become apparent in cases of conflict. The second concept is the meta-discursive rule, introduced by Anna Sfard and refers to “mostly tacit navigational principles that seem to underlie any discursive decision of the interlocutors” (2002, p. 324). Sfard (2008) goes on to claim that a rule is considered a norm only if it fulfils two conditions: it must be widely enacted within the discursive community and it must be endorsed by almost all members of that community, especially those considered as experts. The third concept which seems to be related to the concept of norm is the didactical contract, as introduced by Brousseau (1997): it refers to specific habits of the students that are expected by the teacher and vice-versa. Knowledge construction is seen as a shared responsibility between the teacher and the students, and as a result of interpretations of each other’s actions. However, an established didactical contract can also create problems for the students, especially when they enter a situation where the contract changes considerably, e.g., during the transition from primary to secondary education or from secondary to tertiary education.

2.2. Positioning and neighbouring constructs

Positioning theory, as Herbel-Eisenmann et al. (2015) claim, is another example of an imported theory in mathematics education. Originated in the work of Rom Harré and his colleagues (e.g., Harré and van Langenhove, 1999), the theory has been deployed in mathematics education in order to interpret communicative actions performed by the teachers and the students. Such actions affect the positioning of each other in the establishment of mathematical knowledge, by sometimes challenging the prevailing authority structures. For example, it is assumed that the teacher is an authority of the classroom; what happens if this authority is challenged by the teacher herself? In a next section I will demonstrate how this approach was combined with another sociological approach, in order to analyse interactions. At this point though, I will present another concept, which is neighbouring to positioning: the concept of framing, introduced by Goffman (1974). This concept is part of a wider system of constructs used to interpret how people interactively define the situations they are involved in. Particularly, during any interaction, speakers align between each other, but also in relation to utterances; this creates the space for intersubjective knowledge.

The next neighbouring concept is the interactive frame, proposed by Tannen and Wallat (1993) and refers to "a definition of what is going on in interaction, without which no utterance (or movement or gesture) could be interpreted" (pp. 59–60). By moving to a linguistic approach, we find Gumperz's (1982) notion of speech activity. According to Gumperz, the analysis of framing can be done by the use of contextualisation cues, which are defined as "any feature of linguistic form that contributes to the signalling of contextual presuppositions" (1982, p. 131).

Summing up, the relationships — or the borders — between the aforementioned constructs are not clearly defined; Gordon (2015) presents examples of researchers who consider them as rough synonyms (Tannen, 1994); she also claims that the notion of storyline can bring together positioning and framing:

> framing — growing from a field that has tended to look out into the world (sociology) — and positioning — developing from one that has tended to look within (psychology) — have found a meeting point, where the interactional and the psychological are understood as inseparable in language. (p. 340)

Until now I have claimed that neighbouring concepts, originating in other disciplines, are deployed in various studies, including studies in mathematics education. In the next section, I present another characteristic of studies on language in mathematics education: the use of (supposedly) clearly defined categories to describe and distinguish the relevant phenomena.

3. Distinctions and Dichotomies that We Live By

One may claim that it is apparent for researchers to deploy clearly defined categories in order to better present their results to the scientific community and the general public. If we limit ourselves to the relationship of language and mathematics, we may notice a basic distinction between mathematical and everyday language. The former has specific vocabulary and syntax, minimum use of verbs and an absence of the human agent:

> … the prevailing image of mathematical writing, perpetuated in most of the texts encountered by students in the later years of schooling and at university, is still impersonal, lacking a narrative of human involvement in doing mathematics. (Morgan, 2001, p. 169) [2]

The above distinction, which refers to written texts, might have served research well during the initial attempts to analyse the linguistic phenomena in mathematics education. However, all contemporary researchers agree that language needs to be viewed as a multi-faceted phenomenon:

> Analyses should consider every-day and scientific discourses as interdependent, dialectical, and related rather than assume they are mutually exclusive. (Moschkovich, 2018, p. 40)

Following the above, there have been attempts to overcome the dichotomy of ordinary versus mathematical language. A characteristic example is the categorisation suggested by Pirie (1998) for the means of mathematical communication: ordinary language, mathematics verbal language, symbolic language, visual representations, unspoken shared assumptions, quasi-mathematical language. Pirie (1998) goes on to discuss the problems that occur during the move from ordinary to mathematical language, by presenting relevant examples from solving linear equations to division. She also claims that problems also exist during the move from verbal mathematical to symbolic language. Then she discusses two of the most interesting parts of her framework: the unspoken assumptions shared by the students (which resembles the concept of norm discussed before) and the quasi-mathematical language, which is:

> … carrying meaning for the users that is unorthodox or incompatible with acceptable mathematical language. On the other hand, however, this quasi-mathematical language leads to no difficulties in understanding and can, in fact, often enhance understanding by forming a language-linked image that is of personal relevance to the learner. (p. 22)

The existence of quasi-mathematical language, and its role in establishing shared mathematical meanings has been examined in a series of studies that I have done in the last years. The core of these studies has been a game called "Broken phone", in which the students are asked to either describe (based on a drawn image) or draw (based on written instructions) complex geometrical figures (see Fig. 1).

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[2] The above stereotypical view is challenged by most — if not all — researchers today. For instance, Morgan, in the paper quoted, suggests widening the spectrum of what is considered 'appropriate' and 'mathematical' by the teacher, leaving room for students' descriptions of their investigations and their problem solving processes.

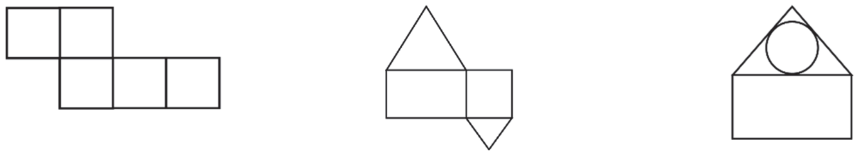

Fig. 1. Figures given in the “broken phone” game (Tatsis, 2007; Tatsis and Moutsios-Rentzos, 2013)

One of the results of these studies has been that the students have sometimes been able to communicate the properties of the given figures by using either everyday or quasi-mathematical language.

A second way to overcome the unproductive distinctions while studying language in mathematics education, is by combining theoretical and even methodological frameworks. This possibility is discussed in the next section.

4. The Need to Combine Approaches

It has been apparent from the previous sections that language in mathematics education is a rich, complex and evolving field of mathematics education. Moreover, due to its focus on language and interactions, it has been importing or adapting theories and methodologies from linguistics, sociology and psychology. Some researchers in the field have realised that each theoretical approach is good enough to shed light on only one (or few) sides of the multi-faceted phenomena of mathematics teaching and learning. That is why the need for combining theories in a systematic way (usually referred to as networking theories, see, e.g., Bikner-Ahsbahs et al., 2014) has come to the fore. The advocates of this approach acknowledge that the variety of theories is a resource that researchers should build upon, but at the same time, a unified “theory of everything” is far from possible (Prediger et al., 2008).

Despite the above suggestions, the instances of combined approaches are still scarce. As I mentioned before, many researchers are focused on introducing a “novel approach” that “will shed new light” on the phenomena of interest, sometimes neglecting the significance of existing theories, which become even more powerful when combined. A characteristic example comes from combining positioning theory, that was mentioned before, with politeness theory (Brown and Levinson, 1987) for the analysis of a mathematics teacher’s interactions with his students. Tatsis and Wagner (2018) have presented two juxtaposed analyses of particular excerpts, which were originally analysed elsewhere (Wagner and Herbel-Eisenmann, 2014) by the lens of positioning, particularly in relation to the authority structures that have been observed: personal authority, discourse as authority, discursive inevitability, and personal latitude. Tatsis and Wagner then enriched these juxtaposed analyses with a combined one. According to this, a student’s question (which, according to positioning theory, was classified as a manifestation of the student’s personal latitude) was also seen as threat to the teacher’s positive face. This, in turn, allowed the authors to interpret the teacher’s reaction as an attempt to protect his face. As a result, the combination of these approaches has allowed the authors to gain the insights from both analytical lenses:

> … the two theories are both interested in the phenomena that occur during classroom interactions; moreover, we see them as complementary since politeness theory helps us consider reasons for teachers and students to choose particular authority structures in their classroom interactions. Thus, we generally believe that in order to fully comprehend the dynamics of the exchanges in the mathematics classroom we need to be able to continuously shift our focus from the participants' acts to the established (or striving-to-be-established) norms and from the participants' positionings to their own and the others' face-wants. (pp. 183–184)

At this point, it is worth mentioning that there have also been attempts to offer juxtaposed (but not combined) analyses of the same data; such attempts may also have a significant contribution to research on language in mathematics education. A characteristic example is the paper of Candia Morgan and her colleagues (Morgan et al., 2007), in which six researchers analysed a given excerpt by Cohors-Fresenborg and Kaune (2007), by deploying six different theoretical and methodological approaches.

5. Discussion and Recommendations for Research

In the present chapter, I have drawn upon various approaches for studying phenomena related to language in mathematics education. I did not aim to cover all approaches, since such works already exist in the literature (e.g., Planas et al., 2018); my aim was to focus the readers' attention on particular issues that I consider significant in the field. The first issue is the neighbouring theoretical concepts, which usually originate from sociology, psychology or linguistics, but all refer to the same phenomenon. I have provided the examples of two such concepts, which are deployed in studies of language and interactions in mathematics education, namely the norm and positioning. I claim that one way to deal with this is to deeply examine the background of such concepts — see their differences and their similarities, be cautious before adopting one and restrain oneself from giving a new name to an already existing construct. Neologisms seem attractive, but most of the times lead to confusions and repetitive studies.

The second issue refers to the distinctions, and eventually the dichotomies that appear in some studies, the most prominent being the one between everyday and mathematical language. Although these can be helpful for particular kind studies, I claim that it is more realistic to view language as a more complex and more dynamic phenomenon. In contemporary multilingual and multimodal mathematics classrooms, such dichotomies fail to grasp the complexity of the verbal and non-verbal interactions. Therefore, we need more holistic approaches — which may come from the combination of existing approaches; such approaches may provide the researchers with the appropriate tools to zoom in and out from the micro-level of face-to-face interactions to the meso-level of the classroom discursive community (and the interpersonal relations) and then to the macro-level of the school and education system and their discursive practices. A second way to deal with such dichotomies is by establishing new categories, eventually placed between the existing ones, in order to cover the "grey zones" that exist in real classroom interactions. I have provided an example of a study, in which quasi-mathematical language was effectively used to communicate geometrical concepts.

The last issue refers to the need to combine approaches in the study of language in mathematics. As I mentioned above, this need is grounded on the need to analyse in a holistic way the phenomena related to language in mathematics education. I have provided an example of such a study, in which positioning theory and politeness theory were effectively combined to offer new insights to an already-analysed episode. There is a movement in mathematics education (networking theories) that focuses on combining particular theoretical approaches. I claim that it is generally a positive thing that more combination studies appear in the field; only by allowing ourselves to view phenomena from different points of view, we can achieve a wider understanding of language and mathematics teaching and learning.

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44

What Matters for Effective Mathematics Educator: Preservice or In-service Training?

Alphonse Uworwabayeho[1]

ABSTRACT According to UNESCO (2015), the equity gap in education is exacerbated by the shortage and uneven distribution of professionally trained teachers, especially in disadvantaged areas. Target 4.c (MOI) of the SDG 4 is therefore aimed at substantially increasing the supply of qualified teachers, including through international cooperation for teacher training in developing countries, especially least developed countries and Small Island developing States by 2030. It further states that teachers are one of the fundamental conditions for guaranteeing quality education and therefore there is need to empower and adequately recruit, remunerate and motivate professionally qualified teachers and educators, and support them within a well-resourced, efficient and effectively governed systems (UNESCO, 2015). The knowledge of teachers in the last three decades was mainly influenced by a well-known scholar Lee Shulman who categorized teacher knowledge into seven categories among which content knowledge is included. However, much research on in-service teachers focused on the pedagogical content knowledge hypothesizing the mastery of content as much as they are graduated from recognized training institutions. Based on this categorization, the present paper presents an analysis of Rwandan mathematics school subject leaders' Content Knowledge (CK). The presentation is based on partnership established between governmental and nongovernmental institutions led to development and implementation of certified Continuous Professional Development (CPD) programs for primary and secondary mathematics teachers. Findings reveal a lack of teachers' preparedness to adopt the new curriculum teaching approaches, there is also lack of appropriate physical facilities in schools to accommodate every leaner's individual needs among other hindering factors. Recommendations include systematic CPD programs for in service teachers to complement preservice training so that they can adapt various reforms and in-service teachers to establish their individual professional development plans.

Keywords: Continuous professional development; In-service training; Teacher education.

1. Introduction

It is not doubtable that teaching is a complex work and pre-service teacher education is rarely sufficient to provide all knowledge and skills necessary to successful teaching and students learning (Killen, 2015). Therefore, a significant portion of teacher

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education can be acquired only on the job. In particular, growing research (e.g., Killen, 2015; Ball et al., 2008) shows that quality induction has a positive impact on beginning teachers' motivation and commitment; beginning teachers' job satisfaction; beginning teachers' teaching practices and learning outcomes of students. Rwanda registered an impressive growth enrolment especially at primary level as result of different education policies especially with the aim of achieving education for all goals. However, the quality was undermined as demonstrated by learners' low performance from national tests especially in mathematics subject. The low performance was attributed to different factors including lack of teaching and learning materials including textbooks, large class size, heavy teaching load for teachers (Mineduc, 2015). This situation was worsened by the introduction of the competence-based curriculum (CBC) at the beginning of 2016 in primary and secondary education. Since then, different strategies to improve students' performance in mathematics and sciences have been an important concern to each level of education system in Rwanda for all stakeholders. These strategies include teachers training on the implementation of CBC, dissemination of books and laboratory equipment, smart classrooms and improving students' welfare by introducing school feeding program across the whole country. However, as per our own experience, it has been observed that most of strategies set by most of educational stakeholders tend to focus on empowering in-service teachers with pedagogical skills with little, if any, focus on improving teachers' subject content knowledge. Though education statistics affirm that 93.6% of primary school teachers are qualified (Mineduc, 2018), there is a little information on link between pre-service training and mathematics syllabus requirements as well as how teachers are motivated to adapt regular changes in terms of content. In this line, the Rwandan Teacher Development and Management (TDM) Policy states that all beginning teachers — defined as teachers in the first three years of their career, have to receive systematic professional support from their head teachers, subject school leaders, school-based mentors, and school inspectors. For the purpose of the present paper, we limit the description on the role of mathematics subject school leaders (MSSLs). MSSLs are the experienced teachers appointed by the school head teachers to support new qualified mathematics teachers through mentoring and coaching process in addition to their ordinary teaching load (REB, 2019). They are called to play key role in leadership of mathematics teaching and learning in their schools for students' achieving learning outcomes. To assure this important role, they are supposed to be expert in mathematics content knowledge, pedagogical content knowledge, knowledge of mathematics curriculum and its requirements such as ensuring inclusive education in mathematics lessons (making sure all students can learn). Since none of MSSLs was prepared to assume these new responsibilities, it is worthy to interrogate ourselves to what extent selected MSSLs are knowledgeable in terms of curriculum syllabus content. This will inform policy makers and educational partners further trajectories for teacher professional development. We review the existing literature on teachers content knowledge with focus on the work of Shulman (1986), the methodological considerations, discussions of findings and conclusions.

2. Literature Review

Research (Darling-Hammond, 2000; Ingvarson et al., 2005) found that students' mathematical achievement would be attributed to teachers and their teaching practices; though more teacher mathematical knowledge does not necessarily imply more student learning (Ball, Loewenberg, Thames, and Phelps, 2008). In other words, having more knowledge of mathematics does not automatically lead to better teaching of mathematics. Shulman (1986) identified different types of knowledge that are required for being effective teacher: mathematics content knowledge, pedagogical content knowledge and curriculum knowledge. Later work in mathematics education built on Shulman's work (e.g., Ball et.al., 2008) made further distinctions within mathematics content knowledge and pedagogical content knowledge. Inside of subject matter knowledge, it is distinguished between common content knowledge (what most people will know in mathematics) and specialized knowledge (what people who have studied mathematics will know) (Ball et al., 2008). In other words, subject matter knowledge is knowledge of mathematics. It's not about knowing about children or how to teach the mathematics to children.

Concerning with pedagogical content knowledge, there are mixtures of knowing the mathematics content and knowing how to teach it. It involves knowledge of the mathematics curriculum that lies ahead and how what is taught has an impact on children's learning of mathematics in their later life. Knowledge of content and curriculum is about what is taught when; and how do learners move through topics. If you are a mathematician, it is quite unclear how things are ordered and why. It is knowledge of when to introduce a concept and how each concept builds on previous knowledge and forms a building block to more advanced knowledge. This overlaps with the knowledge of students and knowledge of content and teaching. For example, a teacher who plans to teach a lesson on multiplying decimals needs to know a lot more than how to do the multiplication: *"The teacher had to know more than how to multiply decimals correctly herself. She had to understand why the algorithm for multiplying decimals works and what might be confusing about it for students."* (Ball, 1990; p. 448). In the first part of the statement, it is the content knowledge for mathematicians while the second concerns with mathematics content knowledge for teachers. Research in developed countries illustrates that a substantial part of the difference in student achievement is attributable to teachers and their teaching practices (Darling-Hammond, 2000; Rice, 2003, Ingvarson et al., 2004). In developing countries, and with specific regard to Lesotho, it was found that teacher characteristics such as gender, class size or years of experience had no influence on students' mathematical achievement (Parke and Kanyongo, 2012); rather teachers' content knowledge seems to be the only certain influence on students' mathematical achievement. Based on the argument that the teacher needs to know more about the topic he/she has to teach, the paper focuses its investigation on in-service teachers' subject matter knowledge in different areas of the mathematics syllabus in place.

3. Methodological Considerations

As mentioned throughout previous sections, the paper concerns with analysis of mathematics subject school leaders (MSSLs) content knowledge. The motivation for this paper draws appointment of these teachers to implement professional development strategies put in place by the Rwanda ministry of education. On one hand, we argue that success for these strategies depends on competencies of MSSLs for the effective delivery of the desired outcomes. On the other hand, we know that none of these MSSLs was prepared to assume these responsibilities added to their normal teaching load. Therefore, this becomes a challenge for not only themselves but also for education local education leaders in terms of the implementation. It is within this context, a partnership between Rwanda Education Board (Rwanda governmental organ in charge of implementation education policies up to secondary school level), the University of Rwanda-College of Education (the unique Rwanda government higher learning institution in charge of education training) and the Flemish Association for Development Cooperation and Technical Assistance (VVOB) organized a formal credited continuous professional development program for MSSLs. The program introduces MSSLs to a variety of aspects of pedagogical content knowledge for teaching mathematics and subject leadership. The overall aim of the program is to equipping them with mentoring and coaching skills of their fellows and new qualified teachers. In this way, to support the implementation of the CBC, the training is also competence-based whereby the focus is on practice based and learning collaboration. Training approaches includes learning collaboration through developing culture of discussion, active participation and micro teaching. In order to make microteaching relevant and meaningful for participants, facilitators judged it better to focus as much as possible on mathematics topic area that seem to be difficult to introduce to learners. But how could we decide which topic was the most difficult? One way was to ask participants to name these topics; but this would lead to individuals' sentiments. Hypothetically, once one does not master a given topic, he is likely either to avoid teaching it or badly teach it, thus students' missing part of the curriculum content. We therefore preferred to use a simple test whereby participants were given a series of questions covering the six topic areas (number and operations, fractions and proportional reasoning, metric measurements, geometry, algebra, statistics and elementary probability) as identified in the national mathematics syllabus (REB, 2015a, p.18). Therefore, the purpose of the test was not evaluating participants' mathematics content knowledge; rather supporting facilitators identifying topics of the primary mathematics curriculum to be used in the training process.

Though the program has to reach all MSSLs in six districts that registered learners' low performance in national mathematics examination ending primary education in addition to high rate of dropouts, the present study concerns 39 teachers (one MSSL per primary school in six selected districts) who were invited to starting the CPD program. It is worthy to mention that an MSSL is appointed by the school leader on the basis to have served at least 3 years in the same school and all MSSLs of the six districts have to benefit of the CPD program at different cohorts. These 39 MSSLs comprised of 8 females and 31 males. They are in range of 5 to 30 years of teaching experience.

The mathematics test took place on the first day of the actual training that consists in face to face sessions in three different centers with two facilitators from URCE per centre. Prior to administer the mathematics test, facilitators (authors of the paper) briefed MSSLs on modalities governing training and the rationale of the training and the test. MSSLs signed individual consent form related to video recording, interviews and any other type of data with educational purpose. It is expected that some videos of good practices may be shared for educational purpose. Participants had one after another two question papers on mathematics content and pedagogical content knowledge; reasonable period of 2hours for each component. But the focus of the paper is mathematics that was composed of 20 mixed open and multiple-choice questions. Answer sheets were marked, outcomes analyzed and mean and standard deviation calculated. Results are hereafter graphically or tabularly presented according to the six topic areas from the primary national mathematics syllabus.

4. Results and Discussions

All 20 questions were grouped under six themes that coincide with the six topic areas. One part presents results as a whole group, that is no consideration of district, while another part takes district into account. The rationale behind the second arrangement consists in depicting any contextual particularity of a given that from the 6 districts whereby 4 (Kirehe, Kayonza, Gatsibo, and Nyagatare) are located in eastern province and 2 others (Rusizi and Nyabihu) in western province.

4.1. Topic area 1: numbers and operations

The topic on numbers and operations is taught from primary 1 (P1) to primary 6 (P6). Specifically, by the end of primary, learners are expected to be able to read, write and compare whole numbers beyond 1,000,000. In this view, mathematics subject school leaders (MSSLs) were tested on writing numbers in figures. They were also asked to write in words as they would say numbers. Findings on these questions are summarized in Fig. 1.

4.1.1. Whole numbers

Fig. 1 shows that writing numbers in figures from words was less confusing than writing them in words from figures. For example, all MSSLs (100%) could write correctly “two thousand and fifty in figures” whereas, 2% of them failed to write 140,000 in words. It also shows that writing numbers in figures seemed difficulty for larger numbers, e.g., 10% of MSSLs could not correctly write “*four hundred thousand*

and seventy-three" in figures. In addition, writing decimal numbers in words seemed difficult to some MSSLs (12%). Looking into different answer sheets, it can be observed that some SSLs don't have skills to translate between written decimal numbers and spoken English. Since, being able to write, read and compare numbers is one of targeted key competences for learners in primary schools, teachers of mathematics should be able to help learners achieve this competence (REB, 2015). Therefore, there is a justifiable need to involve MSSLs in activities where they work together to understand and adopt appropriate ways of teaching and learning whole numbers.

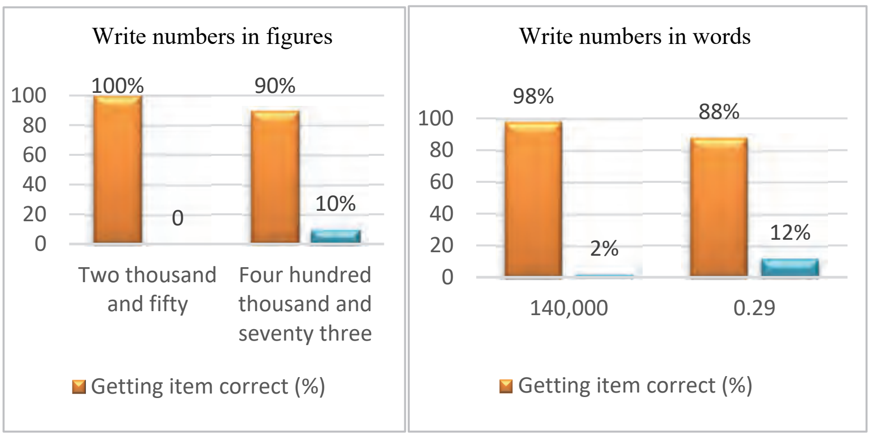

Fig. 1. Write numbers in figures and in words

4.1.2. *Decimal numbers*

The national mathematics competence-based syllabus under implementation in Rwanda prevails teaching decimal numbers in upper primary (REB, 2015a). From P4 through P6, learners should be able to add, subtract, and compare decimal numbers using place values of decimals up to some numbers of decimal places. Fig. 2 (on the next page) summarizes MSSLs competences to write decimals.

Fig. 2 (a) indicates challenges in writing decimal for some MSSLs whereby 76% could not write correctly "*eleven tenths"* in figures. The majority of those who failed could confuse it with "eleven thousandths" or "eleven hundredths". Other MSSLs could not identify the place values for eleven tenths. On the other side in Fig. 2 (b), 48% of MSSLs failed to compare *'four tenths' and 'hundredths'* If teachers are still hesitating writing and comparing decimal numbers, how can they teach learners to correctly read, write and compare decimal numbers in figures and in words?

4.1.3. *Doing mathematics operations on decimal numbers*

Another key competence targeted in the CBC is having primary school learners able to multiply, add and subtract decimal numbers (REB, 2015a). Fig. 3 describes MSSLs' ability to perform mathematics operations on decimal numbers.

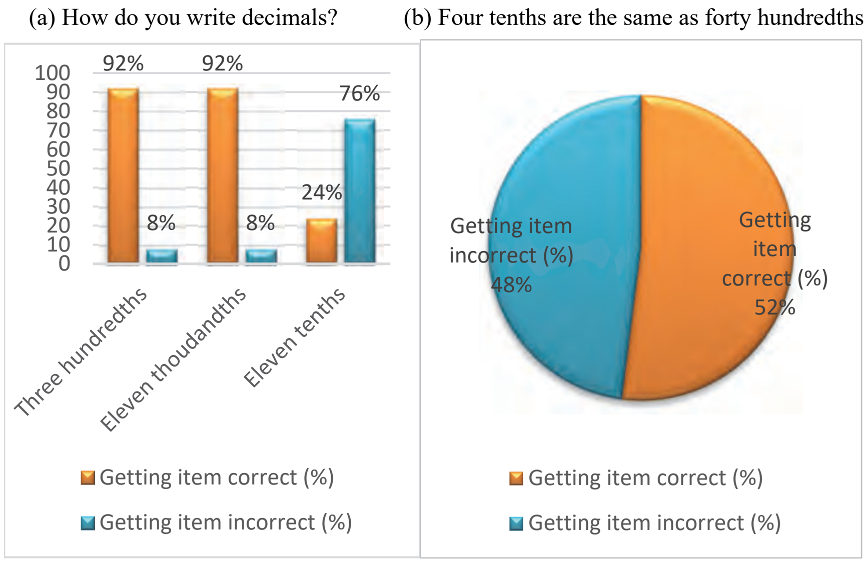

Fig. 2. Write decimal numbers

Though, teachers who plan to teach these operations need to know a lot more than what the CBC foresees for primary school learners (Ball, 1990), Fig. 3 indicates that MSSLs (95%) equally performed in transforming decimal numbers to fractions and in doing addition with decimal numbers. Likewise, MSSLs (86%) did 4 digits subtraction. However, only 43% of MSSLs managed to multiply a decimal number by 100.

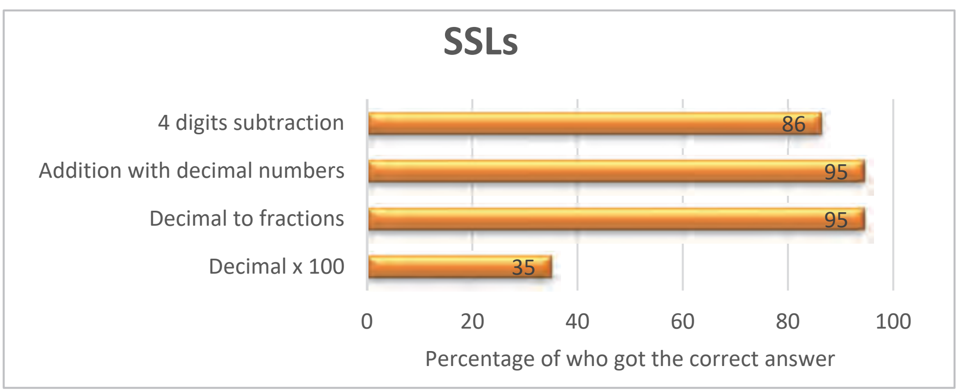

Fig. 3. Mathematics operations on numbers

4.1.4. Rounding of decimal numbers

By the end of primary schools, learners should be able to round off decimals, convert fraction to decimals and *vice versa*. They should be able to solve problems involving rounding and conversion (REB, 2015a). MSSLs were asked to identify nearest numbers in size by estimation. Fig. 4 summarizes the percentage of those who correctly answered questions related to rounding of decimal numbers.

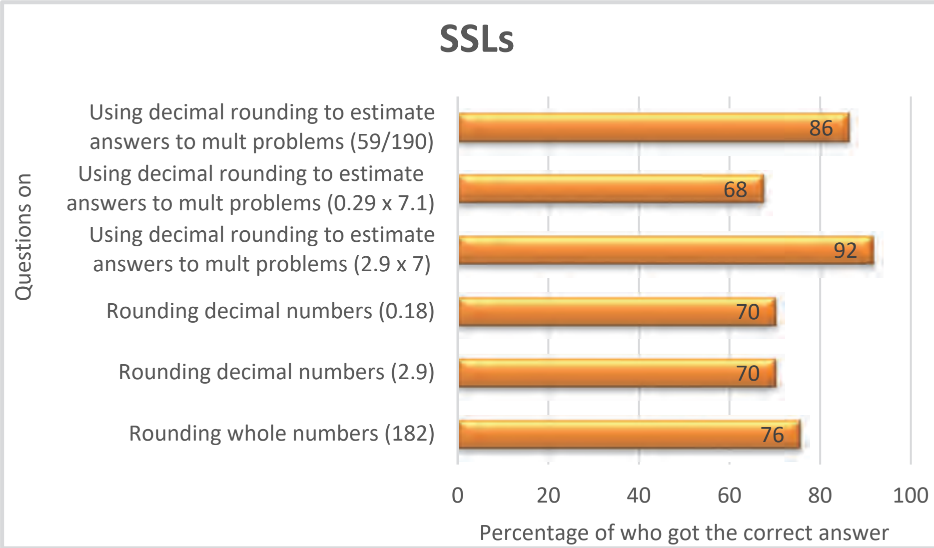

Fig. 4. Using decimal rounding off

Fig. 4 indicates that shows sign of facing difficulties in using decimal rounding to estimate answers to multi-problems, estimating the number to nearest in size to the answer, multiplying with decimal numbers. In addition, only 70% of MSSLs could identify a nearest number in size to 0.18. Again, this shows some weaknesses in manipulating decimal numbers. In general, MSSLs showed weaknesses in rounding off decimal numbers to the nearest tens and hundredths.

4.1.5. Selecting the correct operation for word problems

By the end of primary education, the CBC emphasizes that learners should be able to use numbers and operations to solve real problems (REB, 2015). Fig. 5 describes the

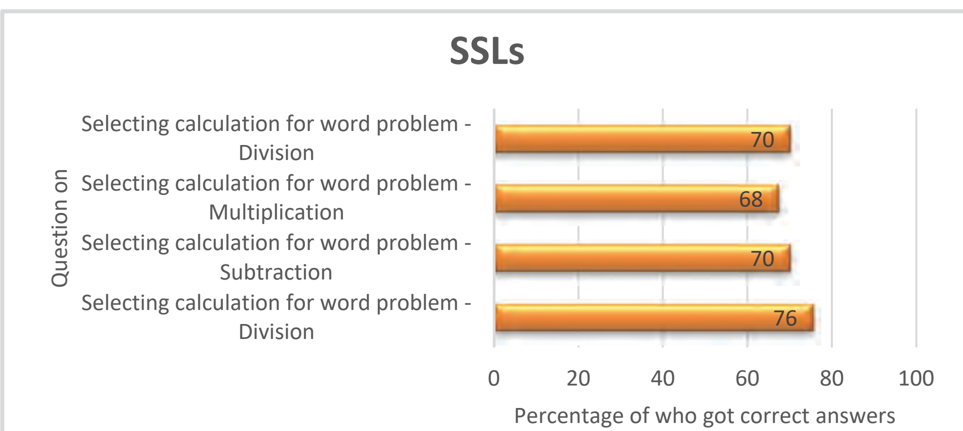

Fig. 5. Selecting calculation for word problem

percent of SSLs who could correctly answer questions that involved addition, multiplication, subtraction and division of whole and decimal numbers.

Fig. 5 indicates that selecting calculation for word problem was not perfect for MSSLs. For a word problem related to addition, they (70%) did it relatively better than they (68%) did for a word problem related to multiplication. It can be inferred that MSSLs need to develop skills related to calculation for word problem which they are teaching.

4.1.6. Division of decimals

As the current curriculum, primary school teachers of mathematics should be able to teach division of numbers. Therefore, they should be conversant with the process of division of decimals up to 3 decimal places. In this view, SSLs were asked a question on dividing two decimal numbers:

12.3 ÷ 0.15 has the same answer as	
A 123 ÷ 0.015	B 123 ÷ 1.5
C 123 ÷ 15	D 123 ÷ 150

Their performance on this question is summarized in the following Fig. 6.

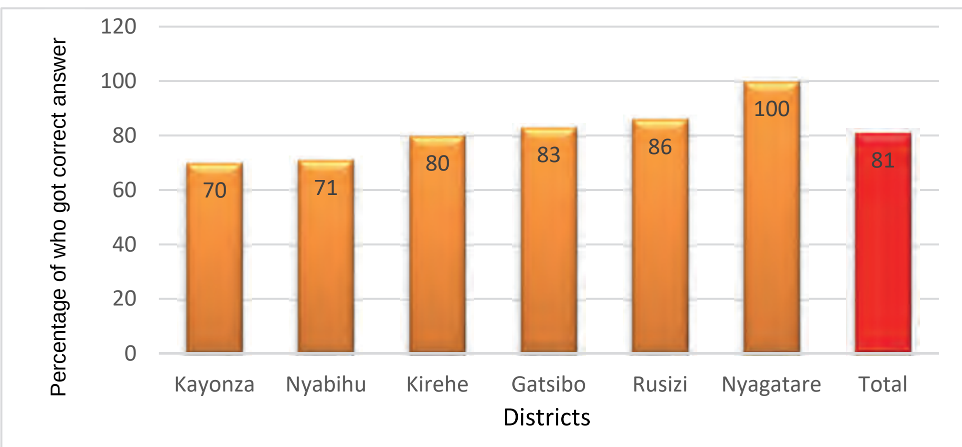

Fig. 6. Performance in division of decimals

Fig. 6 shows that the score is not constant across a given province as Kayonza and Nyagatare located in the same eastern province scored the least and the most respectively.

4.2. Topic area 2: fractions and proportional reasoning

4.2.1. Estimating whole and decimal numbers on a number line

In the CBC framework, teacher of mathematics should be able to involve learners in activities to solve problems related to comparing, ordering, and finding distance between numbers.

From Fig. 7 we can observe that estimating decimal numbers on a number line was more difficult than estimating whole numbers. For examples, only 44% of male SSLs and tutors could estimate a decimal value on an empty interval number line. In addition, only 56% of male SSLs and tutors could estimate decimal value on a grouped interval number line.

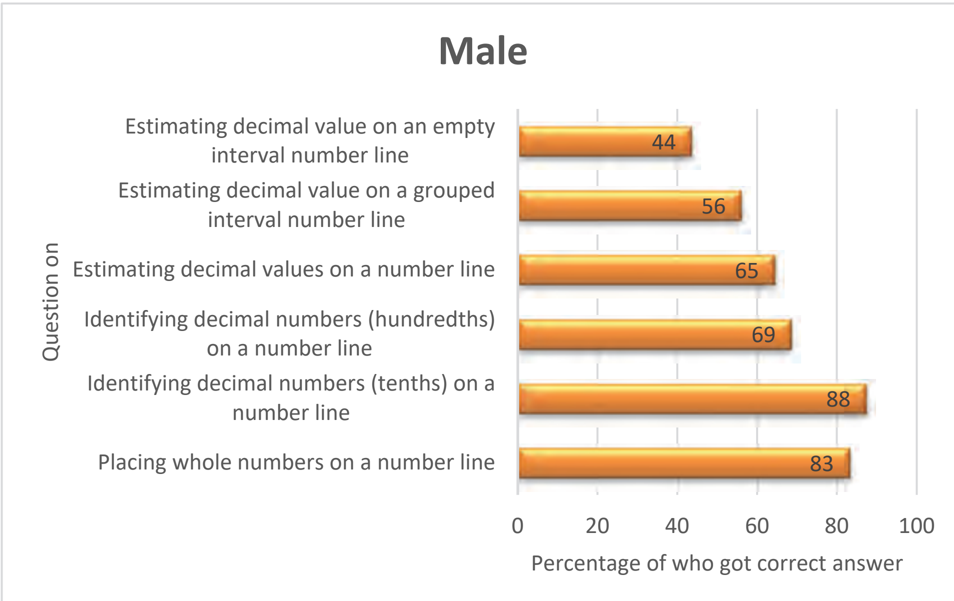

Fig. 7. Estimating whole and decimal numbers on a number line

4.2.2. Proportions and percentage

According to the CBC, teachers should be able to help learners solve simple problems involving proportions, ratios, percentages, mixtures, fractions and decimals (REB, 2015a). As such they should know more on this concept. MSSLs attempted questions on proportions and percentage. Correct answers are summarized in Fig. 8 (on the next page).

Fig. 8 shows that calculating prices after percentage increase and/or reduction was the most challenging activity for teachers and tutors of mathematics in all districts. This was more challenging in Rusizi district where only 50% of teachers and tutors of mathematics could get the right answer. On this activity, all teachers (100%) in Kirehe and Nyagatare did correctly the given question.

4.2.3. Fractions

According to the CBC, teachers and tutors of mathematics in primary schools, should be able to teach learners how to apply fractions in daily life situations and solve related problems (REB, 2015a). As such, they should know more than what learners expect from them. Fig. 9 summarizes correct answers on a question related to manipulating fractions.

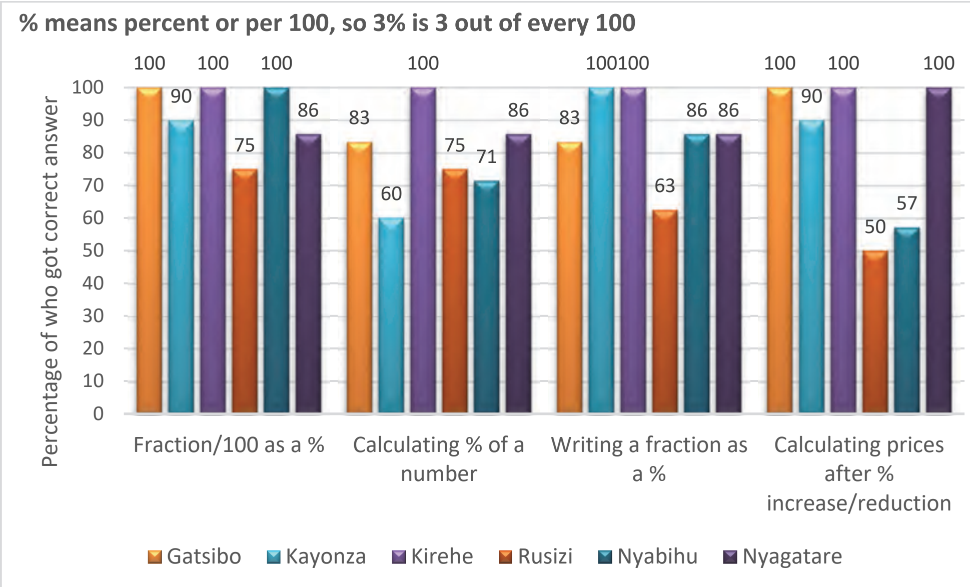

Fig. 8. Word problem on proportions and percentage

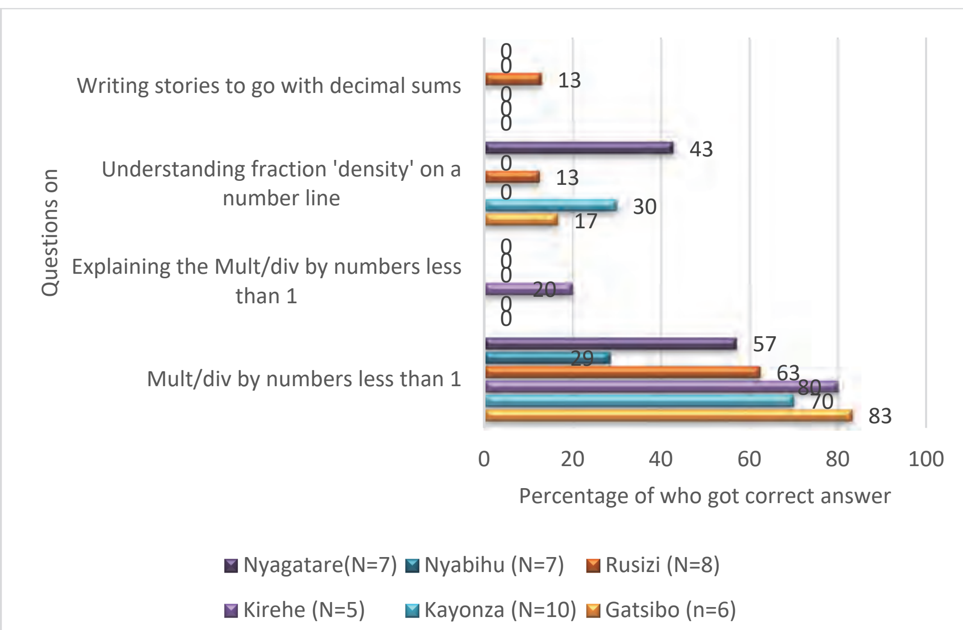

Fig. 9. Working on fractions

Fig. 9 shows that writing stories around decimal sums was challenging for most of teachers. All teachers in Nyagatare, Nyabihu, Kirehe, Kayonza and Gatsibo failed to write an appropriate story around the sum $6.4 + 2.3 = 8.7$.

4.3. *Topic area 3: Algebra*

4.3.1. Algebraic expressions

Primary school teachers of mathematics should be able to guide students on how to perform algebraic expression (MINEDUC, 1985a). So, teachers should have sufficient knowledge and skills to perform related activities. The following figure summarizes correct answers of teachers and tutors on questions about performing algebraic expressions.

> Question: Chip packets cost R8 each and tins of soup cost R6 each. If c stands for the number of chips packets bought and t stands for the number of tins of soup bought.

From Fig. 10 we can observe that in all districts, performing algebraic expression seemed challenging for teachers of mathematics. The situation showed even worse in Rusizi district where all teachers failed to answer the two questions. Solving word problems that involve simple algebraic equation with two unknown was very challenging for teachers and tutors. Specifically, only 14% and 20% of teachers and tutors in Nyagatare and Kirehe districts respectively could get a write answer to $8c + 6t$.

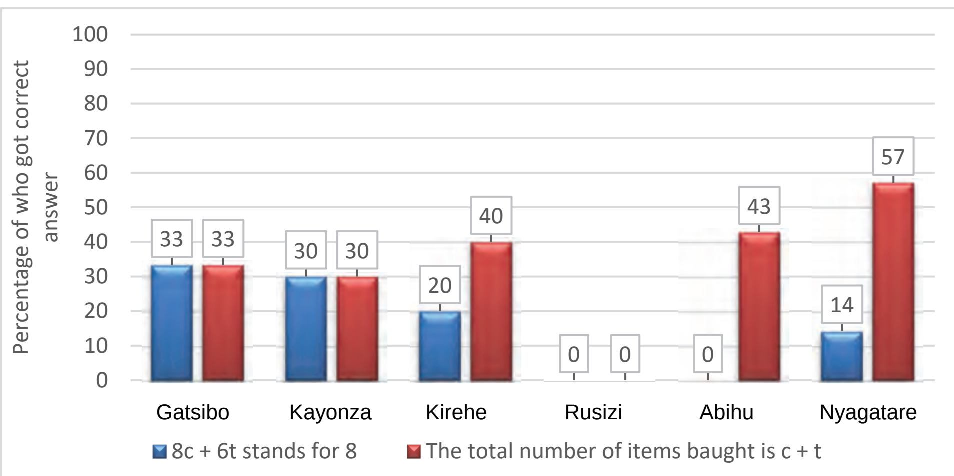

Fig. 10. Performing algebraic expression

4.3.2. Numbers and patterns

According to the CBC, by the end of primary education, learners should be able to describe and generate number patterns following a rule (REB, 2015a). As such, teachers of mathematics in primary schools should know how to determine the pattern for a given expression. They should be able to give learners mathematical word problems to solve by using algebraic methods.

From Fig. 11 we can observe that teachers of mathematics found it difficult to understand repeating pattern structure as well to continue growth patterns from a given examples. Changes are more observed for teachers in Rusizi district than in other districts where only 13% of teachers could correctly answer the question. Furthermore, it was not easy for teachers to extend number patterns to sequences with regularly changing differences. For example, all teachers (100%) in Rusizi district could not express the general term in a growth pattern. In all districts, teachers (100%) failed to provide an equivalent expression when given a machine where to feed numbers to pass out answers.

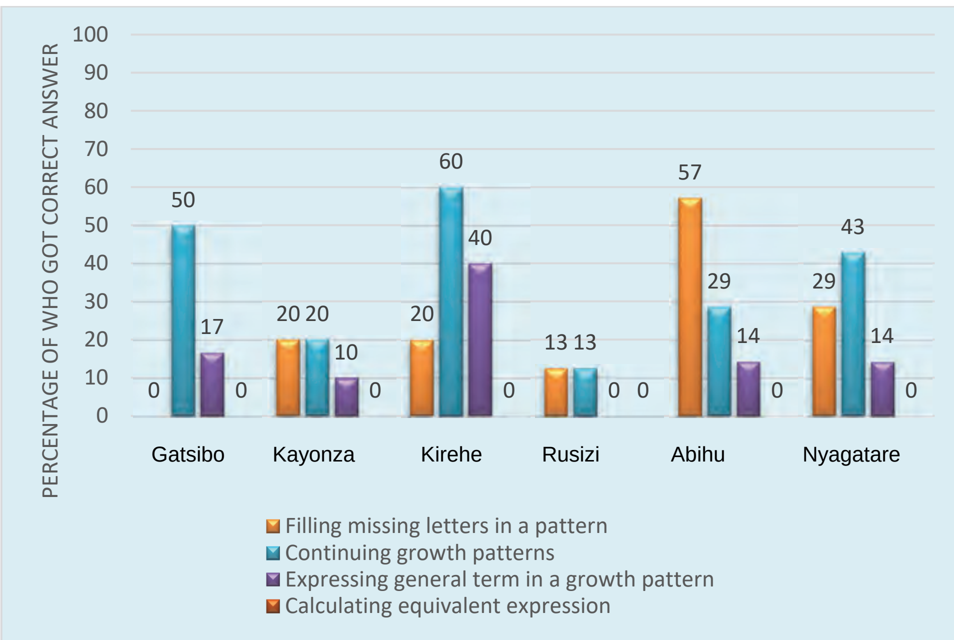

Fig. 11. Determining a pattern for a given expression

4.3.3. Equivalent expressions and number sequences

It is expected that by the end of primary educations, learners should be able to perform operations on algebraic expressions. They should be able to calculate the nth term in a linear expression (REB, 2015a). Therefore, teachers and tutors of mathematics should know more about equivalent expressions and number sequences. In this view, teachers and tutors of mathematics were asked a question to "write down the smallest and the largest number in a number sequence. Correct answers are summarized in Fig. 12 below. Participants were asked to write down the smallest and the largest of these numbers: $n+1$, $n+4$, $n-3$, n, $n-7$.

Fig. 12 indicates that all teachers of mathematics in Gatsibo district (100%) could correctly compare relative size of different linear expressions. In other districts some teachers failed to compare the relative size of different linear expressions. For example, in Kayonza, only 73% of teachers could identify down the smallest and the largest number in sequence.

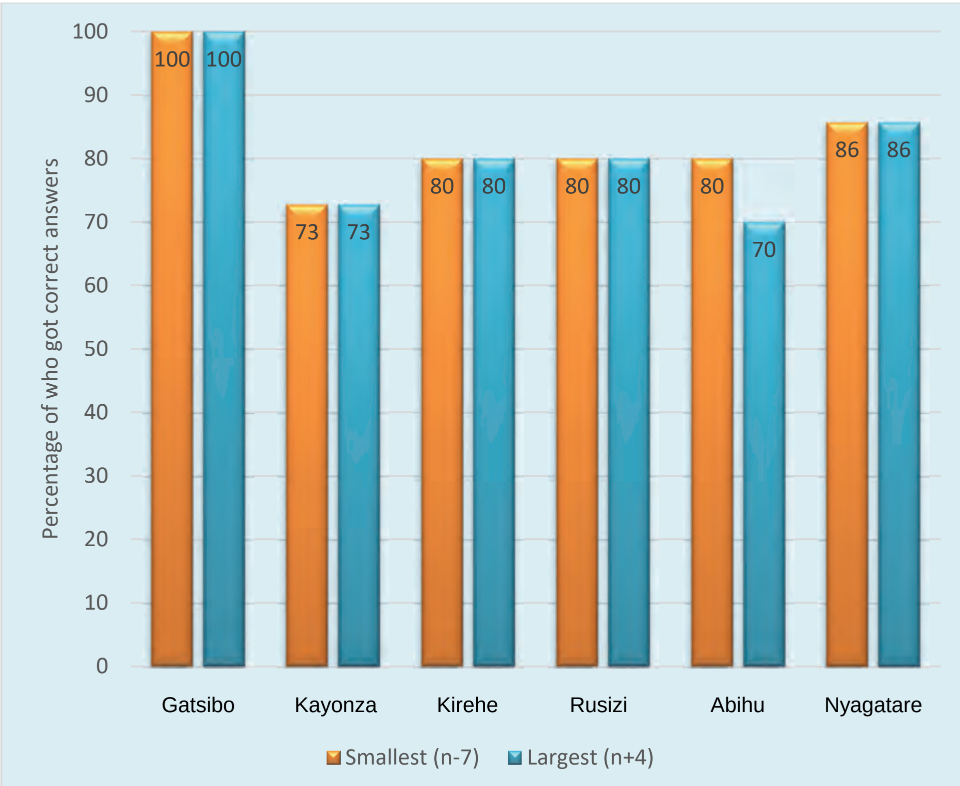

Fig. 12. Comparing the relative size of different linear expressions

4.4. *Topic area 4: Geometry*

Teachers of mathematics have to teach learners how to find different dimensions of geometrical shapes and to solve mathematical problems related to geometrical figures as well as recognizing special quadrilaterals.

4.4.1. Calculating rectangle area and perimeter

According to the CBC teachers of mathematics in primary school should be able to show the origin of formulae and how to use them to calculate area and perimeter of a regular polygon (REB, 2015a). In this view MSSLs were asked to calculate the area and perimeter of a rectangle. Their correct answers are summarized in Figure 13 below. Participants were given different shapes an asked to estimate areas and perimeters.

From Fig. 13 one can observe that rectangle perimeter and area as an algebraic expression was the most challenging activities for teachers. For example, only 14% of teachers and tutors of mathematics could calculate the rectangle perimeter as an algebraic expression of one variable. On the other hand, only 28% of them were able to calculate the rectangle area as an algebraic expression of one variable. Calculating rectangle area given measurements seemed easier than calculating its perimeter. For example, about 96% of teachers and tutors of mathematics could calculate the rectangle area given measurement, whereas only 68% of them were able to calculate the rectangle perimeter given measurement.

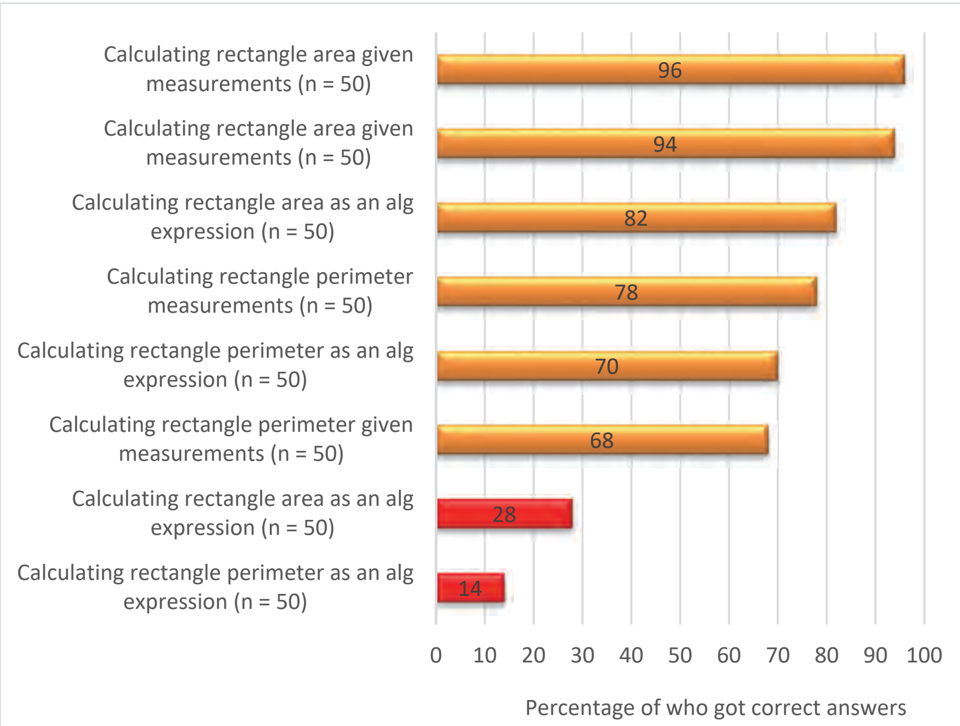

Fig. 13. Calculating rectangle area given measurements

In general, teachers of mathematics lacked basic knowledge on calculating rectangle areas and perimeter. This was more difficulty for calculations that involved. algebraic calculation. Therefore, they need more activity to improve on this knowledge so that they can effectively teach these concepts in primary schools.

4.4.2. Operations on geometry

Whereas teachers of mathematics should teach learners how to solve mathematical problems related to the finding the volume of cuboid and cubes, only 40% of teachers were able to solve a word problem that involved the relationship between cubic cm and liter. In addition, only 66% of mathematics teachers and tutors could be able to identify a trapezium from other shapes such as parallelogram, quadrilateral, rhombs and square.

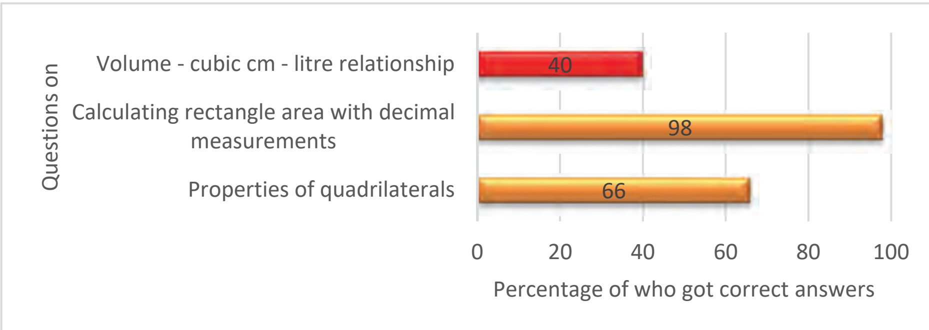

Fig. 14. Working out operations on geometry

4.5. Topic area 5: Statistics and elementary probability

By the end of primary school learners should be able to collect, represent and interpret data (REB, 2015). As such teachers of mathematics should know more on how to collect, represent and interpret data. They were asked the following question:

> in a survey, 40 parents were asked how many children they have. Provided number to present and interpret data.

Percentages of teachers who got correct answers are presented in Fig. 15.

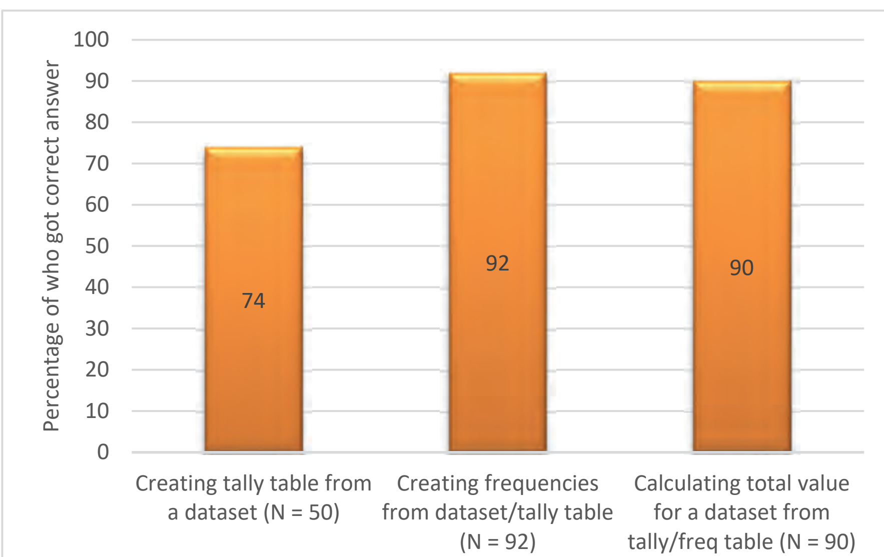

Fig. 15. Data collection, presentation and interpretation

From Fig. 15 we can observe that 74% of teachers of mathematics could create correctly tally table from a data base. At the same time 92% of teachers could create frequencies from dataset/tally; and 90% of them could calculate correctly the total value for a dataset from tally.

4.6. Topic Area 6: Measurement

4.6.1. Measuring time

By the end of primary education, learners should be able to solve problem involving time interval (REB, 2015a). As such their teachers should know and be able to teach how to solve real life problems that involve finding time intervals and conversion. Teachers were asked to solve the problem: *A tray of meringues is placed in the oven at 7:40. The meringues need to bake at a low temperature for 2.5 hours. At what time must they be taken out of the oven?* Figure 16 below represents the percentage of those who got correct answer per district.

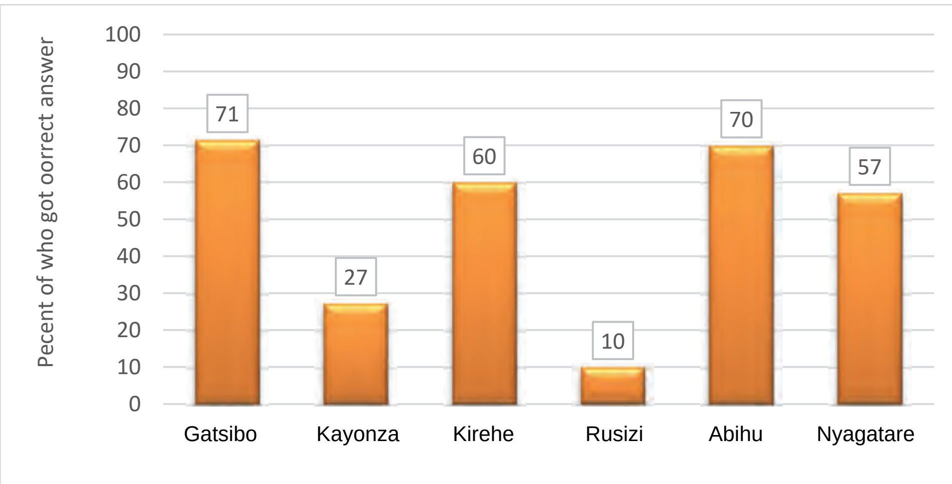

Fig. 16. Calculating time duration

Some teachers failed to calculate time duration with time given in decimal format. In Kayonza and Rusizi district, only 27% and 10% of mathematics teachers could provide correct answers.

4.6.2. Measuring capacity

According to the CBC, by the end of primary education, learners should be able solve mathematical problems involving capacity measurement (REB, 2015). As such

> One litre of petrol costs R10.75: Provide an answer with a method AND an explanation for working out the costs of (i) 0.53 litre (ii) 3 litres.

teachers of mathematics should know more on how to explain the conversion of units and how to do multiplicative scale up and down.

Fig. 17 shows that mathematics teachers face difficulties to explain the multiplicative scale up and down operations involving units of capacity: only 20% of teachers could attach a correct explanation to the multiplication they are able to do; only 60% and 70% of teachers could do the multiplicative scale up down and up respectively.

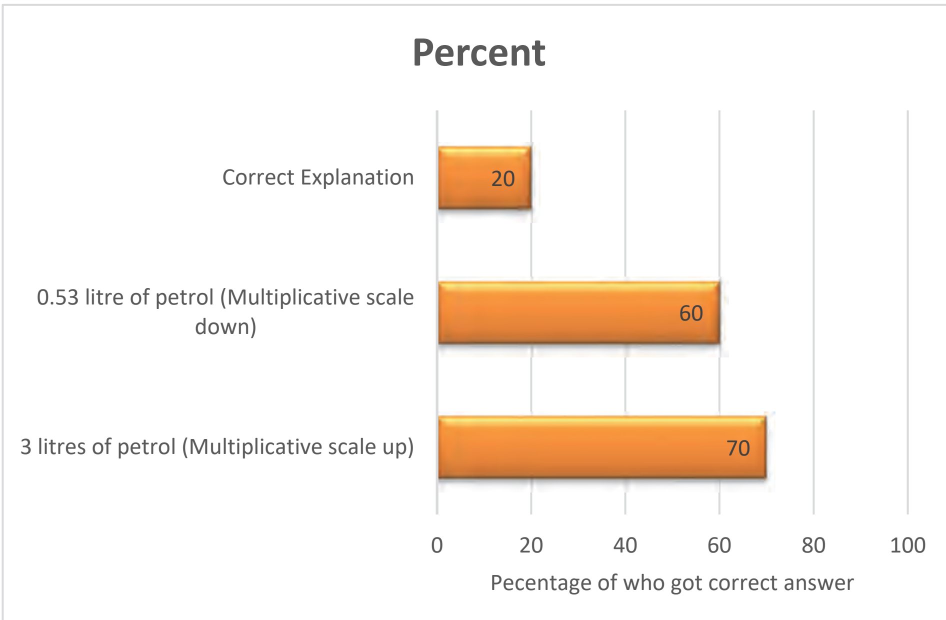

Fig. 17. Provide an answer with a method

4.6.3. Measuring the lengths

Measuring lengths is an important competence targeted for learners by the end of their primary educations. Specifically, as per CBC, learners should be able to convert between units of lengths and apply them in solving mathematical problems related to daily life situations (REB, 2015a).

> When we meaure ther heights with matchsticks, Mr Short's height is four matchsticks. Mr Tall's height is six matchsticks. How many paper clips do we need for Mr Tall's height.

While teachers of mathematics should teach about proportional reasoning, most of them do not have knowledge to manipulate ratios as shown by Fig. 18. In particular, all teachers in Rusizi, could not provide correct answer on determining the ratio of one unit of measurement in terms of another.

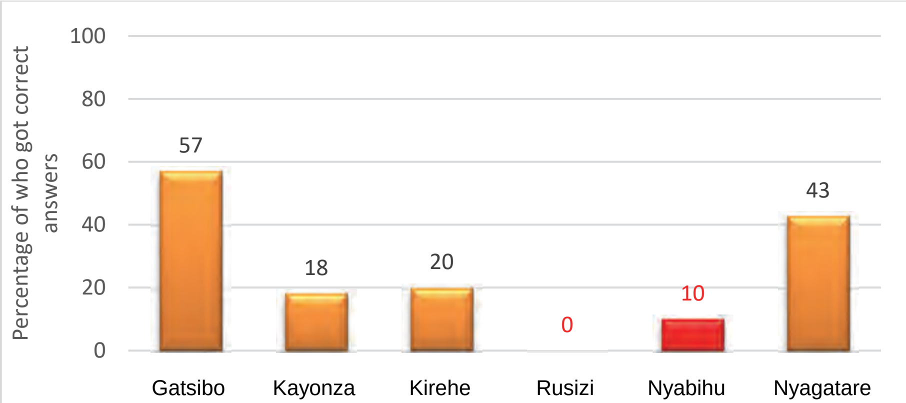

Fig. 18. Ratio

4.7. *General performance per district*

Fig. 19 below presents general performance by district in terms of percentage of teachers who got correct answers to the 20 questions.

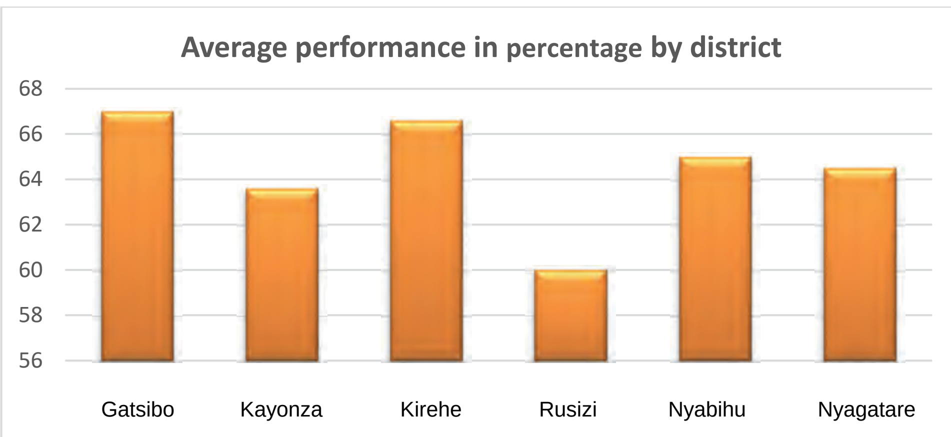

Fig. 19. Average performance per district

Fig. 19 shows that the general performance of mathematics teachers in the test-ranged from 60 to 67 percent; the highest average performance is observed in Gatsibo district where about 67% of teachers could give correct answers to the pre-test questions and the lowest average performance is observed in Rusizi district with 60%.

4.8. *General performance per topic area*

Across grades of primary education, learner learn the content divided into different topic areas such as (1) numbers and operations, (2) fractions, decimals and proportional reasoning, (3) Algebra, (4) Geometry, (5) statistics and elementary probability and (6) measurement (MINEDUC, 2015).

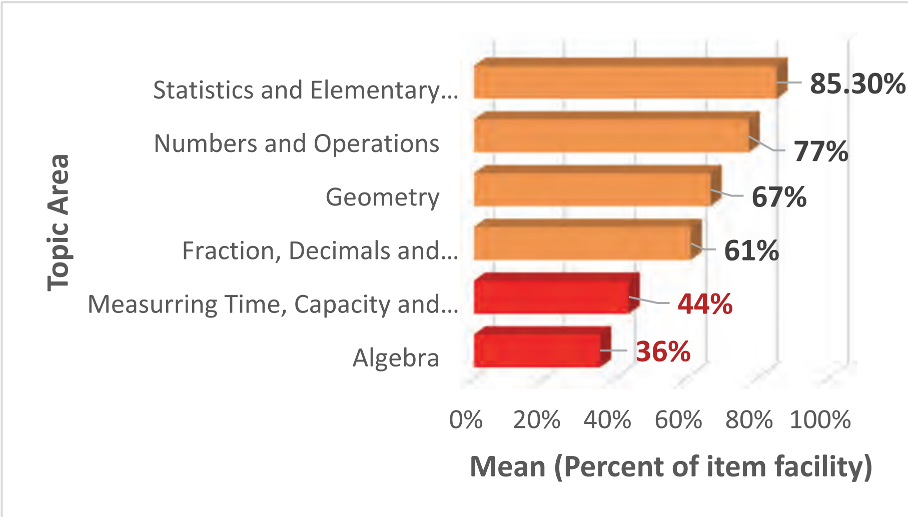

Fig. 20. Average Performance per topic area

The performance in terms of item facility varied across topic areas. This implies that teachers and tutors of mathematics did not have same comprehensive knowledge in all six topic areas indicated in the CBS's syllabus. Mathematics teachers had difficulties in performing on questions related to Algebra and Measurement. On average, only 36% of respondents could perform correctly item related to algebra whereas only 44% could get correct answers for items related to measurement. Statistics and probability seemed the easiest topic area to be performed by teachers and tutors of mathematics since 85% could perform correctly the question related to statistics.

5. Discussions and Conclusions

The Rwanda competence-based curriculum framework (REB, 2015) places numeracy as one of the seven basic competences that are required for all children from preprimary education. Furthermore, literacy and numeracy are considered as basic to accessing learning in other subjects. With regard to numeracy, it is expected that by the end of primary education, all children must be equipped with skills in computing accurately using the four mathematical operations, manipulating numbers, mathematical symbols, quantities, shapes and figures to accomplish a task involving calculations, measurements and estimations. In addition, they should be able to use numerical patterns and relations to solve problems related to everyday activities like commercial context and financial management as well interpreting basic statistical data using tables, diagrams, charts and graphs (REB, 2015). However, despite this emphasis, observations on the ground and national examination tests reveal that children are not yet facilitated to achieve these objectives. Part of the present paper was to explore to what extent teachers are themselves conversant with the content. Though the study used a small number of participants, results can help to rethink the focus of CPD activities and start to think about the link of learners' failure with teachers' content knowledge.

The general consideration shows that MSSLs do face challenges in all topics of the mathematics syllabus while those from Rusizi seem to be the most challenged. For example, understanding fraction 'density' on a number line was challenging for most of teachers and tutors of mathematics. Also, all teachers in Nyabihu, and Kirehe failed to tell that between ¼ and ½ there are infinite number (many) of fractions. In all district, MSSLs could work out the multiplication with a number less than 1, most of them could not explain why dividing 72.4 by 8/7 you get a bigger answer than multiplying the same number by 7/8. However, teachers in Nyabihu district were less competent to perform this activity; all teachers in Nyagatare, Kayonza, Rusizi, Nyabihu and Gatsibo could not give correct explanations.

If the mathematics content knowledge is at lower level, it is hypothetically expected that teachers will not be confident in the teaching, thus hindering children' s learning. It therefore deductible that low level of performance from pupils in the sampled region might have partial explanations in low teacher' s mastery of the subject knowledge.

This shows that their knowledge on algebraic expression needs to be sharpened so that they can teach effectively algebra in primary school. Therefore, teachers of mathematics in primary schools needs a training on the knowledge related to equivalent expressions and number sequence. This implies that they could correctly teach the concept of equivalent expressions and number sequences.

Though MSSLs performed much better in the topic area of statistics and probability, this can not lead us to confirm that the topic is well taught or understood since there was only one question of statistics without any question related to probability. A parallel study (Dushimana and Uworwabayeho, 2021) that is analyzing preservice primary school teachers' performance in national mathematics exams for the period of 2014-2016 shows that the questions related to this topic are the most failed in addition to register general failure in mathematics.

Therefore, we can deduct a lack of teachers' preparedness to adopt the new curriculum teaching approaches, there is also lack of appropriate physical facilities in schools to accommodate every leaner's individual needs among other hindering factors. Strengthening PCK is a key instrument to improve the quality of teaching and learning. PCK develops with teaching experience. However, it doesn't come automatically, but requires continuous professional development and reflection. Through observations, constructions, hands-on manipulations, generalisations, and presentations of information during the learning process, the learner will not only develop deductive and inductive skills but also acquire co-operation, communication, critical thinking and problem-solving skills. This will be realized when learners make presentations leading to inferences and conclusions at the end of the learning unit. This will be achieved if teachers are capable to embed different teaching strategies ranging from simple operations to problem solving through converting fractions into decimals and vice versa, writing numbers in figures and reciprocally.

Recommendations include systematic CPD programmes for in service teachers to complement preservice training so that they can adapt various reforms and in-service teachers to establish their individual professional development plans. As for further area for exploration, we would suggest to extend this research by assessing the impact of the on learners achieving learning outcomes and further issues such as reducing drop outs and improving girls' performance in mathematics.

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45

The Mental Starters Assessment Project: Ambitious Teaching in the South African Context

Hamsa Venkat[1]

ABSTRACT In this paper, I detail the ways in which a South African initiative focused on mental mathematics in the early grades (the Mental Starters Assessment Project — MSAP) can be considered as an intervention aligned with the idea of ambitious instructional practice. In building this argument, I take note of the fact that the materials associated with the MSAP initiative are relatively prescriptive in their format, a feature that has sometimes been argued to work against the goals of ambitious instructional practice. The reasons for considering the MSAP an ambitious instructional practice initiative is linked in the paper with the attention given in the materials to working across the strands of mathematical proficiency, with local conditions and cultures driving the relatively prescriptive format of materials provision.

Keywords: Mental mathematics; Ambitious teaching; South Africa.

1. Introduction

In this paper that follows from my ICME-14 Invited Lecture, I make an argument for why the mathematical content and format of the Mental Starters Assessment Project (MSAP) in South Africa can be seen — in context — as an intervention focused on the idea of "ambitious mathematics teaching practice" (Lampert et al., 2010). Through the details that follow, I argue that this is the case even amidst a format that is relatively prescriptive about content and sequence, with these features being responsive to aspects of the early mathematics education context in the country.

The MSAP initiative, focused on early mental mathematics in South Africa, was rolled out as part of national policy in the opening mental starter section of the advocated mathematics lesson structure in Grade 3 in 2022. The MSAP model is a simple one: there are six mental mathematics units, each focused on a specific strategic mental skill, are taught as two units a term (with each unit taking three weeks of teaching) across the first three terms of the four-term year. Each unit also has a simple routinized structure: a 5 minute written pre-assessment that the teacher sets the class at the start of a unit (usually on a Monday morning) and then marks; eight starter activities — each made up of a quick warm-up task or tasks, a teacher led focus on two problems,

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and individual work on two or three similar problems — that are run at the start of lessons over the course of the interim three weeks; and a 5 minute written post-assessment that the teacher sets the class at the end of the unit (usually on a Friday morning) and again marks. Differences in marks at the individual and class level provide the teacher and children with a sense of the efficacy of the teaching and of improvements in learning related to the unit focus.

But underlying this simplicity, there have been six years of iterative design research that has distilled insights from earlier trials related to the content, format and sequence of mathematical tasks and to the teacher support materials offered alongside the student materials. In this paper, I detail the ways in which — within the simplicity of its model — the MSAP format includes attention to all of the strands of mathematical proficiency as outlined in the work of Kilpatrick, Swafford and Findell (2000). Attention to a holistic focus on mathematical proficiency is commonly invoked as one of the key hallmarks of ambitious instruction (Kazemi et al., 2009), but this is usually coupled with a pedagogic form focused on discussion-based environments, teacher as facilitator, and elicitation of student understandings. In the MSAP model, we have attended extensively to distill mathematical attention to the strands of mathematical proficiency through the provision of a programme of mental mathematics tasks, representations and activities. These are, however, couched within a pedagogic form that reflects much that has been written about the conditions and culture of South African primary classrooms, as predominantly authoritative and teacher-led instructional spaces, where gaps and fragilities in teachers' mathematical knowledge are widespread (Hoadley, 2018). Our approach to the MSAP initiative has therefore been focused on the provision of a programme of teaching materials that are seen as "educative" (Schneider and Krajcik, 2002) in the sense that they are designed to support teachers' attention to teaching for meaning-making and progression in children's mental mathematics working in the terrain of these conditions.

In this paper, I begin by introducing the content and format of the MSAP materials before turning my attention to the ways in which these aspects provide attention to all of Kilpatrick et al.'s (2000) five strands of mathematical proficiency. I then discuss the reasons for a format which can be critiqued for being more prescriptive and more scripted in tone than some of the writing on ambitious instruction would advocate. In the concluding sections, I reflect on the nature of the balance between standardization of content/routines and responsive and flexible teaching in the MSAP materials in the national context, and the possibilities for educative materials in this particular format to bring about a change in the ground of mental mathematics and early number teaching and learning.

2. The MSAP Content and Format

The MSAP initiative came about in a collaboration between two research and development Chairs in South Africa (myself and my colleague, Mellony Graven) and our respective teams, the national Department of Basic Education (DBE), and partners across the education sector: key professional (Association of Mathematics Educators of South Africa — AMESA) and research (Southern African Association for Research in Mathematics, Science and Technology Education — SAARMSTE) organizations and the non-governmental organization sector (OLICO Youth). The DBE were keen to explore formative in-class assessment models for use in early grades' mathematics, and invited the two Chairs to look at options. In both projects, attention had been given in earlier research and development activities to supporting early number learning. The reasons for this focus were two-fold. Firstly, there was extensive evidence in South Africa of the widespread use of highly inefficient counting-based approaches to the four operations, well into the increasing number ranges that children are expected to work in as they progress through the primary grades (Schollar, 2008). These counting approaches are commonly seen in finger counting and in pages of "tally" counts on paper in children's work. Second, number forms the single largest topic area in the mathematics curriculum in the early grades, making up more than 50% of the content distribution between Grades 1–3. Thus, substantively and pragmatically, improving early number learning leverages improvements in mathematics learning overall.

Mental mathematics is widely described in the literature as an important avenue for supporting the building of strong foundations to number working in the idea of number sense, but Beshuizen and Anghileri's (1998) writing points to ongoing differences in the emphasis accorded to mental mathematics in taught and assessed curricula across different countries. This left us with limited examples for models of integrating work on mental mathematics, and particularly so when thinking about the national systemic scale that the Chair projects were devised to attend to. We were influenced by the writing of, and our interactions with, two international experts: Mike Askew and Bob Wright, both of whom had paid attention over an extended period of time to how moves beyond calculating-by-counting could be encouraged. Specifically, we became interested in Askew et al.'s (1997) attention to tasks that emphasized the need for reasoning about number relationships — e.g., example: How does knowing that double 16 is 32 help us to deal with $16 + 17$? Wright, Ellemor-Collins and Tabor's (2012) writing on progression in early number learning was also influential in helping us to consider task sequences and representations that helped to emphasize the idea of "base ten thinking": using the structure of the decimal system to identify and work with multiples of 10 as friendly numbers and relationships between numbers and multiples of 10 as benchmarks to use for the purpose of efficient calculations.

We also worked with the South African Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011) for the early grades in South Africa, which made recurring reference to mental mathematics, but was coupled with an assessment regime that tended to sideline focus on efficient ways of working by marking simply for correct answers without attention to whether these were produced by efficient working with number relationships or by rudimentary counting in ones (Graven and Venkat, 2021). Working with the aspects that were mentioned in the CAPS document (DBE,

2011) in conjunction with the literature on early mental mathematics, we identified six foci for units, presented and exemplified in our final MSAP Teacher Guide document (Graven et al., 2021) as shown in Fig. 1.

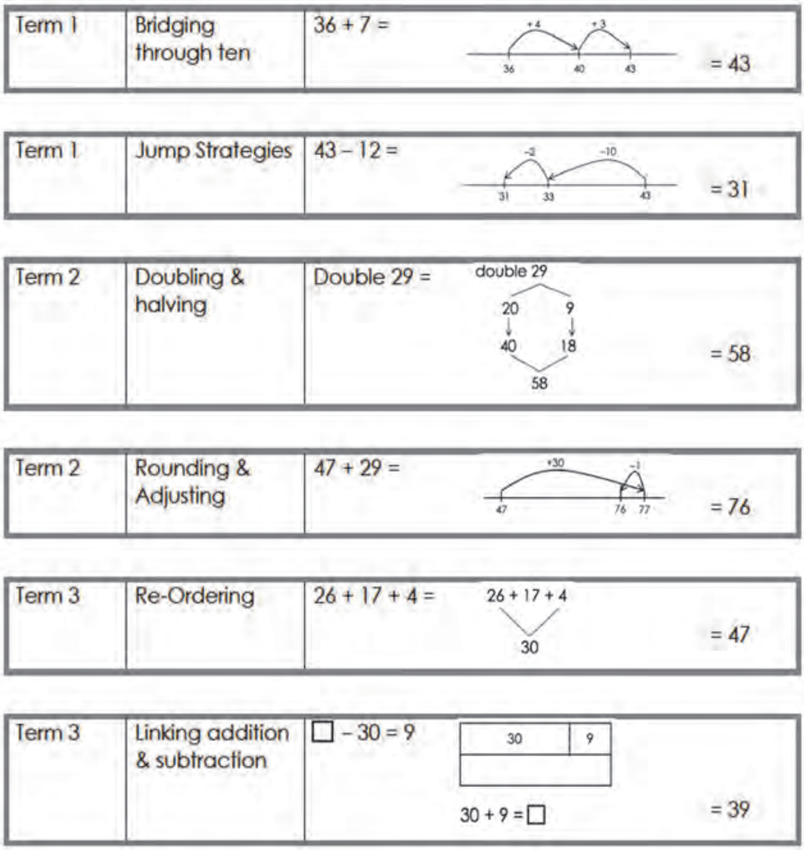


Fig. 1. The six mental mathematics strategy units in the MSAP

The three-week model for each unit (short time-limited pre-assessment, eight lesson starter activities, short time-limited post-assessment) has already been outlined above. In thinking about *how* to focus on mental mathematics, we were mindful of a ground in which there was evidence of very limited attention to the need to establish and grow a bank of basic established results. Venkat and Naidoo’s (2012) writing had drawn attention to the ways in which repeated instruction to children to use concrete resources to count in order to calculate answers sidelined attention to answers produced previously, resulting in a continuous cycle of “first principles” working. Gaps in early grades’ teachers’ understanding of the importance of early number progression were reflected in these instructional approaches, and pointed us to the need to explicate aspects that — at Grade 3 level — children could, in relation to CAPS content, be expected to work with at the level of near automaticity. The strategic focus of each unit was studied and decomposed mathematically for the range of underlying “fluencies” required to work in efficient ways. In some cases, the list of fluencies was edited or expanded based on empirical trialing (see Graven and Venkat, 2021 for more detail on the trials and their outcomes). By way of example, the list of underlying fluencies for working on Jump Strategies that we ended up with consisted of the following aspects:

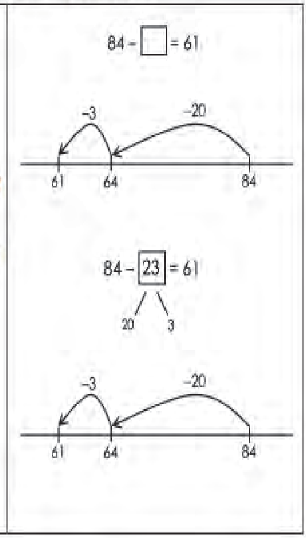
- count on or back in 10s from any number (e.g. 12, 22, 32, or 57, 47, 37, ...)
- add or subtract 10 from any number (e.g. $43 + 10 = 53$ or $89 - 10 = 79$)
- add a multiple of ten to any number (e.g. $61 + 20 = 81$)
- subtract a multiple of ten from any number (e.g. $46 - 30 = 16$)
- jumping to the next multiple of ten after a number (e.g. 32 → 40)
- jumping to the multiple of ten before a number (e.g. 56 → 50)

In wanting to communicate with teachers through the materials in a language that they would be familiar with, we described fluency-oriented tasks as focused on "**Rapid Recall**", and incorporated a 1-minute warm up task sequence into each lesson starter for every unit focused on consistent attention to developing these fluencies in a kick-off whole-class "Warm-up" activity segment.

As noted already, these fluencies were necessary to support children to become more successful with using the strategy in focus in each unit in their calculation, rather than reverting to the unit counting that was so prevalent on the ground. The core tasks in each lesson starter were then focused on "**Strategic Calculating**" — calculating using the focal strategy. In the case of the Jump Strategy unit for example, this involved working with two-digit addition and subtraction tasks through the use of what Beishuizen (1993) calls N10 strategies: where the first number is kept whole and the second number is broken down into its place value decomposition for easier mental addition or subtraction. An extensive evidence base points to this strategy proving particularly useful in surmounting the common errors seen in the widely used column algorithms when "carrying" in addition and "borrowing" in subtraction become necessary. In a carefully graded sequence of starter activities, the complexity of tasks is gradually expanded to include examples that incorporate bridging through ten steps within the use of the jump strategy, and also examples that include missing addend/subtrahend tasks, in which the tens and ones jumps have to be "built up" to find the missing number. There are openings here for conversations about how "building up" numbers using their place value decompositions and "breaking down" numbers into their place value decompositions are related, and as such, tasks like these represent opportunities for focusing on mathematical practices such as "doing and undoing" (Mason, 1988) that is a central and recurrent idea in mathematics. In the South African context where problems with coherent instructional explanations have been widely discussed across all phases, the MSAP materials include illustrations of the instructional talk that can accompany one of the tasks in each starter activity. This is detailed in text for teachers in the Teacher Guide document (Graven et al., 2021a) with a "talking hands" video clip included alongside (a feature included for all the lesson starters) as illustrated in Fig. 2.

Task Sequence

In this lesson we use jump strategies to solve missing number problems.

Problem: 84 – ☐ = 61 Plot '84' on the number line. Teacher: We need to jump back to 61. Mark '61' on the number line. Teacher: What tens jump and what units jump should we make? Learners: Minus 20 to get to 64 and minus 3 to get to 61 Draw these jumps on the number line. Ask: How much did we jump back in total? Learner: 23 Write the answer in the block. Teacher: So 84 – 20 – 3 is the same as 84 – 23 = 61 Write number sentence as shown.	

https://youtu.be/BHC9jDIUdRI

Fig. 2. Jump Strategy Lesson Starter 8 (Graven et al., 2021b)

The tasks in Fig. 2 illustrate also our inclusion of key representations that have been identified as supporting increasingly efficient mental calculation 1–2 the empty number line that has an extensive evidence base for its efficacy in studies located in the Realistic Mathematics Education (RME) (van den Heuvel-Panhuizen, 2000). We also included recurring reference to part-part-whole models as these have been identified as important and useful for sorting out the ways in which given quantities in a problem are related to each other (e.g. Murata, 2008) with Xin's (2012) work illustrating their particular usefulness and importance for students falling behind the mainstream.

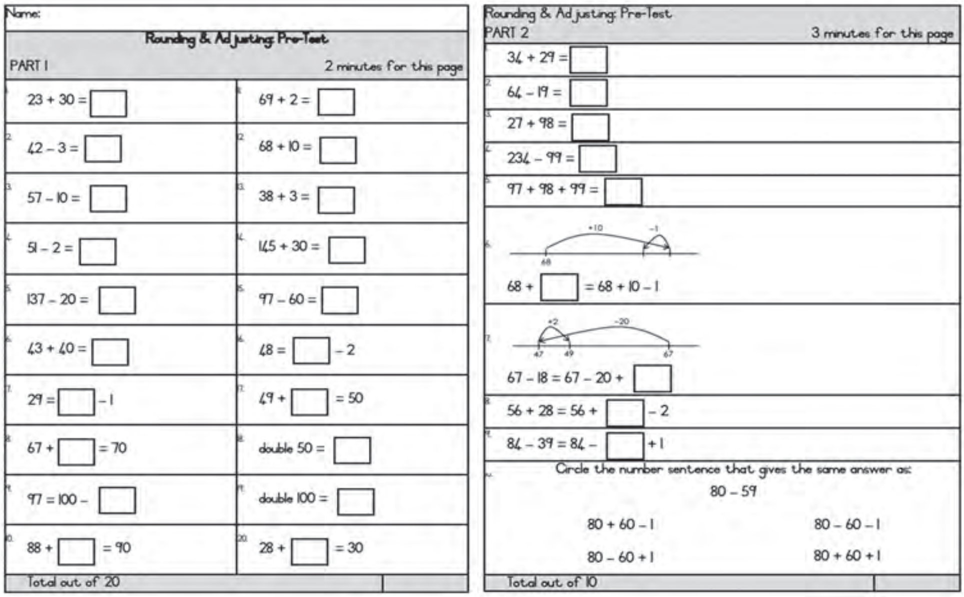
In many ways, moderate aims would have suggested that we stop with these two goals for each unit: improving Rapid Recall and Strategic Calculating. However, we were sensitive to the evidence that improving children's ability to carry out calculations more efficiently could leave aside attention in teaching to a focus on the structural relations that underpin efficient strategies in mental calculation (e.g. Polotskaia and Savard, 2018). Open number sentence formats have been presented in earlier research as one avenue for drawing attention to the representation of structural relationships. An example from the work of Hopkins, Russo and Downton (2019) illustrates items that are focused on the number relationship underlying strategic calculation, rather than on the calculation itself:

$$11 + 3 + 9 = \_ + 3$$

In this example, the strategic calculation skill in focus is reordering, and draws on the associativity property of addition. We referred to items focused on structural relations within the aim of **strategic thinking**, to emphasize that these kinds of questions did not involve calculating. In the MSAP materials, references to strategic thinking included recurring reference to the use of the key representations identified in the excerpts above — part-part-whole and empty number line models — that were used in the context of the rapid recall and strategic calculating tasks.

The pre- and post-assessments linked with each unit included items across all three categories: rapid recall, strategic calculating and strategic thinking. Hopkins et al. (ibid.) note that much of the work focused on strategic efficiency in early number working has been drawn from in-depth one-on-one work with children. While these studies have provided usefully rich illustrations of progression in early number working, the approaches tend to be impractical for larger-scale development activity that includes assessment components. In order to communicate the message about the need for efficient working and an understanding of underlying structure for teaching and assessment at systemic level, we needed a relatively simple model that could be communicated succinctly with early grades' district Subject Advisers nationally who we could provide training for to support policy implementation. Further, the materials had to be suitable for conditions on the ground of large classes and limited classroom resources beyond pencil and paper. The decision we came to on this was to design pre- and post-assessments as double-sided single page time-limited tests for learners. The front side of the test, across all units, features 20 items focused on rapid recall. Children are told they have 2 minutes to answer as many of the questions as they can, with the teacher telling them when to stop writing. Following this, children are asked to turn the test over to the second side, with this page — again across all units, containing 10 items drawn from across the strategic calculating and strategic thinking categories linked to the focal unit. Children are given 3 minutes to complete as many items on the second page as they can. The two pages from the Rounding and Adjusting unit pre-assessment are shown in Fig. 3.

The time-limited format provided a mechanism for communicating the importance of increasingly efficient working, with the low-stakes in-class assessment model allowing us to emphasize that what was important was individual pre- to post-test improvement rather than comparative performance either within or between classes and schools. Stott and Graven's (2013) earlier work had shown that this emphasis had mitigated children's anxiety about the time-limited format, with excitement rather than fear predominating in children's work.


Name:

Rounding & Adjusting Pre-Test

PART 1 2 minutes for this page

1. 23 + 30 = ☐	11. 69 + 2 = ☐
2. 42 – 3 = ☐	12. 68 + 10 = ☐
3. 57 – 10 = ☐	13. 38 + 3 = ☐
4. 51 – 2 = ☐	14. 145 + 30 = ☐
5. 137 – 20 = ☐	15. 97 – 60 = ☐
6. 43 + 40 = ☐	16. 48 = ☐ – 2
7. 29 = ☐ – 1	17. 49 + ☐ = 50
8. 67 + ☐ = 70	18. double 50 = ☐
9. 97 = 100 – ☐	19. double 100 = ☐
10. 88 + ☐ = 90	20. 28 + ☐ = 30

Total out of 20

Rounding & Adjusting Pre-Test

PART 2 3 minutes for this page

1. 34 + 29 = ☐
2. 64 – 19 = ☐
3. 27 + 98 = ☐
4. 234 – 99 = ☐
5. 97 + 98 + 99 = ☐
6. +10 –1 68
68 + ☐ = 68 + 10 – 1
7. +2 –20 47 49 67
67 – 18 = 67 – 20 + ☐
8. 56 + 28 = 56 + ☐ – 2
9. 84 – 39 = 84 – ☐ + 1
10. Circle the number sentence that gives the same answer as:
80 – 59
80 + 60 – 1 80 – 60 – 1
80 – 60 + 1 80 + 60 + 1

Total out of 10

Fig. 3. Rounding and adjusting pre-assessment

I go on now to outline the stran ds of mathematical proficiency, and to discuss the ways in which the mathematical content and its packaging in the MSAP model was designed to address these strands.

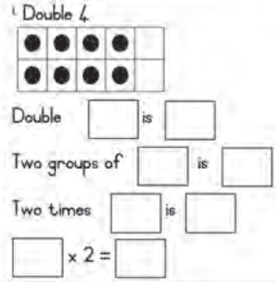
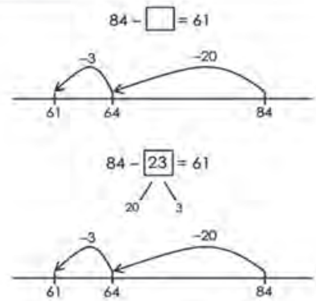
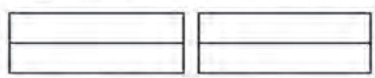
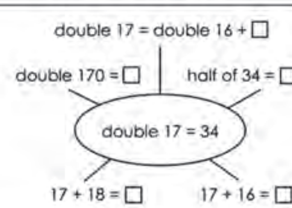
3. Addressing Mathematical Proficiency

Kilpatrick et al. (2001) proposed the idea of mathematical proficiency as the output, or consequence, of taking on: “a composite, comprehensive view of successful mathematics learning” (p. 116). The intertwined strands that they viewed as critical to achieving mathematical proficiency are: conceptual understanding; procedural fluency; strategic competence; adaptive reasoning; and productive disposition. The book: Adding it Up, offers extensive illustration and discussion on each of the strands, so I do not repeat that detail here. Instead, I offer a small number of quotes from this work that offer a sense of the focus of each strand (see Tab. 1 on the next page). This is followed by advice, examples and assessment items drawn from the MSAP materials and assessments that exemplify some of the ways in which the strand was addressed in the project. Key features of these examples are discussed and elaborated below.

The exemplifications from the MSAP materials point to attention to all of the strands in the content and format of the intervention. In this analysis, I use writing on each of the strands to delve into some of the ways in which the strands are incorporated into the materials, and to note if there are specific emphases within the ways in which the strand is presented.

As the quote above makes clear, a key marker of conceptual understanding is fluent moves between representations. In the exemplar in Tab. 1, taken from the Doubling

Tab. 1. Descriptions of the focus of each of the five strands of mathematical proficiency and exemplifications of strands from MSAP materials

Strand	Descriptive quotes	Exemplification from MSAP materials
Conceptual understanding	"being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes" p. 119	
Procedural fluency	"knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately and efficiently" p. 121	
Strategic competence	"the ability to formulate mathematical problems, represent them and solve them" (p. 124)	Write the calculations and empty bar diagrams as shown: □ – 30 = 9 30 – □ = 9  Teacher: Help me to finish the bar diagrams for these calculations. Will the bar diagrams be the same or different?
Adaptive competence	"the capacity to think logically about the relationships among concepts and situations" (p. 129)	
Productive disposition	habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (p. 116) "the tendency … to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (p. 131)	"Each unit begins and ends with a short test for the learners. Marking these tests provides information for you and the learners about how much they have improved in using that particular set of skills during the three weeks" (p. iv)

and Halving unit, I note the attention to translating — not just between concrete and symbolic representations of a double number — but also to different ways of speaking about doubles in everyday language. Variation in representations was built in across all units, with emphasis on translating between these — for example, translating the information in a missing number sentence like $4 + \_ = 23$ into a part-part-whole diagram, and then translating from the part-part-whole diagram into the range of number sentences that could be attached to this structure. Underlying our attention to building in variation, there was writing in variation theory that viewed moves between representations as a key dimension of variation to incorporate (Watson and Mason, 2005) for building flexible conceptual understanding, as well as indications — in a context of limitations in teachers' use of coherent and connected language (Mathews, 2021) — to emphasize variations in language connected to core ideas within a concept in order to familiarize learners with the different forms. Additionally, Gu, Huang and Gu's (2017) noting that careful variation provided useful instructional scaffolding in the context of large classes and predominantly teacher-led instructional settings offered a strong cultural fit with South African classroom settings.

Procedural fluency was important in the South African context given the extensive evidence showing that many children were unable to carry out basic procedures in efficient ways drawing on a bank of known facts. Thus, we needed to communicate the importance of retention of a growing bank of established facts, and also to illustrate how this bank could be used in effective and efficient calculation procedures. In the MSAP model, procedural fluency was incorporated in the attention to a range of underlying basic fluencies in the warm up section of the starter that are then brought together in the execution of efficient calculation procedures linked to a variety of focal strategic calculations for each unit. In the Jump Strategies exemplar in Tab. 1, finding the missing addend using this approach involves fluency with subtracting a multiple of ten and rapid recall of the difference between 64 and 61 in the initial steps, as well as fluency with combining these place value-based decomposed parts into their numerical whole. All of these underlying fluencies are built into the warm up sections of the Jump Strategies unit and then drawn on in the teaching of the focal strategic calculations across the eight lesson starters.

In Kilpatrick et al.'s (2001) writing, strategic competence extends beyond the strategic calculation that was a key feature of the MSAP units. Strategic competence in the Kilpatrick et al. formulation includes attention to formulating and representing situations as well as solving problems. While the MSAP model made strategic calculating a key goal, and while the focus on mental mathematics meant situational or context-oriented problem-solving featured to a more limited extent, there were examples — as illustrated in Tab. 1 — of the need to consider how a given situation could be represented in an alternative form. In particular, given the international evidence of children finding it harder to solve missing start and missing addend/missing subtrahend problems in comparison to missing "result" problems (Carpenter et al., 2000), we incorporated attention — again using ideas of variation and invariance — to supporting teachers and children to focus on the ways in which different number sentences involving the same numbers but in varying relationships implied different part-whole structures. Sorting out the part-whole representation in each case, with discussion of the rationales for deciding how to translate from each number sentence to a part-whole diagram exemplifies a key avenue that was used in the MSAP that expanded our attention to strategic competence beyond the focus on strategic calculation.

Adaptive reasoning has been written about widely as a central feature of mental mathematics underpinned by a strong number sense. Baroody's extended sequence of studies (among these Baroody, 1993; 2003) provide particularly salient evidence of what is gained in what he describes as the "number sense" approach, in which interconnections between results from the primary focus of attention, rather than getting the answer to individual problems. These connections are promoted in the MSAP materials through two key avenues. Firstly, there is consistent and recurring attention to using known results to derive further results. The illustrative example in Tab. 1 provides a direct case of the use of this kind of approach, with the given result (double $17 = 34$) forming the start point for discussing and devising an expanding network of other results connected with this result. A second route through which adaptive expertise is promoted is through the attention to strategic thinking items in the starters and in the assessments. In these items, the focus is on different ways of expressing the structural relation in focus in the given task in ways that link to the focal strategy. For example, in the Bridging through Ten unit, the assessment includes items such as: $98 + 56 = 98 + 2 + \_$. Here, the student is invited to work with how the left hand side expression needs to be adapted to maintain equivalence with a specific bridging through ten action — in this case — coming into play.

Given the South African evidence of examples being treated highly "separately" and with a repeated reversion to first principles counting strategies in early number, the focus within adaptive reasoning on connecting ideas and on leveraging connections to grow the base of known results through constructing further results derived from these, was particularly important.

Finally, and again as exemplified in Tab. 1, the messaging throughout the MSAP booklet is focused on individual learning and improvement, with the emphasis on low-stakes assessments geared towards in-class, developmental use by the class teacher. This messaging was important in a ground where previous early grades assessments (the Annual National Assessments) had been high-stakes for teachers and schools, and which the teacher unions had vociferously opposed and brought to a halt in 2015. The request to teachers to reinforce messages about looking for pre- to post assessment improvement, rather than inter-learner comparisons (or inter-class and inter-school comparisons) has been useful and important for communicating the need for consistent student working with the materials in order to become more skillful and efficient with the focal strategies over time.

Connecting all the strands, the work within the Realistic Mathematics Education (RME) tradition on the use of structured representations that were open to emergent working using increasingly efficient and sophisticated strategies was useful to our thinking. We followed the advice of Askew and Brown (2003) to judiciously select key representations that offered diagrammatic attention to number structure in ways that could underpin subsequent work with number in symbolic forms. The empty number line and part-part-whole models were key to this, and consistently connected with a range of symbolic forms, once again aimed at building connections. These models also provided, as RME advocates, important intermediary devices that function initially as diagrammed models of problem situations that are shared and discussed in class, with advice to teachers that these should, over time, become internalized mental models that function as tools to think with in students' mental mathematical working. Further, these structured models are closely linked with van den Heuvel-Panhuizen's (2008) calculating-by-structuring stage, which forms the key stage that follows the calculation-by-counting stage that South African evidence suggests that students and teachers struggle to transcend.

4. MSAP — Ambitious Instruction?

Lampert et al. (2010) cite Leinhardt and Greeno's (1986) work on the support offered by routines in supporting ambitious instruction. Specifically, the earlier work notes that routines represent ways in which to manage some elements of the background efficiently, allowing the foreground to be occupied by the mathematical goals at hand. This foregrounding of the mathematical goals has a particular urgency in the South African context, given the highly rudimentary strategies being used by children for early number problem-solving, but given too, the evidence of gaps in teachers' mathematical knowledge of how to push for progression in children's approaches. This is coupled with the evidence of teacher-led instruction. This raises questions about what to balance across the prescription-responsive freedom continuum if mental mathematics founded upon number sense is to be supported. Routines in the MSAP are built into a highly consistent structure for each unit and for the activity segments in each starter — with the aim that attention to basic fluencies and to teaching for strategic problem-solving start to become part of the "natural" background of teachers' repertoires of practice. There is also recurrent reference to the empty number line and part-part-whole diagrams as tools for working with and then thinking with — again with a view to helping teachers to see these models as part of the basic toolkit for early mental mathematical working. The lesson starter activities themselves have a relatively prescribed core, but within these activities, the materials draw attention to the need to be responsive to early finishers with more complex tasks that teachers can devise, and to variations that may be seen in children's responses. Thus, while there is more detailing of the content of instructional explaining than is typical in Lampert et al.'s (2010) work, there is room here also for working in ways that are more responsive and more tailored to the needs of the children in particular classes. Our decisions to work in this way are also responsive to the evidence that for education systems in developing country contexts with relatively low levels of performance, more prescription may be necessary in the introductory stages of bringing a system into functionality, and — importantly — there is evidence that this greater prescription is welcomed by teachers (Fleisch et al., 2016).

The outcomes from the early cycles of trials of the MSAP materials reflect Fleisch et al.'s position: that teachers find the materials use-able and useful. We also have evidence that using the materials can produce learning gains (Graven and Venkat, 2021). Thus, taken together, there is therefore the sense of materials in the MSAP package that can be considered both educative for teachers, and ambitious in terms of the mental mathematics learning that they seek to support.

Acknowledgements

My thanks to DBE and the NRF for their support of this work. Thanks to the entire team who grappled with Mellony Graven and myself in developing the MSAP teaching and assessments materials — namely: Lawan Abdulhamid, Mike Askew, Nolunthu Baart, Lynn Bowie, Mark Chetty, Busi Goba, Corin Mathews, Zanele Mofu, Samantha Morrison, Debbie Stott, Thulelah Takane, Herman Tshesane, Lyn Webb, Marie Weitz and Bob Wright.

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https://doi.org/10.1142/9789811287183_0046

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Challenging Deficit Perspectives in Developing Countries: Teachers' Explanations of Fraction Concepts

Debbie Marie Verzosa[1]

ABSTRACT Dominant discourses in teacher development often posit teachers as being lacking in knowledge, beliefs, or skills, thus justifying the "need" for further development and for educational reforms. This perspective shaped the analysis of Filipino teachers' explanations of fraction concepts using the constructs of content knowledge and pedagogical content knowledge, leading to an interpretation that reinforced deficit narratives about teachers. However, there are increasing contestations of these deficit research narratives (Adiredja, 2019) that neither acknowledge the larger context that contributes to the ways teachers perform nor highlight the productive resources that teachers may draw upon in their teaching. This paper aims to illustrate a reconceptualization of the research away from focusing on what teachers lack towards identifying the ways by which teachers' fractional explanations reflect their constructed perception of ideal mathematics teaching as shaped by the broader system where education takes place. This is my attempt to acknowledge my own participation in the deficit perspective and challenge the narrative about education in a developing country.

Keywords: Developing countries; Preservice teachers; Fractions; Anti-deficit.

The scholarly literature is replete with reports of teachers' knowledge of fractions. Olanoff et al. (2014) found at least 43 papers that focused specifically on teachers' fraction knowledge. Since then, further studies on teachers' fraction knowledge were published (Bansilal and Ubah, 2020; Chinnapan and Forrester, 2014; Depaepe et al., 2015; Depaepe et al., 2018; Lee, 2017; Lemonidis et al., 2018; Van Steenbrugge et al., 2014). Majority of the studies were carried out in developed countries or in countries with average to above average mathematics achievement based on international benchmarks such as PISA (OECD, 2019).

The initial objective of the current study was to perform a similar assessment of preservice teachers' content and pedagogical content knowledge of fractions in the context of a developing country (the Philippines). An implicit assumption was that knowledge gaps of teachers in developing countries may be more serious than those in developed countries, which is why this study needed to be conducted. This perspective

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typified the "fixit" approach (Graven, 2012) to professional development. Breen described the prevailing mindset regarding teacher training.

> There is something wrong with mathematics teaching world-wide, and that we, as mathematics educators, must fix it... Mathematics teachers need someone to fix them, and mathematics educators need someone to fix … This culture is based on judging what is right and wrong, paying little attention to what mathematics teachers are actually doing (since it is wrong anyway) in their classrooms, and looking outside themselves for the "right" way, the newest "fix". (Breen, 1999, p. 42)

Deficit master narratives about preservice teachers are much entrenched in the scholarly literature. They exist even in reputable mathematics education journals, and even in carefully designed studies that focused on the development of understanding (Adiredja, 2019). Specifically, deficit master narratives also abound in studies that focus on fraction understanding. Based on teachers' performance in various fraction assessments, they are described in terms of what they "could not" do; their knowledge is often described as "inadequate", "weak", "insufficient", "limited", or as having "gaps" and "misconceptions" (Bansilal and Ubah, 2020; Chinnapan and Forrester, 2014; Depaepe et al., 2015; Klemer et al., 2017; Putra and Winsløw, 2018; Rosli et al., 2020; Şahin et al., 2016; Van Steenbrugge et al., 2014). I also acknowledge my own contribution to these deficit master narratives through claims of teachers' poor reasoning strategies (Verzosa, 2020). Were it not for a wise suggestion and for a further review of the literature, I would probably remain unaware and continue to be entrenched in a deficit view.

Deficit research narratives neither acknowledge the larger context that contributes to the ways teachers perform nor highlight the productive resources that teachers may draw upon in their teaching (Adiredja, 2019). This paper aims to illustrate a reconceptualization of the research away from focusing on what teachers lack towards identifying the ways by which teachers' fractional explanations reflect their constructed perception of ideal mathematics teaching as shaped by the broader system where education takes place. This is my attempt to challenge the narrative about education in the Philippines.

1. Anti-deficit Perspectives

It can be argued that claims of the teachers' weak or inadequate knowledge in fractions in the previous section are based on data. However, the interpretation of the data was shaped by a deficit master-narrative and failed to recognize the teachers' sense making. Previous studies have demonstrated how the same data can be analyzed through a deficit or anti-deficit lens. For example, Lewis (2014) offered two contrasting explanations for the persistent understandings found in her case studies of two children. In the first approach, the two children's persistent understandings could be thought of as an indication of their inability to mentally represent and manipulate numbers. Lewis argues that this interpretation does not provide much value to the identification or remediation of students with mathematical learning disabilities. In the alternative anti-deficit approach, Lewis offered a Vygotskian perspective and interpreted the children's understandings as resulting from the inaccessibility of mediational tools. Within this interpretation, the reason for the two children's persistent understandings was not because they could not mentally represent and manipulate numbers but because they understood the representation of a fractional quantity in unconventional and atypical ways.

In another study, Adiredja (2021), students' claims about the temporal order of epsilon and delta in the formal definition of the limit was investigated. In this study, four questions regarding the temporal order of epsilon and delta were asked—what depends on what? Which one do you think comes first? Which one do you think is set first? How would you arranged the four variables, epsilon, delta, x and $f(x)$ in order? Only one student correctly answered "epsilon first" on all questions. Half the students answered "epsilon first" in at least one question. A deficit perspective would immediately lead to an interpretation that most students did not understand the temporal order. However, the study demonstrated that a single context/question cannot capture students' understanding. The fact that some students answered epsilon first on some, but not all, questions revealed that the students' justifications were context sensitive. Different questions influenced the cueing priority of certain knowledge elements (e.g., functional dependence). These knowledge elements (which are useful in other contexts) were found to be resources that students possessed during their sense-making of the temporal order of the limit definition.

These studies demonstrate that a deficit perspective focuses on what individuals lack (including misconceptions), rather than the resources that they draw upon (Adiredja, 2019). It locates students' knowledge as a problem to be fixed, and not as a resource for learning. Further, a deficit perspective views certain individuals as if they possessed deficiencies, which may be biological or cultural (Settlage, 2011). Some individuals may be thought to be "biologically inferior" or deficient by virtue of being raised in a certain culture. A deficit perspective does not consider that education occurs within a sociopolitical context (Adiredja, 2019) where forces may systematically undo the efforts of teachers or schools (Settlage, 2011).

2. The Philippine Context and Research Questions

Nebres (2009) stated that problems of mathematics education in the Philippines include two types: micro problems or problems internal to mathematical education (curriculum, teacher training, textbooks, etc), and macro problems or issues arising from pressures from other sectors of society. These macro problems very much exist in the Philippines.

In the Philippines, teaching is not an attractive career choice, and entry standards are typically lower than in other degree programs (Tatto et al., 2012). In some cases, education is chosen as a field of study because it is relatively cheap (no equipment is required), or because it accepts students who are unqualified in more attractive programs. Additionally, teachers are known to be overworked and underpaid. Public school teachers teach a maximum of six hours per day, and are also expected to write very detailed lesson plans, fill up official forms, and complete reports (Bautista et al., 2008). They may be asked to perform other non-teaching related duties in a highly centralized environment where information, opinions, and teachers' options are tightly controlled (Bernardo and Garcia, 2006; Nebres, 2009).

Teaching is generally not an option among graduates of the top universities. To encourage the best students to teach, even just for a fixed term such as two years, initiatives such as Teach for the Philippines have been devised (Sodusta and De Leon, 2019). The words of a promising graduate turned public school teacher, sums up the aversion to teaching as a career:

> During my first year as a public school teacher, ahhh, there are three reactions that I always get: surprise, amazement, dismay. It's like, "What? Why are you teaching there?" "Oh you come from a prestigious university but you're a public teacher? What a waste". It's always like that. (Sabrina Ongkiko, quoted by Sodusta and De Leon, 2019, p. 12)

The study reported in this paper is grounded in the assumption that the macro problems emerging from the contextual realities of the educational system can contribute to teachers' training and preparation. It asks the following research questions. How do preservice teachers reason about fraction comparison and operation tasks? What forms of reasoning are associated with correct responses?

3. Method

3.1. Sample

The participants were 405 preservice elementary and 157 preservice secondary teachers. The preservice teachers were enrolled in six universities, spread across the northern, central, southern, and capital region of the Philippines. Moreover, the universities had varied ranking in terms of the passing rate in the September licensure examination for teachers (ranging from 40 to 93% for elementary teachers, and from 54 to 94% for secondary teachers). The preservice teachers were all in their third or fourth year, and had completed all mathematics courses within their program of study. No personal details were collected, but it can be assumed that most preservice teachers in the sample were between 19 and 21 years old.

3.2. Materials

The instrument consisted of four explanatory tasks, three of which involved fractions, and were included in this paper. For each of the three tasks, the respondents were asked to provide an answer and describe their solution to a student. The tasks were as follows: (1) Which is larger, 1/5 or 1/8? (2) What is 2/3 + 1/2? (3) What is 1 ÷ 2/3? For Tasks

2 and 3, the respondents were specifically asked to explain their solutions using a drawing. The concepts (comparison, addition, and division) were chosen because they represented basic knowledge in elementary mathematics, and there was sufficient literature to provide task-specific frameworks to guide this study's analysis (Bansilal and Ubah, 2020; Chinnapan and Forrester, 2014; Geller et al., 2017; Kaasila et al., 2010; Lee, 2017). Although the medium of instruction in mathematics from Grade 4 onwards is English, the teachers could write whole or part of their explanations in the regional language to lessen obstacles brought about by using a second language to explain academic content. Participants were given 30 minutes to complete the tasks, but most were finished before the time allotment.

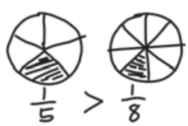
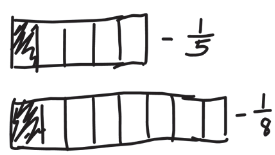
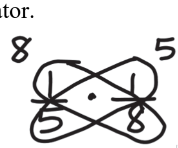
3.3. Data Analysis

Responses were initially coded as correct or incorrect. The preservice teachers' explanation strategies were derived inductively from the responses. From the raw data, repeating responses were noted and assigned codes. Similarities among codes were identified, which resulted in an intermediate list of codes. The data was coded and re-coded until the codes were finalized. With the final set of codes, the data were coded a second time as a manner of checking.

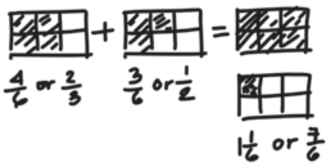
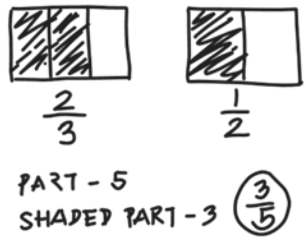
Because I wanted to capture the all the reasoning strategies provided by each preservice teacher, I coded all strategies. If the preservice teacher gave more than one reasoning strategy, responses were coded more than once.

Tab. 1, Tab. 2, and Tab. 3 show the codes for each of the three tasks in this study. Sample responses for each code are also provided.

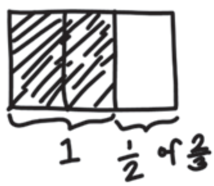
Tab. 1. Response codes for task 1 (which is larger, 1/5 or 1/8?)

Correct picture	Incorrect picture	LCD/decimal
		"8/40 > 5/40; 1/5 is larger." 1/5=0.2; 1/8=0.125; 1/5 is larger
Cross multiply	Denominator	Real life
To know the larger fraction, the two denominators will be multiplied to the opposite numerator. 	As long as the denominator increases, the value decreases. Simply compare the denominator. The larger fraction is the one with smaller denominator.	In our quiz everyday, if we get 1/2 the half of paper is what we need and in 1/4 the half of one-half is what we need then I conclude that the smaller the denominator, the closer is the fraction to the whole so 1/5 is larger.

Tab. 2. Response codes for task 2 (what is 2/3 + 1/2?)

Correct fraction relations	Incorrect fraction relations	Procedure
		First, get the LCD of the two denominators. Divide it from the original ones. 6 divided by 3 is equal to 2. Then 2 multiply by 2 is equal to 4. So the answer is 4/6. Do the same for the other, then add.

Tab. 3. Response codes for task 3 (what is 1 ÷ 2/3?)

Measurement Division	Procedure	Commutativity
	In dividing fractions, the whole number always has 1 in the denominator so the division symbol will be replaced with multiplication. In dividing, we have what we call a reciprocal so reverse 2/3, making it 3/2 then cross multiply.	Any number divided by 1, the answer is also the number.

4. Results

Fig. 1 shows the results in task 1 (which is larger, 1/5 or 1/8?). The white regions represent correct answers. Results show that correct answers were produced through different kinds of explanatory approaches. The use of real-life situations was more prevalent among the preservice elementary teachers, and the use of the least common denominator (LCD) or decimals was more prevalent among the preservice secondary teachers. The strategies LCD/decimal, cross multiply and denominator, considered "non-conceptual" in other studies (Geller et al., 2017), were found to produce correct answers, especially for the preservice secondary group. Most strategies underlying incorrect responses (represented by the dark regions) were incorrect drawings. However, there were 78 preservice teachers who produced an incorrect drawing and a correct answer, implying that an incorrect drawing did not automatically result in an incorrect solution. Further, there were 43 preservice teachers who produced an incorrect drawing but had other resources or strategies from which to draw upon (Fig. 2).

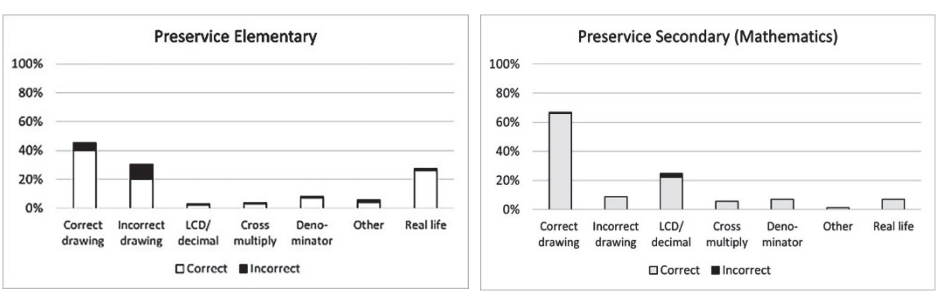


Fig. 1. Explanatory approaches in task 1 (which is larger, 1/5 or 1/8?)

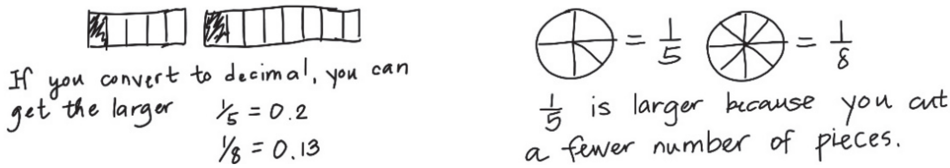


Fig. 2. Sample response with more than one strategy

Fig. 3 shows the results in task 2 (what is 2/3+1/2?). The most common explanation approach for preservice elementary teachers was based on procedures. This approach was effective for the preservice secondary teachers, but had around a 50% success rate among the preservice elementary teachers. For the preservice secondary teachers, the most common explanatory approaches were based on correct fraction relations and procedures. Among those who used correct and incorrect fraction relations, all produced a correct solution. The only exception was one preservice teacher who did not write an answer. The explanation approach based on incorrect fraction relations was only found among the preservice elementary teachers. This strategy of "count the shaded and count the parts" may have been productive in other contexts, such as when recognizing the fraction represented by a drawing, but was incorrectly applied here. Teaching students to understand the contexts where certain strategies are productive remains a learning goal.

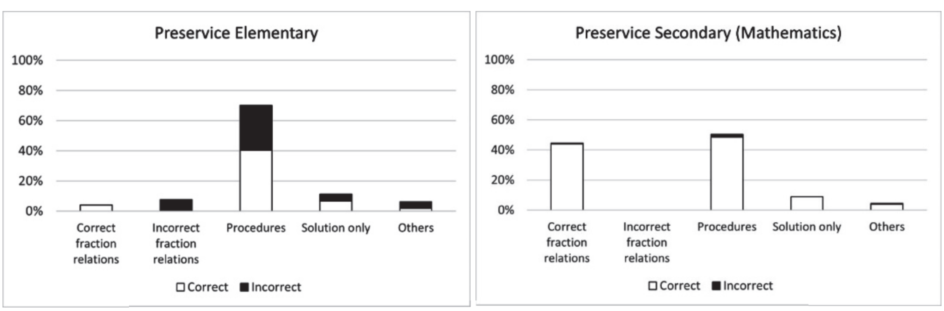


Fig. 3. Explanatory approaches in task 2 (what is 2/3 + 1/2?)

Fig. 4 shows the results in task 3 (what is 1÷2/3?). The most common explanatory approaches for this task mirrored those in the second task. For both cohorts, procedural explanatory approaches dominated. As in the previous task, this approach was effective for the preservice secondary teachers, but had a rather 50% success rate among the preservice elementary teachers. Except for one preservice teacher, everyone who used measurement division as an explanatory approach gave correct answers. Thus, this offers a potential goal for learning. Procedures were not as reliable, as almost half of preservice elementary teachers used an incorrect rule to solve the task. The rules they provided also resembled some of the rules they might have encountered in their mathematical experiences. The explanation approach based on commutativity was only found among the preservice secondary teachers. For these teachers, the strategy of “interchanging the numbers in an operation”, which is productive in addition or multiplication contexts, was incorrectly applied.

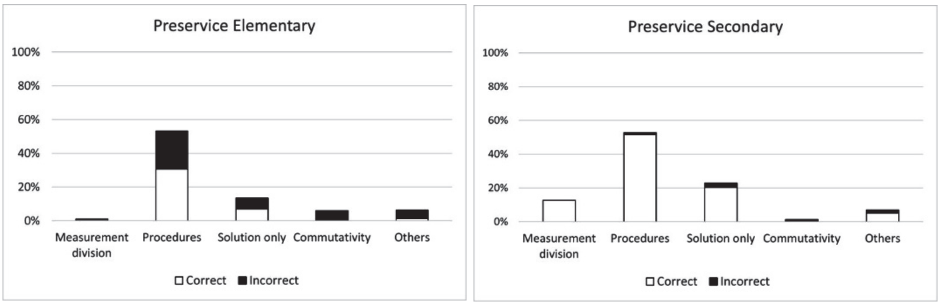


Fig. 4. Explanatory approaches in task 3 (what is 1 ÷ 2/3?)

Although procedures led to correct responses among half of the preservice elementary teachers, most of their explanatory approaches resembled standard rules. To compute 1÷2/3, they utilized elements of the correct solution (such as getting the reciprocal at some point). For example, one preservice teacher said, “We will make 1 the numerator, and then 2/3 the denominator. So that 1 becomes 3/1.”

4.1. Repeating strategies

As indicated in the Data Analysis section, the explanatory approaches were searched for repeating ideas as a means to finalize codes. One repeating pattern in the responses was the use of words such as “just”, “simply” or “easily”. For example, 33 (7%) respondents included these words in their explanatory approaches. These words convey encouragement to students by suggesting that the task is doable.

Several explanations made use of a stated fact. These did not require much thought — these can be directly conveyed to any student who listens attentively. For example, in task 3, 64 preservice teachers explained that to divide 1 by 2/3, one must write 1 as 1/1. They explained that it is “understood” or “automatic” that a whole number has 1 in the denominator, that there is an “imaginary” or “invisible” 1 in the denominator, or that any whole number has 1 in the denominator.

5. Discussion

The data suggests that procedures are reasonably reliable, particularly for the preservice secondary teachers. Most preservice secondary teachers remembered the procedure while around half of the preservice elementary teachers did not. The preservice secondary teachers had more math courses in their training, which may have increased their opportunities to learn. Another possible explanation is that preservice teachers who enrolled in preservice secondary mathematics teaching are themselves already predisposed to math to begin with. Thus, it appears that procedural explanations are reliable only for students who have reached some threshold of exposure to mathematics.

Knowledge of the fundamental fraction concept of equivalence (Smith, 1995) provides a reliable gateway towards a correct solution. Those who reasoned correctly about quantities were able to provide correct answers. The implication is that without the resources offered by thinking about quantities, procedural thinking can be a hit-or-miss — it is either you remember the rule or not. Interviews may reveal the reason why some preservice teachers remembered, while some did not.

The large proportion of explanatory approaches found in this study is consistent with the beliefs that Philippine teachers and teacher educators hold about mathematics learning. In the Teacher Education and Development Study in Mathematics (TEDS-M) study (Tatto et al., 2012), Filipino teachers and teacher educators, on average, believed 90% of the statements related to mathematics as a set of rules and procedures. Bergqvist and Lithner (2012) used Brousseau's (2002) theory of didactic situations to interpret that giving the students a procedure relieves the student of his/her mental work. Whether this is indeed the case for the preservice teachers in this study is up for further investigation. Certainly, the less-than-ideal educational context where students are crammed 80 to rooms built for 50, or where teachers are burdened by too much work makes it reasonable to ensure that rules sounded clear and friendly and to promote the easiest and most straightforward explanatory approach possible.

To highlight the difference and affordances of an anti-deficit approach in the interpretation of the results, I would like to offer two deficit conclusions form the same data set. Admittedly, this deficit model informed my initial analytic framework and the interpretation of the results. The first conclusion stemming from a deficit model is that the preservice teachers, particularly the preservice elementary teachers, lacked content knowledge and pedagogical content knowledge in fractions, given that more than half the teachers did not give a correct answer to Tasks 2 and 3. However, a more detailed look at their explanatory approaches revealed some potential resources from which the preservice teachers drew upon. These included "count the shaded", the notion of "commutativity", and the production of rules that resembled standard rules. Teacher educators may build upon these and elucidate the contexts where such knowledge is reliable. Teacher educators may also design assessments so that these contexts can be clarified. One example to facilitate a discussion of an "equal pieces" model is to ask preservice teachers to draw 1/5 and 1/8 given (a) two similar-sized objects, and (b) given two different-sized objects. These tasks may be a springboard for discussion whether it is possible to compare 1/5 and 1/8 using these models.

A second possible conclusion that perpetuates a deficit master narrative is that preservice teachers promoted learning by rote because most of their strategies relied on procedures. However, this explanation does not recognize the classroom contexts or the broader educational system as major influencers of teachers' actions. Johnson et al. (2000) argued that teachers know more strategies than what they use in the classroom. They argued,

> Whereas the wisdom gained from northern/western contexts suggests it is the teacher that does the selecting, we wish to reverse this wisdom, and suggest that for science teachers who are in educational systems at anything other than the professional stage, it is the environment in which the teacher works that creates the selection. (Johnson et al., 2000, p. 186)

These teacher constraints were expressed by a South African teacher during the time when an innovative assessment was being piloted in South Africa (Bansilal, 2011). This teacher said,

> I found out that you can't do any of the innovative tasks with them, you have to teach the concepts. Because if you don't teach these concepts, then they can't do the activity. You saw for yourself how difficult it was — there are 50 learners in each of my classes. So we have to stand in the front and do as much as we can, even though we ourselves know at the end of the day it doesn't make much of a difference. (Bansilal, 2011, pp. 104–105)

This quote demonstrates how contextual factors such as students' non-readiness for the task or the physical setup of the classroom can constrain teachers' actions. In the Philippines, the same contexts and macro-structures are also present. An interview with a Filipino teacher revealed such pressures (Verzosa et al., 2017). In relation to a mandated focus on critical thinking and exploration which was considerably different from the "spoonfeeding" method that their students were exposed to in their elementary school years, one Filipino teacher mentioned (p. 93), "ikaw na nga gagawa ng activity, ikaw din ang sasagot (you designed the activity but you end up answering it as well)." Further, for the recommended strategies to work, teachers are compelled to provide worksheets and other materials for students at their own expense.

6. Limitations and Future Research

A three-item written assessment cannot access students' full understanding. Interviews with the preservice teachers would have provided the information needed to understand how their choices were shaped by their own experiences. Most studies that focused on students' productions rather than misconceptions utilized interviews in their research design (Adiredja, 2021; Adiredja et al., 2020; Hunt and Empson, 2014; Lewis, 2014; Lewis and Lynn, 2018). As was stated earlier in this paper, this study was initially not designed with an anti-deficit perspective as a supporting framework. If not for the wise guidance of experts in the field, I would have persisted with my initial analysis that focused on errors and downplayed the resources from which the preservice teachers drew upon. By doing so, I would have contributed even further to deficit master narratives about teachers. While I was reconceptualizing my analysis, I began to recognize that many of my initial conclusions produced a deficit story about preservice teachers.

Much of the research framed by anti-deficit perspectives involve students of color or students with mathematical learning disabilities. There are few, if any, published research from non-Western contexts. Thus, I will end this paper by also pushing for the use of anti-deficit perspectives in designing, implementing, and interpreting research, especially in developing countries.

Acknowledgments

The author thanks the University of Southern Mindanao for funding this research under Fund 101. The author also thanks the universities and preservice teachers who participated in this study.

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https://doi.org/10.1142/9789811287183_0047

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Modeling and Digital Technologies: Experiences and Challenges for Teacher Education

Mónica E. Villarreal[1]

ABSTRACT This lecture presents a didactic proposal that combines mathematical modeling and digital technologies in the framework of a teacher education program for future mathematics teachers in a public university of Argentina. After presenting the theoretical assumptions underpinning this proposal, the characteristics of the program and an annual mathematics education course that forms part of its curriculum are described. This course covers topics related to mathematical modeling and the use of digital technologies, among others. Details are given of the characteristics of the modeling scenario created within the framework of this course, for preservice teachers to experience the development of open modeling projects. A synthesis of the modeling experiences developed in 2020 during the COVID-19 pandemic is shown. These experiences were carried out in groups of preservice teachers, allowing them to choose freely a real-life topic of their own interest and the use of various digital technologies. The topics chosen by each group, the role of technologies, the learnings recognized by the preservice teachers and the difficulties and limitations detected are detailed. The text concludes with some reflections on the relevance of this type of experience in teacher education.

Keywords: Preservice teachers; Modeling scenario; Digital technologies.

1. Introduction

Curricular designs for secondary education in various countries and educational contexts present recommendations on the incorporation and use of digital technologies and modeling tasks for the teaching and learning of mathematics. However, the implementation of such recommendations is still scarce in several countries. In particular, this situation prevails in a significant number of regions throughout Argentina. While there are factors related to the lack of technological infrastructure, or the prevalence of a certain conservative academic culture that affects the acceptance of such recommendations, the literature reports that teacher education plays a key role in whether or not the incorporation of technologies (Clark-Wilson et al., 2014) and

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modeling (Blum, 2015; Doerr, 2007) at secondary school level is encouraged. In particular, it seems that if preservice teachers (PSTs) are given the opportunity to experiment with mathematical modeling tasks and technologies during their undergraduate education, it would contribute to making such tasks part of their teaching agendas in the future (Doerr, 2007; Lingefjärd, 2013; Villarreal and Esteley, 2023).

In the analysis of some of the official curriculum designs for the initial mathematics teacher education in Argentina (for example, Ministerio de Educación de la Provincia de Córdoba, 2010), the importance of the use of technology as a powerful auxiliary in the educational task is recognized. At the same time, mathematical modeling tasks are highlighted as a way of linking mathematics with the extra-mathematical real world. Meanwhile, the ways in which modeling and technologies are approached differ from one teacher education program to another. In this lecture, I present a particular pedagogical approach that involves modeling and digital technologies in a synergic way. This approach has been implemented, since 2010, in a mathematics teacher education program for preservice secondary school teachers at a public university from Córdoba (Argentina). The program lasts for 4 years and 66% of the syllabus is composed by mathematics courses, mainly taught by mathematicians, while the remaining 34% of the courses deal with educational issues, taught by pedagogues or mathematics educators. I have been working as a teacher educator in this program since 2006. For some years now, I have been in charge of an annual course called *Didáctica especial y taller de matemática*, which I will later describe and make reference to as the *Mathematics Education* (ME) course. Also, for other years of this period, I have been in charge of another annual course called *Metodología y práctica de la enseñanza* (*Teaching methodology and practice*). Within this course, the PSTs carry out their first teaching practices in secondary schools.

For the last fifteen years, these courses were scenarios in which we have been investigating, in conjunction with other colleagues from the university, different aspects related to: (1) PSTs executing mathematical modeling tasks (as modellers) and (2) PSTs implementing modeling tasks during their first teaching practices at secondary schools. In both cases, we observed that different technologies were important actors for enhancing the modeling processes, and then, the study of the impact of technologies on modeling performed by PSTs became one of the focuses of our research. For this lecture, I focus on PSTs' experiences while doing mathematical modeling (MM) with different types of digital technologies (DTs).

In what follows, after presenting some theoretical ideas on MM and DTs, I describe the modeling scenario we created in the framework of the ME course for PSTs to have MM experiences. Subsequently, I briefly report about topics that the PSTs selected, roles played by digital technologies, types of learning that occurred, difficulties that arose, and limitations that were recognized.

2. Pedagogical and Epistemological Assumptions regarding MM and DTs

In this section, I present the theoretical background that supports our actions as teacher educators and researchers. On one hand, there is the modeling perspective that we adopted in the teacher education program at the university. On the other hand, we have the epistemological perspective we hold concerning the use of DTs in the production of knowledge, and in the teaching and learning of mathematics.

The modeling perspective that we adopted is characterized by the following principles:

- Open nature of the activities posed to the students with no predetermined mathematical knowledge to be taught.
- Interdisciplinary nature of the work.
- Promotion of reflections about mathematics itself, the models created, and the social role of mathematics and mathematical modeling.
- Mastery of the whole modeling process considering all the phases of the cycle.

Given that there isn't any previewed mathematical content to be learned through the modeling projects, the focus is on modeling as a mathematical activity that deserves to be taught and learned for its own sake. Our didactical approach to the teaching of modeling is compatible with the perspective known as *modeling-as-content* (Julie and Mudaly, 2007), with the notion of *active modeling* proposed by Muller and Burkardt (2007), the *socio-critical perspective* of MM discussed by many Brazilian authors such as Araújo (2012, 2010, 2009), Barbosa (2006), Silva and Kato (2012), or the ideas of *project work* described by Ole Skovsmose, in the framework of critical mathematics education (Skovsmose, 1994, 2001).

Regarding the use of technology, we have adopted the idea that knowledge is produced by collectives of *humans-with-media* (Borba and Villarreal, 2005). The word media refers to any kind of tool, device, equipment, instrument, artefact, or material resulting from technological developments. For this lecture, I am interested specifically in DTs, which include the Internet, any kind of mathematical software, and programming languages.

The notion of *humans-with-media* was presented in Borba and Villarreal (2005), and it is associated with two main ideas. One is that cognition is not an individual enterprise, but a social one, which is why the construct explicitly includes humans, in plural. The other key idea is that cognition includes tools, media with which the knowledge is produced. This component of the epistemic subject is not auxiliary or complementary, but essential. Media are constitutive elements of knowledge. The integration of DTs reorganizes the thinking and production of knowledge. This reorganization may imply transformations in educational environments; for example, in the kind of problems that may be addressed, in the ways of solving them and in the manners of validating and communicating results. Borba and Villarreal (2005) also explored the synergic relationship between modeling and technology, and other authors, such as Greefrath (2011) and Doerr et al. (2017), among many others, refer to the role of technology in modeling. Some examples of strong influence of DTs on the development of open modeling projects in different educational contexts can be found in Borba et al. (2016), Villarreal et al. (2010), and Villarreal et al. (2018).

In the next section, I present the features of a *modeling scenario* we designed for our teacher education program and the reasons that justify such design. This scenario was also the focus of several investigations.

3. A Mathematical Modeling Scenario in our Local Context

In Argentina, the official curricular documents for secondary school and the national standards for teacher education programs recommend the introduction of MM and the use of DTs for the teaching and learning of mathematics.

These recommendations imply challenges for teacher education, in general, and particularly for preservice teacher education. Meanwhile, the reality of our university teacher education program is far removed from heeding such recommendations. On one hand, mathematical courses usually give little or no room for active modeling. On the other hand, although technologies are gaining terrain in our program, they usually have a supplementary role, with little use of their potential for enhancing mathematical thinking and learning.

To reverse this situation in our local context, in 2010, we decided to create a *mathematical modeling scenario* within the regular ME course. Since then, at least one of the researchers of our research group is in charge of the course.

The ME course is in the third year of the teacher education program; it extends for 30 weeks with two 4-hour classes per week. In this course, several trends in mathematics education are studied, for example, problem-solving, critical mathematics education, the use of technology in mathematics education, and mathematical modeling.

The notions of model, mathematical model and mathematical modeling process are discussed in the course. The phases of a MM process are studied using different modeling cycles such as those proposed by Bassanezi or Blomhøj. Fig. 1 shows a modeling cycle adapted from Bassanezi (2002) and Fig. 2 shows the one proposed in Blomhøj (2004).

During the classes, experiences of modeling activities in different educational contexts are analyzed, and several modeling tasks are solved. Finally, the PSTs are invited to develop their modeling projects freely using DTs, if they wish. For this, the PSTs are asked to form small groups and select a (non-mathematical) real-world theme of their interest, formulate problems related to this theme, select variables, raise hypotheses, design experiments (if necessary), search for information, collect and process data, and solve the problems. Each group has to write a report and make an oral presentation for the whole class. During such presentations, which last about 40 minutes, the rest of the class ask questions and make comments about the projects. In many cases, the students discuss possible modeling tasks for the secondary school, or they reflect on the role of technology in the modeling process. Further on, I will refer to the projects carried out in 2020 and present a detailed timeline with the activities that guided the development of the modeling projects of that year.

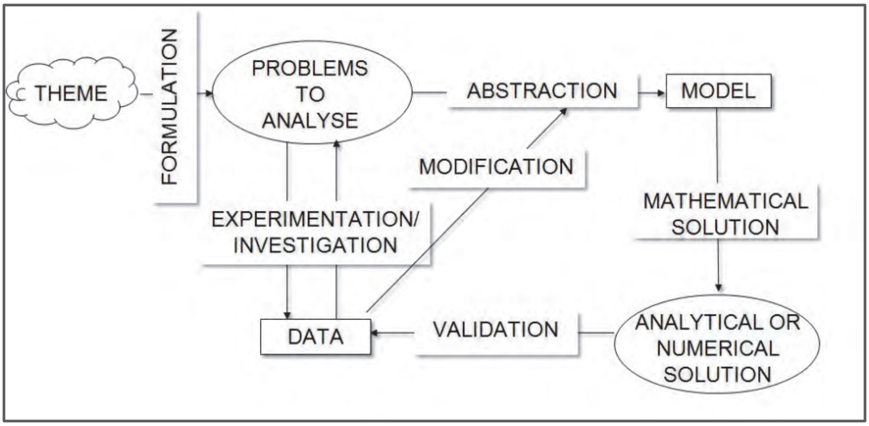

Fig. 1. Modeling cycle adapted with permission from Bassanezi (2002, p. 27)

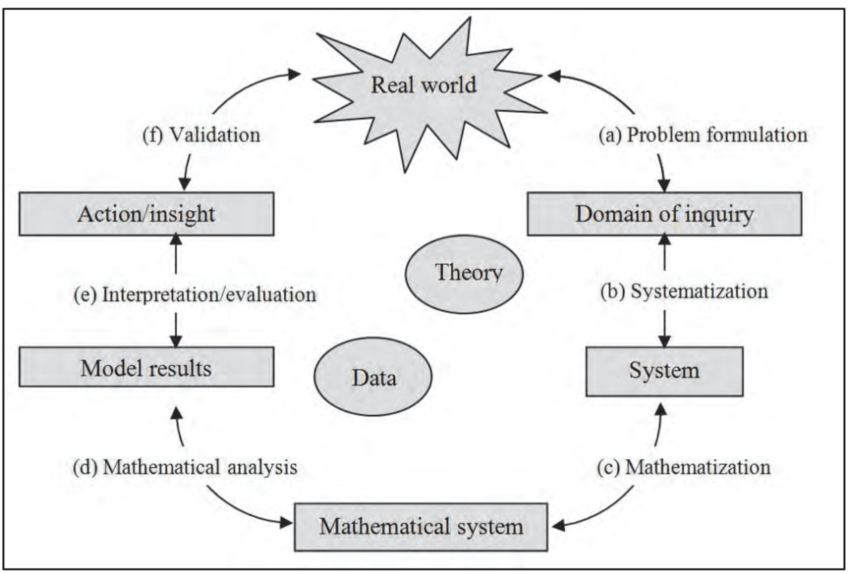

Fig. 2. Modeling cycle proposed in Blomhøj (2004, p. 148)

This scenario offers the PSTs the opportunity of experiencing a complete open modeling process, following the phases of the modeling cycle. Such experiences have been registered in different ways: PTS' final written reports, our field notes during the classes, GeoGebra files, spreadsheets files, Python codes, and videotapes of the final oral presentations. These are the sources that allowed us to develop different studies in the period between 2010−2020. So the MM scenario became a research scenario, in which we analyzed, for example, the themes that the PSTs selected for their modeling projects, the reasons for such selection, and the relation with social concerns. We also analyzed the mathematical contents that the PSTs used in their projects. The results of this analysis were published in Villarreal et al. (2015) and Villarreal (2019).

When focusing on the integration of DTs in modeling processes, we sought to determine which technologies PSTs chose, for which modeling purposes, and in which phases of the modeling process the technologies were significantly used. These questions were addressed in Villarreal et al. (2018). In this case, it was found that PSTs used the Internet, spreadsheets, mathematical software and programming languages. The Internet was the most utilized technology. It was used to find information or data, to select variables, or to formulate problems. The other three technologies significantly influenced the processes of mathematical solution and validation.

The next sections describe the experience in 2020, during the pandemic. First, I will describe, in general, the organization of the ME course in distance mode, followed by the work with modeling projects, in particular. Then, I will present brief descriptions of the MM projects that four groups of PSTs carried out.

4. Mathematical Modeling during the Pandemic in 2020

4.1. The organization of the ME course in distance mode

In 2020 the ME course was held in a distance learning mode, except for the first two classes. Thirteen students attended the course and three teachers were in charge of it. The course started in the week of March 16th, and on Friday of that week, mandatory quarantine was decreed all over the country. Classes at all levels of education moved to an online modality.

In the case of the ME course, for some years now, we have used a virtual classroom on the Moodle platform (Fig. 3, on the next page) in which we upload texts, activities, videos, PowerPoint presentations of the classes, among other materials. In the transition to distance learning, this tool was fundamental, but we realized that this resource alone was not enough. So, a few weeks after starting the quarantine, we decided to start virtual meetings with the students twice a week through Google Meet.

This brought on other issues. Many students did not have good connectivity at home, some did not have a computer with a camera and others could only connect through their mobile phones, with multiple interruptions and with expensive costs for internet connection. There were students whom we rarely saw, given they could not turn on their cameras due to poor connectivity and, at times, even the teachers had connectivity problems. One student and one of the teachers had to manage childcare while in class.

As demonstrated in other levels of the educational system in Argentina, the COVID-19 pandemic revealed the unequal conditions for university students to access education in this public health emergency.

Fig. 3. Snapshot of the ME virtual classroom in Moodle.

4.2. *The organization of work with MM projects*

The thirteen students in the course were divided into four groups: one group of four members and three groups of three members. Fig. 4 shows a timeline with dates and activities related to the development of the MM projects. It shows the events from the first class in which MM was introduced as a trend in mathematics education, to the submission of the final written report of each group.

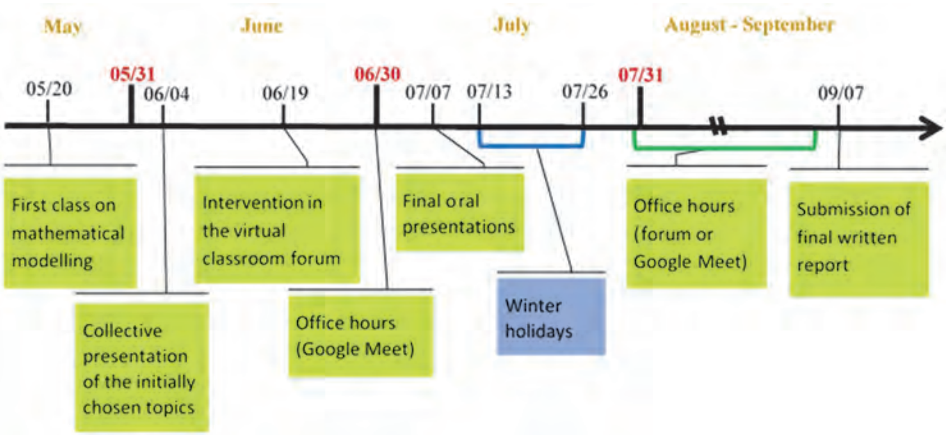

Fig. 4. Timeline of the development of MM projects in 2020

The moment of the collective presentation of the first topics chosen by the groups was fundamental since, based on the exchange with classmates and teachers, each group finally decided which topic to address. After this initial exchange, the activities related to the modeling projects were carried out by the PSTs in an autonomous way in schedules outside the classes. A forum was created in the Moodle virtual classroom and a first intervention on the progress of the projects was requested for June 19th. Each group could add a new discussion topic, whenever needed, to send questions and receive advice from the teachers. Previous to the final oral presentation, office hours were arranged with each group through Google Meet.

The teachers acted as guides that helped to formulate or reformulate the problems, inform about possible sources of data, and suggest new questions to make the students engage in more complex modeling processes.

The final oral presentations of the groups allowed us to have an overview of the state of progress of each project. After the oral presentations, the PSTs had 2 months to complete the written report, considering all the suggestions and observations received during the oral presentations. During this 2-month period, the PSTs also consulted with the teachers to make progress in their writing. They did it via the forum or Google Meet.

In the next section I will report on the four MM projects that PSTs developed in 2020.

5. The Four PMTs' Modeling Projects

To give a comprehensive overview of the four modeling projects, in this section I will briefly report about: the topics that the PSTs selected, the roles played by digital technologies, the types of learning that occurred, and the difficulties and limitations that were recognized.

5.1. The selected topics

I will refer to the topics that each group initially chose and which topic was finally addressed in each one.

Group 1 thought about four possible themes: (1) Lottery games, (2), Canes for the blind (in this group, one of the students is blind), (3) Body mass index, and (4) COVID-19.

After the exchange with teachers and colleagues, and considering the large amount of data available on COVID-19, the group finally chose this topic. The first questions were broad and difficult to address in the available time:

- Taking into account the reliable data, is it possible to predict the evolution of this disease in Argentina?
- Can this possible prediction be compared, or not, with a situation in which we have not taken health prevention measures?
- Is it possible to make a calculation that would allow us to estimate, more or less quickly, whether we are better off, worse off or in the same situation in relation to the evolution of the disease?

These questions were very ambitious and with the advice of the teachers, the group decided to consider data only from our province (Córdoba) and to make a descriptive model of the situation, analyzing the effects of the restriction or relaxation measures in the period between March and July 2020.

Group 2 thought about the following issues: (1) Evolution of prices to build a house from 2015 to 2029, (2) Causes of undergraduate students dropping out of the FAMAF (the faculty in which they are studying), (3) Electronic waste in Córdoba, and (4) Femicides in Argentina.

According to the members of the group, the topic of femicides "was a more current one for us to choose, with elements of analysis that can be enriching when it comes to modeling". They also stressed the importance of sharing the results of their research, as an important way to raise awareness of the severity of this social problem.

As in the case of Group 1, the data available from different official sources for the period 2014–2019 influenced the formulation of the questions. The following are some of these questions:

- Is there a trend in the number of femicides in Córdoba from 2014 to 2019?
- In which province of Argentina was the highest number of victims registered in the period between 2014–2019?
- What was the distribution of the femicide rate in Argentina in 2019?

Group 3 proposed to address a topic related to technology and health. The students were concerned with the amount of time a person spends in front of a screen, taking into account the situation of preventive and mandatory social isolation due to the COVID-19 pandemic. The group posed the following question: How can screen time affect a person's health during this quarantine time? To answer this question, new questions were posed:

- How much time does a person spend in front of a screen per day, per week, during the quarantine period?
- Is screen time the same on a weekday as on a weekend day?
- For what purposes are the devices used?
- What effects can too many hours in front of a screen cause, and do all devices have the same effect?

Group 4 thought about the following themes: (1) The impact of the quarantine on access to education, (2) Gender-based violence in Argentina due to the COVID-19 pandemic, (3) Decrease of contamination after compulsory isolation, and (4) Potato production in Argentina.

The group finally chose the first topic, but a reformulation was necessary since it was a very broad subject, with many aspects to explore. The final decision was conditioned by the possibility of accessing real data from the responses to a survey posted on the Internet, about computer accessibility conditions among first and second-year students of the faculty. This survey had been carried out by the Student Centre to learn the technological needs of undergraduate students to be able to continue their studies at the university. Finally, the topic chosen was: Internet access in first and second-year students of FAMAF during the pandemic. From this topic, the group formulated the following question: How much does the lack of internet access affect the dropout rate of first and second-year FAMAF students during the COVID-19 pandemic?

In all cases, the final decision on the topic to be addressed and the questions posed were significantly influenced by the pandemic and confinement situation we were experiencing, and the consequent profusion of data associated with this situation that were available for use. In addition, the first discussions with the whole class at the time of the collective presentations also played a decisive role in the final decision on the problems to be addressed in each project.

5.2. Roles played by digital technologies

Regarding the technologies used in the MM projects, the most popular were: spreadsheets, GeoGebra and the Internet. Spreadsheets were used to systematize data in tables, make pie or bar graphs to represent data, and count cells using the Excel Filter function and the COUNTIF function to analyze answers in a survey. GeoGebra was used to fit curves to a dataset and represent data in a coordinate system. The Internet was used to search for information on the topics to be studied (e.g. effects of prolonged screen time), watch lectures on different mathematical models associated with the spread of the coronavirus, watch tutorials on how to use Excel functions, have access to official agency websites to collect data on the number of people infected with coronavirus per day in the province of Córdoba, or the number of femicides in Argentina in a given period.

As we had observed in previous years (Villarreal et al., 2018), the use of DTs was significant in the phases of problem formulation, abstraction and mathematical resolution (see phases in Fig. 1).

5.3. Types of learning that occurred according to the PSTs

When the PSTs were asked what they had learned, all the groups reported specific learning related to the selected topics: awareness of the complexity of the health situation caused by COVID-19, the different issues associated with violence against women and femicides, awareness of the negative effects of excessive screen time on eyesight and posture and recommendations to mitigate these effects.

PSTs also highlighted their learning around the use of some spreadsheet functions and GeoGebra commands to fit curves.

Regarding mathematics and its use, PSTs stated that, in the development of their modeling projects, mathematics was a tool to read reality. Sometimes the mathematics used was simple, but it had a clear application. Every index, every rate that was calculated had a meaning. A number could represent a dead woman, a sick person, a student who couldn't access education or the indicator that we need to stop spending so much time in front of a screen.

PSTs also acknowledged that there was learning related to statistical work. For example, the elaboration of summary tables from a dataset, the selection of variables, the creation of categories, the construction of contingency tables to analyze the existence, or not, of relationships between variables, and the interpretation of tables.

In summary, PSTs identified learning in four areas: in relation to the topics addressed, in relation to the use of DTs, in relation to mathematics itself, and in relation to the use of mathematics for critical reading and understanding of the world. These learnings are evidence of a socio-critical modeling perspective in action.

5.4. Difficulties and limitations

One of the major difficulties reported by the PSTs during the modeling process was the impossibility of meeting face-to-face to work. This made it difficult to agree on actions and make decisions smoothly. One group decided to break the quarantine and meet to move forward.

Another difficulty was agreeing on when to have remote meetings through video calls. This was complicated due to the different responsibilities and activities of each student, and due to the poor internet connectivity of some of them.

The impossibility of meeting face-to-face involved situations such as that experienced by the members of the group working with COVID-19 (Group 1), which concerns the use of Excel. They had constructed a table recording the number of new coronavirus infections per day in Córdoba. They also needed to calculate the cumulative number of infections per day as shown in Fig. 5. Doing this calculation using Excel is trivial, but none of the students in the group knew how to do it. One of them tried to make a code using Python, to calculate the sum of the cumulative cases per day, but Python reported that there was an error in his code. This student is blind and was alone. As he could not meet with the colleagues of the group, it was difficult for him to detect the error in the code, because he could not see it. Therefore, they decided to do the calculations by hand, which involved a lot of manual work and a lot of time wasted. The students stated that they were not good users of Excel.

FECHA.	NUEVOS INFECTADOS.	acumulad
2-mar	1	1
3-mar	1	2
4-mar	0	2
5-mar	1	3
6-mar	1	4
7-mar	0	4
8-mar	1	5
9-mar	1	6
10-mar	0	6
11-mar	1	7
12-mar	2	9
13-mar	5	14
14-mar	4	18
15-mar	3	21
16-mar	8	29
17-mar	9	38
18-mar	5	43
19-mar	13	56

Fig. 5. Number of new infected persons per day and cumulative number in Córdoba city, between March 2 and March 19, 2020. Source: students' written report.

It seems that both not being able to meet face-to-face and having poor connectivity were major obstacles to the development of the MM projects.

In addition to the difficulties recognized by the PSTs, we (teachers in charge of the ME course) recognized some deficiencies in the teacher education program. The PSTs do not study differential equations and the study of descriptive statistics is very limited. This implies limitations when working with population growth problems or when statistical analysis of data was needed.

6. Some Final Remarks

Authors such as Doerr (2007), Niss et al. (2007), Blum (2015) and Gastón and Lawrence (2015) refer to the necessity of providing future teachers with opportunities of experiencing the modeling process during their preservice education to place modeling on the agenda of their future teaching activities. In line with these authors, Lingefjärd (2007, 2013) asserts that technology may broaden and enhance PSTs' experiences with modeling processes.

While what I have presented here is merely a description of the MM experiences of four groups of PSTs, it is an example of the kind of work that can be done in a teacher education program around MM in conjunction with DTs.

The modeling projects carried out by the PSTs and their reflections, expressed during oral presentations or written in the final report, provide evidence of the variety of topics addressed and their close link to the COVID-19 pandemic situation, the technologies used, the learning achievements, the difficulties encountered and the limitations recognized. In particular, some students made sense of the experience envisioning possible implications for their future as teachers.

In this lecture I have reported on the positive issues pointed out by the PSTs during the MM process carried out in pandemic times and about the difficulties they experienced. Despite the difficulties, and based on the positive evidence we gained through many years of experience, we argue that the implementation of open modeling activities and the use of technologies are imperative within preservice teacher education for many reasons:

- They can enhance learning at any level of the education system.
- They can contribute to mathematics being seen as a useful tool for describing and analyzing real problems, making informed decisions, and criticizing situations using solid arguments.
- They can make future teachers sensitive towards the different ways of making sense of mathematics.

Finally, it should be noted that while it is true that this type of experience is valuable for PSTs, the implementation of open MM tasks such as the ones described here, in the context of secondary education, implies an important challenge for teachers and students. In this sense, the study of the design and implementation of open MM tasks in secondary school classrooms has also been a topic of research in our research group. These studies permanently lead us to rethink the principles that characterize our modeling perspective and to reflect on different ways of working with MM and the use of DTs in mathematics teacher education.

Acknowledgments

This work was carried out with the financial support of the Secretaría de Ciencia y Técnica (UNC), the Agencia Nacional de Promoción de la Investigación, el Desarrollo Tecnológico y la Innovación, and the Consejo Nacional de Investigaciones Científicas y Técnicas.

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https://doi.org/10.1142/9789811287183_0048

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Gifted Students Education in China — Introduction of Chinese Mathematical Competitions

Bin Xiong[1] and Yijie He[2]

ABSTRACT The development of Chinese mathematics competition activities can be divided into the following stages: Stage I (1956–1964): Considered as the birth of China's competition mathematics, Stage II (1978–1984): A working committee was set up by the Chinese Mathematical Society (CMS) in order to standardize and institutionalize the development of mathematics competition activities in China, Stage III (1985–present): This period sees a flourish of mathematics competition activities. China began to participate in the International Mathematical Olympiad (IMO) and in 1990, the 31st IMO was successfully held in Beijing on an unprecedented scale.

After more than three decades' exploration and practice, an ever-enriching, relatively stable system has evolved in the Chinese high school mathematical Olympiad practice.

Competition mathematics education is beneficial to the development of gifted students' mathematical ability in various aspects.

Competition mathematics was introduced to China in 1956 (cf. Hua, 1956a, 1956b). In the same year, mathematics competitions for senior high school students were held in Beijing, Tianjin, Shanghai and Wuhan respectively. Luogeng Hua (known as Loo-keng Hua in the west) personally served as chairman of the Beijing Competition Committee and engaged in the preparation of test materials. Many famous senior mathematicians, including Luogeng Hua, Zhongsun Fu, Jiangong Chen, Buqing Su, Xuefu Duan, and Zehan Jiang also made lectures during the competitions (Sun and Hu, 1994).

The development of Chinese mathematics competition activities can be divided into the following stages (cf. Chen and Zhang, 2013):

Stage I (1956–1964): Considered as the birth of China's competition mathematics, competitions were mainly advocated and personally directed by senior mathematicians and held in a few key cities of China.

Stage II (1978–1984): After the ten-year political turmoil came to an end, mathematics competitions were resumed. A working committee was set up by the Chinese Mathematical Society (CMS) in order to standardize and institutionalize the development of mathematics competition activities in China.

Stage III (1985–present): This period sees a flourish of mathematics competition activities. China began to participate in the International

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Mathematical Olympiad (IMO) and in 1990, the 31st IMO was successfully held in Beijing on an unprecedented scale. The level of mathematics competitions in China quickly caught up with the international standard and continued to maintain its leading position afterwards. Meanwhile, various kinds of competitions at all levels were launched and a wide and diverse range of learning materials was readily accessible. The competition-oriented training became the "second classroom" for a proportion of students, or even the "second school" for a few top students.

1. Description of the Mathematics Competition Organization in China

Since the establishment of the Popularization Committee of the CMS and after more than three decades' explorations and practices, an ever-enriching, relatively stable system has evolved in the Chinese high school mathematical Olympiad practice. The National Senior High School Mathematics Competition and National Junior High School Mathematics Competition are two important events in this framework. Over the selection of the IMO national team, the CMS has also gradually set up a feasible working procedure.

1.1. National senior high school mathematics competition

In 1980, with a view to popularize mathematics on a national scale, the CMS held its first national conference in Dalian, in which organizing mathematics competitions was confirmed to be a regular task for the CMS and its branches in each province, municipality and autonomous region. In addition, the National Senior High School Mathematics Competition was agreed to be held in October annually (Zhu, 2009). Since 2014, the competition dates have been adjusted to be in September.

The National Senior High School Mathematics Competition is divided into two parts: Test 1 and Test 2 (also called the additional test). Test 1 mainly aims at educational spreading and popularization. Problems are mainly based on the *Mathematics Syllabus for Full-time Senior High Schools* (cf. Ministry of Education PRC, 2000), thus closely related to what students learn in class, but require some problem-solving strategies as well as a flexibility in knowledge applications. On the other hand, problems in Test 2 largely focus on the capability enhancement, some of which are specially designed for testing the mathematical ability and identifying mathematical talents. Contents are linked to the IMO, covering geometry, algebra, number theory and combinatorics. The overall difficulty of such test problems is lower than that of the IMO problems.

Before 2013, the set of test problems of the National Senior High School Mathematics Competition each year was carried out by the CMS in cooperation with a certain province. Since 2013, the CMS has been directly responsible for the test development. The quality of problems has been further improved.

Taking the 2013 National Senior High School Mathematics Competition as an example (cf. National Team Coaching Staff, 2014). Test 1 problems were largely related to the main knowledge topics like functions, inequalities, sequences and analytic geometry, which effectively covered each chapter in the textbooks and were designed at an appropriate difficulty tier. Test 2 focused on the extensible contents specified in the *Syllabus for Senior High School Mathematics Competition* (cf. Popularization Committee of the CMS, 2006a). The following four examples are taken respectively from Problems 1, 3 and 8 of Test 1 as well as Problem 3 of Test 2 of the 2013 National Senior High School Mathematics Competition.

Example 1.1. Let $A=\{2,\ 0,\ 1,\ 3\}$ and $B=\{x \mid -x\in A,\ 2-x^2\notin A\}$. Then the sum of all the elements in B is ______.

This problem only tests the basic knowledge of set representations. The first-year senior high school students may complete it successfully when they know the relevant concepts.

Since the elements in a set are unordered, $\{0,\ 1,\ 2,\ 3\}$ represents the same set as $\{2,\ 0,\ 1,\ 3\}$, but we are usually more used to the former representation because of its ordering from small to large. This problem intentionally disrupts the order of the elements. On the one hand, it is associated with the Year of 2013, which is lively. On the other hand, it is also an attempt to implicitly make the solver realize the unordered nature of set elements, and offer potential materials to the classroom teaching.

Example 1.2. Let ABC be a triangle with $\sin A=10\cdot\sin B\cdot\sin C$ and $\cos A=10\cdot\cos B\cdot\cos C$. Then the value of $\tan A$ is ______ .

The reference solving process of this problem is as follows:

Note that

$$\begin{aligned}\sin A-\cos A&=10(\sin B\cdot\sin C-\cos B\cdot\cos C)\\&=-10\cdot\cos(B+C)=10\cdot\cos A.\end{aligned}$$

Therefore, we have $\sin A=11\cos A$, which is equivalent to $\tan A=11$.

The problem is novel and beautiful with a natural statement. The formulae used to solve this problem are all basic contents of in-class instructions, and the reasoning chain is relatively short, which makes it a high-quality competition problem suitable for classroom teaching.

However, one cannot solve this problem by mechanically applying the formula in the textbook, nor by analyzing the two conditions in isolation. Instead, one should consider the conditions as a whole according to their structural characteristics, and find effective information combinations from them, so as to eliminate irrelevant quantities B and C. Therefore, this problem is also enlightening for the teaching of heuristics of mathematical problem solving.

Example 1.3. Let $a_1, a_2, \cdots, a_9$ be a sequence satisfying $a_1=a_9=1$ and $\dfrac{a_{i+1}}{a_i}\in\left\{2, 1, -\dfrac{1}{2}\right\}$ $(1\le i\le 8)$. The number of sequences with such property is ______.

This is a counting problem designed with a sequence background, which is obviously much more difficult than exercises in textbooks, yet the knowledge and method involved are still within the teaching requirements of combinations and permutations, and the amount of calculation needed is appropriately controlled, so it would not be too challenging for students with clear thinking and good logic judgment.

Example 1.4. There are m problems in a test and n students taking the test, where m and n are integers greater than 1. For each problem, the scoring rule is as follows: If a total of x students fail to get a correct solution to this problem, those presenting correct answers will get x points, and the ones with wrong answers will receive zero points. The grade of each student is the total of the points gained from m problems. If the grades of all the students are written $p_1 \geq p_2 \geq \cdots \geq p_n$ from high to low, find the maximum possible value of $p_1 + p_n$.

Discrete mathematics is a relatively weak part in the current high school mathematics education in China, which leads to the incomplete display of students' mathematical talent. In competition mathematics, however, the materials of discrete mathematics are far richer. Especially, many of the combinatorics problems do not require specialized mathematical knowledge, but need imagination, insight and mathematical wit to some extent. Hence, discrete mathematics has an important value for the discovery of mathematically gifted students and the cultivation of their mathematical thinking.

Example 1.4 is a typical extremal value problem of discrete variables in competitions, but is unconventional for most students, since in-class instruction in senior high schools is mainly concerned with continuous variables. As a Test 2 problem, the knowledge and approach to be applied are both extended to some degree, but the problem is still mainly focused on mathematical thinking rather than specialized background knowledge. Generally, it requires the students to translate the condition into quantity, apply correctly the properties of inequalities and the mean value inequality of n unknowns to complete the upper bound estimation, as well as construct an optimal example.

The National Senior High School Mathematics Competition is a public-oriented extracurricular activity with significant influences. In recent years, there are approximately 50,000 students participating in this competition annually (around 1 million students in total if the preliminary competitions organized by each province or city are included). Moreover, the National Senior High School Mathematics Competition also has another function — to select best contestants around China (currently 350–400 annually) to participate in the China Mathematical Olympiad.

1.2. National junior high school mathematics competition

The National Junior High School Mathematics Competition is another public-oriented competitive event organized by the CMS to popularize mathematics. It aims to arouse students' interest in mathematical learning, develop their innovative awareness and capability, as well as discover and cultivate mathematical talents.

Since 1985, the National Junior High School Mathematics Competition has been held annually, normally in March or April (Zhu, 2009). The organization form is similar to that of the National Senior High School Mathematics Competition.

The contents of the National Junior High School Mathematics Competition cover numbers, algebraic expressions, equations and inequalities, functions, geometry, logic reasoning, etc., in which the contents listed in the mathematics curriculum standard are specified as the basic requirements of the competition (cf. Ministry of Education of the PRC, 2012). High requirements are made with regard to the comprehension level, flexibility in applications as well as the level of proficiency in the grasp of methods and skills. The competition similarly consists of Test 1 and Test 2. The former focuses on the basic knowledge and skills, while the latter focuses on the problem solving and analysis capabilities.

It is worth noting that though the contents of "Viete's theorem of quadratic equations," "criteria and properties of triangle similarity," "angle of circumference," "cyclic quadrilateral" and "tangent length of circle" have been diminished in the test requirements of the currently used curriculum standards, the mathematics competition for junior high schools still keeps these contents as a supplement for the in-class instruction. In the *Syllabus for Junior High School Mathematics Competition* (Popularization Committee of the CMS, 2006b), contents of "four concyclic points" and "circle power theorem" were added in particular besides the above-mentioned knowledge. Take Problems 1 and 2 in Test 2 (Paper A) of the 2014 National Junior High School Mathematics Competition as examples (cf. Xu, 2015):

Example 1.5. Let a, b be real numbers such that $a^2(b^2+1)+b(b+2a)=40$, and $a(b+1)+b=8$. Find the value of $\frac{1}{a^2}+\frac{1}{b^2}$.

Example 1.6. As shown in Fig. 1, in the parallelogram $ABCD$, point E is on diagonal BD such that $\angle ECD = \angle ACB$. The extended line of AC meets the circumcircle of triangle ABD at F. Prove that $\angle DFE = \angle AFB$.

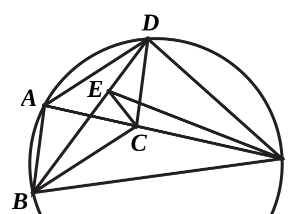

Fig. 1. A geometric problem in 2014 National Junior High School Mathematics Competition

In Example 1.5, the problem can be simplified by changing the variables $a+b=x,\ ab=y$. It is not difficult to see that the structural feature of algebraic expressions in this problem is closely related to "Viete's theorem of quadratic equations." In Example 1.6, the knowledge of "criteria and properties of triangle similarity" and "four concyclic points" is tested.

If the curriculum standards are meant for the fundamental goal of compulsory education, then competition mathematics can be reckoned as an extension, in both breadth and depth, for students who have the capacity and desire for further study, to develop their thinking ability and avoid a large amount of low-level repetition in mathematical learning. The training in algebraic techniques can improve their ability in algebraic calculations; the knowledge of circles and similar triangles can provide abundant mathematical materials for the in-class instruction and after-class study, and further improve students' ability of geometrical reasoning. It should also be noted that, these are directly related to analytic geometry and other contents to be studied in senior high school.

1.3. Chinese Girls' Mathematical Olympiad

In many mathematics competitions, there are always more men and fewer women among the competitors. Traditionally, many people think that boys are generally better than girls in mathematics. Although this statement lacks the support of the actual research data, the fact that the numbers of boys and girls of the mathematical Olympians are out of balance promotes the birth of the "Chinese Girls' Mathematical Olympiad."

In August 2002, China Mathematics Olympic Committee of the CMS held the first Chinese Girls' Mathematical Olympiad in Zhuhai. The participants were girls in senior high schools. The purpose of this activity was to show their mathematical talent and other talents, set up a stage for them, increase their interest in learning mathematics, improve their mathematics learning level, and promote their mutual learning and friendship in different regions.

Academician Wang Yuan, a famous mathematician, inscribed the Chinese Girls' Mathematical Olympiad: "Sophie Germain, Sofya Kovalevskaya, and Emmy Noether, the names of these great women mathematicians and their outstanding achievements are enough to prove that women have very high mathematical talent, which is certainly suitable for studying mathematics."

It has been 18 years since the first Chinese Girls' Mathematical Olympiad was held in 2002. Through the activities of the 18th Session, it provides a platform for many excellent high school girls to appreciate, study and research into mathematics, and also trains a large number of outstanding girls.

The Chinese Girls' Mathematical Olympiad is held once a year and has been held for 18 times. The competition time is in the middle of August every year. There are about 40 teams participating in each competition, each team has 4 participants. Chinese Hong Kong SAR, Macao SAR, and Taiwan Region have teams to join the series of activities. Participants in the Mathematical Olympiad also include teams from foreign countries such as United States, Russia, the Philippines, Singapore, the United Kingdom, Japan, and South Korea. The Chinese Girls' Mathematical Olympiad has a written test for two days which is in line with the IMO. Four problems need to be solved within 4 hours each day.

The scope of test problems for the Chinese Girls' Mathematical Olympiad is the same as the IMO, involving algebra, geometry, combinatorics and number theory, but the difficulty is lower than the IMO. The competition awards the first place in the total score of the group and the gold, silver and bronze medals of the individuals. From 2012, the top 12 students in total score are directly invited to participate in the Chinese Mathematics Olympiad.

In order to enrich the life of the contestants, cultivate their creativity and the team spirit, the Chinese Girls' Mathematical Olympiad has specially designed the girls' aerobics competition, which was planned when we held the first Chinese Girls' Mathematical Olympiad in Zhuhai, and has persisted until today.

1.4. China Western Mathematical Invitation

The China Western Mathematical Invitation began in 2001, shortly after the country launched the western development, with the original name "China Western Mathematical Olympiad".

At the beginning of China's participation in the IMO, there were participating students from Western China almost every year. In 1989, the national team even had four participants from the West. But since 1991, the western region was quite silent for ten years. At the same time, the performance in both the National Senior High School Mathematics Competition and the China Mathematical Olympiad showed obvious gap between Western China and Eastern China.

In order to "maximize the mobilization of schools in the western region to actively participate in the mathematics competition" and promote the improvement of the level of mathematics and science education in the western region, Zonghu Qiu, who has been doing "what I am willing to do," put forward the idea of holding the mathematics competition in the western region.

The first China Western Mathematical Olympiad was held in Xi'an, a city with long history, in early November 2001. The teams of high school students from 11 provinces, cities and districts in the West, and Shanxi, Jiangxi, Hainan and Hong Kong SAR participated in the competition. The participants of the China Western Mathematical Olympiad are mainly grade 11 and grade 10 students. The competition is divided into two days. Four problems need to be solved within 4 hours each day. The overall difficulty is roughly equivalent to that of the additional test of the National Senior High School Mathematics Competition. Before 2012, the first and second place winners in each competition could directly participate in the training of the national training team.

Since 2012, the China Western Mathematical Olympiad has been renamed the China Western Mathematical Invitation. The competition dates have been changed from the first half of October to the middle of August since 2013.

The development of this competition reignited the enthusiasm of Western students for mathematics. Once again, the figure of Western students often appears in the national team.

This competition has also attracted the representatives of Singapore, Indonesia, Malaysia, Philippines and other countries to take part in successively. Kazakhstan, as a strong team in IMO, has participated in each of the following competitions since

2003, and has often sent IMO gold and silver medalists to participate in this competition.

1.5. Selection procedures for the IMO Chinese national team

The following figure (Fig. 2) illustrates the selection procedure for the IMO Chinese national team.

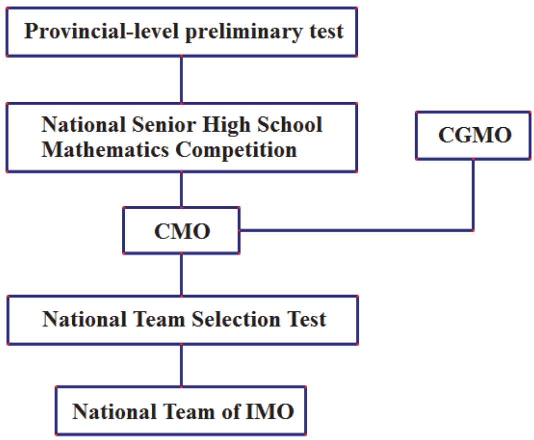

Fig. 2. The selection procedure for the IMO Chinese national team

In the figure, CMO refers to "China Mathematical Olympiad," a mathematical competition organized by the Mathematical Olympic Committee of the CMS aiming to select mathematical talents. It is also the top mathematics competition for high school students in China. CGMO refers to the "China Girls' Mathematical Olympiad."

The participants in the CMO consist of winners of the National Senior High School Mathematics Competition from each province, the CGMO winners and the invited international teams. The top 60 participants from Chinese mainland have qualifications to enter the National Training Team and they also qualify for admissions into any top university in China. The difficulty of the CMO test is roughly the same as the IMO.

As the final round of the entire selection process, the National Team Selection Test aims to select 6 members for the national team of the IMO. Usually this round consists of several sets of extremely difficult problems.

1.6. Performance of the IMO Chinese National Team

China took part in the 26th IMO in 1985 for the first time, when only two students went there. Since 1986, with the exception of the one held in Chinese Taiwan in 1998, the Chinese team has sent six students to participate in the IMO. The following Fig. 3 and Tab. 1 are about China's participation in the IMO.

Fig. 3. Team results of the IMO Chinese national team

Tab. 1. Individual results of the IMO Chinese national team

Year	Team Size			P1	P2	P3	P4	P5	P6	Total	Rank		Awards				Leader	Deputy leader
	All	M	F								Abs.	Rel.	G	S	B	HM		
2019	6	6		40	41	27	41	42	36	227	1	100.00%	**6**	0	0	0	Bin Xiong	Yijie He
2018	6	6		42	37	17	42	42	19	199	3	98.11%	**4**	2	0	0	Zhenhua Qu	Yijie He
2017	6	6		42	25	0	42	19	31	159	2	99.09%	**5**	1	0	0	Yijun Yao	Sihui Zhang
2016	6	6		42	30	20	42	42	28	204	3	98.15%	**4**	2	0	0	Bin Xiong	Qiusheng Li
2015	6	6		42	36	12	42	22	27	181	2	99.03%	**4**	2	0	0	Bin Xiong	Qiusheng Li
2014	6	6		42	42	16	42	35	24	201	1	100.00%	**5**	1	0	0	Yijun Yao	Qiusheng Li
2013	6	6		42	38	30	41	42	15	208	1	100.00%	**5**	1	0	0	Bin Xiong	Qiusheng Li
2012	6	6		42	40	14	31	38	30	195	2	98.99%	**5**	0	1	0	Bin Xiong	Zhigang Feng
2011	6	6		42	12	42	42	42	9	189	1	100.00%	**6**	0	0	0	Bin Xiong	Zhigang Feng
2010	6	5	1	41	42	23	42	24	25	197	1	100.00%	**6**	0	0	0	Bin Xiong	Zhigang Feng
2009	6	6		42	42	42	42	42	11	221	1	100.00%	**6**	0	0	0	Huawei Zhu	Gangsong Leng
2008	6	5	1	42	42	42	42	35	14	217	1	100.00%	**5**	1	0	0	Bin Xiong	Zhigang Feng
2007	6	6		36	42	17	41	42	3	181	2	98.91%	**4**	2	0	0	Gangsong Leng	Huawei Zhu
2006	6	6		42	42	35	41	38	16	214	1	100.00%	**6**	0	0	0	Shenghong Li	Gangsong Leng
2005	6	6		32	42	35	42	42	42	235	1	100.00%	**5**	1	0	0	Bin Xiong	Jianwei Wang
2004	6	6		37	41	21	42	37	42	220	1	100.00%	**6**	0	0	0	Yonggao Chen	Bin Xiong
2003	6	6		42	40	28	42	42	17	211	2	98.77%	**5**	1	0	0	Shenghong Li	Zhigang Feng
2002	6	6		41	41	24	42	42	22	212	1	100.00%	**6**	0	0	0	Yonggao Chen	Shenghong Li
2001	6	6		42	40	23	42	42	36	225	1	100.00%	**6**	0	0	0	Yonggao Chen	Shenghong Li
2000	6	6		42	42	19	39	42	34	218	1	100.00%	**6**	0	0	0	Jie Wang	Yonggao Chen
1999	6	6		40	41	11	42	33	15	182	1	100.00%	**4**	2	0	0	Jie Wang	Jianping Wu
1997	6	6		39	42	38	38	41	25	223	1	100.00%	**6**	0	0	0	Jie Wang	Jianping Wu
1996	6	5	1	34	42	32	36	0	16	160	6	93.24%	**3**	2	1	0	Wuchang Shu	Chuanli Chen
1995	6	5	1	42	42	40	36	42	34	236	1	100.00%	**4**	2	0	0	Zhusheng Zhang	Jie Wang
1994	6	6		42	42	42	42	40	21	229	2	98.53%	**3**	3	0	0	Xuanguo Huang	Xingguo Xia
1993	6	6		35	42	34	39	39	26	215	1	100.00%	**6**	0	0	0	Lu Yang	Xilu Du
1992	6	6		41	42	42	38	35	42	240	1	100.00%	**6**	0	0	0	Chun Su	Zhenjun Yan
1991	6	6		42	42	31	42	42	32	231	2	98.18%	**4**	2	0	0	Yumin Huang	Hongkun Liu
1990	6	6		42	42	35	42	40	29	230	1	100.00%	**5**	1	0	0	Zun Shan	Hongkun Liu
1989	6	5	1	40	42	37	41	42	35	237	1	100.00%	**4**	2	0	0	Xiwen Ma	Zun Shan
1988	6	5	1	42	41	17	42	42	17	201	2	97.92%	**2**	4	0	0	Gengzhe Chang	Wuchang Shu
1987	6	5	1	31	42	27	38	40	22	200	8	82.93%	**2**	2	2	0	Xiangming Mei	Zonghu Qiu
1986	6	5	1	30	42	14	29	30	32	177	4	91.67%	**3**	1	1	0	Shouren Wang	Zonghu Qiu
1985	2	2		8	14	0	4	0	0	26	32	16.22%	**0**	0	1	0	Shouren Wang	Zonghu Qiu

Many outstanding young mathematics talents have emerged in China's mathematics competitors, such as Wei Zhang, Zhiwei Yun, Chenyang Xu, Yifeng Liu, etc. who have won the famous Ramanujan prize. Many scholars, such as Xinwen Zhu, Song Wang, Ruochuan Liu, Hongyu He, Simai He, Xinyi Yuan, Liang Xiao, etc., have been engaged in mathematical teaching and research at well-known universities or scientific research institutions in China and abroad, and made a great professional job. Wei Dongyi, who won the full mark gold medals of the IMO in 2008 and 2009, made good achievements in the first and second year of his postgraduate studies.

In China, 6 people won the Paul Erdös award, including Zonghu Qiu (1994), Wen-Hsien Sun (2002), Shian Leou (2008), Kar-ping Shum (2016), Bin Xiong (2018) and Gangsong Leng (2020).

2. Training Systems of Mathematics Competitions in China

Along with various mathematics competitions, extensive educational activities related to these competitions have been organized all over China and a comprehensive training system has taken shape gradually. As a result, China's performances in mathematics competitions have improved a lot and have gradually formed international influence.

2.1. Organizational forms of school-level training

School-level training is the foundation for Chinese education on competition mathematics. It mainly adopts a form of "second classroom," such as interest groups and extension courses etc., which is accessible by each grade of high school. Such training activities mainly serve as a means of popularization for mathematics competitions and a supplement for the classroom study. However, at some key schools, school-level training is highly specialized and these schools have become home bases for training high-level Mathematical Olympiad participants.

It is generally accepted that gifted education mainly consists of three basic forms: enrichment, differentiation and acceleration (Sheng and Zhou, 2010). Enrichment refers to enhanced study materials for gifted students, so as to extend their scope of knowledge. Differentiation refers to grouping students according to their capabilities with adapted courses to suit the needs of students at different levels. Acceleration means intensive materials or more information for students, so as to accelerate the pace of teaching, and let gifted students acquire the advanced knowledge system as soon as possible. The three forms of training coexist in the school-level training for mathematics competitions.

For enrichment, to be specific, schools offer mathematics extension courses with different themes that include competition mathematics, mathematical problem solving, mathematical modeling and application, history and culture of mathematics. Schools offer a broad variety of choices to students with different interests and at different levels. These are beneficial to enhancing students' mathematical understanding, improving their mathematical literacy and broadening their horizons. Moreover, some schools also organize competition mathematics groups among gifted students by conducting extension lectures or training after normal school hours. Participation in this kind of activities is voluntary and mainly for mathematics competitions.

For differentiation, at some key schools, students with a mathematical talent are assigned to a certain class to receive separate small-size class teaching. With a tradition of specializing in competition mathematics, schools provide accelerated mathematics courses in each grade. The overall teaching plan for the entire semester is mapped out at the time of admissions, and further differentiated teaching is adopted for students in the above accelerated programs. Shanghai High School, for example, has adopted differentiated teaching since 1990 and successfully trained the IMO gold medalists for consecutive years starting from 2008. Every year, the school selects around 40 freshmen who are mathematically talented to form an experimental class, and designs specific courses for them. Since 1998, it began to further select about 10 most gifted students from the experimental class to teach them in small class. Finally, individual instruction is offered to 3–4 exceptionally gifted students who have shown great potentials (Tang and Feng, 2011).

In many instances, differentiation is to accelerate the teaching process for gifted students, i.e., to let students quickly master textbook knowledge as well as the basic knowledge and skills required in mathematics competitions, so as to lay a better foundation. After this is completed, students are at liberty to further study for competition mathematics and improve their skills in solving mathematical problems.

Among the above three forms of gifted education on mathematics, enrichment mainly focuses on popularization of competition mathematics, while differentiation and acceleration serve the functions of both popularization and improvement. The latter two forms do not apply to most students so as to avoid excessive academic burdens. However, they are suitable for students who can easily master mathematical knowledge and skills without adversely affecting the study of other subjects.

For high school mathematics competitions, many schools such as Shanghai High School adopt a tutor teaching model of "$1+n$" (Tang and Feng, 2011), which is to assign a long-term mathematics teacher for a certain group of gifted students. This model consolidates the collective wisdom of the school's mathematics tutoring team and sometimes even a guest expert team. The core teacher must have intimate knowledge of every student to assess their potentials, follow closely their performances in school and formulate individualized instructional plans. He or she must also set up a specific timetable for each milestone and invite professional experts to offer special guidance at appropriate times. This model has several advantages, namely, putting equal emphasis on students' foundation and improvement, showing students how different teachers have different perspectives and thinking styles of mathematical problem solving.

School-level training normally starts from classroom teaching to the popularization and promotion of competition mathematics. This whole process can be roughly divided into four stages, i.e., basic training, thematic training, intensified training and pre-competition training (Feng, 2006). The emphasis for each stage is different.

The basic training stage aims to finish teaching the contents of high school mathematics, thus ensuring that students reach the level required for graduation. Forms and methods of teaching are similar to conventional ones, but contents are intensified and expanded to some degree, with an emphasis on basic thinking skills used in competition problems. Training at this stage enables students with a certain mathematical ability to perform well in public-oriented competitions.

The main task of the "thematic training" stage is to systematically impart knowledge and problem solving skills required in mathematics competitions. Teachers need to encourage students to form good study habits, further improve their self-learning awareness and ability, as well as keep a positive learning attitude. After this stage, students can gain a deeper understanding of various topics in competition mathematics, be able to solve difficult problems such as those in the additional test of the National Senior High School Mathematics Competition.

Intensified training is generally targeted at a few outstanding students (mainly the CMO and higher-level participants). At this stage, challenging problems can be used for exercises, to keep top students moving forward by competing and cooperating with each other.

Pre-competition training is usually aimed at preparing students for a certain competition. Well-prepared simulation tests not only enable students to familiarize themselves with forms of competition, but also reveal students' lack of knowledge and skills so as to provide reference for them and their teachers. A week or two prior to some provincial or higher level mathematics competitions, a lot of schools will organize intensive training for participants during the vacation time and after-class hours. This is also an integral part of school-level training.

Generally speaking, the first stage is closely related to the usual curriculum, while the second and the third mainly put emphasis on skill improvements, and the fourth mainly focus on competitive readiness.

2.2. Provincial and municipal-level competitions and training organizations

There are many high school mathematics competitions at provincial and municipal levels in China. Such competitions are usually hosted by mathematical societies or associations in relevant provinces and cities. In Beijing and Shanghai, these competitions have become traditional events. In some provinces, the provincial mathematics competition is reckoned as preliminary tests for the National Mathematics Competition.

In addition to hosting competitions, provincial and municipal mathematical societies also organize training programs for mathematics competitions.

Many provinces organize mathematics summer camps, and invite university professors, senior teachers and trainers to teach participating students. In Zhejiang province, for example, the Department of Mathematics of Zhejiang University and Zhejiang Mathematical Society jointly hold mathematics summer camps every year, and organize training courses at levels from college entrance exams to the National Senior High School Mathematics Competition, in order to improve students' performances in competitions and university entrance exams.

The various provincial and municipal mathematics summer camps are popular among high school mathematics enthusiasts, and often attract students from other provinces and cities to participate. These camps have turned out thousands of talented mathematics students, among which there were IMO and CMO winners as well as top candidates in the National College Entrance Examinations.

A distinctive model can be found in Shanghai — the Secondary Extracurricular Mathematics School. Thanks to the training of the Secondary Extracurricular Mathematics School, students in Shanghai have got great achievements in domestic and international mathematics competitions in the recent two decades and more. The school is hosted by the Shanghai Mathematical Society and guided by the Teaching and Research Office of the Shanghai Municipal Education Commission. Since its establishment in 1987, the school has been run for more than twenty years, and the late mathematician Chaohao Gu used to be the honorary principal of this school. Students come from the sixth to twelfth grades, with approximately 400 students from each grade, all of whom are Shanghai's students talented in mathematics. The school also invites professors from Fudan University, East China Normal University and the best local trainers for mathematics competitions. Every Sunday, the school offers 2 hours of extracurricular instruction on mathematics.

Courses offered by this school are roughly synchronized with general courses by high schools, conforming to the "enrichment" form of gifted education. Here, students not only have the opportunity to strengthen what they have already learnt in the classroom, but also to learn new knowledge, such as knowledge in elementary number theory and combinatorics, some famous theorems in Euclidean geometry, as well as mathematical problem-solving strategies and skills. Most students can quickly master them after teachers' instructions and group discussions.

Furthermore, entrusted by the Shanghai Mathematical Society, the school has also been responsible for the proposition and organization of competitions and selected some exceptionally gifted students to receive special tutoring. Since 1987, almost all national team members coming from Shanghai have experiences of training in this school.

With a combination of provincial, municipal and school-level training, both mathematically gifted students and their tutoring team are offered the opportunity for further development.

2.3. The training system for mathematics competitions

Generally, the structure of training systems for China's mathematics competitions is of "school level — provincial and municipal level — inter-provincial level" as shown in Fig. 4, balancing popularization and improvement, and covering students of different ages and learning levels. Inter-provincial mathematics competition activities consist of regional mathematics competitions and a variety of summer camps, which can generate many shared resources each year.

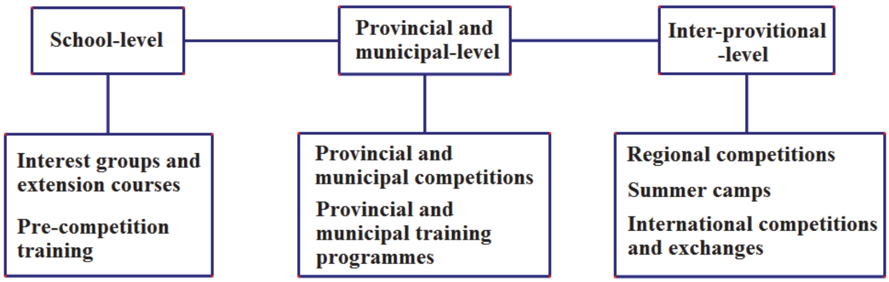

Fig. 4. The training system for China's mathematics competitions

Moreover, the training of tutors is also a part of the overall training system. The Popularization Committee of the CMS established a "China Mathematical Olympiad Tutor Rating System" in 1988. The CMS and its branches in each province organize training workshops for high school teachers who have few experiences in teaching competition mathematics, then evaluate their performances through their problem-solving skills and teaching plans. Tutors will receive some extra learning opportunities before workshops in addition to basic courses and teaching internship on mathematics education. Some normal universities also offer optional courses such as competition mathematics, problem-solving strategies and mathematical methodology to them for future developments.

2.4. Extensions of mathematics competition training

As a way of mathematics gifted education, the mathematics competition training aims at fostering future mathematicians and scientific elites in the long run. Many mathematics educators and practitioners pay attention to encouraging high school students and teachers to conduct research-oriented learning and academic communications in mathematics competition activities.

In this regard, the New Star of Mathematics (NSMATH) is an influential training project in recent years. It was founded in January 2014 by Professor Leng Gangsong, one of the winners of the 2020 Paul Erdös award. Its current official website is http://www.nsmath.cn/.

The website of NSMATH has several columns, among which the most distinctive one is "Problems for Solutions". The problem proposers are mainly high school students, coaches and young mathematicians, many of whom are former contestants, even IMO gold medalists. The selected problems are very challenging and have profound mathematical meaning, with training and research values for the students who take part in the high-level mathematics competitions at home and abroad. From the 13th issue, this column has been in the charge of Mou Xiaosheng (2008 IMO gold medalist, Ph.D. from Harvard University). The novelty and difficulty of the selected problems have been further improved, making this column more acclaimed.

"Students' Works" is another brilliant column of this website. Students contribute enthusiastically with articles full of new ideas and methods, reflecting the strong creativity of high school students. Many of their articles are concerned, discussed and carefully modified by experts and scholars. In this way students' research interests are greatly stimulated and their research ability is also improved. Nowadays students are proud to be able to publish in this column.

By the end of 2019, NSMATH has published 35 issues of "Problems for Solutions" and more than 200 articles. The website has grown into a high-quality mathematics competition network, which not only reflects the mathematics innovation ability of high school students, but also encourages students and teachers to conduct research-oriented learning in mathematics competition activities. NSMATH has also held many mathematics competitions learning camps. The learning camps have first-class tutor resources and a large number of excellent students, which ensures a high level of these activities and provides valuable opportunities for the excellent students and teachers around the country to face up to each other.

The programs such as NSMATH are of long-term significance to the development of high school mathematically gifted students in China, as well as to the improvement of coaches' ability and career development.

3. Developing Ability through Competition Mathematics

Mathematical ability refers to a relatively stable psychological characteristic for the successful completion of mathematics activities. Mathematicians, educators and psychologists at home and abroad have discussed mathematical ability from different aspects.

Krutetskii (1976) has determined the 9 key elements of mathematical ability according to the basic features of mathematical thinking:

1. The ability to formalize mathematical materials, and operate in the formal structure.
2. The ability to summarize mathematical materials.
3. The ability to operate by using numbers and other symbols.
4. The ability to use continuous and rhythmic logical reasoning.
5. The ability to shorten reasoning process.
6. The ability to reverse psychology process (ability of transferring from positive thinking to reverse thinking).
7. Flexibility in thinking.
8. Mathematical memory.
9. Spatial concept.

In the Principles and Standards for School Mathematics released by the American National Council of Teachers of Mathematics (NCTM, 2000), "problem-solving, reasoning and proof, communication, connection and representation" are defined as the five criteria of process ability in mathematics.

In 2003, it was pointed out by the Mathematics Curriculum Standard for Senior High Schools (Experimental) that when people were learning mathematics and using mathematics to solve problems, they constantly underwent thinking process such as intuitive perception, observation and discovery, analogical induction, spatial visualization, abstraction, symbolic representation, operation and problem solving, data processing, deductive reasoning, reflection and construction (Ministry of Education of the PRC, 2003). It also indicated that mathematical thinking played a unique role in the formation of rational thinking.

Competition mathematics education is beneficial to the development of gifted students' mathematical ability in various aspects.

3.1. Mathematical ability and problem solving

We may as well combine with the NCTM standards to give some explanations to the educational value of competition mathematics.

In recent years, the teaching of geometry has been weakened in the compulsory education in China. The lack of training in deductive reasoning hinders the development of students' reasoning skills, which arouses the concern of mathematicians and mathematics educators. Yet in mathematics competitions at all levels, the proportion of geometry problems remains stable, especially in the IMO of recent years, where team leaders from all countries often select some extremely difficult geometric problems as official competition problems. It is somewhat beneficial to maintain the level of reasoning and proving of those mathematically gifted.

In addition, mathematics competition problems require students to recognize and use "connections" with flexible mathematical thinking at a more advanced level. At the same time, they require students to be good at selecting, applying and converting mathematical representations, which contributes to the enhancement of students' ability in mathematical connections and mathematical representations.

Furthermore, mathematics competition problems require students to present their mathematical ideas and problem-solving processes clearly. Students should also evaluate others' mathematical thinking and problem-solving processes by communicating with teachers and classmates. These activities are beneficial for improving students' ability in mathematical communications.

More importantly, competition mathematics provides rich sources for mathematical problem solving. It is pointed out by Luogeng Hua that "the nature of mathematics competition is different from that of an exam in school, and also not the same as the university entrance exam. What we require is that students taking part in the competition can not only apply formulae and theorems, but also show their flexibility in thinking, a good understanding of mathematical principles, and the ability to use these principles to solve problems. They should even be able to discover new methods and principles to solve unfamiliar problems. Such a requirement can exactly test and train the students' mathematical ability" (Hua, 1956b, p. 1).

Example 3.1. Ten numbers $1, 2, \cdots, 10$ are written by order on the circumference initially (10 adjacent to 1). Two types of operations are permitted: (a) swap the locations of two adjacent numbers, (b) allow any two adjacent numbers to plus an integer at the same time. Is it possible to turn all numbers into 10 by finite steps of such operations?

This is a relatively easy competition problem. The authors of this article once put forward this problem to senior high school students who had not undergone competition mathematics training. The students made repeated attempts and found that it was always so close to the "goal" but failed. They hardly caught the key point of the problem. Some of them were confident that the conclusion was "impossible," but they struggled to carry out the mathematical reasoning. As a matter of fact, one might consider the invariance on the whole: No matter operating through (a) or (b), the sum of the ten numbers always maintains the original parity. Teachers often guide students in problem-solving strategy, inspire them to be aware of all-rounded considerations and to look for the quantitative relationship which remains unchanged in the operation.

Example 3.2. Given real numbers m, x, y, with $x, y > 0$ and $x + y < \pi$. Prove that

$$(m^2 - m)\sin(x+y) + m(\sin x - \sin y) + \sin y > 0.$$

This problem is somewhat difficult. The challenge lies in the complex structure (a quadratic form mixed with a trigonometric form), too many parameters (three in total), and not knowing how to use the condition $x, y > 0$ and $x + y < \pi$. Here is an illuminating solution:

Step 1. Construct a triangle ABC with $A = x,\ B = y,\ C = \pi - x - y$. Let $a = BC,\ b = CA,\ c = AB$.

Step 2. Applying the law of sines, one has

$$\frac{\sin x}{a} = \frac{\sin y}{b} = \frac{\sin(\pi - x - y)}{c} = \frac{\sin(x+y)}{c} > 0.$$

Therefore, the initial inequality can be transformed into

$$(m^2 - m)c + m(a - b) + b > 0.$$

Step 3. Since $c > 0$, it suffices to prove that the discriminant of the quadratic form $cm^2 + (a - b - c)m + b$ with respect to m is negative, which is equivalent to $(a - b - c)^2 - 4bc < 0$, or

$$a(a - b - c) + b(b - c - a) + c(c - a - b) < 0,$$

which is an easy consequence of $a < b + c,\ b < c + a,\ c < a + b$.

Throughout the above solution, the original problem has been translated and reduced step by step. The first step is an application of the idea of "construction," which allows the subsequent steps to be removed from a lot of complex calculation. The second step is due to the conditions $x, y > 0,\ x + y < \pi$ and the homogeneity of the law of sines. The third step is due to the quadratic structure of the inequality with respect to m.

When explaining problems such as Example 3.2, teachers should guide students to grasp the structure of the algebraic expression, to make associations and connections with familiar knowledge in order to simplify the original problem. A helpful tip is to apply the idea of "construction," which often enables the problem-solving process to be "simple and delicate."

Solving problems by a constructive method requires comprehensive knowledge, divergent thinking and keen intuition. The following Example 3.3, a problem quite impressive in the authors' teaching experience, is also related to the constructive method.

Example 3.3. For any given positive integer m, prove that there exist $2m+1$ positive integers a_i $(1 \le i \le 2m+1)$ making up an increasing arithmetic progression, so that the product of these integers is a perfect square (Xiong and He, 2012).

The solution can be done in a sentence: Let $a_i = ik$ $(1 \le i \le 2m+1)$, where $k = (2m+1)\,!$, so that $a_1 a_2 \cdots a_{2m+1} = (2m+1)!k^{2m+1} = (k^{m+1})^2$.

However, the brief answer above does not reflect the hidden thinking process. How can this problem be considered? In fact, one can start from an arbitrary increasing arithmetic progression with $2m+1$ positive integer terms $b_1, b_2, \cdots, b_{2m+1}$.. Note that for any positive integer k, $b_i k$ $(1 \le i \le 2m+1)$ also make up an increasing arithmetic progression. Therefore, one can freely select the value of k to satisfy the condition "product is a perfect square". For instance, one can let $k = b_1 b_2 \cdots b_{2m+1}$, and then $(kb_1)(kb_2)\cdots(kb_{2m+1}) = (k^{m+1})^2$.

In brief, we have adopted a frequently used strategy that "relax a condition and then try to re-impose it," which is typically helpful in solving a number of construction problems.

The problem is not as easy as it seems in solution. Once we set this problem to a number of senior high school students with some experience in mathematics competition. We observed that most students considered this issue from the following two perspectives: The first perspective was starting from the simple cases (for example, the case of 3 terms or 5 terms, and then tried to make a generalization). However, it would seem difficult for these students to extract the general rules. The second perspective was to set out the two basic parameters of arithmetic progression (such as the common difference d and the middle term $a = a_{m+1}$), expressed the product of

$2m+1$ terms, and then tried to solve the equation with three unknowns. We noticed that some students gazed longingly at the equation and got stuck because of the complexity of the structure. Only a few of them completed the construction within 20 minutes.

Indeed, in solving such a problem, a great deal of thinking is needed. Students should not only strive to plan their solving processes, but should always monitor their thinking to determine the feasibility of the scheme, and avoid the interference of invalid plans. Such kind of experience is beneficial for students to improve both their problem solving skills and their metacognitive monitoring strategies.

For ease of comparison with Example 3.2 and Example 3.3, the following is a list of three problems relating to the same knowledge point:

Example 3.4. Determine all possible real numbers k such that $kx^2-(k-2)x+k>0$ holds for any real number x.

Example 3.5. In triangle ABC, the circumcircle radius is $\sqrt{2}$, and

$$2\sqrt{2}\left(\sin^2 A-\sin^2 C\right)=(a-b)\sin B.$$

Find the degree of angle C and the maximum value of the area of triangle ABC.

Example 3.6. Let $\{a_n\}$ be an infinite arithmetic progression. For any positive integer n,, denote the sum of $a_1, a_2, \cdots, a_n$ by S_n.

(1) If $a_1=\dfrac{3}{2}$ and the common difference of $\{a_n\}$ is 1, find the positive integer k that meets $S_{k^2}=(S_k)^2$.

(2) Find all $\{a_n\}$ such that $S_{k^2}=(S_k)^2$ for every positive integer k.

The three problems above are selected from the book Mathematics Review Guide for College Entrance Examination (Office of High Schools of the REP, 2006). They roughly correspond to the difficult problems in classroom teaching. The knowledge and method involved in Example 3.4 and Example 3.5 are close to that of Example 3.2, but the direction in problem solving is somewhat clear. In Example 3.6, one can apply a routine method to find the answer of problem (1). The corresponding groups (a_1, d) of problem (2) can be inferred from $k=1, 2$ before a complete verification, or one can also directly write the quadratic equation of k, and solve it by the principle of polynomial identity. Both solutions require strong computational and reasoning skills, however, a specific formula and method can be applied at each step. In contrast, although no more knowledge is needed in Examples 3.2 and 3.3, the two problems are not conventional in classroom teaching and require much more thinking. Moreover, Example 3.1 only relates to the knowledge of addition and subtraction, however, it is still good material for high school students to enrich their thinking patterns.

For those who have a certain understanding of competition mathematics, most feel that even if one is able to read the methods and principles for 100 problems, one could still feel helpless when facing the 101st problem. In a sense, many mathematics competition problems are special or even unique, thus simple imitation does not always work. In fact, students should be able to transfer the thinking method and problem-solving strategy in one case to other cases by imitating and practicing, in which way they will not be at a loss when solving new problems in mathematics or even other disciplines. At the same time, they should also be able to generate new ideas and try different methods, which is in line with the basic spirit of problem solving.

3.2. Competition mathematics and teaching of open-ended mathematics problems

Teaching of open-ended problems is another major issue related to problem-solving and creativity. It was first introduced from Japan by Zaiping Dai who was a strong advocate of it. Since then, abundant theoretical achievements have been obtained in China (cf. Dai, 2002).

Roughly speaking, open-ended mathematics problems are those with non-exclusive answers. In most cases, there is only one correct answer to a high school mathematics problem; even for "a problem with several solutions," there are only a few which would be easy to arrive at. However, it seems that problems posed in higher-level mathematics competitions are more likely to be open-ended in terms of the problem-solving approach, with solutions often beyond the problem proposers' anticipation.

For example, among the four problems in the additional test of the 2013 National Senior High School Mathematics Competition, the solutions of three problems were simplified afterwards. In the 2014 National Team Selection Test, the examining committee noted some brilliant solutions after going over the answer sheets, and the given reference answers to several other problems were also simplified, which accounted for more than half of the total. In the Southeast China Mathematical Olympiad in 2014, a much simpler solution, unexpected by all experts from the examining committee, was discovered by a few of participants to unlock the most difficult problem. Historically, in the 1980 joint competition of Finland, the United Kingdom, Hungary and Sweden, the solution of a certain problem was rather complicated where mathematical induction was used four times, the Chinese translation of which took around 4,000 characters. Later Chinese experts gave some simpler solutions, the lengths of which were just a dozen lines (cf. Zhu, 2009). However, the relevant algebraic skills were deeply concealed and difficult to be discovered.

In all previous sessions of the IMO, the examining committee would present a special award to those participants who gave particularly brilliant solutions and non-trivial generalizations. To date, this special prize has been awarded to more than 40 participants.

The following is a case from the authors' personal teaching experience:

Example 3.7. Let ABC be a triangle with area S and circumcircle radius R.

(1) Prove that $S = 2R^2 \sin A \sin B \sin C.$.

(2) Prove that $S = \dfrac{R^2}{2}(\sin 2A + \sin 2B + \sin 2C)$.

This problem was designed for a test just after the completion of teaching, which was intended to test students' flexible application of the formula of triangle's area, the law of sines and the sum-to-product formula. The anticipated method was to use the formula of triangle's area $S = \dfrac{1}{2}ab\sin C \left(\text{or } S = \dfrac{abc}{4R}\right)$ and $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ (the law of sines) to finish problem (1), and further to solve problem (2) by deducting the identity $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$ in the triangle.

One of the students came up with a different approach to solve problem (2):

Denote the circumcenter of triangle ABC by O. The student first assumed that ABC was an acute triangle. In this case,

$$S_{\Delta AOB} = \frac{1}{2} OA \cdot OB \cdot \sin \angle AOB = \frac{1}{2} R^2 \sin 2C;$$

likewise, $S_{\Delta BOC} = \dfrac{1}{2}R^2 \sin 2A$, $S_{\Delta COA} = \dfrac{1}{2}R^2 \sin 2B$. Hence

$$S = S_{\Delta AOB} + S_{\Delta BOC} + S_{\Delta COA} = \frac{R^2}{2}(\sin 2A + \sin 2B + \sin 2C)..$$

This equality also holds though the area of one part is zero when ABC is a right triangle.

When ABC is an obtuse triangle, the student assumed $A > \dfrac{\pi}{2}$ without loss of generality so that $S = S_{\Delta AOB} + S_{\Delta COA} - S_{\Delta BOC}$.

Note that $\angle BOC = 2\pi - 2A$. Then

$$S_{\Delta BOC} = \frac{1}{2} R^2 \sin(2\pi - 2A) = -\frac{1}{2} R^2 \sin 2A.$$

Therefore, the conclusion could be derived by imitating the above approach.

When asked how he discovered such an ingenious proof, the student said he was inspired by the derivation of another area formula of triangle $S = S_{\Delta ABC} = \dfrac{1}{2}(a+b+c)r$, where a, b, c are the opposite sides of A, B, C, and r is the radius of the inscribed circle of triangle ABC. He thought of "calculating the total area using the sum of parts," considering that the conclusion was related to the circumcircle radius R, thus triangle ABC could naturally be divided into triangles AOB, BOC and COA.

He was also aware of discussing the situation of an obtuse triangle. After completing his reasoning, he made an analogy of this case with the derivation of the area formula $S = S_{\Delta ABC} = \frac{1}{2}(b+c-a)r_a$, where r_a is the radius of the escribed circle of triangle ABC with respect to A.

This solution showed a kind of unexpected yet reasonable artistic charm.

Definitely, many competition problems that have been studied and discussed are still highly "open in solutions." Other solutions or non-trivial relevant problems may also be obtained under the "attack" of new solution seekers (which is of great educational value although less so in academic value).

Meanwhile, there are plenty of explorative problems in competition mathematics, of which the target of problem-solvers is uncertain. As there is no ready method to follow, the solving process often requires thinking and exploration from multiple perspectives.

Therefore, although competition mathematics problems are different in content from open-ended mathematics problems, it is possible to partially achieve the effect of open-ended mathematics problem teaching for the development of gifted students' mathematical ability and creativity by taking advantage of competition mathematics problems.

4. Disputes over China's Competition Mathematics Education

Although quite a few Chinese and foreign mathematicians support and promote mathematics competitions by asserting their educational value, they also hold a very prudent attitude towards these competitions. At the very beginning of the launch of mathematics competitions in China, Luogeng Hua had such a concern: "Will this work (mathematics competition) affect our school education negatively? Will it affect the students' overall development? It might happen if the job is not done properly (Hua, 1956a, p. 2)." Nowadays the involvement of the "general public" and very young children in mathematics competitions have been disputed continuously, which echo Professor Hua's concerns to a certain extent.

In the respect of universalization of competition mathematics education, opinions are widely divided among Chinese scholars. Some argue that competition mathematics education should be oriented to the entire high school student population in order to inspire creative thinking and cultivate problem-solving ability. On the other hand, others argue that competition mathematics education should only apply to a small group of students (e.g., 5%) and should have the scale of their effects controlled. To use an analogy, sports for the general public are for keeping fit, whereas sports practiced by athletes aim to achieve excellency and breakthroughs. Likewise, in terms of the function of popularization, competition mathematics can be targeted at a large population of students. But in terms of talent identification and selection, competition mathematics is only suitable for a small group of students with a strong interest and aptitude for mathematics. Krutetskii (1976) studied three groups of students with different mathematical abilities. Analysis showed that students with mediocre mathematical ability needed more time and efforts than students with strong ability to make mathematical achievements; they tended to feel very troubled when solving new types of mathematical problems and needed assistance to understand general methods; only through rote practice is it possible for them to shorten the reasoning process. Students with strong mathematical ability could work intensively on mathematical activities for a long period of time, without exhibiting any tiredness. On the contrary, mathematically-weak students were prone to feeling more tired after a short period of time in studying mathematics than in studying other subjects. Thus, it can be concluded that it is against the interests of education to require the involvement of too many students in high-level mathematical training.

With regard to the involvement of younger children in mathematics competitions, actually younger-grade award winners are by no means rare in the IMO of previous years. One example is the Fields Medal winner Terence Tao who obtained his IMO gold medal when he was just 12 years old. The launch of mathematics competitions is instrumental for discovering the mathematical gift of such child prodigies. As the eminent mathematician Kolmogorov wrote in his preface to the book 1st -50th Moscow Mathematical Olympiad (Гальперин and Толпыго, 1986): “At the very beginning, the Moscow Mathematical Olympiad was only for Grade 9 and 10 students; from 1940, it started to invite students of Grade 7 and 8. This change in the age of the participants was because students of such grades had already started to show their interest and talent in mathematics.” However, the involvement of overly young students has shortcomings and disadvantages. Kolmogorov further expressed that, “Although it is possible to invite even younger competitors, we could not help but notice that most of these Grade 5 and 6 students who had solved competition problems later lost their problem-solving ability and interest in mathematics as they progressed to higher grades.” It shows that Kolmogorov had his reservations about the involvement of younger participants. According to the research of mathematicians and psychologists, such as V. A. Krutetskii’s tracking study on the 26 gifted children in mathematics (Krutetskii, 1976), M. A. Clements’ case study on the child Terence Tao (Clements, 1984), to name a few, talent in mathematics is formed in early childhood, and gifted children demonstrate a remarkable talent for mathematics and learning speed. So, how to make such students develop their mathematical ability while still maintaining their interest in mathematics is an issue that deserves a great attention when conducting mathematics competitions.

In recent years, as more and younger children are attending competition mathematics lessons, a substantial improvement in problem solving abilities among younger participants has been observed. On the other hand, the difficulty and complexity of the problems have increased and are sometimes out of touch with the classroom teaching. Take the first item in the 2014 National Junior High School Mathematics Competition for example (cf. Xu, 2015):

Example 4.1. Let x, y be integers such that

$$\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)=-\frac{2}{3}\left(\frac{1}{x^4}-\frac{1}{y^4}\right).$$

Then the number of possible values of $x+y$ is (   ).

(A) 1 (B) 2 (C) 3 (D) 4

The essence of the problem is to find out, by observation, that $\frac{1}{x}+\frac{1}{y}$, $\frac{1}{x^2}+\frac{1}{y^2}$ on the left side of the equation are 2 factors of $\frac{1}{x^4}-\frac{1}{y^4}$ on the right side, thus the problem could be simplified; yet after that, a quadratic indeterminate equation should be solved, whether x, y are zero in value also be considered and the case in $\frac{1}{x}+\frac{1}{y}=0$ cannot be neglected. There are too many details to be considered. Moreover, the validity of the option set is also questionable. As the first item in a test, would it not be more advisable to lower the difficulty and try to design problems with the purpose of promoting in-class instruction?

There is an opinion that "when the new contents that appears in mathematics competitions are familiar to and within the grasp of high school students and teachers, the competitions would then have successfully fulfilled their Olympiad mission — to integrate into high school mathematics, or in other words, to have popularized and disseminated mathematical knowledge (Chen and Zhang, 2013, p. 15)." Actually, two challenges are behind the successful popularization. For one, more and more extracurricular contents will be integrated into in-class instruction, resulting in excessive information for students to learn, thus adding to their burden in schoolwork, thereby raising the problem of controlling this tendency. For another, as the contents of competition mathematics become increasingly familiar with the general public, the function of selecting talents may be lost as it becomes more difficult to differentiate between participants, so the cooperation of experts in various fields including mathematicians is required in order to prepare the test problems conscientiously, to ensure the novelty and validity in the problems given. In competition mathematics, we should try to avoid proposing those complicated problems in such fields that are known to all or systematically studied.

In conclusion, China's competition mathematics education serves as a supplement and enhancement for in-class instruction and is also a means to identify and cultivate mathematically gifted students. However, many practices and explorations in the long run are essential for kicking a balance between popularization and selection, as well as the public education and gifted education.

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https://doi.org/10.1142/9789811287183_0049

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Automatic Reports to Support Students with Inquiry Learning: Initial Steps in the Development of Content Specific Learning Analytics

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ABSTRACT When students are asked to examine their understanding individually or in small groups, information can become part of a feedback process that supports students' learning. As designers of technology to support learning, we are interested in supporting such feedback processes in the context of guided inquiry instruction. This paper explores the potential of automatically associating mathematical descriptions with student submissions created with interactive diagrams. The paper focuses on the feedback processes that occur when students use the descriptions provided by the technology as resources for reflection and learning. We discuss the design of personal feedback processes where students reflect on and communicate their own learning, utilizing individually-reported multi-dimensional automatic analysis of their submissions in response to example-eliciting tasks. While there is much research and development work to be done, we consider mathematical descriptions of student work as an important contribution to broader developments in learning analytics.

Keywords: Inquiry learning; Feedback; Learning analytics; Technological supports for learning; Example eliciting tasks.

1. Introduction

In this chapter, we suggest that automated descriptions of student work are a new strategy that can be designed into technology for providing students with support for inquiry learning. We illustrate this strategy with a set of tasks designed in the STEP platform (Olsher et al., 2016) that are intended to support students in developing conjectures about the intersection points of perpendicular bisectors to the sides of quadrilaterals. We argue that these tasks have the potential to support reflection on commonly used classifications of geometric shapes for which a conjecture holds.

The chapter is organized in five sections. We begin with two sections that examine key aspects of relevant literature about using technology to support student inquiry

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learning in mathematics and about feedback processes in mathematics classrooms. The next two sections focus on the strategy of describing students' mathematical work in words as a way to provide students with feedback on their work, and then illustrate this new strategy in the context of a particular set of tasks on perpendicular bisectors of the sides of quadrilaterals. In this illustration, we describe the kinds of characteristics that are used to support feedback processes, as well as how the reporting on those characteristics is organized. We conclude with thoughts about how describing students' work mathematically can contribute to the growing field of learning analytics and about directions towards the design and use of reports in other types of learning settings and at other stages of inquiry learning.

2. Coordination of Examples and Concepts in the Context of Inquiry Learning

We view inquiry learning in mathematics as centrally involving the coordination of examples and of concepts, where the definition of a concept identifies criteria for classifying instances as examples or non-examples of that concept. In taking this stance, we follow Hershkowitz's theory of fundamental concepts with its focus on definitions and examples (Hershkowitz, 1990; Tall and Vinner, 1981; Vinner, 1991). Therefore, by concept, we refer to a combination of characteristics; by concept definition, to the minimal combination of critical characteristics (necessary and sufficient) to define the concept; and, by concept image, to the collection of examples and the derived properties reflected in students' work. Naftaliev and Hershkowitz (2021) stress as an important implication of their study of concept construction, that a learning trajectory, in which learners should focus and examine the relations among definition, examples, and critical characteristics (which these authors refer to as attributes) will reduce the use of prototypical examples (Presmeg, 1992) and strengthen the coherence between concept images and concepts definitions held by learners.

Given this perspective, example-eliciting-tasks (EETs) are an important component of our approach to guided inquiry with a digital environment (Yerushalmy et al., 2017). Having students construct examples of a concept, or having students interact with carefully designed repertoires of examples, are important kinds of activity for the development of mathematical ideas and concepts and can provide a window into the nature of learners' understanding (Zaslavsky and Zodik, 2014). These scholars further suggest that example generation can also be a catalyst for enhancing students' understanding and expanding students' concept image. However, each particular example is limited in what it may manifest about students' understandings. To overcome this obstacle and to gain insight into the breadth of students' concept image and the nature of their concept definition, as is illustrated in Section 4, tasks can prompt students to produce multiple submissions that are as different as possible from one another.

The discussion in this section has not yet focused on learning goals and on how learning goals shape inquiry learning in school environments. As suggested by the didactical contract (Brousseau, 1997), meaningful interactions that are part of learning in school with technology-based tools involve meeting learning goals. When technology is designed for experimentation, exemplifying, conjecturing, and arguing, it can be an important component in creating environments where situations of inquiry may occur, but there are other layers to consider, including the nature of the feedback processes involving teacher, student, and tool, that are necessary to create educative experiences.

3. Feedback in Technology Supported Mathematics Instruction

The traditional view of feedback in instruction emphasizes the role of verbal feedback by a teacher at a stage when the learner has already finished a task, or a part of the task, and the information provided assists the learner toward the next learning goal. This feedback usually focuses on evaluation, and it is often followed by verbal explanations. When students perform a task well, there is not much to the feedback process, beyond the acknowledgment of a job well done.

This type of attitude is apparent in early forms of technology supported assessment that were concerned with the need for efficiency. Yet, warning flags were raised early on. The case of Benny by Erlwanger (1973) describes a student using the Individually Prescribed Instruction (IPI) mathematics curriculum, which provided students with automated adaptive feedback about their individual progress. This case study has become a classic example of the possible effects of programmed automated feedback. Erlwanger and the many reflections and studies on independent programmed learning have pointed to the possible wrong conceptions students can develop as a result of automated feedback that later might have damaging effects on what students think about the concept and about the logic of mathematics. A main concern is with the assumption that assessment of a sequence of performances focused on evaluation without elaboration could be sufficient as constructive feedback for learners. Students may focus on reaching a correct answer without understanding why it is correct, or worse, try to get what is "correct for the teacher," which is not necessarily aligned with what the student thinks is mathematically correct.

Despite these potential pitfalls, automated assessment remains commonly used in association with multiple-choice questions (MCQ). When well-constructed, MCQs present distractors based on experience or research about student learning of the topic. Nevertheless, students might use answering strategies that do not require actually solving the MCQ, such as guessing or validating the various possible answers, moving away from what was meant to be assessed (Sangwin, 2013). Our information-rich world is moving toward extended types of mathematical competence, which require assessment that does not rely on proxies, such as multiple-choice questions, but assesses student competence directly (Stacey and William, 2012).

In the literature on learning with technology, feedback is often used to describe the information that technology presents regarding aspects of a learner's performance or understanding. These aspects may include corrective information, an alternative strategy, information to clarify ideas, encouragement, or simply the correct answer. Generalizing the types of information provided, Sadler (1989) noted that feedback needs to provide information specifically relating to the task or process of learning that fills a gap between what is understood and what is aimed to be understood. Hattie and Timperley (2007) conceptualized feedback to be "[I]nformation provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (p. 81).

Shifting from teaching-centered processes to learning-centered inquiry has implications, in particular, for the meaning and quality of feedback processes. In an environment in which self-reflections, dialogic interactions, and whole-class discussions are the means for learning, we need to consider the quality of the whole process of feedback, including the quality of the teacher's contributions, the role of technology and the active role of students. Considering the feedback process as involving only information or comments given by the teacher is not helpful. Carless (2015) described feedback in the classroom as "a dialogic process in which learners make sense of information from varied sources and use it to enhance the quality of their work or learning strategies" (p. 192). We are interested in conceptualizing feedback in a similar way, as an ongoing process. In the literature we cite, the term "feedback" is used both to describe processes in the classroom and to identify the information that technology can present for use as part of such processes. Although we conceptualize feedback as a process, to be true to our citations, we use 'feedback' in these two different ways.

We focus on personal feedback processes, exploring the uses of technological design to support students' reflection on their mathematical understandings, and addressing the key question" *What makes information part of an effective feedback process*?" to guide our thinking about automatically analyzed student work as a resource for learning.

A common practice in the study of feedback processes is to examine whether and how feedback helps students close the gap between their current and expected performance. The literature assumes that feedback is a process that "counts" only when it makes a difference to what students do, and that the information communicated to learners is intended to modify their thinking or actions for the purpose of improving learning (Shute, 2008). Yet, studies agree that the perceived effectiveness of feedback is inextricably dependent on the goals of instruction, which may remain tacit. Thus, the effectiveness of feedback is highly contextual. Researchers have reached conclusions similar to Sadler's (2010), that "feedback is capable of making a difference to learning, but the mere provision of feedback does not necessarily lead to improvement… the general picture is that the relationship between its form, timing and effectiveness is complex and variable, with no magic formulas" (p. 536).

4. Automatic Mathematical Characterizations: A Different Sort of Intellectual Mirroring Strategy

In this section, we discuss the potential affordances for students' learning of technology that automatically associates textual descriptions of mathematical characteristics with students' submissions. We do so by reflecting on how the identification of predetermined characteristics provided as linguistic resources has the potential to make a new contribution to supporting individual student learning in the context of student inquiry. We consider the potential and pitfalls of including automatic characterization of student examples in words as a way for students to interact with points of view that may be different from their point of view while engaging in inquiry. We argue that through reports constructed for individual learners, such communication could be useful, enabling students to further their learning by reflecting on their understanding. We now describe the strategy for providing students with information about their work: describing students' submissions using mathematical properties.

Our illustrations of this strategy involve the use of STEP (Olsher et al., 2016), an online platform designed to support teachers' work in assessing various open-ended example-eliciting tasks, and to support technology-enhanced didactical situations that involve learning processes with non-judgmental individual feedback. The platform was designed to support evidence-based formative assessment practices (Mislevy, 2017) that go beyond whether or not students' work is correct. The tasks include interactive diagrams (applets) in GeoGebra (Hohenwarter et al., 2009), and usually prompt students to submit examples and non-examples to support or contradict a mathematical claim, or to create examples under given constraints (Yerushalmy, 2020). In this section, we focus on how STEP is designed to provide students information that describes their work back to them. The theory of conceptions of fundamental concepts (Tall and Vinner,1981; Hershkowitz, 1990) is at the core of the way in which STEP carries out automatic analysis of students' submissions.

With the STEP platform, potential characterizations of student work are programmed into tasks and then are automatically associated with student work by the machine. Task designers can make these linguistic descriptions of the mathematical characteristics of student examples available to learners throughout their work process with an interactive diagram (Naftaliev and Yerushalmy, 2017), as well as a report after they have submitted their work. Students must learn through interaction with the words used to describe their work, to appreciate the information provided to them as a resource for their inquiry efforts. In particular, in the context of conjecturing, students can explore the implications of having shared sets of characteristics in the examples that they submit; when used in this way, characteristics can be considered as conditions that a set of examples meet and that may influence results. We have been exploring this strategy through design research studies focused on particular characteristics for particular tasks (For a description of this process in the context of a particular task, see Harel and Yerushalmy, 2021).

We conceptualize the strategy of automatically characterizing student work as a departure from current technology supported feedback practices in two ways. First, providing mathematical characterizations of student work is an alternative to the information more commonly provided to learners in standard online learning platforms: evaluation of the correctness of their responses.

Second, in the context of technology supported inquiry learning, information is often presented through multiple linked representations where a user's action in one representation is reflected in another. The phrase "intellectual mirror," coined by Schwartz (1989), articulates the essence of this strategy for supporting technology-based inquiry: linked representations support a process of self-reflection where the implications of actions taken in one representation are reflected back to the user as in a mirror.

The information offered by STEP can operate like the intellectual mirror described by Schwartz in that a user receives feedback on actions that they take and can reflect on what STEP offers. On the other hand, the kind of mirroring that STEP provides is different from the mirroring provided by multiple linked representations (MLR). While MLRs are mute and do not speak, STEP uses words to characterize, rather than evaluate, student work. In this sense, STEP, like the mirror in the Snow-White tale, speaks. This sort of mirror has a perspective that a designer has developed, and offers that perspective on learners' submissions (whether or not students like what they hear).

When information about how students' work relates to a task is provided to learners while they are working, there can be a feedback process that is like providing hints or clues. The information provided by STEP is not evaluative and does not attempt to bring the student closer to a predetermined solution, but enhances students' potential interaction with key characteristics of the task and in that way can support a more compatible solution based on what is required in the task.

However, STEP offers an additional perspective or voice, the mathematical perspective of the designer, that may be in conflict with students' own understandings of the words that STEP offers (Yerushalmy et al., 2022). When STEP offers its characterizations, there can be conflicts between the usage of the student and the usage of the software; and as a result, a student may feel that the software's characterization is mistaken (Rezat et al., 2021). Thus, feedback processes with STEP may involve reflection that is generated by commognitive conflicts of the sort Sfard (2007) identifies between classroom participants.

While STEP does not evaluate student work, once students have completed their work, the information provided by STEP can also be used for evaluative processes by communicating the degree to which the student's work meets requirements of a task. Although this is a well-defined function for closed tasks, it is not always easy to define an algorithm when tasks are more open. Of particular interest is when a task asks that students submit more than one example and that those examples be different from another. Such tasks are meant to support students' development of broad personal example spaces by asking that there be a variety of examples in their submission for an example-eliciting task. To support student work with such tasks, the characteristics that STEP speaks of should enable students to compare and distinguish between their examples, suggesting mathematical descriptions or contextual descriptions of mathematical phenomena. Mathematical characteristics that allow students to distinguish between their examples define a relevant domain to start a meaningful inquiry process that goes beyond mere trial and error.

5. Illustration of How the Design of STEP Supports Students in Feedback Process

To illustrate how mathematical characterization of student work can be a strategy for supporting inquiry learning, we now turn to an activity in the STEP platform in which students use an interactive diagram to submit examples of quadrilaterals whose perpendicular bisectors meet in a point, as well as examples of quadrilaterals whose perpendicular bisectors do not meet in a single point. To organize this illustration of mathematical characterization of student work, we present the following subsections:

- We begin with a pedagogical challenge related to conjecturing and provide a rationale for an inquiry-based set of tasks about the shapes formed by the intersections of perpendicular bisectors of a quadrilateral.
- Next, in the context of these learning goals, we explore how to characterize students' submissions mathematically in an effort to support students in their inquiry learning.
- We then present three STEP tasks and provide a rationale for the design of the tasks. These tasks depend on the same interactive diagram which provides students with some initial automatically provided information for their immediate use.
- Next, we illustrate how automatic characterization can be used by STEP to support student inquiry after students have submitted responses to these tasks. Each student receives a report on their submission that can be used for subsequent classroom activity.
- Finally, we describe how classroom activity can build on the information provided to students in the reports.

5.1. A pedagogical challenge when designing to support conjecturing

A traditional high school geometry problem to prove reads: The four perpendicular bisectors in any quadrilateral where the sum of opposite angles is 180 intersect at a point. Alternatively, the problem to prove can be stated as: *A convex quadrilateral is cyclic if and only if the four perpendicular bisectors to the sides are concurrent.* Soldano and Luz (2018) studied this version of the problem as part of their approach of using dynamic construction (of a dynamic quadrilateral that can only be dragged at a point while the other three points are circumscribed), as the basis for generating sets of examples and non-examples that eventually lead to conjectures about cyclic quadrilaterals and why their perpendicular bisectors meet at a point. Marrades and Gutierrez (2000), also using a dynamic environment that allows to drag a single vertex of a quadrilateral, used this task to study types of justifications that high-school students offer when working on a given construction and the statement "A, B, and C are three fixed points. *What conditions have to be satisfied by point D for the perpendicular bisectors to the sides of ABCD to meet in a single point?"*

The results of these two studies suggest a more general challenge related to conjecturing: Students often face challenges understanding that the geometric shapes for which a conjecture holds need not always be a set of shapes that has a commonly used name. In addition, the names used for sets of shapes do not always identify the critical characteristics that are relevant to a particular conjecture. This has implications for students' efforts to prove and argue about the truth of a claim; students may not identify the critical conditions that are relevant to the desired conclusions (Haj-Yayha, 2020).

With the tasks and feedback processes that we illustrate in this section, we do not seek to help students develop a proof. Rather, we attempt to organize a meaningful autonomous inquiry experience for students that help them prepare for a teacher-guided whole class (or a group) discussion and will help them come to that discussion with an inkling that the set of quadrilaterals for which the perpendicular bisectors meet at a point (the "if" part of the conjecture) is not one of the standardly named sets of quadrilateral (e.g. trapezoids, parallelograms, …).

We accomplish this goal by stating the task as an example-eliciting-task for exploration. We ask students to create different examples, in which the perpendicular bisectors to the sides of ABCD meet in a single point. The exploration is based on dragging the four vertices of a quadrilateral of the construction in a GeoGebra-based interactive diagram which is unconstrained (any vertex of a quadrilateral can be dragged anywhere).

5.2. Characteristics for describing the conjecture

As part of authoring a STEP task, designers provide the platform with direction about the characteristics of student submissions to be checked and associated with examples. When a conjecture is focal, there are two types of characteristics to consider. One set of characteristics of the interactive diagrams are associated with the *if* part, and the other with the *then* part of a conjecture.

In the case of the tasks described here, the desired outcome for the interactive diagram, the then part of the conjecture, is that all 4 of the perpendicular bisectors of the side meet in a point. One could make other choices for the *then* part of the conjecture, like for example describing the geometric shape created by the points of intersection of the angle bisectors.

The second set of characteristics to consider is the characteristics of the quadrilateral in the interactive diagram on whose sides the perpendicular bisectors are constructed, an element of the *if* part of the statement of a conjecture. For use as support for having students consider the *if* component of the conjecture, we considered three competing sets of characteristics from which to choose, one focused on angles and relationships between angles, another on the lengths of sides and one on prototypical types of quadrilaterals. These three choices each involve linguistic terms that may help students to formulate conjectures about the quadrilaterals for which the perpendicular bisectors of the sides meet in one point. To summarize the choice, we faced as designers of this activity, we developed the following list of characteristics:

Tab.1. Three sets of characteristics for describing the quadrilaterals in student submissions

Types of quadrilaterals	Angle relationships	Side relationships
Parallelogram	All angles equal	All sides equal
Trapezoid	No angles equal	Two pairs of equal sides
Kite	Two pairs of equal angles	One pair of equal side
Rectangle	Sum of adjacent is 180	One pair of opposite equal sides
Square	Sum of opposite is 180	No sides are congruent.
Rhombus	More than two angles equal	

These characteristics are candidates for use to support the identification and classification of examples and non-examples, not only by referring to types of quadrilaterals, but by identifying the characteristics of quadrilaterals for which the perpendicular bisectors intersect, using side and angle relations. These relations may guide the exploration of quadrilaterals to focus on the use of critical characteristics as the if part of the conjecture statement.

5.3. Three STEP tasks using one interactive diagram

Task 1 is stated as follows: "A, B, C and D make quadrilateral ABCD (see Fig. 1). They are all dynamic and can be dragged. If it is possible, create 3 examples that are as different as possible from each other, in which the perpendicular bisectors to the sides of ABCD to meet in a single point."

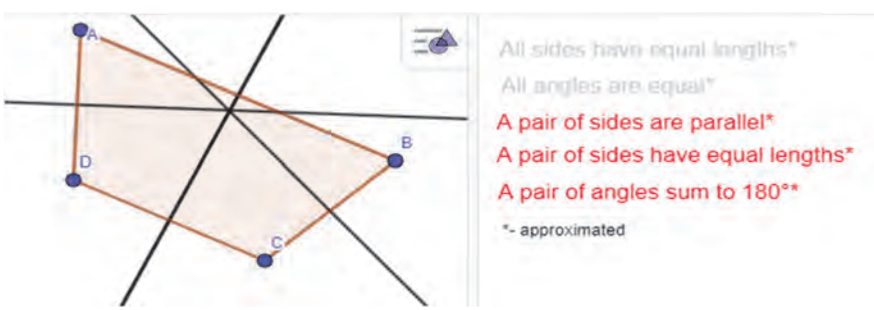

Fig. 1. An example for task 1 constructed with an interactive diagram in STEP

The main reason that this exploration in this task did not include measurements was to enable the students to identify qualitatively the possible variation between submitted examples. Students were encouraged to explore various situations before they decided which state of the interactive diagram they would like to include in their submission. STEP allows them to change their decisions before submitting.

Task 1 was for the students to create different examples and explore possible shapes that meet the condition. On the right-hand side of the screen, the GeoGebra interactive diagram shown in Fig. 1 provides characterization in words of the quadrilateral ABCD.

Of the 11 characteristics of sides and angles we listed in Table 1, we decided to provide three characteristics related to sides and two related to angles. Our decision was influenced by four pedagogically grounded considerations. First, we wanted to allow students to be able to explore familiar quadrilaterals. These characteristics allow students to recognize quadrilaterals by relations between the measures of their sides and of their angles. Second, the interactive list does not include the shapes names themselves, so recognizing a shape requires attending to the properties that characterize the shape (e.g. watching the diagram one may realize that there is a single intersection point when the shape looks like a rectangle, but one recognizes this by focusing on characteristics of pairs of sides and angles). Third, the characteristics we chose include those that may appear in examples but are not critical ones (e.g., 4 equal angles). And fourth, none of the five directly lead to the correct general answer (e.g., a pair of angles sum to 180 could refer to adjacent angles and thus is not sufficient nor necessary for the claim to be true.)

Task 2, using the same interactive diagram, engages students in creating a personal example space of examples and non-examples. As opposed to Task 1, Task 2 has students begin to think about non-examples and how to distinguish examples and non-examples focusing on the critical characteristics. It directs students' attention to the characteristics provided in the interactive diagram and asks students to think about subsets of those conditions and whether or not there are examples and non-examples for states of the interactive diagram that fit the same subset of the list of characteristics.

The task is an existential EET that requires finding a subset for which they can find an example and a non-example. In Fig. 2 (on the next page), the student has chosen a subset that includes three characteristics; the quadrilateral on the left is an isosceles trapezoid where two opposite angles sum to a 180 and it exemplifies the claim. The quadrilateral on the right is a right angle trapezoid and the lines intersect in 4 points. This state of the interactive diagram is a non-example as it does not fulfill the *then* part of the claim.

Task 3 requires an answer to a universal question that would require convincing arguments to why all quadrilaterals with those characteristics will demonstrate the truth of the claim: *Choose another subset from the same 5 characteristics, a subset for which you believe there can only be examples. Submit two examples and explain why you think that any quadrilaterals with these characteristics must have perpendicular bisectors that meet at a point.*

Fig. 3 offers two examples that are different in their position, but both look like rectangles (approximately). Any quadrilateral that is characterized by the condition "all angles are equal" will also be characterized by having a pair of parallel sides, a pair of equal length sides and a pair of angles that sum to 180 degrees. Under these conditions the four lines will meet at a point.

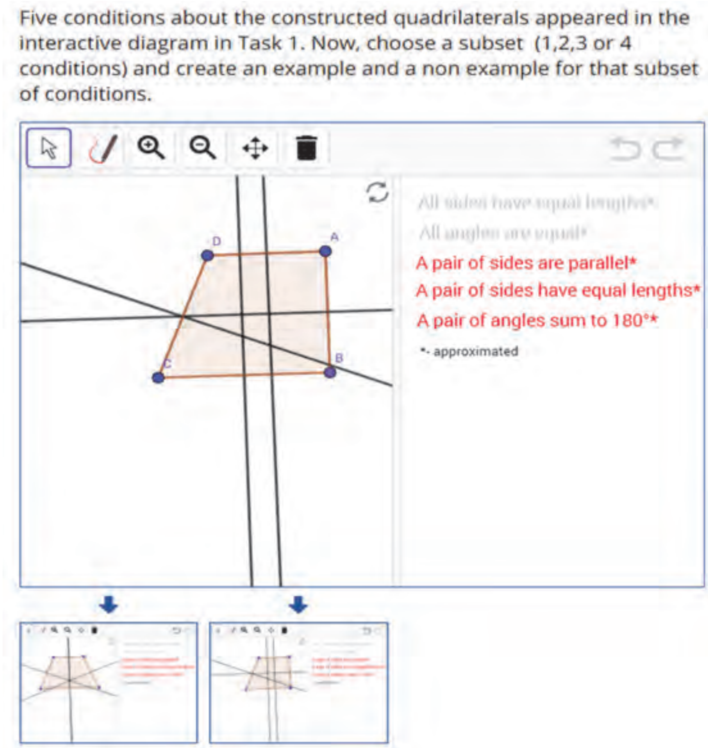

Fig. 2. An example and non-example in Task 2 for a set of three characteristics

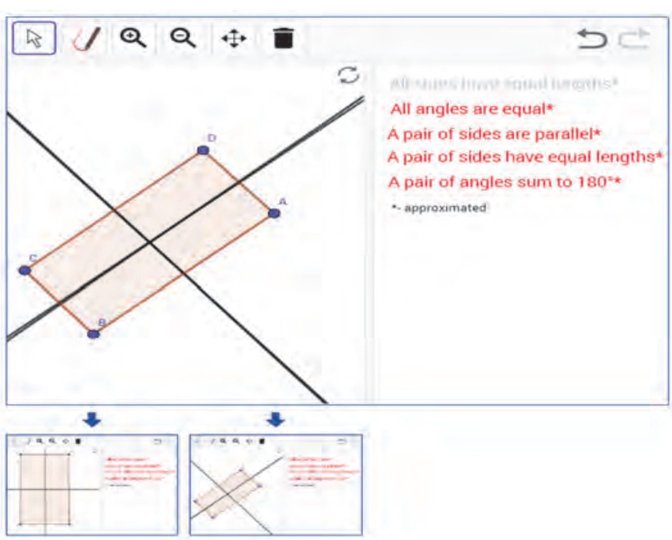

Fig. 3. Two examples in Task 3 that meet the same 4 characteristics

During the exploration phase, whenever an element of the quadrilateral has a particular characteristic, GeoGebra highlights the relevant text on the list. In this way, students can begin to think about characteristics of the quadrilaterals they are creating and whether particular conditions or sets of conditions consistently only produce examples (e.g., Fig. 3 seems to suggest that the perpendicular bisectors of all rectangles meet in one point).

5.4. Supporting inquiry by presenting post-submission mathematical characterizations to students

To provide students with support for understanding their personal example space, part of the task design process in STEP includes *a priori* definitions of mathematical characteristics of submissions that appear in the students' submitted examples and could provide useful feedback to them. Thus, in addition to information provided during exploration, students' submissions are stored and automatically analyzed by STEP (on a larger set of conditions than what appears in the interactive diagram) to produce post-submission individual reports for the students (hereinafter, post-submission report) (Olsher et al., 2016). Figure 4 presents a sample post-submission report.

The post-submission report includes three parts: Part 1 addresses the relationship between a student submission and the requirements of the task. For example, in the student submission for Task 3 (as represented in Fig. 3), the top part of the report addresses two questions: Do the 4 perpendicular bisectors intersect at a point for both examples and are both examples characterized by the same subset of characteristics? Information about which subset of conditions is held in common is found in the submitted states of the interactive diagram. Part 1 helps the student to know whether their submission meets the task requirements and will thus help indicate how to interpret the remaining parts of the report.

Part 2 of the report supports a comparative view across submissions. It includes the submitted states of the interactive diagram with the characteristics found for each (multiple examples as in Task 3 or examples and non-examples as in Task 2).

The characteristics in Part 3 (appearing below each of the submitted diagrams) offer another set of characteristics that has not been introduced to the student so far. The use of the set of shapes that have common names is designed to challenge the common attention to prototypical quadrilaterals. Such information goes beyond whether students' submissions are right or wrong, have the potential to challenge the students' current perspectives, and can be used to describe where in the space of possible subsets of characteristics and the space of examples and non-examples the submission resides.

In Fig. 4, both states of the interactive diagram fulfill the task requirements of Task 2 as (i) they constrain by the same chosen subset of 3 characteristics and (ii) one is an example where under these characteristics the perpendicular bisectors meet at a point and the second is a non-example (There are four points of intersection). Finally, the

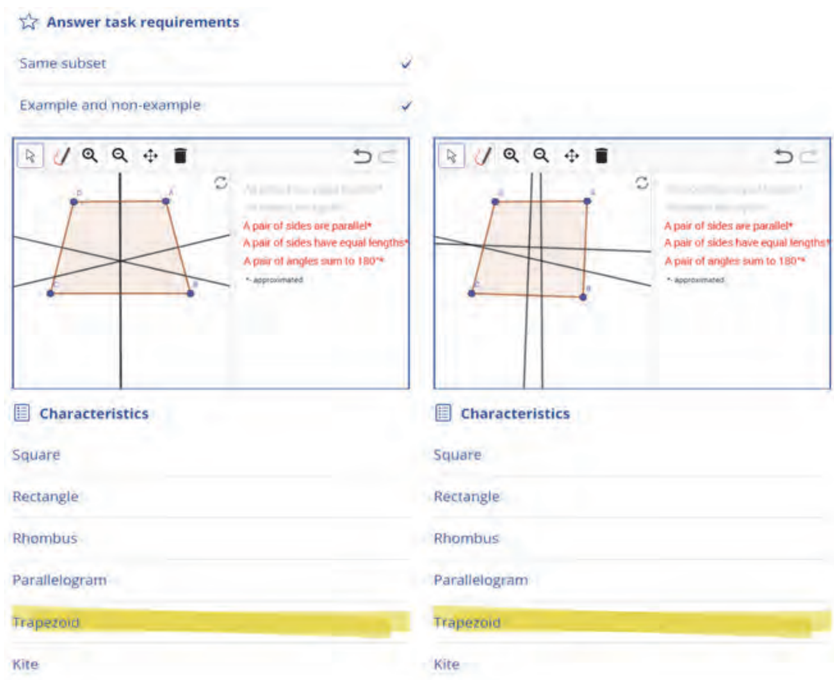

Fig. 4. Post submission information reported to the student's submissions to task 2

quadrilaterals submitted with these characteristics are both examples of the same shape: trapezoids.

5.5. Working with a class when each student has a post-submission report

Group discussion, in their small groups or whole group settings, is at the core of inquiry based learning. Such discussions, done in small groups following the personal submission and getting the personal information, have potential to advance learning. Olsher (2022) describes the construction of a dialogic space by pairs of students who were working on the same task in STEP. Whether working as pairs or small size groups participants may discuss the various choices of subsets of characteristics and the resulting shapes. This would lead to discussions of critical and non-critical characteristics of the quadrilaterals in this task and the effects of the choice of the subset of characteristics on the space of examples and non-examples (similar to the group discussions described by Naftaliev and Hershkowitz, 2021).

A whole class discussion can be based on the information that both the teacher and the students receive from STEP. The teacher can already be familiar with students' submissions — answers, frequent mistakes, characteristics that dominate the choice of subset etc. The students arriving prepared for the lesson with ideas and possible conjectures to discuss based on their examples and personal or group reflection on their post submission reports. The discussion then takes the form of a meta-feedback process based on the rich information that each of the students and the teacher have. For illustrative purposes, ideas that might be raised in such discussions might include:

1. What do the two sets of characteristics appearing in the report tell us? What are the ramifications of characterizing examples and non-examples of quadrilaterals by familiar names? What other information might you suggest to be reported? For what purposes?
   Such collective analysis, similar to the one demonstrated in Olsher (2019), might lead to a discussion about the limitations of thinking about quadrilaterals only by the familiar names and to analyze the distinctions between critical and non-critical characteristics.
2. Do you find any of the characteristics helpful in answering related questions? If usually the intersections of the perpendicular bisectors create a quadrilateral, what can be said about relations between the dragged quadrilateral and the one created by the intersections (Schwartz and Yerushalmy, 1987)? Why and when does the shape collapse into a single point? And, what determines where the point of intersection: When is it outside, inside, or on the perimeter of the quadrilateral?

6. Conclusion

In this chapter, we have illustrated a new strategy for feedback processes involving teacher, student, and tool during inquiry learning. We illustrated this new strategy in the context of a particular task related to the learning of geometry that has both specific learning goals related to the content of this task (the intersection of the perpendicular bisectors of a quadrilateral) and to what it means to develop a conjecture (that the set of mathematical objects for which a conjecture holds need not be a specific named set of objects, like square, but instead can be described by their characteristics, quadrilaterals that have an opposite pair of angles that sum to 180 degrees).

As this illustration suggests, the strategy of characterizing student submissions with mathematical descriptions points the field of learning analytics in two potentially useful directions. First, the illustration suggests that the data used in learning analytics can be content-specific and directly related to learning goals (to complement other sorts of measures like time on task, correctness, and more). Second, we also are intrigued by the potential of engaging students in analysis of their own learning. We have illustrated how information can be shared directly with students and can help shape their learning both in the midst of doing their work, as well as during reflection on the work they have submitted.

We also think there is still important development work to do as well as we learn to carry out this strategy. As designers, we continue to be interested in improving the nature of the feedback that STEP provides to support student inquiry. We close by outlining three future directions we have begun exploring and implementing in STEP. We are interested in providing other kinds of reports for students at a different stage of the inquiry process, and we are eager to explore the use of other types of learning settings and tools that can support students' learning with the reported information through meta-cognitive self-reflection processes.

One direction we intend to follow is to design means for learning through an activity that includes different tasks. To do so, we are designing an activity report. This report attempts to articulate and characterize changes across related tasks and in this way help students deepen their mathematical discourse about the activity. We envision the activity reports as tools that will help students communicate the progress they have made in developing their reasoning skills and strategies of inquiry beyond working on one task, but rather on a series of tasks. The report would include more explicit indications of changes with regard to the key aspects of the problem (e.g., "more directions as activity progressed", "fewer speed changes as activity progressed"). We will study whether it is helpful for students to follow new elements that were incorporated as they moved along in the activity. Another characterization we are using now on teacher reports and analytics is descriptive statistics (e.g., the number of students who had a certain characteristic in their submissions, Abu-Raya and Olsher, 2021). These numeric results can also be communicated interactively upon the student's request by providing ways to interact with the report, using the characteristics as filter conditions. The reports could in turn introduce students to descriptions of the progress of their work, providing a broader view than is achieved when analyzing individual tasks.

Another path that might be integrated as part of enhancing the opportunities for meaningful student mathematical inquiry in classrooms is working in small groups that create, share, and reflect on their collective example space. As described in Abdu et al. (2022), we grouped students according to the analysis of their respective examples with a content-specific objective to foster students' development of their personal example spaces. This diversity of personal example spaces indicates the potential of example-eliciting tasks and associated analytics in deploying content-specific grouping of students (Olsher, 2022). We are studying ways to provide such groups of students' collective information about their work before the grouping and in the course of the group work. This will give students opportunities to interact with the descriptions that led to their grouping and provide them with additional means to describe mathematically and distinguish between their respective submissions, and to find commonalities in their work.

Finally, deepening the meaning of feedback processes in inquiry learning can combine the meta-cognitive practice of self-assessment with automated assessment. *Self-reflection* or *self-assessment by students* is important for supporting their meta-cognitive skills, and places learners in a key position where they can develop responsibility and ownership for their learning (Ruchniewicz and Barzel, 2019). We continue to study how automated online information reports can become part of the meta-cognitive feedback process (Kadan-Tabaja and Yerushalmy, 2022.)

As these examples illustrate, we have only just begun to identify the potential of automatically characterized student examples as part of feedback processes, but there is much room for continued innovation in the design of such feedback.

Acknowledgments

This research was supported by the Israel Science Foundation (grant 147/18).

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Freudenthal Ideas Continues in Indonesia: From ICMI 1994 to ICME-14 in Shanghai

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ABSTRACT This paper shares several realistic mathematics education (RME) projects designed and implemented at the Department of Mathematics Education, Universitas Sriwijaya, Indonesia. These projects are developed using the design research method. They are based on the foundational work of Freudenthal and his successors. The development of the PMRI approach (the Indonesian version of RME) started at an ICMI Regional Conference in Shanghai in 1994, where Robert Sembiring met Jan de Lange. We will also briefly reflect on how RME was adapted in Indonesia, inspiring research and development in mathematics education. The focus of this paper will be on (1) designing a learning environment within the PMRI approach to support students' learning mathematics literacy; (2) designing an international journal on mathematics education; (3) creating PISA-like tasks on mathematics using the Indonesian context; and (4) our way of surviving the current COVID-19 context. I will discuss these issues and illustrates their examples from PMRI practices.

Keywords: Design research; PISA-like; PMRI; COVID-19 context.

1. Introduction

1.1. From ICMI 1994 to ICME 2021

Indonesia started reforming mathematics education by adapting RME in 1994. As the head of the reform team, Robert Sembiring, an Indonesian mathematician, saw Jan de Lange give a speech about RME as the plenary speaker at the International Commission of Mathematics Instruction (ICMI) conference in Shanghai. As one of the successors of Freudenthal, Jan agreed to introduce RME in Indonesia. At that time, like many countries, Indonesia changed its teaching and learning approach from Modern Mathematics influenced by New Math. Finally, Jan visited Indonesia twice, in 1998 and 2000.

The scenario continued in 1998 when six lecturers, including myself, were selected to go to the Netherland to learn RME at both University of Twente and Freudenthal Institute for Mathematics and Science Education Utrecht University. We know what,

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why, and how RME as an instructional theory, design research method, and curriculum development. The Ph.D. program spans four years, and we conduct research in schools in Indonesia. After graduating in 2002, we returned to Indonesia and joined a team that started the project called PMRI.

In 2000, Jan de Lange gave the keynote presentation at the Institute Teknologi Bandung as a keynote at the Tenth National Conference on Mathematics. He presented the audience with two essential new topics: RME and Programme for International Student Assessment (PISA). In 2000, the first PISA was administered. As the PISA Expert Group on Mathematics leader, he stated that PISA and RME have a solid mathematical connection (Zulkardi, 2020a).

1.2. Freudenthal and Jan de lange ideas

Three well-known ideas of Freudenthal were described in this paper that are RME, developmental research, and the journal Educational Studies in Mathematics. Moreover, the idea of a PISA-like task was inspired by Jan de Lange, the successor of Freudenthal. As the head team of PISA Mathematics and Ph.D. promotor, Jan guided that PISA study needs many suitable tasks at all difficulty levels.

The primary purpose of this paper is to report on several projects relating to PMRI at the department doctoral study program in mathematics education at Universitas Sriwijaya in Palembang, Indonesia. Freudenthal inspired three ideas in this paper described here are (1) use of design research approach in order to produce learning environments and RME learning materials on various mathematical contents using relevant Indonesian and global contexts; (2) creating PISA-like tasks on mathematics using Indonesian contexts and COVID-19 context; and (3) developing an academic journal to publish results of research primarily related PMRI to all over the world.

2. PMRI Continues

2.1. PMRI and design research continues in Indonesia

PMRI approach and Design Research method continue in Indonesia. At Universitas Sriwijaya, PMRI is taught in all levels of mathematics education study programs, namely undergraduate and postgraduate (master and doctoral) programs. In these courses, pre-service teachers collaborate with teachers at PMRI partner schools to design teaching materials that can be used in classroom learning.

After the IMPoME end in 2015, research on PMRI in Universitas Sriwijaya continues in the magister and Ph.D. programs. The thesis uses design research as a research methodology. Up to now, seven dissertations have been published in the International journal. The following is the summary of the Ph.D. theses which used PMRI and design research as an umbrella for research projects.

1) *Task design on modelling in senior high school mathematics* (Riyanto et al., 2019)

This research produced valid and practical high school mathematics modeling tasks, a lesson plan, and a student worksheet. The products also have potentially effective. According to the findings of this study, students were highly passionate about studying mathematics using mathematical modeling, and students could create mathematical models using their strategies. Students can use the modeling process, which increases their mathematics literacy.

2) *Developing mathematics worksheet using futsal context* (Effendi et al., 2019)

This study supports one of the government's School Literacy Movement efforts to improve students' literacy skills. This study generates reading texts in the context of futsal that will be presented in student activities and have potential effects on mathematics learning. Learning that begins with the activity of reading texts linked to the mathematical subject increases students' interest in learning and improves their literacy skills because the given context is attractive to them, and learning becomes more diversified.

3) *Developing PMRI learning environment through lesson study for pre-service primary school teachers* (Fauziah et al., 2020)

This study's development process resulted in a learning environment based on a Campus-School model (CS). The learning environment consisted of the first and second training on campus and implementation in the school. PMRI and lesson study materials were used in training and two PMRI learning simulations via lesson study, discussion, and development of learning tools, peer teaching, application of learning to lesson study model schools, final discussions, and tests. The PMRI learning environment is a valid, practical criterion and enhances the pedagogical abilities of pre-service primary school teachers.

4) *Learning integers with RME approach based on Islamic values* (Muslimin et al., 2020)

The integer learning trajectory based on Islamic values proved helpful in helping students comprehend numbers. The established learning pathways include four phases: beginning with a presentation of the context-based on Islamic principles, Iqra (literacy) to grasp the situation, resolving the context individually and in groups to construct an informal model to formal, and communicating with the presentation. Furthermore, the learning trajectory of integers with the context of the starting point based on Islamic values sharpens students' reasoning power and forms good character.

5) *On creativity through mathematization in solving non-routine problems* (Arifin et al., 2021)

This study aimed to analyze and compare students' fluency, flexibility, and originality in solving non-routine tasks in the Palembang context. The data analysis revealed that the answers supplied by the high-ability students were unique and tended to employ formal mathematics in the form of formulae, symbols, and operations. Meanwhile, moderate-ability students preferred to begin solving issues by simplifying them and then visualizing them. This study found that low-ability students had difficulty grasping the questions and made several errors in completing them.

6) *Designing geometrical learning activities for supporting students' higher order thinking skills* (Meryansumayeka et al., 2022)

The study aimed to develop a cuboid volume learning trajectory in ICT-assisted learning to improve students' higher-order thinking skills. The cuboid volume learning trajectory includes activities about the relationship between visible and invisible parts for determining the cuboid volume, the relationship between the cuboid's sides, making cuboids with a specific volume size and determining the size of the cuboid's sides, and activities to solve related problems with the cuboid volume. The ICT media employed in this study are critical in enhancing students' higher-order thinking skills.

7) *Curious mind uses mini games for early childhood in early mathematics learning* (Rahayu et al., 2022)

This study explicitly discusses how the PMRI mathematics introduction learning track uses mini-games to support children's curiosity. This study uses a Realistic Mathematics Approach at the food learning process stage called "pempek" as a starting point in introducing basic mathematics learning materials. The goal is to design learning activities and observe children's curiosity. This research resulted in a learning trajectory for introducing length and volume measurements using mini-games. The learning trajectory for submitting length measurements and volume measurements can help early childhood learn the introduction of measures and activities and games designed to support children's curiosity about introducing mathematics to measurement materials.

2.2. The first international journal on mathematics education in Indonesia

Inspired by Freudenthal's idea of the importance of academic journals in publishing research results, we started an international journal in mathematics education called JME (Journal on Mathematics Education) in 2010. This idea is easily supported by the Indonesian Mathematical Society (IndoMS), not because I am the vice president of IndoMS, but because there is no international journal in mathematics education (Zulkardi, 2019). JME was launched on July 31, 2010, at the beginning of the Fifteenth National Conference on Mathematics (KNM15) at the University of Manado, North Sulawesi. JME is dedicated to publishing research articles on mathematics education by school mathematics teachers, teacher educators, and university students.

A number of authors contribute and publish their articles in JME. They are P. Y. Lee (2010) and B. Kaur (2014) from Singapore, K. Stacey (2011) and T. Lowrie (2018) from Australia, K. Gravemeijer (2011), Galen and Eerde (2013) from the Netherlands, C. Kaune, and E. Nowinska (2013) from Germany, and F. L. Lin (2014) from China (Taiwan Region). They were a part of the beginning of this journal pioneered. Interestingly, up to Volume 7, published in January 2016, 47 (58 percent) of the 81 published articles concerned RME or PMRI. JME might alternatively be referred to as "JRME" (Journal on Realistic Mathematics Education). Furthermore, the growth in RME papers indicates the continuity of RME research in Indonesia. The journal is also indexed in DOAJ, ERIC Database, Google Scholar, and Scopus. All articles are freely available at http://ejournal.unsri.ac.id/index.php/jme.

2.3. Designing PISA-like tasks on mathematics using Indonesian context

COVID-19 has implications for crises and disruptions in education, including significant changes in mathematics education (Bakker et al., 2021). This change in teaching and learning includes alignment with learning objectives, teaching approaches, teaching materials (activities and assessments), and an emphasis on the achievement of student competencies (Gravemeijer et al., 2017; Chan et al., 2021). In Indonesia, the Minister of Education and Culture has taken a bold step by launching the "Free Learning Program." One of them is replacing the National Examination with a Minimum Competency Assessment (MCA) which focuses on numeracy and literacy (MoEC, 2019); (MoEC, 2020a). MCA questions refer to international level assessments such as PISA (MoEC, 2020a, 2020b).

Based on the low PISA results of Indonesian students, there is an urgent need to provide problems that fit the PISA criteria by adjusting the situation for Indonesian students (Zulkardi and Kohar, 2018). The COVID-19 Pandemic is an ongoing widespread issue affecting all sectors of global life and students' academic activities (Bakker and Wagner, 2020). Every day, this situation is emphasized and reported in various media to remind and increase awareness of the essentials of health procedures (Nusantara et al., 2021). The COVID-19 situation, officially verified cases, death counts, and transmission categorization are published daily in the form of a map, epidemic curve, and table, allowing readers to think mathematically about the most recent numbers and trends at the global, national, and regional levels. As a result, this unfortunate circumstance might be exploited to teach mathematics.

Research on PISA items is still being developed, with the Bangka (Dasaprawira et al., 2019) and Asian Games as contexts (Putri and Zulkardi, 2020). However, no study has used COVID-19 to create a PISA-like task on mathematics (PISAComat). Fig. 1 depicts the PISACOmat shown below.

PISAComat was created by modifying the original PISA items and changing the context from "Climbing Mount Fuji" to "Daily Data on COVID-19 in Indonesia". Based on PISA content, "Quantity" is associated with 2013 curriculum topics, such as arithmetic operations and rounding decimal numbers. Furthermore, the cognitive level of thinking in PISAComat is at level 3, which is reasoning.

DAILY COVID-19 DATA IN INDONESIA

Given the information in the following picture.

Source: https://covid19.go.id/

COVID-19 Response Acceleration Task Force showed the following picture and said:

"On September 11, 2020, an average of four people died every hour in Indonesia due to COVID-19"

Question

Do you agree with the statement? Give an explanation to support your answer.

Fig. 1. PISAComat on quantity content

PISAComat (as shown in Fig. 1) design process continued with task experiment and followed a formative evaluation (Zulkardi, 2002; Bakker, 2018). The following is a sample of PISAComat student responses.

Fig. 2(a) and Fig. 2(b) show the solutions given by students to answer PISAComat. (S)he understands that 106 people have died from COVID-19 in one day. Furthermore, (s)he calculates using the concept of a division of numbers, namely 106 divided by 24

Pernyataannya benar, ada penambahan kasus meninggal sebanyak 106 orang dalam 1 hari.
Selisih waktu = 24 jam. (12 − 11 september = 1 hari)
Dengan demikian, banyaknya kasus meninggal akibat COVID-19 tiap jamnya adalah :
$\frac{106}{24} = 4{,}4 \approx 4$ orang/jam (pembulatan)
Rata-rata ada 4 orang orang tiap jamnya.

(a)

The statement is true; there are additional death cases of as many as 106 people in 1 day.
Time difference = 24 hours (12−11 September = 1 day)
Thus, the number of deaths due to COVID-19 every hour is:
106/24 = 4.4 ≈ 4 people/hour (rounded up)
On average, there are four people (who die) every hour

Salah, karena 8.650 : 24 = 360,4
Artinya, dalam 1 hari ada 360,4 orang yang meninggal akibat COVID-19.

(b)

It's a wrong statement because 8,650:24 = 360.4
This means, in 1 day, there are 360.4 people who died from COVID-19

Fig. 2. Students' solutions on PISAComat

(the number of hours in a day), to get the result of 4.4 people. (S)he then rounds the decimal number to integer (4). In contrast to student 2a, student 2b does not understand the symbol's meaning for adding pictures. (S)he counts the total number of deaths (8,650) and divides it by 24 (the number of hours in a day) so that it gets 360,4. Furthermore, (s)he cannot round decimal numbers and only interpret cases in the form of decimal numbers.

2.4. Ways of surviving the current COVID-19 context

Uncertainty and COVID-19 data as starting point in learning mathematics

Various COVID data and uncertain situations are spread across various media. COVID data in the form of infographics is very interesting to teach students to interpret mathematically. This uncertain situation needs to be taught by students to invite students' awareness to survive and coexist with COVID-19. Various mathematical contexts are related to COVID-19, including COVID-19 Data and Making Hand Sanitizer (Zulkardi et al., 2020b), Physical Distancing in Public Place (Nusantara et al., 2020a), COVID-19 Data Interpretation on TV (Nusantara et al., 2020b) Large Scale Social Restriction and Panic Buying Context (Nusantara et al., 2021a), COVID-19 Transmission Map (Nusantara et al., 2021b).

Opportunity to learn new knowledge as props in learning mathematics

The effects of COVID-19 provide an opportunity to learn new knowledge. Various situations demand interesting adjustments for students to learn mathematics. Before the COVID-19 Pandemic, teachers taught measurements using teaching props such as thermometers, rulers, and scales. However, during the COVID-19 Pandemic, teachers can teach measurements using a pulse oximeter and heat thermometer gun, CT-value of PCR test, etc.

The Art of mathematics in the form of posters

For three years in a row, the Doctoral Program in Mathematics Education has celebrated Pi Day (March 14) in conjunction with its anniversary celebration. The International Mathematical Union started this activity by conducting a Poster Challenge in 2021. Students from the Sriwijaya University mathematics education study program participated in the poster challenge. Zulkardi and Meryansumayeka mentored three teams who placed in the top 90 posters out of over 2,100 competitors. All poster items may be seen at https://www.idm314.org/2021-poster-challenge-gallery.html#.

Fig. 3 depicts three selected posters from Indonesia still on exhibit on the International Day of Mathematics (IDM) website. The three posters introduce elements

Fig. 3. Winning posters at IDM 2020 designed by student teachers mathematics department Universitas Sriwijaya

of mathematics and COVID-19 in the same concept. Students may use this poster to learn mathematics and, at the same time, make sense of COVID-19 data.

3. Conclusion

Three great ideas from Freudenthal (RME, Design Research, and ESM) and Jan de Lange (PISA study) are continually adapted and implemented by Zulkardi as the head of the PMRI team at Universitas Sriwijaya, Indonesia. RME and design research have been settled in the courses and research at the master and doctoral programs in Universitas Sriwijaya. Also, the new study, PISA-like tasks designed on mathematics, which is used the global context, COVID-19. Finally, the development of an International Journal on Mathematics Education (JME) has reached the top journal in Asia or the top ten globally, ranked by Scimago journal.

Acknowledgments

This article is supported by the research project PNBP Universitas Sriwijaya 2020-2021. Thanks to Ratu Ilma and Duano Nusantara for helping and reading this paper.

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https://doi.org/10.1142/9789811287183_0051

Appendix

Name List of Invited Lecturers

This Appendix lists all participants who actually gave invited lectures in the ICME-14 (on-line or on-site), according to the conference video recording, and all lecturers who contributed to this volume (indicated with asterisks), together with his/her co-authors, if any.

No.	Lecturer	Gender	Country	Co-author(s)
1	*Dor Abrahamson	Male	USA	
2	*Takuya Baba	Male	Japan	
3	*Nicolas Balacheff	Male	France	
4	*Richard Barwell	Male	Canada	
5	*Robert. Q. Berry III	Male	USA	Basil M. Conway IV, Brian R. Lawler, John W. Staley
6	Kim Beswick	Female	Australia	
7	*Jill Patricia Brown	Female	Australia	
8	*Yiming Cao	Male	China	
9	*Cheng Meng Chew	Male	Malaysia	Huan Chin
10	*Anna Chronaki	Female	Greece	
11	*Alison Clark-Wilson	Female	UK	
12	*Jaguthsing Dindyal	Male	Singapore	
13	*Lianghuo Fan	Male	China	
14	*Ahmad Fauzan	Male	Indonesia	Rafki Nasuha, Afifah Zafira
15	Patricio Felmer	Male	Chile	
16	*Claudia Regina Flores	Female	Brazil	
17	Megan Franke	Female	USA	
18	Maisie Gholson	Female	USA	
19	*Keiko Hino	Female	Japan	

20	*Rongjin Huang	Male	USA	
21	Roberta Hunter	Female	New Zealand	
22	Chunlian Jiang	Female	China (Macao, SAR)	
23	*Houssam Kasti	Male	Lebanon	
24	*Miheso O'Connor Marguerite Khakasa	Female	Kenya	
25	*Tinne Hoff Kjeldsen	Female	Denmark	
26	*Oleksandr Kryzhanovskiy	Male	Ukraine	
27	*Ngan Hoe Lee	Male	Singapore	
28	Shuk-kwan Leung	Female	China (Taiwan Region)	
29	*Jun Li	Female	Australia	Xinfeng Huang, Hua Huang
30	*Zhongru Li	Male	China	Chaoran Gou
31	*Di Liu	Female	China	Yan Feng, Ziyi Chen, Rui Kang, Yifan Zuo
32	*Po-Hung Liu	Male	China (Taiwan Region)	
33	*Rachel Ka Wai Lui	Female	China (Hong Kong, SAR)	
34	Fernand Malonga	Male	Congo (Brazzaville)	
35	*Mirko Maracci	Male	Italy	
36	*Salomé Martínez	Female	Chile	Farzaneh Saadati, Paulina Araya, Eugenio Chandía, Daniela Rojas
37	*Pietro Di Martino	Male	Italy	
38	*Vilma Mesa	Female	USA	
39	*Judit N. Moschkovich	Female	USA	
40	*Reidar Mosvold	Male	Norway	
41	Chi Thanh Nguyen	Male	Vietnam	
42	*Núria Planas	Female	Spain	
43	*Susanne Prediger	Female	Germany	
44	*Ana Isabel Sacristan	Female	Mexico	
45	*Veronica Sarungi	Female	Tanzania	
46	*Baruch Schwarz	Male	Israel	Nadav Marco

47	*Björn Schwarz	Male	Germany	
48	*Hyunyong Shin	Male	Korea	
49	*Sophie Soury-Lavergne	Female	France	
50	*Marilyn. E. Strutchens	Female	USA	Brea Ratliff
51	*Francis Edward Su	Male	USA	
52	*Konstantinos Tatsis	Male	Greece	
53	*Alphonse Uworwabaheyo	Male	Rwanda	
54	*Hamsa Venkat	Female	South Africa	
55	*Debbie Marie Verzosa	Female	Philippines	
56	*Mónica. E. Villarreal	Female	Argentina	
57	*Bin Xiong	Female	China	Yijie He
58	Xinrong Yang	Male	China	
59	*Michal Yerushalmy	Female	Israel	Daniel Chazan, Shai Olsher
60	*Zulkardi Zulkardi	Male	Indonesia	

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https://doi.org/10.1142/9789811287183_bmatter

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